

Global existence of solutions to the compressible Navier–Stokes equation around parallel flows

Kagei, Yoshiyuki
Faculty of Mathematics, Kyushu University

<https://hdl.handle.net/2324/18411>

出版情報 : MI Preprint Series. 2010-33, 2011-12-01. Academic Press
バージョン :
権利関係 : (C) 2011 Elsevier Inc.



MI Preprint Series

**Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub**

Global existence of solutions to the compressible Navier-Stokes equation around parallel flows

Yoshiyuki Kagei

MI 2010-33

(Received October 20, 2010)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Global existence of solutions to the compressible Navier-Stokes equation around parallel flows

Yoshiyuki Kagei

Faculty of Mathematics, Kyushu University

Fukuoka 819-0395, JAPAN

Abstract

The initial boundary value problem for the compressible Navier-Stokes equation is considered in an infinite layer of \mathbf{R}^n . It is proved that if $n \geq 3$, then strong solutions to the compressible Navier-Stokes equation around parallel flows exist globally in time for sufficiently small initial perturbations, provided that the Reynolds and Mach numbers are sufficiently small. The proof is given by a variant of the Matsumura-Nishida energy method based on a decomposition of solutions associated with a spectral property of the linearized operator.

Mathematics Subject Classification (2000). 35Q30, 76N15.

Keywords. Compressible Navier-Stokes equation, global existence, parallel flow.

1. Introduction

This paper is concerned with the initial boundary value problem for the compressible Navier-Stokes equation

$$(1.1) \quad \partial_t \rho + \operatorname{div}(\rho v) = 0,$$

$$(1.2) \quad \rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \operatorname{div} v + \nabla P(\rho) = \rho g$$

in an n dimensional infinite layer $\Omega_\ell = \mathbf{R}^{n-1} \times (0, \ell)$:

$$\Omega = \{x = (x', x_n); x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}, 0 < x_n < \ell\} \quad (n \geq 3).$$

Here $\rho = \rho(x, t)$ and $v = {}^\top(v^1(x, t), \dots, v^n(x, t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega$, respectively; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$P'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are the viscosity coefficients that are assumed to be constants and satisfy $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$; and $\mathbf{g} = \mathbf{g}(x_n)$ is an external force which has the form

$$\mathbf{g} = {}^\top(g^1(x_n), 0, \dots, 0, g^n(x_n)),$$

where g^j ($j = 1, n$) are given smooth function of x_n . Here and in what follows the superscript ${}^\top$ stands for the transposition.

The system (1.1)–(1.2) is considered under the boundary condition

$$(1.3) \quad v|_{x_n=0} = 0, \quad v|_{x_n=\ell} = V^1 \mathbf{e}_1,$$

and the initial condition

$$(1.4) \quad {}^\top(\rho, v)|_{t=0} = {}^\top(\rho_0, v_0).$$

Here V^1 is a given constant and \mathbf{e}_1 is the unit vector $\mathbf{e}_1 = {}^\top(1, 0, \dots, 0) \in \mathbf{R}^n$.

Under suitable smallness conditions on g^n , problem (1.1)–(1.3) has a stationary solution ${}^\top(\bar{\rho}_s, \bar{v}_s)$ of parallel flow with properties:

$$\bar{\rho}_s = \bar{\rho}_s(x_n), \quad \bar{v}_s = {}^\top(\bar{v}_s^1(x_n), 0, \dots, 0),$$

$$\|\rho_s - \rho_*\|_{C^0[0,\ell]} \leq C \frac{\ell \|g^n\|_{C^0[0,\ell]}}{P'(\rho_*)},$$

$$\|v_s^1\|_{C^0[0,\ell]} \leq C \left(\frac{\rho_* \ell^2 \|g^1\|_{C^0[0,\ell]}}{\mu} + |V^1| \right).$$

Here and in what follows $\|\cdot\|_{C^k[a,b]}$ denotes the usual C^k norm of functions on the interval $[a, b]$. Typical examples are the plane Couette flow:

$$\rho_s = \rho_*, \quad v_s^1 = \frac{V^1}{\ell} x_n$$

when $\mathbf{g} = 0$ and $V^1 \neq 0$; and the Poiseuille flow:

$$\rho_s = \rho_*, \quad v_s^1 = \frac{\rho_* g^1}{2\mu} x_n (\ell - x_n)$$

when $\mathbf{g} = g^1 \mathbf{e}_1$ with a constant $g^1 \neq 0$ and $V^1 = 0$. If g^n is sufficiently small, then ${}^\top(\bar{\rho}_s, \bar{v}_s)$ can be obtained as a perturbation of the superposition of the plane Couette flow and Poiseuille flow.

The purpose of this paper is to show the global existence of solutions to (1.1)–(1.4) when the initial value ${}^\top(\rho_0, v_0)$ is sufficiently close to a parallel flow ${}^\top(\rho_s, v_s)$.

To state our result more precisely, we set $V = \frac{\rho_* \ell^2 \|g^1\|_{C^0[0,\ell]}}{\mu} + |V^1| > 0$ and introduce the parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \gamma = \frac{\sqrt{P'(\rho_*)}}{V}.$$

We note that $Re = 1/\nu$ and $Ma = 1/\gamma$ are the Reynolds and Mach numbers. We will prove that if $\nu \gg 1$, $\gamma \gg 1$ and $\|\mathbf{g}\|_{C^m[0,\ell]} \ll 1$ for some $m \in \mathbf{N}$ satisfying $m \geq [n/2] + 1$, then (1.1)–(1.4) has a unique global solution ${}^\top(\rho, v)$ such that ${}^\top(\rho - \bar{\rho}_s, v - \bar{v}_s) \in C([0, \infty); H^m)$, provided that ${}^\top(\rho_0 - \bar{\rho}_s, v_0 - \bar{v}_s) \in H^m$ is small enough. Here H^m denotes the L^2 Sobolev space on Ω_ℓ of order m .

In the case of the plane Couette flow, the results mentioned above was proved in [3]. Therefore, the result of this paper is an extension of that of [3] to the case of general parallel flows for $n \geq 3$. The main difference to [3] arises in the following point. A non-dimensional form of the equations for the perturbation ${}^\top(\phi, w) = {}^\top(\frac{\gamma^2}{\rho_*}(\rho - \bar{\rho}_s), \frac{1}{V}(v - \bar{v}_s))$ is written in the form

$$(1.5) \quad \partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \operatorname{div}(\rho_s w) = f^0,$$

$$(1.6) \quad \begin{aligned} \partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w + \nabla \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi \right) \\ + \left(\frac{\nu}{\gamma^2 \rho_s^2} \Delta v_s \right) \phi + v_s \cdot \nabla w + w \cdot \nabla v_s = \mathbf{f}, \end{aligned}$$

$$(1.7) \quad w|_{x_n=0,1} = 0,$$

$$(1.8) \quad {}^\top(\phi, w)|_{t=0} = {}^\top(\phi_0, w_0).$$

Here $\tilde{\nu} = \nu + \nu'$; f^0 and \mathbf{f} denote the nonlinearities; the domain Ω_ℓ is transformed into Ω_1 . The equation (1.6) has a lower order term $\left(\frac{\nu}{\gamma^2 \rho_s^2} \Delta v_s \right) \phi$ which is absent in the case of the plane Couette flow [3]. We note that the Poincaré inequality holds for w but not for ϕ in general. So, due to the appearance of this term, a direct application of the Matsumura-Nishida energy method [9] does not work even though the coefficient $\left(\frac{\nu}{\gamma^2 \rho_s^2} \Delta v_s \right)$ is assumed to be sufficiently small. To overcome this, we employ a decomposition of solutions essentially introduced in the linearized analysis in [7]. We write (1.5)–(1.8) in the form

$$(1.9) \quad \begin{cases} \partial_t u + Lu = \mathbf{F}, & w|_{x_n=0,1} = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $u = {}^\top(\phi, w)$; L is the linearized operator; \mathbf{F} denotes the nonlinearity; and $u_0 = {}^\top(\phi_0, w_0)$. The Fourier transform of (1.9) in $x' \in \mathbf{R}^{n-1}$ can be written as

$$(1.10) \quad \begin{cases} \partial_t \hat{u} + \hat{L}_{\xi'} \hat{u} = \hat{\mathbf{F}}, & \hat{w}|_{x_n=0,1} = 0, \\ \hat{u}|_{t=0} = \hat{u}_0, \end{cases}$$

where $\xi' = (\xi_1, \dots, \xi_{n-1}) \in \mathbf{R}^{n-1}$ is the dual variable. When $\xi' = 0$, the operator \hat{L}_0 (as an operator on $H^1(0, 1) \times L^2(0, 1)$) has a one dimensional kernel spanned by a function $u^{(0)} = {}^\top(\phi^{(0)}(x_n), w^{(0)}(x_n))$ with $\int_0^1 \phi^{(0)}(x_n) dx_n = 1$. We then define a projection P_1 by

$$P_1 u = \mathcal{F}^{-1} \left(\hat{\chi}_1(\xi') \int_0^1 \hat{\phi}(\xi', x_n) dx_n \right) u^{(0)}(x_n)$$

for $u = {}^\top(\phi, w)$. Here \mathcal{F}^{-1} denotes the inverse Fourier transform and $\hat{\chi}_1$ is a cut off function: $\hat{\chi}_1(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\chi}_1(\xi) = 0$ for $|\xi| > 1$. We decompose the solution $u(t)$ of (1.9) into

$$(1.11) \quad u = P_1 u + P_\infty u \equiv \sigma_1 u^{(0)} + u_\infty,$$

where $P_\infty = I - P_1$ and $P_1 u = \sigma_1 u^{(0)}$ with

$$\sigma_1 = \sigma_1(x', t) = \mathcal{F}^{-1} \left(\hat{\chi}_1(\xi') \int_0^1 \hat{\phi}(\xi', x_n, t) dx_n \right).$$

We can show, by a variant of the Matsumura-Nishida energy method, that

$$(1.12) \quad \|u(t)\|_{H^m}^2 + \int_0^t \|\partial_{x'} \sigma_1\|_{L^2}^2 + \|\partial_x \phi_\infty\|_{H^{m-1}}^2 + \|\partial_x w_\infty\|_{H^m}^2 d\tau \leq C \|u_0\|_{H^m}^2$$

for sufficiently small $\|u_0\|_{H^m}$, provided that $\nu \gg 1$, $\gamma \gg 1$ and $\|\mathbf{g}\|_{C^m[0,\ell]} \ll 1$. An advantage of the decomposition (1.11) is that the Poincaré inequality $\|u_\infty\|_{L^2} \leq C \|\partial_x u_\infty\|_{L^2}$ holds for u_∞ -part and that differentiations of any order in x variable are bounded operators on the subspace $\text{Range}(P_1)$, i.e., $\|\partial_x^k(\sigma_1 u^{(0)})\|_{L^2} \leq C_k \|\sigma_1 u^{(0)}\|_{L^2}$ for any $k = 1, 2, \dots$. Using these properties we can establish the a priori estimate by a variant of the Matsumura-Nishida energy method.

Once H^m -energy bound (1.12) is obtained, then the following decay and asymptotic behavior can be shown as in the case of the plane Couette flow [3] by using the linearized analysis in [7]:

$$\|u(t)\|_{L^2} = O(t^{-\frac{n-1}{4}}),$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2} = O(t^{-\frac{n-1}{4}-\frac{1}{2}}L(t))$$

as $t \rightarrow \infty$. Here $\sigma = \sigma(x', t)$ is a function given by

$$\sigma(x', t) = \mathcal{F}^{-1}(e^{-(ia_0\xi_1 + \kappa_0|\xi_1|^2 + \kappa_1|\xi''|^2)t} \int_0^1 \hat{\phi}_0(\xi', x_n) dx_n)$$

with some constants $a_0 \in \mathbf{R}$, $\kappa_0 > 0$ and $\kappa_1 > 0$, where $\xi'' = (\xi_2, \dots, \xi_{n-1}) \in \mathbf{R}^{n-2}$; and $L(t) = \log(1+t)$ when $n = 3$ and $L(t) = 1$ when $n \geq 4$. In this paper we concentrate on the proof of the global existence of solutions and do not consider the decay and asymptotic behavior of perturbations.

We remark that in contrast to the case of the plane Couette flow [3], we here restrict ourselves to the case $n \geq 3$. The case $n = 2$ is different from the case $n \geq 3$; and we will study the case $n = 2$ elsewhere.

This paper is organized as follows. In section 2, we rewrite problem (1.1)–(1.4) into a non-dimensional form and show the existence of parallel stationary solutions. In section 3, we state the main result of this paper. In section 4, we introduce the decomposition (1.11) of the solution and examine some properties of P_1 . Section 5 is devoted to deriving the a priori estimate.

2 Non-dimensionalization and parallel flows

In this section we rewrite the problem into a non-dimensional form and show the existence of stationary parallel flows.

2.1 Notations

We first introduce some notation which will be used throughout the paper. For a domain D and $1 \leq p \leq \infty$ we denote by $L^p(D)$ the usual Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let m be a nonnegative integer. The symbol $H^m(D)$ denotes the m -th order L^2 Sobolev space on D with norm $\|\cdot\|_{H^m(D)}$. $C_0^m(D)$ stands for the set of all C^m functions which have compact support in D . We denote by $H_0^1(D)$ the completion of $C_0^1(D)$ in $H^1(D)$.

We simply denote by $L^p(D)$ (resp., $H^m(D)$) the set of all vector fields $w = {}^\top(w^1, \dots, w^n)$ on D with $w^j \in L^p(D)$ (resp., $H^m(D)$), $j = 1, \dots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(D)}$ (resp., $\|\cdot\|_{H^m(D)}$). For $u = {}^\top(\phi, w)$ with $\phi \in H^k(D)$ and $w = {}^\top(w^1, \dots, w^n) \in H^m(D)$, we define $\|u\|_{H^k(D) \times H^m(D)}$ by $\|u\|_{H^k(D) \times H^m(D)} = \|\phi\|_{H^k(D)} + \|w\|_{H^m(D)}$. When $k = m$, we simply write $\|u\|_{H^k(D) \times H^k(D)} = \|u\|_{H^k(D)}$.

Later we will transform the problem into a non-dimensional form; and then Ω_ℓ will be transformed into $\Omega \equiv \Omega_1 = \mathbf{R}^{n-1} \times (0, 1)$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as L^p (resp., H^m). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$.

In case D is the interval $(0, 1)$ we denote the norm of $L^p(0, 1)$ by $|\cdot|_p$. The norm of $H^m(0, 1)$ is denoted by $|\cdot|_{H^m}$.

The inner product of $L^2(\Omega)$ is denoted by

$$(f, g) = \int_{\Omega} f(x)g(x) dx, \quad f, g \in L^2(\Omega).$$

We also denote the inner product of $L^2(0, 1)$ by

$$(f, g) = \int_0^1 f(x_n)g(x_n) dx_n, \quad f, g \in L^2(0, 1),$$

if no confusion occurs. We further introduce a weighted inner product $\langle\langle \cdot, \cdot \rangle\rangle$ defined by

$$\langle\langle u_1, u_2 \rangle\rangle = \int_{\Omega} \phi_1 \phi_2 \frac{\tilde{P}'(\rho_s)}{\gamma^4 \rho_s} dx + \int_{\Omega} w_1 \cdot w_2 \rho_s dx$$

for $u_j = {}^{\top}(\phi_j, w_j) \in L^2(\Omega)^{n+1}$ ($j = 1, 2$); and, also, $\langle \cdot, \cdot \rangle$ defined by

$$\langle u_1, u_2 \rangle = \int_0^1 \phi_1 \phi_2 \frac{\tilde{P}'(\rho_s)}{\gamma^4 \rho_s} dx_n + \int_0^1 w_1 \cdot w_2 \rho_s dx_n$$

for $u_j = {}^{\top}(\phi_j, w_j) \in L^2(0, 1)^{n+1}$ ($j = 1, 2$). Here $\rho_s = \rho_s(x_n)$ denotes the density of the parallel flow whose existence will be proved in Proposition 2.1 below. We note that $\langle\langle \cdot, \cdot \rangle\rangle$ and $\langle \cdot, \cdot \rangle$ define inner product of $L^2(\Omega)^{n+1}$ and $L^2(0, 1)^{n+1}$, respectively, since $0 < \rho_1 \leq \rho_s \leq \rho_2$ and $\tilde{P}'(\rho_s) > 0$ for $\rho_1 \leq \rho_s \leq \rho_2$ from Proposition 2.1. Furthermore, we denote the mean value of $f \in L^1(0, 1)$ by $\langle f \rangle$:

$$\langle f \rangle = \int_0^1 f(x_n) dx_n.$$

For $u = {}^{\top}(\phi, w) \in L^1(0, 1)$ with $w = {}^{\top}(w^1, \dots, w^n)$ we define $\langle u \rangle$ by

$$\langle u \rangle = \langle \phi \rangle + \langle w^1 \rangle + \dots + \langle w^n \rangle.$$

We denote the $k \times k$ identity matrix by I_k . We also define $(n+1) \times (n+1)$ diagonal matrices Q_0 , Q_n and \tilde{Q} by

$$Q_0 = \text{diag}(1, 0, \dots, 0), \quad Q_n = \text{diag}(0, \dots, 0, 1)$$

and

$$\tilde{Q} = \text{diag}(0, 1, \dots, 1).$$

Note that

$$\langle Q_0 u \rangle = \langle \phi \rangle \quad \text{for } u = {}^\top(\phi, w).$$

We often write $x \in \Omega$ as

$$x = {}^\top(x', x_n), \quad x' = {}^\top(x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}.$$

Partial derivatives of a function u in x, x', x_n and t are denoted by $\partial_x u, \partial_{x'} u, \partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$.

For a function $f = f(x')$ ($x' \in \mathbf{R}^{n-1}$), we denote its Fourier transform by \hat{f} or $\mathcal{F}f$:

$$\hat{f}(\xi') = (\mathcal{F}f)(\xi') = \int_{\mathbf{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} dx'.$$

The inverse Fourier transform is denoted by \mathcal{F}^{-1} :

$$(\mathcal{F}^{-1}f)(x) = (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} d\xi'.$$

2.2 Non-dimensional form of equations

We introduce the following dimensionless variables:

$$x = \ell \tilde{x}, \quad t = \frac{\ell}{V} \tilde{t}, \quad v = V \tilde{v}, \quad \rho = \rho_* \tilde{\rho}, \quad P = \rho_* V^2 \tilde{P}$$

with

$$V = \frac{\rho_* \ell^2 |g^1|_\infty}{\mu} + |V^1| > 0.$$

Then the problem (1.1)–(1.4) is transformed into the following dimensionless problem on the layer $\Omega \equiv \Omega_1 = \mathbf{R}^{n-1} \times (0, 1)$:

$$(2.1) \quad \partial_{\tilde{t}} \tilde{\rho} + \operatorname{div}(\tilde{\rho} \tilde{v}) = 0,$$

$$(2.2) \quad \tilde{\rho}(\partial_{\tilde{t}} \tilde{v} + \tilde{v} \cdot \nabla \tilde{v}) - \nu \Delta \tilde{v} - (\nu + \nu') \nabla \operatorname{div} \tilde{v} + \tilde{P}'(\tilde{\rho}) \nabla \tilde{\rho} = \tilde{\rho} \tilde{\mathbf{g}},$$

$$(2.3) \quad \tilde{v}|_{\tilde{x}_n=0} = 0, \quad \tilde{v}|_{\tilde{x}_n=1} = \frac{V^1}{V} \mathbf{e}^1,$$

$$(2.4) \quad {}^\top(\tilde{\rho}, \tilde{v})|_{\tilde{t}=0} = {}^\top(\tilde{\rho}_0, \tilde{v}_0).$$

Here div , ∇ and Δ are the divergence, gradient and Laplacian with respect to \tilde{x} ; $\tilde{\mathbf{g}}(\tilde{x}_n) = \frac{\ell}{V^2} \mathbf{g}(\ell \tilde{x}_n)$; and ν , ν' and γ are the non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \gamma = \sqrt{\tilde{P}'(1)} = \frac{\sqrt{P'(\rho_*)}}{V}.$$

Remark. The Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively.

2.3 Existence of stationary parallel flow

One can see that if $|\tilde{g}^n|_\infty$ is small enough, then a stationary solution ${}^\top(\rho_s, v_s) = {}^\top(\rho_s(x_n), v_s^1(x_n) \mathbf{e}_1)$ exists. More precisely, substituting $(\tilde{\rho}, \tilde{v}) = (\rho_s(x_n), v_s^1(x_n) \mathbf{e}_1)$ into (2.1)–(2.3), we have

$$(2.5) \quad -\nu \partial_{\tilde{x}_n}^2 v_s^1 = \rho_s \tilde{g}^1,$$

$$(2.6) \quad \partial_{\tilde{x}_n}(\tilde{P}(\rho_s)) = \rho_s \tilde{g}^n,$$

$$(2.7) \quad v_s^1|_{\tilde{x}_n=0} = 0, \quad v_s^1|_{\tilde{x}_n=1} = \frac{V^1}{V}.$$

We will look for solutions of (2.5)–(2.7) with

$$(2.8) \quad \int_0^1 \rho_s(\tilde{x}_n) d\tilde{x}_n = 1.$$

Proposition 2.1. *Assume that $\tilde{P}'(\rho) > 0$ for $\rho_1 \leq \rho \leq \rho_2$ with some $0 < \rho_1 < 1 < \rho_2$. Let $\Phi(\rho) = \int_1^\rho \frac{\tilde{P}'(\eta)}{\eta} d\eta$ for $\rho_1 \leq \rho \leq \rho_2$ and let $\Psi(r) = \Phi^{-1}(r)$ for $r_1 \leq r \leq r_2$. Here Φ^{-1} denotes the inverse function of Φ and $r_j = \Phi(\rho_j)$ ($j = 1, 2$).*

If

$$|\tilde{g}^n|_\infty \leq C \min \left\{ |r_1|, r_2, \frac{1}{4\gamma^2 \|\Psi''\|_{C^0[r_1, r_2]}} \right\},$$

then there exists a smooth stationary solution ${}^\top(\rho_s, v_s) = {}^\top(\rho_s(x_n), v_s^1(x_n) \mathbf{e}_1)$ of (2.5)–(2.8) satisfying

$$\rho_1 \leq \rho_s(\tilde{x}_n) \leq \rho_2, \quad |\rho_s - 1|_\infty \leq C \frac{|\tilde{g}^n|_\infty}{\gamma^2},$$

$$|\partial_{x_n}^k v_s^1|_\infty \leq C(1 + \rho_2), \quad k = 0, 1, 2.$$

Furthermore, if $\|\tilde{g}^n\|_{C^{k-1}[0,1]} \leq \eta$, then

$$|\partial_{\tilde{x}_n}^k \rho_s|_\infty \leq C_k \|\tilde{g}^n\|_{C^{k-1}[0,1]} \quad \text{for } k = 1, 2, \dots,$$

and

$$|\partial_{x_n}^k v_s|_\infty \leq \frac{C_k}{\nu} \|\tilde{\mathbf{g}}\|_{C^{k-2}[0,1]} \quad \text{for } k = 3, 4, \dots.$$

Here C_k are positive constants depending on k , η , $\|\Psi\|_{C^k[r_1, r_2]}$ and ρ_2 .

