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The minimal entropy martingale measures for exponential additive processes revisited

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Abstract. The mimimal entropy martingale measure for the stochastic process defined as the exponential of an additive process with the structure of semimartingale will be investigated. Special attention will be paid to the case when the underlying additive process has fixed times of discontinuity. The investigation of this paper will establish a unified way that is applicable both to the case of Lévy processes and that of the sums of independent random variables.

Keywords. additive process, process with independent increments, semimartingale, minimal entropy martingale measure, exponential moment, Laplace cumulant, modified Laplace cumulant

1. Introduction

Let $(X_t = (X_t^1, \ldots, X_t^d))_{t\in[0,T]}$, $T > 0$, be an $\mathbb{R}^d$-valued additive process with the structure of semimartingale, in other words, a d-dimensional semimartingale with independent increments. According to [7], we will also call such a stochastic process as (X_t) a d-dimensional PII-semimartingale. We always suppose that $X_0 = 0$.

Let $(S_t = (S_t^1, \ldots, S_t^d))_{t\in[0,T]}$ be a stochastic process defined as the exponential of (X_t):

$$S_t^i := S_0^i \,\mathrm{e}^{X_t^i}, \quad i = 1, \ldots, d$$

where we suppose that each S_0^i is a positive constant. We call such a stochastic process as (S_t) a d-dimensional exponential additive process based on the additive process (X_t) or, for simplicity, an exponential PII-semimartingale (based on (X_t)).

The purpose of this paper is to propose a condition under which the mimimal entropy martingale measure (MEMM) for (S_t) exists and to represent the MEMM explicitly by the characteristics of (X_t). In a series of previous papers, we have discussed this problem in the case when

1. (X_t) is a Lévy process in [5] and [2];
2. (X_t) is a stochastically continuous PII-semimartingale in [3],

respectively. In this paper, we are interested in the case when (X_t) is a PII-semimartingale that is not necessarily stochastically continuous, in other words, that may have fixed times of discontinuity. Hence, we can say that the aim of this paper is to give a final answer to the problem described above in the framework of exponential PII-semimartingales.

In Section 2, we will review several properties of PII-semimartingales. In particular, Theorem 1 plays fundamental role in showing Corollary 1 that ensures the existence of exponential moments of integrals of deterministic processes based on (X_t) and that gives the representation of them by the characteristics of (X_t).

In Section 3, we will precisely state our main result of this paper, Theorem 2, where the existence and the representation of the MEMM for (S_t) will be shown under a mild condition (C). Owing to removing the restriction of stochastic continuity of the underlying PII-semimartingale (X_t), we can also treat the case when it is defined by a sum of independent random variables in a unified framework. See Corollary 2.

In Section 4, we will give a proof of Theorem 2. It follows on the stream proposed in the proof of Theorem 3.1 in [5]. However, we will see that suitable modification of discussions and deeper consideration will be needed to overcome the difficulties arising from the existence of fixed times of discontinuity.

2. Additive processes and exponential additive processes

Additive process with the structure of semimartingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)$ that satisfies the usual conditions. See [7] I.1.2 (p.2) for the definition of the usual conditions. Let $(X_t = (X_t^1, \ldots, X_t^d))_{t\in[0,T]}$, $T > 0$, be an $\mathbb{R}^d$-valued additive process with the structure of semimartingale defined on the probability space $(\Omega, \mathcal{F}, P)$ with $(\mathcal{F}_t)$. To be precise, (X_t) is an $\mathbb{R}^d$-valued adapted càdlàg process with $X_0 = 0$ that has the following properties:

1. (X_t) is a **p**rocess with **i**ndependent **i**ncrements: for all $s \le t$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s$;

2. (X_t) is a **semimartingale** with respect to the filtration $(\mathcal{F}_t)$.

According to [7], we will also call such a stochastic process as (X_t) a d-dimensional **PII-semimartingale**.

We would like to emphasize that throughout this paper we do *not necessarily* assume that (X_t) is stochastically continuous. Note that, in our scheme, the stochastic continuity is equivalent to the property of having no fixed time of discontinuity and also to the quasi-left continuity. See [6] Corollary 11.28 (p.308) and [7] Theorem II.4.18 (p.107).

Let $(C_t, n(dtdx), B_t)$ be the characteristics of (X_t) associated with the truncation function $h(x) := xI_{\{|x|\le 1\}}(x)$ on $\mathbb{R}^d$.

In other words, let the canonical representation of (X_t) (associated with h) be given as follows ([7] Theorem II.2.34 (p.84)):

$$X_t = X_t^c + B_t + \int_{(0,t]}\int_{\mathbb{R}_0^d} h(x)\,\widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}_0^d} \check{h}(x)\,N(dudx), \tag{1}$$

where $\mathbb{R}_0^d := \mathbb{R}^d \backslash \{0\}$. Here,

- (X_t^c) is a continuous (local) martingale with $X_0^c = 0$ and $C_t^{ij} = \left\langle X^{c,i}, X^{c,j}\right\rangle_t$ for $i, j = 1, \ldots, d$.

- $N(dudx)$ denotes the counting measure of the jumps of (X_t):

  $$N((0,t], A) := \sharp\{u \in (0,t];\ \Delta X_u := X_u - X_{u-} \in A\}$$

  for $A \in \mathcal{B}(\mathbb{R}_0^d)$, where $X_{u-} := \lim_{v\uparrow u} X_v$ and $\mathcal{B}(\mathbb{R}_0^d)$ is the Borel σ-field on $\mathbb{R}_0^d$.

  We denote by $\widetilde{N}(dudx) := N(dudx) - n(dudx)$ the compensated measure of $N(dudx)$, where $n(dudx)$ is the compensator of $N(dudx)$. Also, we set $\check{h}(x) := x - h(x)$.

- Each component (B_t^i) $(i = 1, \ldots, d)$ is a càdlàg function with finite variation on $[0, T]$. ([7] Definition II.2.6 (p.76))

As fundamental properties of characteristics, the folllowing facts are known:

- $(C_t, n(dtdx), B_t)$ are deterministic, since (X_t) has independent increments. ([7] Theorem II.4.15 (p.106))

- $\int_{(0,T]}\int_{\mathbb{R}_0^d}(|x|^2 \wedge 1)\,n(dudx) < \infty$, where $\alpha \wedge \beta := \min\{\alpha, \beta\}$ for $\alpha, \beta \in \mathbb{R}$, and $n(\{u\}, \mathbb{R}_0^d) \le 1$. ([7] II.2.13 (p.77))

- $\Delta B_u = \int_{\mathbb{R}_0^d} h(x)\,n(\{u\}, dx)$. ([7] II.2.14 (p.77))

Also, note that the law of (X_t) is characterized by the Lévy-Khinchin formula ([7] Theorem II.4.15 (p.106)):

$$\begin{aligned} &E^P[\mathrm{e}^{\sqrt{-1}\,\xi\cdot(X_t - X_s)}] \\ &= \exp\Big[-\frac{1}{2}\xi\cdot(C_t - C_s)\xi + \sqrt{-1}\,\xi\cdot(B_t - B_s) \\ &\qquad + \int_{(0,t]}\int_{\mathbb{R}_0^d}\big(\mathrm{e}^{\sqrt{-1}\xi\cdot x} - 1 - \sqrt{-1}\,\xi\cdot h(x)\big) \\ &\qquad\qquad \times I_{J^c}(u)\,n(dudx)\Big] \\ &\quad \times \prod_{u\in(s,t]}\Big\{\mathrm{e}^{-\sqrt{-1}\,\xi\cdot\Delta B_u} \\ &\qquad\qquad \times \big[1 + \int_{\mathbb{R}_0^d}\big(\mathrm{e}^{\sqrt{-1}\xi\cdot x} - 1\big)n(\{u\}, dx)\big]\Big\}, \end{aligned}$$

where $a\cdot b$ denotes the inner product of $a, b \in \mathbb{R}^d$; J denotes the set of all fixed times of discontinuity of (X_t), that is, $J := \{t > 0;\ n(\{t\}, \mathbb{R}_0^d) > 0\}$.

