# 整数値指標である有限群の表現の交代テンソル積表現の研究

田村, 朋之

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# Study of exterior powers of representations of finite groups with integer-valued characters

Tomoyuki Tamura
Graduate school of Mathematics
Kyushu University
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Four years have passed since I stared my research about relations between representations of groups and of $\lambda$-rings. Initially, I had not understood anything about $\lambda$-rings. I was worried about what kind of result could be obtained.

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# Contents

# Preface

For a representation $\rho : G \to GL(V)$ of a group $G$ on a finite dimensional complex vector space $V$, we define the $i$-th exterior power of the representation $\rho$ by

$$\Lambda^i \rho : G \to GL(\bigwedge^i(V)),$$
$$\Lambda^i \rho(g)(v_1 \wedge \cdots \wedge v_i) := (\rho(g)v_1) \wedge \cdots \wedge (\rho(g)v_i)$$

for any integer $i \geq 0$, where $g \in G$ and $v_1, \ldots, v_i \in V$.

This paper consists of part I and part II, and both parts are motivated by the common problem to find reasonable algorithms for calculating the exterior powers of representations for a finite group $G$.

This problem was raised by Knutson [Knu]. Knutson discovered the method using $\lambda$-rings to calculate the character of $\Lambda^i \rho$ (For $\lambda$-rings, see §2.2, [Knu] or [Yau]). This method is as follows: Let $CF(G)$ be the set of all class functions $f : G \to \mathbb{C}$, which satisfies $f(g^{-1}xg) = f(x)$ for any $x, g \in G$. The set $CF(G)$ is a commutative $\mathbb{C}$-algebra, in particular, is a commutative $\mathbb{Q}$-algebra. Moreover, the set $CF(G)$ has a $\lambda$-ring structure where the $n$-th Adams operation $\psi^n$ satisfies $\psi^n(f)(g) := f(g^n)$ for any $f \in CF(G)$, $g \in G$ and integer $n \geq 1$. Knutson [Knu, p.84] showed that the character of $\Lambda^i \rho$ equals $\lambda^i(\chi)$ for any integer $i \geq 0$ where $\chi$ is the character of a representation $\rho$ and $\lambda^i$ is a $\lambda$-operation of $CF(G)$.

We note the relation between this result in [Knu] and representation rings and previous studies of representation rings. Let $R(G)$ be a representation ring of $G$, which has a pre-$\lambda$-ring structure whose $\lambda$-operation is defined via exterior powers of representations. The map $X : R(G) \to CF(G)$ defined by that $X([V])$ is the character of $V$, is an injective ring homomorphism where $[V]$ is an isomorphism class of a representation $V$. From this result in [Knu], the map $X$ can preserve both of $\lambda$-operations. Thus, the study of a character of the exterior powers of representations is reduced to the study of the $\lambda$-ring structure on representation rings.

We recall previous studies on a representation ring of a group as a $\lambda$-ring after [Knu]. For a set of generators of the representation ring as a $\lambda$-ring, Boorman [Boo] or Bryden [Bry] found the case of symmetric groups or Weyl groups associated with the root systems of type $B_n$ and $D_n$ in 1975 or 1995, respectively. For continuous groups, Osse [Oss] characterized a representation ring of a compact connected Lie group in terms of a $\lambda$-ring. Yau [Yau] arises the problem to characterize a representation ring of a finite group in terms of a $\lambda$-ring.

Because the set of all virtual characters is isomorphic to a representation ring as a $\lambda$-ring, the study of the $\lambda$-ring structure of representation rings is equivalent to the study of the $\lambda$-ring structure of the set of virtual characters. In this paper, we will focus on virtual characters, not representation rings.

In part I, we will focus on a representation with an integer-valued character, which satisfies that $\chi(g)$ is an integer for any $g \in G$, and will discuss relations among integer-valued characters, $\lambda$-operations and truncated operations on necklace rings. The notion of necklace rings was introduced by N.Metropolis and G.-C.Rota [MR]. We will introduce necklace rings in §2.1, and truncated operations in §3.2. The discussion in part I is based on [Knu]. For more details, see §1.