In particular,

$$|\partial_{x_n} \rho_s|_\infty \leq C \frac{1}{\gamma^2} \left(1 + \frac{1}{\gamma^2}\right) |\tilde{g}^n|_\infty,$$

$$|\tilde{P}'(\rho_s) - \gamma^2|_\infty \leq C \|\tilde{P}''\|_{C^0[\rho_1, \rho_2]} \frac{|\tilde{g}^n|_\infty}{\gamma^2}.$$

Outline of proof. We proceed as in [8, Proof of Lemma 2.1]. In the proof we omit "tilde" of \tilde{x}_n . We also denote $\|g\|_\infty$ by $\|g\|$. We first observe that for Φ and Ψ there hold

$$\Phi(1) = 0, \quad \Psi(0) = \Phi^{-1}(0) = 1, \quad r_1 < 0 < r_2,$$

$$\Phi'(1) = \tilde{P}'(1) = \gamma^2, \quad \Psi'(0) = \frac{1}{\Phi'(0)} = \frac{1}{\gamma^2},$$

$$\left| \Psi'(r) - \frac{1}{\gamma^2} \right| \leq \|\Psi\|_{C^0[r_1, r_2]} r \quad (r_1 \leq r \leq r_2),$$

$$|\Psi'(r)| \leq \|\Psi\|_{C^0[r_1, r_2]} r + \frac{1}{\gamma^2} \quad (r_1 \leq r \leq r_2).$$

We set $g(x_n) = \int_0^{x_n} \tilde{g}^n(y_n) dy_n$. By (2.6), we have

$$\frac{\tilde{P}'(\rho_s)}{\rho_s} \rho_s' = \tilde{g}^n.$$

It then follows that ρ_s is given by

$$(2.9) \quad \int_1^{\rho(x_n)} \frac{\tilde{P}'(\eta)}{\eta} d\eta = \alpha + g(x_n)$$

with some constant α which is determined by g through (2.8). In terms of Ψ , (2.9) is written as

$$(2.10) \quad \rho_s(x_n) = \Psi(\alpha + g(x_n)).$$

By (2.8) and (2.10), problem (2.6)–(2.8) is reduced to

$$(2.11) \quad G(\alpha, g) = 0,$$

where $G : \mathbf{R} \times C^0[r_1, r_2] \rightarrow \mathbf{R}$ defined by

$$G(\alpha, g) = \int_0^1 (\Psi(\alpha + g(x_n)) - 1) dx_n.$$

Observe that

$$\begin{aligned} \partial_\alpha G(\alpha, g) &= \int_0^1 \Psi'(\alpha + g(x_n)) dx_n, \\ \partial_g G(\alpha, g)h &= \int_0^1 \Psi'(\alpha + g(x_n))h(x_n) dx_n, \end{aligned}$$

$$G(0, 0) = 1, \quad \partial_\alpha G(0, 0) = \frac{1}{\gamma^2}, \quad \partial_g G(0, 0) = \frac{1}{\gamma^2} \int_0^1 h(x_n) dx_n.$$

We can show a unique existence of a solution $\alpha = \alpha(g)$ of (2.11) for a suitably given g , together with estimate on $\alpha = \alpha(g)$ in terms of g , which leads to the desired estimates for ρ_s .

Let us solve (2.11) by contraction mapping principle. We define $\Gamma(\alpha, g)$ by

$$\Gamma(\alpha, g) = \alpha - (\partial_\alpha G(0, 0))^{-1} G(\alpha, g) = \alpha - \gamma^2 G(\alpha, g).$$

Note that (2.11) is equivalent with

$$\alpha = \Gamma(\alpha, g);$$

and that $\Gamma(0, 0) = 0$.

One can show that $\Gamma : X \times Y \rightarrow X$ is a uniform contraction with $X = \{\alpha \in \mathbf{R}; |\alpha| \leq \delta\}$ and $Y = \{g \in C^0[0, 1]; \|g\| \leq \delta\}$, where $\delta = \min\{|r_1|, r_2, 1/(4\gamma^2\|\Psi''\|_{C^0[r_1, r_2]})\}$; and hence, for each $g \in Y$, Γ has a unique fixed point $\alpha = \alpha(g) \in X$. Furthermore, it can be seen that $\alpha(0) = 0$ and $|\alpha(g)| \leq \|g\|$.

The estimates for $\rho_s(x_n)$ are obtained as follows. We see from (2.10)

$$\rho_s(x_n) - 1 = \Psi(\alpha + g(x_n)) - \Psi(0) = \int_0^1 \Psi'(\theta(\alpha + g(x_n)))(\alpha + g(x_n)) d\theta.$$

This implies

$$\begin{aligned} |\rho_s(x_n) - 1| &\leq \left(\|\Psi''\|_{C^0[r_1, r_2]} (|\alpha(g)| + \|g\|) + \frac{1}{\gamma} \right) (|\alpha(g)| + \|g\|) \\ &\leq \frac{3}{2\gamma^2} \|g\| \leq C \frac{|\tilde{g}^n|_\infty}{\gamma^2}. \end{aligned}$$

Moreover,

$$|\partial_{x_n}^k \rho_s(x_n)| = |\partial_{x_n}^k (\Psi(\alpha + g(x_n)))| \leq C \|\tilde{g}^n\|_{C^{k-1}[0,1]} (1 + \|\tilde{g}^n\|_{C^{k-1}[0,1]})^{k-1},$$

where C depends on $\|\Psi\|_{C^k[r_1, r_2]}$.

Once $\rho_s(x_n)$ is obtained, then $v_s^1(x_n)$ is given by

$$(2.13) \quad v_s^1(x_n) = \tilde{v}_{s,C}^1(x_n) + \tilde{v}_{s,P}^1(x_n),$$

where

$$(2.14) \quad v_{s,C}^1 = \frac{V^1}{V} x_n, \quad v_{s,P}^1(x_n) = \frac{1}{\nu} \int_0^1 G(x_n, y_n) \rho_s(y_n) \tilde{g}^1(y_n) dy_n$$

with

$$(2.15) \quad G(x_n, y_n) = \begin{cases} (1 - x_n)y_n & (0 < y_n < x_n), \\ x_n(1 - y_n) & (x_n < y_n < 1). \end{cases}$$

Since $|V^1|/V \leq 1$ and $|\tilde{g}^1|_\infty/\nu \leq 1$, we see from (2.13)–(2.15) that $|\partial_{x_n}^k v_s^1|_\infty \leq C(1 + \rho_2)$ for $k = 0, 1, 2$. The estimates for $k \geq 3$ can be obtained by differentiating (2.5). This completes the proof. \square

3. Main Result

In this section we rewrite the equations into the ones for the perturbation and state the main result of this paper.

We first rewrite the system (2.1)–(2.4) into the one for the perturbation. We set $\tilde{\rho} = \rho_s + \gamma^{-2}\phi$ and $\tilde{v} = v_s + w$ in (2.1)–(2.4). Then omitting "tildes" in \tilde{t} , \tilde{x} we arrive at the initial boundary value problem for the perturbation $u = {}^\top(\phi, w)$:

$$(3.1) \quad \partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \operatorname{div}(\rho_s w) = f^0,$$

$$(3.2) \quad \begin{aligned} \partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w + \nabla \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi \right) \\ + \left(\frac{\nu}{\gamma^2 \rho_s^2} \Delta v_s \right) \phi + v_s \cdot \nabla w + w \cdot \nabla v_s = \mathbf{f}, \end{aligned}$$

$$(3.3) \quad w|_{x_n=0,1} = 0,$$

$$(3.4) \quad u|_{t=0} = u_0 = {}^\top(\phi_0, w_0).$$

Here $\tilde{\nu} = \nu + \nu'$; and f^0 and \mathbf{f} denote the nonlinearities:

$$\begin{aligned} f^0 &= -\operatorname{div}(\phi w), \\ \mathbf{f} &= -w \cdot \nabla w + \frac{\nu\phi}{(\phi+\gamma^2\rho_s)\rho_s} \left(-\Delta w + \left(\frac{1}{\gamma^2\rho_s} \Delta v_s \right) \phi \right) \\ &\quad - \frac{\tilde{\nu}\phi}{(\phi+\gamma^2\rho_s)\rho_s} \nabla \operatorname{div} w \\ &\quad + \frac{\phi}{\gamma^2\rho_s} \nabla \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s} \phi \right) - \frac{1}{2\gamma^4\rho_s} \nabla \left(\tilde{P}''(\rho_s) \phi^2 \right) \\ &\quad + \tilde{P}^3(\rho_s, \phi, \partial_x \phi), \end{aligned}$$

where

$$\begin{aligned} \tilde{P}_3 &= \frac{\phi^3}{\gamma^4(\phi+\gamma^2\rho_s)\rho_s^3} \nabla \tilde{P}(\rho_s) - \frac{1}{2\gamma^6\rho_s} \nabla (\phi^3 P_3(\rho_s, \phi)) \\ &\quad + \frac{\phi}{2\gamma^6\rho_s^2} \nabla \left(\tilde{P}''(\rho_s) \phi^2 + \frac{1}{\gamma^2} \phi^3 P_3(\rho, \phi) \right) \\ &\quad - \frac{\phi^2}{\gamma^2(\phi+\gamma^2\rho_s)\rho_s} \nabla \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2} \phi + \frac{1}{2\gamma^4} \tilde{P}''(\rho_s) \phi^2 + \frac{1}{2\gamma^6} \phi^3 P_3(\rho_s, \phi) \right) \end{aligned}$$

with

$$P_3(\rho_s, \phi) = \int_0^1 (1-\theta)^2 P'''(\theta\gamma^{-2}\phi + \rho_s) d\theta.$$

Before stating the main result we mention the compatibility condition for $u_0 = {}^\top(\phi_0, w_0)$. We will look for a solution $u = (\phi, w)$ of (3.1)–(3.4) in $\cap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, \infty); H^{m-2j})$ satisfying $\int_0^t \|\partial_x w\|_{H^m}^2 d\tau < \infty$ for all $t \geq 0$ with $m \geq \lfloor n/2 \rfloor + 1$. Therefore, we need to require the compatibility condition for the initial value $u_0 = {}^\top(\phi_0, w_0)$, which is formulated as follows.

Let $u = {}^\top(\phi, w)$ be a smooth solution of (3.1)–(3.4). Then $\partial_t^j u = {}^\top(\partial_t^j \phi, \partial_t^j w)$ ($j \geq 1$) is inductively determined by

$$\partial_t^j \phi = -v_s \cdot \nabla \partial_t^{j-1} \phi - \gamma^2 \operatorname{div}(\rho_s \partial_t^{j-1} w) + \partial_t^{j-1} f^0$$

and

$$\begin{aligned} \partial_t^j w &= \frac{\nu}{\rho_s} \Delta \partial_t^{j-1} w + \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} \partial_t^{j-1} w - \nabla \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s} \partial_t^{j-1} \phi \right) \\ &\quad - \left(\frac{\nu}{\gamma^2\rho_s^2} \Delta v_s \right) \partial_t^{j-1} \phi - v_s \cdot \nabla \partial_t^{j-1} w - \partial_t^{j-1} w \cdot \nabla v_s + \partial_t^{j-1} \mathbf{f}. \end{aligned}$$

From these relations we see that $\partial_t^j u|_{t=0} = {}^\top(\partial_t^j \phi, \partial_t^j w)|_{t=0}$ is inductively given by $u_0 = {}^\top(\phi_0, w_0)$ in the following way:

$$\partial_t^j u|_{t=0} = {}^\top(\partial_t^j \phi, \partial_t^j w)|_{t=0} = {}^\top(\phi_j, w_j) = u_j,$$

where

$$\begin{aligned}
\phi_j &= -v_s \cdot \nabla \phi_{j-1} - \gamma^2 \operatorname{div}(\rho_s w_{j-1}) \\
&\quad + f_{j-1}^0(u_0, \dots, u_{j-1}, \partial_x w_0, \dots, \partial_x w_{j-1}), \\
w_j &= \frac{\nu}{\rho_s} \Delta w_{j-1} + \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w_{j-1} - \nabla \left(\frac{\bar{P}'(\rho_s)}{\gamma^2 \rho_s} \phi_{j-1} \right) \\
&\quad - \left(\frac{\nu}{\gamma^2 \rho_s^2} \Delta v_s \right) \phi_{j-1} - v_s \cdot \nabla w_{j-1} - w_{j-1} \cdot \nabla v_s \\
&\quad + \mathbf{f}_{j-1}(u_0, \dots, u_{j-1}, \dots, \partial_x u_{j-1}, \dots, \partial_x^2 w_{j-1}).
\end{aligned}$$

Here $f_l^0(u_0, \dots, u_l, \dots)$ is a certain polynomial in u_0, \dots, u_l, \dots ; \dots , and so on.

By the boundary condition $w|_{x_n=0,1} = 0$ in (3.3), we necessarily have $\partial_t^j w|_{x_n=0,1} = 0$, and hence,

$$w_j|_{x_n=0,1} = 0.$$

Assume that $u = {}^\top(\phi, w)$ is a solution of (3.1)–(3.4) in $\cap_{j=0}^{[\frac{m}{2}]} C^j([0, T_0]; H^{m-2j})$ for some $T_0 > 0$. Then, from the above observation, we need the regularity $u_j = {}^\top(\phi_j, w_j) \in H^{m-2j}$ for $j = 0, \dots, [m/2]$, which, indeed, follows from the fact that $u_0 = {}^\top(\phi_0, w_0) \in H^m$ with $m \geq [n/2] + 1$. Furthermore, it is necessary to require that $u_0 = {}^\top(\phi_0, w_0)$ satisfies the \hat{m} -th order compatibility condition:

$$w_j \in H_0^1 \quad \text{for } j = 0, 1, \dots, \hat{m} = \left\lfloor \frac{m-1}{2} \right\rfloor.$$

We are ready to state our main result of this paper.

Theorem 3.1. *Assume that $n \geq 3$. Let m be an integer satisfying $m \geq [n/2] + 1$. Then there are positive numbers ν_0, γ_0 and $\tilde{\omega}$ such that if $\nu \geq \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ and $\|\tilde{\mathbf{g}}\|_{C^m[0,1]} \leq \tilde{\omega}$, then the following assertion holds. There is a positive number ε_0 such that if $u_0 = {}^\top(\phi_0, w_0) \in H^m$ satisfies $\|u_0\|_{H^m} \leq \varepsilon_0$ and the \hat{m} -th compatibility condition, then there exists a unique global solution $u(t) = {}^\top(\phi(t), w(t))$ of (3.1)–(3.4) in $\cap_{j=0}^{[\frac{m}{2}]} C^j([0, \infty); H^{m-2j})$ which satisfies*

$$(3.5) \quad \|u(t)\|_{H^m}^2 + \int_0^t \|\partial_{x'} \sigma_1\|_2^2 + \|\partial_x \phi_\infty\|_{H^{m-1}}^2 + \|\partial_x w_\infty\|_{H^m}^2 d\tau \leq C \|u_0\|_{H^m}^2$$

uniformly for $t \geq 0$. Here $\sigma_1 = \sigma_1(x', t) = \mathcal{F}^{-1}(\hat{\chi}_1 \langle \hat{\phi}(t) \rangle)$; $\hat{\chi}_1(\xi') = 1$ for $|\xi'| \leq 1$ and $\hat{\chi}_1(\xi') = 0$ for $|\xi'| > 1$; $u^{(0)} = u^{(0)}(x_n)$ is the function given in Lemma 4.1 below; and $u_\infty = {}^\top(\phi_\infty, w_\infty) = u - \sigma_1 u^{(0)}$.

Remark. Once (3.5) is obtained for $m \geq [n/2] + 2$, then one can establish the decay estimates

$$\begin{aligned} \|u(t)\|_2 &= O(t^{-\frac{n-1}{4}}), \\ \|u(t) - (\sigma u^{(0)})(t)\|_2 &= O(t^{-\frac{n-1}{4} - \frac{1}{2}} L(t)) \end{aligned}$$

as $t \rightarrow \infty$, provided that $u_0 = (\phi_0, w_0) \in H^m \cap L^1$ with $\|u_0\|_{H^m \cap L^1} \ll 1$. Here $\sigma = \sigma(x', t) = \mathcal{F}^{-1}(e^{-(ia_0\xi_1 + \kappa_0|\xi_1|^2 + \kappa_1|\xi''|^2)t} \langle \hat{\phi}_0 \rangle)$ with some constants $a_0 \in \mathbf{R}$, $\kappa_0 > 0$ and $\kappa_1 > 0$; and $L(t) = 1$ when $n \geq 4$ and $L(t) = \log(1+t)$ when $n = 3$. In fact, this can be proved in a similar manner to the case of the plane Couette flow [3] by using (3.5) and the decay estimates for the linearized problem given in [7]. We note that in [7] it is considered the special case of a Poiseuille type flow, but one can easily see that the argument in [7] is valid for general parallel flows given in Proposition 2.1 if $\|\tilde{\mathbf{g}}\|_{C^1[0,1]}$ is sufficiently small.

Theorem 3.1 is proved by showing the local existence of solutions and the a priori estimate such as (3.5). The local existence is proved by applying the local solvability result in [4]. The a priori estimate will be derived in section 5. To do so, in section 4, we introduce a decomposition of solutions; and then, in section 5, we establish the necessary a priori estimate in Proposition 5.1 by a variant of the Matsumura-Nishida energy method.

4. Decomposition of solutions

We write (3.1)–(3.4) as

$$(4.1) \quad \partial_t u + Lu = \mathbf{F}, \quad w|_{x_n=0,1} = 0, \quad u|_{t=0} = u_0.$$

Here L is the operator of the form

$$L = A + B + C_0,$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\nu}{\rho_s} \Delta I_n - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} \end{pmatrix}, \quad B = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & v_s^1 \partial_{x_1} I_n \end{pmatrix}, \\ C_0 &= \begin{pmatrix} 0 & 0 \\ \frac{\nu \Delta v_s^1}{\gamma^2 \rho_s^2} \mathbf{e}_1 & (\partial_{x_n} v_s^1) \mathbf{e}_1^\top \mathbf{e}_n \end{pmatrix}, \end{aligned}$$

and $\mathbf{F} = {}^\top(f^0, \mathbf{f})$ is the nonlinearity. Note that

$$(4.2) \quad \langle \langle Au, u \rangle \rangle = \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2, \quad \langle \langle Bu_1, u_2 \rangle \rangle = -\langle \langle u_1, Bu_2 \rangle \rangle$$

for $u, u_1, u_2 \in H^1 \times (H^2 \cap H_0^1)$.

In the analysis of this paper we will decompose the solution by a projection operator associated with the linearized operator. To do so, we consider the Fourier transform of (3.1)–(3.4) in $x' \in \mathbf{R}^{n-1}$:

$$(4.3) \quad \partial_t \hat{\phi} + i\xi_1 v_s^1 \hat{\phi} + i\gamma^2 \xi' \cdot (\rho_s \hat{w}') + \gamma^2 \partial_{x_n} (\rho_s \hat{w}^n) = \hat{f}^0,$$

$$(4.4) \quad \begin{aligned} & \partial_t \hat{w}' + \nu(|\xi'|^2 - \partial_{x_n}^2) \hat{w}' - i\tilde{\nu} \xi' (i\xi' \cdot \hat{w}' + \partial_{x_n} \hat{w}^n) + i\xi' \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \hat{\phi} \right) \\ & + \frac{\nu \Delta v_s^1}{\gamma^2 \rho_s^2} \hat{\phi} \mathbf{e}_1' + i\xi_1 v_s^1 \hat{w}' + (\partial_{x_n} v_s^1) \hat{w}^n \mathbf{e}_1' = \hat{\mathbf{f}}', \end{aligned}$$

$$(4.5) \quad \begin{aligned} & \partial_t \hat{w}^n + \nu(|\xi'|^2 - \partial_{x_n}^2) \hat{w}^n - \tilde{\nu} \partial_{x_n} (i\xi' \cdot \hat{w}' + \partial_{x_n} \hat{w}^n) \\ & + \partial_{x_n} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \hat{\phi} \right) + i\xi_1 v_s^1 \hat{w}^n = \hat{f}^n, \end{aligned}$$

$$(4.6) \quad \hat{w}|_{x_n=0,1} = 0,$$

$$(4.7) \quad \hat{u}|_{t=0} = \hat{u}_0 = {}^\top(\hat{\phi}_0, \hat{w}_0).$$

Here $\hat{\phi} = \hat{\phi}(\xi', x_n, t)$ and $\hat{w} = \hat{w}(\xi', x_n, t)$ are the Fourier transform of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbf{R}^{n-1}$ with $\xi' \in \mathbf{R}^{n-1}$ being the dual variable. We thus arrive at the following problem

$$(4.8) \quad \partial_t \hat{u} + \hat{L}_{\xi'} \hat{u} = \hat{\mathbf{F}}, \quad \hat{u}|_{t=0} = \hat{u}_0$$

with a parameter $\xi' \in \mathbf{R}^{n-1}$. Here $\hat{L}_{\xi'}$ is the operator on $H^1(0, 1) \times L^2(0, 1)$ of the form

$$\hat{L}_{\xi'} = \hat{A}_{\xi'} + \hat{B}_{\xi'} + \hat{C}_0,$$

where

$$\begin{aligned} \hat{A}_{\xi'} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_s} \xi'^\top \xi' & -i\frac{\tilde{\nu}}{\rho_s} \xi' \partial_{x_n} \\ 0 & -i\frac{\tilde{\nu}}{\rho_s} \xi'^\top \partial_{x_n} & \frac{\nu}{\rho_s}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_s} \partial_{x_n}^2 \end{pmatrix}, \\ \hat{B}_{\xi'} &= \begin{pmatrix} i\xi_1 v_s^1 & i\gamma^2 \rho_s^\top \xi' & \gamma^2 \partial_{x_n} (\rho_s \cdot) \\ i\xi' \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} & i\xi_1 v_s^1 I_{n-1} & 0 \\ \partial_{x_n} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & 0 & i\xi_1 v_s^1 \end{pmatrix}, \end{aligned}$$

$$\hat{C}_0 = C_0$$

with domain

$$D(\hat{L}_{\xi'}) = H^1(0, 1) \times (H^2(0, 1) \cap H_0^1(0, 1)),$$

We will make use of a spectral property of the linearized operator $\hat{L}_0 = \hat{A}_0 + \hat{B}_0 + \hat{C}_0$ concerned with problem (4.3)–(4.7) with $\xi' = 0$.