1-dimensional PII-semimartingale and the exponential moment

Let $(Y_t)_{t\in[0,T]}$, $T > 0$, be a 1-dimensional PII-semimartingale and $(C_t^Y, n^Y(dtdy), B_t^Y)$ the characteristics of (Y_t) associated with the truncation function $h_1(y) := yI_{\{|y|\le 1\}}(y)$ on $\mathbb{R}$.

In [4], we have shown the following result with an explicit proof. See Theorem 1 therein.

Theorem 1. *Suppose that*

$$\int_{(0,T]}\int_{\{y>1\}} \mathrm{e}^y\, n^Y(dudy) < \infty. \tag{2}$$

Then, $(\mathrm{e}^{Y_t - K^Y(1)_t})_{t\in(0,T]}$ *is a uniformly integrable martingale with mean 1, where* $(K^Y(1)_t)$ *is the modified Laplace cumulant of* (Y_t) *at 1:*

$$\begin{aligned} &K^Y(1)_t \\ &= \frac{1}{2}C_t^Y + B_t^Y \\ &\quad + \int_{(0,t]}\int_{\mathbb{R}_0}\big(\mathrm{e}^y - 1 - h_1(y)\big)\,n^Y(dudy) \\ &\quad + \sum_{u\in(0,t]}\Big\{\log\big(1 + \int_{\mathbb{R}_0}(\mathrm{e}^y - 1)\,n^Y(\{u\}, dy)\big) \\ &\qquad\qquad - \int_{\mathbb{R}_0}(\mathrm{e}^y - 1)\,n^Y(\{u\}, dy)\Big\}. \end{aligned} \tag{3}$$

In particular,

$$E[\mathrm{e}^{Y_t}] = \mathrm{e}^{K^Y(1)_t}. \tag{4}$$

Integral based on the d-dimensional PII-semimartingale (X_t):

Let $(\theta_u = (\theta_u^1, \ldots, \theta_u^d))$ be an $\mathbb{R}^d$-valued Borel measurable function. Note that it is deterministic. We say that (θ_u) is integrable with respect to (X_t) if the following conditions (i)∼(iii) are satisfied:

(i) $\int_{(0,T]} \theta_u\, dC_u \theta_u := \sum_{i,j=1}^{d} \int_{(0,T]} \theta_u^i\, dC_u^{ij} \theta_u^j < \infty$,

(ii) $\sum_{i=1}^{d} \int_{(0,T]} |\theta_u^i|\, d(\mathrm{Var}(B^i))_u < \infty$, where $\mathrm{Var}(A)_t$ denotes the total variation of the function (A_u) on the interval $[0,t]$,

(iii) $\int_{(0,T]} \int_{\mathbb{R}_0^d} |\theta_u \cdot h(x)|^2\, n(dudx) < \infty$.

We denote by $L(X)$ the set of all integrable functions with respect to (X_t). Note that an arbitrary bounded measurable function belongs to $L(X)$.

Let $(\theta_u) \in L(X)$. Then, we can define an integral $(\int_{(0,t]} \theta_u \cdot dX_u)$ of (θ_u) based on (X_t) by

$$\begin{aligned}\int_{(0,t]} \theta_u \cdot dX_u := &\int_{(0,t]} \theta_u \cdot dX_u^c + \int_{(0,t]} \theta_u \cdot dB_u \\ &+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot h(x)\, \widetilde{N}(dudx) \\ &+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot \check{h}(x)\, N(dudx).\end{aligned} \tag{5}$$

The following result is shown as Proposition 1 in [4]:

Proposition 1. *Let $(\theta_u) \in L(X)$. Then $(Y_t := \int_{(0,t]} \theta_u \cdot dX_u)$ is a 1-dimensional PII-semimartingale; the characteristics (C^Y, n^Y, B^Y) (associated with h_1 on $\mathbb{R}$)) are given by*

$$\begin{aligned} C_t^Y &= \int_{(0,t]} \theta_u dC_u \theta_u; \\ n^Y((0,t], A) &= \int_{(0,t]} \int_{\mathbb{R}_0} I_A(\theta_u \cdot x)\, n(dudx),\ A \in \mathcal{B}(\mathbb{R}_0); \\ B_t^Y &= \int_{(0,t]} \theta_u \cdot dB_u \\ &\quad + \int_{(0,t]} \int_{\mathbb{R}_0^d} \big(h_1(\theta_u \cdot x) - \theta_u \cdot h(x)\big)\, n(dudx) \end{aligned}$$

EXPONENTIAL MOMENT OF $(\int_{(0,t]} \theta_u \cdot dX_u)$:

The following result is shown as Corollary 1 in [4]:

Corollary 1. *Let $(\theta_u) \in L(X)$ and suppose that*

$$\int_{(0,T]} \int_{\{\theta_u \cdot x > 1\}} \mathrm{e}^{\theta_u \cdot x}\, n(dudx) < \infty. \tag{6}$$

Then, $(\mathrm{e}^{\int_{(0,t]} \theta_u \cdot dX_u - K^X(\theta)_t})_{t \in (0,T]}$ is a uniformly integrable martingale with mean 1, where $(K^X(\theta)_t)$ is the modified Laplace cumulant of (X_t) at (θ_u):

(7)
$$\begin{aligned} K^X(\theta)_t = &\frac{1}{2} \int_{(0,t]} \theta_u\, dC_u\, \theta_u + \int_{(0,t]} \theta_u \cdot dB_u \\ &+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \big\{\mathrm{e}^{\theta_u \cdot x} - 1 - \theta_u \cdot h(x)\big\}\, n(dudx) \\ &+ \sum_{u \in (0,t]} \Big\{ \log\Big(1 + \int_{\mathbb{R}_0^d} \big(\mathrm{e}^{\theta_u \cdot x} - 1\big)\, n(\{u\}, dx)\Big) \\ &\qquad - \int_{\mathbb{R}_0^d} \big(\mathrm{e}^{\theta_u \cdot x} - 1\big)\, n(\{u\}, dx)\Big\}. \end{aligned}$$

In particular,

$$E[\mathrm{e}^{\int_{(0,t]} \theta_u \cdot dX_u}] = \mathrm{e}^{K^X(\theta)_t}. \tag{8}$$

Remark 1. If (θ_u) is bounded, the condition (6) can be replaced by the one

$$\int_{(0,T]} \int_{\{|x|>1\}} \mathrm{e}^{\theta_u \cdot x}\, n(dudx) < \infty.$$

Remark 2. The sum of the first three terms in the right hand side of (7) is called the Laplace cumulant of (X_t) at (θ_u) and denoted by $\widetilde{K}^X(\theta)_t$:

(9)
$$\begin{aligned} \widetilde{K}^X(\theta)_t = &\frac{1}{2} \int_{(0,t]} \theta_u\, dC_u\, \theta_u + \int_{(0,t]} \theta_u \cdot dB_u \\ &+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \big\{\mathrm{e}^{\theta_u \cdot x} - 1 - \theta_u \cdot h(x)\big\}\, n(dudx). \end{aligned}$$

The modified Laplace cumulant and the Laplace cumulant are related to each other through the following relation:

$$\mathrm{e}^{K^X(\theta)_t} = \mathcal{E}(\widetilde{K}^X(\theta))_t.$$

See [8] and [7] for fundamental properties of the (modified) Laplace cumulant in the framework of the theory of semimartingales.