In part II, we will focus on not representations but actions. We will define "$|A|$-colored $N$-nested $G$-set" and use it for calculation of $\lambda^i(\chi)$, where $\chi$ is a permutation character. Moreover, we will calculate the number of primitive colorings on some objects on polyhedrons as an application. For more details, see §5.

Here, we use the following notations and definitions in this paper.

(i) Let $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ be the set of all positive integers, integers or rational integers, respectively.

(ii) For any integers $i$ and $j$, a symbol $i \mid j$ stands for that $i$ divides $j$ and $i \nmid j$ stands for that $i$ does not divide $j$.

(iii) For any integers $i$ and $j$, a symbol $(i, j)$ stands for the greatest common divisor of $i$ and $j$, and a symbol $[i, j]$ stands for the least common multiple of $i$ and $j$.

(iv) We denote the Möbius function by $\mu$.

(v) For any set $X$, we denote the identity map on $X$ by $\mathrm{id}_X$ defined by $\mathrm{id}_X(x) = x$ for any $x \in X$.

(vi) Let $\mathbb{C}$ be the complex field, and let $M(k, \mathbb{C})$ be the set of all complex matrices whose size is $k \times k$.

(vii) We denote the unit element of a finite group by 1.

(viii) For each element $g$ of a finite group $G$, we denote by $\langle g \rangle$ the subgroup of $G$ generated by $g \in G$, and by $O(g)$ the order of $g$. We define the exponent of $G$ by the least common multiple of all $O(g)$'s for $g \in G$.

(ix) We assume that every ring and semiring have the unit element, written by 1.

# Part I

# Exterior powers of representations of finite groups and integer-valued characters

## 1 Introduction

Let $G$ be a finite group. For any character $\chi$ of a representation $\rho : G \to GL(V)$, Knutson showed that $\lambda^i(\chi)$ is the character of the $i$-th exterior power of $\rho$. We define a power series $\lambda_t(\chi)(g)$ by

$$\lambda_t(\chi)(g) := \sum_{i=0}^{\infty} \lambda^i(\chi)(g)t^i \tag{1.1}$$

for any $g \in G$. It has the following form:

$$\lambda_t(\chi)(g) = \det(I_m + \rho(g)t) = \exp\Big(\sum_{i=1}^{\infty} -\frac{\chi(g^i)}{i}(-t)^i\Big)$$

where $I_m$ is the unit matrix whose size is $m = \deg(\rho)$.

**Example 1.1.** Let $X$ be a finite $G$-set and let $\chi$ be the permutation character associated with $X$. It is known that $\chi(g) = |\mathrm{Fix}(g)|$ holds where

$$\mathrm{Fix}(g) := \{x \in X \mid gx = x\}$$

for any $g \in G$. Then we have

$$\lambda_t(\chi)(g) = \exp\Big(\sum_{i=1}^{\infty} -\frac{|\mathrm{Fix}(g^i)|}{i}(-t)^i\Big).$$

In this part, we discuss relations between integer-valued characters of finite groups and necklace rings via (1.1). An integer-valued character is a character $\chi$ such that $\chi(g) \in \mathbb{Z}$ holds for any $g \in G$.

The necklace ring over a commutative ring $R$ was introduced by Metropolis and Rota in 1983 to investigate the notion of Witt rings as a commutative ring. For any commutative ring $R$, the set of all infinite vectors

$$a = (a_1, a_2, a_3, \ldots), \quad a_i \in R$$

has a commutative ring structure whose addition "$+_{Nr}$" is defined by componentwise and multiplication $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) := x \cdot_{Nr} y$ is defined by

$$\alpha_n := \sum_{[i,j]=n} (i,j)x_i y_j \tag{1.2}$$

where $x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots) \in R^{\mathbb{N}}$. With definitions of the addition "$+_{Nr}$" and the multiplication "$\cdot_{Nr}$", we call the set of all infinite vectors the necklace ring over $R$, written by $Nr(R)$.

