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Idèlic class field theory for 3-manifolds

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Abstract. We study a topological analogue of idèlic class field theory for 3-manifolds, in the spirit of arithmetic topology. We firstly introduce the notion of a very admissible link $\mathcal{K}$ in a 3-manifold M, which plays a role analogous to the set of primes of a number field. For such a pair $(M, \mathcal{K})$, we introduce the notion of idèles and define the idèle class group. Then, getting the local class field theory for each knot in $\mathcal{K}$ together, we establish analogues of the global reciprocity law and the existence theorem of idèlic class field theory.

Contents

Notation and convention 4

Chapter 1. Introduction 5

Chapter 2. Basic analogies 7
- 2.1. M^2KR dictionary 7
- 2.2. Expanded dictionary (a preview) 8
- 2.3. Review of Hilbert theory for number fields 9
- 2.4. Hilbert theory for 3-manifolds 10

Chapter 3. Local class field theory for tori 12
- 3.1. Review of local class field theory for local fields 12
- 3.2. Local class field theory for tori 13

Chapter 4. Idèlic class field theory for 3-manifolds 15
- 4.1. Review of global class field theory for number fields 15
- 4.2. Very admissible links 17
- 4.3. The universal $\mathcal{K}$-branched cover 21
- 4.4. Idèle class group for 3-manifolds 23
- 4.5. The topologies on idèle class group and the existence theorems 27
- 4.6. The norm residue symbols 30

Acknowledgments 32

Bibliography 33

Notation and convention

We denote the empty set by $\emptyset$.

We denote the ring of integers by $\mathbb{Z}$, the rational number field by $\mathbb{Q}$, the real number field by $\mathbb{R}$, and the complex field by $\mathbb{C}$.

The symbol $\mathbb{F}_q$ denotes the field with q-elements.

For a connected topological space X (respectively a field k), we denote the maximal abelian covering of X (respectively the maximal abelian extension of k) by X^{ab} (respectively k^{ab}).

We write $\pi_1(X)$ for the fundamental group of X omitting a base point and $H_n(X)$ simply for the n-th homology group with coefficients in $\mathbb{Z}$.

For a Galois covering $h : Y \to X$ (respectively a Galois extension F/k), we denote the Galois group by $\mathrm{Gal}(Y/X)$ (respectively $\mathrm{Gal}(F/k)$).

A branched cover of a 3-manifold means one branched over a link.

CHAPTER 1

Introduction

In the middle of 20th century, J. Tate, M. Artin, and J. L. Verdier interpreted class field theory for number fields as an analogue of 3-dimensional Poincare duality in Galois/étale cohomology ([**Tat63**], [**AV64**]). The analogies between knots and primes were initially pointed out by B. Mazur ([**Maz64**]). After a long silence, M. Kapranov and A. Reznikov took up the analogies between 3-manifolds and number rings again ([**Kap95**], [**Rez97**], [**Rez00**]), and M. Morishita investigated the foundational analogies systematically ([**Mor02**], [**Mor10**], [**Mor12**]). This area of mathematics is now called *arithmetic topology*.

It is known that there is an analogy between the Hurewicz isomorphism and unramified class field theory, where the 1st homology group corresponds to the ideal class group. In number theory, the Takagi-Artin class field theory describes abelian ramified extensions of number fields by generalized ideal class groups, and Chevalley introduced the notion of idèles by which global class field theory is obtained by getting all local theories together. One of the most important open problems in arithmetic topology is to study a topological analogue of idèlic class field theory. This thesis addresses this problem and presents our attempt to construct idèlic class field theory for 3-manifolds. This thesis is based on the papers [**Nii14**] and [**NU**].

Now, we describe our main results in this thesis. Based on the local analogies in the dictionary in §2.1, we first develop a local theory for each knot in a 3-manifold, which is an analogue of local class field theory. Let K be a knot in a solid torus V_K. A topological analogue of the local reciprocity homomorphism is simply given by the Hurewicz homomorphism

$$\rho_K : H_1(\partial V_K) \longrightarrow \mathrm{Gal}(\partial V_K^{\mathrm{ab}}/\partial V_K).$$

Let M be an oriented, connected, closed 3-manifold. We introduce a certain infinite components link $\mathcal{K}$ called *a very admissible link* of M, which may be regarded as an analogue of the set of primes in a number ring. We prove its existence (Theorem 4.2.8). For such a pair $(M, \mathcal{K})$, we introduce *the idèle group* $I_{M,\mathcal{K}}$ as a restricted product of $H_1(\partial V_K)$ over all the knots K in $\mathcal{K}$. In addition, we introduce *the principal idèle group* $P_{M,\mathcal{K}}$ as the image of a natural homomorphism

$$\Delta : H_2(M, \mathcal{K}) \longrightarrow I_{M,\mathcal{K}}.$$

We put $\mathrm{Gal}(M, \mathcal{K})^{\mathrm{ab}} = \varprojlim_L \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$, where L runs over all the finite sublinks of $\mathcal{K}$, X_L denotes $M - L$, and X_L^{ab} denotes the maximal abelian covering of X_L. We regard it as an analogue of the Galois group of the maximal abelian extension of a number field. Getting ρ_K together over all K in $\mathcal{K}$, we define a natural

homomorphism

$$\widetilde{\rho}_{M,\mathcal{K}} : I_{M,\mathcal{K}} \longrightarrow \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}.$$

We prove $\mathrm{Ker}(\widetilde{\rho}_{M,\mathcal{K}}) = \mathrm{Im}(\Delta)$ (Theorem 4.4.3), which yields the global reciprocity homomorphism $\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \to \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$. The first part of our main results is stated as follows.

Theorem A (The global reciprocity law for 3-manifolds. Cf. Theorem 4.4.6)**.** *There is a canonical isomorphism*

$$\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \xrightarrow{\cong} \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$$

called the global reciprocity map which satisfies the following properties:
(i) *For any finite abelian cover $h : N \to M$ branched over a finite link L in $\mathcal{K}$, ρ_M induces an isomorphism*

$$C_{M,\mathcal{K}}/h_*(C_{N,h^{-1}(\mathcal{K})}) \cong \mathrm{Gal}(N/M).$$

(ii) *For each knot K in $\mathcal{K}$, there is a commutative diagram:*

$$\begin{tikzcd} H_1(\partial V_K) \arrow[r, "\cong", "\mathrm{Hur}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Gal}(\widetilde{\partial V_K}/\partial V_K) \arrow[d] \\ C_{M,\mathcal{K}} \arrow[r, "\rho_{M,\mathcal{K}}"'] & \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}, \end{tikzcd}$$

where the vertical maps are induced by the natural inclusions.

Next, we introduce *the standard topology* and *the norm topology* on the idèle class group. The second part of our main results is stated as follows.

Theorem B (The existence theorem. Cf. Theorem 4.5.7)**.** *The correspondence*

$$(h : N \to M) \mapsto h_*(C_{N,h^{-1}(\mathcal{K})})$$

gives a bijection between the set of (isomorphism classes of) finite abelian covers of M branched over finite links L in $\mathcal{K}$ and the set of open subgroups of finite indices in $C_{M,\mathcal{K}}$ with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of $C_{M,\mathcal{K}}$ with respect to the norm topology.

These theorems above may be regarded as an analogue of the fundamental theorem of global class field theory for number fields.

We note that idèlic class field theory for 3-manifolds was initially studied by A. Sikora ([**Sik03**], [**Sik0s**], [**Sik11**]). Our approach is different from his and elementary.

Here are the contents of this thesis. In Chapter 2, we review the basic analogies in arithmetic topology. We give a description of the Hilbert ramification theory for 3-manifolds. In Chapter 3, we review the local class field theory for local fields, and describe its analogue for 2-dimensional tori. In Chapter 4, we recall the idèlic global class field theory for number fields, and we develop the idèlic class field theory for 3-manifolds.

CHAPTER 2

Basic analogies

In this chapter, we introduce some basic analogies in arithmetic topology. Next, we present a new dictionaries which we develop in the later chapters of this thesis. We review the Hilbert ramification theory in number theory, which discribes a decomposition of a prime in a finite Galois extension of number fields. Based on the analogies, we review an analogue of the Hilbert ramification theory for coverings of 3-manifolds. We consult [**Mor12**] as a basic reference in this chapter.

2.1. M^{2}KR dictionary

In this section, we introduce the analogies between knots and primes, 3-manifolds, and number rings.

There is an analogy between the fundamental group of 1-dimensional sphere S^1 and of a finite field $\mathbb{F}_q$. Let $\overline{\mathbb{F}}_q$ be the separable closure of $\mathbb{F}_q$, $\hat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$.

$\pi_1(S^1) = \mathrm{Gal}(\mathbb{R}/S^1) \cong \mathbb{Z}$	$\pi_1(\mathrm{Spec}(\mathbb{F}_q)) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$

Furthermore, S^1 is the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$ and $\mathrm{Spec}(\mathbb{F}_q)$ is regarded as an étale homotopical analogue $K(\hat{\mathbb{Z}}, 1)$.

$S^1 = K(\mathbb{Z}, 1)$	$\mathrm{Spec}(\mathbb{F}_q) = K(\hat{\mathbb{Z}}, 1)$

Secondly, we introduce some analogies between tori and local fields (see §3.2).

tubular neighborhood V_K of knot K	$\mathfrak{p}$-adic integers $\mathrm{Spec}(\mathcal{O}_\mathfrak{p})$
boundary of V_K	$\mathfrak{p}$-adic local field $\mathrm{Spec}(k_\mathfrak{p})$

Finally, we introduce some analogies between 3-manifolds and number fields. For a number field k, we denote the ring of integers of k by $\mathcal{O}_k$.

3-manifold M	number ring $\mathrm{Spec}(\mathcal{O}_k)$
knot K in M	prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$
link $L = \{K_1, \ldots, K_r\}$	set of primes $\{\mathfrak{p}_1, \ldots \mathfrak{p}_r\}$
unbranched covering $N \to M$	unramified extension K/k
branched covering $N \to M$	ramified extension K/k
fundamental group $\pi_1(M)$	ètale fundamental group $\pi_1(\mathrm{Spec}(\mathcal{O}_k))$
link group $\pi_1(M - L)$	$\pi_1(\mathrm{Spec}(\mathcal{O}_k) - S)$

1-cycles in M generate the singular cycle group $Z_1(M)$ of 1-cycles of M. The boundaries ∂D of 2-chains $D \in C_2(M)$ generate the subgroup $B_1(M)$ of $Z_1(M)$. The first homology group $H_1(M)$ is defined by the quotient group:

$$H_1(M) = Z_1(M)/B_1(M).$$

2-chains D with $\partial D = 0$ form the 2nd homology group of M.

On the other hand, prime ideals of the integer ring $\mathcal{O}_k$ generate the ideal group $I(k)$. The principal ideals (a) generated by numbers $a \in k^\times$ generate the subgroup $P(k)$ of $I(k)$. The ideal class group is defined by the quotient group:

$$\mathrm{Cl}(k) = I(k)/P(k).$$

Numbers $a \in k^\times$ with $(a) = \mathcal{O}_k$ form the unit group $\mathcal{O}_k^\times$. We have the following dictionary.

1st cycle group $Z_1(M)$	ideal group $I(k)$
$C_2(M) \to Z_1(M)$ $D \mapsto \partial D$	$k^\times \to I(k)$ $a \mapsto (a)$
1st boundary group $B_1(M)$	principal ideal group $P(k)$
1st homology group $H_1(M) = Z_1(M)/B_1(M)$	ideal class group $\mathrm{Cl}(k) = I(k)/P(k)$
2nd homology group $H_2(M)$	unit group $\mathcal{O}_k^\times$

There is also an analogy between Hurewics isomorphism and unramified class field theory.

$H_1(M) \cong \mathrm{Gal}(M^{\mathrm{ab}}/M)$	$\mathrm{Cl}(k) \cong \mathrm{Gal}(k_{\mathrm{ur}}^{\mathrm{ab}}/k)$

Here M^{ab} (respectively $k_{\mathrm{ur}}^{\mathrm{ab}}$) denotes the maximal abelian covering of M (respectively the maximal unramified abelian extension of k).

The purpose of this thesis is to construct an idèlic theoretic form of class field theory for 3-manifold and extended these analogies for branched covering.

2.2. Expanded dictionary (a preview)

In this section, we present a new dictionaries which we develop in the later chapters of this thesis. Let M be a connnected oriented colosed 3-manifold. Let k be a number field, $\mathfrak{p}$ be a prime of k.

First, for a knot K in M, let V_K be a tubular neighborhood of K. Then, the natural inclusion $\partial V_K \to V_K$ induces the homomorphism $v_K : H_1(\partial V_K) \to H_1(V_K)$, which is an analogue of $\mathfrak{p}$-adic valuations (§3.2).

K-adic valuation $v_K : H_1(\partial V_K) \to \mathbb{Z}$	$\mathfrak{p}$-adic valuation $v_{\mathfrak{p}} : k_{\mathfrak{p}}^\times \to \mathbb{Z}$

Secondly, in §4.2.3, we introduce the notion of a very admissible link $\mathcal{K}$ in M, which may be regarded as the set of all primes of k. For an M equipped with a very admissible link $\mathcal{K}$, we present the notion of *a universal $\mathcal{K}$-branched cover* in §4.3, which is an analogue of an algebraic closure of a number field. We also present an analogy between a base point of M and a geometric point of a number field in §4.3.