EXPONENTIAL ADDITIVE PROCESS

Let $S_t = (S_t^1, \ldots, S_t^d)$ be an $\mathbb{R}^d$-vauled stochastic process defined by

$$S_t^i := S_0^i \mathrm{e}^{X_t^i}, \quad i = 1, \ldots, d, \tag{10}$$

where we will asuume that all of S_0^i are positive constants. We call (S_t) the exponential additive process based on the additive process (X_t). For simplicity, we will also call it the exponential PII-semimartingale (based on (X_t)).

Then, it follows from Itô's formula that

$$S_t^i = S_0^i + \int_{(0,t]} S_{u-}^i d\widehat{X}_u^i,$$

where

$$\widehat{X}_t^i := X_t^i + \frac{1}{2}\langle X^{i,c}\rangle_t + \sum_{u \in (0,t]} \big\{\mathrm{e}^{\Delta X_u^i} - 1 - \Delta X_u^i\big\}. \tag{11}$$

Combining this definition with (1), we see that the canonical representation of $(\widehat{X}_t)$ (associated with h) is given as follows:

Proposition 2.

(12)
$$\begin{aligned}\widehat{X}_t = X_t^c + \Big\{B_t + \frac{1}{2}\overline{C}_t \\ + \int_{(0,t]}\int_{\mathbb{R}_0^d}\big\{h(E(x)-I) - h(x)\big\}\,n(dudx)\Big\} \\ + \int_{(0,t]}\int_{\mathbb{R}_0^d} h(E(x)-I)\,\widetilde{N}(dudx) \\ + \int_{(0,t]}\int_{\mathbb{R}_0^d} \check{h}(E(x)-I)\,N(dudx),\end{aligned}$$

where we set $E(x) := (\mathrm{e}^{x^1}, \dots, \mathrm{e}^{x^d})$ *for* $x = (x^1, \dots, x^d)$, $I := (\underbrace{1, \dots, 1}_{d}) \in \mathbb{R}^d$ *and*

$$\overline{C}_t^i := C_t^{ii}. \tag{13}$$

It is also immediate from Proposition 2 that the following proposition holds:

Proposition 3. *The characteristics* $(\widehat{C}, \widehat{n}, \widehat{B})$ *of* $(\widehat{X}_t)$ *(associated with* h*) are given by*

$$\widehat{C}_t = C_t; \tag{14}$$

$$\widehat{n}((0,t], G) = \int_{(0,t]}\int_{\mathbb{R}_0^d} I_G(E(x)-I)\,n(dudx), \quad G \in \mathcal{B}(\mathbb{R}_0^d); \tag{15}$$

$$\widehat{B}_t = B_t + \frac{1}{2}\overline{C}_t + \int_{(0,t]}\int_{\mathbb{R}_0^d}\big\{h(E(x)-I) - h(x)\big\}\,n(dudx). \tag{16}$$

3. The minimal entropy martingale measures for exponential additive processes

We will use the same notation as in Section 2. In particular, recall that $(X_t)_{t\in[0,T]}$ is a d-dimensional PII-semimartingale, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions, with characteristics $(C_t, n(dtdx), B_t)$ (associated with the truncation function $h(x)$). Moreover, (S_t) denotes the exponential PII-semimartingale defined by (10).

In this section, we will precisely state our main result (Theorem 2 below). To this end, we prepare some notion.

Minimal entropy martingale measure

For a probability measure Q on the measurable space $(\Omega, \mathcal{G})$, where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$, the relative entropy of Q on $\mathcal{G}$ with respect to P is defined as follows:

$$\mathbb{H}_{\mathcal{G}}(Q|P) := \begin{cases} E^Q\Big[\log\Big(\frac{dQ}{dP}\Big|_{\mathcal{G}}\Big)\Big], & \text{if } Q \text{ is absolutely continuous with respect to } P \text{ on } \mathcal{G} \\ +\infty, & \text{otherwise,}\end{cases}$$

where $\frac{dQ}{dP}\big|_{\mathcal{G}}$ is the Radon-Nikodym derivative of Q with respect to P on $\mathcal{G}$.

Next, we denote by ALMM(P) the set of all absolutely continuous probability measures on $(\Omega, \mathcal{F}_T)$ with respect to P such that (S_t) is an $(\mathcal{F}_t)$-local martingale under Q. An element of ALMM(P) is called an absolutely continuous local martingale measure for (S_t).

For a class $\mathcal{D}$ of probability measures on $(\Omega, \mathcal{F}_T)$, we call an equivalent martingale measure **the (equivalent) minimal entropy martingale measure** (**MEMM**) for (S_t) in $\mathcal{D}$ if it minimizes the values of the function: $Q \in \mathcal{D} \longmapsto \mathbb{H}_{\mathcal{F}_T}(Q|P)$.

Main result

We are now in a position to state our main result. Our main objective is to show the existence and the representation of the MEMM for (S_t) in ALMM(P). To establish this, we propose the following condition (C) for (S_t), which is described by the characteristics $(C_t, n(dtdx), B_t)$ of the underlying PII-semimartingale (X_t).

Condition (**C**):
There exists an $\mathbb{R}^d$-valued bounded Borel measurable function (θ_u^*), $u \in [0,T]$, that satisfies the following (i) and (ii):

(i) for each $i = 1, \dots, d$,

$$\int_{(0,T]}\int_{\{|x|>1\}} \mathrm{e}^{x^i}\,\mathrm{e}^{\theta_u^*\cdot(E(x)-I)}\,n(dudx) < \infty; \tag{17}$$

(ii) for all $t \in [0,T]$,

$$\begin{aligned} B_t^c + \frac{1}{2}\overline{C}_t + \int_{(0,t]} dC_u\,\theta_u^* \\ + \int_{(0,t]}\int_{\mathbb{R}_0^d}\Big\{\big(E(x)-I\big)\mathrm{e}^{\theta_u^*\cdot(E(x)-I)} \\ - h(x)\Big\}I_{J^c}(u)\,n(dudx) = 0;\end{aligned} \tag{18}$$

$$\int_{\mathbb{R}_0^d}\big(E(x)-I\big)\mathrm{e}^{\theta_t^*\cdot(E(x)-I)}\,n(\{t\},dx) = 0. \tag{19}$$

Here, (B_t^c) denotes the continuous part of the function with finite variation (B_t): $B_t^c := B_t - \sum_{u\in(0,t]}\Delta B_u$.

Theorem 2. *Suppose that the condition (C) holds. Then we have the following (I)∼(III).*

(I)

$$\Big(\exp\Big[\int_{(0,t]}\theta_u^*\cdot d\widehat{X}_u - K^{\widehat{X}}(\theta^*)_t\Big]\Big)_{t\in[0,T]}$$

is a true maringale under the probability P, where $(\widehat{X}_t)$ is the stochastic process of (11) and $(K^{\widehat{X}}(\theta^*)_t)$ is the modified Laplace cumulant of $(\widehat{X}_t)$ at θ^* :

(20)
$$K^{\widehat{X}}(\theta^*)_t = \frac{1}{2}\int_{(0,t]} \theta^*_u\, dC_u\, \theta^*_u + \frac{1}{2}\int_{(0,t]} \theta^*_u \cdot d\overline{C}_u + \int_{(0,t]} \theta^*_u \cdot dB_u + \int_{(0,t]}\int_{\mathbb{R}^d_0} \{e^{\theta^*_u\cdot(E(x)-I)} - 1 - \theta^*_u \cdot h(x)\}\, n(dudx) + \sum_{u\in(0,t]} \Big\{ \log\Big(1 + \int_{\mathbb{R}^d_0} \{e^{\theta^*_u\cdot(E(x)-I)} - 1\}\, n(\{u\},dx)\Big) - \int_{\mathbb{R}^d_0} \{e^{\theta^*_u\cdot(E(x)-I)} - 1\}\, n(\{u\},dx)\Big\}.$$