We will relate characters of $G$ and elements of necklace rings as follows: Let $\Lambda(R) = 1 + tR[[t]]$ be a universal $\lambda$-ring with an indeterminate variable $t$. Knutson [Knu] showed that $\Lambda(R)$ has a $\lambda$-ring structure where the addition "$+_\Lambda$" is defined by the multiplication of $R[[t]]$. If $R$ is a binomial ring, which is a $\mathbb{Z}$-torsion free $\lambda$-ring whose all Adams operations of $R$ are identity maps, then for each $f \in \Lambda(R)$ there exists a unique element $(a_1, a_2, a_3, \ldots) \in Nr(R)$ such that

$$f(t) = \prod_{i=1}^{\infty} (1 - (-t)^i)^{a_i}. \tag{1.3}$$

holds. Yau [Yau, §5.6] showed that the map $E_{Nr} : \Lambda(R) \to Nr(R)$ defined by $E_{Nr}(f) := (a_1, a_2, a_3, \ldots)$ is a ring isomorphism.

For any finite group $G$, let $CF(G)$ be the set of all class functions from $G$ to $\mathbb{C}$. The set $CF(G)$ also has a binomial ring structure. When $f(t)$ is the generating function $\lambda_t(\chi) \in \Lambda(CF(G))$ in (1.3), it is sufficient to calculate $E_{Nr}(\lambda_t(\chi))$ in order to calculate $\lambda^i(\chi)$. We will relate characters of $G$ and elements of necklace rings using the map $E_{Nr} \circ \lambda_t$.

Now, we outline this part. In §2, we will state basic definitions of necklace rings, $\lambda$-rings, representations of finite groups and the relation among them.

In §3, we will show main theorems of this part. Let $e$ be the exponent of $G$. Then, a character $\chi$ is an integer-valued character if and only if $a_n = 0$ for any integer $n \geq 1$ with $n \nmid e$ where $(a_1, a_2, a_3, \ldots) = E_{Nr}(\lambda_t(\chi))$ (Theorem 3.20).

Next, we will discuss the support of an element $(a_1, a_2, a_3, \ldots) \in Nr(R)$. We will say that $a = (a_1, a_2, a_3, \ldots)$ has the finite support if the number of integers $n \geq 1$ such that $a_n \neq 0$ is finite. By Theorem 3.20, the element $E_{Nr}(\lambda_t(\chi))$ has the finite support for any integer-valued character $\chi$. We will show that if $E_{Nr}(\lambda_t(\chi))$ has the finite support, then the character $\chi$ is an integer-valued character by Theorem 3.26 from an arbitrary $\mathbb{Z}$-torsion free commutative ring.

Finally, we will discuss the case of a product group of two finite groups $G_1$ and $G_2$. Let $\chi_1$ be a virtual character of $G_1$ and let $\chi_2$ be a virtual character of $G_2$. Then we define a virtual character $\chi_1\chi_2$ of $G_1 \times G_2$ by $\chi_1\chi_2((g_1, g_2)) := \chi_1(g_1)\chi_2(g_2)$ for any $g_1 \in G_1$ and $g_2 \in G_2$. We will show that

$$\lambda_t(\chi_1\chi_2)((g_1, g_2)) = \lambda_t(\chi_1)(g_1) \cdot_\Lambda \lambda_t(\chi_2)(g_2) \tag{1.4}$$

holds in Theorem 3.34, where "$\cdot_\Lambda$" is the multiplication of the universal $\lambda$-ring $\Lambda\mathbb{C}$.