Thirdly, in §4.4, we introduce analogues of the idèle group, the principal idèle group, the idèle class group. We present an analogue of the Galois group of the maximal abelian extension $\mathrm{Gal}(k^{\mathrm{ab}}/k)$, which is defined by $\varprojlim_L \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ where L

3-manifold with very admissible link $(M, \mathcal{K})$	number ring $\operatorname{Spec} \mathcal{O}_k$
universal $\mathcal{K}$-branched cover $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$	algebraic closure $\overline{k}/k$
base point $b_M : \{\mathrm{pt}\} \hookrightarrow M$	geometric point $x : \operatorname{Spec} \Omega \to \operatorname{Spec} \mathcal{O}_k$

runs through all the finite links of $\mathcal{K}$ and X_L denotes $M - L$. We give an analogue of the global reciprocity map.

idèle group $I_{M,\mathcal{K}}$	idèle group I_k
$\Delta : H_2(M, \mathcal{K}) \to I_{M,\mathcal{K}}$	$\Delta : k^\times \to I_k$
principal idèle group $P_{M,\mathcal{K}} := \operatorname{Im} \Delta$	principal idèle group $P_k := \operatorname{Im} \Delta$
idèle class group $C_{M,\mathcal{K}} := I_{M,\mathcal{K}}/P_{M,\mathcal{K}}$	idèle class group $C_k := I_k/P_k$
$\operatorname{Gal}(M, \mathcal{K})^{\mathrm{ab}} := \varprojlim_L \operatorname{Gal}(X_L^{\mathrm{ab}}/X_L)$	$\operatorname{Gal}(k^{\mathrm{ab}}/k) = \varprojlim_F \operatorname{Gal}(F/k)$
global reciprocity map $\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \to \operatorname{Gal}(M, \mathcal{K})^{\mathrm{ab}}$	global reciprocity map $\rho_k : C_k \to \operatorname{Gal}(k^{\mathrm{ab}}/k)$

Finally, in §4.6, we present an analogy between the linking number and the Legendre symbol. We introduce an analogue of the norm residue symbol.

linking number $\operatorname{lk}(K_1, K_2) \pmod 2$	Legendre symbol $\left(\dfrac{p}{q}\right)$
norm residue symbol $(\ , h)$	norm residue symbol $(\ , F/k)$

2.3. Review of Hilbert theory for number fields

Let $k/\mathbb{Q}$ be a finite Galois extension with degree n. Let $S_p = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be a set of prime ideals in $\mathcal{O}_k$ over p. Then, the Galois group $\operatorname{Gal}(k/\mathbb{Q})$ acts on S_p transitively. We call the stabilizer $D_{\mathfrak{p}_i}$ of $\mathfrak{p}_i$ *the decomposition group* of $\mathfrak{p}_i$:

$$D_{\mathfrak{p}_i} := \{g \in \operatorname{Gal}(k/\mathbb{Q}) \mid g(\mathfrak{p}_i) = \mathfrak{p}_i\}.$$

Since we have the bijection $\operatorname{Gal}(k/\mathbb{Q})/D_{\mathfrak{p}_i} \cong S_p$, $\#D_{\mathfrak{p}_i} = n/r$ is independent of $\mathfrak{p}_i$. Indeed, if $\mathfrak{p}_j = g(\mathfrak{p}_i)$ $(g \in \operatorname{Gal}(k/\mathbb{Q}))$, we have $D_{\mathfrak{p}_j} = gD_{\mathfrak{p}_i}g^{-1}$. Since $g \in \operatorname{Gal}(k/\mathbb{Q})$ induces an isomorphism $\hat{g} : k_{\mathfrak{p}_i} \cong k_{g(\mathfrak{p}_i)}$, $\hat{g}$ is in $\operatorname{Gal}(k_{\mathfrak{p}_i}/\mathbb{Q}_p)$ if $g \in D_{\mathfrak{p}_i}$, where $k_{\mathfrak{p}_i}$ (respectively $k_{g(\mathfrak{p}_i)}$) is the $\mathfrak{p}_i$-adic local field (respectively the $g(\mathfrak{p}_i)$-adic local field), and the correspondence $g \mapsto \hat{g}$ induces the isomorphism

$$D_{\mathfrak{p}_i} \cong \operatorname{Gal}(k_{\mathfrak{p}_i}/\mathbb{Q}).$$

The subfield of k corresponding to $D_{\mathfrak{p}_i}$ is called *the decomposition field* of $\mathfrak{p}_i$ and is denoted by $Z_{\mathfrak{p}_i}$. Furthermore, $g \in D_{\mathfrak{p}_i}$ induces the isomorphism $\bar{g}$ of $\mathbb{F}_{\mathfrak{p}_i} := \mathcal{O}_k/\mathfrak{p}_i$ over $\mathbb{F}_p$ defined by $\bar{g}(x \mod \mathfrak{p}_i) := g(x) \mod \mathfrak{p}_i$, for $x \in \mathcal{O}_k$. The map $g \mapsto \bar{g}$ induces the homomorphism

$$D_{\mathfrak{p}_i} \longrightarrow \operatorname{Gal}(\mathbb{F}_{\mathfrak{p}_i}/\mathbb{F}_p),$$

whose kernel is called *the inertia group* of $\mathfrak{p}_i$ and is denoted by $I_{\mathfrak{p}_i}$:

$$I_{\mathfrak{p}_i} := \{g \in D_{\mathfrak{p}_i} \mid \bar{g} = \mathrm{id}_{\mathbb{F}_{\mathfrak{p}_i}}\}.$$

If $\mathfrak{p}_j = g(\mathfrak{p}_i)$ $(g \in \operatorname{Gal}(k/\mathbb{Q}))$, we obtain $I_{\mathfrak{p}_j} = gI_{\mathfrak{p}_i}g^{-1}$ and hence $\#I_{\mathfrak{p}_i}$ is independent of $\mathfrak{p}_i$. Set $e = e_p := \#I_{\mathfrak{p}_i}$. The subfield k corresponding to $I_{\mathfrak{p}_i}$ is called *the inertia field* of $\mathfrak{p}_i$ and denoted by $T_{\mathfrak{p}_i}$:

$$k \text{ ——— } T_{\mathfrak{p}_i} \text{ ——— } Z_{\mathfrak{p}_i} \text{ ——— } \mathbb{Q}$$

$$\{1\} \overset{e}{\text{———}} I_{\mathfrak{p}_i} \overset{f}{\text{———}} D_{\mathfrak{p}_i} \overset{r}{\text{———}} \mathrm{Gal}(k/\mathbb{Q})\ .$$

By the isomorphism $D_{\mathfrak{p}_i} \cong \mathrm{Gal}(k_{\mathfrak{p}_i}/\mathbb{Q}_p)$, we see that the homomorphism $D_{\mathfrak{p}_i} \to \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}_i}/\mathbb{F}_p)$ is surjective. Thus, we have the following exact sequence:

$$1 \longrightarrow I_{\mathfrak{p}_i} \longrightarrow D_{\mathfrak{p}_i} \longrightarrow \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}_i}/\mathbb{F}_p) \longrightarrow 1.$$

Then we have the equalities

$$\#D_{\mathfrak{p}_i} = ef,\ \#I_{\mathfrak{p}_i} = e,\ \#\,\mathrm{Gal}(\mathbb{F}_{\mathfrak{p}_i}/\mathbb{F}_p) =: f.$$

Suppose $k/\mathbb{Q}$ is an abelian extension. Then $D_{\mathfrak{p}_i}$ and $I_{\mathfrak{p}_i}$, are independent of $\mathfrak{p}_i$ lying over p and so we denote them by D_p and I_p respectively.

THEOREM 2.3.1 ([**Mor12**]). *Let the notations be as above and suppose $k/\mathbb{Q}$ is an abelian extension. Then there is an exact sequence*

$$1 \longrightarrow I_p \longrightarrow D_p \longrightarrow \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}_i}/\mathbb{F}_p) \longrightarrow 1$$

and the equality

$$n = efr.$$

2.4. Hilbert theory for 3-manifolds

In this section, we review the Hilbert ramification theory for 3-manifolds according to [**Mor12**] Chap.5. We also show a relation between the linking number and the decomposition law of a knot in a finite abelian covering, which generalizes a result in [**Mor12**].

Let M be an integral homology 3-sphere, namely M be a oriented closed 3-manifold and $H_i(M) \cong H_i(S^3)$ for each $i \in \mathbb{Z}$, and let $h : N \to M$ be a finite Galois covering of connected oriented closed 3-manifolds branched over a link $L \subset M$. Let $X_L := M - L$, $Y_L := N - h^{-1}(L)$, and let n denote the covering degree of Y_L over X_L so that $n = \#\mathrm{Gal}(Y_L/X_L) = \#\mathrm{Gal}(N/M)$. Let K be a knot in M which is a component of L or disjoint from L, and suppose $h^{-1}(K) = K_1 \cup \cdots \cup K_r$. Then $\mathrm{Gal}(N/M)$ acts transitively on the set of knots $S_K := \{\, K_1, \ldots, K_r \,\}$ lying over K. We call the stabilizer D_{K_i} of K_i the *decomposition group* of K_i:

$$D_{K_i} := \{\, g \in \mathrm{Gal}(N/M) \mid g(K_i) = K_i \,\}.$$

Since we obtain the bijection $\mathrm{Gal}(N/M)/D_{K_i} \cong S_K$ for each i, $\#D_{K_i} = n/r$ is independent of K_i.

Since each $g \in \mathrm{Gal}(N/M)$ induces a homeomorphism $g|_{\partial V_{K_i}} : \partial V_{K_i} \to \partial V_{g(K_i)}$, $g|_{\partial V_{K_i}}$ is a covering transformation of ∂V_{K_i} over ∂V_K, so we have the following isomorphism,

$$D_{K_i} \cong \mathrm{Gal}(\partial V_{K_i}/\partial V_K).$$

The Fox completion of the subcovering space of Y_L over X_L corresponding to D_{K_i} is called the *decomposition covering space* of K_i and this space is denoted by Z_{Ki}. The map $g \mapsto \bar{g} := g|_{\partial V_{K_i}}$ induces the homomorphism

$$D_{K_i} \to \mathrm{Gal}(K_i/K)$$

whose kernel is called the *inertia group* of K_i and is denoted by I_{K_i}:

$$I_{K_i} := \{\, g \in D_{K_i} \mid \bar{g} = \mathrm{id}_{K_i} \,\}.$$

If $K_j = g(K_i)$ ($g \in \mathrm{Gal}(N/M)$), we obtain $I_{K_j} = gI_{K_i}g^{-1}$ and hence $\#I_{K_i}$ is independent of K_i. Set $e = e_K := \#I_{K_i}$. The Fox completion of the subcovering space of Y_L over X_L corresponding to I_{K_i} is called the *inertia covering space* of K_i and denoted by T_{K_i}:

$$N \longrightarrow T_{K_i} \longrightarrow Z_{K_i} \longrightarrow M$$

$$\{1\} \overset{e}{\text{———}} I_{K_i} \overset{f}{\text{———}} D_{K_i} \overset{r}{\text{———}} \mathrm{Gal}(N/M)\ .$$

By the isomorphism $D_{K_i} \cong \mathrm{Gal}(\partial V_{K_i}/\partial V_K)$, we see that the homomorphism $D_{K_i} \to \mathrm{Gal}(K_i/K)$ is surjective:

$$1 \longrightarrow I_{K_i} \longrightarrow D_{K_i} \longrightarrow \mathrm{Gal}(K_i/K) \longrightarrow 1.$$

Then we have the equalities

$$\#D_{K_i} = ef,\ \#I_{K_i} = e,\ \#\mathrm{Gal}(K_i/K) =: f,$$

where f is called the *covering degree* of K.

Suppose $h : N \to M$ is an abelian covering. Then D_{K_i} and I_{K_i}, are independent of K_i lying over K and so we denote them by D_K and I_K respectively.

THEOREM 2.4.1 ([**Mor12**]). *Let the notations be as above and suppose $h : N \to M$ is an abelian covering. Then there is an exact sequence*

$$1 \longrightarrow I_K \longrightarrow D_K \longrightarrow \mathrm{Gal}(K_i/K) \longrightarrow 1$$

and the equality

$$n = efr.$$

Finally, let us extend the relation between linking number and the decomposition law of a knot in a finite abelian covering.