In particular,

(21)
$$E^P[e^{\int_{(0,t]} \theta^*_u \cdot d\widehat{X}_u}] = e^{K^{\widehat{X}}(\theta^*)_t}.$$

Therefore, a probability measure P^* on $\mathcal{F}_T$ is consistently determined by

(22)
$$\frac{dP^*}{dP}\Big|_{\mathcal{F}_t} := \frac{e^{\int_{(0,t]} \theta^*_u \cdot d\widehat{X}_u}}{E^P[e^{\int_{(0,t]} \theta^*_u \cdot d\widehat{X}_u}]}.$$

(II) Under the probability measure P^* of (22), the stochastic process (X_t) of (2.1) is an additive process; the characteristics $(C^*_t, n^*(dtdx), B^*_t)$ (associated with the truncation function $h(x) := xI_{\{|x|\le 1\}}(x)$) are given by

(23)
$$C^*_t = C_t;$$

(24)
$$n^*(dtdx) = \frac{e^{\theta^*_t(E(x)-I)}}{1 + \Delta\widetilde{K}^{\widehat{X}}(\theta^*)_t}\, n(dtdx);$$

(25)
$$B^*_t = B_t + \int_{(0,t]} dC_u\theta^*_u + \int_{(0,t]}\int_{\mathbb{R}^d_0} h(x)\Big(\frac{e^{\theta^*_u(E(x)-I)}}{1 + \Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u} - 1\Big)\, n(dudx),$$

where $(\widetilde{K}^{\widehat{X}}(\theta^*)_t)$ is the Laplace cumulant of $(\widehat{X}_t)$ at θ^*:

(26)
$$\widetilde{K}^{\widehat{X}}(\theta^*)_t = \frac{1}{2}\int_{(0,t]} \theta^*_u\, dC_u\, \theta^*_u + \int_{(0,t]} \theta^*_u \cdot dB_u + \frac{1}{2}\int_{(0,t]} \theta^*_u \cdot d\overline{C}_u + \int_{(0,t]}\int_{\mathbb{R}^d_0} \{e^{\theta^*_u\cdot(E(x)-I)} - 1 - \theta^*_u \cdot h(x)\}\, n(dudx)$$

and

(27)
$$\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_t = \int_{\mathbb{R}^d_0} \{e^{\theta^*_t\cdot(E(x)-I)} - 1\}\, n(\{t\},dx).$$

Furthermore, P^* is an equivalent martingale measure for (S_t) of (10).

(III) The probability measure P^* of (22) attains the minimal entropy in the class ALMM(P):

(28)
$$\min_{Q\in ALMM(P)} \mathbb{H}_{\mathcal{F}_T}(Q|P) = \mathbb{H}_{\mathcal{F}_T}(P^*|P) = -K^{\widehat{X}}(\theta^*)_T.$$

DISCRETE CASE

Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of $\mathbb{R}^d$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{G}_k)$ of sub-σ-fields of $\mathcal{F}$; suppose that ξ_k is adapted to $\mathcal{G}_k$ for each $k \in \mathbb{N}$ and that, for all $j < k$, ξ_k is independent of $\mathcal{G}_j$.

Let

(29)
$$X_t := \sum_{k=1}^{[t]} \xi_k,$$

where $[t]$ denotes the greatest integer that does not exceed the real number t. Then (X_t) of (29) can be regarded as a PII-semimartingale with respect to the filtration $(\mathcal{F}_t := \mathcal{G}_{[t]})$.

The canonical representation of (X_t) associated with h is given as follows:

$$X_t = B_t + \int_{(0,t]}\int_{\mathbb{R}^d_0} h(x)\, \widetilde{N}(dudx) + \int_{(0,t]}\int_{\mathbb{R}^d_0} \check{h}(x)\, N(dudx),$$

where $B_t = \sum_{k=1}^{[t]} E[h(\xi_k)]$; $N((0,t], A) := \sharp\{k \in \mathbb{N} \cap (0,t];\ \xi_k \in A\}$ for $A \in \mathcal{B}(\mathbb{R}^d_0)$; the compensator of $N(dudx)$ is given by $n(\{k\}, A) = P[\xi_k \in A]$.

Note that $J = \mathbb{N}$ and that $I_{J^c}(u)n(dudx) = 0$.

In the setting above, the condition (C) is reduced to the following one:

Condition $(\mathbf{C})_d$:
There exisits an $\mathbb{R}^d$-valued function (θ^*_k), $k = 1, \dots, [T]$, that satisfies the following: for each $k = 1, \dots, [T]$ and $i = 1, \dots, d$,

(i)
$$\int_{\{|x|>1\}} e^{x^i}\, e^{\theta^*_k\cdot(E(x)-I)}\, n(\{k\},dx) = E[e^{\xi^i_k}\, e^{\theta^*_k\cdot(E(\xi_k)-I)}; |\xi_k| > 1] < \infty;$$

(ii)
$$\int_{\mathbb{R}^d_0} (e^{x^i} - 1)e^{\theta^*_k\cdot(E(x)-I)}\, n(\{k\},dx) = E[(e^{\xi^i_k} - 1)\, e^{\theta^*_k\cdot(E(\xi_k)-I)}] = 0.$$

Then it is easy to obtain the following result from Theorem 2.

Corollary 2. *Suppose that the condition* $(C)_d$ *holds. Then the probability measure* P^* *defined by*

$$\frac{dP^*}{dP}\Big|_{\mathcal{F}_t} := \frac{e^{\sum_{k=1}^{[t]} \theta_u^* \cdot (E(\xi_k)-I)}}{\prod_{k=1}^{[t]} E[e^{\theta_k^* \cdot (E(\xi_k)-I)}]}, \quad t \in [0,T].$$

is the mimimal entropy martingale measure for $(S_t := S_0\, e^{\sum_{k=1}^{[t]} \xi_k})$ *in the class* ALMM(P)*;*

$$\min_{Q \in ALMM(P)} \mathbb{H}_{\mathcal{F}_T}(Q|P) = \mathbb{H}_{\mathcal{F}_T}(P^*|P) = -\sum_{k=1}^{[t]} \log\left(E[e^{\theta_k^* \cdot (E(\xi_k)-I)}]\right).$$

4. Proof of Theorem 2

In this section, we will give a proof of Theorem 2. Roughly speaking, it follows on the stream proposed in the proof of Theorem 3.1 in [5]; however, we need to consider more finely to overcome the difficulties arising from the existence of fixed times of discontinuity.

Proof of (I)

We will apply Corollary 1 to the stochastic integral $(\int_{(0,t]} \theta_u^* \cdot d\widehat{X}_u)$. Recall that, as we have shown in Proposition 3, the characteristics $(\widehat{C}, \widehat{n}, \widehat{B})$ of $(\widehat{X}_t)$ (associated with h) are given by (14), (15) and (16).

Then, it follows from the condition (C)-(i) that

$$\begin{aligned}&\int_{(0,T]}\int_{\{|x|>1\}} e^{\theta_u^* \cdot x}\,\widehat{n}(dudx)\\ &= \int_{(0,T]}\int_{\{|E(x)-I|>1\}} e^{\theta_u^* \cdot (E(x)-I)}\, n(dudx)\\ &\le \int_{(0,T]}\int_{\{|x|>1/\alpha\}} e^{\theta_u^* \cdot (E(x)-I)}\, n(dudx)\\ &< \infty,\end{aligned}$$

because $\{|E(x)-I|>1\} \subset \{|x|>1/\alpha\}$ for some $\alpha > 0$.

Therefore, we see from Corollary 1 and Remark 1 that

$$(e^{\int_{(0,t]} \theta_u^* \cdot d\widehat{X}_u - K^{\widehat{X}}(\theta^*)_t})_{t\in(0,T]}$$

is a uniformly integrable martingale with mean 1 and hence

$$E[e^{\int_{(0,t]} \theta_u^* \cdot d\widehat{X}_u}] = e^{K^{\widehat{X}}(\theta^*)_t},$$

where $(K^{\widehat{X}}(\theta^*)_t)$ is the modified Laplace cumulant of $(\widehat{X}_t)$ at (θ_u^*):

$$\begin{aligned}&K^{\widehat{X}}(\theta^*)_t\\ &= \frac{1}{2}\int_{(0,t]} \theta_u^*\, d\widehat{C}_u\, \theta_u^* + \int_{(0,t]} \theta_u^* \cdot d\widehat{B}_u\\ &\quad + \int_{(0,t]}\int_{\mathbb{R}_0^d} \left\{e^{\theta_u^* \cdot x} - 1 - \theta_u^* \cdot h(x)\right\} \widehat{n}(dudx)\\ &\quad + \sum_{u\in(0,t]} \Big\{ \log\Big(1 + \int_{\mathbb{R}_0^d} \left\{e^{\theta_u^* \cdot x} - 1\right\} \widehat{n}(\{u\}, dx)\Big)\\ &\qquad - \int_{\mathbb{R}_0^d} \left\{e^{\theta_u^* \cdot x} - 1\right\} \widehat{n}(\{u\}, dx)\Big\}.\end{aligned}$$

Furthermore, it is easy to see from Proposition 3 that (20) holds.

Proof of (II)

We will first show that (26) and (27) hold.

By the definition of the Laplace cumulant,

$$\begin{aligned}\widetilde{K}^{\widehat{X}}(\theta^*)_t &= \frac{1}{2}\int_{(0,t]} \theta_u^*\, d\widehat{C}_u\, \theta_u^* + \int_{(0,t]} \theta_u^* \cdot d\widehat{B}_u\\ &\quad + \int_{(0,t]}\int_{\mathbb{R}_0^d} \left(e^{\theta^* \cdot x} - 1 - \theta_u^* \cdot h(x)\right) \widehat{n}(dudx).\end{aligned}$$

Hence, as in the proof of (20), it follows from Proposition 3 that (26) holds.

Moreover, since (C_t) is continuous and $\Delta B_t = \int_{\mathbb{R}_0^d} h(x)\, n(\{t\}, dx)$, we see that

$$\begin{aligned}&\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_t\\ &= \theta_t^* \cdot \Delta B_t + \int_{\mathbb{R}_0^d} \left\{e^{\theta_t^* \cdot (E(x)-I)} - 1 - \theta_t^* \cdot h(x)\right\} n(\{t\}, dx)\\ &= \int_{\mathbb{R}_0^d} \left\{e^{\theta_t^* \cdot (E(x)-I)} - 1\right\} n(\{t\}, dx).\end{aligned}$$

Next, we will show the following proposition:

Proposition 4. *Let* (D_t^*) *be the density process of* P^* *against* P*, that is,* $D_t^* = \frac{dP^*}{dP}\Big|_{\mathcal{F}_t} = e^{\check{M}_t^*}$*, where we set*

$$\check{M}_t^* := \int_{(0,t]} \theta_u^* \cdot d\widehat{X}_u - K^{\widehat{X}}(\theta^*)_t. \tag{30}$$

Then,

$$D_t^* = e^{\check{M}_t^*} = \mathcal{E}(M^*)_t, \tag{31}$$

where

$$M_t^* := \int_{(0,t]} \theta_u^* \cdot dX_u^c + \int_{(0,t]}\int_{\mathbb{R}_0^d} \frac{e^{\theta_u^* \cdot (E(x)-I)} - 1}{1 + \Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u} \widetilde{N}(dudx). \tag{32}$$

Before giving a proof of this proposition, we note the following result, which ensures that the second term in the right hand side of (32) makes sense as a martingale.

Lemma 1. *Let*

$$V_t := 1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_t.$$

Then $(1/V_t)_{t\in(0,T]}$ *is uniformly bounded.*

Proof. For each $n \in \mathbb{N}$, let

$$u_{n+1} := \begin{cases} \inf\{u \in (0,T]; u > u_n, V_u \le 1/2\} \\ \qquad \text{if } \{u \in (0,T]; u > u_n, V_u \le 1/2\} \ne \emptyset; \\ T \quad \text{if } \{u \in (0,T]; u > u_n, V_u \le 1/2\} = \emptyset. \end{cases}$$

We will show that there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = T$.

To see this, suppose that for all $n \in \mathbb{N}$, $u_n < T$. Then, $V_{u_n} \le 1/2$, and hence $1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_{u_n} \le 1/2$. Therefore, for all n, $-1 < \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_{u_n} \le -1 + 1/2 = -1/2$, which implies that $|\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_{u_n}| \ge 1/2$. Hence we have

$$\sum_{u\in(0,T]} |\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u| \ge \sum_{n=1}^{\infty} |\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_{u_n}| \ge \sum_{n=1}^{\infty} \frac{1}{2} = +\infty.$$

However, this result contradicts to the fact that $(\widetilde{K}^{\widehat{X}}(\theta^*)_t)$ is a function with finite variation on $[0,T]$. Hence the hypothesis that $u_n < T$ for all $n \in \mathbb{N}$ is rejected. Thus, it holds that there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = T$. This means that the number of times $t \in (0,T]$ such that $V_t \le 1/2$ is finite. Also, since $V_t > 0$ for each t, we see that $\sup_{t\in(0,T]}(1/V_t) < \infty$. □

Proof of Proposition 4. By Proposition 2 and the representation (20), we see that

(33)
$$\begin{aligned} \check{M}^*_t &= \int_{(0,t]} \theta^*_u \cdot d\widehat{X}_u - K^{\widehat{X}}(\theta^*)_t \\ &= \int_{(0,t]} \theta^*_u \cdot dX^c_u + \int_{(0,t]}\int_{\mathbb{R}^d_0} \theta^*_u \cdot (E(x) - I)\, \widetilde{N}(dudx) \\ &\quad - \frac{1}{2}\int_{(0,t]} \theta^*_u \, dC_u \, \theta^*_u \\ &\quad - \int_{(0,t]}\int_{\mathbb{R}^d_0} \{ e^{\theta^*_u \cdot (E(x)-I)} - 1 \\ &\qquad - \theta^*_u \cdot (E(x) - I)\}\, n(dudx) \\ &\quad - \sum_{u\in(0,t]} \{ \log\left(1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u\right) - \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u \}, \end{aligned}$$

where we have used (27) to represent the last term in the right hand side. Also, note that

(34) $$\Delta \check{M}^*_u = \theta^*_u \cdot (E(\Delta X_u) - I) - \log\left(1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u\right).$$

Hence, it follows from Itô's formula ([7](p.57)) that

(35)
$$\begin{aligned} e^{\check{M}^*_t} - 1 &= \int_{(0,t]} e^{\check{M}^*_{u-}} \theta^*_u \cdot dX^c_u \\ &\quad + \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \theta^*_u \cdot (E(x) - I)\, \widetilde{N}(dudx) \\ &\quad - \frac{1}{2}\int_{(0,t]} e^{\check{M}^*_{u-}} \theta^*_u \, dC_u \, \theta^*_u \\ &\quad - \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \{ e^{\theta^*_u \cdot (E(x)-I)} - 1 \\ &\qquad - \theta^*_u \cdot (E(x) - I)\}\, n(dudx) \\ &\quad - \sum_{u\in(0,t]} e^{\check{M}^*_{u-}} \{ \log\left(1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u\right) - \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u \} \\ &\quad + \frac{1}{2}\int_{(0,t]} e^{\check{M}^*_{u-}} \theta^*_u \, dC_u \, \theta^*_u \\ &\quad + \sum_{u\in(0,t]} e^{\check{M}^*_{u-}} \{ \Delta \check{M}^*_u - 1 - \Delta \check{M}^*_u \} \\ &= \int_{(0,t]} e^{\check{M}^*_{u-}} \theta^*_u \cdot dX^c_u \\ &\quad + \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \theta^*_u \cdot (E(x) - I)\, \widetilde{N}(dudx) \\ &\quad - \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \{ e^{\theta^*_u \cdot (E(x)-I)} - 1 \\ &\qquad - \theta^*_u \cdot (E(x) - I)\}\, n(dudx) \\ &\quad + \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \Big\{ e^{\theta^*_u \cdot (E(x)-I)} \times \frac{1}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} \\ &\qquad - 1 - \theta^*_u \cdot (E(x) - I)\Big\} N(dudx) \\ &\quad + \sum_{u\in(0,t]} e^{\check{M}^*_{u-}} \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u \\ &= \Big\{ \int_{(0,t]} e^{\check{M}^*_{u-}} \theta^*_u \cdot dX^c_u \\ &\qquad + \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \Big(\frac{e^{\theta^*_u \cdot (E(x)-I)} - 1}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} \Big) \widetilde{N}(dudx) \Big\} \\ &\quad - \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \Big(e^{\theta^*_u \cdot (E(x)-I)} - 1 \Big) \\ &\qquad \times \Big(\frac{\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} \Big) n(dudx) \\ &\quad - \int_{(0,t]}\int_{\mathbb{R}^d_0} e^{\check{M}^*_{u-}} \Big(\frac{\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} \Big) N(dudx) \\ &\quad + \sum_{u\in(0,t]} e^{\check{M}^*_{u-}} \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u. \end{aligned}$$

Note that

(36) $$\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u = 0 \quad \text{for } u \in J^c.$$

Therefore, we see that the sum of the last three terms in the right hand side is equal to

(37)

$$
\begin{aligned}
&-\int_{(0,t]}\int_{\mathbb{R}^d_0} \mathrm{e}^{\check{M}^*_{u-}}\left(\mathrm{e}^{\theta^*_u\cdot(E(x)-I)}-1\right)\\
&\qquad\times\Big(\frac{\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}\Big) I_J(u)\, n(dudx)\\
&-\int_{(0,t]}\int_{\mathbb{R}^d_0} \mathrm{e}^{\check{M}^*_{u-}}\Big(\frac{\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}\Big) I_J(u) N(dudx)\\
&+\sum_{u\in(0,t]} \mathrm{e}^{\check{M}^*_{u-}}\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u\\
&=-\sum_{u\in(0,t]}\int_{\mathbb{R}^d_0} \mathrm{e}^{\check{M}^*_{u-}}\left(\mathrm{e}^{\theta^*_u\cdot(E(x)-I)}-1\right)\\
&\qquad\times\Big(\frac{\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}\Big) n(\{u\},dx)\\
&-\sum_{u\in(0,t]} \mathrm{e}^{\check{M}^*_{u-}}\Big(\frac{\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}\Big)\\
&+\sum_{u\in(0,t]} \mathrm{e}^{\check{M}^*_{u-}}\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u\\
&=\sum_{u\in(0,t]} \mathrm{e}^{\check{M}^*_{u-}}\Big\{-\Big(\frac{\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}\Big)\times\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u\\
&\qquad-\frac{\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u\Big\}\\
&=0.
\end{aligned}
$$

Combining (35) and (37), we obtain

$$
\begin{aligned}
\mathrm{e}^{\check{M}^*_t} &= 1+\int_{(0,t]} \mathrm{e}^{\check{M}^*_{u-}}\theta^*_u\cdot dX^c_u\\
&\quad+\int_{(0,t]}\int_{\mathbb{R}^d_0} \mathrm{e}^{\check{M}^*_{u-}}\Big(\frac{\mathrm{e}^{\theta^*_u\cdot(E(x)-I)}-1}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}\Big)\widetilde{N}(dudx).
\end{aligned}
$$

Therefore, if we take M^*_t of (32), we have

$$
\mathrm{e}^{\check{M}^*_t}=1+\int_{(0,t]}\mathrm{e}^{\check{M}^*_{u-}}dM^*_u,
$$

that is to say, we obtain (31). Thus, we have completed the proof of Proposition 4. □

Proposition 5. *Under the probability P^*, the canonical representation of (X_t) (associated with h) is given as follows:*

$$
\begin{aligned}
(38)\qquad X_t &= X^{*,c}_t+B^*_t+\int_{(0,t]}\int_{\mathbb{R}^d_0} h(x)\,\widetilde{N}^*(dudx)\\
&\quad+\int_{(0,t]}\int_{\mathbb{R}^d_0}\check{h}(x)\,N(dudx),
\end{aligned}
$$

where

$$
(39)\qquad X^{*,c}_t := X^c_t-\int_{(0,t]} dC_u\theta^*_u,
$$

$$
(40)\qquad \widetilde{N}^*(dudx):=N(dudx)-n^*(dudx),
$$

and (B^*_t) is the deterministic process defined by (25).

Proof. Recall that, under P, (X^c_t) is a continuous (local) martingale with $\langle X^{c,i},X^{c,j}\rangle_t=C^{ij}_t$. Hence, it follows from the representation (31) of the density process (D^*_t) that

$$
\begin{aligned}
&\int_{(0,t]}\frac{1}{D^*_{u-}}d\langle X^c,D^*\rangle_u\\
&=\int_{(0,t]}\frac{1}{D^*_{u-}}d\left\langle X^c,\int_{(0,\cdot]}D^*_{u-}\theta^*_u\cdot dX^c_u\right\rangle_u\\
&=\int_{(0,t]}dC_u\theta^*_u.
\end{aligned}
$$

Therefore, by Theorem 49 in [1], we see that $(X^{*,c}_t)$ of (39) is a continuous (local) martingale under P^*. It is also clear that $\langle X^{*,c,i},X^{*,c,j}\rangle_t=C^{ij}_t$.

Next, we will show that the compensator $\widehat{N}^*(dudx)$ of $N(dudx)$ under P^* is equal to $n^*(dudx)$ of (24).

Let A be any set in $\mathcal{B}(\mathbb{R}^d\backslash\{0\})$ satisfying $n^*((0,T],A)<\infty$ and set $A_\varepsilon:=A\cap\{\varepsilon<|x|<(1/\varepsilon)\}$ for an arbitrary $\varepsilon\in(0,1)$. Also, let

$$
N_t:=\int_{(0,t]}\int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\,\widetilde{N}(dudx).
$$

Then, it follows from the representation (31) of the density process (D^*_t) that

$$
\begin{aligned}
&\langle N,D^*\rangle_t\\
&=\int_{(0,t]}\int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\times D^*_{u-}\frac{\mathrm{e}^{\theta^*_u\cdot(E(x)-I)}-1}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}n(dudx)\\
&\quad-\sum_{u\in(0,t]}\int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\,n(\{u\},dx)\\
&\qquad\times\int_{\mathbb{R}^d_0} D^*_{u-}\frac{\mathrm{e}^{\theta^*_u\cdot(E(x)-I)}-1}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}n(\{u\},dx),
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_{(0,t]}\frac{1}{D^*_{u-}}d\langle N,D^*\rangle_u\\
&=\int_{(0,t]}\int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\frac{\mathrm{e}^{\theta^*_u\cdot(E(x)-I)}-1}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}n(dudx)\\
&\quad-\sum_{u\in(0,t]}\int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\,n(\{u\},dx)\times\frac{\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u}.
\end{aligned}
$$

Here again, we have used the relation (27) to get the last term. Therefore, we see that

$$
\begin{aligned}
&N_t - \int_{(0,t]} \frac{1}{D^*_{u-}} d\langle N, D^* \rangle_u \\
&= \Big\{ N((0,t], A_\varepsilon) \\
&\qquad - \int_{(0,t]} \int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x) \frac{\mathrm{e}^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} n(dudx) \Big\} \\
&\quad - \int_{(0,t]} \int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x) \frac{\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} n(dudx) \\
&\quad + \sum_{u \in (0,t]} \int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\, n(\{u\}, dx) \times \frac{\Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u}.
\end{aligned}
$$

Since the sum of the last two terms in the right hand side is equal to 0, we obtain

$$
\begin{aligned}
&N_t - \int_{(0,t]} \frac{1}{D^*_{u-}} d\langle N, D^* \rangle_u \\
&= N((0,t], A_\varepsilon) - \int_{(0,t]} \int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x) \frac{\mathrm{e}^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} n(dudx) \\
&= N((0,t], A_\varepsilon) - \int_{(0,t]} \int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\, n^*(dudx).
\end{aligned}
$$

On the other hand, it follows from Theorem 49 in [1] that $(N_t - \int_{(0,t]} \frac{1}{D^*_{u-}} d\langle N, D^* \rangle_u)$ is a local martingale under P^*. Therefore, we see that $(N((0,t], A_\varepsilon) - \int_{(0,t]} \int_{\mathbb{R}^d_0} I_{A_\varepsilon}(x)\, n^*(dudx))$ is a local martingale under P^* for any $\varepsilon \in (0,1)$. Letting $\varepsilon \downarrow 0$, we can conclude that $(N((0,t], A) - \int_{(0,t]} \int_{\mathbb{R}^d_0} I_A(x)\, n^*(dudx))$ is a local martingale under P^* (see p.82 in [3] for a precise argument), which implies that the compensator $\widehat{N}^*(dudx)$ of $N(dudx)$ under P^* is equal to $n^*(dudx)$ of (24). By the discussion above, the compensated measure $\widetilde{N}^*(dudx)$ of $N(dudx)$ under P^* is given by (40).

Hence, the canonical representation of (X_t) (associated with h) under P^* is given as follows:

$$
\begin{aligned}
X_t &= X^c_t + B_t + \int_{(0,t]} \int_{\mathbb{R}^d_0} h(x)\, \widetilde{N}(dudx) \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}^d_0} \check{h}(x)\, N(dudx) \\
&= X^{*,c}_t + \Big\{ B_t + \int_{(0,t]} dC_u \theta^*_u \\
&\qquad + \int_{(0,t]} \int_{\mathbb{R}^d_0} h(x) \Big(\frac{\mathrm{e}^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} - 1 \Big) n(dudx) \Big\} \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}^d_0} h(x)\, \widetilde{N}^*(dudx) + \int_{(0,t]} \int_{\mathbb{R}^d_0} \check{h}(x)\, N(dudx).
\end{aligned}
$$

Therefore, the third component (B^*_t) of the characteristics of (X_t) under P^* is given by (25).

Thus, we have completed the proof of Proposition 5. □

By Proposition 5, it is clear that the characteristics $(C^*_t, n^*(dtdx), B^*_t)$ (associated with h) of (X_t) under P^* are given by (23), (24) and (25). Also, since they are deterministic and (B^*_t) is a process with finite variation on $[0,T]$, (X_t) is a PII-semimartingale under P^*.

In the remainder of this part (Proof of (II)), we will show that P^* is an equivalent martingale measure for (S_t) of (10).

For each $i (= 1, \ldots, d)$, let $\theta^{(i)}_u := (0, \ldots, 0, 1, 0, \ldots, 0) (\in \mathbb{R}^d)$, where 1 is on the i-th component, and

$$
Y^{(i)}_t := \mathrm{e}^{\int_{(0,t]} \theta^{(i)}_u \cdot dX_u - (K^*)^X (\theta^{(i)})_t}.
$$

Note that we have regarded (X_t) as a semimartingale with respect to P^* and have denoted by $((K^*)^X(\theta)_t)$ the modified Laplace cumulant of (X_t) at (θ_u). Hence,

(41)

$$
\begin{aligned}
&(K^*)^X (\theta^{(i)})_t \\
&= \frac{1}{2} \int_{(0,t]} \theta^{(i)}_u\, dC^*_u\, \theta^{(i)}_u + \int_{(0,t]} \theta^{(i)}_u \cdot dB^*_u \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}^d_0} \big\{ \mathrm{e}^{\theta^{(i)}_u \cdot x} - 1 - \theta^{(i)}_u \cdot h(x) \big\}\, n^*(dudx) \\
&\quad + \sum_{u \in (0,t]} \Big\{ \log \Big(1 + \int_{\mathbb{R}^d_0} \big(\mathrm{e}^{\theta^{(i)}_u \cdot x} - 1 \big)\, n^*(\{u\}, dx) \Big) \\
&\qquad\qquad - \int_{\mathbb{R}^d_0} \big(\mathrm{e}^{\theta^{(i)}_u \cdot x} - 1 \big)\, n^*(\{u\}, dx) \Big\}.
\end{aligned}
$$

By Lemma 1 and the condition (C)-(i), we can check that

$$
\int_{(0,T]} \int_{\{|x|>1\}} \mathrm{e}^{\theta^{(i)}_u \cdot x}\, n^*(dudx) < \infty.
$$

Therefore, it follows from Corollary 1 that $(Y^{(i)}_t)_{t \in (0,T]}$ is a uniformly integrable martingale under P^*.

Next, we will show that

$$
(K^*)^X (\theta^{(i)})_t \equiv 0. \tag{42}
$$

Since the characteristics $(C^*_t, n^*(dtdx), B^*_t)$ (associated with h) of (X_t) under P^* are given by (23), (24) and (25), it follows from (41) that

(43)

$$
\begin{aligned}
&(K^*)^X (\theta^{(i)})_t \\
&= B^i_t + \frac{1}{2} \overline{C}^i_t + \Big(\int_{(0,t]} dC_u\, \theta^*_u \Big)^i \\
&\quad + \int_{(0,t]} \int_{\mathbb{R}^d_0} \Big\{ (\mathrm{e}^{x^i} - 1) \frac{\mathrm{e}^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} - h^i(x) \Big\}\, n(dudx) \\
&\quad + \sum_{u \in (0,t]} \Big\{ \log \Big(1 + \int_{\mathbb{R}^d_0} \big(\mathrm{e}^{x^i} - 1 \big)\, n^*(\{u\}, dx) \Big) \\
&\qquad\qquad - \int_{\mathbb{R}^d_0} \big(\mathrm{e}^{x^i} - 1 \big)\, n^*(\{u\}, dx) \Big\}.
\end{aligned}
$$

Here, note that it follows from (19) that

$$
\int_{\mathbb{R}^d_0} \big(E(x) - I \big) \frac{\mathrm{e}^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u}\, n(\{u\}, dx) = 0, \tag{44}
$$

which implies that the last term in the right hand side of (43) is equal to 0.

Furthermore, we can see that the condition (C)-(ii) implies that the sum of the first four terms in the right hand side of (43) is also equal to 0 as follows:

Lemma 2. *The condition (C)-(ii) implies that*

$$(45)\quad B_t + \frac{1}{2}\overline{C}_t + \int_{(0,t]} dC_u\, \theta^*_u + \int_{(0,t]}\int_{\mathbb{R}^d_0} \Big\{ \big(E(x) - I\big) \frac{e^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} - h(x) \Big\}\, n(dudx) = 0.$$

Proof. By the fact that $\Delta B_u = \int_{\mathbb{R}^d_0} h(x)\, n(\{u\}, dx)$ and (44), we have

$$\Delta B_u + \int_{\mathbb{R}^d_0} \Big\{ \big(E(x) - I\big) \frac{e^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} - h(x) \Big\}\, n(\{u\}, dx) = 0.$$

Hence, we obtain

$$(46)\quad \sum_{u \in (0,t]} \Delta B_u + \int_{(0,t]}\int_{\mathbb{R}^d_0} \Big\{ \big(E(x) - I\big) \frac{e^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} - h(x) \Big\} I_J(u)\, n(dudx) = 0.$$

On the other hand, due to the property (36), we see from (18) that

$$(47)\quad B^c_t + \frac{1}{2}\overline{C}_t + \int_{(0,t]} dC_u\, \theta^*_u + \int_{(0,t]}\int_{\mathbb{R}^d_0} \Big\{ \big(E(x) - I\big) \frac{e^{\theta^*_u \cdot (E(x)-I)}}{1 + \Delta \widetilde{K}^{\widehat{X}}(\theta^*)_u} - h(x) \Big\} I_{J^c}(u)\, n(dudx) = 0.$$

Thus, combining (46) and (47), we see that (45) holds. □

Thus, we have shown that (42) holds and hence we see that $(e^{X^i_t})_{t\in(0,T]}$ is a true martingale under P^*, in other words, P^* is an equivalent martingale measure for (S_t).

Proof of (III)

Let Q be an arbitrary absolutely continuous martingale measure for (S_t) satisfying $\mathbb{H}_{\mathcal{F}_T}(Q|P) < \infty$. Then, $\Big(\widehat{X}^i_t = \int_{(0,t]} \frac{1}{S^i_{u-}} dS^i_u\Big)$ is a local martingale with respect to Q. See Theorem III.29 in [9] (p.128). Hence, $(\int_{(0,t]} \theta^*_u \cdot d\widehat{X}_u)$ is a local martingale with respect to Q. See Theorem IV.29 in [9] (p.171). Therefore, there exists an increasing sequence $\{\tau_n\}_n$ of stopping times such that $\tau_n \nearrow \infty$ as $n \to \infty$ and that $(\int_{(0,t\wedge\tau_n]} \theta^*_u \cdot d\widehat{X}_u)$ is a martingale with respect to Q for each $n \in \mathbb{N}$. In particular, $\int_{(0,T\wedge\tau_n]} \theta^*_u \cdot d\widehat{X}_u$ is integrable with respect to Q;

$$\log\Big(\frac{dP^*}{dP}\Big|_{\mathcal{F}_{T\wedge\tau_n}}\Big) = \int_{(0,T\wedge\tau_n]} \theta^*_u \cdot d\widehat{X}_u - K^{\widehat{X}}(\theta^*)_{T\wedge\tau_n}.$$

Note that $K^{\widehat{X}}(\theta^*)_{T\wedge\tau_n}$ is uniformly bounded with respect to n and ω, since $(K^{\widehat{X}}(\theta^*)_t)$ is actually a function with finite variation on $[0,T]$. Hence, $\log\Big(\frac{dP^*}{dP}\Big|_{\mathcal{F}_{T\wedge\tau_n}}\Big)$ is integrable with respect to Q. Therefore, we see from Lemma 2.1 in [5] that

$$\begin{aligned} \mathbb{H}_{\mathcal{F}_T}(Q|P) &\geq \mathbb{H}_{\mathcal{F}_{T\wedge\tau_n}}(Q|P) \\ &\geq E^Q[\log\Big(\frac{dP^*}{dP}\Big|_{\mathcal{F}_{T\wedge\tau_n}}\Big)] \\ &= E^Q[\int_{(0,T\wedge\tau_n]} \theta^*_u \cdot d\widehat{X}_u] - E^Q[K^{\widehat{X}}(\theta^*)_{T\wedge\tau_n}] \\ &= 0 - E^Q[K^{\widehat{X}}(\theta^*)_{T\wedge\tau_n}]. \end{aligned}$$

On the other hand, since $\tau_n \nearrow \infty$ as $n \to \infty$, it follows from the bounded convergence theorem that

$$\lim_{n\to\infty} E^Q[K^{\widehat{X}}(\theta^*)_{T\wedge\tau_n}] = E^Q[K^{\widehat{X}}(\theta^*)_T] = K^{\widehat{X}}(\theta^*)_T.$$

Thus, we have shown that

$$\mathbb{H}_{\mathcal{F}_T}(Q|P) \geq -K^{\widehat{X}}(\theta^*)_T.$$

Next, we will show that

$$\mathbb{H}_{\mathcal{F}_T}(P^*|P) = -K^{\widehat{X}}(\theta^*)_T.$$

To this end, we note the following fact:

Proposition 6.

$$(48)\qquad \widehat{X}_t = X^{*,c}_t + \int_{(0,t]}\int_{\mathbb{R}^d_0} (E(x) - I)\, \widetilde{N}^*(dudx).$$

Proof. As we have shown in the step (2) that , under P^*, (X_t) is a PII-semimartingale with characteristics $(C^*_t, n^*(dtdx), B^*_t)$, we see from Proposition 2 that the corresponding canonical representation is given by

$$\begin{aligned} \widehat{X}_t = X^{*,c}_t &+ \Big\{ B^*_t + \frac{1}{2}\overline{C}_t \\ &+ \int_{(0,t]}\int_{\mathbb{R}^d_0} \big\{ h(E(x) - I) - h(x) \big\}\, n^*(dudx) \Big\} \\ &+ \int_{(0,t]}\int_{\mathbb{R}^d_0} h(E(x) - I)\, \widetilde{N}^*(dudx) \\ &+ \int_{(0,t]}\int_{\mathbb{R}^d_0} \check{h}(E(x) - I)\, N(dudx). \end{aligned}$$

Hence, we have

$$\begin{aligned}\widehat{X}_t = X_t^{*,c} &+ \Big\{ B_t^* + \frac{1}{2}\overline{C}_t \\ &+ \int_{(0,t]}\int_{\mathbb{R}_0^d} \big\{h(E(x)-I) - h(x)\big\}\, n^*(dudx) \\ &+ \int_{(0,t]}\int_{\mathbb{R}_0^d} \check{h}(E(x)-I)\, n^*(dudx)\Big\} \\ &+ \int_{(0,t]}\int_{\mathbb{R}_0^d} (E(x)-I)\, \widetilde{N}^*(dudx).\end{aligned}$$

Note that the sum of the second term through the fifth term in the right hand side is equal to

$$\begin{aligned}&B_t^* + \frac{1}{2}\overline{C}_t + \int_{(0,t]}\int_{\mathbb{R}_0^d} \big\{E(x) - I - h(x)\big\}\, n^*(dudx) \\ &= B_t + \frac{1}{2}\overline{C}_t + \int_{(0,t]} dC_u \theta_u^* \\ &\quad + \int_{(0,t]}\int_{\mathbb{R}_0^d} \Big\{(E(x)-I)\frac{\mathrm{e}^{\theta_u^*\cdot(E(x)-I)}}{1+\Delta\widetilde{K}^{\widehat{X}}(\theta^*)_u} - h(x)\Big\}\, n(dudx) \\ &= 0,\end{aligned}$$

where we have used Lemma 2 to get the last equality.

Thus, we have obtained (48). □

By Proposition 6, we see that

$$E^{P^*}[\int_{(0,T]} \theta_u^* \cdot d\widehat{X}_u] = 0,$$

and hence

$$\begin{aligned}\mathbb{H}_{\mathcal{F}_T}(P^*|P) &= E^{P^*}[\int_{(0,T]} \theta_u^* \cdot d\widehat{X}_u - K^{\widehat{X}}(\theta^*)_T] \\ &= -K^{\widehat{X}}(\theta^*)_T.\end{aligned}$$

Thus, we have the conclusion of the step (3): for any absolutely continuous martingale measure Q for (S_t) satisfying $\mathbb{H}_{\mathcal{F}_T}(Q|P) < \infty$,

$$\mathbb{H}_{\mathcal{F}_T}(Q|P) \geq -K^{\widehat{X}}(\theta^*)_T = \mathbb{H}_{\mathcal{F}_T}(P^*|P).$$

We have at last completed our proof of Theorem 2.

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