In addition, we study an equation obtained from (1.4) using the map $E_{Nr}$. Under the assumption that characters $\chi_1$ and $\chi_2$ are integer-valued characters, we calculate the multiplication of two elements $E_{Nr}(\lambda_t(\chi_1))$ and $E_{Nr}(\lambda_t(\chi_2))$ in order to calculate the left side of (1.4). Note that $E_{Nr}(\lambda_t(\chi_1))$ and $E_{Nr}(\lambda_t(\chi_2))$ have the finite support. Moreover, we will give another form of the multiplication of two elements of necklace rings which have finite support with Frobenius operations of necklace rings.

In §4, we consider the character $\chi$ of a representation of $S_n$ which will be defined with an $R$-matrix, and the calculation method of $E_{Nr}(\lambda_t(\chi))$ as an example.

# 2 Preliminaries

In this section, we will define some notations to state main results of this part in §3. In
S 2.1, we will discuss necklace rings and operations. In §2.2, we will state definitions and properties of pre-$\lambda$-rings and $\lambda$-rings. In §2.3, we will discuss a relation between a character of exterior powers of representations of finite groups and $\lambda$-rings.

## 2.1 Necklace rings and operations

Main references of §2.1 are [MR] and [VW]. Given a commutative ring $R$, let $R^{\mathbb{N}}$ be the set of all infinite vectors whose elements belong to $R$. The set $R^{\mathbb{N}}$ has a commutative ring structure whose addition and multiplication are defined by componentwise. The zero element of $R^{\mathbb{N}}$ is $(0, 0, 0, \ldots)$ and the unit element of $R^{\mathbb{N}}$ is $(1, 1, 1, \ldots, )$. In addition, we define another commutative ring structure on $R^{\mathbb{N}}$. Its addition "$+_{Nr}$" is defined by componetwise, and multiplication "$\cdot_{Nr}$" by

$$\alpha_n := \sum_{[i,j]=n} (i, j) x_i y_j$$

where $x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots) \in R^{\mathbb{N}}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = x \cdot_{Nr} y$. If we regard $R^{\mathbb{N}}$ as a commutative ring with such operations, then we call $R^{\mathbb{N}}$ the necklace ring over $R$, written by $Nr(R)$. The zero element of $Nr(R)$ is $(0, 0, 0, \ldots)$ and the unit element of $Nr(R)$ is $(1, 0, 0, \ldots)$.

Next, we define a map $\phi : Nr(R) \to R^{\mathbb{N}}$ by $\phi(x) := (a_1, a_2, a_3, \ldots)$ where

$$a_n := \sum_{d|n} d x_d, \quad x = (x_1, x_2, x_3, \ldots)$$

for any $x \in Nr(R)$ and integer $n \geq 1$.

**Proposition 2.1.** *The map $\phi$ is a ring homomorphism. If $R$ is $\mathbb{Z}$ torsion-free, then the map $\phi$ is injective. If $R$ is a $\mathbb{Q}$-algebra, then the map $\phi$ is bijective.*

*Proof.* The fact that $\phi$ is a ring homomorphism was proved in [Yau, Lemma 5.4].

We assume that $R$ is $\mathbb{Z}$-torsion free, and prove that the map $\phi$ is injective. Let $x = (x_1, x_2, x_3, \ldots)$ be an element of $Nr(R)$ and assume $(a_1, a_2, a_3, \ldots) = \phi(x) = 0$. We prove $x_n = 0$ by induction on $n \geq 1$. By the definition of the map $\phi$, we have $x_1 = a_1 = 0$. Let $n \geq 2$ be an integer and we assume that $x_k = 0$

holds for any integer $k$ with $k < n$. We use the definition of the map $\phi$ again, so we have

$$na_n = -\sum_{d|n, d\neq n} dx_d = 0.$$

Since $R$ is $\mathbb{Z}$-torsion free, we have $x_n = 0$. Hence, we have $x = 0$.