We also introduce a formal adjoint operator \hat{L}_0^* of \hat{L}_0 with respect to the inner product $\langle \cdot, \cdot \rangle$:

$$\hat{L}_0^* = \hat{A}_0 - \hat{B}_0 + \hat{C}_0^*,$$

with domain $D(\hat{L}_0^*) = D(\hat{L}_0)$, where

$$\hat{C}_0^* = \begin{pmatrix} 0 & \frac{\gamma^2 \nu \Delta v_s^1}{\rho_s \bar{P}'(\rho_s)}^\top \mathbf{e}'_1 & 0 \\ 0 & 0 & 0 \\ 0 & (\partial_{x_n} v_s^1)^\top \mathbf{e}'_1 & 0 \end{pmatrix}.$$

Lemma 4.1. ([7]) *Let ${}^\top(\rho_s, v_s)$ be a stationary solution obtained in Proposition 2.1. Then the following assertions hold.*

(i) $\lambda = 0$ is a simple eigenvalue of \hat{L}_0 and \hat{L}_0^* .

(ii) The eigenspaces for $\lambda = 0$ of \hat{L}_0 and \hat{L}_0^* are spanned by $u^{(0)}$ and $u^{(0)*}$ respectively, where

$$u^{(0)} = {}^\top(\phi^{(0)}, w^{(0),1} \mathbf{e}'_1, 0)$$

and

$$u^{(0)*} = {}^\top(\phi^{(0)*}, 0, 0)$$

with

$$\phi^{(0)}(x_n) = \alpha_0 \frac{\gamma^2 \rho_s(x_n)}{\bar{P}'(\rho_s(x_n))}, \quad \alpha_0 = \left(\int_0^1 \frac{\gamma^2 \rho_s}{\bar{P}'(\rho_s)} dx_n \right)^{-1},$$

$$w^{(0),1}(x_n) = \frac{1}{\nu \gamma^2} \int_0^1 G(x_n, y_n) \tilde{g}^1(y_n) \phi^{(0)}(y_n) dy_n,$$

$$\phi^{(0)*}(x_n) = \frac{\gamma^2}{\alpha_0} \phi^{(0)}(x_n).$$

Here

$$G(x_n, y_n) = \begin{cases} (1 - x_n)y_n & (0 < y_n < x_n) \\ x_n(1 - y_n) & (x_n < y_n < 1). \end{cases}$$

(iii) The eigenprojection $\hat{\Pi}^{(0)}$ for $\lambda = 0$ of \hat{L}_0 is given by

$$\hat{\Pi}^{(0)} u = \langle u, u^{(0)*} \rangle u^{(0)} = \langle \phi \rangle u^{(0)} \quad \text{for } u = {}^\top(\phi, w).$$

(iv) Let $u^{(0)}$ be written as $u^{(0)} = u_0^{(0)} + u_1^{(0)}$, where

$$u_0^{(0)} = {}^\top(\phi^{(0)}, 0, 0), \quad u_1^{(0)} = {}^\top(0, w^{(0),1} \mathbf{e}'_1, 0).$$

Then $u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$ and

$$\langle u, u^{(0)} \rangle = \frac{\alpha_0}{\gamma^2} \langle \phi \rangle + (w^1, w^{(0),1} \rho_s)$$

for $u = {}^\top(\phi, w', w^n)$.

Remark 4.2. We note that if $|\tilde{g}^n|_\infty$ is sufficiently small, then

$$\phi^{(0)} = O(1) > 0, \quad \alpha_0 = O(1) > 0, \quad w^{(0),1} = O(1/\gamma^2).$$

Since $\rho_s > 0$ and $\tilde{P}'(\rho_s)$, Lemma 4.1 can be proved in the same way as the proof of [7, Lemma 4.3].

We now introduce a projection operator P_1 . We define the projection P_1 by

$$P_1 u = \mathcal{F}^{-1}(\hat{\chi}_1 \hat{I}^{(0)} \hat{u}) = \mathcal{F}^{-1}(\hat{\chi}_1 \langle \hat{Q}_0 u \rangle) u^{(0)},$$

i.e.,

$$P_1 u = \mathcal{F}^{-1}(\hat{\chi}_1 \langle \hat{\phi} \rangle) u^{(0)} \quad \text{for } u = {}^\top(\phi, w),$$

and P_∞ by

$$P_\infty = I - P_1.$$

Here $\hat{\chi}_1(\xi') = 1$ for $|\xi'| \leq 1$ and $\hat{\chi}_1(\xi') = 0$ for $|\xi'| > 1$.

We state several properties of P_1 and P_∞ which are easily seen but useful in the subsequent analysis. One can easily see that differentiations in x of any order are bounded on the range of P_1 . As for the P_∞ -part, one can see that the Poincaré inequality holds. In fact, let $P_\infty u = {}^\top(\phi_\infty, w_\infty)$. Since $\langle \hat{\phi}_\infty(\xi', \cdot) \rangle = 0$ for $|\xi'| \leq 1$, we have $|\hat{\phi}_\infty(\xi', \cdot)|_2 \leq C |\partial_{x_n} \hat{\phi}_\infty(\xi', \cdot)|_2$ for $|\xi'| \leq 1$. We also have $\int_{|\xi'| > 1} |\hat{\phi}_\infty(\xi', \cdot)|_2^2 d\xi' \leq \int_{\mathbf{R}^{n-1}} |\xi'|^2 |\hat{\phi}_\infty(\xi', \cdot)|_2^2 d\xi'$. Therefore, by the Plancherel Theorem,

$$\|\phi_\infty\|_2 \leq C \|\hat{\phi}_\infty\|_2 \leq C \{ \|\partial_{x_n} \hat{\phi}_\infty\|_2 + \|(\widehat{\partial_{x'} \phi_\infty})\|_2 \} \leq C \|\partial_x \phi_\infty\|_2.$$

As for w_∞ , we note that if $w|_{x_n=0,1} = 0$, then $w_\infty|_{x_n=0,1} = 0$, since $w^{(0),1}|_{x_n=0,1} = 0$. Therefore, we have $\|w_\infty\|_2 \leq \|\partial_{x_n} w_\infty\|_2$ if $w|_{x_n=0,1} = 0$. We write these properties in the following lemma.

Lemma 4.3. *There hold the following assertions.*

- (i) $P_j^2 = P_j$ ($j = 1, \infty$).
- (ii) $\|\partial_{x'}^k P_1 u\|_2 \leq \|P_1 u\|_2$ for all $k = 0, 1, 2, \dots$.
- (iii) $\|\partial_{x_n}^l P_1 u\|_2 \leq C_l \|P_1 u\|_2$ for all $l = 0, 1, 2, \dots$.
- (iv) Let $u = {}^\top(\phi, w)$ and let $P_\infty u = {}^\top(\phi_\infty, w_\infty)$. Then

$$\|\phi_\infty\|_2 \leq C \|\partial_x \phi_\infty\|_2.$$

Furthermore, if $w|_{x_n=0,1} = 0$, then

$$\|w_\infty\|_2 \leq C \|\partial_x w_\infty\|_2.$$

Therefore, if $w|_{x_n=0,1} = 0$, then

$$\|P_\infty u\|_2 \leq C \|\partial_x P_\infty u\|_2.$$

We now decompose the solution $u(t)$ of (3.1)–(3.4) into the P_1 and P_∞ parts. Let $T > 0$ and let $u(t) = {}^\top(\phi(t), w(t))$ be a solution of (3.1)–(3.4), i.e., a solution of (4.1), in $\cap_{j=0}^{\lfloor \frac{m}{2} \rfloor} C^j([0, T]; H^{m-2j})$ with $\partial_t^j w \in L^2(0, T; H^{m+1-2j})$, $0 \leq j \leq \lfloor (m+1)/2 \rfloor$. We decompose $u(t)$ as

$$u(t) = (\sigma_1 u^{(0)})(t) + u_\infty(t),$$

where

$$(\sigma_1 u^{(0)})(t) = P_1 u(t), \quad \sigma_1 = \sigma_1(x', t) = \mathcal{F}^{-1}(\hat{\chi}_1 \langle \hat{\phi}(t) \rangle),$$

$$u_\infty(t) = P_\infty u(t).$$

Note that σ_1 is a function of x' and t and $u^{(0)}$ is a function of x_n . Furthermore, as for u_∞ -part, we have the Poincaré inequality $\|u_\infty\|_2 \leq \|\partial_x u_\infty\|_2$ by Lemma 4.3.

We now deduce the equations for $\sigma_1(t)$ and $u_\infty(t)$. To do so, we introduce some notations. We define $\langle \cdot \rangle_1$ by

$$\langle f \rangle_1 = \mathcal{F}^{-1}(\hat{\chi}_1 \langle \hat{f} \rangle).$$

We also denote

$$\tilde{\mathcal{M}} = L - \hat{L}_0 = \tilde{A} + \tilde{B}$$

with

$$\begin{aligned}\tilde{A} = A - \hat{A}_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s} \Delta' I_{n-1} - \frac{\tilde{\nu}}{\rho_s} \nabla'^\top \nabla' & -\frac{\tilde{\nu}}{\rho_s} \nabla' \partial_{x_n} \\ 0 & -\frac{\tilde{\nu}}{\rho_s} \partial_{x_n}^\top \nabla' & -\frac{\nu}{\rho_s} \Delta' \end{pmatrix}, \\ \tilde{B} = B - \hat{B}_0 &= \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \rho_s^\top \nabla' & 0 \\ \nabla' \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & v_s^1 \partial_{x_1} I_{n-1} & 0 \\ 0 & 0 & v_s^1 \partial_{x_1} \end{pmatrix},\end{aligned}$$

where $\nabla' = {}^\top(\partial_{x_1}, \dots, \partial_{x_{n-1}})$, $\Delta' = \partial_{x_1}^2 + \dots + \partial_{x_{n-1}}^2$. We note the relations

$$P_1 L = P_1 \tilde{\mathcal{M}}, \quad L(\sigma_1 u^{(0)}) = \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), \quad Q_0 \tilde{\mathcal{M}} = Q_0 \tilde{B},$$

and

$$(4.9) \quad \langle \langle \tilde{B} u_1, u_2 \rangle \rangle = -\langle \langle u_1, \tilde{B} u_2 \rangle \rangle.$$

Applying P_1 and P_∞ to (4.1), we have

$$(4.10) \quad \partial_t \sigma_1 + \langle Q_0 \tilde{\mathcal{M}}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 = \langle Q_0 \mathbf{F} \rangle_1,$$

$$(4.11) \quad \partial_t u_\infty + L u_\infty + \tilde{\mathcal{M}}(\sigma_1 u^{(0)}) - \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u^{(0)} = \mathbf{F}_\infty,$$

$$(4.12) \quad w_\infty|_{x_n=0,1} = 0,$$

$$(4.13) \quad \sigma_1|_{t=0} = \sigma_{1,0}, \quad u_\infty|_{t=0} = u_{\infty,0}.$$

Here $u_\infty = {}^\top(\phi_\infty, w_\infty) = P_\infty u$, $\sigma_{1,0} = \langle \phi_0 \rangle_1$, and $u_{\infty,0} = P_\infty u_0$. In what follows we will denote $\mathbf{F}_\infty = {}^\top(f_\infty^0, \mathbf{f}_\infty)$.

We note that if u is a solution of (4.1) in $\cap_{j=0}^{[m/2]} C([0, T]; H^{m-2j})$, then $w_\infty|_{x_n=0,1} = 0$ since $w^{(0),1}|_{x_n=0,1} = 0$; and furthermore, $u_\infty \in \cap_{j=0}^{[m/2]} C([0, T]; H^{m-2j})$ by Lemma 4.3.

We close this section with several lemmas, which will be used in the next section to derive the a priori estimate.

Lemma 4.4. (i) For $\sigma = \sigma(x')$, $\langle \sigma f \rangle_1 = \sigma \langle f \rangle_1$.

- (ii) $\langle \partial_{x'}^k f \rangle_1 = \partial_{x'}^k \langle f \rangle_1$ and $\|\partial_{x'} \langle f \rangle_1\|_2 \leq \|\langle f \rangle_1\|_2$.
 (iii) Let $\sigma = \sigma(x')$ with $\text{supp}(\hat{\sigma}) \subset \{|\xi'| \leq 1\}$. Then

$$(\langle Q_0 \tilde{B}u \rangle_1, \sigma) = -\frac{\gamma^2}{\alpha_0} \langle \langle u, \tilde{B}(\sigma u_0^{(0)}) \rangle \rangle.$$

- (iv) $\langle \langle \langle f \rangle_1 u_0^{(0)}, u_\infty \rangle \rangle = 0$ for $u_\infty \in \text{Range}(P_\infty)$.

Proof. It is easy to see (i) and (ii). Let us prove (iii). Since $\langle u \rangle = \langle u, u^{(0)*} \rangle$ and $Q_0 u^{(0)*} = u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$, we have, by the Plancherel theorem and (4.9),

$$\begin{aligned} (\langle Q_0 \tilde{B}u \rangle_1, \sigma) &= (2\pi)^{-(n-1)} \langle \langle \mathcal{F} Q_0 \tilde{B}u, \hat{\sigma} u^{(0)*} \rangle \rangle \\ &= \langle \langle Q_0 \tilde{B}u, \sigma u^{(0)*} \rangle \rangle \\ &= -\langle \langle u, \tilde{B}(\sigma u^{(0)*}) \rangle \rangle \\ &= -\frac{\gamma^2}{\alpha_0} \langle \langle u, \tilde{B}(\sigma u_0^{(0)}) \rangle \rangle. \end{aligned}$$

This proves (iii). As for (iv), since $u_0^{(0)} = {}^\top(\phi^{(0)}, 0)$ and $\hat{\chi}_1 \langle \hat{\phi}_\infty \rangle = 0$, we have

$$\begin{aligned} \langle \langle \langle f \rangle_1 u_0^{(0)}, u_\infty \rangle \rangle &= (2\pi)^{-(n-1)} \frac{\alpha_0}{\gamma^2} (\hat{\chi}_1 \langle \hat{f} \rangle, \hat{\phi}_\infty) \\ &= (2\pi)^{-(n-1)} \frac{\alpha_0}{\gamma^2} (\hat{\chi}_1 \langle \hat{f} \rangle, \langle \hat{\phi}_\infty \rangle)_{L^2(\mathbf{R}^{n-1})} \\ &= 0. \end{aligned}$$

This completes the proof. \square

Lemma 4.5. *There hold the following assertions.*

- (i) $\|\langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1\|_2^2 \leq C(\|\partial_{x'} \sigma_1\|_2^2 + \|\partial_{x'} \phi_\infty\|_2^2 + \gamma^4 \|\partial_{x'} w_\infty\|_2^2)$.
 (ii) If $w_\infty^n|_{x_n=0,1} = 0$, then $\langle Q_0 \tilde{B}u_\infty \rangle_1 = \langle Q_0 B u_\infty \rangle_1 = \langle v_s^1 \partial_{x_1} \phi_\infty + \gamma^2 \text{div}(\rho_s w_\infty) \rangle_1$.
 (iii) If $w_\infty^n|_{x_n=0,1} = 0$, $\|\tilde{\mathbf{g}}\|_{C^m[0,1]} \leq \tilde{\omega}$ and $2j + k \leq m$, then

$$\begin{aligned} &\|\partial_{x'}^k \partial_t^j \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1\|_2^2 \\ &\leq C\{\|\partial_{x'}^p \partial_t^j \sigma_1\|_2^2 + \|\partial_{x'}^q \partial_t^j \phi_\infty\|_2^2 + \gamma^4 \|\text{div}(\partial_{x'}^r \partial_t^j w_\infty)\|_2^2 + \gamma^4 \tilde{\omega}^2 \|\partial_{x'}^s \partial_t^j w_\infty\|_2^2\} \end{aligned}$$

for $0 \leq p, q \leq k+1$, $0 \leq r, s \leq k$.

Proof. A direct computation gives (i), By integration by parts, we see $\langle \operatorname{div}(\rho_s w_\infty) \rangle_1 = \langle \nabla' \cdot (\rho_s w'_\infty) \rangle_1$. This gives (ii). By using (ii) and Lemma 4.4, one can obtain (iii) by a direct computation. This completes the proof. \square

5. A priori estimate

In this section we establish the a priori estimate such as (3.5). We introduce the following notations:

$$\begin{aligned} \llbracket f(t) \rrbracket_m &= \left(\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \|\partial_t^j f(t)\|_{H^{m-2j}}^2 \right)^{\frac{1}{2}}, \\ |||Df(t)|||_m &= \begin{cases} \|\partial_x f(t)\|_2 & \text{for } m = 0, \\ (\llbracket \partial_x f(t) \rrbracket_m^2 + \llbracket \partial_t f(t) \rrbracket_{m-1}^2)^{\frac{1}{2}} & \text{for } m \geq 1. \end{cases} \end{aligned}$$

We will prove the following estimate.

Proposition 5.1. *Let m be an integer satisfying $m \geq [n/2] + 1$. There are positive numbers ν_0 , γ_0 , $\tilde{\omega}$ and ε_1 such that the following assertion holds.*

Let $u(t)$ be a solution of (4.1) in $\cap_{j=0}^{\lfloor m/2 \rfloor} C^j([0, T]; H^{m-2j})$ with $\int_0^T |||Dw_\infty|||_m^2 d\tau < \infty$. If $\llbracket u(t) \rrbracket_m \leq \varepsilon_1$ for $t \in [0, T]$, then there holds the inequality

$$\llbracket u(t) \rrbracket_m^2 + \int_0^t (|||D\sigma_1|||_{m-1}^2 + |||D\phi_\infty|||_{m-1}^2 + |||Dw_\infty|||_m^2) d\tau \leq C \|u_0\|_{H^m}^2$$

for $t \in [0, T]$ with $C > 0$ independent of T , provided that $\nu \geq \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$ and $\|\tilde{\mathbf{g}}\|_{C^m[0,1]} \leq \tilde{\omega}$.

To prove Proposition 5.1, we first derive fundamental estimates in the energy method (section 5.1); we then combine them to obtain the H^m -energy inequality (section 5.2); and we finally estimate the nonlinearities to complete the a priori estimate given in Proposition 5.1 (section 5.3).

Throughout this section we assume that $u(t) = (\sigma_1 u^{(0)})(t) + u_\infty(t)$ is a solution of (4.1) in $\cap_{j=0}^{\lfloor m/2 \rfloor} C([0, T]; H^{m-2j})$ with $\int_0^T |||Dw_\infty||| d\tau < \infty$ for an arbitrarily fixed $T > 0$.

We also assume that $\|\tilde{\mathbf{g}}\|_{C^m[0,1]} \leq \tilde{\omega}$ for a constant $\tilde{\omega}$.

5.1. Fundamental estimates

We introduce some quantities. Let $E^{(0)}[u]$ and $D^{(0)}[w]$ be defined by

$$E^{(0)}[u] = \frac{\alpha_0}{\gamma^2} \|\sigma_1\|_2^2 + \frac{1}{\gamma^2} \left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} \phi_\infty \right\|_2^2 + \|\sqrt{\rho_s} w_\infty\|_2^2$$

for $u = \sigma_1 u^{(0)} + u_\infty$ with $u_\infty = {}^\top(\phi_\infty, w_\infty)$; and

$$D^{(0)}[w] = \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2.$$

Note that

$$\langle \langle Au, u \rangle \rangle = D^{(0)}[w]$$

for $u = {}^\top(\phi, w)$ with $w|_{x_n=0,1} = 0$.

In what follows we will denote the tangential derivatives $\partial_t^j \partial_{x'}^k$ by $T_{j,k}$:

$$T_{j,k}u = \partial_t^j \partial_{x'}^k u;$$

and, for operators A and B , we will denote the commutator $AB - BA$ by $[A, B]$.

We begin with the L^2 energy estimates for tangential derivatives.

Proposition 5.2. *There is a constant $\nu_0 > 0$ such that if $\nu \geq \nu_0$ and $\tilde{\omega} \leq 1$, then the following estimate holds for $0 \leq 2j + k \leq m$:*

$$(5.1) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} E^{(0)}[T_{j,k}u] + \frac{3}{4} D^{(0)}[T_{j,k}w_\infty] \\ & \leq R_{j,k}^{(1)} + C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) \|\tilde{T}_{j,k} \phi_\infty\|_2^2 \right\}, \end{aligned}$$

where

$$R_{j,k}^{(1)} = \frac{\alpha_0}{\gamma^2} (\langle Q_0 T_{j,k} \mathbf{F} \rangle_1, T_{j,k} \sigma_1) + \langle \langle T_{j,k} \mathbf{F}_\infty, T_{j,k} u_\infty \rangle \rangle,$$

and

$$\tilde{T}_{j,k} \phi_\infty = \begin{cases} \partial_x \phi_\infty & (j = k = 0), \\ T_{j,k} \phi_\infty & (2j + k \geq 1). \end{cases}$$

Proof. We consider the case $j = k = 0$. Recall that $u^{(0)} = u_0^{(0)} + u_1^{(0)}$ with $u_0^{(0)} = {}^\top(\phi^{(0)}, 0)$ and $u_1^{(0)} = {}^\top(0, w^{(0),1} \mathbf{e}_1)$. (See Lemma 4.1.)

We take the inner product of (4.10) with σ_1 to obtain

$$\frac{1}{2} \frac{d}{dt} \|\sigma_1\|_2^2 + (\langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1, \sigma_1) = (\langle Q_0 \mathbf{F} \rangle_1, \sigma_1).$$

By integration by parts, we have

$$\begin{aligned} (\langle Q_0 \tilde{B}(\sigma_1 u^{(0)}) \rangle_1, \sigma_1) &= (\langle v_s^1 \partial_{x_1} \sigma_1 \phi^{(0)} \rangle_1 + \langle \gamma^2 \rho_s \partial_{x_1} \sigma_1 w^{(0),1} \rangle_1, \sigma_1) \\ &= (\langle v_s^1 \phi^{(0)} \rangle + \langle \gamma^2 \rho_s w^{(0),1} \rangle) (\partial_{x_1} \sigma_1, \sigma_1) \\ &= 0. \end{aligned}$$

We also see from Lemma 4.4 (iii) that

$$(\langle Q_0 \tilde{B} u_\infty \rangle_1, \sigma_1) = -\frac{\gamma^2}{\alpha_0} \langle \langle u_\infty, \tilde{B}(\sigma_1 u_0^{(0)}) \rangle \rangle.$$

We thus obtain

$$(5.2) \quad \frac{1}{2} \frac{d}{dt} \|\sigma_1\|_2^2 - \frac{\gamma^2}{\alpha_0} \langle \langle u_\infty, \tilde{B}(\sigma_1 u_0^{(0)}) \rangle \rangle = (\langle Q_0 \mathbf{F} \rangle_1, \sigma_1).$$

We next take the inner product of (4.11) with u_∞ to obtain

$$(5.3) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} \phi_\infty \|_2^2 + \| \sqrt{\rho_s} w_\infty \|_2^2) + \langle \langle L u_\infty, u_\infty \rangle \rangle + \langle \langle \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), u_\infty \rangle \rangle \\ & - \langle \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u^{(0)}, u_\infty \rangle \rangle = \langle \langle \mathbf{F}_\infty, u_\infty \rangle \rangle. \end{aligned}$$

By (4.2), we see that

$$\langle \langle L u_\infty, u_\infty \rangle \rangle = D^{(0)}[w_\infty] + \langle \langle C_0 u_\infty, u_\infty \rangle \rangle.$$

By Lemma 4.4 (iv), we have

$$\langle \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u^{(0)}, u_\infty \rangle \rangle = \langle \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u_1^{(0)}, u_\infty \rangle \rangle.$$

It then follows from (5.3) that

$$(5.4) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} \phi_\infty \|_2^2 + \| \sqrt{\rho_s} w_\infty \|_2^2) \\ & + D^{(0)}[w_\infty] + \langle \langle C_0 u_\infty, u_\infty \rangle \rangle + \langle \langle \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), u_\infty \rangle \rangle \\ & - \langle \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u_1^{(0)}, u_\infty \rangle \rangle = \langle \langle \mathbf{F}_\infty, u_\infty \rangle \rangle. \end{aligned}$$

We add $\frac{\alpha_0}{\gamma^2} \times (5.2)$ to (5.3). Then, since

$$\begin{aligned} \langle \langle \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), u_\infty \rangle \rangle &= \langle \langle \tilde{A}(\sigma_1 u^{(0)}), u_\infty \rangle \rangle + \langle \langle \tilde{B}(\sigma_1 u_0^{(0)}), u_\infty \rangle \rangle \\ &+ \langle \langle \tilde{B}(\sigma_1 u_1^{(0)}), u_\infty \rangle \rangle, \end{aligned}$$

the term $\langle \langle \tilde{B}(\sigma_1 u_0^{(0)}), u_\infty \rangle \rangle$ is cancelled, and hence, we have

$$(5.5) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} E^{(0)}[T_{j,k} u] + D^{(0)}[T_{j,k} w_\infty] + \langle \langle C_0 u_\infty, u_\infty \rangle \rangle \\ & + \langle \langle \tilde{A}(\sigma_1 u^{(0)}), u_\infty \rangle \rangle + \langle \langle \tilde{B}(\sigma_1 u_1^{(0)}), u_\infty \rangle \rangle, \\ & - \langle \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u_1^{(0)}, u_\infty \rangle \rangle \\ & = \frac{\alpha_0}{\gamma^2} (\langle Q_0 \mathbf{F} \rangle_1, \sigma_1) + \langle \langle \mathbf{F}_\infty, u_\infty \rangle \rangle. \end{aligned}$$

A direct calculation gives, if $\nu \geq \nu_0$, then

$$(5.6) \quad |\langle\langle C_0 u_\infty, u_\infty \rangle\rangle| \leq \frac{1}{16} D^{(0)}[w_\infty] + C \frac{\nu}{\gamma^4} \|\phi_\infty\|_2^2.$$

Since $w^{(0),1} = O(\gamma^{-2})$, by integration by parts, we have

$$(5.7) \quad \begin{aligned} |\langle\langle \tilde{A}(\sigma_1 u^{(0)}), u_\infty \rangle\rangle| &\leq \frac{C}{\gamma^2} (\nu \|\nabla' \sigma_1\|_2 \|\nabla w_\infty\|_2 + \tilde{\nu} \|\partial_{x_1} \sigma_1\|_2 \|\operatorname{div} w_\infty\|_2) \\ &\leq \frac{1}{16} D^{(0)}[w_\infty] + C \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_{x'} \sigma_1\|_2^2, \end{aligned}$$

and, also,

$$(5.8) \quad \begin{aligned} |\langle\langle \tilde{B}(\sigma_1 u_1^{(0)}), u_\infty \rangle\rangle| &\leq \frac{C}{\gamma^2} \|\nabla' \sigma_1\|_2 (\|\phi_\infty\|_2 + \|w_\infty\|_2) \\ &\leq \frac{1}{16} D^{(0)}[w_\infty] + C \left\{ \left(\frac{1}{\gamma^2} + \frac{1}{\nu \gamma^4} \right) \|\partial_{x'} \sigma_1\|_2^2 + \frac{1}{\gamma^2} \|\phi_\infty\|_2^2 \right\}. \end{aligned}$$

By Lemma 4.5, we find that

$$\begin{aligned} &\langle\langle\langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u_1^{(0)}, u_\infty \rangle\rangle \\ &\leq \frac{C}{\gamma^2} \|\langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1\|_2 \|w_\infty\|_2 \\ &\leq \frac{C}{\gamma^2} \{ \|\partial_{x'} \sigma_1\|_2^2 + \|\partial_{x'} \phi_\infty\|_2^2 + \gamma^2 \|\operatorname{div} w_\infty\|_2 + \gamma^2 \tilde{\omega}^2 \|w_\infty\|_2 \} \|w_\infty\|_2. \end{aligned}$$

So, if $\nu \geq \nu_0$ for some $\nu_0 > 0$, then

$$(5.9) \quad \begin{aligned} &\langle\langle\langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u_1^{(0)}, u_\infty \rangle\rangle \\ &\leq \frac{1}{16} D^{(0)}[w_\infty] + \frac{C}{\gamma^4} (\|\partial_{x'} \sigma_1\|_2^2 + \|\partial_{x'} \phi_\infty\|_2^2). \end{aligned}$$

By using the Poincaré inequality $\|\phi_\infty\|_2 \leq \|\partial_x \phi_\infty\|_2$, we deduce from (5.5)–(5.9) that

$$(5.10) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} E^{(0)}[u] + \frac{3}{4} D^{(0)}[w_\infty] \\ &\leq R_{0,0}^{(1)} + C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\partial_{x'} \sigma_1\|_2^2 + \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) \|\partial_x \phi_\infty\|_2^2 \right\}. \end{aligned}$$

This proves (5.1) for $j = k = 0$. The case $2j + k \geq 1$ can be proved similarly by applying $T_{j,k}$ to (4.10) and (4.11) and using Lemma 4.4. This completes the proof. \square

We next derive the H^1 -parabolic estimates for w_∞ . We define $J[u]$ by

$$J[u] = -2\langle \langle \sigma_1 u^{(0)} + u_\infty, B\tilde{Q}u_\infty \rangle \rangle \quad \text{for } u = \sigma_1 u^{(0)} + u_\infty.$$

It is easy to see that if $\gamma^2 \geq 1$ and $\tilde{\omega} \leq 1$, then

$$\begin{aligned} |J[u]| &\leq C\left\{ \frac{1}{\gamma^2}(\|\sigma_1\|_2 + \|\phi_\infty\|_2)\|\gamma^2 \operatorname{div}(\rho_s w_\infty)\|_2 + \left(\frac{1}{\gamma^2}\|\sigma_1\|_2 + \|\phi_\infty\|_2\right)\|w_\infty\|_2 \right\} \\ &\leq \frac{b_0\gamma^2}{\nu}E^{(0)}[u] + \frac{1}{2}D^{(0)}[w_\infty] \end{aligned}$$

for some constant $b_0 > 0$.

Let b_1 be a positive constant (to be determined later) and define $E^{(1)}[u]$ by

$$E^{(1)}[u] = \left(1 + \frac{b_1\gamma^2}{\nu}\right)E^{(0)}[u] + D^{(0)}[w_\infty] + J[u].$$

Note that if $b_1 \geq 2b_0$, then $E^{(1)}[u]$ is equivalent to $E^{(0)}[u] + D^{(0)}[w_\infty]$.

Proposition 5.3. *There exists $b_1 \geq 2b_0$ such that if $\nu \geq \nu_0$, $\gamma^2 \geq 1$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq 1$ and $\tilde{\omega} \leq 1$, then the following estimate holds for $0 \leq 2k+j \leq m-1$:*

$$\begin{aligned} (5.11) \quad &\frac{1}{2} \frac{d}{dt} E^{(1)}[T_{j,k}u] + \frac{b_1\gamma^2}{\nu} D^{(0)}[T_{j,k}w_\infty] + \frac{1}{2} \|\sqrt{\rho_s} \partial_t T_{j,k}w_\infty\|_2^2 \\ &\leq R_{j,k}^{(2)} + C\left\{ \left(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\gamma^2}\right) \|\partial_{x'} T_{j,k}\sigma_1\|_2^2 \right. \\ &\quad \left. + \left(\frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4}\right) \|\tilde{T}_{j,k}\phi_\infty\|_2^2 + \frac{1}{\gamma^2} \|\partial_{x'} T_{j,k}\phi_\infty\|_2^2 \right\}, \end{aligned}$$

where

$$\begin{aligned} R_{j,k}^{(2)} &= \frac{2b_1\alpha_0}{\nu} (\langle Q_0 T_{j,k} \mathbf{F} \rangle_1, T_{j,k}\sigma_1) + \frac{2b_1\gamma^2}{\nu} \langle \langle T_{j,k} \mathbf{F}_\infty, T_{j,k}u_\infty \rangle \rangle \\ &\quad + C \left\{ \frac{1}{\gamma^2} \|T_{j,k}Q_0 \mathbf{F}\|_2^2 + \|T_{j,k}\tilde{Q}\mathbf{F}\|_2^2 \right\}. \end{aligned}$$

Proof. We consider the case $j = k = 0$. We take the inner product of (4.11) with $\partial_t \tilde{Q}u_\infty$ to obtain

$$\begin{aligned} (5.12) \quad &\|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 + \langle \langle Lu_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle + \langle \langle \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle \\ &- \langle \langle \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + T_{j,k}u_\infty) \rangle_1, \partial_t \tilde{Q}u_\infty \rangle \rangle = \langle \langle \mathbf{F}_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle. \end{aligned}$$

Let us first consider $\langle \langle Lu_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle$ on the left of (5.12) which is written as

$$(5.13) \quad \langle \langle Lu_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle = \langle \langle Au_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle + \langle \langle Bu_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle + \langle \langle C_0 u_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle.$$

The first term on the right of (5.13) is written as

$$(5.14) \quad \langle \langle Au_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle = \frac{1}{2} \frac{d}{dt} D^{(0)}[w_\infty].$$

As for the second term,

$$\begin{aligned} \langle \langle Bu_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle &= -\frac{d}{dt} \langle \langle u_\infty, B\tilde{Q}u_\infty \rangle \rangle + \langle \langle \partial_t u_\infty, B\tilde{Q}u_\infty \rangle \rangle \\ &= -\frac{d}{dt} \langle \langle u_\infty, B\tilde{Q}u_\infty \rangle \rangle + \langle \langle \partial_t Q_0 u_\infty, B\tilde{Q}u_\infty \rangle \rangle \\ &\quad + \langle \langle \partial_t \tilde{Q}u_\infty, B\tilde{Q}u_\infty \rangle \rangle. \end{aligned}$$

By (4.11), we have

$$\begin{aligned} \partial_t \phi_\infty &= -\{v_s^1 \partial_{x_1} \phi_\infty + \gamma^2 \operatorname{div}(\rho_s w_\infty) + (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1}) \partial_{x_1} \sigma_1 \\ &\quad - \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 \phi^{(0)}\} + f_\infty^0. \end{aligned}$$

Therefore, using the Poincaré inequality, if $\tilde{\omega} \leq 1$, then we have

$$\begin{aligned} &|\langle \langle \partial_t Q_0 u_\infty, B\tilde{Q}u_\infty \rangle \rangle| \\ &\leq C\{\|\partial_{x_1} \phi_\infty\|_2 + \gamma^2 \|\partial_x w_\infty\|_2 + \|\partial_{x'} \sigma_1\|_2\} \|\operatorname{div}(\rho_s w_\infty)\|_2 \\ &\quad + C\|Q_0 \mathbf{F}_\infty\|_2 \|\operatorname{div}(\rho_s w_\infty)\|_2 \\ &\leq C\left\{\frac{\gamma^2}{\nu} D^{(0)}[w_\infty] + \frac{1}{\gamma^2} (\|\partial_{x_1} \phi_\infty\|_2^2 + \|\partial_{x'} \sigma_1\|_2^2 + \|Q_0 \mathbf{F}_\infty\|_2^2)\right\}. \end{aligned}$$

We also have

$$\begin{aligned} |\langle \langle \partial_t \tilde{Q}u_\infty, B\tilde{Q}u_\infty \rangle \rangle| &\leq \|\sqrt{\rho_s} \partial_t w_\infty\|_2 \|\sqrt{\rho_s} v_s^1 \partial_{x_1} w_\infty\|_2 \\ &\leq \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 + \frac{C}{\nu} D^{(0)}[w_\infty]. \end{aligned}$$

We thus obtain, if $\gamma^2 \geq 1$, then

$$\begin{aligned} (5.15) \quad &\langle \langle Bu_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle \\ &\geq -\frac{d}{dt} \langle \langle u_\infty, B\tilde{Q}u_\infty \rangle \rangle - \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 \\ &\quad - C\left\{\frac{\gamma^2}{\nu} D^{(0)}[w_\infty] + \frac{1}{\gamma^2} (\|\partial_{x_1} \phi_\infty\|_2^2 + \|\partial_{x'} \sigma_1\|_2^2 + \|Q_0 \mathbf{F}_\infty\|_2^2)\right\}. \end{aligned}$$

As for the third term on the right of (5.13), we have

$$\begin{aligned} (5.16) \quad &\langle \langle C_0 u_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle \leq C\left\{\frac{\nu}{\gamma^2} \|\phi_\infty\|_2 + \|w_\infty\|_2\right\} \|\sqrt{\rho_s} \partial_t w_\infty\|_2 \\ &\leq \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 + C\left\{\frac{\nu^2}{\gamma^4} \|\phi_\infty\|_2^2 + \|w_\infty\|_2^2\right\} \\ &\leq \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 + C\left\{\frac{\nu^2}{\gamma^4} \|\phi_\infty\|_2^2 + \frac{1}{\nu} D^{(0)}[w_\infty]\right\}. \end{aligned}$$

It follows from (5.13)–(5.16) that if $\gamma^2 \geq 1$, then

$$\begin{aligned}
(5.17) \quad \langle \langle Lu_\infty, \partial_t \tilde{Q}u_\infty \rangle \rangle &\geq \frac{1}{2} \frac{d}{dt} (D^{(0)}[w_\infty] - 2\langle \langle u_\infty, B\tilde{Q}u_\infty \rangle \rangle) - \frac{1}{4} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 \\
&\quad - C \left\{ \frac{\gamma^2}{\nu} D^{(0)}[w_\infty] + \frac{\nu^2}{\gamma^4} \|\phi_\infty\|_2^2 \right. \\
&\quad \left. + \frac{1}{\gamma^2} (\|\partial_{x_1} \phi_\infty\|_2^2 + \|\partial_{x'} \sigma_1\|_2^2 + \|Q_0 \mathbf{F}_\infty\|_2^2) \right\}.
\end{aligned}$$

We next consider $\langle \langle \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle$ in (5.12):

$$\langle \langle \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle = \langle \langle \tilde{A}(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle + \langle \langle \tilde{B}(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle.$$

Each term on the right is estimated as follows. Since $w^{(0),1} = O(\gamma^{-2})$, we have

$$\begin{aligned}
|\langle \langle \tilde{A}(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle| &\leq C \frac{\nu + \tilde{\nu}}{\gamma^2} (\|\partial_{x'} \sigma_1\|_2 + \|\partial_{x'}^2 \sigma_1\|_2) \|\sqrt{\rho_s} \partial_t w_\infty\|_2 \\
&\leq \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 + C \frac{(\nu + \tilde{\nu})^2}{\gamma^4} \|\partial_{x'} \sigma_1\|_2^2.
\end{aligned}$$

By Lemma 4.1 (ii), we have $\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} = \alpha_0$, and so $\partial_{x_n} (\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)}) = 0$. This gives $\hat{B}_0 u^{(0)} = 0$, which, in turn, gives $\hat{B}_0(\sigma_1 u^{(0)}) = \sigma_1 \hat{B}_0 u^{(0)} = 0$. It follows that $\tilde{B}(\sigma_1 u^{(0)}) = \tilde{B}(\sigma_1 u^{(0)}) + \hat{B}_0(\sigma_1 u^{(0)}) = B(\sigma_1 u^{(0)})$. Therefore, we see, by (4.2),

$$\begin{aligned}
\langle \langle \tilde{B}(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle &= \langle \langle B(\sigma_1 u^{(0)}), \partial_t \tilde{Q}u_\infty \rangle \rangle \\
&= -\frac{d}{dt} \langle \langle \sigma_1 u^{(0)}, B\tilde{Q}u_\infty \rangle \rangle + \langle \langle \partial_t \sigma_1 u^{(0)}, B\tilde{Q}u_\infty \rangle \rangle.
\end{aligned}$$

As for the second term on the right, we see from (4.10)

$$\partial_t \sigma_1 = -\langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 + \langle Q_0 \mathbf{F} \rangle_1.$$

This, together with Lemma 4.5 (ii), implies

$$\begin{aligned}
|\langle \langle \partial_t \sigma_1 u^{(0)}, B\tilde{Q}u_\infty \rangle \rangle| &\leq C \{ \|\partial_{x'} \sigma_1\|_2 + \|\partial_{x'} \phi_\infty\|_2 + \gamma^2 \|\operatorname{div}(\rho_s w_\infty)\|_2 + \|\langle Q_0 \mathbf{F} \rangle_1\|_2 \} \\
&\quad \times \{ \|\operatorname{div}(\rho_s w_\infty)\|_2 + \frac{1}{\gamma^2} \|\partial_{x_1} w_\infty\|_2 \} \\
&\leq C \left\{ \frac{\gamma^2}{\nu} D^{(0)}[w_\infty] + \frac{1}{\gamma^2} (\|\partial_{x'} \sigma_1\|_2^2 + \|\partial_{x'} \phi_\infty\|_2^2 + \|\langle Q_0 \mathbf{F} \rangle_1\|_2^2) \right\}.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
(5.18) \quad & \langle \langle \tilde{\mathcal{M}}(\sigma_1 u^{(0)}), \partial_t \tilde{Q} u_\infty \rangle \rangle \\
& \geq -\frac{d}{dt} \langle \langle \sigma_1 u^{(0)}, B \tilde{Q} u_\infty \rangle \rangle - \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 \\
& \quad - C \left\{ \frac{\gamma^2}{\nu} D^{(0)}[w_\infty] + \frac{1}{\gamma^2} (\|\partial_{x'} \sigma_1\|_2^2 + \|\partial_{x'} \phi_\infty\|_2^2 + \|\langle Q_0 \mathbf{F} \rangle_1\|_2^2) \right. \\
& \quad \left. + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} \|\partial_{x'} \sigma_1\|_2^2 \right\}.
\end{aligned}$$

We also have

$$\begin{aligned}
(5.19) \quad & |\langle \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 u^{(0)}, \partial_t \tilde{Q} u_\infty \rangle \rangle| \\
& \leq \frac{C}{\gamma^2} \{ \|\partial_{x'} \sigma_1\|_2 + \|\partial_{x'} \phi_\infty\|_2 + \gamma^2 \|\operatorname{div}(\rho_s w_\infty)\|_2 \} \|\sqrt{\rho_s} \partial_t w_\infty\|_2 \\
& \leq \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 + C \left\{ \frac{1}{\nu} D^{(0)}[w_\infty] + \frac{1}{\gamma^4} (\|\partial_{x'} \sigma_1\|_2^2 + \|\partial_{x_1} \phi_\infty\|_2^2) \right\}
\end{aligned}$$

and

$$(5.20) \quad |\langle \langle \mathbf{F}_\infty, \partial_t \tilde{Q} u_\infty \rangle \rangle| \leq \frac{1}{12} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 + C \|\tilde{Q} \mathbf{F}_\infty\|_2^2.$$

It then follows from (5.12), (5.17)–(5.20) that

$$\begin{aligned}
(5.21) \quad & \frac{1}{2} \frac{d}{dt} (D^{(0)}[w_\infty] + J[u]) + \frac{1}{2} \|\sqrt{\rho_s} \partial_t w_\infty\|_2^2 \\
& \leq C R_{0,0}^{(2)} + C \frac{\gamma^2}{\nu} D^{(0)}[w_\infty] \\
& \quad + C \left\{ \left(\frac{1}{\gamma^2} + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} \right) \|\partial_{x'} \sigma_1\|_2^2 + \left(\frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) \|\partial_{x'} \phi_\infty\|_2^2 \right\}.
\end{aligned}$$

We take $b_1 > 0$ in such a way that $b_1 \geq \max\{2b_0, 2C\}$. Adding $\frac{2b_1\gamma^2}{\nu} \times (5.1)$ to (5.21), we obtain (5.11) for $j = k = 0$. The case $2j + k \geq 1$ can be obtained by replacing u_∞ with $T_{j,k} u_\infty$. This completes the proof. \square

We next derive the dissipative estimates for x_n -derivatives of ϕ_∞ .

Proposition 5.4. *If $\tilde{\omega} \leq \min\{1, \frac{(\nu + \tilde{\nu})^2}{\gamma^4}\}$, then the following estimate holds for $0 \leq 2j + k + l \leq m - 1$:*

$$\begin{aligned}
(5.22) \quad & \frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^2} \left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|_2^2 + \frac{1}{2(\nu + \tilde{\nu})} \left\| \frac{\tilde{P}'(\rho_s)}{\gamma^2} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|_2^2 \\
& \leq R_{j,k,l}^{(3)} + C \frac{\nu + \tilde{\nu}}{\gamma^4} \|K_{j,k,l}\|_2^2,
\end{aligned}$$

where

$$R_{j,k,l}^{(3)} = \left| \frac{1}{2\gamma^2} (\operatorname{div} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} w \right), |T_{j,k} \partial_{x_n}^{l+1} \phi_\infty|^2) \right| + \frac{\nu + \tilde{\nu}}{\gamma^4} \|H_{j,k,l}\|_2^2$$

with

$$\|H_{j,k,l}\|_2^2 \leq C\{\|[T_{j,k}\partial_{x_n}^{l+1}, w] \cdot \nabla \phi_\infty\|_2^2 + \|T_{j,k}\partial_{x_n}^{l+1}\tilde{f}_\infty^0\|_2^2 + \|\frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^l f_\infty^n\|_2^2\},$$

and

$$\tilde{f}_\infty^0 = -\phi \operatorname{div} w - w \cdot \nabla(\sigma_1 \phi^{(0)}) - \langle Q_0 \mathbf{F} \rangle_1 \phi^{(0)};$$

and $K_{j,k,l}$ is estimated as

$$\begin{aligned} & \frac{\nu+\tilde{\nu}}{\gamma^4} \|K_{j,k,l}\|_2^2 \\ & \leq C\left\{ \frac{\nu^2}{\nu+\tilde{\nu}} \|T_{j,k+1}\partial_{x_n}^l \partial_x w_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\sqrt{\rho_s} \partial_t T_{j,k} \partial_{x_n}^l w_\infty\|_2^2 \right. \\ & \quad + (\nu + \tilde{\nu}) \tilde{\omega}^2 \left(\sum_{p=0}^{l-1} \|T_{j,k+1}\partial_{x_n}^p \partial_x w_\infty\|_2^2 + \sum_{p=0}^l \|T_{j,k}\partial_{x_n}^p \partial_x w_\infty\|_2^2 \right) \\ & \quad + \frac{1}{\nu+\tilde{\nu}} \sum_{p=0}^l \|T_{j,k+1}\partial_{x_n}^p w_\infty\|_2^2 \\ & \quad \left. + \frac{\nu+\tilde{\nu}}{\gamma^4} \left(\sum_{p=0}^l \|T_{j,k}\partial_{x_n}^p \partial_x \phi_\infty\|_2^2 + \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 \right) \right\}. \end{aligned}$$

Proof. The first equation of (4.11) is written as

$$\begin{aligned} & \partial_t \phi_\infty + (v_s + w) \cdot \nabla \phi_\infty + \gamma^2 \rho_s \nabla' \cdot w'_\infty + \gamma^2 \partial_{x_n} (\rho_s w_\infty^n) \\ & \quad + (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1}) \partial_{x_1} \sigma_1 - \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 \phi^{(0)} \\ & = \tilde{f}_\infty^0. \end{aligned}$$

Applying $T_{j,k}\partial_{x_n}^{l+1}$ to this equation, we have

$$\begin{aligned} & \partial_t (T_{j,k}\partial_{x_n}^{l+1} \phi_\infty) + (v_s + w) \cdot \nabla (T_{j,k}\partial_{x_n}^{l+1} \phi_\infty) + \gamma^2 \rho_s T_{j,k}\partial_{x_n}^{l+2} w_\infty^n \\ & = -[T_{j,k}\partial_{x_n}^{l+1}, v_s + w] \cdot \nabla \phi_\infty - \gamma^2 \partial_{x_n}^{l+1} (\rho_s \nabla' \cdot T_{j,k} w'_\infty) \\ (5.23) \quad & - \gamma^2 [\partial_{x_n}^{l+2}, \rho_s] T_{j,k} w_\infty^n - \partial_{x_n}^{l+1} (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1}) \partial_{x_1} T_{j,k} \sigma_1 \\ & + \langle Q_0 \tilde{B}(T_{j,k} \sigma_1 u^{(0)} + T_{j,k} u_\infty) \rangle_1 \partial_{x_n}^{l+1} \phi^{(0)} + T_{j,k} \partial_{x_n}^{l+1} \tilde{f}_\infty^0. \end{aligned}$$

The $(n+1)$ th equation of (4.11) is written as

$$\begin{aligned} & -\frac{\nu+\tilde{\nu}}{\rho_s} \partial_{x_n}^2 w_\infty^n + \partial_{x_n} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi_\infty \right) \\ & = -\left\{ \partial_t w_\infty - \frac{\nu}{\rho_s} \Delta' w_\infty^n - \frac{\tilde{\nu}}{\rho_s} \partial_{x_n} \nabla' \cdot w'_\infty + v_s^1 \partial_{x_1} w_\infty^n - \frac{\tilde{\nu}}{\rho_s} (\partial_{x_n} w^{(0),1}) \partial_{x_1} \sigma_1 \right\} + f_\infty^n. \end{aligned}$$

Applying $T_{j,k}\partial_{x_n}^l$ to this equation, we have

$$\begin{aligned}
(5.24) \quad & -\frac{\nu+\tilde{\nu}}{\rho_s}T_{j,k}\partial_{x_n}^{l+2}w_\infty^n + \frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \\
& = (\nu + \tilde{\nu})[\partial_{x_n}^l, \frac{1}{\rho_s}]\partial_{x_n}^2 T_{j,k}w_\infty^n - [\partial_{x_n}^{l+1}, \frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}]T_{j,k}\phi_\infty \\
& \quad - \left\{ \partial_t T_{j,k}\partial_{x_n}^l w_\infty - T_{j,k}\partial_{x_n}^l \left(\frac{\nu}{\rho_s}\Delta' w_\infty^n + \frac{\tilde{\nu}}{\rho_s}\partial_{x_n}\nabla' \cdot w'_\infty \right) \right. \\
& \quad \left. + T_{j,k}\partial_{x_n}^l (v_s^1\partial_{x_1}w_\infty^n) - \partial_{x_n}^l \left(\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}w^{(0),1} \right)\partial_{x_1}T_{j,k}\sigma_1 + T_{j,k}\partial_{x_n}^l f_\infty^n \right\}.
\end{aligned}$$

Adding $\frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}} \times (5.24)$ to (5.23), we have

$$\begin{aligned}
(5.25) \quad & \partial_t(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty) + (v_s + w) \cdot \nabla(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty) + \frac{\rho_s\tilde{P}'(\rho_s)}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \\
& = H_{j,k,l} + K_{j,k,l},
\end{aligned}$$

where

$$\begin{aligned}
H_{j,k,l} &= -[T_{j,k}\partial_{x_n}^{l+1}, w] \cdot \nabla\phi_\infty + T_{j,k}\partial_{x_n}^{l+1}\tilde{f}_\infty^0 + \frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^l f_\infty^n, \\
K_{j,k,l} &= -[\partial_{x_n}^{l+1}, v_s^1] \cdot \nabla T_{j,k}\phi_\infty - \gamma^2[\partial_{x_n}^{l+1}, \rho_s]\nabla' \cdot T_{j,k}w'_\infty \\
& \quad - \gamma^2[T_{j,k}\partial_{x_n}^{l+2}, \rho_s]w_\infty^n - \partial_{x_n}^{l+1}(v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1})\partial_{x_1}T_{j,k}\sigma_1 \\
& \quad + \langle Q_0\tilde{B}(T_{j,k}\sigma_1 u^{(0)} + T_{j,k}u_\infty) \rangle_1 \partial_{x_n}^{l+1}\phi^{(0)} \\
& \quad + \gamma^2\rho_s^2[\partial_{x_n}^l, \frac{1}{\rho_s}]\partial_{x_n}^2 T_{j,k}w_\infty - \frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}[\partial_{x_n}^{l+1}, \frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}]T_{j,k}\phi_\infty \\
& \quad - \frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\left\{ \partial_t T_{j,k}\partial_{x_n}^l w_\infty - \frac{\nu}{\rho_s}\Delta' T_{j,k}\partial_{x_n}^l w_\infty^n + \frac{\nu}{\rho_s}\partial_{x_n}^{l+1}\nabla' \cdot T_{j,k}w'_\infty \right. \\
& \quad \left. - \nu[\partial_{x_n}^l, \frac{\nu}{\rho_s}]\Delta' T_{j,k}w_\infty^n - \tilde{\nu}[\partial_{x_n}^l, \frac{\nu}{\rho_s}]\partial_{x_n}\nabla' \cdot T_{j,k}w'_\infty \right. \\
& \quad \left. + T_{j,k}\partial_{x_n}^l (v_s^1\partial_{x_1}w_\infty^n) - \partial_{x_n}^l \left(\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}w^{(0),1} \right)\partial_{x_1}T_{j,k}\sigma_1 \right\}.
\end{aligned}$$

Multiplying (5.25) by $\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty$ and integrating over Ω , we have

$$\begin{aligned}
(5.26) \quad & \frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2}\left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \right\|_2^2 + \frac{1}{\nu+\tilde{\nu}}\left\| \frac{\tilde{P}'(\rho_s)}{\gamma^2}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \right\|_2^2 \\
& = -((v_s + w) \cdot \nabla(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty), T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}) \\
& \quad + (H_{j,k,l}, T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}) + (K_{j,k,l}, T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}).
\end{aligned}$$

Since $v_s = v_s^1 e_1$, an integration by parts gives

$$\begin{aligned}
(5.27) \quad & (v_s \cdot \nabla(T_{j,k}\partial_{x_n}^{l+1}\phi_\infty), T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}) = \frac{1}{2}(v_s, \partial_{x_1}(|T_{j,k}\partial_{x_n}^{l+1}\phi_\infty|^2) \frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}) = 0.
\end{aligned}$$

Also, we have

(5.28)

$$(w \cdot \nabla(T_{j,k} \partial_{x_n}^{l+1} \phi_\infty), T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \frac{\tilde{P}'(\rho_s)}{\gamma^4 \rho_s}) = -\frac{1}{2\gamma^2} (\operatorname{div}(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} w), |T_{j,k} \partial_{x_n}^{l+1} \phi_\infty|^2)$$

Since $\tilde{P}'(\rho_s) = O(\gamma^2)$ and $\rho_s \geq \rho_1(> 0)$, the last two terms on the right of (5.26) are estimates as

$$\begin{aligned} & |(H_{j,k,l}, T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \frac{\tilde{P}'(\rho_s)}{\gamma^4 \rho_s}) + (K_{j,k,l}, T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \frac{\tilde{P}'(\rho_s)}{\gamma^4 \rho_s})| \\ (5.29) \quad & \leq \frac{C}{\gamma^2} \|\frac{\tilde{P}'(\rho_s)}{\gamma^2} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty\|_2 (\|H_{j,k,l}\|_2 + \|K_{j,k,l}\|_2) \\ & \leq \frac{1}{2(\nu+\tilde{\nu})} \|\frac{\tilde{P}'(\rho_s)}{\gamma^2} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty\|_2^2 + C \frac{\nu+\tilde{\nu}}{\gamma^4} (\|H_{j,k,l}\|_2^2 + \|K_{j,k,l}\|_2^2). \end{aligned}$$

One can see that $K_{j,k,l}$ has the desired estimate if $\tilde{\omega}^2 \leq \min\{1, \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\}$, and hence, estimate (5.22) follows from (5.26)–(5.29). This completes the proof. \square

In order to obtain the dissipative estimates for higher order x_n -derivatives of w_∞ and the tangential derivatives of ϕ_∞ , we consider the material derivative of ϕ_∞ . We denote the material derivative of ϕ_∞ by $\dot{\phi}_\infty$:

$$\dot{\phi}_\infty = \partial_t \phi_\infty + (v_s + w) \cdot \nabla \phi_\infty.$$

We have the following estimates.

Proposition 5.5. (i) *If $\tilde{\omega}^2 \leq \min\{1, \frac{(\nu+\tilde{\nu})^2}{\gamma^4}\}$ then the following estimate holds for $0 \leq 2j + k + l \leq m - 1$:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \frac{1}{\gamma^2} \left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|_2^2 + \frac{1}{4(\nu+\tilde{\nu})} \left\| \frac{\tilde{P}'(\rho_s)}{\gamma^2} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty \right\|_2^2 \\ (5.30) \quad & + c_0 \frac{\nu+\tilde{\nu}}{\gamma^4} \|T_{j,k} \partial_{x_n}^{l+1} \dot{\phi}_\infty\|_2^2 \\ & \leq R_{j,k,l}^{(3)} + C \frac{\nu+\tilde{\nu}}{\gamma^4} \|K_{j,k,l}\|_2^2, \end{aligned}$$

where c_0 is a positive constant; and $R_{j,k,l}^{(3)}$ and $K_{j,k,l}$ are the same as in Proposition 5.3.

(ii) *Let $0 \leq q \leq k + 1$. Then*

$$\begin{aligned} (5.31) \quad & \frac{\nu+\tilde{\nu}}{\gamma^4} \|T_{j,k} \dot{\phi}_\infty\|_2^2 \leq C \{ R_{j,k}^{(4)} + D^{(0)}[T_{j,k} w_\infty] + (\nu + \tilde{\nu}) \tilde{\omega}^2 \|T_{j,k} w_\infty\|_2^2 \\ & + \frac{\nu+\tilde{\nu}}{\gamma^4} \|\partial_{x_1} T_{j,k} \sigma_1\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} \|\partial_{x_1} T_{j,q} \phi_\infty\|_2^2 \}, \end{aligned}$$

where $R_{j,k}^{(4)} = \|T_{j,k}\tilde{f}_\infty^0\|_2^2$.

Proof. By (5.25), we have

$$T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty = -\frac{\rho_s\tilde{P}'(\rho_s)}{\nu+\tilde{\nu}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty + \tilde{H}_{j,k,l} + \tilde{K}_{j,k,l},$$

where

$$\tilde{H}_{j,k,l} = [T_{j,k}\partial_{x_n}^{l+1}, w] \cdot \nabla \phi_\infty + H_{j,k,l},$$

$$\tilde{K}_{j,k,l} = [\partial_{x_n}^{l+1}, v_s] \cdot \nabla T_{j,k}\phi_\infty + K_{j,k,l}.$$

It follows that

$$(5.32) \quad \begin{aligned} & \frac{\nu+\tilde{\nu}}{\gamma^4} \|T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty\|_2^2 \\ & \leq C \left\{ \frac{1}{\nu+\tilde{\nu}} \left\| \frac{\tilde{P}'(\rho_s)}{\gamma^2} T_{j,k}\partial_{x_n}^{l+1}\phi_\infty \right\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} (\|\tilde{H}_{j,k,l}\|_2^2 + \|\tilde{K}_{j,k,l}\|_2^2) \right\}, \end{aligned}$$

Since $\tilde{H}_{j,k,l}$ and $\tilde{K}_{j,k,l}$ have the same estimates as those for $H_{j,k,l}$ and $K_{j,k,l}$, respectively, we obtain (5.30) by adding (5.22) to $c_0 \times (5.32)$ with $c_0 > 0$ satisfying $c_0 C \leq \frac{1}{4}$. This proves (i).

As for (ii), we see from the first equation of (4.11) that

$$\begin{aligned} T_{j,k+p}\dot{\phi}_\infty &= -\rho_s\gamma^2 \operatorname{div}(T_{j,k+p}w_\infty) - \gamma^2(\partial_{x_n}\rho_s)T_{j,k+p}w_\infty^n \\ &\quad - (v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1})\partial_{x_1}T_{j,k+p}\sigma_1 \\ &\quad - \langle Q_0\tilde{B}(T_{j,k+p}\sigma_1 u^{(0)} + T_{j,k+p}u_\infty) \rangle_1 \phi^{(0)} + T_{j,k+p}\tilde{f}_\infty^0, \end{aligned}$$

from which one can obtain (5.31). This completes the proof. \square

We next derive the dissipative estimates for σ_1 .

Proposition 5.6. *There are positive constants ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\tilde{\nu}) \geq \gamma_0^2$, then the following estimate holds for $0 \leq 2j+k \leq m-1$:*

$$(5.33) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} \|T_{j,k}\sigma_1\|_2^2 + \frac{\alpha_1}{2(\nu+\tilde{\nu})} \|\partial_{x'}T_{j,k}\sigma_1\|_2^2 \\ & \leq R_{j,k}^{(5)} + C \left\{ \frac{1}{\nu+\tilde{\nu}} \|\partial_t T_{j,k}w_\infty\|_2^2 + D^{(0)}[T_{j,k}w_\infty] \right. \\ & \quad \left. + \left(\frac{1}{\nu+\tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) \|T_{j,p}\partial_{x_n}\phi_\infty\|_2^2 \right\}, \end{aligned}$$

where $\alpha_1 > 0$ is a constant; p is any integer satisfying $0 \leq 2j+p+1 \leq m$ and $0 \leq p \leq k$; and

$$R_{j,k}^{(5)} = \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} (Q_0 T_{j,k} \mathbf{F}, T_{j,k} \sigma_1) - \frac{1}{\nu+\tilde{\nu}} (\rho_s (-\Delta)^{-1} (\rho_s \nabla' \cdot T_{j,k} \mathbf{f}'_\infty), T_{j,k} \sigma_1).$$

Here $(-\Delta)^{-1}$ is the inverse of $-\Delta$ on $L^2(\Omega)$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. It is convenient to consider the Fourier transform of (4.10) in x' -variable:

$$\partial_t(\widehat{T_{j,k}\sigma_1}) + \langle Q_0 \tilde{B}_{\xi'}(\widehat{T_{j,k}\sigma_1} u^{(0)} + \widehat{T_{j,k}u_\infty}) \rangle = \langle Q_0 \widehat{T_{j,k}\mathbf{F}} \rangle,$$

where $|\xi'| \leq 1$ and

$$\tilde{B}_{\xi'} = \hat{B}_{\xi'} - \hat{B}_0 = \begin{pmatrix} i\xi_1 v_s^1 & i\gamma^2 \rho_s^\top \xi' & 0 \\ i\xi' \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} & i\xi_1 v_s^1 I_{n-1} & 0 \\ 0 & 0 & i\xi_1 v_s^1 \end{pmatrix}.$$

We write this equation as

$$\begin{aligned} (5.34) \quad & \partial_t(\widehat{T_{j,k}\sigma_1}) + \gamma^2 \langle \rho_s i\xi' \cdot \widehat{T_{j,k}w'_\infty} \rangle + \langle Q_0 \tilde{B}_{\xi'}(\widehat{T_{j,k}\sigma_1} u^{(0)}) \rangle \\ & = \langle Q_0 \widehat{T_{j,k}\mathbf{F}} \rangle - \langle i\xi_1 v_s^1 \widehat{T_{j,k}\phi_\infty} \rangle \end{aligned}$$

for $|\xi'| \leq 1$.

We further rewrite the second term $\gamma^2 \langle \rho_s i\xi' \cdot \widehat{T_{j,k}w'_\infty} \rangle$ on the left of (5.34). We introduce $(n-1) \times (n-1)$ matrix operators \tilde{A}' , \tilde{B}' and \tilde{C}'_0 :

$$\begin{aligned} \tilde{A}' &= \begin{pmatrix} 0 & -\frac{\nu}{\rho_s} \Delta' I_{n-1} - \frac{\tilde{\nu}}{\rho_s} \nabla'^\top \nabla' & -\frac{\tilde{\nu}}{\rho_s} \nabla' \partial_{x_n} \end{pmatrix}, \\ \tilde{B}' &= \begin{pmatrix} \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \nabla' & v_s^1 \partial_{x_1} I_{n-1} & 0 \end{pmatrix}, \\ \tilde{C}'_0 &= \begin{pmatrix} \frac{\nu \Delta v_s^1}{\gamma^2 \rho_s^2} \mathbf{e}'_1 & 0 & (\partial_{x_n} v_s^1) \mathbf{e}'_1 \end{pmatrix}. \end{aligned}$$

Since $\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} = \alpha_0$ (see Lemma 4.1 (ii)), we have

$$(5.35) \quad \tilde{B}'(\sigma_1 u^{(0)}) = \tilde{B}'(\sigma_1 u_0^{(0)}) + \tilde{B}'(\sigma_1 u_1^{(0)}) = \alpha_0 \nabla' \sigma_1 + \tilde{B}'(\sigma_1 u_1^{(0)}).$$

Furthermore,

$$(5.36) \quad \tilde{A}'(\sigma_1 u^{(0)}) = \tilde{A}'(\sigma_1 u_1^{(0)}).$$

Therefore, by (4.11), (5.35) and (5.36), we have

$$\begin{aligned} & \partial_t w'_\infty - \frac{\nu}{\rho_s} \Delta w'_\infty - \frac{\tilde{\nu}}{\rho_s} \nabla' \operatorname{div} w_\infty + \tilde{B}' u_\infty + \tilde{C}'_0 u_\infty \\ & + \tilde{A}'(\sigma_1 u_1^{(0)}) + \alpha_0 \nabla' \sigma_1 + \tilde{B}'(\sigma_1 u_1^{(0)}) - \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 w^{(0),1} \mathbf{e}'_1 \\ & = \mathbf{f}'_\infty. \end{aligned}$$

This gives

$$(5.37) \quad -\Delta T_{j,k} w'_\infty = -\frac{\alpha_0}{\nu} \rho_s \nabla' T_{j,k} \sigma_1 + T_{j,k} \tilde{\mathbf{f}}'_\infty + \frac{\rho_s}{\nu} T_{j,k} \mathbf{f}'_\infty,$$

where

$$\begin{aligned} \tilde{\mathbf{f}}'_\infty &= -\frac{\rho_s}{\nu} \left\{ \partial_t w'_\infty - \frac{\tilde{\nu}}{\rho_s} \nabla' \operatorname{div} w_\infty + \tilde{B}' u_\infty + \tilde{C}'_0 u_\infty \right. \\ &\quad \left. + \tilde{A}'(\sigma_1 u_1^{(0)}) + \tilde{B}'(\sigma_1 u_1^{(0)}) - \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 w^{(0),1} \mathbf{e}'_1 \right\}. \end{aligned}$$

We take the Fourier transform of (5.37) to obtain

$$(5.38) \quad \begin{aligned} &(|\xi'|^2 + \mathcal{A}) \widehat{T_{j,k} w'_\infty} \\ &= -\frac{\alpha_0}{\nu} (i \xi' \widehat{T_{j,k} \sigma_1}) \rho_s + \mathcal{F}(T_{j,k} \tilde{\mathbf{f}}'_\infty) + \frac{\rho_s}{\nu} \mathcal{F}(T_{j,k} \mathbf{f}'_\infty). \end{aligned}$$

Here \mathcal{A} is an operator on $L^2(0, 1)$ with domain $D(\mathcal{A}) = H^2(0, 1) \cap H_0^1(0, 1)$ defined by

$$\mathcal{A}v = -\partial_{x_n}^2 v \quad \text{for } v \in D(\mathcal{A}).$$

It follows from (5.38) that

$$(5.39) \quad \widehat{T_{j,k} w'_\infty} = -\frac{\alpha_0}{\nu} (i \xi' \widehat{T_{j,k} \sigma_1}) (|\xi'|^2 + \mathcal{A})^{-1} \rho_s + (|\xi'|^2 + \mathcal{A})^{-1} \widehat{\mathbf{h}}'_{j,k}$$

for $|\xi'| \leq 1$, where

$$\widehat{\mathbf{h}}'_{j,k} = \mathcal{F}(T_{j,k} \tilde{\mathbf{f}}'_\infty) + \frac{\rho_s}{\nu} \mathcal{F}(T_{j,k} \mathbf{f}'_\infty).$$

Substituting (5.39) into (5.34), we obtain

$$(5.40) \quad \begin{aligned} &\partial_t (\widehat{T_{j,k} \sigma_1}) + \frac{\alpha_0 \gamma^2}{\nu} \langle \rho_s (|\xi'|^2 + \mathcal{A})^{-1} \rho_s \rangle |\xi'|^2 \widehat{T_{j,k} \sigma_1} + \langle Q_0 \tilde{B}_{\xi'} (\widehat{T_{j,k} \sigma_1} u^{(0)}) \rangle \\ &= \langle Q_0 \widehat{T_{j,k} \mathbf{F}} \rangle - \langle i \xi_1 v_s^1 \widehat{T_{j,k} \phi_\infty} \rangle - \gamma^2 \langle \rho_s i \xi' \cdot (|\xi'|^2 + \mathcal{A})^{-1} \widehat{\mathbf{h}}'_{j,k} \rangle \end{aligned}$$

for $|\xi'| \leq 1$.

We multiply (5.40) by $\overline{\widehat{T_{j,k} \sigma_1}}$. Here \bar{z} denotes the complex conjugate of z . Since

$$\langle Q_0 \tilde{B} u^{(0)} \rangle = i \xi_1 \langle v_s^1 \phi^{(0)} \rangle + \gamma^2 \rho_s w^{(0),1} \in i\mathbf{R},$$

and $\operatorname{supp}(\widehat{T_{j,k} \sigma_1}) \subset \{|\xi'| \leq 1\}$, integrating the resulting equation in ξ' and taking its real part, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\widehat{T_{j,k} \sigma_1}\|_2^2 + \frac{\alpha_0 \gamma^2}{\nu} \|\sqrt{\langle \rho_s (|\xi'|^2 + \mathcal{A})^{-1} \rho_s \rangle} |\xi'|^2 \widehat{T_{j,k} \sigma_1}\|_2^2 \\ &= \operatorname{Re} \{ (Q_0 \widehat{T_{j,k} \mathbf{F}}, \widehat{T_{j,k} \sigma_1}) - (i \xi_1 v_s^1 \widehat{T_{j,k} \phi_\infty}, \widehat{T_{j,k} \sigma_1}) \\ &\quad - \gamma^2 (\rho_s i \xi' \cdot (|\xi'|^2 + \mathcal{A})^{-1} \widehat{\mathbf{h}}'_{j,k}, \widehat{T_{j,k} \sigma_1}) \}. \end{aligned}$$

We here note that there is a constant $c > 0$ such that

$$\langle \rho_s(|\xi'|^2 + \mathcal{A})^{-1} \rho_s \rangle = |(|\xi'|^2 + \mathcal{A})^{-1/2} \rho_s|_2^2 \geq c \quad \text{for } |\xi'| \leq 1.$$

This implies

$$\frac{\alpha_0 \gamma^2}{\nu} \|\sqrt{\langle \rho_s(|\xi'|^2 + \mathcal{A})^{-1} \rho_s \rangle |\xi'|^2} \widehat{T_{j,k} \sigma_1}\|_2^2 \geq \frac{\alpha_1 \gamma^2}{\nu} \|i \xi' \widehat{T_{j,k} \sigma_1}\|_2^2$$

for some constant $\alpha_1 > 0$. It then follows, through the Plancherel theorem, that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|T_{j,k} \sigma_1\|_2^2 + \frac{\alpha_1 \gamma^2}{\nu} \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 \\ (5.41) \quad & = \left\{ (Q_0 T_{j,k} \mathbf{F}, T_{j,k} \sigma_1) - (v_s^1 \partial_{x_1} T_{j,k} \phi_\infty, T_{j,k} \sigma_1) \right. \\ & \quad \left. - \gamma^2 (\rho_s (-\Delta)^{-1} \nabla' \cdot \mathbf{h}'_{j,k}, T_{j,k} \sigma_1) \right\}. \end{aligned}$$

The second term on the right of (5.41) is estimated as

$$\begin{aligned} |(v_s^1 \partial_{x_1} T_{j,k} \phi_\infty, T_{j,k} \sigma_1)| &= |(v_s^1 T_{j,k} \phi_\infty, \partial_{x_1} T_{j,k} \sigma_1)| \\ (5.42) \quad &\leq \frac{\alpha_1 \gamma^2}{4\nu} \|\partial_{x_1} T_{j,k} \sigma_1\|_2^2 + \frac{C}{\gamma^2} \|T_{j,k} \phi_\infty\|_2^2. \end{aligned}$$

As for the last term on the right of (5.41), we have

$$\begin{aligned} (5.43) \quad \gamma^2 (\rho_s (-\Delta)^{-1} \nabla' \cdot \mathbf{h}'_{j,k}, T_{j,k} \sigma_1) &= -\gamma^2 (\rho_s (-\Delta)^{-1} T_{j,k} \tilde{\mathbf{f}}'_\infty, \nabla' T_{j,k} \sigma_1) \\ &\quad + \frac{\gamma^2}{\nu} (\rho_s (-\Delta)^{-1} (\rho_s \nabla' \cdot T_{j,k} \mathbf{f}'_\infty), T_{j,k} \sigma_1), \end{aligned}$$

$$\begin{aligned} & \gamma^2 |(\rho_s (-\Delta)^{-1} T_{j,k} \tilde{\mathbf{f}}'_\infty, \nabla' T_{j,k} \sigma_1)| \\ & \leq \frac{\alpha_1 \gamma^2}{4\nu} \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + C \nu \gamma^2 \|\rho_s (-\Delta)^{-1} T_{j,k} \tilde{\mathbf{f}}'_\infty\|_2^2 \\ (5.44) \quad & \leq \frac{\alpha_1 \gamma^2}{4\nu} \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + C \frac{\gamma^2}{\nu} \left\{ \|\partial_t T_{j,k} w_\infty\|_2^2 + \tilde{\nu}^2 \|\operatorname{div} T_{j,k} w_\infty\|_2^2 \right. \\ & \quad \left. + \|T_{j,k} w_\infty\|_2^2 + \|\partial_{x'} T_{j,k} w_\infty\|_2^2 + \left(1 + \frac{1}{\gamma^4} + \frac{\nu^2}{\gamma^4}\right) \|T_{j,k} \phi_\infty\|_2^2 \right. \\ & \quad \left. + \left(\frac{1}{\gamma^4} + \frac{(\nu + \tilde{\nu})^2}{\gamma^4}\right) \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 \right\}. \end{aligned}$$

The desired estimate (5.33) now follows from (5.41)–(5.44) and Lemma 4.3 (iv), provided that $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2$ for some $\nu_0 > 0$ and $\gamma_0 > 0$. This completes the proof. \square

We next apply estimates for the Stokes system to obtain the estimates for higher order derivatives.

Proposition 5.7. *If $\tilde{\omega}^2 \leq 1$, then there holds the following estimate for $0 \leq 2j + k + l \leq m - 1$:*

$$\begin{aligned}
(5.45) \quad & \frac{\nu^2}{\nu+\tilde{\nu}} \|\partial_x^{l+2} T_{j,k} w_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_x^{l+1} T_{j,k} \phi_\infty\|_2^2 \\
& \leq CR_{j,k,l}^{(6)} + C \left\{ \left(\frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{1}{\nu+\tilde{\nu}} \right) \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 \right. \\
& \quad + \left(\frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{\tilde{\omega}^2}{\nu+\tilde{\nu}} \right) \|T_{j,k} \phi_\infty\|_{H^l}^2 + \frac{1}{\gamma^4(\nu+\tilde{\nu})} \|\partial_{x'} T_{j,k} \phi_\infty\|_2^2 \\
& \quad + \frac{\nu+\tilde{\nu}}{\gamma^4} \|T_{j,k} \dot{\phi}_\infty\|_{H^{l+1}}^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_t T_{j,k} w_\infty\|_{H^l}^2 \\
& \quad \left. + \left(\frac{1}{\nu+\tilde{\nu}} + \nu \tilde{\omega}^2 \right) \|T_{j,k} w_\infty\|_{H^{l+1}}^2 + D^{(0)}[T_{j,k} w_\infty] \right\},
\end{aligned}$$

where

$$R_{j,k,l}^{(6)} = \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \tilde{f}_\infty^0\|_{H^{l+1}}^2 \frac{1}{\nu + \tilde{\nu}} \|T_{j,k} \mathbf{f}_\infty\|_{H^l}^2.$$

Proof. To prove Proposition 5.6, we employ estimates for the Stokes system. Let ${}^\top(\tilde{\phi}, \tilde{w})$ be the solution of the Stokes system

$$\begin{aligned}
\operatorname{div} \tilde{\phi} &= F \quad \text{in } \Omega, \\
-\Delta \tilde{w} + \nabla \tilde{\phi} &= G \quad \text{in } \Omega, \\
\tilde{w}|_{\partial\Omega} &= 0.
\end{aligned}$$

Then for any $l \in \mathbf{Z}$, $l \geq 0$, there exists a constant $C > 0$ such that

$$(5.46) \quad \|\partial_x^{l+2} \tilde{w}\|_2^2 + \|\partial_x^{l+1} \tilde{\phi}\|_2^2 \leq C \{ \|F\|_{H^{l+1}}^2 + \|G\|_{H^l}^2 + \|\partial_x \tilde{w}\|_2^2 \}.$$

(See, e.g., [1], [2, Appendix].)

By (4.11), we have

$$\begin{aligned}
\operatorname{div} (T_{j,k} w_\infty) &= F_{j,k} \quad \text{in } \Omega, \\
-\Delta (T_{j,k} w_\infty) + \nabla \left(\frac{\tilde{P}'(\rho_s)}{\nu \gamma^2} T_{j,k} \phi_\infty \right) &= G_{j,k} \quad \text{in } \Omega, \\
T_{j,k} w_\infty|_{\partial\Omega} &= 0,
\end{aligned}$$

where

$$\begin{aligned}
F_{j,k} &= \frac{1}{\gamma^2 \rho_s} T_{j,k} \tilde{f}_\infty^0 - \frac{\partial_{x_n} \rho_s}{\rho_s} T_{j,k} w_\infty^n \\
&\quad - \frac{1}{\gamma^2 \rho_s} \{ T_{j,k} \dot{\phi}_\infty + (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1}) \partial_{x_1} T_{j,k} \sigma_1 \\
&\quad - \langle Q_0 \tilde{B}(T_{j,k} \sigma_1 u^{(0)} + T_{j,k} u_\infty) \rangle_1 \phi^{(0)} \},
\end{aligned}$$

$$\begin{aligned}
G_{j,k} &= -\frac{\rho_s}{\nu} T_{j,k} \mathbf{f}_\infty + [\nabla, \frac{\rho_s}{\nu}] \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} T_{j,k} \phi_\infty \\
&\quad - \frac{\rho_s}{\nu} \left\{ \partial_t T_{j,k} w_\infty - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} T_{j,k} w_\infty + v_s^1 \partial_{x_1} T_{j,k} w_\infty - \frac{\nu \Delta v_s^1}{\gamma^2 \rho_s^2} T_{j,k} \phi_\infty \mathbf{e}_1 \right. \\
&\quad \left. + (\partial_{x_n} v_s^1) T_{j,k} w_\infty \mathbf{e}_1 - \frac{\nu}{\rho_s} w^{(0)} \Delta' T_{j,k} \sigma_1 - \frac{\tilde{\nu}}{\rho_s} \nabla (w^{(0),1} \partial_{x_1} T_{j,k} \sigma_1) \right. \\
&\quad \left. + \alpha_0 \nabla T_{j,k} \sigma_1 + v_s^1 w^{(0)} \partial_{x_1} T_{j,k} \sigma_1 - \langle Q_0 \tilde{B}(T_{j,k} \sigma_1 u^{(0)} + T_{j,k} u_\infty) \rangle_1 w^{(0)} \right\}.
\end{aligned}$$

We now apply (5.46) to obtain

$$\begin{aligned}
(5.47) \quad & \|\partial_x^{l+1} T_{j,k} w_\infty\|_2^2 + \frac{1}{\nu^2} \|\partial_x^{l+1} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2} T_{j,k} \phi_\infty \right)\|_2^2 \\
& \leq C \left\{ \|F_{j,k}\|_{H^{l+1}}^2 + \|G_{j,k}\|_{H^l}^2 + \|\partial_x T_{j,k} w_\infty\|_2^2 \right\}.
\end{aligned}$$

The desired estimate (5.45) now follows from $\frac{\nu^2}{\nu+\tilde{\nu}} \times (5.47)$ by a direct computation. This completes the proof. \square

We finally estimate the time derivatives of σ_1 and ϕ_∞ .

Proposition 5.8. (i) *If $0 \leq 2j + k \leq m - 1$, then there holds the following estimate:*

$$\begin{aligned}
(5.48) \quad & \|\partial_t T_{j,k} \sigma_1\|_2^2 \\
& \leq C \left\{ R_{j,k}^{(7)} + \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + \|\partial_{x'} T_{j,k} \phi_\infty\|_2^2 + \gamma^4 \|\partial_{x'} T_{j,k} w_\infty\|_2^2 \right\}.
\end{aligned}$$

Here $R_{j,k}^{(7)} = \|\langle Q_0 T_{j,k} \mathbf{F} \rangle_1\|_2^2$.

(ii) *If $0 \leq 2j \leq m - 1$ and $\tilde{\omega}^2 \leq 1$, then there holds the following estimate:*

$$\begin{aligned}
(5.49) \quad & \|\partial_t^{j+1} \phi_\infty\|_{H^{m-1-2j}}^2 \\
& \leq C \left\{ R_j^{(8)} + \|\partial_{x'} \partial_t^j \phi_\infty\|_{H^{m-1-2j}}^2 + \gamma^4 \|\partial_{x'} \partial_t^j w_\infty\|_{H^{m-1-2j}}^2 + \|\partial_{x_1} \partial_t^j \sigma_1\|_2^2 \right\}.
\end{aligned}$$

Here $R_{j,k}^{(8)} = \|\partial_t^j Q_0 \mathbf{F}_\infty\|_{H^{m-1-2j}}^2$.

Proof. We see from (4.10)

$$\partial_t T_{j,k} \sigma_1 = \langle Q_0 T_{j,k} \mathbf{F} \rangle_1 - \langle Q_0 \tilde{B}(T_{j,k} \sigma_1 u^{(0)} + T_{j,k} u_\infty) \rangle_1,$$

which gives the estimate in (i).

As for (ii), we see from (4.11)

$$\begin{aligned}
\partial_t^{j+1} \phi_\infty &= \partial_t^j f_\infty^0 - \{v_s^1 \partial_{x_1} \partial_t^j \phi_\infty + \gamma^2 \operatorname{div}(\rho_s \partial_t^j w_\infty) \\
&\quad + (v_s^1 \phi^{(0)} + \gamma^2 \rho_s w^{(0),1}) \partial_{x_1} \partial_t^j \sigma_1 \\
&\quad - \langle Q_0 \tilde{B}(\partial_t^j \sigma_1 u^{(0)} + \partial_t^j u_\infty) \rangle_1 \phi^{(0)}\},
\end{aligned}$$

which gives the estimate in (ii). This completes the proof. \square

5.2 H^m -energy bound: induction argument

In this subsection we establish H^m -energy bound by using Propositions 5.2–5.8.

We will show the following estimate.

Proposition 5.9. *There are positive constants ν_0 , γ_0 and $\tilde{\omega}$ such that if $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2$ and $\|\tilde{g}\|_{C^m[0,1]} \leq \tilde{\omega}$, then*

$$\begin{aligned} & \|u(t)\|_m^2 + \int_0^t (\|D\sigma_1\|_{m-1}^2 + \|D\phi_\infty\|_{m-1}^2 + \|Dw_\infty\|_m^2) d\tau \\ & \leq C\{\|u_0\|_{H^m}^2 + \sum_{j=0}^8 \int_0^t R^{(8)}(\tau) d\tau\}, \end{aligned}$$

where

$$\begin{aligned} R^{(1)} &= \sum_{2j+k \leq m} R_{j,k}^{(1)}, \quad R^{(p)} = \sum_{2j+k \leq m-1} R_{j,k}^{(p)} \quad (p = 2, 5, 7), \quad R^{(4)} = \sum_{\substack{2j+k \leq m \\ 2j \neq m}} R_{j,k}^{(1)}, \\ R^{(p)} &= \sum_{2j+k+l \leq m-1} R_{j,k,l}^{(p)} \quad (p = 3, 6), \quad R^{(8)} = \sum_{2j \leq m-1} R_j^{(8)}. \end{aligned}$$

Proof. We set

$$\begin{aligned} \tilde{E}^{(0)}(t) &= \sum_{\substack{2j+k \leq m \\ 2j \neq m}} E^{(0)}[T_{j,k}u(t)], \quad E^{(1)}(t) = \sum_{2j+k \leq m-1} E^{(1)}[T_{j,k}u(t)] \\ \tilde{D}^{(0)}(t) &= \sum_{\substack{2j+k \leq m \\ 2j \neq m}} D^{(0)}[T_{j,k}w_\infty(t)] \end{aligned}$$

and

$$D^{(1)}(t) = \sum_{2j+k \leq m-1} \left(\frac{2b_1\gamma^2}{\nu(\nu+\tilde{\nu})} D^{(1)}[T_{j,k}w_\infty(t)] + \frac{1}{\nu+\tilde{\nu}} \|\sqrt{\rho_s} \partial_t T_{j,k}w_\infty(t)\|_2^2 \right).$$

By (5.1) with $2j+k \leq m-1$ and $\frac{1}{\nu+\tilde{\nu}} \times (5.11)$, we have

$$\begin{aligned} & \frac{d}{dt} \left(\tilde{E}^{(0)}(t) + \frac{1}{\nu+\tilde{\nu}} E^{(1)}(t) \right) + \frac{3}{2} \tilde{D}^{(0)}(t) + D^{(1)}(t) \\ & \leq C(R^{(1)} + R^{(2)}) \\ (5.50) \quad & + C \sum_{\substack{2j+k \leq m \\ 2j \neq m}} \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + \left(\frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) \|\tilde{T}_{j,k} \phi_\infty\|_2^2 \right\} \\ & + C \sum_{2j+k \leq m-1} \left\{ \left(\frac{1}{\nu(\nu+\tilde{\nu})} + \frac{1}{\gamma^2} \right) \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 \right. \\ & \left. + \left(\frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) \frac{1}{\nu+\tilde{\nu}} \|\tilde{T}_{j,k} \phi_\infty\|_2^2 + \frac{1}{\gamma^2(\nu+\tilde{\nu})} \|\partial_{x'} T_{j,k} \phi_\infty\|_2^2 \right\}. \end{aligned}$$

We set

$$E^{(2)}(t) = \sum_{2j+k \leq m-1} \frac{1}{\gamma^2} \left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x_n} T_{j,k} \phi_\infty(t) \right\|_2^2,$$

$$D^{(2)}(t) = \sum_{2j+k \leq m-1} \left(\frac{1}{\nu + \tilde{\nu}} \left\| \frac{\tilde{P}'(\rho_s)}{\gamma^2} \partial_{x_n} T_{j,k} \phi_\infty(t) \right\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} \|T_{j,k} \dot{\phi}_\infty\|_{H^1}^2 \right).$$

By (5.30) with $l = 0$ and (5.31) with $2j + k \leq m$, $2j \neq m$, we have

$$\begin{aligned} (5.51) \quad & \frac{d}{dt} E^{(2)}(t) + D^{(2)}(t) \\ & \leq C(R^{(3)} + R^{(4)}) + C \sum_{2j+k \leq m-1} \frac{\nu + \tilde{\nu}}{\gamma^4} \|K_{j,k,0}\|_2^2 \\ & \quad + C \sum_{\substack{2j+k \leq m \\ 2j \neq m}} \left\{ D^{(0)}[T_{j,k} w_\infty] + (\nu + \tilde{\nu}) \tilde{\omega}^2 \|T_{j,k} w_\infty\|_2^2 \right. \\ & \quad \left. + \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_{x_1} T_{j,k} \sigma_1\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} \|\tilde{T}_{j,k} \phi_\infty\|_2^2 \right\}. \end{aligned}$$

Let b_2 be a positive number with $b_2 \leq 1$ to be determined later. It then follows from (5.50) and $b_2 \times (5.51)$ that

$$\begin{aligned} (5.52) \quad & \frac{d}{dt} (\tilde{E}^{(0)}(t) + \frac{1}{\nu + \tilde{\nu}} E^{(1)}(t) + b_2 E^{(2)}(t)) + \frac{3}{2} \tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) \\ & \leq C \sum_{j=1}^4 R^{(j)} + C \sum_{\substack{2j+k \leq m \\ 2j \neq m}} \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) (\|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + \|\tilde{T}_{j,k} \phi_\infty\|_2^2) \\ & \quad + C \sum_{2j+k \leq m-1} \left\{ \left(\frac{1}{\nu(\nu + \tilde{\nu})} + \frac{1}{\gamma^2} \right) \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + \left(\frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) \frac{1}{\nu + \tilde{\nu}} \|\tilde{T}_{j,k} \phi_\infty\|_2^2 \right. \\ & \quad \left. + \frac{1}{\gamma^2(\nu + \tilde{\nu})} \|\partial_{x'} T_{j,k} \phi_\infty\|_2^2 \right\} + C b_2 \sum_{2j+k \leq m-1} \frac{\nu + \tilde{\nu}}{\gamma^4} \|K_{j,k,0}\|_2^2. \end{aligned}$$

We take $b_2 > 0$ so small that $2C b_2 (1 + \frac{1}{\nu(\nu + \tilde{\nu})}) \leq \frac{1}{2}$. Furthermore, $\tilde{\omega}$ is taken so small that $2C b_2 (\nu + \tilde{\nu}) \tilde{\omega}^2 \leq \frac{1}{2}$. Then the terms $\|T_{j,k+1} \partial_x w_\infty\|_2^2$, $\|\sqrt{\rho_s} \partial_t T_{j,k} w_\infty\|_2^2$ and $\|T_{j,k} \partial_x w_\infty\|_2^2$ in $\frac{\nu + \tilde{\nu}}{\gamma^4} \|K_{j,k,0}\|_2^2$ on the right of (5.52) are absorbed in the left. We thus arrive at

$$\begin{aligned} (5.53) \quad & \frac{d}{dt} (\tilde{E}^{(0)}(t) + \frac{1}{\nu + \tilde{\nu}} E^{(1)}(t) + b_2 E^{(2)}(t)) + \frac{3}{2} \tilde{D}^{(0)}(t) + \frac{3}{4} D^{(1)}(t) + b_2 D^{(2)}(t) \\ & \leq C \sum_{j=1}^4 R^{(j)} + C \sum_{\substack{2j+k \leq m \\ 2j \neq m}} \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) (\|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + \|\tilde{T}_{j,k} \phi_\infty\|_2^2) \\ & \quad + C \sum_{2j+k \leq m-1} \left\{ \left(\frac{1}{\nu(\nu + \tilde{\nu})} + \frac{1}{\gamma^2} \right) \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 + \left(\frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) \frac{1}{\nu + \tilde{\nu}} \|\tilde{T}_{j,k} \phi_\infty\|_2^2 \right. \\ & \quad \left. + \left(\frac{1}{\gamma^2(\nu + \tilde{\nu})} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\partial_x T_{j,k} \phi_\infty\|_2^2 \right\}. \end{aligned}$$

We set

$$E^{(3)}(t) = \sum_{2j+k \leq m-1} \frac{\nu}{\gamma^2(\nu + \tilde{\nu})} \|T_{j,k} \sigma_1\|_2^2$$

and

$$D^{(3)}(t) = \sum_{2j+k \leq m-1} \frac{\alpha_1}{\nu + \tilde{\nu}} \|\partial_{x'} T_{j,k} \sigma_1\|_2^2.$$

Let b_3 be a positive number to be determined later. We add $b_3 \times \sum_{2j+k \leq m-1}$ (5.33) to (5.53). It then follows

$$\begin{aligned}
& \frac{d}{dt} (\tilde{E}^{(0)}(t) + \frac{1}{\nu + \tilde{\nu}} E^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t)) \\
& + \frac{3}{2} \tilde{D}^{(0)}(t) + \frac{3}{4} D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t) \\
(5.54) \quad & \leq C \sum_{j=1}^5 R^{(j)} + C b_3 \{D^{(1)}(t) + \tilde{D}^{(0)}(t) + D^{(2)}(t)\} \\
& + C \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} + \frac{1}{\nu} + \frac{1}{\gamma^2} \right) D^{(3)}(t) \\
& + C \sum_{\substack{2j+k \leq m \\ 2j \neq m}} \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\tilde{T}_{j,k} \phi_\infty\|_2^2 \\
& + C \sum_{2j+k \leq m-1} \frac{1}{\nu(\nu + \tilde{\nu})} \|\tilde{T}_{j,k} \phi_\infty\|_2^2.
\end{aligned}$$

We take $b_3 > 0$ so small that $C b_3 \leq \frac{1}{4}$. We then take ν , $\tilde{\nu}$ and γ^2 so large that $C \left(\frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} + \frac{1}{\nu} + \frac{1}{\gamma^2} \right) \leq \frac{b_3}{2}$. The terms $\tilde{D}^{(0)}(t)$, $D^{(j)}(t)$ ($j = 1, 2, 3$) on the right of (5.54) are absorbed in the left. We thus arrive at

$$\begin{aligned}
(5.55) \quad & \frac{d}{dt} \tilde{E}^{(4)}(t) + \frac{1}{2} (\tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t)) \\
& \leq C \sum_{\substack{2j+k \leq m \\ 2j \neq m}} \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\tilde{T}_{j,k} \phi_\infty\|_2^2 + C \sum_{2j+k \leq m-1} \frac{1}{\nu(\nu + \tilde{\nu})} \|\tilde{T}_{j,k} \phi_\infty\|_2^2.
\end{aligned}$$

Here

$$\tilde{E}^{(4)}(t) = \tilde{E}^{(0)}(t) + \frac{1}{\nu + \tilde{\nu}} E^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t).$$

Let b_4 be a positive number to be determined later and set

$$\begin{aligned}
\tilde{D}^{(4)}(t) &= \frac{1}{2} (\tilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t)) \\
&+ b_4 \sum_{2j+k \leq m-1} \left(\frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^2 T_{j,k} w_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_x T_{j,k} \phi_\infty\|_2^2 \right).
\end{aligned}$$

It then follows from $b_4 \times \{(5.45) \text{ with } l = 0\}$ and (5.55) that

$$\begin{aligned}
(5.56) \quad & \frac{d}{dt} \tilde{E}^{(4)}(t) + \tilde{D}^{(4)}(t) \\
& \leq C \sum_{j=1}^6 R^{(j)}(t) + C b_4 \left\{ \left(1 + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} \right) D^{(3)}(t) + D^{(2)}(t) + D^{(1)}(t) + \tilde{D}^{(0)}(t) \right\} \\
& + C b_4 \sum_{2j+k \leq m-1} \left\{ \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{\tilde{\omega}^2}{\nu + \tilde{\nu}} \right) \|T_{j,k} \phi_\infty\|_2^2 + \frac{1}{\gamma^4(\nu + \tilde{\nu})} \|\partial_{x'} T_{j,k} \phi_\infty\|_2^2 \right\} \\
& + C \sum_{\substack{2j+k \leq m \\ 2j \neq m}} \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\tilde{T}_{j,k} \phi_\infty\|_2^2 + C b_4 \sum_{2j+k \leq m-1} \frac{1}{\nu(\nu + \tilde{\nu})} \|\tilde{T}_{j,k} \phi_\infty\|_2^2.
\end{aligned}$$

We may assume $\frac{\nu+\tilde{\nu}}{\gamma^2} \leq 1$. We take $b_4 > 0$ and $\tilde{\omega}$ so small that $Cb_4 \leq \frac{1}{4}$ and $Cb_4\tilde{\omega}^2 \leq \frac{1}{4}$. We also take $\nu, \tilde{\nu}$ and γ^2 so large that $Cb_4 \left(\frac{(\nu+\tilde{\nu})^2}{\gamma^4} + \frac{1}{\gamma^4} \right) \leq \frac{1}{8}$ and $C \left(\frac{\nu+\tilde{\nu}}{\gamma^2} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4} + \frac{1}{\nu} \right) \leq \frac{1}{8}$. It then follows that the term on the right of (5.56) except $R^{(j)}$ are absorbed in the left. We thus arrive at

$$(5.57) \quad \frac{d}{dt} \tilde{E}^{(4)}(t) + \tilde{D}^{(4)}(t) \leq C \sum_{j=1}^6 R^{(j)}(t).$$

By (5.48) and (5.49) we see

$$(5.58) \quad \begin{aligned} & \sum_{2j \leq m-2} (\|\partial_t^{j+1} \sigma_1(t)\|_2^2 + \|\partial_t^{j+1} \phi_\infty(t)\|_2^2) \\ & \leq C(R^{(7)} + R^{(8)}) + C\{(\nu + \tilde{\nu})\tilde{D}^{(4)}(t) + \frac{\gamma^4}{\nu} \tilde{D}^{(0)}(t)\}. \end{aligned}$$

We add $\frac{\nu}{\gamma^4} b_5 \times (5.58)$ to (5.57) to obtain

$$\begin{aligned} & \frac{d}{dt} \tilde{E}^{(4)}(t) + \tilde{D}^{(4)}(t) + \frac{\nu b_5}{\gamma^4} \sum_{2j \leq m-2} (\|\partial_t^{j+1} \sigma_1(t)\|_2^2 + \|\partial_t^{j+1} \phi_\infty(t)\|_2^2) \\ & \leq C \sum_{j=1}^8 R^{(j)}(t) + Cb_5 \left\{ \frac{(\nu+\tilde{\nu})^2}{\gamma^4} \tilde{D}^{(4)}(t) + \tilde{D}^{(0)}(t) \right\}. \end{aligned}$$

Take $b_5 > 0$ so small that $Cb_5 \left(\frac{(\nu+\tilde{\nu})^2}{\gamma^4} + 1 \right) \leq \frac{1}{2}$. We then obtain

$$(5.59) \quad \begin{aligned} & \frac{d}{dt} \tilde{E}^{(4)}(t) + \frac{1}{2} \tilde{D}^{(4)}(t) + \frac{\nu b_5}{\gamma^4} \sum_{2j \leq m-2} (\|\partial_t^{j+1} \sigma_1(t)\|_2^2 + \|\partial_t^{j+1} \phi_\infty(t)\|_2^2) \\ & \leq C \sum_{j=1}^8 R^{(j)}(t). \end{aligned}$$

Let b_6 be a positive number to be determined later. We set

$$E^{(4)}(t) = \tilde{E}^{(4)}(t) + \frac{b_6 \nu}{\gamma^2} E^{(0)}[\partial_t^{\frac{m}{2}} u(t)]$$

and

$$D^{(4)} = \tilde{D}^{(4)}(t) + \frac{b_6 \nu}{\gamma^2} D^{(0)}(t)[\partial_t^{\frac{m}{2}} u(t)] + \frac{\nu b_5}{\gamma^4} \sum_{2j \leq m-2} (\|\partial_t^{j+1} \sigma_1(t)\|_2^2 + \|\partial_t^{j+1} \phi_\infty(t)\|_2^2).$$

Since $\|\partial_x T_{j,0} \sigma_1\|_2^2 \leq \|T_{j,0} \sigma_1\|_2^2$, it follows from (5.1) with $2j = m$ and (5.59) that

$$\begin{aligned} & \frac{d}{dt} E^{(4)}(t) + D^{(4)}(t) \\ & \leq C \sum_{j=1}^8 R^{(j)}(t) + Cb_6 \frac{\nu}{\gamma^4} \left(1 + \frac{\nu+\tilde{\nu}}{\gamma^2} \right) (\|\partial_t^{\frac{m}{2}} \sigma_1\|_2^2 + \|\partial_t^{\frac{m}{2}} \phi_\infty\|_2^2). \end{aligned}$$

Taking $b_6 > 0$ so small that $2Cb_6 \leq \frac{b_5}{2}$, we see that

$$(5.60) \quad \frac{d}{dt}E^{(4)}(t) + D^{(4)}(t) \leq C \sum_{j=1}^8 R^{(j)}(t).$$

Let b_7 be a positive number to be determined later. For $1 \leq l \leq m-1$, we set

$$E_l^{(4)}(t) = \sum_{2j+k \leq m-1-l} \frac{1}{\gamma^2} \left\| \sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty(t) \right\|_2^2$$

and

$$\begin{aligned} D_l^{(4)}(t) &= \frac{1}{2} \sum_{2j+k \leq m-1-l} \left(\frac{1}{\nu+\tilde{\nu}} \left\| \frac{\tilde{P}'(\rho_s)}{\gamma^2} T_{j,k} \partial_{x_n}^{l+1} \phi_\infty(t) \right\|_2^2 + \frac{c_0(\nu+\tilde{\nu})}{\gamma^4} \|T_{j,k} \partial_{x_n}^{l+1} \dot{\phi}_\infty(t)\|_2^2 \right) \\ &\quad + \frac{b_7}{2} \sum_{2j+k \leq m-1-l} \left(\frac{\nu^2}{\nu+\tilde{\nu}} \left\| \partial_x^{l+2} T_{j,k} w_\infty(t) \right\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \left\| \partial_x^{l+1} T_{j,k} \phi_\infty(t) \right\|_2^2 \right). \end{aligned}$$

We will show that there exists a number $b_8 > 0$ such that the following estimate holds for $1 \leq l \leq m-1$:

$$(5.61) \quad \begin{aligned} &\frac{d}{dt} (2^l E^{(4)}(t) + \sum_{p=1}^l 2^{l+1-p} b_8 E_p^{(4)}(t)) + \sum_{p=0}^l b_8^p D_p^{(4)}(t) \\ &\leq 2^{l+2} C_1 \sum_{j=1}^8 R^{(j)}(t), \end{aligned}$$

where $D_0^{(4)}(t) = D^{(4)}(t)$.

By (5.30) and (5.45), we have

$$\begin{aligned} &\frac{d}{dt} E_l^{(4)}(t) + 2D_l^{(4)}(t) \\ &\leq C_1 \{R^{(3)}(t) + R^{(5)}(t)\} + C_2 F_l^{(1)}(t) + C_3 b_7 F^{(2)}(t) \\ &\quad + \frac{\nu+\tilde{\nu}}{\gamma^4} \sum_{2j+k \leq m-1-l} (C_2 \|T_{j,k} \partial_x^{l+1} \phi_\infty\|_2^2 + C_3 b_7 \|T_{j,k} \partial_{x_n}^{l+1} \dot{\phi}_\infty\|_2^2), \end{aligned}$$

where

$$\begin{aligned} F_l^{(1)} &= \sum_{2j+k \leq m-1-l} \left\{ \frac{\nu^2}{\nu+\tilde{\nu}} \|T_{j,k+1} \partial_x^{l+1} w_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\sqrt{\rho_s} \partial_t T_{j,k} \partial_{x_n}^l w_\infty\|_2^2 \right. \\ &\quad \left. + (\nu + \tilde{\nu}) \tilde{\omega}^2 \left(\sum_{p=0}^{l-1} \|T_{j,k+1} \partial_x^{p+1} w_\infty\|_2^2 + \sum_{p=0}^l \|T_{j,k} \partial_x^{p+1} w_\infty\|_2^2 \right) \right. \\ &\quad \left. + \frac{\nu+\tilde{\nu}}{\gamma^4} \left(\sum_{p=0}^{l-1} \|T_{j,k} \partial_x^{p+1} \phi_\infty\|_2^2 + \|\partial_{x'} T_{j,k} \sigma_1\|_2^2 \right) \right\}, \\ F_l^{(2)} &= \sum_{2j+k \leq m-1-l} \left\{ \left(\frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{\tilde{\omega}^2}{\nu+\tilde{\nu}} \right) \|T_{j,k} \phi_\infty\|_{H^l}^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_t T_{j,k} w_\infty\|_{H^l}^2 \right. \\ &\quad \left. + \left(\frac{1}{\nu+\tilde{\nu}} + \nu \tilde{\omega}^2 \right) \|T_{j,k} w_\infty\|_{H^{l+1}}^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} (\|T_{j,k+1} \dot{\phi}_\infty\|_{H^l}^2 + \|T_{j,k} \dot{\phi}_\infty\|_{H^l}^2) \right\} \\ &\quad + \left(1 + \frac{(\nu+\tilde{\nu})^2}{\gamma^4} + \frac{1}{\gamma^2} \right) D^{(4)}(t) \end{aligned}$$

We take b_7 and ν , $\tilde{\nu}$ and γ^2 so that $C_2 \frac{\nu+\tilde{\nu}}{\gamma^4} \leq \frac{1}{2(\nu+\tilde{\nu})}$ and $C_3 b_7 \leq \frac{c_0}{2}$. It then follows

$$(5.62) \quad \frac{d}{dt} E_l^{(4)}(t) + 2D_l^{(4)}(t) \leq C_1(R^{(4)} + R^{(6)}) + C_2 F_l^{(1)} + C_3 b_7 F_l^{(2)}.$$

We here note that if ν , $\tilde{\nu}$, γ^2 and $\tilde{\omega}$ are in the range restricted above, then there exists a constant $C_4 > 0$ such that

$$C_2 F_l^{(1)}(t) + C_3 b_7 F_l^{(2)}(t) \leq C_4 \sum_{p=0}^{l-1} D_p^{(4)}(t),$$

which, together with (5.62), yields

$$(5.63) \quad \frac{d}{dt} E_l^{(4)}(t) + 2D_l^{(4)}(t) \leq C_1 \sum_{j=1}^8 R^{(j)} + C_4 \sum_{p=0}^{l-1} D_p^{(4)}(t).$$

Let b_8 be a positive number satisfying $C_4 b_8 \leq \frac{1}{2}$ and $b_8 \leq \frac{1}{2}$. We now prove (5.61) by induction on l . Consider first $l = 1$. By adding (5.60) to $b_8 \times \{(5.63) \text{ with } l = 1\}$, we have

$$\frac{d}{dt} (E^{(4)}(t) + b_8 E_1^{(4)}(t)) + \sum_{p=0}^1 b_8^p D_p^{(4)}(t) \leq (1 + b_8) C_1 \sum_{j=1}^8 R^{(j)} + b_8 C_4 D_0^{(4)}(t).$$

Since $C_4 b_8 \leq \frac{1}{2}$, we have

$$\frac{d}{dt} (E^{(4)}(t) + b_8 E_1^{(4)}(t)) + \frac{1}{2} \sum_{p=0}^1 b_8^p D_p^{(4)}(t) \leq (1 + b_8) C_1 \sum_{j=1}^8 R^{(j)},$$

and, since $2b_8 \leq 1$, we arrive at

$$\frac{d}{dt} (2E^{(4)}(t) + 2b_8 E_1^{(4)}(t)) + \sum_{p=0}^1 b_8^p D_p^{(4)}(t) \leq (2 + 1) C_1 \sum_{j=1}^8 R^{(j)}.$$

This shows that (5.61) holds for $l = 1$.

Let $1 \leq l \leq m - 2$ and suppose that (5.61) holds up to l . We will prove that (5.61) holds with l replaced by $l + 1$. By adding (5.61) to $b_8^{l+1} \times \{(5.63) \text{ with } l \text{ replaced by } l + 1\}$, we have

$$\begin{aligned} & \frac{d}{dt} (2^l E^{(4)}(t) + \sum_{p=1}^{l+1} 2^{l+1-p} b_8 E_p^{(4)}(t)) + \sum_{p=0}^{l+1} b_8^p D_p^{(4)}(t) \\ & \leq (\sum_{p=0}^l 2^p + b_8^{l+1}) C_1 \sum_{j=1}^8 R^{(j)}(t) + b_8^{l+1} C_4 \sum_{p=0}^l D_p^{(4)}(t). \end{aligned}$$

Since $b_8^{l+1}C_4 \leq \frac{1}{2}b_8^l \leq \frac{1}{2}b_8^p$ for $0 \leq p \leq l$, we have

$$\begin{aligned} & \frac{d}{dt} \left(2^l E^{(4)}(t) + \sum_{p=1}^{l+1} 2^{l+1-p} b_8 E_p^{(4)}(t) \right) + \frac{1}{2} \sum_{p=0}^{l+1} b_8^p D_p^{(4)}(t) \\ & \leq \left(\sum_{p=0}^l 2^p + b_8^{l+1} \right) C_1 \sum_{j=1}^8 R^{(j)}(t) \end{aligned}$$

We thus obtain

$$\begin{aligned} & \frac{d}{dt} \left(2^{l+1} E^{(4)}(t) + \sum_{p=1}^{l+1} 2^{l+2-p} b_8 E_p^{(4)}(t) \right) + \sum_{p=0}^{l+1} b_8^p D_p^{(4)}(t) \\ & \leq \left(\sum_{p=0}^{l+1} 2^p \right) C_1 \sum_{j=1}^8 R^{(j)}(t). \end{aligned}$$

This proves (5.61) with l replaced by $l+1$. Therefore, we conclude that (5.61) holds for all $1 \leq l \leq m-1$. We thus arrive at

$$(5.64) \quad \frac{d}{dt} \tilde{E}(t) + 2D(t) \leq C\tilde{R}(t),$$

where

$$\begin{aligned} \tilde{E}(t) &= 2^{m-1} E^{(4)}(t) + \sum_{p=1}^{m-1} 2^{m-p} b_8 E_p^{(4)}(t), \quad D(t) = \frac{1}{2} \sum_{p=0}^{l+1} b_8^p D_p^{(4)}(t), \\ \tilde{R}(t) &= \sum_{j=1}^8 R^{(j)}(t). \end{aligned}$$

Integrating (5.64) in t , we obtain

$$(5.65) \quad \tilde{E}(t) + 2 \int_0^t D(\tau) d\tau \leq \tilde{E}(0) + C \int_0^t \tilde{R}(\tau) d\tau.$$

To complete the H^m -energy estimate, it remains to estimate $\|\partial_{x_n}^2 w_\infty(t)\|_{m-2}^2$. This can be estimated by the equations

$$\begin{aligned} -\nu \partial_{x_n}^2 w'_\infty &= \mathbf{f}'_\infty - \rho_s \left\{ \partial_t w'_\infty - \frac{\nu}{\rho_s} \Delta' w'_\infty - \frac{\tilde{\nu}}{\rho_s} \nabla' \operatorname{div} w_\infty + \nabla' \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi_\infty \right) \right. \\ &\quad + v_s^1 \partial_{x_1} w'_\infty + \frac{\nu \Delta v_s^1}{\gamma^2 \rho_s^2} \phi_\infty \mathbf{e}'_1 + (\partial_{x_n} v_s^1) \partial_{x_1} w_\infty^n \mathbf{e}'_1 \\ &\quad - \frac{\nu}{\rho_s} w^{(0)'} \Delta' \sigma_1 - \frac{\tilde{\nu}}{\rho_s} \nabla' (w^{(0),1} \partial_{x_1} \sigma_1) + \alpha_0 \nabla' \sigma_1 + v_s^1 w^{(0)'} \partial_{x_1} \sigma_1 \\ &\quad \left. - \langle Q_0 \tilde{B}(\sigma_1 u^{(0)} + u_\infty) \rangle_1 w^{(0),1} \mathbf{e}'_1 \right\}, \\ -(\nu + \tilde{\nu}) \partial_{x_n}^2 w_\infty^n &= f_\infty^n - \rho_s \left\{ \partial_t w_\infty^n - \frac{\nu}{\rho_s} \Delta' w_\infty^n - \frac{\tilde{\nu}}{\rho_s} \nabla' \operatorname{div} w_\infty + \partial_{x_n} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi_\infty \right) \right. \\ &\quad \left. + v_s^1 \partial_{x_1} w_\infty^n - \frac{\tilde{\nu}}{\rho_s} (\partial_{x_n} w^{(0),1}) \partial_{x_1} \sigma_1 \right\}, \end{aligned}$$

which follows from (4.11). Applying $\partial_{x_n}^l T_{j,k}$ with $2j+k+l \leq m-2$ to these relations, one can show the following estimate by a direct computation:

$$(5.66) \quad \frac{\nu^2}{\nu + \tilde{\nu}} \llbracket \partial_{x_n}^2 w_\infty(t) \rrbracket_{m-2}^2 \leq CE^{(4)}(t) + CR^{(9)}(t),$$

where $R^{(9)}(t) = \llbracket \mathbf{f}_\infty(t) \rrbracket_{m-2}^2$.

It then follows from (5.65) and (5.66) that

$$(5.67) \quad E(t) + 2 \int_0^t D(\tau) d\tau \leq \tilde{E}(0) + C \int_0^t R(\tau) d\tau,$$

where

$$E(t) = \tilde{E}(t) + \frac{\nu^2}{\nu + \tilde{\nu}} \llbracket \partial_{x_n}^2 w_\infty(t) \rrbracket_{m-2}^2, \quad R(t) = \tilde{R}(t) + R^{(9)}(t).$$

Clearly,

$$(5.68) \quad |||D\sigma_1|||_{m-1}^2 + |||D\phi_\infty|||_{m-1}^2 + |||Dw_\infty|||_m^2 \leq D(t).$$

Since $E^{(1)}[u]$ is equivalent to $E^{(0)}[u] + D^{(0)}[w]$ with our choice of b_1 in Proposition 5.3, we also have

$$(5.69) \quad \llbracket u(t) \rrbracket_m^2 \leq CE(t).$$

Therefore, by (5.67)–(5.69) we obtain the estimate in Proposition 5.9. This completes the proof of Proposition 5.9. \square

5.3 Estimates on the nonlinearities

It remains to estimate $R(t)$. The points in the estimates on the nonlinearities are as follows. Compared with the standard Matsumura-Nishida energy method, we have more terms which involve σ_1 . By Lemma 4.4 (ii), we have

$$(5.70) \quad \|\partial_{x'}^k \sigma_1\|_2 \leq \|\sigma_1\|_2 \quad \text{for } k = 1, 2, \dots.$$

This, together with the Gagliardo-Nirenberg-Sobolev inequality, implies

$$(5.71) \quad \|\sigma_1\|_\infty \leq C\|\sigma_1\|_2.$$

Also, if $n \geq 3$, then we have

$$(5.72) \quad \|\sigma_1\|_4 \leq C\|\sigma_1\|_2^{1/2} \|\partial_{x'} \sigma_1\|_2^{1/2}.$$

See Lemma 5.11 below. As for u_∞ -part, by Lemma 4.3, we have the Poincaré inequality

$$(5.73) \quad \|u_\infty\|_2 \leq C \|\partial_x u_\infty\|_2.$$

Using these inequalities, one can control the terms involving σ_1 , which are classified as

$$\begin{aligned} \text{in } \langle Q_0 \mathbf{F} \rangle_1 : & \quad O(\sigma_1 \partial_{x_1} \sigma_1), \quad O(\sigma_1 \partial_{x'} u_\infty), \quad O(u_\infty \partial_{x'} \sigma_1), \\ \text{in } \mathbf{F}_\infty : & \quad O(\sigma_1 \partial_{x_1} \sigma_1), \quad O(\sigma_1 \partial_x u_\infty), \quad O(u_\infty \partial_{x'} \sigma_1), \quad O(\sigma_1 \partial_{x'}^2 \sigma_1), \\ & \quad O(\phi_\infty \partial_{x'}^2 \sigma_1), \quad O(\sigma_1 \partial_x^2 w_\infty), \quad O(\sigma_1 u_\infty), \quad O(\sigma_1^2), \\ & \quad \text{and terms in } O(\phi^3), \quad O(\phi^2 \nabla \phi). \end{aligned}$$

For example, as for the term $O(\sigma_1^2)$ in \mathbf{F}_∞ , which comes from the term $-\frac{1}{2\gamma^4 \rho_s} \partial_{x_n} (P''(\rho_s) \sigma_1^2 \{\phi^{(0)}\}^2)$, we see from (5.72) and (5.73)

$$\begin{aligned} \left| \left(-\frac{1}{2\gamma^4 \rho_s} \partial_{x_n} (P''(\rho_s) \sigma_1^2 \{\phi^{(0)}\}^2), w_\infty \rho_s \right) \right| & \leq C \|\sigma_1\|_4^2 \|w_\infty\|_2 \\ & \leq C \|\sigma_1\|_2 \|\partial_{x'} \sigma_1\|_2 \|\partial_x w_\infty\|_2 \\ & \leq C \|u\|_m (\|D\sigma_1\|_{m-1}^2 + \|Dw_\infty\|_m^2). \end{aligned}$$

As for the term $O(\sigma_1 \partial_{x_1} \sigma_1)$ in $\langle Q_0 \mathbf{F} \rangle_1$, we have, by integration by parts,

$$\begin{aligned} (\langle \sigma_1 \partial_{x_1} \sigma_1 \phi^{(0)} w^{(0),1} \rangle_1, \sigma_1) & = (\sigma_1 \partial_{x_1} \sigma_1 \phi^{(0)} w^{(0),1}, \sigma_1) \\ & = -\frac{1}{2} (\sigma_1^2 \phi^{(0)} w^{(0),1}, \sigma_1, \partial_{x_1} \sigma_1), \end{aligned}$$

and thus,

$$(\langle \sigma_1 \partial_{x_1} \sigma_1 \phi^{(0)} w^{(0),1} \rangle_1, \sigma_1) = 0.$$

The terms involving only u_∞ can be treated similarly to the standard Matsumura-Nishida energy method. We also note that the Poincaré inequality (5.73) is effectively used in the estimates on the terms involving only u_∞ .

We summarize inequalities to estimate the nonlinearities.

Lemma 5.10. *Let $2 \leq p \leq \infty$ and let j and k be integers satisfying*

$$0 \leq j < k, \quad k > j + n \left(\frac{1}{2} - \frac{1}{p} \right).$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^j f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^2(\mathbf{R}^n)}^{1-a} \|\partial_x^k f\|_{L^2(\mathbf{R}^n)}^a,$$

where $a = \frac{1}{k}(j + \frac{n}{2} - \frac{n}{p})$.

Lemma 5.10 can be proved by using Fourier transform.

Combining Lemma 5.10, the extension argument and the Poincaré inequality, we have the following inequalities for u_∞ -part.

Lemma 5.11. *Let p, k and j be as in Lemma 5.10 and let $u_\infty = {}^\top(\phi_\infty, w_\infty)$ be in $\text{Range}(P_\infty)$. Then*

$$\|\phi_\infty\|_p \leq C \|\partial_x \phi_\infty\|_{H^{k-1}}.$$

The same inequality also holds for w_∞ if $w_\infty|_{x_n=0,1} = 0$.

As for (5.71) and (5.72), Lemma 4.4 and Lemma 5.10 with n replaced by $n - 1$ yields the following inequalities.

Lemma 5.12.

- (i) $\|\langle f \rangle_1\|_\infty \leq C \|\langle f \rangle_1\|_2$.
- (ii) $\|\langle f \rangle_1\|_4 \leq C \|\langle f \rangle_1\|_2^{1/2} \|\partial_{x'} \langle f \rangle_1\|_2^{1/2}$.

Proof. We first note that $\langle f \rangle_1$ is a function of $x' \in \mathbf{R}^{n-1}$. Let k be an integer satisfying $k > \frac{n-1}{2}$. Then, Lemma 5.10 with n replaced by $n - 1$ implies

$$\|\langle f \rangle_1\|_\infty \leq C \|\langle f \rangle_1\|_{H^k}.$$

Lemma 4.4 thus gives (i).

As for (ii), let k be an integer satisfying $k > \frac{n-1}{4}$. Then we have

$$(5.74) \quad \|\langle f \rangle_1\|_4 \leq C \|\langle f \rangle_1\|_2^{1-a} \|\partial_{x'}^k \langle f \rangle_1\|_2^a$$

with $a = \frac{n-1}{4k}$. We see that $\frac{1}{2} \leq a < 1$ if and only if $\frac{n-1}{4} < k \leq \frac{n-1}{2}$. Since $\frac{n-1}{2} \geq 1$ for $n \geq 3$, we find an integer $k \geq 1$ for which (5.74) holds with $\frac{1}{2} \leq a < 1$. With this k , we see from (5.74) and Lemma 4.4

$$\begin{aligned} \|\langle f \rangle_1\|_4 &\leq C \|\langle f \rangle_1\|_2^{1-a} \|\partial_{x'}^k \langle f \rangle_1\|_2^{(a-\frac{1}{2})+\frac{1}{2}} \\ &\leq C \|\langle f \rangle_1\|_{H^k}^{\frac{1}{2}} \|\partial_{x'}^k \langle f \rangle_1\|_2^{\frac{1}{2}} \\ &\leq C \|\langle f \rangle_1\|_2^{\frac{1}{2}} \|\partial_{x'} \langle f \rangle_1\|_2^{\frac{1}{2}}. \end{aligned}$$

This completes the proof. \square

We next state estimate on composite functions.

Lemma 5.13. (i) *Let m and m_k ($k = 1, \dots, \ell$) be nonnegative integers and let α_k ($k = 1, \dots, \ell$) be multi-indices. Suppose that*

$$m \geq [\frac{n}{2}] + 1, \quad 0 \leq |\alpha_k| \leq m_k \leq m + |\alpha_k| \quad (k = 1, \dots, \ell)$$

and

$$m_1 + \dots + m_\ell \geq (\ell - 1)m + |\alpha_1| + \dots + |\alpha_\ell|.$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^{\alpha_1} f_1 \cdots \partial_x^{\alpha_\ell} f_\ell\|_2 \leq C \prod_{1 \leq k \leq \ell} \|f_k\|_{H^{m_k}}.$$

(ii) *Let $1 \leq k \leq m$. Suppose that $F(x, t, y)$ is a smooth function on $\Omega \times [0, \infty) \times I$, where I is a compact interval in \mathbf{R} . Then for $|\alpha| + 2j = k$ there hold*

$$\begin{aligned} & \left\| [\partial_x^\alpha \partial_t^j, F(x, t, f_1)] f_2 \right\|_2 \\ & \leq \begin{cases} C_0(t, f_1(t)) \|f_2\|_{k-1} + C_1(t, f_1(t)) \{1 + \|Df_1\|_{m-1}^{|\alpha|+j-1}\} \|Df_1\|_{m-1} \|f_2\|_k, \\ C_0(t, f_1(t)) \|f_2\|_{k-1} + C_1(t, f_1(t)) \{1 + \|Df_1\|_{m-1}^{|\alpha|+j-1}\} \|Df_1\|_m \|f_2\|_{k-1}. \end{cases} \end{aligned}$$

Here

$$C_0(t, f_1(t)) = \sum_{\substack{(\beta, l) \leq (\alpha, j) \\ (\beta, l) \neq (0, 0)}} \sup_x |(\partial_x^\beta \partial_t^l F)(x, t, f_1(t))|$$

and

$$C_1(t, f_1(t)) = \sum_{\substack{(\beta, l) \leq (\alpha, j) \\ 1 \leq p \leq j + |\alpha|}} \sup_x |(\partial_x^\beta \partial_t^l \partial_y^p F)(x, t, f_1(t))|.$$

The proof of Lemma 5.12 can be found in [5, 6].

Using inequalities mentioned above, one can obtain the following estimates for the nonlinearities.

Before stating the estimates for the nonlinearities, we observe that since $m \geq [n/2] + 1$ we have the Sobolev inequality

$$\|f\|_\infty \leq C_2 \|f\|_{H^m}.$$

Let $\varepsilon_2 > 0$ be a number such that $C_2 \varepsilon_2 \leq \frac{\gamma^2 \rho_1}{2}$. Then whenever $\|u(t)\|_m \leq \varepsilon_2$, we have

$$\|\phi(t)\|_\infty \leq C_2 \|u(t)\|_m \leq C_2 \varepsilon_2 \leq \frac{\gamma^2 \rho_1}{2},$$

and hence,

$$\rho(x, t) = \rho_s(x_n) + \gamma^{-2}\phi(x, t) \geq \rho_1 - \gamma^{-2}\|\phi(t)\|_\infty \geq \frac{\rho_1}{2} (> 0).$$

We thus see that $\mathbf{F}(t)$ is smooth, whenever $\llbracket u(t) \rrbracket_m \leq \varepsilon_2$. So, we assume that $\llbracket u(t) \rrbracket_m \leq \varepsilon_2$ for $t \in [0, T]$.

Proposition 5.14. *Let j be an integer satisfying $0 \leq j \leq m-1$ and let $u(t)$ be a solution of (4.1) in $\cap_{j=0}^{[m/2]} C^j([0, T]; H^{m-2j})$ with $\int_0^T |||Dw_\infty|||_m^2 d\tau < \infty$. Assume that $\llbracket u(t) \rrbracket_m \leq \min\{1, \varepsilon_2\}$ for $t \in [0, T]$. Then the following estimates hold for $t \in [0, T]$ with $C > 0$ independent of T .*

$$(i) \quad \begin{aligned} & \|\partial_t^j(\phi \operatorname{div} w)\|_{H^{m-2j}} + \|\partial_t^j(w \cdot \nabla(\sigma_1 \phi^{(0)}))\|_{H^{m-2j}} + \|\partial_t^j \langle Q_0 \mathbf{F} \rangle_1\|_{H^{m-2j}} \\ & \leq CE(t)^{1/2} D(t)^{1/2}, \end{aligned}$$

$$(ii) \quad \|\partial_t^j(w \cdot \nabla \phi_\infty)\|_{H^{m-1-2j}} + \|\partial_t^j \mathbf{f}\|_{H^{m-1-2j}} \leq CE(t)^{1/2} D(t)^{1/2}.$$

Proposition 5.15. *Let $u(t)$ be a solution of (4.1) in $\cap_{j=0}^{[m/2]} C^j([0, T]; H^{m-2j})$ with $\int_0^T |||Dw_\infty|||_m^2 d\tau < \infty$. Assume that $\llbracket u(t) \rrbracket_m \leq \min\{1, \varepsilon_2\}$ for $t \in [0, T]$. Then there hold the following estimates for $t \in [0, T]$ with $C > 0$ independent of T .*

(i) *Let $0 \leq 2j + k \leq m$. Then*

$$|(T_{j,k} \langle Q_0 \mathbf{F} \rangle_1, T_{j,k} \sigma_1)| + |\langle \langle T_{j,k} \mathbf{F}_\infty, T_{j,k} \rangle \rangle| \leq CE(t)^{1/2} D(t).$$

(ii) *Let $0 \leq 2j + k \leq m-1$. Then*

$$|\langle \langle \rho_s(-\Delta)^{-1}(\rho_s \nabla' \cdot T_{j,k} f'_\infty) \rangle_1, T_{j,k} \sigma_1 \rangle| \leq CE(t)^{1/2} D(t).$$

(iii) *Let $0 \leq 2j + k + l \leq m-1$. Then*

$$\begin{aligned} & \| [T_{j,k} \partial_{x_n}^{l+1}, w] \cdot \nabla \phi_\infty \|_2 \leq CE(t)^{1/2} D(t)^{1/2}, \\ & \left| \left(\operatorname{div} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} w \right), |T_{j,k} \partial_{x_n}^{l+1} \phi_\infty|^2 \right) \right| \leq CE(t)^{1/2} D(t). \end{aligned}$$

We are now in a position to derive the a priori estimate. We deduce from Propositions 5.9, 5.14 and 5.15 that

$$\begin{aligned} & \llbracket u(t) \rrbracket_m^2 + \int_0^t (|||D\sigma_1|||_{m-1}^2 + |||D\phi_\infty|||_{m-1}^2 + |||Dw_\infty|||_m^2) d\tau \\ & \leq C_3 \{ \|u_0\|_{H^m}^2 + \llbracket u(t) \rrbracket_m \int_0^t (|||D\sigma_1|||_{m-1}^2 + |||D\phi_\infty|||_{m-1}^2 + |||Dw_\infty|||_m^2) d\tau \}, \end{aligned}$$

provided that $\|u(t)\|_m \leq \min\{1, \varepsilon_2\}$. The desired a priori estimate in Proposition 5.1 now follows by taking $\varepsilon_1 = \min\{1, \varepsilon_2, C_3/2\}$.

References

- [1] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Vol. 1, Springer-Verlag New York (1994).
- [2] Y. Kagei, Large time behavior of solutions to the compressible Navier-Stokes equation in an infinite layer, *Hiroshima Math. J.*, **38** (2008), pp. 95–124.
- [3] Y. Kagei, Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow, *MI Preprint Series 2009-7*, Kyushu University, accepted for publication in *J. Math. Fluid Mech.*, 2009.
- [4] Y. Kagei and S. Kawashima, Local solvability of initial boundary value problem for a quasilinear hyperbolic-parabolic system, *J. Hyperbolic Differential Equations*, **3** (2006), pp. 195–232.
- [5] Y. Kagei and S. Kawashima, Stability of planar stationary solutions to the compressible Navier-Stokes equation on the half space, *Comm. Math. Phys.*, **266** (2006), pp. 401–430.
- [6] Y. Kagei, and T. Kobayashi, Asymptotic behavior of solutions to the compressible Navier-Stokes equations on the half space, *Arch. Rational Mech. Anal.*, **177** (2005), pp. 231–330.
- [7] Y. Kagei, Y. Nagafuchi and T. Sudo, Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow, *J. Math-for-Ind.*, **2A** (2010), pp. 39–56.
- [8] A. Matsumura and T. Nishida, Initial boundary value problems for the equations of motion of general fluids, *Computing Methods in Appl. Sci. Engineering*, V (1982), pp. 389–406.
- [9] A. Matsumura and T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, *Comm. Math. Phys.*, **89** (1983), pp. 445–464.

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

- MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata
- MI2008-2 Eiji ONODERA
The initial value problem for a third-order dispersive flow into compact almost Hermitian manifolds
- MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space
- MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme
- MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in infinite extensions over a p -adic field
- MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields
- MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited
- MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition
- MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

- MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE
- MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds
- MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator
- MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials
- MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality
- MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions
- MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings
- MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors
- MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization
- MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions
- MI2009-5 Toshiro HIRANOUCI & Yuichiro TAGUCHI
Flat modules and Groebner bases over truncated discrete valuation rings

- MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations
- MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow
- MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization
- MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity
- MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption
- MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve
- MI2009-12 Tetsu MASUDA
Hypergeometric q -functions of the q -Painlevé system of type $E_8^{(1)}$
- MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination
- MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications
- MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

- MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring
- MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions
- MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force
- MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application
- MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions
- MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations
- MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces
- MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions
- MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map
- MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2 -projection

- MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine's property (Pm)
- MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space
- MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme
- MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance
- MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis
- MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke's zeros and higher depth determinants
- MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type
- MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect
- MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type $(A_2 + A_1)^{(1)}$
- MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

- MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes
- MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model
- MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes
- MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations
- MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models
- MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons
- MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine
- MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization
- MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation
- MI2010-8 Takahiro ITO
Abstract collision systems on groups
- MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

- MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models
- MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems
- MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow
- MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates
- MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight
- MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency
- MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres
- MI2010-17 Kazunori YASUTAKE
On the classification of rank 2 almost Fano bundles on projective space
- MI2010-18 Toshimitsu TAKAESU
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field
- MI2010-19 Reiichiro KAWAI & Hiroki MASUDA
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling
- MI2010-20 Yasuhide FUKUMOTO, Makoto HIROTA & Youichi MIE
Lagrangian approach to weakly nonlinear stability of an elliptical flow

- MI2010-21 Hiroki MASUDA
Approximate quadratic estimating function for discretely observed Lévy driven SDEs with application to a noise normality test
- MI2010-22 Toshimitsu TAKAESU
A Generalized Scaling Limit and its Application to the Semi-Relativistic Particles System Coupled to a Bose Field with Removing Ultraviolet Cutoffs
- MI2010-23 Takahiro ITO, Mitsuhiro FUJIO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Composition, union and division of cellular automata on groups
- MI2010-24 Toshimitsu TAKAESU
A Hardy's Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra
- MI2010-25 Toshimitsu TAKAESU
On the Essential Self-Adjointness of Anti-Commutative Operators
- MI2010-26 Reiichiro KAWAI & Hiroki MASUDA
On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling
- MI2010-27 Chikashi ARITA & Daichi YANAGISAWA
Exclusive Queueing Process with Discrete Time
- MI2010-28 Jun-ichi INOBUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Motion and Bäcklund transformations of discrete plane curves
- MI2010-29 Takanori YASUDA, Masaya YASUDA, Takeshi SHIMOYAMA & Jun KOGURE
On the Number of the Pairing-friendly Curves
- MI2010-30 Chikashi ARITA & Kohei MOTEGI
Spin-spin correlation functions of the q -VBS state of an integer spin model
- MI2010-31 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and spike detection via Gaussian basis expansions

MI2010-32 Nobutaka NAKAZONO

Hypergeometric τ functions of the q -Painlevé systems of type $(A_2 + A_1)^{(1)}$

MI2010-33 Yoshiyuki KAGEI

Global existence of solutions to the compressible Navier-Stokes equation around parallel flows