PROPOSITION 2.4.2. *Let $L := K_1 \cup \cdots \cup K_r$ be an r-component link in an integral homology 3-sphere M. For given integers $n_i \geq 2$, let $\psi : \pi_1(X_L) \to \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$ be the homomorphism sending a each meridian of K_i to $(0, \ldots, 0, \overset{i}{\check{1}}, 0, \ldots, 0)$. Let $Y_L \to X_L$ be the covering corresponding to $\mathrm{Ker}(\psi)$, whose covering degree is $n := n_1 n_2 \cdots n_r$, and let $h : N \to M$ denotes its Fox completion. Then, for a knot K in M disjoint from L, the covering degree of K in $h : N \to M$ coincides with the order of $(\mathrm{lk}(K, K_1) \bmod n_1, \ldots, \mathrm{lk}(K, K_i) \bmod n_i, \ldots, \mathrm{lk}(K, K_r) \bmod n_r)$ in $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$.*

proof. Let K' be a component of $h^{-1}(K)$. Since $I_{K'} = I_K = \{1\}$, by Theorem 2.3.1, the covering degree of K in $h : N \to M$ is the order of a generator σ_K of $\mathrm{Gal}(K'/K) \cong D_K$ in $\mathrm{Gal}(N/M)$, where σ_K corresponds to a loop K. Since $[K]$ is sent to $(\mathrm{lk}(K, K_1) \bmod n_1, \ldots, \mathrm{lk}(K, K_r) \bmod n_r)$ by the natural homomorphism $H_1(X_L) \to \mathrm{Gal}(N/M) \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$ given by the Hurewicz map and Galois theory, our assertion follows. □

In particular, suppose K is not a component of L, so that K is unbranched in N. Then the equality $fr = n$ implies that K is decomposed completely in N (i.e. decomposed into an n-component link) if and only if for each i, $\mathrm{lk}(K_i, K) \equiv 0 \bmod n_i$.

CHAPTER 3

Local class field theory for tori

In this chapter, we review the local class field theory for local fields and describe its analogue for 2-dimensional tori.

3.1. Review of local class field theory for local fields

We consult [**Neu99**] as a reference for this section. Let k be a number field of finite degree over the rational number field $\mathbb{Q}$. We denote the ring of integers of k by $\mathcal{O}_k$. A prime $\mathfrak{p}$ of k is a class of equivalent valuations of k. Finite primes belong to the maximal ideals of $\mathcal{O}_k$. Infinite primes fall into two classes, real primes and complex primes. Here real primes correspond to the embeddings $k \to \mathbb{R}$, and complex primes correspond to the pairs of conjugate non-real embeddings $k \to \mathbb{C}$. For a finite prime $\mathfrak{p}$, let $v_\mathfrak{p}$ be the corresponding additive valuation of k, and $|a|_\mathfrak{p} := (N\mathfrak{p})^{-v_\mathfrak{p}(a)}$ for $a \in k$ where $N\mathfrak{p} = \#(\mathcal{O}_k/\mathfrak{p})$. For a real prime $\mathfrak{p}$ with corresponding embedding $\iota : k \to \mathbb{R}$, let $|a|_\mathfrak{p} := |\iota(a)|$ for $a \in k$, and complex prime $\mathfrak{p}$ with corresponding $\iota : k \to \mathbb{C}$, let $|a|_\mathfrak{p} := |\iota(a)|^2$ for $a \in k$.

Let $\mathfrak{p}$ be a finite prime of k, and let $k_\mathfrak{p}$ be the local field obtained as the completion of a number field k with respect to the metric $|\cdot|_\mathfrak{p}$. Then $k_\mathfrak{p}$ is a non-archimedean local field, which is a finite extension of the p-adic field $\mathbb{Q}_p$ for a prime number p. Let $v_\mathfrak{p} : k_\mathfrak{p} \to \mathbb{Z}$ be the discrete normalized valuation. We denote by $\mathcal{O}_\mathfrak{p}$ the valuation ring and by $\mathfrak{p}$ the unique maximal ideal of $\mathcal{O}_\mathfrak{p}$. Let $\mathbb{F}_\mathfrak{p}$ be the residue field $\mathcal{O}_\mathfrak{p}/\mathfrak{p}$, which is a finite extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We have $\mathcal{O}_\mathfrak{p}^\times = \mathrm{Ker}(v_\mathfrak{p})$. The valuation map v_p yields the following splitting exact sequence:

$$1 \longrightarrow \mathcal{O}_\mathfrak{p}^\times \longrightarrow k_\mathfrak{p}^\times \xrightarrow{v_\mathfrak{p}} \mathbb{Z} \longrightarrow 0. \tag{1}$$

Let $k_\mathfrak{p}^{\mathrm{ab}}$ be the maximal abelian extension of $k_\mathfrak{p}$. When $k_\mathfrak{p}$ is non-archimedean, we denote the maximal unramified extension of $k_\mathfrak{p}$ by $k_\mathfrak{p}^{\mathrm{ur}}$. A main theorem of local class field theory for the non-archimedean local field $k_\mathfrak{p}$ is stated as follows.

Theorem 3.1.1 (Local class field theory). *There is a homomorphism,*

$$\rho_{k_\mathfrak{p}} : k_\mathfrak{p}^{\mathrm{ab}} \longrightarrow \mathrm{Gal}(k_\mathfrak{p}^{\mathrm{ab}}/k_\mathfrak{p})$$

called the local reciprocity map, which satisfies the following conditions.
(i). *For any finite abelian extension $F/k_\mathfrak{p}$, the homomorphism $\rho_{k_\mathfrak{p}}$ induces the following isomorphism*

$$\rho_{k_\mathfrak{p}} : k_\mathfrak{p}^\times / N_{F/k_\mathfrak{p}}(F^\times) \longrightarrow \mathrm{Gal}(F/k_\mathfrak{p})$$

where $N_{F/k_\mathfrak{p}}$ denotes the norm map for $F/k_\mathfrak{p}$.

(ii). *There is commutative diagram with exact horizontal sequences:*

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_{\mathfrak{p}}^{\times} \arrow[r] \arrow[d] & k_{\mathfrak{p}}^{\times} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{ur}}) \arrow[r] & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}) \arrow[r] & \mathrm{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_p) \arrow[r] & 1
\end{tikzcd}$$

By the above theorem, there is an isomorphism $\mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ur}}/k_{\mathfrak{p}}) \cong \mathrm{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) \cong \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$.

THEOREM 3.1.2. *There is a bijection between the set of finite unramified extensions of $k_{\mathfrak{p}}$ and the set of open subgroups of finite indices in $k_{\mathfrak{p}}^{\times}$ containing $\mathcal{O}_{\mathfrak{p}}^{\times}$.*

The local theory of an infinite prime $\mathfrak{p} : k \overset{\widetilde{\mathfrak{p}}}{\hookrightarrow} \mathbb{C} \to \mathbb{R}_{\geq 0}; x \mapsto |\widetilde{\mathfrak{p}}(x)|$ is described as follows. If $\mathfrak{p}$ is real, then $v_{\mathfrak{p}} : k^{\times} \to \mathbb{R}; x \mapsto \log|\widetilde{\mathfrak{p}}(x)|$ yields an exact sequence $1 \to \{\pm 1\} \to \mathbb{R}^{\times} \overset{v_{\mathfrak{p}}}{\to} \mathbb{R} \to 0$. By taking Hausdorffication with respect to the local norm topology, we obtain an exact sequence $1 \to \{\pm 1\} \to \{\pm 1\} \to 0 \to 0$. If $\mathfrak{p}$ is complex, then we have an exact sequence $1 \to S^1 \to \mathbb{C}^{\times} \overset{v_{\mathfrak{p}}}{\to} \mathbb{R} \to 0$, and obtain an exact sequence $1 \to 1 \to 1 \to 0 \to 0$ of trivial terms in a similar way. We put $\mathcal{O}_{\mathfrak{p}}^{\times} = \{\pm 1\}$ or 1 according as $\mathfrak{p}$ is real or complex. In both cases, there are commutative diagrams similar to the case of finite primes.

3.2. Local class field theory for tori

Let K be a fixed knot in an orientable 3-manifold M and let V_K be a tubular neighborhood of K. Then, the boundary of the tubular neighborhood ∂V_K is a 2-dimensional torus. The inclusion $\partial V_K \to V_K$ induces the homomorphism $v_K : H_1(\partial V_K) \to H_1(V_K)$. This homomorphism v_K is an analogue of $\mathfrak{p}$-adic valuation. *The meridian* $\mu \in H_1(\partial V_K)$ of K is the generator of $\mathrm{Ker}(v_K)$ corresponding to the orientation of K. A *longitude* $\lambda \in H_1(\partial V_L)$ of K is an element satisfying that μ and λ form a basis of $H_1(\partial V_K)$. We denote the image of $\lambda \in H_1(V_K)$ also by λ. We fix a longitude of K.

We have the following exact sequence:

$$0 \longrightarrow \langle \mu \rangle \longrightarrow H_1(\partial V_K) \xrightarrow{v_K} H_1(V_K) = \langle \lambda \rangle \longrightarrow 0.$$

This exact sequence is an analogue of (1).

According to the local dictionary, ∂V_K and V_K is an analogues of $\mathfrak{p}$-adic local field $k_{\mathfrak{p}}$ and the integer ring $\mathcal{O}_{\mathfrak{p}}$. In our context, the local theory for tori is nothing but the Galois theory for the covers of ∂V_K. For each manifold X, we denote the universal covering of X by $\widetilde{X}$. Then we have the following theorem.

THEOREM 3.2.1 (Local class field theory for tori). *There is a canonical isomorphism*

$$\rho_K : H_1(\partial V_K) \longrightarrow \mathrm{Gal}(\partial V_K^{\mathrm{ab}}/\partial V_K)$$

which satisfies the following conditions.
(i). *For any finite abelian covering $h : Y \to \partial V_K$, the homomorphism ρ_K induces the isomorphism*

$$\rho_{Y/\partial V_K} : H_1(\partial V_L)/h_*(H_1(Y)) \cong \mathrm{Gal}(Y/\partial V_K).$$

(ii). *There is a commutative diagram with exact horizontal sequences:*

$$\begin{tikzcd}
0 \arrow[r] & \langle\mu\rangle \arrow[r] \arrow[d, "\rho_K|_{\langle\mu\rangle}"] & H_1(\partial V_K) \arrow[r, "v_K"] \arrow[d, "\rho_K"] & \langle\lambda\rangle \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathrm{Gal}(\widetilde{\partial V_K}/\widetilde{\partial V_K}) \arrow[r] & \mathrm{Gal}(\widetilde{\partial V_K}/\partial V_K) \arrow[r] & \mathrm{Gal}(\widetilde{V_K}/V_K) \arrow[r] & 1.
\end{tikzcd}$$

THEOREM 3.2.2. *There is a bijection between the set of finite unbranched covers of V_K branched over K and the set of subgroups of finite indices in $H_1(\partial V_K)$ containing $\langle\mu\rangle$.*

proof. These theorem is nothing but Galois theory for covering spaces. □

Summing up all the results above, we have the following dictionarry.

K-adic valuation $v_K : H_1(\partial V_K) \to \mathbb{Z}$	$\mathfrak{p}$-adic valuation $v_{\mathfrak{p}} : k_{\mathfrak{p}}^{\times} \to \mathbb{Z}$

CHAPTER 4

Idèlic class field theory for 3-manifolds

In this chapter, we present an analogue of idèlic class field theory. First, we review the idelic class field theory for number fields in §4.1. In §4.2, we introduce the notion of a very admissible link $\mathcal{K}$ in a 3-manifold M which is regarded as an analogous object of the set of all the primes in a number field. After that, we introduce the definitions of *the idèle group*, *the principal idèle group*, and *the idèle class group*, and we construct an analogue of the grobal reciprocity law of idèlic class field theory. Moreover, we introduce certain topologies on our idèle class group, and present an analogue of the exsistence theorem of class field theory.

4.1. Review of global class field theory for number fields

In this section, we review the idelic class field theory for number fields, whose analogue will be described in the later section. We define the notions of the idele groups, the principal idele groups, and the idele class groups, together with the global reciprocity map. Then we state the fundamental theorem of global class field theory. Finally, we recall the notion of the norm residue symbol. We consult [**Neu99**] and [**KKS11**] as basic references for this section.

4.1.1. the idèle class groups. Let k be a number field. We define the idèle group I_k of k by the following restricted product of $k_{\mathfrak{p}}^{\times}$ with respect to the local unit group $\mathcal{O}_{\mathfrak{p}}^{\times}$ over all finite and infinite primes $\mathfrak{p}$ of k:

$$I_k := \prod_{\mathfrak{p}} k_{\mathfrak{p}}^{\times} = \Big\{ (a_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}} k_{\mathfrak{p}}^{\times} \mid v_{\mathfrak{p}}(a_{\mathfrak{p}}) = 0 \text{ for almost all finite primes } \mathfrak{p} \Big\}.$$

This group is the restricted products with respect to the local topology on $k_{\mathfrak{p}}^{\times}$ (see §4.1.2) and the family of open subgroups $\{\mathcal{O}_{\mathfrak{p}}^{\times} \subset k_{\mathfrak{p}}^{\times}\}$. Since we have $v_{\mathfrak{p}}(a) = 0$ for $a \in k^{\times}$ and for almost all finite primes $\mathfrak{p}$, $k^{\times}$ is embedded into I_k diagonally. We define the principal idèle group P_k by the image of the diagonal embedding $\Delta : k^{\times} \to I_k$. Then, we denote the idèle class group of k by

$$C_k := I_k / P_k.$$

Let $I(k)$ and $\mathrm{Cl}(k)$ denote the ideal group and the ideal class group of k respectively. Consider the natural homomorphism $\varphi : I_k \to I(k)$; $(a_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(a_{\mathfrak{p}})}$. We define the unit idele group U_k by $\mathrm{Ker}(\varphi)$, which is equal to $\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^{*}$. We have

Proposition 4.1.1. *The homomorphism φ induces a natural isomorphism*

$$I_k/(U_k \cdot P_k) \cong \mathrm{Cl}(k).$$

4.1.2. Topologies of idèle class groups. The idèle class group C_k admits the *standard topology* and the *norm topology*.