Next, we assume that $R$ is a $\mathbb{Q}$-algebra and prove that the map $\phi$ is bijective. It is enough to prove that the map $\phi$ is surjective, because a $\mathbb{Q}$-algebra commutative ring is $\mathbb{Z}$-torsion free. For any $a = (a_1, a_2, a_3, \ldots) \in R^{\mathbb{N}}$, put $x = (x_1, x_2, x_3, \ldots) \in Nr(R)$ satisfying that

$$x_n = \frac{1}{n}\sum_{d|n} \mu\Big(\frac{n}{d}\Big) a_d$$

for any integer $n \geq 1$. We have $\phi(x) = a$ by the Möbius inversion formula, that is, the map $\phi$ is surjective. □

For any integer $r \geq 1$, we define maps

$$\begin{aligned} V_r &: Nr(R) \to Nr(R), \\ V_r &: R^{\mathbb{N}} \to R^{\mathbb{N}}, \\ F_r &: Nr(R) \to Nr(R), \\ F_r &: R^{\mathbb{N}} \to R^{\mathbb{N}} \end{aligned}$$

as follows: Let $x = (x_1, x_2, x_3, \ldots) \in Nr(R)$. Then $(y_1, y_2, y_3, \ldots) = V_r(x)$ and $(z_1, z_2, z_3, \ldots) = F_r(x)$ satisfy

$$\begin{aligned} y_n &:= \begin{cases} x_{n/r} & \text{if } r \mid n, \\ 0 & \text{if } r \nmid n, \end{cases} \\ z_n &:= \sum_{[j,r]=nr} \frac{j}{n} x_j \end{aligned}$$

for any integer $n \geq 1$.

Let $a = (a_1, a_2, a_3, \ldots) \in R^{\mathbb{N}}$. Then $(b_1, b_2, b_3, \ldots) = V_r(a)$ and
$(c_1, c_2, c_3, \ldots) = F_r(a)$ satisfy

$$\begin{aligned} b_n &:= \begin{cases} ra_{n/r} & \text{if } r \mid n, \\ 0 & \text{if } r \nmid n, \end{cases} \\ c_n &:= a_{nr} \end{aligned}$$

for any integer $n \geq 1$.

In this paper, we call $V_r$ the $r$-th Verschiebung operation, and $F_r$ the $r$-th Frobenius operation, respectively [MR] and [VW].

**Proposition 2.2.** *The operations $V_r$ and $F_s$ satisfy the following relations.*

(1) $V_r \circ V_s = V_{rs}$,

(2) $F_r \circ F_s = F_{rs}$,

(3) $F_r \circ V_r = r\mathrm{id}_R$,

(4) $F_r \circ V_s = V_s \circ F_r$ if $(r, s) = 1$,

(5) $F_r \circ V_s = (r, s)F_{r/(r,s)} \circ V_{s/(r,s)}$.

*Proof.* For necklace ring $Nr(R)$, identities (1), (2) and (3) were proved in [MR, Theorem 1], (4) was proved in [VW, Theorem 5.4] and (5) was proved in [VW, Proposition 5.5]. So, we consider this proposition in the case of $R^{\mathbb{N}}$. Let $a = (a_1, a_2, a_3, \ldots)$ be an element of $R^{\mathbb{N}}$ and let $n \geq 1$ be an integer.

(1) Let $b = (b_1, b_2, b_3, \ldots) = V_r(a)$ and $c = (c_1, c_2, c_3, \ldots) = V_s(b)$. If $rs \mid n$, then $c_n = b_{n/s} = a_{n/rs}$ holds. If $rs \nmid n$, then we have $c_n = 0$. Hence, we have $V_r \circ V_s = V_{rs}$.

(2) Let $b = (b_1, b_2, b_3, \ldots) = F_r(a)$ and $c = (c_1, c_2, c_3, \ldots) = F_s(b)$. Then, we have $c_n = b_{sn} = a_{rsn}$, that is, we have $F_r \circ F_s = F_{rs}$.

(3) Let $b = (b_1, b_2, b_3, \ldots) = V_r(a)$ and $c = (c_1, c_2, c_3, \ldots) = F_r(b)$. Then, we have $c_n = b_{rn} = ra_n$, that is, we have $F_r \circ V_r = r\mathrm{id}_R$.

(4) