Determinants of Matrices over Group Algebras

山口, 尚哉

https://doi.org/10.15017/1806831

出版情報：九州大学, 2016, 博士（数理学）, 課程博士
バージョン：

Determinants of Matrices over Group Algebras

NAOYA YAMAGUCHI

To my family, Hisamitsu, Haruko, and Atsushi.

Preface

We research the determinants of matrices over group algebras. Firstly, we give an extension and a generalization of Dedekind's theorem. Secondly, we give a further extension of the above theorem. Thirdly, we give a generalization of Frobenius' theorem. Fourthly, we give Capelli elements of the group algebra of any finite group. Finally, we give a natural interpretation of the transfers in group theory in terms of noncommutative determinants.

Probably, readers think that "the determinant" is the ordinary determinant which is used for solving a system of linear equations. However, there are various types of determinants, such as Study, Dieudonné, row, column, and double determinants. Our main concern in this paper is these noncommutative determinants for matrices over group algebras on representations.

To research the determinants of matrices on representations was important subject. In the late 19th century, Georg Ferdinand Frobenius and Julius Wilhelm Richard Dedekind built a representation theory of finite groups in the process of obtaining the irreducible factorizations of the group determinants. The group determinant $\Theta(G)$ is the determinant of the regular representation $L : \mathbb{C}G \to \mathrm{Mat}(n, \mathbb{C})$ of G. The irreducible factorizations of $\Theta(G)$ is the following.

$$\Theta(G) = \prod_{\varphi \in \widehat{G}} \det \left(\sum_{g \in G} \varphi(g) x_g \right)^{\deg \varphi}.$$

As a result, we obtain theorems on the representations of finite groups. However, Frobenius and Dedekind's method became obsolete after the abstraction of the representation theory put forth by Issai Schur and Amalie Emmy Noether et al.

Nevertheless, Frobenius and Dedekind's idea (method) remains in the quaternions. In the early 20th, Eduard Study researched the determinant of a quaternionic matrix. In this research, Study defined the Study determinant, which uses an injective algebra homomorphism of quaternions. This homomorphism is a regular representation of quaternions. So, the Study determinant is similar to the group determinant.

In Chapter 1, 2, and 3, we generalize Frobenius and Dedekind's method. Specifically, we consider the determinant of the regular representation

$$L_H : \mathbb{C}G \to \mathrm{Mat}([G : H], \mathbb{C}H)$$

where H is a subgroup of G and $[G : H]$ is the index of H in G. Let e be the unit element. If $H = \{e\}$, we can regard the regular representation L as L_H. That is, we can regard the group determinant as a special case of determinants of regular representations of associative algebras.

In Chapter 1, we research the eigenvalues of $\det \circ L_H$ when G is any finite abelian group. However, note that the determinant does not appear explicitly in Chapter 1. We define operators on the group algebra, and research the eigenvalues of L_H by using the operators. As a result, we give an extension and a generalization of a special case of Frobenius' theorem.

In the next chapter, we research the noncommutative determinant $\det \circ L_H$ when G is a finite group and H is an abelian subgroup of G. Consequently, we give an extension and a generalization of Dedekind's theorem. The generalization in turn leads to a corollary on irreducible representations of finite groups. In addition, if a finite group has an index-two abelian subgroup, we define a conjugation of elements of the group algebra by using the further extension of Dedekind's theorem. In this process, we see the comparison between the Study determinant and $\det \circ L_H$ everywhere.

Let $L' : \mathrm{Mat}([G : H], \mathbb{C}H) \to \mathrm{Mat}(n, \mathbb{C}\{e\})$ be a regular representation of $\mathrm{Mat}([G : H], \mathbb{C}H)$. Then we have

$$\det \circ L = \det \circ L' \circ L_H.$$

In Chapter 3, we give a generalization of Frobenius' theorem by using the above equation on the determinant.

In the remaining chapters, we are inspired by the research of Professor Tôru Umeda. He suggested that a transfer can be derived as a noncommutative determinant, and gave Capelli identities for group determinants.

A transfer is defined by Schur as a group homomorphism from a group to an abelian quotient group of a subgroup of finite index. In Chapter 4, we develop Umeda's idea in order to explain the properties of the transfers by using noncommutative determinants. As a result, we give a natural interpretation of the transfers in group theory in terms of noncommutative determinants. The determinants are a hybrid of the Study determinant and Dieudonné determinant.

The Capelli identity is analogous to the product formula for the determinant in the Weyl algebra. The identity leads to the Capelli element. It is known that the Capelli elements is a central element in the universal enveloping algebra of $\mathfrak{gl}_n$. Umeda is one of the pioneers in Capelli identities. In recent years, he give Capelli identities for group determinants. There are Capelli identities for irreducible representations in the background of the Capelli identities for group determinants. In the last Chapter, we give a basis of the center of the group algebra of any finite group by using Capelli identities for irreducible representations. These identities lead to Capelli elements of the group algebra, and these elements construct a basis. These elements are defined by using row, column, or double determinants.

Contents

Chapter 1

An extension and a generalization of Dedekind's theorem

1.1 Introduction

In this chapter, we give factorizations of the group determinant for any given finite abelian group G in the group algebra of subgroups. The factorizations are an extension of Dedekind's theorem. The extension leads to a generalization of Dedekind's theorem and a simple expression for inverse elements in the group algebra.

The group determinant $\Theta(G)$ is the determinant of a matrix whose elements are independent variables x_g corresponding to $g \in G$. Dedekind gave the following theorem about the irreducible factorization of the group determinant for any finite abelian group.

Theorem 1.1.1 (Dedekind's theorem [4]). *Let G be a finite abelian group and $\widehat{G}$ the group of characters of G. Then we have*

$$\Theta(G) = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g.$$

Frobenius gave the following theorem about the irreducible factorization of the group determinant for any finite group; thus, Frobenius gave a generalization of Dedekind's theorem.

Theorem 1.1.2 (Frobenius' theorem [4]). *Let G be a finite group and $\widehat{G}$ a complete set of irreducible representations of G over $\mathbb{C}$. Then we have*

$$\Theta(G) = \prod_{\varphi \in \widehat{G}} \det \left(\sum_{g \in G} \varphi(g) x_g \right)^{\deg \varphi}.$$

The main results of this chapter are an extension and a generalization of Dedekind's theorem that are different from Frobenius' theorem.

1.1.1 Main results

We give an extension and a generalization of Dedekind's theorem.

Let G be a finite abelian group, $\mathbb{C}G$ the group algebra of G over $\mathbb{C}$, $R = \mathbb{C}[x_g] = \mathbb{C}\left[x_g; g \in G\right]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$, $RG = R \otimes \mathbb{C}G = \left\{\sum_{g \in G} A_g g \mid A_g \in R\right\}$ the group algebra of G over R, H a subgroup of G, and $[G : H]$ the index of H in G. Then we have the following theorem that is an extension of Dedekind's theorem.

Theorem 1.1.3 (Extension of Dedekind's theorem)**.** *Let G be a finite abelian group, e the unit element of G, H a subgroup of G, and $\widehat{H}$ the dual group of H. For every $h \in H$, there exists a homogeneous polynomial $A_h \in R$ such that* $\deg A_h = [G : H]$ *and*

$$\Theta(G)e = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Note that the equality in Theorem 1.1.3 is the equality in RH. Theorem 1.1.3 leads to the following theorem.

Theorem 1.1.4 (Generalization of Dedekind's theorem)**.** *Let G be a finite abelian group and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in R$ such that* $\deg A_h = [G : H]$ *and*

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Theorem 1.1.4 is a generalization of Dedekind's theorem. In fact, let $H = G$ and $A_h = x_h$. Then we have Dedekind's theorem.

Moreover, we obtain the following formula for inverse elements in the group algebra $\mathbb{C}G$ from Theorem 1.1.3. However, only now the situation is that x_g is a complex number for any $g \in G$. Hence, we assume that $\sum_{g \in G} x_g g \in \mathbb{C}G$ and $\Theta(G) = \det\left(x_{gh^{-1}}\right)_{g,h \in G} \in \mathbb{C}$.

Corollary 1.1.5. *Let G be a finite abelian group, χ_1 the trivial representation of G, and $\sum_{g \in G} x_g g \in \mathbb{C}G$ such that $\Theta(G) \neq 0$. Accordingly, we have*

$$\left(\sum_{g \in G} x_g g\right)^{-1} = \frac{1}{\Theta(G)} \prod_{\chi \in \widehat{G} \setminus \{\chi_1\}} \left(\sum_{g \in G} \chi(g) x_g g\right).$$

1.2 Irreducible factorization of group determinant

In this section, we recall the definition of the group determinant and its irreducible factorization.

1.2.1 Irreducible factorization of group determinant

Let G be a finite group and $\{x_g \mid g \in G\}$ independent commuting variables. Below, we define the group determinant $\Theta(G)$ of G.

Definition 1.2.1. *The group determinant $\Theta(G)$ of G is given by*

$$\Theta(G) = \det \left(x_{gh^{-1}}\right)_{g,h \in G}$$

where we give a numbering to the element of G.

Namely, the group determinant $\Theta(G)$ is a homogeneous polynomial of degree $|G|$ in $\{x_g \mid g \in G\}$, where $|G|$ is the order of G.

In general, the matrix $\left(x_{gh^{-1}}\right)_{g,h \in G}$ is a covariant under change of a numbering to the element of G. However, the group determinant $\Theta(G)$ is an invariant.

Example 1.2.2. *Let $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. Then we have*

$$\Theta(G) = \det \begin{bmatrix} x_0 & x_2 & x_1 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{bmatrix}.$$

Dedekind proved the following theorem about the irreducible factorization of the group determinant for any finite group.

Theorem 1.2.3 (Chapter 1, Theorem 1.1.1)**.** *Let G be a finite abelian group and $\widehat{G}$ the group of characters of G. Then we have*

$$\Theta(G) = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g.$$

Example 1.2.4. *Let $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. Then we have*

$$\begin{aligned} \Theta(G) &= \det \begin{bmatrix} x_0 & x_2 & x_1 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{bmatrix} \\ &= (x_0 + x_1 + x_2)(x_0 + x_1\omega + x_2\omega^2)(x_0 + x_1\omega^2 + x_2\omega) \end{aligned}$$

where ω is a primitive third root of unity.

1.3 An extension and a generalization of Dedekind's theorem

In this section, we give an extension and a generalization of Dedekind's theorem.

1.3.1 Degree one representations

In this subsection, we describe two lemmas needed later.

Let G be a finite group, $\overline{G}$ the set of degree one representations, H a subgroup of G and

$$\overline{G}_H = \left\{\chi \in \overline{G} \mid \chi(h) = 1, h \in H\right\}.$$

Then, $\overline{G}_H$ is a subgroup of $\overline{G}$.

Let $\widehat{G}$ be a complete set of irreducible representations of G. If G is an abelian group, since the degree of irreducible representations of G is one, we have $\overline{G} = \widehat{G}$.

The following lemmas are well known.

Lemma 1.3.1. *Let G be a finite group and H a normal subgroup of H such that G/H is an abelian group. Then we have*

$$\overline{G}_H = \left\{\varphi \circ \pi \mid \varphi \in \widehat{G/H}\right\}$$

where $\pi : G \to G/H$ is a natural projection.

Proof. Clearly, $\left\{\varphi \circ \pi \mid \varphi \in \widehat{G/H}\right\} \subset \overline{G}_H$. We show that $\overline{G}_H \subset \left\{\varphi \circ \pi \mid \varphi \in \widehat{G/H}\right\}$. Let $\chi \in \overline{G}_H$. We define the map $\varphi : G/H \to \mathbb{C}^\times$ by $\varphi(gH) = \chi(g)$. It is easy to see that φ is well defined and $\chi = \varphi \circ \pi$. This completes the proof. □

Lemma 1.3.2. *Let G be a finite abelian group, and suppose that $g \in G$ is not the unit element of G. Then, there exists $\chi \in \widehat{G}$ such that $\chi(g) \neq 1$.*

Proof. From the structure theorem for finite abelian groups, there exist cyclic groups $\mathbb{Z}/m_i\mathbb{Z}\,(1 \leq i \leq r)$ and a group isomorphism

$$f : G \to \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_r\mathbb{Z}.$$

Therefore, for all $g \in G$, there exists $\overline{a_i} \in \mathbb{Z}/m_i\mathbb{Z}$ such that

$$f(g) = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}).$$

For all $x_i \in \mathbb{N}\,(1 \leq i \leq r)$ where we assume that $0 \in \mathbb{N}$, we define the map $\chi : G \to \mathbb{C}^\times$ by

$$\chi(g) = \xi_1^{x_1 a_1} \xi_2^{x_2 a_2} \cdots \xi_r^{x_r a_r}$$

where ξ_i is a primitive m_i-th root of unity$(1 \leq i \leq r)$. Then, the map χ is a degree one representation of G. Since g is not the unit element, there exists $i \neq 0$ such that $a_i \neq 0$. Let $x_i = 1$ and $x_j = 0\,(1 \leq i \neq j \leq r)$. Then, χ is a degree one representation of G such that $\chi(g) \neq 1$. This completes the proof. □

Lemma 1.3.3. *Let G be a finite group and H a normal subgroup of G such that G/H is an abelian group. If $g \notin H$, there exists $\chi \in \overline{G}_H$ such that $\chi(g) \neq 1$.*

Proof. From Lemma 1.3.2, there exists $\varphi \in \widehat{G/H}$ such that $\varphi(gH) \neq 1$ where $g \notin H$. Let $\pi : G \to G/H$ be the natural projection. By Lemma 1.3.1, $\chi = \varphi \circ \pi \in \overline{G}_H$. This completes the proof. □

1.3.2 Operators on group algebras

In this subsection, we define operators on group algebras that are used in the proof of the main theorem.

Definition 1.3.4. *Let G be a finite group and $\chi \in \overline{G}$. We define the map $T_\chi : RG \to RG$ by*

$$T_\chi \left(\sum_{g \in G} A_g g \right) = \sum_{g \in G} \chi(g) A_g g$$

where $A_g \in R$.

Let $\chi, \chi' \in \overline{G}$ and $\alpha, \beta \in RG$. It is easy to see that $T_\chi \circ T_{\chi'} = T_{\chi \circ \chi'}$ and $T_\chi(\alpha\beta) = T_\chi(\alpha)T_\chi(\beta)$, where $(\chi \circ \chi')(g) = \chi(g)\chi'(g)$.

We give a necessary and sufficient condition for T_χ-invariance for all $\chi \in \overline{G}_H$.

Lemma 1.3.5. *Let G be a finite group, H a normal subgroup of G such that G/H is an abelian group and $\alpha \in RG$. For all $\chi \in \overline{G}_H$, $T_\chi(\alpha) = \alpha$ if and only if $\alpha \in RH$.*

Proof. Let $\alpha \in RH$. Obviously, $T_\chi(\alpha) = \alpha$ for all $\chi \in \overline{G}_H$. Let $\alpha = \sum_{g \in G} A_g g$. If $T_\chi(\alpha) = \alpha$ for all $\chi \in \overline{G}_H$, then we have $\chi(g)A_g g = A_g g$ for all $g \in G$. From this condition and Lemma 1.3.3, if $g \notin H$, there exists $\chi \in \overline{G}_H$ such that $\chi(g) \neq 1$. Therefore, $A_g = 0$. Namely, $\alpha = \sum_{h \in H} A_h h$. This completes the proof. □

Let G be a finite abelian group, $\widehat{G}_H = \overline{G}_H$, S a subgroup of $\widehat{G}$, and $S|_H$ the set of restrictions of $\chi \in S$ on H.

Lemma 1.3.6. *Let G be a finite abelian group, H a subgroup of G, and $\widehat{G} = \chi_1 \widehat{G}_H \sqcup \chi_2 \widehat{G}_H \sqcup \cdots \sqcup \chi_k \widehat{G}_H$. Then we have $k = |H|$ and $\widehat{H} = \{\chi_1, \chi_2, \ldots, \chi_k\}|_H$.*

Proof. First, we show that $k = |H|$. From $|G| = |\widehat{G}| = k|\widehat{G}_H|$ and Lemma 1.3.1, we have $|\widehat{G}_H| = |\widehat{G/H}| = \frac{|G|}{|H|}$. Therefore, $k = |H|$. Next, we show that $\widehat{H} = \{\chi_1, \chi_2, \ldots, \chi_k\}|_H$. Since the restriction of elements of $\widehat{G}_H$ is the trivial representation on H, $\widehat{G}|_H = \{\chi_1, \chi_2, \ldots, \chi_k\}|_H \subset \widehat{H}$. From $|\widehat{H}| = |H|$, we can show that $\chi_1, \chi_2, \ldots, \chi_k$ are different on H. If $\chi_i(h) = \chi_j(h)$ $(1 \leq i \neq j \leq k)$ for all $h \in H$, $\left(\chi_i^{-1} \circ \chi_j\right)(h) = 1$. Therefore, $\chi_i^{-1} \circ \chi_j \in \widehat{G}_H$. This is a contradiction for the left $\widehat{G}_H$-coset decomposition of $\widehat{G}$. Namely, we have $\chi_i \neq \chi_j$. This completes the proof. □

1.3.3 An extension and a generalization of Dedekind's theorem

In this subsection, we give the extension and generalization of Dedekind's theorem.

Lemma 1.3.7. *Let G be a finite abelian group, e the unit element of G, and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in R$ such that* $\deg A_h = [G : H]$ *and*

$$\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g = \sum_{h \in H} A_h h$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Proof. For all $\chi' \in \widehat{G}_H$,

$$\begin{aligned} T_{\chi'} \left(\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \right) &= \prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \left(\chi' \circ \chi \right) (g) x_g g \\ &= \prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g. \end{aligned}$$

From Lemma 1.3.5, we have $\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \in RH$. Clearly, $\deg A_h = |\widehat{G}_H| = [G : H]$. If $H = G$, $\widehat{G}_H$ is the trivial group. This completes the proof. $\square$

Definition 1.3.8. *Let $F : RG \to R$ be the R-algebra homomorphism such that $F(g) = 1$ for all $g \in G$. We call the map F the fundamental RG-function.*

Now, we give factorizations of the group determinant for any given finite abelian group in the group algebra of subgroups. The factorizations are the extension of Dedekind's theorem.

Theorem 1.3.9 (Chapter 1, Theorem 1.1.3)**.** *Let G be a finite abelian group, e the unit element of G, and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in R$ such that $\deg A_h = [G : H]$ and*

$$\Theta(G)e = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Proof. Clearly,

$$T_{\chi} \left(\prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g \right) = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g$$

for all $\chi \in \widehat{G}$. From this, $\widehat{G} = \widehat{G}_{\{e\}}$ and Lemma 1.3.5, there exists $C \in R$ such that

$$\begin{aligned} \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g &= \prod_{\chi \in \widehat{G}_{\{e\}}} \sum_{g \in G} \chi(g) x_g g \\ &= Ce. \end{aligned}$$

Let F be the fundamental RG-function. By applying F to this equation and Theorem 1.2.3, we have $C = \Theta(G)$. Namely, we have

$$\prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g = \Theta(G)e.$$

Let $\widehat{G} = \chi_1 \widehat{G}_H \sqcup \chi_2 \widehat{G}_H \sqcup \cdots \sqcup \chi_k \widehat{G}_H$. Then we have

$$\begin{aligned}\prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g g &= \prod_{i=1}^{k} \prod_{\chi \in \chi_i \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \\ &= \prod_{i=1}^{k} T_{\chi_i} \left(\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \right).\end{aligned}$$

There exists a homogeneous polynomial $A_h \in R$ for each $h \in H$ such that

$$\begin{aligned}\prod_{i=1}^{k} T_{\chi_i} \left(\prod_{\chi \in \widehat{G}_H} \sum_{g \in G} \chi(g) x_g g \right) &= \prod_{i=1}^{k} T_{\chi_i|_H} \left(\sum_{h \in H} A_h h \right) \\ &= \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h h\end{aligned}$$

from Lemmas 1.3.6 and 1.3.7. This completes the proof. □

As a corollary, we obtain the following formula for inverse elements in the group algebra $\mathbb{C}G$ from Theorem 1.3.9. However, only now the situation is that x_g is a complex number for any $g \in G$. Hence, we assume that $\sum_{g \in G} x_g g \in \mathbb{C}G$ and $\Theta(G) = \det\left(x_{gh^{-1}}\right)_{g,h \in G} \in \mathbb{C}$.

Corollary 1.3.10 (Chapter 1, Corollary 1.1.5)**.** *Let G be a finite abelian group, χ_1 the trivial representation of G, and $\sum_{g \in G} x_g g \in \mathbb{C}G$ such that $\Theta(G) \neq 0$. Then we have*

$$\left(\sum_{g \in G} x_g g \right)^{-1} = \frac{1}{\Theta(G)} \prod_{\chi \in \widehat{G} \setminus \{\chi_1\}} \left(\sum_{g \in G} \chi(g) x_g g \right).$$

We give factorizations of the group determinant for any given finite abelian group. The factorizations are the generalization of Dedekind's theorem.

Theorem 1.3.11 (Chapter 1, Theorem 1.1.4)**.** *Let G be a finite abelian group and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $A_h \in R$ such that $\deg A_h = [G : H]$ and*

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$.

Proof. From Theorem 1.3.9 and the fundamental RG-function, we have

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) A_h.$$

This completes the proof. □

Chapter 2

Factorizations of group determinant in group algebra for any abelian subgroup

2.1 Introduction

In this chapter, we give an extension and generalization of Dedekind's theorem over those presented in Chapter 1. The generalization in turn leads to a corollary on irreducible representations of finite groups. In addition, if a finite group has an index-two abelian subgroup, we can define a conjugation of elements of the group algebra by using the further extension of Dedekind's theorem.

Let G be a finite group, $\widehat{G}$ a complete set of irreducible representations of G over $\mathbb{C}$, and $\Theta(G)$ the group determinant of G. The group determinant $\Theta(G)$ is the determinant of a matrix whose elements are independent variables x_g corresponding to $g \in G$. Dedekind proved the following theorem about the irreducible factorization of the group determinant for any finite abelian group.

Theorem 2.1.1 (Chapter 1, Theorem 1.1.1)**.** *Let G be a finite abelian group. We have*

$$\Theta(G) = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g.$$

Frobenius proved the following theorem about the irreducible factorization of the group determinant for any finite group; thus, he gave a generalization of Dedekind's theorem.

Theorem 2.1.2 (Chapter 1, Theorem 1.1.2)**.** *Let G be a finite group. Then we have the irreducible factorization*

$$\Theta(G) = \prod_{\varphi \in \widehat{G}} \det \left(\sum_{g \in G} \varphi(g) x_g \right)^{\deg \varphi}.$$

Let $\mathbb{C}G$ be the group algebra of G over $\mathbb{C}$, $R = \mathbb{C}[x_g; g \in G]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$, $RG = R \otimes \mathbb{C}G = \left\{\sum_{g \in G} A_g g \mid A_g \in R\right\}$ the group algebra of G over R, H an abelian subgroup of G, and $[G : H]$ the index of H in G. Chapter 1 gives the following extension and generalization of Dedekind's theorem that are different from the theorem by Frobenius.

Theorem 2.1.3 ([Chapter 1, Theorem 1.3.9])**.** *Let G be a finite abelian group, e the unit element of G, and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $a_h \in R$ such that* $\deg a_h = [G : H]$ *and*

$$\Theta(G)e = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) a_h h.$$

If $H = G$, we can take $a_h = x_h$ for each $h \in H$.

Theorem 2.1.4 ([Chapter 1, Theorem 1.3.11])**.** *Let G be a finite abelian group and H a subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $a_h \in R$ such that* $\deg a_h = [G : H]$ *and*

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) a_h.$$

If $H = G$, we can take $a_h = x_h$ for each $h \in H$.

Here, we give a further extension of Theorem 2.1.3 and generalization of Theorem 2.1.4.

2.1.1 Results

The following theorem is the further extension of Dedekind's theorem.

Theorem 2.1.5 (Further extension of Dedekind's theorem)**.** *Let G be a finite group, e the unit element of G, and H an abelian subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $a_h \in R$ such that* $\deg a_h = [G : H]$ *and*

$$\Theta(G)e = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) a_h h.$$

If H is normal and h is a conjugate of h' on G, then $a_h = a_{h'}$.

Note that the equality in Theorem 2.1.5 is an equality in RH. Theorem 2.1.5 is proved using an extension of the group determinant $\Theta(G : H)$. The group determinant $\Theta(G : H)$ is an element of RH, and it is defined by using a left regular representation of RH. The left regular representation is reviewed in Section 2.2. In addition, Section 2.2 gives two expressions for the regular representation and shows that composition of regular representations is a regular representation. These expressions are helpful for describing some of the properties of $\Theta(G : H)$.

Above, we said that the group determinant is defined by using a left regular representation. In more detail, we define a noncommutative determinant by using a left regular representation and define the group determinant by using the noncommutative determinant. We know that the noncommutative determinant is analogous to the Study determinant [3]. The Study determinant is a quaternionic determinant, defined by using the regular representation ψ of the quaternions. In Sections 2.3 and 2.4, we describe the relationship between the noncommutative determinant and the Study determinant and their properties.

In the next section, we define the extension of the group determinant $\Theta(G : H)$ and give some properties of $\Theta(G : H)$.

In Section 2.6, we prove the further extension and generalization of Dedekind's theorem. In particular, Theorem 2.1.5 leads to the following theorem that is the further generalization of Dedekind's theorem.

Theorem 2.1.6 (Further generalization of Dedekind's theorem)**.** *Let G be a finite group and H an abelian subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $a_h \in R$ such that $\deg a_h = [G : H]$ and*

$$\Theta(G) = \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) a_h.$$

If H is normal and h is a conjugate of h' on G, then $a_h = a_{h'}$.

From Theorem 2.1.6, we have the following corollary on irreducible representations of finite groups.

Corollary 2.1.7. *Let G be a finite group and H an abelian subgroup of G. For all $\varphi \in \widehat{G}$, we have*

$$\deg \varphi \leq [G : H].$$

In the last section, we define a conjugation of the group algebra of the group which has an index two abelian subgroup. The conjugation comes from the noncommutative determinant. By applying the conjugation, we arrive at an inverse formula of 2×2 matrix.

2.2 Regular representation

Here, we describe the left regular representation of the group algebra and give two expressions for the representation. In addition, we show that a composition of regular representations is a regular representation.

Let R be a commutative ring, G a group, H a subgroup of G of finite index, and RG the group algebra of G over R whose elements are all possible finite sums of the form $\sum_{g \in G} a_g g$, where $a_g \in R$. We take a complete set $T = \{t_1, t_2, \ldots, t_{[G:H]}\}$ of left coset representatives of H in G, where $[G : H]$ is the index of H in G.

Definition 2.2.1 (Left regular representation)**.** *For all $A \in \mathrm{Mat}(m, RG)$, there exists a unique $L_T(A) \in \mathrm{Mat}(m[G:H], RH)$ such that*

$$A(t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]} I_m) = (t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]} I_m) L_T(A).$$

We call the map $L_T : \mathrm{Mat}(m, RG) \ni A \mapsto L_T(A) \in \mathrm{Mat}(m[G:H], RH)$ the left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$ with respect to T.

Obviously, L_T is an injective R-algebra homomorphism.

Example 2.2.2. *Let $G = \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$, $H = \{\overline{0}\}$, and $\alpha = x\overline{0} + y\overline{1} \in RG$. Then, we have*

$$\alpha(\overline{0} \quad \overline{1}) = (\overline{0} \quad \overline{1}) \begin{bmatrix} x\overline{0} & y\overline{0} \\ y\overline{0} & x\overline{0} \end{bmatrix}.$$

To give an expression for L_T when H is a normal subgroup of G, we define the map $\dot{\chi}$ by

$$\dot{\chi}(g) = \begin{cases} 1 & g \in H, \\ 0 & g \notin H \end{cases}$$

for all $g \in G$ and we denote (i, j) the $m \times m$ block element of an $(mn) \times (mn)$ matrix M by $M_{(i,j)}$. We can now prove the following theorem.

Lemma 2.2.3. *Let H be a normal subgroup of G, $L_T : \mathrm{Mat}(m, RG) \to \mathrm{Mat}(m[G:H], RH)$ the left regular representation with respect to T, and $A = \sum_{t \in T} t A_t \in \mathrm{Mat}(m, RG)$, where $A_t \in \mathrm{Mat}(m, RH)$. Then we have*

$$L_T(A)_{(i,j)} = \sum_{t \in T} \dot{\chi}(t_i^{-1} t t_j) t_i^{-1} t A_t t_j.$$

Proof. Let $r = [G:H]$. Then we have

$$\begin{aligned}
&(t_1 I_m \quad t_2 I_m \quad \cdots \quad t_r I_m) \left(\sum_{t \in T} \dot{\chi}\left(t_i^{-1} t t_j\right) t_i^{-1} t A_t t_j \right)_{1 \le i \le r, 1 \le j \le r} \\
&= \left(\sum_{i=1}^{r} \sum_{t \in T} \dot{\chi}(t_i^{-1} t t_1) t A_t t_1 \quad \sum_{i=1}^{r} \sum_{t \in T} \dot{\chi}(t_i^{-1} t t_2) t A_t t_2 \quad \cdots \quad \sum_{i=1}^{r} \sum_{t \in T} \dot{\chi}(t_i^{-1} t t_m) t A_t t_r \right) \\
&= \left(\sum_{t \in T} t A_t \right) (t_1 I_m \quad t_2 I_m \quad \cdots \quad t_r I_m).
\end{aligned}$$

This completes the proof. $\square$

To get another expression for L_T when H is a normal subgroup of G, we recall the Kronecker product. Let $A = (a_{ij})_{1\le i\le m_1, 1\le j\le n_1}$ be an $m_1 \times n_1$ matrix and $B = (b_{ij})_{1\le i\le m_2, 1\le j\le n_2}$ be an $m_2 \times n_2$ matrix. The Kronecker product $A \otimes B$ is the $(m_1 m_2) \times (n_1 n_2)$ matrix,

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2n_1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_1 1}B & a_{m_1 2}B & \cdots & a_{m_1 n_1}B \end{bmatrix}.$$

Let e be the unit element of G and $|G|$ the order of G. If $G = \{g_1, g_2, \ldots, g_{|G|}\}$ is a finite group. Then the restriction of the left regular representation $L_G : G \to \mathrm{Mat}(|G|, R\{e\})$ with respect to G is

$$L_G(g)_{ij} = \dot{\chi}(g_i^{-1} g g_j)e$$

from Lemma 2.2.3. We often assume that $R\{e\} = R$; thus, we often assume that $e = 1 \in R$. So, we can see that L_G is a matrix form of the left regular representation of the group G.

Let

$$P = \begin{bmatrix} t_1 I_m & & & \\ & t_2 I_m & & \\ & & \ddots & \\ & & & t_{[G:H]} I_m \end{bmatrix}.$$

Thus, we have the following lemma.

Lemma 2.2.4. *Let H be a normal subgroup of G, L_T the left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$ with respect to T, $L_{G/H}$ the left regular representation from $R(G/H)$ to $\mathrm{Mat}(|G/H|, R\{eH\})$ with respect to G/H, and $A = \sum_{t\in T} tA_t \in \mathrm{Mat}(m, RG)$, where $A_t \in \mathrm{Mat}(m, RH)$. Accordingly, we have*

$$L_T(A) = P^{-1}\left(\sum_{t\in T} L_{G/H}(tH) \otimes tA_t\right) P.$$

Proof. From Lemma 2.2.3, we have

$$\begin{aligned} \left(P^{-1}\left(\sum_{t\in T} L_{G/H} \otimes tA_t\right) P\right)_{(i,j)} &= t_i^{-1} I_m \left(\sum_{t\in T} \left(L_{G/H}(tH)\right)_{ij} tA_t\right) t_j I_m \\ &= \sum_{t\in T} \dot{\chi}(t_i^{-1} t t_j) t_i^{-1} t A_t t_j \\ &= L_T(A)_{(i,j)}. \end{aligned}$$

This completes the proof. □

We now show that a composition of regular representations is a regular representation. Theorem 2.6.5 requires the following lemma.

Lemma 2.2.5. *Let $K \subset H \subset G$ be a sequence of groups, $G = t_1H \cup t_2H \cup \cdots \cup t_{[G:H]}H$, $H = u_1K \cup u_2K \cup \cdots \cup u_{[H:K]}K$, $L_T : \mathrm{Mat}(m, RG) \to \mathrm{Mat}(m[G:H], RH)$ the representation with respect to T, and $L_U : \mathrm{Mat}(m[G:H], RH) \to \mathrm{Mat}(m[G:K], RK)$ the representation with respect to U. Then there exists a unique representation L_V from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:K], R\{e\})$ with respect to V such that*

$$L_V = L_U \circ L_T$$

where $V = \{v_1, v_2, \ldots, v_{[G:K]}\}$ is a complete set of the left coset representatives of K in G

Proof. Let $A \in \mathrm{Mat}(m, RG), r = [G:H]$, and $s = [H:K]$. By definition, we have

$$\begin{aligned} A(t_1I_m \quad t_2I_m \quad \cdots \quad t_rI_m) &= (t_1I_m \quad t_2I_m \quad \cdots \quad t_rI_m)L_T(A), \\ L_T(A)(u_1I_{mr} \quad u_2I_{mr} \quad \cdots \quad u_sI_{mr}) &= (u_1I_{mr} \quad u_2I_{mr} \quad \cdots \quad u_sI_{mr})L_U(L_T(A)). \end{aligned}$$

Let $(a_{ij})_{1\le i\le r, 1\le j\le r} = L_T(A)$ and $(b_{ij})_{1\le i\le s, 1\le j\le s} = L_U(L_T(A))$, where $a_{ij} \in \mathrm{Mat}\,(m, RH)$ and $b_{ij} \in \mathrm{Mat}\,(mr, RH)$. Then we have

$$(a_{ij})_{1\le i\le r, 1\le j\le r}u_pI_{mr} = \sum_{q=1}^{s} u_q b_{qp}.$$

We obtain

$$a_{ij}u_p = \sum_{q=1}^{s} u_q (b_{qp})_{ij}.$$

Therefore, we have

$$\begin{aligned} (At_i)u_j &= \left(\sum_{p=1}^{r} t_p a_{pi}\right) u_j \\ &= \sum_{p=1}^{r} t_p (a_{pi}u_j) \\ &= \sum_{p=1}^{r} t_p \left(\sum_{q=1}^{s} u_q (b_{qj})_{pi}\right) \\ &= \sum_{p=1}^{r}\sum_{q=1}^{s} t_p u_q (b_{qj})_{pi}. \end{aligned}$$

On the other hand, obviously $V = \{t_pu_q \mid 1 \le p \le r, 1 \le q \le s\}$ is a complete set of left coset representatives of K in G. From

$$At_iu_j = \sum_{p=1}^{r}\sum_{q=1}^{s} t_p u_q (b_{qj})_{pi},$$

we have

$$\begin{aligned}&A(t_1u_1I_m \quad \cdots \quad t_ru_1I_m \quad t_1u_2I_m \quad \cdots \quad t_ru_2I_m \quad \cdots \quad t_ru_sI_m)\\&\quad = (t_1u_1I_m \quad \cdots \quad t_ru_1I_m \quad t_1u_2I_m \quad \cdots \quad t_ru_2I_m \quad \cdots \quad t_ru_sI_m)L_U(L_T(A)).\end{aligned}$$

This completes the proof. □

2.3 Characteristics of image of representation when quotient group is abelian

In this section, we assume that G/H is a finite abelian group. Let

$$L_T\left(\mathrm{Mat}(m, RG)\right) = \{L_T(A) \mid A \in \mathrm{Mat}(m, RG)\}$$

and

$$J_t = P^{-1}\left(L_{G/H}(tH) \otimes I_m\right)P$$

for all $t \in T$. The following lemma will be used to show that $B \in \mathrm{Mat}(m[G:H], RH)$ is an image of L_T if and only if B commutes with J_t.

Lemma 2.3.1. *Let G/H be a finite abelian group and L_T the left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$ with respect to T. Then, the elements of $L_T(\mathrm{Mat}(m, RG))$ and J_t for all $t \in T$ are commutative.*

Proof. Suppose $A = \sum_{t\in T} tA_t \in \mathrm{Mat}(m, RG)$, where $A_t \in \mathrm{Mat}(m, RH)$. From Lemma 2.2.4, we have $L_T(A) = P^{-1}\left(\sum_{t\in T} L_{G/H}(tH) \otimes tA_t\right)$. Therefore, we have

$$\begin{aligned}L_T(A)J_{t'} &= P^{-1}\left(\sum_{t\in T} L_{G/H}(tH) \otimes tA_t\right)PP^{-1}\left(L(t'H) \otimes I_m\right)P\\&= P^{-1}\left(\sum_{t\in T} L_{G/H}(tt'H) \otimes tA_t\right)P\\&= P^{-1}\left(\sum_{t\in T} L_{G/H}(t'tH) \otimes tA_t\right)P\\&= J_{t'}L_T(A)\end{aligned}$$

for all $t' \in T$. This completes the proof. □

Now we are in a position to prove the following theorem.

Theorem 2.3.2. *Let G/H be a finite abelian group and L_T the left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$ with respect to T. We have*

$$L_T(\mathrm{Mat}(m, RG)) = \{B \in \mathrm{Mat}(m[G:H], RH) \mid J_tB = BJ_t, t \in T\}.$$

Proof. From Lemma 2.3.1, we have

$$L_T(\mathrm{Mat}(m, RG)) \subset \{B \in \mathrm{Mat}(m[G:H], RH) \mid J_t B = B J_t, t \in T\}.$$

We will show that

$$\{B \in \mathrm{Mat}(m[G:H], RH) \mid J_t B = B J_t, t \in T\} \subset L_T(\mathrm{Mat}(m, RG)).$$

For all $B \in \mathrm{Mat}(m[G:H], RH)$, there exists $A \in \mathrm{Mat}(m, RG)$ and $B_{ij} \in \mathrm{Mat}(m, RH)$ such that

$$B = L_T(A) + B'$$

where

$$B' = \begin{bmatrix} 0 & B_{12} & B_{13} & \cdots & B_{1[G:H]} \\ 0 & B_{22} & B_{23} & \cdots & B_{2[G:H]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & B_{[G:H]2} & B_{[G:H]3} & \cdots & B_{[G:H][G:H]} \end{bmatrix}.$$

From Lemma 2.3.1, we have $B' J_t = J_t B'$. For all $p \in \{2, 3, \ldots, [G:H]\}$, there exists $t \in T$ such that $J_{t(p,1)} = t_p I_m t_1^{-1}$ and $J_{t(i,1)} = 0$ for all $i \neq p$. Therefore, we have

$$\begin{aligned} B_{qp} t_p I_m t_1^{-1} &= (B' J_t)_{(q,1)} \\ &= (J_t B')_{(q,1)} \\ &= 0 \end{aligned}$$

for all $q \in \{1, 2, \ldots, [G:H]\}$. Thus, we have $B = L_T(A) \in L_T(\mathrm{Mat}(m, RG))$. This completes the proof. □

Theorem 2.3.2 is similar to a property of a left regular representation of the quaternions $\mathbb{H}$. Let $C + jD \in \mathrm{Mat}(m, \mathbb{H})$, where $C, D \in \mathrm{Mat}(m, \mathbb{C})$, and $\overline{C}$ the complex conjugation matrix of C. Then we have $(C + jD)(I_m \quad jI_m) = (I_m \quad jI_m)\psi(C + jD)$, where

$$\psi(C + jD) = \begin{bmatrix} C & -\overline{D} \\ D & -\overline{C} \end{bmatrix}.$$

Hence, $\psi : \mathrm{Mat}(m, \mathbb{H}) \ni C + jD \mapsto \psi(C + jD) \in \mathrm{Mat}(2m, \mathbb{C})$ is a left regular representation. The following is known for the image of ψ [3].

$$\psi(\mathrm{Mat}(m, \mathbb{H})) = \left\{B \in \mathrm{Mat}(2m, \mathbb{C}) \mid JB = \overline{B}J\right\}$$

where

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}.$$

2.4 Noncommutative determinant and some properties

In this section, we give a noncommutative determinant and describe its properties. This determinant is analogous to the Study determinant. Hence, we will define the determinant by using the regular representation of the group algebra.

Before defining the noncommutative determinant, we explain that we do not have to distinguish between the left and right inverses. Let H be an abelian subgroup of G.

Lemma 2.4.1 (Invertibility)**.** *For all $A, B \in \mathrm{Mat}(m, RG)$, $AB = I_m$ if and only if $BA = I_m$.*

Proof. Let $AB = I_m$. We have $L_T(A)L_T(B) = I_{m[G:H]}$. The elements of $L_T(A)$ and the elements of $L_T(B)$ are elements of a commutative ring. Hence, $L_T(B)L_T(A) = I_{m[G:H]}$. Therefore, $L_T(BA - I_m) = 0$. Since L_T is an injective, we have $BA = I_m$. □

The noncommutative determinant is as follows.

Definition 2.4.2. *Let H be an abelian subgroup of G and L be a left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$. We define the map $\mathrm{Det} : \mathrm{Mat}(m, RG) \to RH$ by*

$$\mathrm{Det} = \det \circ L.$$

Let $T' = \{t'_1, t'_2, \ldots, t'_m\}$ be another complete set of left coset representatives of H in G. Then, there exists $Q \in \mathrm{Mat}(m, RH)$ such that $L_T = Q^{-1}L_{T'}Q$. Therefore, we have

$$\begin{aligned}\mathrm{Det} &= \det \circ L_T \\ &= \det \circ L_{T'}.\end{aligned}$$

Thus, Det is an invariant under a change of the left regular representation; hence, Det is well-defined.

Det has the following properties.

Theorem 2.4.3. *For all $A, B \in \mathrm{Mat}(m, RG)$,*

(1) $\mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B)$.

(2) $A \in \mathrm{Mat}(m, RG)$ *is invertible if and only if* $\mathrm{Det}(A) \in RH$ *is invertible.*

Proof. Det is a multiplicative map, because L_T and det are multiplicative maps. Therefore, the equation (1) holds. Now let us prove (2). If A is invertible, there exists $B \in \mathrm{Mat}(m, RG)$ such that $AB = I_m$. Hence, $L_T(A)L_T(B) = I_{m[G:H]}, L_T(A)$ is invertible. Conversely, if $\mathrm{Det}(A)$ is invertible, there exists $B \in \mathrm{Mat}(m[G:H], RH)$ such that $L_T(A)B = I_{m[G:H]}$. Therefore,

$$A(t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]} I_m)B = (t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]} I_m) I_{m[G:H]}.$$

Thus, A is invertible. □

Next let us define characteristic polynomial of $A \in \mathrm{Mat}(m, KG)$.

Definition 2.4.4 (Characteristic polynomial)**.** *Let H be an abelian subgroup of G and L be a left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$. For all $A \in \mathrm{Mat}(m, RG)$, we define $\Phi_A(X)$ by*

$$\begin{aligned}\Phi_A(X) &= \mathrm{Det}(XI_m - A)\\ &= \det(XI_{m[G:H]} - L_T(A))\end{aligned}$$

where X is an independent variable such that $L_T(XB) = XL_T(B)$ and $\alpha X = X\alpha$ for any $B \in \mathrm{Mat}(m, RG)$ and $\alpha \in RH$.

We have the following lemma.

Lemma 2.4.5. *Let H be a normal abelian subgroup of G and $\Phi_A(X)$ the characteristic polynomial of A over RH. Then we have $\Phi_{g^{-1}Ag}(X) = \Phi_A(X)$ for all $g \in G$.*

Proof. Since $f_g : G/H \ni t_iH \mapsto gt_iH \in G/H$ is a bijection for all $g \in G$, for all $g \in G$, there exists $P \in \mathrm{Mat}(m[G:H], RH)$ such that

$$g(t_1I_m \quad t_2I_m \quad \cdots \quad t_{[G:H]}I_m) = (t_1I_m \quad t_2I_m \quad \cdots \quad t_{[G:H]}I_m)P.$$

Therefore, we have

$$\begin{aligned}\Phi_{g^{-1}Ag}(X) &= \det\left(XI_{m[G:H]} - L_T(g^{-1}Ag)\right)\\ &= \det\left(XI_{m[G:H]} - P^{-1}L_T(A)P\right)\\ &= \Phi_A(X).\end{aligned}$$

Here, we should remark that $P^{-1}L_T(A)P \in \mathrm{Mat}(m[G:H], RH)$, since H is a normal subgroup of G. This completes the proof. $\square$

We denote the center of the ring R by $Z(R)$. The following corollary will be used in the proof of Theorem 2.6.5.

Corollary 2.4.6. *Let H be a normal abelian subgroup of G and*

$$\Phi_A(X) = X^{m[G:H]} + a_{m[G:H]-1}X^{m[G:H]-1} + \cdots + a_0$$

the characteristic polynomial of A over RH. Then we have $a_i \in Z(RG) \cap RH$ for all $0 \leq i \leq m[G:H]-1$. In particular, $a_0 = \mathrm{Det}(A)$ and $a_{m[G:H]-1} = \mathrm{Tr}(L(A)) \in Z(RG) \cap RH$.

Next let us prove a Cayley-Hamilton type theorem for $\Phi_A(X)$.

Theorem 2.4.7 (Cayley-Hamilton type theorem)**.** *Let*

$$\Phi_A(X) = X^{m[G:H]} + a_{m[G:H]-1}X^{m[G:H]-1} + \cdots + a_0$$

be the characteristic polynomial of A over RH. We have

$$\begin{aligned}\Phi_A(A) &= A^{m[G:H]} + a_{m[G:H]-1}A^{m[G:H]-1} + \cdots + a_0I_m\\ &= 0.\end{aligned}$$

Proof. From the Cayley-Hamilton theorem for commutative rings,

$$L_T(A)^{m[G:H]} + a_{m[G:H]-1}L_T(A)^{m[G:H]-1} + \cdots + a_0 I_m = 0$$

and $A(t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]}I_m) = (t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]}I_m)L_T(A)$, we have

$$\begin{aligned}\Phi_A(A)(t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]}I_m) &= (t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]}I_m)0 \\ &= (0 \quad 0 \quad \cdots \quad 0)\end{aligned}$$

Thus, we have $\Phi_A(A) = 0$. This completes the proof. □

The noncommutative determinant Det is analogous to the Study determinant. Therefore, these determinant have similar properties.

The Study determinant Sdet is defined by $\det \circ \psi : \mathrm{Mat}(m, \mathbb{H}) \to \mathbb{C}$. The Study determinant has the following properties [3]. For all $A, B \in \mathrm{Mat}(m, \mathbb{H})$,

(1) $\mathrm{Sdet}AB = \mathrm{Sdet}A\,\mathrm{Sdet}B$.

(2) $A \in \mathrm{Mat}(m, \mathbb{H})$ is invertible if and only if $\mathrm{Sdet}A \neq 0$.

(3) $\mathrm{Sdet}A \in \mathbb{R}$. Hence, $\mathrm{Sdet}A$ is a central element of $\mathbb{H}$.

That is, Theorem 2.4.3 and Corollary 2.4.6 are similar to the above properties.

2.5 Extension of the group determinant in the group algebra for any abelian subgroup

Here, we extend the group determinant in the group algebra for any subgroup and show that the extension determines invertibility in $\mathrm{Mat}(m, RG)$. First, let us recall the group determinant.

Let G be a finite group, $\{x_g \mid g \in G\}$ be independent commuting variables, and $R = \mathbb{C}[x_g; g \in G]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$. The group determinant $\Theta(G) \in R$ is the determinant of a $|G| \times |G|$ matrix $(x_{g,h})_{g,h \in G}$, where $x_{g,h} = x_{gh^{-1}}$ for $g, h \in G$, and is thus a homogeneous polynomial of degree $|G|$ in x_g.

Now let us extend the group determinant in the group algebra for any abelian subgroup.

Definition 2.5.1 (Extension of the group determinant)**.** *Let G be a finite group, H an abelian subgroup of G, $\alpha = \sum_{g \in G} x_g g \in RG$, and $L : RG \to \mathrm{Mat}([G : H], RH)$ a left regular representation. We define*

$$\Theta(G : H) = (\det \circ L)(\alpha).$$

We call $\Theta(G : H)$ an extension of the group determinant in the group algebra RH.

If $H = \{e\}$, we know that $\Theta(G : H) = \Theta(G)e$. Thus, we can prove the following lemma.

Lemma 2.5.2. *Let G be a finite group, $\Theta(G)$ the group determinant of G, $\alpha = \sum_{g \in G} x_g g \in RG$, and $L : RG \to \mathrm{Mat}(|G|, R\{e\})$ a left regular representation. We have*

$$\begin{aligned}\Theta(G : \{e\}) &= (\det \circ L)(\alpha) \\ &= \Theta(G)e.\end{aligned}$$

Proof. Let L_G be the left regular representation from RG to $\mathrm{Mat}(|G|, R\{e\})$ with respect to G. From Lemma 2.2.3, we have

$$\begin{aligned}L_G\left(\sum_{g \in G} x_g g\right)_{ij} &= \sum_{g \in G} \dot{\chi}(g_i^{-1} g g_j) x_g g_i^{-1} g g_j \\ &= \begin{cases} x_g e & g_i^{-1} g g_j = e, \\ 0 & g_i^{-1} g g_j \neq e. \end{cases}\end{aligned}$$

Therefore, we have

$$L_G(\alpha) = \left(x_{g_i g_j^{-1}} e\right)_{1 \leq i \leq |G|, 1 \leq j \leq |G|}.$$

This completes the proof. □

Let us explain how the extension of the group determinant determines invertibility. Now the situation is that x_g is an element of R for any $g \in G$. Hence, we assume that $\sum_{g \in G} x_g g \in RG$ and $\Theta(G) = \det (x_{gh^{-1}})_{g,h \in G} \in R$. Accordingly, we get the following theorem from Theorem 2.4.3.

Theorem 2.5.3. *Let $\alpha = \sum_{g \in G} x_g g \in RG$. Then α is invertible if and only if $\Theta(G : H)$ is invertible.*

Obviously, $\Theta(G : \{e\}) = \Theta(G)e$ is invertible if and only if $\Theta(G) \neq 0$. Therefore, we get the following corollary.

Corollary 2.5.4. *Let $\alpha = \sum_{g \in G} x_g g \in RG$. Then α is invertible if and only if $\Theta(G) \neq 0$.*

2.6 Factorizations of the group determinant in the group algebra for any abelian subgroup

In this section, we give factorizations of the group determinant in the group algebra of abelian subgroups. The factorizations compose a further extension of Dedekind's theorem upon the one presented in Chapter 1. This extension in turn leads to a further generalization of Dedekind's theorem. Moreover, the generalization leads to a corollary on irreducible representations of finite groups.

First, we give a number of lemmas that will be needed later. The following theorem is well known.

Theorem 2.6.1 ([15, Theorem 4.4.4]). *Let G be a finite group, $\widehat{G} = \{\varphi_1, \varphi_2, \ldots, \varphi_s\}$ a complete set of inequivalent irreducible representations of G, $d_i = \deg \varphi_i$, and L_G the left regular representation of G. We have*

$$L_G \sim d_1\varphi_1 \oplus d_2\varphi_2 \oplus \cdots \oplus d_s\varphi_s.$$

Let $\mathrm{Mul}(G,R)$ be the set of multiplicative maps from G to R and $\chi \in \mathrm{Mul}(G,R)$. We define $F_\chi^{(m)} : \mathrm{Mat}(m, RG) \to \mathrm{Mat}(m, RG)$ by

$$F_\chi^{(m)}\left(\left(\sum_{g\in G} x_{ij}(g)g\right)_{1\le i\le m, 1\le j\le m}\right) = \left(\sum_{g\in G} \chi(g)x_{ij}(g)g\right)_{1\le i\le m, 1\le j\le m}$$

where $x_{ij}(g) \in R$. Now we have the following lemmas.

Lemma 2.6.2. *Let G be an abelian group, $\chi \in \mathrm{Mul}(G,R)$, and $A = \sum_{g\in G} A_g g \in \mathrm{Mat}(m, RG)$, where $A_g \in \mathrm{Mat}(m,R)$. If $\det A = \sum_{g\in G} a_g g$, where $a_g \in R$, we have*

$$\det\left(\sum_{g\in G} \chi(g)A_g g\right) = \sum_{g\in G} \chi(g)a_g g.$$

Hence, we have

$$\det \circ F_\chi^{(m)} = F_\chi^{(1)} \circ \det.$$

Proof. Let $A = \left(\sum_{g\in G} a_{ij}(g)g\right)_{1\le i\le m, 1\le j\le m}$, where $a_{ij}(g) \in R$. Then we have

$$\det A = \sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\left(\prod_{i=1}^{m}\sum_{g\in G} a_{\sigma(i)i}(g)g\right)$$

Therefore, we have

$$\begin{aligned}\sum_{g\in G} \chi(g)a_g g &= F_\chi^{(1)}(\det A)\\ &= F_\chi^{(1)}\left(\prod_{i=1}^{m}\sum_{g\in G} a_{\sigma(i)i}(g)g\right).\end{aligned}$$

From $\chi \in \mathrm{Mul}(G,R)$, we have

$$\begin{aligned}F_\chi^{(1)}\sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\left(\prod_{i=1}^{m}\sum_{g\in G} a_{\sigma(i)i}(g)g\right) &= \sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\left(\prod_{i=1}^{m}\sum_{g\in G} a_{\sigma(i)i}(g)\chi(g)g\right)\\ &= \det\left(\sum_{g\in G} a_{ij}(g)\chi(g)g\right)_{1\le i,j\le m}\\ &= \det\left(F_\chi^{(m)}(A)\right).\end{aligned}$$

This completes the proof. $\square$

Lemma 2.6.3. *Let G be an abelian group, H a subgroup of G, L a left regular representation from* $\mathrm{Mat}(m, RG)$ *to* $\mathrm{Mat}(m[G:H], RH)$*, and $\sum_{t\in T} tA_t \in \mathrm{Mat}(m, RG)$, where $A_t \in \mathrm{Mat}(m, RH)$. We have*

$$(\det \circ L)\left(\sum_{t\in T} tA_t\right) = \prod_{\chi\in\widehat{G/H}} \det\left(\sum_{t\in T} \chi(tH)tA_t\right).$$

Proof. From Lemma 2.2.4 and Theorem 2.6.1,

$$\begin{aligned}
(\det \circ L)\left(\sum_{t\in T} tA_t\right) &= \det\left(P^{-1}\left(\sum_{t\in T} L_{G/H}(tH)\otimes tA_t\right)P\right)\\
&= \det\left(\sum_{t\in T} L_{G/H}(tH)\otimes tA_t\right)\\
&= \prod_{\chi\in\widehat{G/H}} \det\left(\sum_{t\in T} \chi(tH)\otimes tA_t\right)\\
&= \prod_{\chi\in\widehat{G/H}} \det\left(\sum_{t\in T} \chi(tH)tA_t\right).
\end{aligned}$$

This completes the proof. $\square$

Lemma 2.6.4. *Let G be an abelian group, H a subgroup of G, L_1 a left regular representation from* $\mathrm{Mat}(m, RG)$ *to* $\mathrm{Mat}(m[G:H], RH)$*, and L_2 a left regular representation from RG to* $\mathrm{Mat}([G:H], RH)$*. Then the following diagram is commutative.*

$$\begin{tikzcd}
& RG \arrow[dr, "L_2"] & \\
\mathrm{Mat}(m, RG) \arrow[ur, "\det"] \arrow[d, "L_1"] & & \mathrm{Mat}([G:H], RH) \arrow[d, "\det"] \\
\mathrm{Mat}(m[G:H], RH) \arrow[rr, "\det"] & & RH
\end{tikzcd}$$

Proof. Let $A = \sum_{t\in T} tA_t$ and $\det A = \sum_{t\in T} ta_t$, where $A_t \in \mathrm{Mat}(m, RH)$ and $a_t \in RH$. From Lemma 2.6.3, we have

$$(\det \circ L_1)(A) = \prod_{\chi\in\widehat{G/H}} \det\left(\sum_{t\in T} \chi(tH)tA_t\right).$$

and

$$
\begin{aligned}
(\det \circ L_2 \circ \det)(A) &= (\det \circ L_2)\left(\sum_{t\in T} t a_t\right) \\
&= \prod_{\chi\in\widehat{G/H}} \left(\sum_{t\in T} \chi(tH) t a_t\right).
\end{aligned}
$$

We regard $\chi : G/H \to R$ as $\chi : G \ni g \mapsto \chi(gH) \in R$. Accordingly, we have

$$
\begin{aligned}
\prod_{\chi\in\widehat{G/H}} \det\left(\sum_{t\in T} \chi(tH) t A_t\right) &= \prod_{\chi\in\widehat{G/H}} (\det \circ F_\chi^{(m)})(A) \\
&= \prod_{\chi\in\widehat{G/H}} (F_\chi^{(1)} \circ \det)(A) \\
&= \prod_{\chi\in\widehat{G/H}} F_\chi^{(1)}\left(\sum_{t\in T} t a_t\right) \\
&= \prod_{\chi\in\widehat{G/H}} \sum_{t\in T} \chi(tH) t a_t
\end{aligned}
$$

by Lemma 2.6.2. This completes the proof. □

Now we are ready to state and prove the further extension of Dedekind's theorem.

Theorem 2.6.5 (Chapter 2, Theorem 2.1.5)**.** *Let G be a finite group and H be an abelian subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $a_h \in R$ such that* $\deg a_h = [G : H]$ *and*

$$\Theta(G)e = \prod_{\chi\in\widehat{H}} \sum_{h\in H} \chi(h) a_h h.$$

If H is normal and h is a conjugate of h' on G, then $a_h = a_{h'}$.

Proof. Let L_1 be a left regular representation from RG to $\mathrm{Mat}([G : H], RH)$, L_2 a left regular representation from $\mathrm{Mat}([G : H], RH)$ to $\mathrm{Mat}(|G|, R\{e\})$, L_3 a left regular representation from RH to $\mathrm{Mat}(|H|, R\{e\})$, and $(\det \circ L_1)\left(\sum_{g\in G} x_g g\right) = \sum_{h\in H} a_h h$, where $a_h \in R$. From Lemmas 2.2.5 and 2.5.2, we have

$$(\det \circ L_2 \circ L_1)\left(\sum_{g\in G} x_g g\right) = \Theta(G)e.$$

On the other hand, we have

$$\begin{aligned}(\det \circ L_3 \circ \det \circ L_1)\left(\sum_{g\in G} x_g g\right) &= (\det \circ L_3)\left(\sum_{h\in H} a_h h\right)\\ &= \prod_{\chi\in\widehat{H}}\left(\sum_{h\in H}\chi(h)a_h h\right)\end{aligned}$$

by Lemma 2.6.3. From Lemma 2.6.4, we can build the following commutative diagram.

$$\begin{tikzcd}
RG \arrow[dr, "L_1"] & & RH \arrow[dr, "L_3"] & \\
& \mathrm{Mat}([G:H], RH) \arrow[ur, "\det"] \arrow[d, "L_2"] & & \mathrm{Mat}(|H|, R\{e\}) \arrow[d, "\det"] \\
& \mathrm{Mat}(|G|, R\{e\}) \arrow[rr, "\det"] & & R\{e\}
\end{tikzcd}$$

Therefore, we have

$$\Theta(G)e = \prod_{\chi\in\widehat{H}}\sum_{h\in H}\chi(h)a_h h.$$

If H is a normal subgroup of G, we have

$$(\det \circ L_1)\left(\sum_{g\in G} x_g g\right) = \sum_{h\in H} a_h h \in Z(RG)$$

by Corollary 2.4.6. Hence, $a_h = a_{h'}$ when h is a conjugate h' on G. This completes the proof. □

Now we are in a position to state and prove the further generalization of Dedekind's theorem. Let $F : RG \to R$ be the R-algebra homomorphism such that $F(g) = 1$ for all $g \in G$. We call the map F the fundamental RG-function.

Theorem 2.6.6 (Chapter 2, Theorem 2.1.6)**.** *Let G be a finite group and H an abelian subgroup of G. For every $h \in H$, there exists a homogeneous polynomial $a_h \in R$ such that $\deg a_h = [G : H]$ and*

$$\Theta(G) = \prod_{\chi\in\widehat{H}}\sum_{h\in H}\chi(h)a_h.$$

If H is normal and h is a conjugate of h' on G, then $a_h = a_{h'}$.

Proof. From Theorem 2.6.5 and the fundamental RG-function, we have

$$\Theta(G) = \prod_{\chi\in\widehat{H}}\sum_{h\in H}\chi(h)a_h.$$

This completes the proof. □

From Theorems 2.1.2 and 2.6.6, we have the following corollary.

Corollary 2.6.7 (Chapter 2, Corollary 2.1.7)**.** *Let G be a finite group and H an abelian subgroup of G. For all $\varphi \in \widehat{G}$, we have*

$$\deg \varphi \leq [G : H].$$

Remark that Corollary 2.6.7 follows from Frobenius reciprocity[12].

2.7 Conjugate of the group algebra of the groups which have an index two abelian subgroup

In this section, we define a conjugation of elements of the group algebra of the group which has an index-two abelian subgroup.

First, we recall the conjugation of elements of $\mathbb{H}$. Let $z = x+jy \in \mathbb{H}$. The conjugation of z is defined as $\overline{z} = x - jy$. The conjugation has the following properties.

(1) $\mathrm{Sdet} z = z\overline{z}$.

(2) $\overline{\overline{z}} = z$.

(3) $z + \overline{z}$ and $z\overline{z} = \overline{z}z \in \mathbb{R} = Z(\mathbb{H})$.

(4) $\overline{zw} = \overline{w}\,\overline{z} \quad (w \in \mathbb{H})$.

(5) $z = \overline{z}$ if and only if $z \in \mathbb{R}$.

We refer to the above for definition of the conjugation of elements of the group algebra.

Let R be a commutative ring, G a group, H an index-two abelian subgroup of G, $T = \{e, t\}$ a complete set of left coset representatives of H in G, L_T the left regular representation from RG to $\mathrm{Mat}(2, RH)$ with respect to T, and $A = \alpha + t\beta \in RG$. We have

$$L_T(A) = \begin{bmatrix} \alpha & t\beta t \\ \beta & t^{-1}\alpha t \end{bmatrix}$$

and

$$\begin{aligned} \Phi_A(X) &= \det \begin{bmatrix} X - \alpha & t\beta t \\ \beta & X - t^{-1}\alpha t \end{bmatrix} \\ &= X^2 - (\alpha + t^{-1}\alpha t)X + \alpha t^{-1}\alpha t - \beta t\beta t. \end{aligned}$$

We notice that

$$\begin{aligned} &(X - (\alpha + t\beta))(X - (t^{-1}\alpha t - t\beta)) \\ &\quad = X^2 - (t^{-1}\alpha t - t\beta)X - (\alpha + t\beta)X + (\alpha + t\beta)(t^{-1}\alpha t - t\beta) \\ &\quad = X^2 - (\alpha + t^{-1}\alpha t)X + \alpha t^{-1}\alpha t - \alpha t\beta + (t\beta t^{-1})\alpha t - (t\beta t)\beta \\ &\quad = X^2 - (\alpha + t^{-1}\alpha t)X + \alpha t^{-1}\alpha t - \alpha t\beta + \alpha(t\beta t^{-1})t - \beta(t\beta t) \\ &\quad = X^2 - (\alpha + t^{-1}\alpha t)X + \alpha t^{-1}\alpha t - \beta t\beta t \\ &\quad = \Phi_A(X). \end{aligned}$$

Therefore, we define the conjugate of $A = \alpha + t\beta$ by

$$\overline{A} = t^{-1}\alpha t - t\beta.$$

The following theorem follows from Corollary 2.4.6 and a direct calculation:

Theorem 2.7.1. *For all $A, B \in RG$,*

(1) $(\det \circ L_T)(A) = A\overline{A}$.

(2) $\overline{\overline{A}} = A$.

(3) $A + \overline{A}$ *and* $A\overline{A} = \overline{A}A \in Z(RG)$.

(4) $\overline{AB} = \overline{B}\,\overline{A}$.

(5) $A = \overline{A}$ *if and only if* $A \in Z(RG)$.

We give the inverse formula of 2×2 matrix by conjugation.

Theorem 2.7.2. *Let $A, B, C, D \in RG$. Then we have*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} (\alpha\overline{\alpha} - \beta\gamma)^{-1} \begin{bmatrix} \overline{\alpha} & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

where

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \overline{\alpha} \end{bmatrix} = \begin{bmatrix} A\overline{D} + B\overline{C} & A\overline{B} + B\overline{A} \\ C\overline{D} + D\overline{C} & D\overline{A} + C\overline{B} \end{bmatrix},$$

and we assume that $\alpha\overline{\alpha} - \beta\gamma$ is invertible.

Proof. We have

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \overline{\alpha} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix}.$$

From $\beta, \gamma \in Z(RG)$, these elements $\alpha, \overline{\alpha}, \beta, \gamma$ are interchangeable. Therefore, we can use the inverse formula for 2×2 matrix whose elements are in commutative ring. This completes the proof. $\square$

Chapter 3

Generalization of Frobenius' theorem for group determinants

3.1 Introduction

In this chapter, we give a generalization of Frobenius' theorem. In addition, the generalization leads to a corollary on irreducible representations of finite groups.

Let G be a finite group, $\widehat{G}$ a complete set of irreducible representations of G over $\mathbb{C}$, and $R = \mathbb{C}[x_g] = \mathbb{C}[x_g; g \in G]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$. The group determinant $\Theta(G) \in R$ is the determinant of a matrix whose elements are independent variables x_g corresponding to $g \in G$. Frobenius proved the following theorem about the irreducible factorization of the group determinant.

Theorem 3.1.1 (Chapter 1, Theorem 1.1.2)**.** *Let G be a finite group. Then we have the irreducible factorization*

$$\Theta(G) = \prod_{\varphi \in \widehat{G}} \det\left(\sum_{g \in G} \varphi(g) x_g\right)^{\deg \varphi}.$$

Frobenius built a representation theory of finite groups in the process of obtaining Theorem 3.1.1. Here, we give a generalization of Theorem 3.1.1, i.e., a generalization of Frobenius' theorem. The theorem is as follows. However, we will explain $F_{[G:H]}$ in Section 3.4.

Theorem 3.1.2 (Generalization of Frobenius' theorem)**.** *Let G be a finite group, H a subgroup of G, L a left regular representation from RG to $\mathrm{Mat}([G:H], RH)$, $\alpha = \sum_{g \in G} x_g g \in RG$, and $L(\alpha) = \sum_{h \in H} C_h h$, where $C_h \in \mathrm{Mat}([G:H], R\{e\})$. Then, we have*

$$\Theta(G) = \prod_{\psi \in \widehat{H}} \det\left(\sum_{h \in H} \psi(h) \otimes C_h^{F_{[G:H]}}\right)^{\deg \psi}.$$

Theorem 3.1.2 leads to the following corollary.

Corollary 3.1.3. *Let G be a finite group and H a subgroup of G. For all $\varphi \in \widehat{G}$, we have*

$$\deg \varphi \leq [G:H] \times \max \left\{ \deg \psi \mid \psi \in \widehat{H} \right\}.$$

Theorem 3.1.2 is obtained by using left regular representations of the group algebra. In Section 3.3, we review the left regular representation and properties of the left regular representation needed for proving Theorem 3.1.2. The last section proves a generalization of Theorem 3.1.1.

3.2 Group determinant

Let G be a finite group, $\{x_g \mid g \in G\}$ be independent commuting variables, and $R = \mathbb{C}[x_g] = \mathbb{C}[x_g; g \in G]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$. The group determinant $\Theta(G)$ is the determinant of the $|G| \times |G|$ matrix $(x_{g,h})_{g,h \in G}$, where $x_{g,h} = x_{gh^{-1}}$ for $g, h \in G$, and it is thus a homogeneous polynomial of degree $|G|$ in x_g. Frobenius proved the following theorem about the factorization of the group determinant.

Theorem 3.2.1 (Chapter 3, Theorem 3.1.1)**.** *Let G be a finite group. Then we have the irreducible factorization*

$$\Theta(G) = \prod_{\varphi \in \widehat{G}} \det \left(\sum_{g \in G} \varphi(g) x_g \right)^{\deg \varphi}.$$

The above equation holds from the following theorem.

Theorem 3.2.2 ([15, Theorem 4.4.4])**.** *Let G be a finite group, $\{\varphi_1, \varphi_2, \ldots, \varphi_s\}$ a complete set of inequivalent irreducible representations of G, $d_i = \deg \varphi_i$, and L_G the left regular representation of G. Then,*

$$L_G \sim d_1\varphi_1 \oplus d_2\varphi_2 \oplus \cdots \oplus d_s\varphi_s.$$

3.3 Preparation for the main result

Here, we review the left regular representation of the group algebra and describe some of the properties of the left regular representation that will be needed later.

Let R be a commutative ring, G a group, H a subgroup of G of finite index, and RG the group algebra of G over R whose elements are all possible finite sums of the form $\sum_{g \in G} a_g g$, where $a_g \in R$. We take a complete set $T = \{t_1, t_2, \ldots, t_{[G:H]}\}$ of left coset representatives of H in G, where $[G:H]$ is the index of H in G.

Definition 3.3.1 (Left regular representation). *For all $A \in \mathrm{Mat}(m, RG)$, there exists a unique $L_T(A) \in \mathrm{Mat}(m[G:H], RH)$ such that*

$$A(t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]} I_m) = (t_1 I_m \quad t_2 I_m \quad \cdots \quad t_{[G:H]} I_m) L_T(A).$$

We call the map $L_T : \mathrm{Mat}(m, RG) \ni A \mapsto L_T(A) \in \mathrm{Mat}(m[G:H], RH)$ the left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$ with respect to T.

Obviously, L_T is an injective R-algebra homomorphism.

To give an expression for L_T when H is a normal subgroup of G, we will use the Kronecker product. Let $A = (a_{ij})_{1 \le i \le m_1, 1 \le j \le n_1}$ be an $m_1 \times n_1$ matrix and $B = (b_{ij})_{1 \le i \le m_2, 1 \le j \le n_2}$ an $m_2 \times n_2$ matrix. The Kronecker product $A \otimes B$ is the $(m_1 m_2) \times (n_1 n_2)$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2n_1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_1 1}B & a_{m_1 2}B & \cdots & a_{m_1 n_1}B \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} t_1 I_m & & & \\ & t_2 I_m & & \\ & & \ddots & \\ & & & t_{[G:H]} I_m \end{bmatrix}.$$

Now, we have the following lemma.

Lemma 3.3.2 ([21, Lemma 12]). *Let H be a normal subgroup of G, L_T the left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$ with respect to T, $L_{G/H}$ the left regular representation from $R(G/H)$ to $\mathrm{Mat}(|G/H|, R\{eH\})$ with respect to G/H, and $A = \sum_{t \in T} tA_t \in \mathrm{Mat}(m, RG)$, where $A_t \in \mathrm{Mat}(m, RH)$. Accordingly, we have*

$$L_T(A) = P^{-1} \left(\sum_{t \in T} L_{G/H}(tH) \otimes tA_t \right) P.$$

Let $K \subset H \subset G$ be a sequence of groups, $H = u_1 K \cup u_2 K \cup \cdots \cup u_{[H:K]} K$ and $U = \{u_1, u_2, \ldots, u_{[H:K]}\}$. We can now prove the following lemma.

Lemma 3.3.3 ([21, Lemma 13]). *Let $L_T : \mathrm{Mat}(m, RG) \to \mathrm{Mat}(m[G:H], RH)$ the representation with respect to T and $L_U : \mathrm{Mat}(m[G:H], RH) \to \mathrm{Mat}(m[G:K], RK)$ the representation with respect to U. Then there exists a unique representation L_V from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:K], R\{e\})$ with respect to V such that*

$$L_V = L_U \circ L_T$$

where $V = \{v_1, v_2, \ldots, v_{[G:K]}\}$ is a complete set of left coset representatives of K in G

The following lemma connects the left regular representation with the group determinant.

Lemma 3.3.4 ([21, Lemma 24])**.** *Let G be a finite group, $\Theta(G)$ the group determinant of G, $\alpha = \sum_{g \in G} x_g g \in RG$, and $L : RG \to \mathrm{Mat}(|G|, R\{e\})$ a left regular representation. We have*

$$(\det \circ L)(\alpha) = \Theta(G)e.$$

3.4 Generalization of Frobenius' theorem

Here, we prove the generalization of Frobenius' theorem. In addition, the proof leads to a corollary on irreducible representations of finite groups.

We define $F_m : \mathrm{Mat}(m, RG) \to \mathrm{Mat}(m, R)$ by

$$F_m\left(\left(\sum_{g \in G} x_{ij}(g)g\right)_{1 \le i \le m, 1 \le j \le m}\right) = \left(\sum_{g \in G} x_{ij}(g)\right)_{1 \le i \le m, 1 \le j \le m}$$

where $x_{ij}(g) \in R$. We denote $F_m(A)$ by A^{F_m} for all $A \in \mathrm{Mat}(m, RG)$.

Let G be a finite group and K a normal subgroup of G and H. The lemmas will be needed later.

Lemma 3.4.1. *Let H be a normal subgroup of G, L a left regular representation from $\mathrm{Mat}(m, RG)$ to $\mathrm{Mat}(m[G:H], RH)$, and $A = \sum_{t \in T} tA_t$, where $A_t \in \mathrm{Mat}(m, RH)$. We have*

$$(\det \circ F_{m[G:H]} \circ L)(A) = \prod_{\varphi \in \widehat{G/H}} \det\left(\sum_{t \in T} \varphi(tH) \otimes A_t^{F_m}\right).$$

Proof. Let $L \sim \varphi_1' \oplus \varphi_2' \oplus \cdots \oplus \varphi_{s'}'$ where φ_i' is an irreducible representation of G. From Lemma 3.3.2 and Theorem 3.2.2, we find that

$$\begin{aligned}
&(\det \circ F_{m[G:H]} \circ L)(A) \\
&= \det\left(P^{-1}\left(\sum_{t \in T} L_{G/H}(tH) \otimes tA_t\right)P\right)^{F_{m[G:H]}} \\
&= \det\left(\sum_{t \in T}\left(\begin{bmatrix} \varphi_1'(tH) & & & \\ & \varphi_2'(tH) & & \\ & & \ddots & \\ & & & \varphi_{s'}'(tH) \end{bmatrix} \otimes A_t^{F_m}\right)\right) \\
&= \prod_{\varphi \in \widehat{G/H}} \det\left(\sum_{t \in T} \varphi(tH) \otimes A_t^{F_m}\right)^{\deg \varphi}.
\end{aligned}$$

This completes the proof. □

Lemma 3.4.2. *Let* $L : \mathrm{Mat}(m, RG) \to \mathrm{Mat}(m[G:H], RH)$ *be a left regular representation,* $A = \sum_{v \in V} vB_v$*, and* $L(A) = \sum_{u \in U} uC_u$*, where* $B_v \in \mathrm{Mat}(m, RK)$ *and* $C_u \in \mathrm{Mat}(m[G:H], RK)$*. We have*

$$\prod_{\varphi \in \widehat{G/K}} \det\left(\sum_{v \in V} \varphi(vK) \otimes B_v^{F_m}\right)^{\deg\varphi} = \prod_{\psi \in \widehat{H/K}} \det\left(\sum_{u \in U} \psi(uK) \otimes C_u^{F_{m[G:H]}}\right)^{\deg\psi}.$$

Proof. From Lemmas 3.3.3 and 3.4.1, we have

$$\begin{aligned}
\prod_{\varphi \in \widehat{G/K}} \det\left(\sum_{v \in V} \varphi(vK) \otimes B_v^{F_m}\right)^{\deg\varphi} &= (\det \circ F_{m[G:K]} \circ L_V)(A) \\
&= (\det \circ F_{m[G:K]} \circ L_U \circ L_T)(A) \\
&= (\det \circ F_{m[G:K]} \circ L_U \circ L)(A) \\
&= (\det \circ F_{m[G:K]} \circ L_U)\left(\sum_{u \in U} uC_u\right) \\
&= \prod_{\psi \in \widehat{H/K}} \det\left(\sum_{u \in U} \psi(uK) \otimes C_u^{F_{m[G:H]}}\right)^{\deg\psi}.
\end{aligned}$$

This completes the proof. ∎

The following is the proof of the generalization of Frobenius' theorem.

Theorem 3.4.3 (Chapter 3, Theorem 3.1.2)**.** *Let* G *be a finite group,* $\Theta(G)$ *the group determinant of* G*,* H *a subgroup of* G*,* L *a left regular representation from* RG *to* $\mathrm{Mat}([G:H], RH)$*,* $\alpha = \sum_{g \in G} x_g g \in RG$*, and* $L(\alpha) = \sum_{h \in H} C_h h$*, where* $C_h \in \mathrm{Mat}([G:H], R\{e\})$*. We have*

$$\Theta(G) = \prod_{\psi \in \widehat{H}} \det\left(\sum_{h \in H} \psi(h) \otimes C_h^{F_{[G:H]}}\right)^{\deg\psi}.$$

Proof. For all $v \in V$, there exists $B_v \in \mathrm{Mat}(m, R\{e\})$ such that

$$\begin{aligned}
\Theta(G) &= (\Theta(G)e)^{F_1} \\
&= \prod_{\varphi \in \widehat{G/\{e\}}} \det\left(\sum_{v \in V} \varphi(vK) \otimes B_v^{F_1}\right)^{\deg\varphi} \\
&= \prod_{\psi \in \widehat{H/\{e\}}} \det\left(\sum_{u \in U} \psi(u\{e\}) \otimes C_u^{F_{[G:H]}}\right)^{\deg\psi} \\
&= \prod_{\varphi \in \widehat{H}} \det\left(\sum_{h \in H} \psi(h) \otimes C_u^{F_{[G:H]}}\right)^{\deg\psi}
\end{aligned}$$

from Lemmas 3.3.4, 3.4.1 and 3.4.2. □

The polynomial ring R is a unique factorization domain. Therefore, we have the following corollary from Theorems 3.2.1 and 3.4.3.

Corollary 3.4.4 (Chapter 3, Corollary 3.1.3)**.** *Let G be a finite group and H a subgroup of G. For all $\varphi \in \widehat{G}$, we have*

$$\deg \varphi \leq [G:H] \times \max \left\{ \deg \psi \mid \psi \in \widehat{H} \right\}.$$

Proof. We have

$$\begin{aligned}
\deg \varphi &= \deg \left(\det \left(\sum_{g \in G} \varphi(g) x_g \right) \right) \\
&\leq \max \left\{ \deg \left(\det \left(\sum_{h \in H} \psi(h) \otimes C_u^{F_{[G:H]}} \right) \right) \mid \psi \in \widehat{H} \right\} \\
&= \max \left\{ \deg \psi \times [G:H] \mid \psi \in \widehat{H} \right\} \\
&= [G:H] \times \max \left\{ \deg \psi \mid \psi \in \widehat{H} \right\}.
\end{aligned}$$

This completes the proof. □

Remark that Corollary 3.4.4 follows from Frobenius reciprocity[12].

Chapter 4

Proof of some properties of transfer using noncommutative determinants

4.1 Introduction

A transfer is defined by Issai Schur [14] as a group homomorphism from a group to an abelian quotient group of a subgroup of finite index. In finite group theory, transfers play an important role in transfer theorems. Transfer theorems include, for example, Alperin's theorem [1, Theorem 4.2], Burnside's theorem [8, Hauptsatz], and Hall-Wielandt's theorem [6, Theorem 14.4.2].

On the other hand, Eduard Study defined the determinant of a quaternionic matrix [3]. The Study determinant uses a regular representation from $\mathrm{Mat}(n, \mathbb{H})$ to $\mathrm{Mat}(2n, \mathbb{C})$, where $\mathbb{H}$ is the quaternions. Similarly, we define a noncommutative determinant. It is similar to the Dieudonné determinant [2].

Tôru Umeda suggested that a transfer can be derived as a noncommutative determinant [16, Footnote 7]. In this paper, we develop his ideas in order to explain the properties of the transfers by using noncommutative determinants. As a result, we give a natural interpretation of the transfers in group theory in terms of noncommutative determinants.

Let G be a group, H a subgroup of G of finite index, K a normal subgroup of H, and the quotient group H/K of K in H an abelian group. The transfer of G into H/K is a group homomorphism $V_{G \to H/K} : G \to H/K$. The definition of the transfer $V_{G \to H/K}$ uses the left (or right) coset representatives of H in G. We can show that a transfer has the following properties.

(1) A transfer is a group homomorphism from G to H/K.

(2) A transfer is an invariant under a change of coset representatives.

(3) A transfer by left coset representatives equals a transfer by right coset representatives.

Let R be a commutative ring with unity and RG the group algebra of G over R whose elements are all possible finite sums of the form $\sum_{g\in G} x_g g, x_g \in R$. The noncommutative determinant uses a left (or right) regular representation from RG to $\mathrm{Mat}(m, RH)$, where m is the index of H in G. We can show that the noncommutative determinant has the following properties.

($1'$) The determinant is a multiplicative map from RG to $R(H/K)$.

($2'$) The determinant is an invariant under a change of a regular representation.

($3'$) Any left regular representation is equivalent to any right regular representation.

Here, our objective is to obtain the properties of transfers (1), (2), and (3) by using the properties of noncommutative determinants ($1'$), ($2'$), and ($3'$).

4.2 Definition of the transfer

Here, we define the left and right transfer of G into H/K.

Let $G = t_1H \cup t_2H \cup \cdots \cup t_mH$. That is, we take a complete set $\{t_1, t_2, \ldots, t_m\}$ of left coset representatives of H in G. We define $\overline{g} = t_i$ for all $g \in t_iH$.

Definition 4.2.1 (Left transfer). *We define the map $V_{G\to H/K} : G \to H/K$ by*

$$V_{G\to H/K}(g) = \prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} gt_i \right\} K.$$

We call the map $V_{G\to H/K}$ the left transfer of G into H/K.

Next, we define the right transfer of G into H/K. Let $G = Hu_1 \cup Hu_2 \cup \cdots \cup Hu_m$. That is, we take a complete set $\{u_1, u_2, \ldots, u_m\}$ of right coset representatives of H in G. We define $\widetilde{g} = u_i$ for all $g \in Hu_i$.

Definition 4.2.2 (Right transfer). *We define the map $\widetilde{V}_{G\to H/K} : G \to H/K$ by*

$$\widetilde{V}_{G\to H/K}(g) = \prod_{i=1}^{m} \left\{ u_i g \left(\widetilde{u_i g}\right)^{-1} \right\} K.$$

We call the map $\widetilde{V}_{G\to H/K}$ the right transfer of G into H/K.

The definitions of the left and right transfers use the coset representatives of H in G. But, we can show that the left and right transfers are invariant under a change of coset representatives. Furthermore, we can show that a transfer is a group homomorphism from G to H/K and a transfer by left coset representatives equals a transfer by right coset representatives.

4.3 Definition of the noncommutative determinant

Here, we define the noncommutative determinant.

First, we define the left regular representation of RG. We take a complete set $T = \{t_1, t_2, \ldots, t_m\}$ of left coset representatives of H in G.

Definition 4.3.1 (Left regular representation)**.** *For all $\alpha \in RG$, there exists a unique $L_T(\alpha) \in \mathrm{Mat}(m, RH)$ such that*

$$\alpha(t_1 \quad t_2 \quad \cdots \quad t_m) = (t_1 \quad t_2 \quad \cdots \quad t_m) L_T(\alpha).$$

We call the map $L_T : RG \ni \alpha \mapsto L_T(\alpha) \in \mathrm{Mat}(m, RH)$ the left regular representation with respect to T.

Obviously, L_T is an R-algebra homomorphism.

Let $T' = \{t'_1, t'_2, \ldots, t'_m\}$ be another complete set of left coset representatives of H in G. Then, there exists $P \in \mathrm{Mat}(m, RH)$ such that $L_T = P^{-1} L_{T'} P$.

Example 4.3.2. *Let $G = \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$, $H = \{\overline{0}\}$, and $\alpha = x\overline{0} + y\overline{1} \in RG$. Then, we have*

$$\alpha(\overline{0} \quad \overline{1}) = (\overline{0} \quad \overline{1}) \begin{bmatrix} x\overline{0} & y\overline{0} \\ y\overline{0} & x\overline{0} \end{bmatrix}.$$

To get an expression for L_T, we define the map $\dot{\chi}$ by

$$\dot{\chi}(g) = \begin{cases} 1 & g \in H, \\ 0 & g \notin H \end{cases}$$

for all $g \in G$.

Lemma 4.3.3. *Let $\alpha = \sum_{g \in G} x_g g$. Then, we have*

$$L_T(\alpha)_{ij} = \sum_{g \in G} \dot{\chi}\left(t_i^{-1} g t_j\right) x_g t_i^{-1} g t_j.$$

Proof. We have

$$\begin{aligned}
&(t_1 \quad t_2 \quad \cdots \quad t_m) \left(\sum_{g \in G} \dot{\chi}\left(t_i^{-1} g t_j\right) x_g t_i^{-1} g t_j \right)_{1 \le i \le m, 1 \le j \le m} \\
&= \left(\sum_{i=1}^{m} \sum_{g \in G} \dot{\chi}(t_i^{-1} g t_1) x_g g t_1 \quad \sum_{i=1}^{m} \sum_{g \in G} \dot{\chi}(t_i^{-1} g t_2) x_g g t_2 \quad \cdots \quad \sum_{i=1}^{m} \sum_{g \in G} \dot{\chi}(t_i^{-1} g t_m) x_g g t_m \right) \\
&= \left(\sum_{g \in G} x_g g \right) (t_1 \quad t_2 \quad \cdots \quad t_m).
\end{aligned}$$

This completes the proof. $\square$

From Lemma 4.3.3, we have

$$\begin{aligned}L_T(g)_{ij} &= \dot{\chi}\left(t_i^{-1}gt_j\right)t_i^{-1}gt_j\\ &= \begin{cases} t_i^{-1}gt_j & t_i^{-1}gt_j \in H,\\ 0 & t_i^{-1}gt_j \notin H.\end{cases}\end{aligned}$$

From $t_i^{-1}gt_j \in H$ if and only if $\overline{gt_j} = t_i$, we have

$$L_T(g)_{ij} = \begin{cases} \left(\overline{gt_j}\right)^{-1} gt_j & t_i^{-1}gt_j \in H,\\ 0 & t_i^{-1}gt_j \notin H.\end{cases}$$

As for the definition of the noncommutative determinant, let $\psi : \mathrm{Mat}(m, RH) \to \mathrm{Mat}\left(m, R\left(H/K\right)\right)$ be a map such that

$$\psi\left(\left(x_{ij}\right)\right)_{1\leq i\leq m, 1\leq j\leq m} = \left(x_{ij}K\right)_{1\leq i\leq m, 1\leq j\leq m}.$$

Obviously, ψ is an R-algebra homomorphism.

Definition 4.3.4. *We define the map* $\mathrm{Det} : \mathrm{Mat}(m, RG) \to R\left(H/K\right)$ *by*

$$\mathrm{Det} = \det \circ \psi \circ L_T.$$

Since there is P such that $L_T = P^{-1}L_{T'}P$, we have

$$\begin{aligned}\mathrm{Det} &= \det \circ \psi \circ L_T\\ &= \det \circ \psi \circ L_{T'}.\end{aligned}$$

Thus, the determinant is an invariant under a change of left regular representations, so the determinant Det is well-defined. If K is the commutator subgroup of H, the determinant is similar to the Dieudonné determinant.

Obviously, the map Det is a multiplicative map. That is, $\mathrm{Det}(\alpha\beta) = \mathrm{Det}(\alpha)\mathrm{Det}(\beta)$ for all $\alpha, \beta \in RG$. Therefore, we obtain properties ($1'$) and ($2'$).

Remark 4.3.5. *In general, that $\alpha \in RG$ is invertible is not equivalent to that* $\mathrm{Det}(\alpha) \in R(H/K)$ *is invertible. For example, let $R = \mathbb{C}$, $\mathbb{Z}/2\mathbb{Z} = \left\{\overline{0}, \overline{1}\right\}$, dihedral group $D_3 = \langle a, b \,|\, a^3 = b^2 = e, ab = ba^{-1}\rangle$ where e is the unit element of D_3, $G = \mathbb{Z}/2\mathbb{Z} \times D_3$, $H = D_3$ and $K = [D_3, D_3]$ the commutator subgroup of H. Then $\alpha = \left(\overline{0}, e\right) + \left(\overline{0}, a\right) + \left(\overline{0}, a^2\right)$ is not invertible. But,* $\mathrm{Det}(\alpha) = 9K$ *is invertible.*

4.4 Proof of the properties

Here, we prove the transfer properties by using the noncommutative determinant's properties.

For all $g \in G$ and for all $t \in T$, there exists a unique $t_j \in T$ such that $t_i^{-1} g t_j \in H$. Therefore, there exists $\mathrm{sgn}(g) \in \{-1, 1\}$ such that

$$\begin{aligned}
\mathrm{Det}(g) &= \det\left(\psi\left(L_T(g)\right)\right) \\
&= \mathrm{sgn}(g) \prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} g t_i \right\} K \\
&= \mathrm{sgn}(g) V_{G \to H/K}(g).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathrm{sgn}(gh) V_{G \to H/K}(gh) &= \mathrm{Det}(gh) \\
&= \mathrm{Det}(g)\mathrm{Det}(h) \\
&= \mathrm{sgn}(g)\mathrm{sgn}(h) V_{G \to H/K}(g) V_{G \to H/K}(h).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathrm{sgn}(gh) &= \mathrm{sgn}(g)\mathrm{sgn}(h), \\
V_{G \to H/K}(gh) &= V_{G \to H/K}(g) V_{G \to H/K}(h).
\end{aligned}$$

Therefore, from property $(1')$ that Det is a multiplicative map, the left transfer $V_{G \to H/K}$ is a group homomorphism (Assuming, that is, $R = \mathbb{F}_2$, and we do not consider the signature).

Next, we show that the left transfer is an invariant under a change of coset representatives by using property $(2')$ that the determinant is an invariant under a change of regular representations. That is, we show that

$$\prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} g t_i \right\} K = \prod_{i=1}^{m} \left\{ \left(\overline{\overline{gt'_i}}\right)^{-1} g t'_i \right\} K$$

where we define $\overline{\overline{g}} = t'_i$ for all $g \in t'_i H$. From property $(2')$, there exists $\mathrm{sgn}'(g) \in \{-1, 1\}$ such that

$$\begin{aligned}
\prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} g t_i \right\} K &= \mathrm{sgn}(g)\mathrm{Det}(g) \\
&= \mathrm{sgn}(g)\mathrm{sgn}'(g) \prod_{i=1}^{m} \left\{ \left(\overline{\overline{gt'_i}}\right)^{-1} g t'_i \right\} K.
\end{aligned}$$

Therefore, we have $\mathrm{sgn}(g)\mathrm{sgn}'(g) = 1$ and

$$\prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} g t_i \right\} K = \prod_{i=1}^{m} \left\{ \left(\overline{\overline{gt'_i}}\right)^{-1} g t'_i \right\} K.$$

Hence, the left transfer is an invariant under a change of coset representatives.

Now let us prove property (3) that $V_{G \to H/K} = \widetilde{V}_{G \to H/K}$ from property $(3')$ that any left regular representation is equivalent to any right regular representation.

Let $G = Hu_1 \cup Hu_2 \cup \cdots \cup Hu_m$. That is, we take a complete set $U = \{u_1, u_2, \ldots, u_m\}$ of right coset representatives of H in G.

Definition 4.4.1. *For all $\alpha \in RG$, there exists $R_U(\alpha) \in \mathrm{Mat}(m, RH)$ such that*

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \alpha = R_U(\alpha) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

We call the map $R_U : RG \ni \alpha \mapsto R_U(\alpha) \in \mathrm{Mat}(m, RH)$ the right regular representation.

The same as the left transfer, we can show that the following lemma.

Lemma 4.4.2. *Let $\alpha = \sum_{g \in G} x_g g$. Then, we have*

$$R_U(\alpha)_{ij} = \sum_{g \in G} \dot{\chi}(u_i g u_j^{-1}) x_g u_i g u_j^{-1}.$$

Therefore, there exists $\widetilde{\mathrm{sgn}}(g) \in \{-1, 1\}$ such that

$$(\det \circ \psi \circ R_U)(g) = \widetilde{\mathrm{sgn}}(g) \widetilde{V}_{G \to H/K}(g)$$

and $\widetilde{V}_{G \to H/K}$ is an invariant under a change of coset representatives of H in G. We have properties (1) and (2).

Since T is a complete set of left coset representatives of H in G, we can take a complete set of $T^{-1} = \{t_1^{-1}, t_2^{-1}, \ldots, t_m^{-1}\}$ of right coset representatives of H in G. Therefore,

$$\begin{aligned} R_{T^{-1}}(\alpha)_{ij} &= \sum_{g \in G} \dot{\chi}\left(t_i^{-1} g (t_j^{-1})^{-1}\right) x_g t_i^{-1} g (t_j^{-1})^{-1} \\ &= L_T(\alpha)_{ij}. \end{aligned}$$

We obtain property $(3')$. As a result,

$$(\det \circ \psi \circ R_U)(g) = (\det \circ \psi \circ L_T)(g).$$

Therefore, we have

$$\begin{aligned} \widetilde{\mathrm{sgn}}(g) &= \mathrm{sgn}(g), \\ \widetilde{V}_{G \to H/K} &= V_{G \to H/K}. \end{aligned}$$

We obtain property (3).

Chapter 5

Capelli elements of the group algebra

5.1 Introduction

The Capelli identity is analogous to the product formula for the determinant in the Weyl algebra. The identity leads to the Capelli element. It is known that the Capelli element is a central element in the universal enveloping algebra of $\mathfrak{gl}_n$.

In recent years, An Huang gave Capelli-type identities associated with the quaternions and the octonions [7]. Inspired by his results, Tôru Umeda gave Capelli identities for group determinants [17]. There are Capelli identities for irreducible representations in the background of the Capelli identities for group determinants.

In this paper, we give a basis of the center of the group algebra of any finite group by using Capelli identities for irreducible representations. These identities lead to Capelli elements of the group algebra. These elements construct a basis.

First, we explain our motivation.

5.1.1 Motivation

Let G be a finite group, $\widehat{G}$ a complete set of irreducible representations of G over $\mathbb{C}$, $\mathbb{C}G = \left\{\sum_{g \in G} x_g g \mid x_g \in \mathbb{C}\right\}$ the group algebra, and $Z(\mathbb{C}G)$ the center of $\mathbb{C}G$. The following theorem is easily proved from Schur's orthogonal relations.

Theorem 5.1.1. *Let χ_φ be the character of $\varphi \in \widehat{G}$. The set*

$$\left\{\mathrm{Tr}\left(\sum_{g \in G} \varphi(g) g\right) \mid \varphi \in \widehat{G}\right\} = \left\{\sum_{g \in G} \chi_\varphi(g) g \mid \varphi \in \widehat{G}\right\}$$

is a basis of $Z(\mathbb{C}G)$ where we omit the numbering of the element of the basis.

At this point, we have a simple question. Is the set $\left\{\det\left(\sum_{g \in G} \varphi(g) g\right) \mid \varphi \in \widehat{G}\right\}$ a basis of $Z(\mathbb{C}G)$? Our main result gives an answer.

5.1.2 Main result

Let z be a complex variable, $|G|$ the order of G, $\varphi \in \widehat{G}$, $m = \deg \varphi$, $\alpha = \frac{|G|}{m}$, $u_i(z) = \alpha(m-i) - z$, $u^{(i)}(z) = u_m(z)u_{m-1}(z) \cdots u_{m-i+1}(z)$, det the column determinant, and the Capelli element for φ of the group algebra

$$\overline{C}^{\varphi}(z) = \det \left(\sum_{g \in G} \varphi(g) g + \alpha \left(\begin{bmatrix} m-1 & & & \\ & m-2 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \right) - zI_m \right) \in \mathbb{C}[z] \otimes \mathbb{C}G.$$

Then we can prove the following relation.

Theorem 5.1.2. *We have*

$$\overline{C}^{\varphi}(z) = u^{(m)}(z) + \mathrm{Tr} \left(\sum_{g \in G} \varphi(g) g \right) u^{(m-1)}(z).$$

The above relation leads to the following corollary.

Corollary 5.1.3. *Suppose $k_\varphi \in \mathbb{C}$ such that $u^{(m-1)}(k_\varphi) \neq 0$. Then,*

$$\left\{ \overline{C}^{\varphi}(k_\varphi) \mid \varphi \in \widehat{G} \right\}$$

is a basis of $Z(\mathbb{C}G)$.

This is our answer. We provide some sections for the details.

5.2 Capelli identity and Capelli element

Here, we review the Capelli identity and the Capelli element.

5.2.1 Column determinant

First, we explain the column determinant. Let R be an associative algebra.

Definition 5.2.1 (Column determinant)**.** *Let $A = (a_{ij})_{1 \le i,j \le m} \in \mathrm{Mat}(m, R)$. We define the column determinant of A by*

$$\det A = \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(m)m}.$$

Hence, we have $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$.

5.2.2 Weyl algebra

The Capelli identity is analogous to the product formula for the determinant in the Weyl algebra. Next, we explain the Weyl algebra $\mathbb{C}[x_{ij}, \partial_{kl} \mid 1 \le i,j,k,l \le m]$.

Let x_{ij} $(1 \le i,j \le m)$ be variables, $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ $(1 \le i,j \le m)$ partial differential operators, and $\alpha \in \mathbb{C}$. We assume that these variables and operators are related as follows.

For all $1 \le i,j,k,l \le m$, we have

$$[x_{ij}, x_{kl}] = 0, \quad [\partial_{ij}, \partial_{kl}] = 0, \quad [\partial_{ij}, x_{kl}] = \alpha\delta_{ik}\delta_{jl}$$

where δ is the Kronecker delta. Usually, we take $\alpha = 1$. Here, we will not assume that $\alpha = 1$. The Weyl algebra is generated by these variables and operators.

5.2.3 Capelli identity

Next, we explain the Capelli identity. Let

$$X = (x_{ij})_{1\le i\le m, 1\le j\le m}, \qquad \partial = (\partial_{ij})_{1\le i\le m, 1\le j\le m},$$
$$\Pi = {}^t X\partial, \qquad \natural_m = \mathrm{diag}(m-1, m-2, \ldots, 0).$$

The Capelli identity is as follows.

Theorem 5.2.2 (Capelli identity)**.** *We have*

$$\det(\Pi + \alpha\natural_m) = \det X \det\partial.$$

Example 5.2.3. *Let $m = 2$ and $\alpha = 1$. We have*

$$\det\begin{bmatrix} x_{11}\partial_{11} + x_{21}\partial_{21} + 1 & x_{11}\partial_{12} + x_{21}\partial_{22} \\ x_{12}\partial_{11} + x_{22}\partial_{21} & x_{21}\partial_{12} + x_{22}\partial_{12} \end{bmatrix} = \det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \det\begin{bmatrix} \partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22} \end{bmatrix}.$$

5.2.4 Capelli element

The Capelli element is a characteristic polynomial of Π. Let z be a variable.

Definition 5.2.4 (Capelli element)**.** *We define the Capelli element $C(z)$ by*

$$C(z) = \det(\Pi + \alpha\natural_m - zI_m).$$

The Capelli identity is conjugation invariant.

Theorem 5.2.5. *For all $P \in GL(m, \mathbb{C})$, we have*

$$\det\left(P\Pi P^{-1} + \alpha\natural_m - zI_m\right) = C(z).$$

The following theorem plays an important role in what follows.

Theorem 5.2.6. *For all $1 \le i,j \le m$, we have*

$$[\Pi_{ij}, C(z)] = 0.$$

Theorem 5.2.5 and 5.2.6 are obtained only by the following relations.

For all $1 \le i,j,k,l \le m$, $[\Pi_{ij}, \Pi_{kl}] = \alpha(\delta_{jk}\Pi_{il} - \delta_{il}\Pi_{kj})$.

5.3 Capelli identity for irreducible representations

Here, we explain the Capelli identities for irreducible representations.

Let G be a finite group, x_g ($g \in G$) variable and $\partial_g = \frac{\partial}{\partial x_g}$ ($g \in G$) partial differential operator. We assume that the following relations hold.

For all $g, h \in G$,

$$[x_g, x_h] = 0, \quad [\partial_g, \partial_h] = 0, \quad [\partial_g, x_h] = \delta_{gh}.$$

Then, we have the Weyl algebra $\mathbb{C}[x_g, \partial_h]$. Next, we construct Weyl subalgebras of the Weyl algebra by using irreducible unitary representations of G.

Let $|G|$ be the cardinality of the set G (that is, $|G|$ is the order of the group G), φ a unitary matrix form of an irreducible representation of G,

$$\alpha_m = \frac{|G|}{m}, \quad X^\varphi = \sum_{g \in G} \overline{\varphi(g)} x_g, \quad \partial^\varphi = \sum_{g \in G} \varphi(g) \partial_g, \quad \Pi^\varphi = {}^t X^\varphi \partial^\varphi$$

where $\overline{\varphi(g)}$ is the complex conjugate matrix of $\varphi(g)$. Then, we have the following relations.

For all $1 \leq i, j, k, l \leq m$,

$$[X^\varphi_{ij}, X^\varphi_{kl}] = 0, \quad [\partial^\varphi_{ij}, \partial^\varphi_{kl}] = 0, \quad [\partial^\varphi_{ij}, X^\varphi_{kl}] = \alpha_m \delta_{ik} \delta_{jl}.$$

This leads us to the following identity.

Theorem 5.3.1 (Capelli identity for irreducible representations)**.** *We have*

$$\det(\Pi^\varphi + \alpha_m \natural_m) = \det X^\varphi \det \partial^\varphi.$$

Let $C^\varphi(z) = \det(\Pi^\varphi + \alpha_m \natural_m - z I_m)$ be the Capelli element. From Theorem 5.2.5, the Capelli element is invariant under a change of a matrix form of the irreducible representation. This enables us to redefine the Capelli element as follows.

Definition 5.3.2 (Capelli element for irreducible representations)**.** *Let $\varphi \in \widehat{G}$ and $m = \deg \varphi$. We define $C^\varphi(z)$ by*

$$C^\varphi(z) = \det(\Pi^\varphi + \alpha_m \natural_m - z I_m).$$

We call $C^\varphi(z)$ the Capelli element for φ.

5.4 Capelli element of the group algebra

Let $\mathbb{C}G = \left\{ \sum_{g \in G} x_g g \mid x_g \in \mathbb{C} \right\}$ the group algebra of G, $\widetilde{G}$ a complete set of irreducible unitary matrix representations of G, $\varphi \in \widetilde{G}$, and

$$E^\varphi = \sum_{g \in G} \varphi(g) g \in \mathrm{Mat}(\deg \varphi, \mathbb{C}G).$$

From Schur's orthogonal relations, we have the following lemmas.

Lemma 5.4.1. *$\{E^{\varphi}_{ij} \mid 1 \le i, j \le \deg\varphi, \varphi \in \widetilde{G}\}$ is a basis of $\mathbb{C}G$.*

Lemma 5.4.2. *Let $\varphi, \psi \in \widetilde{G}$, where φ is not equivalent to ψ. For all $1 \le i, j \le \deg\varphi$ and $1 \le s, t \le \deg\psi$, we have*

$$E^{\varphi}_{ij}E^{\varphi}_{kl} = \alpha_{\deg\varphi}\delta_{jk}E^{\varphi}_{il}, \quad E^{\varphi}_{ij}E^{\psi}_{st} = 0.$$

In particular, we have

$$[E^{\varphi}_{ij}, E^{\varphi}_{kl}] = \alpha_{\deg\varphi}(\delta^{\varphi}_{jk}E_{il} - \delta_{il}E^{\varphi}_{kj}), \tag{5.1}$$

$$[E^{\varphi}_{ij}, E^{\psi}_{kl}] = 0. \tag{5.2}$$

Let

$$\overline{C}^{\varphi}(z) = \det(E^{\varphi} + \alpha_m\natural_m - zI_m) \in \mathbb{C}[z] \otimes \mathbb{C}G.$$

Recall that Theorems 5.2.5 and 5.2.6 are obtained only by the relations $[\mathit{\Pi}_{ij}, \mathit{\Pi}_{kl}] = \alpha(\delta_{jk}\mathit{\Pi}_{il} - \delta_{il}\mathit{\Pi}_{kj})$. Hence, $\overline{C}^{\varphi}(z)$ is conjugation invariant from the relations (5.1), and we have

$$[E^{\varphi}_{ij}, \overline{C}^{\varphi}(z)] = 0$$

for any $1 \le i, j \le \deg\varphi$.

Using the above conjugation invariance, we redefine $\overline{C}^{\varphi}(z)$.

Definition 5.4.3 (Capelli element of the group algebra)**.** *Let $\varphi \in \widehat{G}$. We define the Capelli element for φ of the group algebra by*

$$\overline{C}^{\varphi}(z) = \det(E^{\varphi} + \alpha_m\natural_m - zI_m).$$

From Lemma 5.4.1, conjugation invariance of $\overline{C}^{\varphi(z)}$, and relations (5.1) and (5.2), we can prove the following Lemma.

Lemma 5.4.4. *For all $\varphi \in \widehat{G}$, $\overline{C}^{\varphi}(z) \in Z(\mathbb{C}G[z])$. That is, $\overline{C}^{\varphi}(z)$ is a central element of the group algebra.*

Let $u_i(z) = \alpha_m(m-i) - z$, $u^{(i)}(z) = u_m(z)u_{m-1}(z)\cdots u_{m-i+1}(z)$, $E^{\varphi}_{ij}(u_j(z)) = E^{\varphi}_{ij} + \delta_{ij}u_j(z)$, and $[m] = \{1, 2, \ldots, m\}$. The following is the main theorem.

Theorem 5.4.5 (Chapter 5, Theorem 5.1.2)**.** *We have*

$$\overline{C}^{\varphi}(z) = u^{(m)}(z) + \mathrm{Tr}(E^{\varphi})u^{(m-1)}(z).$$

Proof. From the definition of $\overline{C}^{\varphi}(z)$, we have

$$\begin{aligned}
\overline{C}^{\varphi}(z) &= \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma)E^{\varphi}_{\sigma(1)1}(u_1(z))E^{\varphi}_{\sigma(2)2}(u_2(z))\cdots E^{\varphi}_{\sigma(m)m}(u_m(z)) \\
&= \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma)(E^{\varphi}_{\sigma(1)1} + \delta_{\sigma(1)1}u_1(z))\cdots(E^{\varphi}_{\sigma(m)m} + \delta_{\sigma(m)m}u_m(z)) \\
&= \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \sum_{\emptyset \ne T \subset [m]} \prod_{t \in T} E^{\varphi}_{\sigma(t)t} \prod_{s \in [m]\setminus T} \delta_{\sigma(s)s}u_s(z) + u^{(m)}(z)
\end{aligned}$$

where $\prod_{t\in T} E^{\varphi}_{\sigma(t)t} = E^{\varphi}_{\sigma(t_1)t_1} E^{\varphi}_{\sigma(t_2)t_2} \cdots E^{\varphi}_{\sigma(t_{|T|})t_{|T|}}$ and $T = \{t_1 < t_2 < \cdots < t_{|T|}\}$. We fix $T = \{t_1 < t_2 < \cdots < t_{|T|}\}(\neq \emptyset) \subset [m]$. From Lemma 5.4.2, we have

$$
\begin{aligned}
&\sum_{\sigma\in S_m} \operatorname{sgn}(\sigma) \sum_{T\subset[m]} \prod_{t\in T} E^{\varphi}_{\sigma(t)t} \prod_{s\in[m]\setminus T} \delta_{\sigma(s)s} u_s(z) \\
&\quad = \alpha_m^{|T|-1} \sum_{\sigma\in S_m} \operatorname{sgn}(\sigma) \delta_{\sigma(t_2)t_1} \delta_{\sigma(t_3)t_2} \cdots \delta_{\sigma(t_{|T|})t_{|T|-1}} E^{\varphi}_{\sigma(t_1)t_{|T|}} \prod_{s\in[m]\setminus T} \delta_{\sigma(s)s} u_s(z) \\
&\quad = (-\alpha_m)^{|T|-1} E^{\varphi}_{t_{|T|}t_{|T|}} \prod_{s\in[m]\setminus T} u_s(z).
\end{aligned}
$$

Therefore, there exists $a_i \in \mathbb{C}[z]$ $(1 \leq i \leq m)$ such that

$$
\overline{C}^{\varphi}(z) = \sum_{i=1}^{m} a_i E^{\varphi}_{ii} + u^{(m)}(z).
$$

We show that $a_p = a_q$ for all $p, q \in [m]$. From Lemma 5.4.2, we have

$$
\begin{aligned}
E^{\varphi}_{pq} \overline{C}^{\varphi}(z) &= E^{\varphi}_{pq} \left(\sum_{i=1}^{m} a_i E^{\varphi}_{ii} + u^{(m)}(z) \right) \\
&= \sum_{i=1}^{m} a_i \delta_{iq} \alpha_m E^{\varphi}_{pi} + u^{(m)}(z) E^{\varphi}_{pq} \\
&= a_q \alpha_m E^{\varphi}_{pq} + u^{(m)}(z) E^{\varphi}_{pq}, \\
\overline{C}^{\varphi}(z) E^{\varphi}_{pq} &= \left(\sum_{i=1}^{m} a_i E^{\varphi}_{ii} + u^{(m)}(z) \right) E^{\varphi}_{pq} \\
&= \sum_{i=1}^{m} a_i \delta_{ip} \alpha_m E^{\varphi}_{iq} + u^{(m)}(z) E^{\varphi}_{pq} \\
&= a_p \alpha_m E^{\varphi}_{pq} + u^{(m)}(z) E^{\varphi}_{pq}.
\end{aligned}
$$

From Lemma 5.4.4, we have $a_p = a_q$ for all $p, q \in [m]$. We calculate a_1. From $\overline{C}^{\varphi}(z) = \sum_{T\subset[m]} (-\alpha_m)^{|T|-1} E^{\varphi}_{t_{|T|}t_{|T|}} \prod_{s\in[m]\setminus T} u_s(z)$, we have

$$
\begin{aligned}
a_1 E^{\varphi}_{11} &= (-\alpha_m)^{\{1\}-1} E^{\varphi}_{11} \prod_{s\in[m]\setminus\{1\}} u_s(z) \\
&= u^{(m-1)}(z) E^{\varphi}_{11}.
\end{aligned}
$$

This completes the proof. □

In addition, we have the following corollary.

Corollary 5.4.6 (Chapter 5, Corollary 5.1.3)**.** *Suppose $k_\varphi \in \mathbb{C}$ such that $u^{(m-1)}(k_\varphi) \neq 0$. Then,*

$$
\left\{ \overline{C}^{\varphi}(k_\varphi) \mid \varphi \in \widehat{G} \right\}
$$

is a basis of $Z(\mathbb{C}G)$.

5.5 Relationship between column, row and double determinant

In this last section, we explain the relationship between column, row, and double determinants. The row and double determinants are as follows.

Definition 5.5.1 (Row determinant). *Let $A = (a_{ij})_{1\le i,j\le m} \in \mathrm{Mat}(m,R)$. We define the row determinant of A is defined as*

$$\mathrm{rdet}A = \sum_{\sigma\in S_m} \mathrm{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)}.$$

Definition 5.5.2 (Double determinant). *Let $A = (a_{ij})_{1\le i,j\le m} \in \mathrm{Mat}(m,R)$. The double determinant of A is defined as*

$$\mathrm{Det}A = \frac{1}{m!} \sum_{\sigma,\tau\in S_m} \mathrm{sgn}(\sigma\tau) a_{\sigma(1)\tau(1)} a_{\sigma(2)\tau(2)} \cdots a_{\sigma(m)\tau(m)}.$$

Reference [10] describes that the relationship between column, row, and double determinants. Let

$$\natural^* = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & m-1 \end{bmatrix}, \quad \natural_\sigma = \begin{bmatrix} \sigma(m) & & & \\ & \sigma(m-1) & & \\ & & \ddots & \\ & & & \sigma(1) \end{bmatrix} \quad (\sigma \in S_m),$$

and $E \in \mathrm{Mat}(m,R)$, where we assume that $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$ for all $1 \le i,j,k,l \le m$. We can prove the following theorem.

Theorem 5.5.3 ([10]). *For all $\sigma \in S_m$, we have*

$$\begin{aligned} \det(E + \natural_m - zI_m) &= \mathrm{rdet}(E + \natural^* - zI_m) \\ &= \mathrm{Det}(E + \natural_\sigma - (z+1)I_m). \end{aligned}$$

The above has the following implication.

Corollary 5.5.4. *Let $\overline{C}^\varphi(z)$ be the Capelli element for φ of the group algebra. For all $\sigma \in S_m$, we have*

$$\begin{aligned} \overline{C}^\varphi(z) &= \mathrm{rdet}(E^\varphi + \alpha_m \natural^* - zI_m) \\ &= \mathrm{Det}(E^\varphi + \alpha_m \natural_\sigma - (z+1)I_m). \end{aligned}$$

Bibliography

[1] ALPERIN, Jonathan L. Sylow intersections and fusion. *Journal of Algebra*, 1967, 6.2: 222–241.

[2] ARTIN, Emil. *Geometric Algebra*. Interscience Publishers, Inc., 1957.

[3] ASLAKSEN, Helmer. Quaternionic determinants. *The Mathematical Intelligencer*, 1996, 18.3: 57–65.

[4] CONRAD, Keith. On the origin of representation theory. *Enseignement Mathematique*, 1998, 44: 361–392.

[5] FORMANEK, Edward; SIBLEY, David. The group determinant determines the group. *Proceedings of the American Mathematical Society*, 1991, 112.3: 649–656.

[6] HALL, Marshall. *The Theory of Groups*. American Mathematical Society, 1976.

[7] HUANG, An. Noncommutative multiplicative norm identities for the quaternions and the octonions. *arXiv preprint arXiv:1102.2657*, 2011.

[8] HUPPERT, Bertram. *Endliche Gruppen I*. Springer-Verlag, 2013.

[9] ISAACS, I. Martin. *Character Theory of Finite Groups*. Courier Corporation, 2013.

[10] ITOH, Minoru; UMEDA, Tôru. On central elements in the universal enveloping algebras of the orthogonal Lie algebras. Compositio Mathematica, 2001, 127.03: 333–359.

[11] JOHNSON, Kenneth W. On the group determinant. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Cambridge University Press, 1991, 109.02: 299–311.

[12] KONDO, Takeshi. *Group Theory*, Iwanami Shoten, 2002 (in Japanese).

[13] LAM, T. Y. Representations of finite groups: A hundred years, part II. *Notices of the AMS*, 1998, 45.4: 465–474.

[14] SCHUR, I. Neuer Beweis eines Satzes über endliche Gruppen, *Sitzungsberichteder Königlich Preussischen Akademie der Wissenschaften zu Berlin*, 1902, 1013–1019.

[15] STEINBERG, Benjamin. *Representation Theory of Finite Groups: An Introductory Approach*. Springer Science & Business Media, 2011.

[16] UMEDA, Tôru, On some variants of induced representations, Symposium on Representation Theory 2012, 2012, 7–17 (in Japanese).

[17] UMEDA, Tôru. Remarks on the Capelli identities for reducible modules, preprint (2016).

[18] VAN DER WAERDEN, Bartel L. *A History of Algebra: From al-Khwārizmī to Emmy Noether*. Springer-Verlag, 1985.

[19] YAMAGUCHI, Naoya. An extension and a generalization of Dedekind's theorem. *arXiv preprint arXiv:1601.03170*, 2016.

[20] YAMAGUCHI, Naoya. Capelli elements of the group algebra. *arXiv preprint arXiv:1611.00662*, 2016.

[21] YAMAGUCHI, Naoya. Factorizations of group determinant in group algebra for any abelian subgroup. *arXiv preprint arXiv:1610.06047*, 2016.

[22] YAMAGUCHI, Naoya. Factorization of group determinant in some group algebras. *arXiv preprint arXiv:1405.1900*, 2014.

[23] YAMAGUCHI, Naoya. Generalization of Frobenius' theorem for group determinants. *arXiv preprint arXiv:1610.06489*, 2016.

[24] YAMAGUCHI, Naoya. Proof of some properties of transfer using noncommutative determinants. *arXiv preprint arXiv:1602.08667*, 2016.

[25] YOSHIDA, Tomoyuki. Character-theoretic transfer. *Journal of Algebra*, 1978, 52.1: 1–38.

Acknowledgments

I am deeply grateful to Professor Hiroyuki Ochiai. He provided me with helpful comments and suggestions at Kyushu University for five years, and he also helped me with travel expenses for my talks. Thanks to him, I became a mathematician.

I would also like to express my gratitude to Professor Minoru Itoh. He provided me with helpful comments and suggestions at Kagoshima University for three years. He gave me the opportunity to become a mathematician.

In addition, I would like to thank Professor Tôru Umeda who motivated me for my research. He also gave me the chance to present my talk. By this chance, I got to know Professor Hideaki Morita, who gave me courage and helped me with travel expenses for my talks. I am also grateful to Professor Yoshimichi Ueda, who gave me the opportunity to study the representation theory of finite groups. I also wish to thank Professors Hitoshi Furusawa, Takuya Konno, Naoyuki Kamiyama, Shinpei Fujii and Yoshihiro Mizoguchi.

Further, I want to thank friends at Kyushu University and Kagoshima University, especially Cid Reyes Bustos, Daisuke Shimada, Hiroki Orihashi, Homare Tadano, Katsuya Hashizume, Sukuse Abe, Takehiro Higa, Tomoyuki Tamura, Yoshihide Kubo, and Yuka Suzuki, who studied together with me.

Special thanks also to Takuya Nakayama who gave me a piece of friendly advice.

My academic career would have been impossible without generous funding by a grant from the Japan Society for the Promotion of Science (JSPS KAKENHI Grant Number 15J06842).

Finally, I would like to give my heartfelt thanks to my father Hisamitsu Yamaguchi and to Haruko Yamaguchi and Atushi Yamaguchi for their longtime support. I would also like to posthumously thank my grandparents.

Naoya Yamaguchi
Graduate School of Mathematics
Kyushu University
Nishi-ku, Fukuoka 819-0395
Japan
n-yamaguchi@math.kyushu-u.ac.jp