First, we introduce the definition of standard topology which is the quotient topology of the restricted product topology on the idèle group I_k of the local topologies, defined as follows.

We firstly define the local topology on a local field $k_{\mathfrak{p}}^{\times}$. For a local field $k_{\mathfrak{p}}$, the multiplicative group $k_{\mathfrak{p}}^{\times}$ equips the *norm topology*, so that it is a topological group, and the family of $N_{k_{\mathfrak{P}}/k_{\mathfrak{p}}}(k_{\mathfrak{P}}^{\times})$ is a fundamental system of neighborhoods of 0, where $k_{\mathfrak{P}}/k_{\mathfrak{p}}$ runs through all the finite abelian extensions of $k_{\mathfrak{p}}$. Then, we consider on $\mathcal{O}_{\mathfrak{p}}^{\times}$ the relative topology of the local norm topology of $k_{\mathfrak{p}}^{\times}$, and re-define the local topology on $k_{\mathfrak{p}}^{\times}$ as the unique topology such that the inclusion $\mathcal{O}_{\mathfrak{p}}^{\times} \to k_{\mathfrak{p}}^{\times}$ is open and continuous.

Next, for each finite set of primes T which includes all the infinite primes, we consider the product topology on $G(T) = \prod_{\mathfrak{p}\in T} k_{\mathfrak{p}}^{\times} \times \prod_{\mathfrak{p}\notin T} \mathcal{O}_{\mathfrak{p}}^{\times}$. Then, we define the *standard topology* on I_k so that each subgroup $H \subset I_k$ is open if and only if $H \cap G(T)$ is open for every T.

Secondly, we introduce the definition of the norm topology on idèle class group. For a finite abelian extension F/k, the norm map $N_{F/k} : C_F \to C_k$ is defined as follows. Let $\mathfrak{p}$ be a prime of k and $F_{\mathfrak{p}}^{\times} := \prod_{\mathfrak{P}|\mathfrak{p}} F_{\mathfrak{P}}^{\times}$. Each $\alpha_p \in F_{\mathfrak{p}}^{\times}$ defines a $k_{\mathfrak{p}}$-linear automorphism $\alpha_{\mathfrak{p}} : F_{\mathfrak{p}}^{\times} \to F_{\mathfrak{p}}^{\times}; x \mapsto \alpha_{\mathfrak{p}} x$, and the norm of $\alpha_{\mathfrak{p}}$ is defined by $N_{F_{\mathfrak{p}}/k_{\mathfrak{p}}}(\alpha_{\mathfrak{p}}) = \det(\alpha_{\mathfrak{p}})$. It induces a homomorphism $N_{F_{\mathfrak{p}}/k_{\mathfrak{p}}} : F_{\mathfrak{p}}^{\times} \to k_{\mathfrak{p}}^{\times}$, and the norm homomorphism $N_{F/k} : I_F \to I_k$ on the idèle groups. Since $N_{F/k}$ sends the principal idèles to principal idèles, it also induces the norm homomorphism $N_{F/k} : C_F \to C_k$ on the idèle class groups. For a number field k, the idèle class group C_k equips the *norm topology*, so that it is a topological group, and the family of $N_{F/k}(C_F)$ is a fundamental system of neighborhoods of 0, where F/k runs through all the finite abelian extensions of k.

There is a relation between the standard topology and the norm topology.

PROPOSITION 4.1.2. *A subgroup $H \subset C_k$ is open and of finite index with respect to the standard topology if and only if it is open with respect to the norm topology.*

4.1.3. Global class field theory for number fields. A main theorem of global class field theory for a number field k is stated as follows.

THEOREM 4.1.3 (Global class field theory). *There is a canonical surjective homomorphism*

$$\rho_k : C_k \longrightarrow \mathrm{Gal}(k^{\mathrm{ab}}/k)$$

called the global reciprocity map satisfying the following properties:
(i) *For any finite abelian extension F/k, the homomorphism ρ_k induces the following isomorphism*

$$\rho_{F/k} : C_k/N_{F/k}(C_F) \longrightarrow \mathrm{Gal}(F/k)$$

where $N_{F/k}$ denotes the norm map for F/k.
(ii) *For a prime $\mathfrak{p}$ of k, there is a commutative diagram:*

$$\begin{tikzcd} k_{\mathfrak{p}}^{\times} \arrow[r, "\rho_{k_{\mathfrak{p}}}"] \arrow[d, "\iota_{\mathfrak{p}}"'] & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}) \arrow[d] \\ C_k \arrow[r, "\rho_k"'] & \mathrm{Gal}(k^{\mathrm{ab}}/k) \end{tikzcd}$$

where $\iota_{\mathfrak{p}}$ is the map induced by the natural inclusion $k_{\mathfrak{p}}^{\times} \to I_k$.

THEOREM 4.1.4 (The existence theorem). *The correspondence*

$$F \mapsto \mathcal{N} = N_{F/k}(C_F)$$

gives a bijection between the set of finite abelian extensions F/k in $\mathbb{C}$ and the set of open subgroups N of finite indices in C_k with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of C_k with respect to the norm topology.

4.1.4. The norm residue symbols. In this section, we introduce the norm residue symbol for number fields. We also explain the Legendre symbol $\left(\frac{p}{q}\right)$.

DEFINITION 4.1.5. For a finite abelian extension F/k, the *norm residue symbol* $(\ ,F/k) : C_k \twoheadrightarrow \mathrm{Gal}(F/k)$ is defined as the composite of $\rho_k : C_k \twoheadrightarrow \mathrm{Gal}(k^{\mathrm{ab}}/k)$ and $\mathrm{Gal}(k^{\mathrm{ab}}/k) \twoheadrightarrow \mathrm{Gal}(F/k)$. For this map, we have $\mathrm{Ker}(\ ,F/k) = N_{F/k}(C_F)$.

The relation with Legendre's quadratic residue symbol can be seen as follows: Let p and q be distinct primes in $k = \mathbb{Q}$, and let $F = \mathbb{Q}(\sqrt{q})$ be the quadratic extension of $\mathbb{Q}$ ramified at q. Then [**KKS11**] Lemma 5.19 states the following equivalences:

$$\left(\frac{q}{p}\right) = 1 \quad \underset{\text{iff}}{\Longleftrightarrow} \quad (p) = \mathfrak{p}_1\mathfrak{p}_2 \text{ with two primes } \mathfrak{p}_1, \mathfrak{p}_2 \text{ in } \mathcal{O}_F \text{ (decomposed)},$$

$$\left(\frac{q}{p}\right) = -1 \quad \underset{\text{iff}}{\Longleftrightarrow} \quad (p) \text{ is a prime in } \mathcal{O}_F \text{ (inert)}.$$

On the other hand, under the identification $\mathrm{Gal}(F/k) \cong \{\pm 1\}$, there are the following equivalences:

$$((p), F/k) = 1 \underset{\text{iff}}{\Longleftrightarrow} (p) \in N_{F/k}(C_{F/k}) \underset{\text{iff}}{\Longleftrightarrow} (p) \text{ is decomposed in } F/k.$$

Therefore, we have $\left(\frac{q}{p}\right) = ((p), \mathbb{Q}(\sqrt{q})/\mathbb{Q})$ in $\{\pm 1\}$.

4.2. Very admissible links

In this section, we introduce the notion of a very admissible link of a 3-manifold M, which may be regarded as an analogue of the set of primes in a number ring. We first recall the notion of tame knots and finite/infinite tame links. Next, we study several properties about infinite tame links, which will be used later. Finally, we define the notion of a very admissible link of M and prove a theorem on the existence of very admissible links.

4.2.1. Infinite tame links. We first recall the definition of a tame knot and links. We fix a connected, oriented, closed 3-manifold M. We may assume that M is an orientable 3-dimensional C^{∞}-manifold. We fix a finite C^{∞}-triangulation T on M.

PROPOSITION 4.2.1. *For a knot $K : S^1 \to M$, the following conditions are equivalent.*
(1). *There is a self-homeomorphism h of M such that $h(K)$ is a subcomplex of some refinement of T.*

(2). *There is a self-homeomorphism h of M such that $h(K)$ is a C^∞-submanifold of M.*
(3). *There is a tubular neighborhood of K, that is, a topological embedding $\iota_K : S^1 \times D^2 \to M$ with $\iota_K(S^1 \times 0) = K$.*

Furthermore, we note that (♯) if a neighborhood V of K is given, then h in (i) and (ii) can be taken so that it has a support in V (i.e., it coincides with identity map on $M - V$).

proof. (1) $\Longrightarrow$ (2) : We may assume that K itself is a subcomplex of some refinement T' of T. For each 0-simplex v of T' on K, by a self-homeomorphism of M with support in a small neighborhood of v, we can modify K so that K is C^∞ in a neighborhood of v. Doing the similar for every v, we obtain (2).
(2) $\Longrightarrow$ (3): We may assume that K itself is a C^∞-submanifold of M. A tubular neighborhood of V is the total space of a D^2-bundle on $K \cong S^1$. Since M is oriented, V is orientable and hence is the trivial bundle. Hence (3).
(3) $\Longrightarrow$ (1): We use [**Moi52**] Theorem 5: Let M be a metrized 3-manifold with a fix triangulation T and let K be a closed subset of M. Suppose that there is a neighborhood V of K in M and a topological embedding $\iota : V \to M$ so that $\iota(K)$ is a subcomplex of a refinement of T. Then, there is a self-homeomorphism $h : M \to M$ such that $h(K)$ is a subcomplex of a refinement of T. In addition, for a given $\varepsilon > 0$, there is some h with its support in the ε-neighborhood of K. Moreover, we can take h as closer to id as we want $\cdots$ $(*)$. If we apply this theorem to our M with a metric, $T, K, V := \iota_K(S^1 \times D^2)$, and the inclusion ι, then we obtain (1).

By noting $(*)$ and the construction in (1)$\Longrightarrow$ (2), we see (♯). □

A knot K in M is said to be *tame* if it satisfies the above equivalent conditions.

Proposition 4.2.2. *For a finite link $L : \sqcup S^1 \to M$, the following conditions are equivalent.*
(1). *There is a self-homeomorphism h of M such that $h(L)$ is a subcomplex of some refinement of T.*
(2). *There is a self-homeomorphism h of M such that $h(L)$ is a C^∞-submanifold of M.*
(3). *Each component $K : S^1 \to M$ of L is tame.*

proof. The non-trivial part of this equivalence is to prove that (3) implies (1). We can prove it by (3) $\Longrightarrow$ (1) for the knot case and the condition (♯) on the support of a self-homeomorphism h. □

A finite link L in M is said to be *tame* if it satisfies the above equivalent conditions. A finite link consisting of tame components always equips a tubular neighborhood as a link.

A link L in M is called *an infinite tame link* if it consists of countably infinitely many tame components. An infinite tame link L equips a tubular neighborhood as a link if and only if it has no accumulation point. We do not eliminate the cases with accumulation points.

For a tame knot K in M, we denote by V_K a tubular neighborhood of K, which is unique up to ambient isotopy. For a link L in M consisting of countably many tame components, we consider the formal (or infinitesimal) tubular neighborhood $V_L := \sqcup_{K \subset L} V_K$, where K runs through all the components of L. We fix a longitude for each K. For a finite branched cover $h : N \to M$ and for each component of

$h^{-1}(K)$ in N, we fix a longitude which is a component of the preimage of that of K.

4.2.2. Lemmas on infinite tame links. We study several properties on an infinite tame link. Let $\mathcal{K}$ be an infinite tame link in M.

PROPOSITION 4.2.3 (the Sielpinski theorem, [**Eng89**] Theorem 6.1.27). *If a compact Hausdorff connected space X and a countable family $\{X_i\}_{i\in\mathbb{N}}$ of pairwise disjoint closed subsets satisfy $X = \cup_i X_i$, then at most one of X_i is non-empty.*

By virtue of Proposition 4.2.3, the notion of a *component* of $\mathcal{K}$ makes sense in a natural way, that is, each connected component of its image is the image of some S^1 in the domain.

The set of finite sublinks of $\mathcal{K}$ is a directed set with respect to the inclusions. In addition, if we take an inclusion sequence $\cdots \subset L_i \subset L_{i+1} \subset \cdots$ of finite sublinks of $\mathcal{K}$ indexed by $i \in \mathbb{N}$, then $\mathcal{K} = \cup_i L_i$ and any finite sublink L of $\mathcal{K}$ is contained in some L_i.

For each finite sublink L of $\mathcal{K}$, we put $X_L = M - L$. Then $H_1(X_L)$'s form an inverse system indexed by $L \subset \mathcal{K}$ with respect to the natural surjections induced by the inclusion maps of the exteriors. We put $X_{\mathcal{K}} = M - \mathcal{K}$.

LEMMA 4.2.4. *There is a natural isomorphism $H_1(X_{\mathcal{K}}) \cong \varprojlim_L H_1(X_L)$.*

proof. We have a Milnor exact sequence ([**Mil62**]) $0 \to \varprojlim^1_L H_2(X_L) \to H_1(X_{\mathcal{K}}) \to \varprojlim H_1(X_L) \to 0$. Since $H_2(X_L)$ is a surjective system and satisfies the Mittag-Leffer condition, we have $\varprojlim^1_L H_2(X_L) = 0$. Thus $H_1(X_{\mathcal{K}}) \cong \varprojlim_L H_1(X_L)$. □

Let L and L' be finite sublinks of $\mathcal{K}$ with $L \subset L'$. Then, the natural surjection $C_*(M, L) \to C_*(M, L')$ induces the natural injection $j_* : H_*(M, L) \to H_*(M, L')$. We obtain the following proposition.

LEMMA 4.2.5. *There is a natural isomorphism*

$$H_n(M, \mathcal{K}) \cong \varinjlim_L H_n(M, L),$$

where L runs through all the finite sublinks of $\mathcal{K}$ and the transition maps are the natural map j_*'s.

proof. By Sielpinski's theorem (Proposition 4.2.3), the singular chain groups satisfy $C_n(\mathcal{K}) = \varinjlim_{L\subset\mathcal{K}} C_n(L)$ for each $n \in \mathbb{N}$. The exact sequence $0 \to C_n(L) \to C_n(M) \to C_n(M, L) \to 0$ yields the exact sequence $0 \to C_n(\mathcal{K}) \to C_n(M) \to \varinjlim C_n(M, L) \to 0$. The exact sequence $0 \to C_n(\mathcal{K}) \to C_n(M) \to C_n(M, \mathcal{K}) \to 0$ induces the natural isomorphism $C_n(M, \mathcal{K}) \to \varinjlim_{L\subset\mathcal{K}} C_n(M, L)$. By taking the long exact sequences and using the five lemma, we obtain the natural isomorphism $H_n(M, \mathcal{K}) \cong \varinjlim_{L\subset\mathcal{K}} H_n(M, L)$. □

4.2.3. Very admissible links.

DEFINITION 4.2.6. Let M be a closed, oriented, connected 3-manifold. Let $\mathcal{K}$ be a link in M consisting of countably many (finite or infinite) tame components. We say $\mathcal{K}$ is an *admissible link* of M if the components of $\mathcal{K}$ generates $H_1(M)$. We

say $\mathcal{K}$ is a *very admissible link* of M if for any finite cover $h : N \to M$ branched over a finite link in $\mathcal{K}$, the components of the link $h^{-1}(\mathcal{K})$ generates $H_1(N)$.

For each K in $\mathcal{K}$, we denote the meridian of K by μ_K and fix a longitude λ_K of K.

LEMMA 4.2.7. *Let M be a closed, oriented, connected 3-manifold and let L be a link in M consisting of countably many tame components. Then there is a link $\mathcal{L}$ in M containing L, consisting of countably many tame components, and satisfying that for any finite cover $h : N \to M$ branched over a finite sublink of L, $H_1(N)$ is generated by the components of the preimage $h^{-1}(\mathcal{L})$.*

proof. The set of all the finite branched covers of M branched over finite sublinks of L is countable, and can be written as $\{h_i : N_i \to M\}_{i\in\mathbb{N}}$, where $h_0 = id_M$. Indeed, for each finite sublink $L' \subset L$, finite branched covers of M branched over L' corresponds to subgroups of $\pi_1(M - L')$ of finite indices. Since $\pi_1(M - L')$ is finitely generated group, such subgroups are countable.

We construct an inclusion sequence $L_0 \subset L_1 \subset \ldots \subset L_i \subset \ldots$ of links consisting of countably many tame components as follows. First, we put $L_0 = L$. Next, for $i \in \mathbb{N}_{>0}$, let L_{i-1} be given. We *claim* that there is a link L_i in M including L_{i-1}, consisting of countably many tame components, and satisfying that the components of the preimage $h_i^{-1}(L_i)$ generates $H_1(N_i)$. By putting $\mathcal{L} := \cup_i L_i$, we obtain an expected link.

The *claim* above can be deduced immediately from the following assertion: *For any finite branched cover $h : N \to M$ and the preimage $\widetilde{L}$ of any link in M consisting of countably many tame components, there is a finite link L' in $N - \widetilde{L}$ consisting of tame components and the image $h(L')$ being also a link.*

Note that N is again a closed, oriented, connected 3-manifold. We may assume that N is a C^∞-manifold. On the space $C^\infty(S^1, N)$ of maps, since S^1 is compact, the well-known two topologies called *the compact open topology (the weak topology)* and *the Whitney topology (the strong topology)* coincide. It is completely metrizable space and satisfies *the Baire property*, that is, *for any countable family of open and dense subsets, their intersection is again dense.* (We refer to [**Hir94**] for the terminologies and the general facts stated here.)

Let $\{K_j\}_j$ denote the set of components of $\widetilde{L}$. Since $F_j := \{K \in C^\infty(S^1, N) \mid K \cap K_j = \emptyset\}$ is open and dense, by the Baire property, the intersection $F := \cap_j F_j$ is dense. Put $H_1(N) = \langle a_1, \ldots, a_r\rangle$, and let A_1 denote the set of tame knots $K \in C^\infty(S^1, N)$ satisfying $[K] = a_1$ whose image $h(K)$ in M is also a tame knot. Then A_1 is open and non-empty. Therefore $A_1 \cap F$ is non-empty, and we can take an element K'_1 of it. For $1 \le k \le r$, if we replace $\widetilde{L}$ by $\widetilde{L} \cup K'_1 \cup \ldots \cup K'_k$ and do a similar construction for a_{k+1} successively, then we complete the proof. □

THEOREM 4.2.8. *Let M be a closed, oriented, connected 3-manifold, and L a link in M. Then, there is a very admissible link $\mathcal{K}$ containing L.*

proof. We construct an inclusion sequence of links $\{\mathcal{K}_i\}_i$ as follows: First, we take a link $\mathcal{K}_0$ which includes L and generates $H_1(M)$. Next, for $i \in \mathbb{N}_{>0}$, let $\mathcal{K}_{i-1}$ be given, and let $\mathcal{K}_i$ be a link obtained from $\mathcal{K}_{i-1}$ by Lemma 4.2.7. Then the union $\mathcal{K} := \cup\mathcal{K}_i$ is a very admissible link. □

Links $\mathcal{L}$ and $\mathcal{K}$ in the Lemma and Theorem above may be taken smaller than in the constructions. It may be interesting to ask whether they can be finite. Let

M be a 3-dimensional sphere S^3. The unknot is a very admissible link. If L is the trefoil knot, by taking branched 2-cover, we see that $\mathcal{K}$ is greater than L. We expect that $\mathcal{K}$ has to be infinite. Next, let M be a 3-manifold, and L a minimum admissible link (L can be empty). For an integral homology 3-sphere M, we have $\mathcal{K} = L = \emptyset$. For a lens space $M = L(p, 1)$ or $M = S^2 \times S^1$, we can take a knot (the core loop) $\mathcal{K} = L = K$.

In the later sections of this thesis, we assume that a very admissible link $\mathcal{K}$ is an infinite link. However, our arguments are applicable also for finite $\mathcal{K}$.

Remark 4.2.9. According to [**Mor12**], counterparts of infinite primes are *ends* of 3-manifolds. In this thesis, since we deal with closed manifolds, the counterpart of the set of infinite primes is empty.

4.3. The universal $\mathcal{K}$-branched cover

Class field theory deals with all the abelian extensions of a number field k in a fixed algebraic closure $\overline{k}$ of k. For a 3-manifold M equipped with an infinite (very admissible) link $\mathcal{K}$, we introduce the notion of the universal $\mathcal{K}$-branched cover, which is an analogue of an algebraic closure of a number field. We also discuss the role of base points.

In the following, we discuss an analogue of an/the algebraic closure of a number field. If we say branched covers, unless otherwise mentioned, we consider *branched covers endowed with base points*, that is, we fix base points in all spaces that are compatible with covering maps. For a space X, we denote by b_X the base point.

First, we recall the notion of an isomorphism of branched covers. For covers $h : N \to M$ and $h' : N' \to M$ branched over L, we say they are *isomorphic* (as branched covers endowed with base points) and denote by $h \cong h'$ if there is a (unique) homeomorphism $f : (N, b_N) \overset{\cong}{\to} (N', b_{N'})$ such that $h = h' \circ f$. Let $\underline{h} : Y_L \to X_L$ and $\underline{h}' : Y'_L \to X_L$ denote the restrictions to the exteriors. Then, $h \cong h'$ is equivalent to that $\underline{h}_*(\pi_1(Y_L, b_{Y_L})) = \underline{h}'_*(\pi_1(Y'_L, b_{Y_{L'}}))$ in $\pi_1(X_L, b_{X_L})$.

Such notion is extended to the class of branched pro-covers, which are objects obtained as inverse limits of finite branched covers.

Next, we introduce an analogue notion of an algebraic closure of a number field. For a finite link L in a 3-manifold, a branched pro-cover $h_L : \widetilde{M_L} \to M$ is a *universal L-branched cover* of M if it satisfies a certain universality: $h_L : \widetilde{M_L} \to M$ is a minimal object such that any finite cover of M branched over L factor through it. It is unique up to the canonical isomorphisms, and it can be obtained by Fox completion of a universal cover of the exterior $\underline{h_L} : \widetilde{X_L} \to X_L$. (Note that Fox completion is defined for a spread of locally connected T_1-spaces in general. ([**Fox57**]))

Now, let M be a 3-manifold equipped with an infinite (very admissible) link $\mathcal{K}$. A branched pro-cover $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$ is a *universal $\mathcal{K}$-branched cover* of M if it satisfies a certain universality: $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$ is a minimal object such that any finite cover of M branched over a finite link L in $\mathcal{K}$ factor through it.

It can be obtained as the inverse limit of a family of universal L-branched covers, as follows: For each finite link L in $\mathcal{K}$, let $h_L : \widetilde{M_L} \to M$ be a universal L-branched cover of M. By the universality, for each $L \subset L'$, we have a unique map $f_{L,L'} : \widetilde{M_{L'}} \to \widetilde{M_L}$ such that $h_{L'} = h_L \circ f_{L,L'}$. Thus $\{h_L\}_{L \subset \mathcal{K}}$ forms an inverse

system. By putting $\widetilde{M_{\mathcal{K}}} = \varprojlim_{L \subset \mathcal{K}} \widetilde{M_L}$, we obtained a universal $\mathcal{K}$-branched cover $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$ as the composite of the natural map $\widetilde{M_{\mathcal{K}}} \to \widetilde{M_L}$ and h_L.

For the universal $\mathcal{K}$-branched cover, the inverse limit $\pi_1(X_{\mathcal{K}})$ of the fundamental groups of exteriors $\pi_1(X_L)$ $(L \subset \mathcal{K})$ acts on it in a natural way. The finite branched covers of M obtained as quotients of $h_{\mathcal{K}}$ by subgroups of $\pi_1(X_{\mathcal{K}})$ form a complete system of representatives of the isomorphism classes of covers of M branched over links in $\mathcal{K}$.

Therefore, in the later section of this thesis, if we take $(M, \mathcal{K})$, we silently fix a universal $\mathcal{K}$-branched cover, call it "the" universal $\mathcal{K}$-branched cover, and restrict our argument to the branched subcovers obtained as its quotients.

Finally, we discuss an analogue of a base point. The following facts explain the role of base points in branched covers:

Proposition 4.3.1. *(1) For $(M, \mathcal{K})$, we fix a universal $\mathcal{K}$-branched cover $h_{\mathcal{K}}$. Then, for a branched cover $h : N \to M$ whose base point is forgotten, taking a branched pro-cover $f : \widetilde{M_{\mathcal{K}}} \to N$ such that $h \circ f = h_{\mathcal{K}}$ is equivalent to fixing a base point in N such that $h(b_N) = b_M$.*
(2) Let $h : N \to M$ be a branched cover. Then, a base point of a universal $\mathcal{K}$-branched cover $h_{\mathcal{K}}$ defines a branched pro-cover $f : \widetilde{M_{\mathcal{K}}} \to N$ such that $h_{\mathcal{K}} = h \circ f$.

An analogue of a base point in a 3-manifold is a geometric point of a number field. Let Ω be a sufficiently large field which includes $\mathbb{Q}$, for instance, $\Omega = \mathbb{C}$. Then, for a number field k, choosing a geometric point $x : \operatorname{Spec} \Omega \to \operatorname{Spec} \mathcal{O}_k$ is equivalent to choosing an inclusion $k \hookrightarrow \Omega$. Moreover, choosing base points in a cover $h : N \to M$ which are compatible with the covering map is an analogue of choosing inclusion $k \subset F \hookrightarrow \Omega$ for an extension F/k. For an algebraic closure $\overline{k}/k$ and an extension F/k of a number field k, we have following facts:

Proposition 4.3.2. *(1) If we fix $\overline{k}/k$ in Ω, taking an inclusion $F \hookrightarrow \overline{k}$ is equivalent to taking an inclusion $F \hookrightarrow \Omega$.*
(2) For an extension F/k in Ω, an inclusion $\overline{k} \hookrightarrow \Omega$ defines $F \hookrightarrow \overline{k}$.

In addition, we have $\operatorname{Spec} \mathcal{O}_k = \{\text{finite primes}\} \cup \operatorname{Spec} k$, and $(\operatorname{Spec} k)(\Omega) = \{\Omega\text{-rational points of } \operatorname{Spec} k\} := \operatorname{Hom}(\operatorname{Spec} \Omega, \operatorname{Spec} k) \cong \operatorname{Hom}(k, \Omega)$. Accordingly, choosing a geometric point (an injection) $k \hookrightarrow \Omega$ is an analogue of choosing a base point in the exterior of $\mathcal{K}$ in M. If $k/\mathbb{Q}$ is Galois, we have a non canonical isomorphism $\{\text{the choices of a geometric point of } k\} = \operatorname{Hom}(k, \Omega) \cong \operatorname{Gal}(k/\mathbb{Q})$. This map depends on the fact that an inclusion of $\mathbb{Q}$ into a field is unique. In order to state an analogue for $(M, \mathcal{K})$, we need to fix an analogue of $k/\mathbb{Q}$. If we fix a Galois branched cover $h_M : M \to S^3$ whose base point is forgotten, an infinite link $\underline{\mathcal{K}}$ in S^3 such that $h^{-1}(\underline{\mathcal{K}}) = \mathcal{K}$, and a base point b_0 in S^3, then we have a non-canonical map $\{\text{the choices of base points in } M\} \cong \operatorname{Gal}(M/S^3)$.

Thereby, we obtained the following dictionary:

3-manifold with very admissible link $(M, \mathcal{K})$	number ring $\operatorname{Spec} \mathcal{O}_k$
universal $\mathcal{K}$-branched cover $h_{\mathcal{K}} : \widetilde{M_{\mathcal{K}}} \to M$	algebraic closure $\overline{k}/k$
base point $b_M : \{\mathrm{pt}\} \hookrightarrow M$	geometric point $x : \operatorname{Spec} \Omega \to \operatorname{Spec} \mathcal{O}_k$

In this thesis, since we consider only regular (Galois) covers, we can forget base points. Then weaker equivalence classes of branched covers should be considered.

4.4. Idèle class group for 3-manifolds

In this section, we develop the idèlic class field theory for 3-manifolds, and present the global reciprocity law over a 3-manifold equipped with a very admissible link.

Let M be a closed, oriented, connected 3-manifold. Let $\mathcal{K}$ be a link in M consisting of countably many tame components with a formal tubular neighborhood $V_{\mathcal{K}} = \sqcup_{K\subset\mathcal{K}} V_K$. For a sublink L of $\mathcal{K}$, we put $V_L = \sqcup_{K\subset L} V_K$.

Definition 4.4.1 (idèle group). We define the idèle group of $(M,\mathcal{K})$ by the restricted product of $H_1(\partial V_K)$ with respect to the subgroups $\{\mu_K\}_{K\subset\mathcal{K}} = \{\mathrm{Ker}(v_K)\}_{K\subset\mathcal{K}}$:

$$I_{M,\mathcal{K}} := \prod_{K\in\mathcal{K}} H_1(\partial V_K) = \Big\{ (a_K)_K \in \prod_K H_1(\partial V_K) \mid v_K(a_K) = 0 \text{ for almost all } K \Big\}.$$

This is the restricted product with respect to the local topology on $H_1(\partial V_K)$ (see later) and the family of open subgroups $\{\mu_K \subset H_1(\partial V_K)\}_K$.

For each finite link L, let $\mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ denote the Galois group of the maximal abelian cover over its exterior X_L. Then $\mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$'s form an inverse system in a natural way. We put $\mathrm{Gal}(M,K)^{\mathrm{ab}} := \varprojlim_L \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ and regard it as an analogue of $\mathrm{Gal}(k^{\mathrm{ab}}/k) = \varprojlim_F \mathrm{Gal}(F/k)$, where F runs finite abelian extensions of k. We have $\mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} \cong H_1(X_{\mathcal{K}})$ by Lemma 4.2.4 and the Hurewicz isomorphism $\pi_1^{\mathrm{ab}}(X_{\mathcal{K}}) \cong H_1(X_{\mathcal{K}})$.

For a finite link L and a knot $K \not\subseteq L$ in $\mathcal{K}$, take an ambient isotopy h fixing K and L so that $h(V_K) \subset X_L$ if needed. Then the composite $\partial V_K \to h(\partial V_K) \to X_L$ with the inclusion induces a natural map $H_1(\partial V_K) \to H_1(X_L)$ commuting with the Hurewicz maps.

$$\begin{tikzcd} H_1(\partial V_L) \arrow[r, "\cong", "\mathrm{Hur}"'] \arrow[d] & \mathrm{Gal}(\widetilde{\partial V_K}/\partial V_K) \arrow[d] \\ H_1(X_L) \arrow[r, "\cong", "\mathrm{Hur}"'] & \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L). \end{tikzcd}$$

Let $\rho_{K,L} : H_1(\partial V_K) \to \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ denote their composite, and we consider the map $\rho_L : I_{M,\mathcal{K}} \to \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L) : (a_K)_K \mapsto \sum_{K\subset\mathcal{K}} \rho_{K,L}(a_K)$ where K runs through all the knots in $\mathcal{K}$. This sum makes sense, because it is actually a finite sum for each $(a_K)_K \in I_{M,\mathcal{K}}$, by the definition of the restricted product. Since $(\rho_L)_L$ is compatible with the inverse system, the following homomorphism is induced

$$\widetilde{\rho}_{M,\mathcal{K}} : I_{M,\mathcal{K}} \longrightarrow \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}.$$

If $\mathcal{K}$ is an admissible link, then this map is surjective.

We give a definition of the principal idèle group and idèle class group by introducing the natural homomorphism $\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$ in the following.

For each finite sublink L of $\mathcal{K}$, let V'_L be a (usual) tubular neighborhood of L and put $X_L^\circ = M - \mathrm{Int}(V'_L)$. The inclusions $(M,L) \to (M,V'_L)$ and $(X_L^\circ, \partial X_L^\circ) \to (M,V'_L)$ induce isomorphisms $H_2(M,L) \cong H_2(M,V'_L) \cong H_2(X_L^\circ, \partial X_L^\circ)$. We denote by ∂_L the homomorphism $H_2(M,L) \to H_1(\partial V_L)$ given as the composite of $\partial_* : H_2(M,L) \cong H_2(X_L^\circ, \partial X_L^\circ) \to H_1(\partial X_L^\circ)$ and a natural isomorphism $H_1(\partial X_L^\circ) = H_1(\partial V'_L) \cong H_1(\partial V_L)$. We also consider the homomorphism $H_1(\partial V_L) \cong H_1(\partial V'_L) \to$

$H_1(X_L)$. For each finite sublinks L and L' of $\mathcal{K}$ with $L \subset L'$, there is a commutative diagran

$$\begin{array}{ccc} H_2(M,L') & \xrightarrow{\partial_{L'}} & H_1(\partial V_{L'}) \\ \uparrow{\scriptstyle j_*} & & \downarrow{\scriptstyle \mathrm{pr}} \\ H_2(M,L) & \xrightarrow{\partial_L} & H_1(\partial V_L). \end{array}$$

where pr denotes the projection to the L-components. Thus, a natural map form $\varinjlim_{L\subset\mathcal{K}} H_2(M,L)$ to $\varprojlim_{L\subset\mathcal{K}} H_1(\partial V_L) = \prod_{K\subset\mathcal{K}} H_1(\partial V_K)$ is induced. Since longitudinal component does not added by j_*, the image of this map is induced in $I_{M,\mathcal{K}}$. Thus, we obtain the natural homomorphism $\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$. If M is a rational homology sphere, then ∂_L is injective for each finite sublink L of $\mathcal{K}$, and so is Δ.

Definition 4.4.2. We define the *principal idèle group* by $P_{M,\mathcal{K}} := \mathrm{Im}(\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}})$, and the *idèle class group* by $C_{M,\mathcal{K}} := I_{M,\mathcal{K}}/P_{M,\mathcal{K}}$.

Theorem 4.4.3. *There is an equality $P_{M,\mathcal{K}} = \mathrm{Ker}(\widetilde{\rho}_{M,\mathcal{K}})$. Furthermore, the homomorphism $\rho_{M,\mathcal{K}}$ induces a natural isomorphism*

$$\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \cong \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}.$$

proof. The assertion $\mathrm{Im}(\Delta) \subset \mathrm{Ker}(\widetilde{\rho}_{M,\mathcal{K}})$ holds in a natural way. Indeed, for any $x \in H_2(M,\mathcal{K})$, there is some $L_0 \subset \mathcal{K}$ and some $x_0 \in H_2(M,L_0)$ such that x is the image of x_0 under the natural injective map $j : H_2(M,L_0) \to H_2(M,\mathcal{K})$. For any finite link L with $L_0 \subset L \subset \mathcal{K}$, there is a commutative diagram

$$\begin{array}{ccccccc} H_2(M,\mathcal{K}) & \xrightarrow{\Delta} & I_{M,\mathcal{K}} & \xrightarrow{\widetilde{\rho}_{M,\mathcal{K}}} & \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} & \longrightarrow & 0 \\ \uparrow{\scriptstyle j} & & \downarrow & & \downarrow & & \\ H_2(M,L) & \xrightarrow{\partial_L} & H_1(\partial V_L) & \longrightarrow & H_1(X_L) & \longrightarrow & 0 \end{array}$$

and the image of x_0 in $H_1(X_L)$ is zero. Thus the image of x in $\mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$ is zero, and $\Delta(x) \in \mathrm{Ker}(\widetilde{\rho}_{M,\mathcal{K}})$ holds.

We prove $\mathrm{Ker}(\widetilde{\rho}_{M,\mathcal{K}}) \subset \mathrm{Im}\,\Delta$ in the following. Let $(a_K)_K \in \mathrm{Ker}(\widetilde{\rho}_{M,\mathcal{K}})$. Then there is a finite sublink $L \subset \mathcal{K}$ such that the longitudinal component of $(a_K)_K$ is zero outside L and that components of L generates $H_1(M)$. Let a denote the image of $(a_K)_K$ in $H_1(\partial V_L)$. The image of a in $H_1(X_L)$ coincides with that of $(a_K)_K$ and hence it is zero. By the exact sequence $H_2(M,L) \to H_1(\partial V_L) \to H_1(X_L) \to 0$, there is some $A \in H_2(M,L)$ with $\partial A = a$. We put $(a'_K)_K = \Delta(j(A))$. Then it is sufficient to prove $(a_K)_K = (a'_K)_K$.

Let L' be any finite link with $L \subset L' \subset \mathcal{K}$, and let b and b' denote the images of $(a_K)_K$ and $(a'_K)_K$ in $H_1(\partial V_{L'})$ respectively. Then it is sufficient to prove $b = b'$. Note that b is the image of A under $H_2(M,L) \to H_2(M,L') \to H_1(\partial V_{L'})$. Now b and b' are both included in $H_1(\partial V_L) \oplus \langle \mu_K \rangle_{K\subset L'-L}$, their images in $H_1(X_{L'})$ are zero, and their images in $H_1(\partial V_L)$ are a.

$$\begin{tikzcd} H_2(M,L') \arrow[r, "\partial_{L'}"] & H_1(\partial V_{L'}) \arrow[r] \arrow[d, "\mathrm{pr}"] & H_1(X_{L'}) \arrow[d, "\iota_*"] \\ H_2(M,L) \arrow[u, "j"] \arrow[r, "\partial_L"] & H_1(\partial V_L) \arrow[r] & H_1(X_L) \end{tikzcd}$$

We put $c := b' - b$. Then we have $c \in \langle \mu_K \rangle_{K \subset L'-L}$. We regard $Z_2(M,L)$ and $Z_2(M,L')$ as subgroups of $C_2(M)$ with $Z_2(M) \subset Z_2(M,L) \subset Z_2(M,L') \subset C_2(M)$, and denote by ∂ the boundary map on $C_*(M)$. Since the image of c in $H_1(X_{L'})$ is zero, there is some $C \in Z_2(M,L')$ with $\partial_{L'}([C]) = c$.

Let $V'_{L'}$ be a tubular neighborhood of L'. Then $\partial_* : H_2(M,L') \to H_1(L')$ factors as $H_2(M,L') \to H_1(\partial V_{L'}) \to H_1(\partial V'_{L'}) \to H_1(V'_{L'}) \to H_1(L')$ with $\langle \mu_K \rangle_{K \subset L'} = \mathrm{Ker}(H_1(\partial V_{L'}) \to H_1(L'))$. Since $\partial_{L'}([C]) \in \langle \mu_K \rangle_{K \subset L'-L}$, we have $\partial_*[C] = 0$, and we can regard $C \in Z_2(M)$.

Let $I : H_2(M) \times H_1(M) \to \mathbb{Z}$ denote the intersection form of M. It is a bilinear form defined by counting the intersection points of transversely intersecting representatives with signs. By the universal coefficient theorem, $H_2(M)$ is torsion-free, and I is right-non-degenerate.

Now $H_1(M)$ is generated by components of L by assumption. Since $\partial_{L'}([C]) \in \langle \mu \rangle_{K \subset L'-L}$, we have $\partial_L([C]) = 0$ by regarding $C \in Z_2(M,L)$, and each component K_i of L satisfies $I([C],[K_i]) = 0$. This implies $[C] = 0$ and hence $c = \partial_{L'}([C]) = 0$. Therefore we have $b = b'$, and $\Delta : H_2(M,\mathcal{K}) \to \mathrm{Ker}\, \widetilde{\rho}_{M,\mathcal{K}}$ is a surjection. □

Theorem 4.4.3 expands the M^{2}KR-dictionary as follows, k is a number field.

idèle group $I_{M,\mathcal{K}}$	idèle group I_k
$\Delta : H_2(M,\mathcal{K}) \to I_{M,\mathcal{K}}$	$\Delta : k^\times \to I_k$
principal idèle group $P_{M,\mathcal{K}} := \mathrm{Im}\,\Delta$	principal idèle group $P_k := \mathrm{Im}\,\Delta$
idèle class group $C_{M,\mathcal{K}} := I_{M,\mathcal{K}}/P_{M,\mathcal{K}}$	idèle class group $C_k := I_k/P_k$

Let $h : N \to M$ be a finite branched cover branched over a finite link L in $\mathcal{K}$. Then the preimage $h^{-1}(\mathcal{K})$ of $\mathcal{K}$ is a link in N, and the covering map h induces the *norm maps* $h_* : I_{N,h^{-1}(\mathcal{K})} \to I_{M,\mathcal{K}}$, $h_* : P_{N,h^{-1}(\mathcal{K})} \to P_{M,\mathcal{K}}$, and $h_* : C_{N,h^{-1}(\mathcal{K})} \to C_{M,\mathcal{K}}$. They satisfy the transitivity (functoriality) in a natural way. If $\mathcal{K}$ is very admissible, then so is $h^{-1}(\mathcal{K})$.

Definition 4.4.4. We define the *unit idèle group* of $(M,\mathcal{K})$ by the meridian group

$$U_{M,\mathcal{K}} := \{(a_K)_K \in I_{M,\mathcal{K}} \mid v_K(a_K) = 0 \text{ in } H_1(V_K), \text{ for all } K \text{ in } \mathcal{K}\},$$

that is, a subgroup of the "infinite linear combinations" $\sum_{K \subset \mathcal{K}} m_K \mu_K$ $(m_K \in \mathbb{Z})$ of the meridians of $\mathcal{K}$ with $\mathbb{Z}$-coefficients.

Proposition 4.4.5. *Let M be a closed, oriented, connected 3-manifold equipped with an admissible link $\mathcal{K}$, and L be a finite link in $\mathcal{K}$. We write $U_{M,\mathcal{K}} = U_L \oplus U_{\mathrm{non}L}$, where U_L is the subgroup generated by the meridians of L, and $U_{\mathrm{non}L} := \mathrm{Ker}(\mathrm{pr}_L : U_{M,\mathcal{K}} \twoheadrightarrow U_L)$. Then there is an isomorphism*

$$I_{M,\mathcal{K}}/(P_{M,\mathcal{K}} + U_{\mathrm{non}L}) \cong H_1(X_L).$$

Especially, if we put $L = \emptyset$, there is an isomorphism

$$I_{M,\mathcal{K}}/(P_{M,\mathcal{K}} + U_{M,\mathcal{K}}) \cong H_1(M).$$

Moreover, if M is an integral homology 3-sphere, there is an isomorphism

$$I_{M,\mathcal{K}} = P_{M,\mathcal{K}} \oplus U_{M,\mathcal{K}}.$$

In the proofs, we abbreviate ${}_{M,\mathcal{K}}$ by ${}_M$, and ${}_{N,h^{-1}(\mathcal{K})}$ by ${}_N$ for simplicity.

proof. For a map $\varphi_L : I_M \to H_1(X_L)$, we prove $\operatorname{Ker}\varphi_L = P_M + U_{\mathrm{non}L}$. Consider the composite $\varphi_L : I_M \twoheadrightarrow I_M/P_M = C_M \cong \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} \cong \varprojlim_{L'} H_1(X_{L'}) \twoheadrightarrow H_1(X_L)$. For each $L \subset L' \subset \mathcal{K}$, it factorizes as $\varphi_L : I_M \underset{\varphi_{L'}}{\twoheadrightarrow} H_1(X_{L'}) \underset{pr}{\twoheadrightarrow} H_1(X_L)$. For the meridian μ_K of K in I_M, the Mayer–Vietoris exact sequence proves the equality $\operatorname{Ker}(\mathrm{pr} : H_1(X_{L'}) \twoheadrightarrow H_1(X_L)) = \langle \varphi_{L'}(\mu_K) \mid K \subset L' - L\rangle$. Hence we have $U_{\mathrm{non}L} \mod P_M \cong \varprojlim_{L'} \langle \varphi_{L'}(\mu_K) \mid K \subset L' - L\rangle = \varprojlim_{L'} \operatorname{Ker}(\mathrm{pr} : H_1(X_{L'}) \twoheadrightarrow H_1(X_L)) \cong \operatorname{Ker}(\varprojlim_{L'} H_1(X_{L'}) \twoheadrightarrow H_1(X_L)) \cong \operatorname{Ker}(I_M/P_M \twoheadrightarrow H_1(X_L))$. Therefore $\operatorname{Ker}\varphi_L = P_M + U_{\mathrm{non}L}$ holds. □

This lemma is an analogue of Proposition 4.1.1.

THEOREM 4.4.6 (The global reciprocity law for 3-manifolds). *Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$. Then, there is a canonical isomorphism called the* global reciprocity map

$$\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \xrightarrow{\cong} \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$$

which satisfies the following properties:
(i) *For any finite abelian cover $h : N \to M$ branched over a finite link L in $\mathcal{K}$, ρ_M induces an isomorphism*

$$C_{M,\mathcal{K}}/h_*(C_{N,h^{-1}(\mathcal{K})}) \cong \mathrm{Gal}(N/M).$$

(ii) *For each knot K in $\mathcal{K}$, there is a commutative diagram:*

$$\begin{tikzcd} H_1(\partial V_K) \arrow[r, "\cong", "\mathrm{Hur}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Gal}(\widetilde{\partial V_K}/\partial V_K) \arrow[d] \\ C_{M,\mathcal{K}} \arrow[r, "\rho_{M,\mathcal{K}}"'] & \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}, \end{tikzcd}$$

where the vertical maps are induced by the natural inclusions.

proof. Since there are isomorphisms

$$C_M/h_*(C_N) \cong (I_M/P_M)/h_*(I_N/P_N) \cong I_M/(P_M + h_*(I_N)),$$

we consider the natural surjection $\varphi' : I_M \underset{\varphi_L}{\twoheadrightarrow} H_1(X_L) \twoheadrightarrow H_1(X_L)/h_*(H_1(Y_L))$. Since $\mathcal{K}$ is very admissible, there is a surjection $I_N \twoheadrightarrow H_1(Y_L)$, and hence a surjection $h_*(I_N) \twoheadrightarrow h_*(H_1(Y_L))$. Then, we have a following commutative diagram.

$$\begin{tikzcd} h_*(I_N) \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & h_*(H_1(Y_L)) \arrow[d, hook] \\ I_M \arrow[r, two heads, "\varphi_L"'] & H_1(X_L) \end{tikzcd}$$

Since $\operatorname{Ker}\varphi_L = P_M + U_{\mathrm{non}L} < P_M + h_*(I_N)$, we have $\operatorname{Ker}\varphi' = \operatorname{Ker}\varphi_L + h_*(I_N) = P_M + h_*(I_N)$, and hence $I_M/(P_M + h_*(I_N)) \cong H_1(X_L)/h_*(H_1(Y_L)) \cong \mathrm{Gal}(N/M)$. □

4.5. The topologies on idèle class group and the existence theorems

In this section, we introduce *the standard topology* and *the norm topology* on the idèle class group, and we show an analogue of the existence theorem in number theory.

4.5.1. The standard topology and the existence theorem. We introduce *the standard topology* on the idèle class group of a 3-manifold, and prove the existence theorem.

Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$. For each group $\pi_1(\partial V_K) \cong H_1(\partial V_K) = \langle \mu_K \rangle \oplus \langle \lambda_K \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ of the boundary of a tubular neighborhood of each knot K in $\mathcal{K}$, we define an analogue of the local topology of $k_{\mathfrak{p}}^{\times}$. Here μ_K and λ_K denote the meridian and the fixed longitude of K respectively. We first consider the *local norm topology* on $H_1(\partial V_K)$, whose open subgroups correspond to the finite abelian covers of ∂V_K. This topology is equal to the *Krull topology*, whose open subgroups are the subgroups of finite indices. Then we consider the relative topology on the local inertia group $\langle \mu_K \rangle < H_1(\partial V_K)$, and re-define the *local topology* on $H_1(\partial V_K)$ as the unique topology such that the inclusion $\iota : \langle \mu_K \rangle \hookrightarrow H_1(\partial V_K)$ is open and continuous. For this topology, under the identification $\mathbb{Z} \cong \langle \mu_K \rangle \hookrightarrow H_1(\partial V_K) = \langle \mu_K \rangle \oplus \langle \lambda_K \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, the open subgroup of $H_1(\partial V_K)$ has the form $\langle (a, 0), (b, c) \rangle$ with some $a, b, c \in \mathbb{Z}$, $a \neq 0$. Then, the local existence theorem is stated as the 1-1 correspondence between the open subgroups of finite indices and the finite abelian covers.

With this local topology, $I_{M,\mathcal{K}}$ is the restricted product with respect to the open subgroups $\langle \mu_K \rangle < H_1(\partial V_K)$, and $I_{M,\mathcal{K}}$ equips the *restricted product topology* as follows. For each finite link L in $\mathcal{K}$, let $G(L) := \prod_{K \subset L} H_1(\partial V_K) \times \prod_{K \not\subset L} \langle \mu_K \rangle$, and consider the product topology on $G(L)$. Then a subgroup $H < I_{M,\mathcal{K}}$ is open if and only if $H \cap G(L)$ is open for every L.

Definition 4.5.1. We endow $C_{M,\mathcal{K}}$ with the quotient topology of the restricted product topology of $I_{M,\mathcal{K}}$ and call it the *standard topology.*

We study a factorization of $I_{M,\mathcal{K}} \twoheadrightarrow C_{M,\mathcal{K}}$ which helps us to deal with open subgroups of $C_{M,\mathcal{K}}$. We fix a finite sublink L_0 of $\mathcal{K}$ whose components generate $H_1(M)$. For each sublink $L \subset \mathcal{K}$, we put $J_L := \prod_{K \subset L_0} H_1(\partial V_K) \times \prod_{K \subset L - L_0} \langle \mu_K \rangle$. Note that $J_{\mathcal{K}} = G(L_0)$ is an open subgroup of $I_{M,\mathcal{K}}$.

For finite sublinks L and L' with $L_0 \subset L \subset L' \subset \mathcal{K}$, the natural maps form the following commutative diagram.

$$\begin{tikzcd} J_{L'} \arrow[r, two heads] \arrow[d, two heads, "\mathrm{pr}"'] \arrow[dr, phantom, "\circlearrowright"] & H_1(X_{L'}) \arrow[d, two heads] \\ J_L \arrow[r, two heads] & H_1(X_L) \end{tikzcd}$$

The natural map $\operatorname{Ker}(J_{L'} \twoheadrightarrow H_1(X_{L'})) \to \operatorname{Ker}(J_L \twoheadrightarrow H_1(X_L))$ is surjective. Indeed, let $x \in \operatorname{Ker}(J_L \twoheadrightarrow H_1(X_L))$ and let x also denote its image by $J_L \hookrightarrow$

$J_L \oplus \prod_{K \subset L'-L} \langle \mu_K \rangle = J_{L'}; x \mapsto x+0$. Since $\mathrm{Ker}(H_1(X_{L'}) \twoheadrightarrow H_1(X_L))$ is generated by the meridians of $L'-L$, there is some $a \in \prod_{K \subset L'-L} \langle \mu_K \rangle$ such that the images of x and a in $H_1(X_{L'})$ coincide. If we put $y = x - a$, then $y \in \mathrm{Ker}(J_{L'} \twoheadrightarrow H_1(X_{L'}))$ and its image in J_L is x. Since $\{\mathrm{Ker}(J_L \twoheadrightarrow H_1(X_L))\}_L$ forms a surjective system with respect to the natural maps and satisfies the Mittag-Leffler condition, we have a natural surjection $J_{\mathcal{K}} = \varprojlim_{\mathrm{pr},L} J_L \twoheadrightarrow C_{M,\mathcal{K}}$.

For each knot K' in $\mathcal{K}$ with $K' \not\subset L_0$, we take an element $x_{K'} \in J_{\mathcal{K}}$ satisfying $\lambda_{K'} - x_{K'} \in P_{M,\mathcal{K}} = \ker \rho_{M,\mathcal{K}}$. Put $Q := \langle \lambda_{K'} - x_{K'} \mid K' \not\subset L_0 \rangle < I_{M,\mathcal{K}}$. Then $J_{\mathcal{K}} \hookrightarrow I_{M,\mathcal{K}} \twoheadrightarrow C_{M,\mathcal{K}}$ factors through $I' := I_{M,\mathcal{K}}/Q \cong (\prod_{K \subset \mathcal{K}} \mathbb{Z}) \times (\prod_{K \subset L_0} \mathbb{Z})$.

Let I' be endowed with the quotient topology of the standard topology of $I_{M,\mathcal{K}}$. Since $J_{\mathcal{K}}$ is open, the induced group isomorphism $J_{\mathcal{K}} \xrightarrow{\cong} I'$ is a homeomorphism.

Proposition 4.5.2. *Let $C_{M,\mathcal{K}}$ be endowed with the standard topology. If M is a rational homology 3-sphere, then an open subgroup of $C_{M,\mathcal{K}}$ is of finite index.*

proof. Put $P' = \mathrm{Ker}(I' \twoheadrightarrow C_{M,\mathcal{K}})$. Then we have $I'/(\prod_K \langle \mu_K \rangle + P') \cong H_1(M)$. The assumption on M means that $H_1(M)$ is a finite group, and hence $\prod_K \langle \mu_K \rangle + P' < I'$ is of finite index. Recall $G(L_0) = J_{\mathcal{K}} \cong I'$ as topological groups. If V is an open subgroup of I', then $V \cap \prod_K \langle \mu_K \rangle < \prod_K \langle \mu_K \rangle$ is of finite index. Let U be an open subgroup of $C_{M,\mathcal{K}}$ and let V denote the preimage of U in I'. Then V is an open subgroup of I' containing P'. Therefore $V < I'$ is of finite index, and so is $U < C_{M,\mathcal{K}}$. □

Theorem 4.5.3 (The existence theorem 1/2). *Let $C_{M,\mathcal{K}}$ be endowed with the standard topology. Then the correspondence*

$$(h : N \to M) \mapsto h_*(C_{N,h^{-1}(\mathcal{K})})$$

gives a bijection between the set of (isomorphism classes of) finite abelian covers of M branched over finite links L in $\mathcal{K}$ and the set of open subgroups of finite indices in $C_{M,\mathcal{K}}$ with respect to the standard topology.

proof. For each finite link L with $L_0 \subset L \subset \mathcal{K}$, let Cov_L denote the set of finite abelian covers $h : N \to M$ branched over sublinks of L, and let $\mathcal{O}_L$ denote the set of open subgroups of finite indices in $C_{M,\mathcal{K}}$ containing $\mathrm{Ker}(C_{M,\mathcal{K}} \twoheadrightarrow H_1(X_L))$.

Let U be an open subgroup of $C_{M,\mathcal{K}}$ of finite index and let V denote the preimage of U by $I' \twoheadrightarrow C_{M,\mathcal{K}}$. Since V is an open subgroup of I' of finite index, there is some finite link L with $L_0 \subset L \subset \mathcal{K}$ such that V contains a subgroup $\prod_{K \not\subset L} \langle \mu_K \rangle \times \prod_{K \subset L} a_K \langle \mu_K \rangle$ ($a_K \in \mathbb{N}$) and hence contains $\prod_{K \not\subset L} \langle \mu_K \rangle \times \prod_{K \subset L} 0$. Therefore, U contains the image of $\prod_{K \not\subset L} \langle \mu_K \rangle \times \prod_{K \subset L} 0$, which coincides with $\mathrm{Ker}(C_{M,\mathcal{K}} \twoheadrightarrow H_1(X_L))$. Thus the union $\cup_L \mathcal{O}_L$ coincides with the set of all the open subgroups of finite indeces in $C_{M,\mathcal{K}}$.

Conversely, if U is a subgroup of $C_{M,\mathcal{K}}$ of finite index containing $\mathrm{Ker}(C_{M,\mathcal{K}} \twoheadrightarrow H_1(X_L))$ for a finite link L with $L_0 \subset L \subset \mathcal{K}$, then U is open.

For each finite link L with $L_0 \subset L \subset \mathcal{K}$, we have a natural bijection $\mathrm{Cov}_L \to \mathcal{O}_L$ by the Galois correspondence. In addition, for each finite links L and L' with $L_0 \subset L \subset L' \subset \mathcal{K}$, the inclusions $\mathrm{Cov}_L \subset \mathrm{Cov}_{L'}$ and $\mathcal{O}_L \subset \mathcal{O}_{L'}$ are compatible with the Galois correspondences.

The union $\cup_L \mathrm{Cov}_L$ is the set of all the finite abelian covers branched over finite sublinks of $\mathcal{K}$. Since the inductive limit of bijective maps is again bijective, we obtain the desired bijection. □

4.5.2. The norm topology and the existence theorem. We introduce *the norm topology* on the idèle class group, and present the existence theorem.

Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$ as before. In the proofs, we use the abbreviations $C_M = C_{M,\mathcal{K}}$ and $C_N = C_{N,h^{-1}(\mathcal{K})}$ for a branched cover $h : N \to M$.

Definition 4.5.4. We define the *norm topology* on C_M to be the topology of topological group generated by the family $\mathcal{V} := \{h_*(C_{N,h^{-1}(\mathcal{K})})\}$, where $h : N \to M$ runs through all the finite abelian covers of M branched over finite links in $\mathcal{K}$.

Lemma 4.5.5. *$\mathcal{V}$ is a fundamental system of neighborhoods of 0.*

proof. For any $V_1, V_2 \in \mathcal{V}$, it is suffice to prove $\exists V_3 \in \mathcal{V}$ such that $V_3 \subset V_1 \cap V_2$. However, we prove $V_3 := V_1 \cap V_2 \in \mathcal{V}$.

Let $h_i : N_i \to M$ be a finite abelian cover branched over L_i in $\mathcal{K}$ for $i = 1, 2$. Let $L := L_1 \cup L_2$, and let $G_L := \mathrm{Gal}(X_L^{\mathrm{ab}}/X_L)$ denote the Galois group of the maximal abelian cover over the exterior $X_L = M - \mathrm{Int}(V_L)$. Then, if a cover $h : N \to M$ is unbranched outside L, the map $C_M \twoheadrightarrow \mathrm{Gal}(N/M)$ factors through the natural map $\varphi_L : C_M \twoheadrightarrow G_L$.

Let $G_i := \mathrm{Ker}(G_L \twoheadrightarrow \mathrm{Gal}(N_i/M)) < G_L$ for $i = 1, 2$, and let $G_3 := G_1 \cap G_2$. Since G_3 is also a subgroup of G_L of finite index, the ordinary Galois theory for branched covers gives a cover $h_3 : N_3 \to M$ such that $G_3 = \mathrm{Ker}(G_L \twoheadrightarrow \mathrm{Gal}(N_3/M))$. (This cover h_3 should be called the "composition cover" of h_1 and h_2, because it is an analogue of the composition field k_1k_2 of k_1 and k_2 in number theory.)

Now, Theorem 4.4.6 (the global reciprocity law) implies $h_{i*}(C_{N_i}) = \varphi_L^{-1}(G_i)$ for $i = 1, 2, 3$, and therefore $h_{3*}(C_{N_3}) = \varphi_L^{-1}(G_3) = \varphi_L^{-1}(G_1 \cap G_2) = \varphi_L^{-1}(G_1) \cap \varphi_L^{-1}(G_2) = h_{1*}(C_{N_1}) \cap h_{2*}(C_{N_2})$. □

Proposition 4.5.6. *Let $C_{M,\mathcal{K}}$ be endowed with the norm topology. A subgroup V of $C_{M,\mathcal{K}}$ is open if and only if it is closed and of finite index.*

proof. Let V be an open subgroup of C_M. The coset decomposition of C_M by V proves that V is closed. Lemma 4.5.5 gives a finite abelian branched cover $h : N \to M$ such that $h_*(C_N) < V$. Then Theorem 4.4.6 implies $(h_*(C_N) : V)(V : C_M) = (h_*(C_N) : C_M) = \#\,\mathrm{Gal}(N/M)$, and hence V is of finite index.

The converse is also clear by the coset decomposition. □

Now we present the existence theorem for 3-manifolds with respect to both the standard topology and the norm topology, which is the counter part of **Theorem 4.1.3** (2).

Theorem 4.5.7 (The existence theorem). *Let M be a closed, oriented, connected 3-manifold equipped with a very admissible link $\mathcal{K}$. Then the correspondence*

$$(h : N \to M) \mapsto h_*(C_{N,h^{-1}(\mathcal{K})})$$

gives a bijection between the set of (isomorphism classes of) finite abelian covers of M branched over finite links L in $\mathcal{K}$ and the set of open subgroups of finite indices in $C_{M,\mathcal{K}}$ with respect to the standard topology. Moreover, the latter set coincides with the set of open subgroups of $C_{M,\mathcal{K}}$ with respect to the norm topology.

proof. The former part is done by Theorem 4.5.3. We prove the theorem for the norm topology. For a finite abelian cover $h : N \to M$ branched over a finite link in $\mathcal{K}$, the isomorphism $C_M/h_*(C_N) \cong \mathrm{Gal}(N/M)$ in Theorem 4.4.6 (the global reciprocity law) gives the following bijections.

$$\{C' \mid h_*(C_N) < C' < C_M\} \longleftrightarrow \{H \mid H < C_M/h_*(C_N) \cong \mathrm{Gal}(N/M)\}$$
$$\longleftrightarrow \{\text{subcovers of } h\}$$

(Injectivity) For covers h_1 and h_2, this bijections proves that $h_{1*}(C_{N_1}) < h_{2*}(C_{N_2}) \underset{\text{iff}}{\Longleftrightarrow} h_2$ is a subcover of h_1, and hence $h_{1*}(C_{N_1}) = h_{2*}(C_{N_2}) \underset{\text{iff}}{\Longleftrightarrow} h_2 = h_1$.
(Surjectivity) For an open subgroup $C' < C_M$, Lemma 4.5.5 gives a cover $h : N \to M$ such that $h_*(C_N) < C'$, and then the above bijection gives a cover h' which corresponds to C'. □

Corollary 4.5.8. *If M is a rational homology 3-sphere, the standard topology and the norm topology on $C_{M,\mathcal{K}}$ coincide.*

proof. By Proposition 4.5.2, it follows immediately from the existence theorem. □

4.6. The norm residue symbols

In this section, we introduce the norm residue symbol for 3-manifolds, as an analogue of the norm residue symbol for number fields. We also explain that they generalize the linking number $\mathrm{lk}(K_1, K_2)$ and the Legendre symbol $\left(\frac{p}{q}\right)$.

Definition 4.6.1. Let M be a 3-manifold equipped with a very admissible link $\mathcal{K}$. For a finite abelian cover $h : N \to M$ branched over a finite link L in $\mathcal{K}$, the *norm residue symbol* $(\ ,h) : C_{M,\mathcal{K}} \twoheadrightarrow \mathrm{Gal}(N/M)$ is defined as the composite of $\rho_{M,\mathcal{K}} : C_{M,\mathcal{K}} \twoheadrightarrow \mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}}$ and $\mathrm{Gal}(M,\mathcal{K})^{\mathrm{ab}} \twoheadrightarrow \mathrm{Gal}(N/M)$. For this map, we have $\mathrm{Ker}(\ ,h) = h_*(C_{N,h^{-1}(\mathcal{K})})$.

The relation with the linking number can be seen as follows: Let $h_2 : N \to M$ be the double cover of $M = S^3$ branched over a knot K_2 in a two component link $K_1 \sqcup K_2$. We identify $\mathrm{Gal}(N/M) \cong \mathbb{Z}/2\mathbb{Z}$. Then, for a longitude λ_1 of K_1 in $C_{M,\mathcal{K}}$, we have $(\lambda_1, h_2) = \mathrm{lk}(K_1, K_2) \pmod 2$. Moreover, there are the following equivalences:

$$(\lambda_1, h_2) = 0 \underset{\text{iff}}{\Longleftrightarrow} h^{-1}(K_1) = K_1' \sqcup K_1'' \text{ with knots } K_1', K_1'' \text{ in } N \text{ (decomposed)},$$
$$(\lambda_1, h_2) = 1 \underset{\text{iff}}{\Longleftrightarrow} h^{-1}(K_1) = \widetilde{K_1} \text{ is a knot in } N \text{ (inert)}.$$

Thus, we have obtained an extension of the dictionary of analogies.

linking number $\mathrm{lk}(K_1, K_2)$ (mod 2)	Legendre symbol $\left(\frac{p}{q}\right)$
norm residue symbol $(\ ,h)$	norm residue symbol $(\ ,F/k)$

Let p and q be distinct odd primes and $q^* := (-1)^{\frac{q-1}{2}} q$. Then the quadratic reciprocity law $\left(\frac{q^*}{p}\right) = \left(\frac{p}{q}\right)$ follows from Artin's global reciprocity law (Theorem 4.1.3) (See [**KKS11**] Chapter 5). Similarly, for knots K_1 and K_2 in S^3, we can

give an alternative proof of $\mathrm{lk}(K_1, K_2) \equiv \mathrm{lk}(K_2, K_1) \mod 2$ by using our global reciprocity law (Theorem 4.4.6). This fact extends an analogy described in [**Mor12**] Chapters 4 and 5.

In the proof of the existence theorem for number fields (Theorem 4.1.4), the norm residue symbol plays an essential role ([**Neu99**]). By using the norm residue symbol for 3-manifolds, we can also give a parallel proof for the existence theorem for 3-manifolds (Theorem 4.5.7), although it becomes a little more complicated-looking than our proof in this thesis.

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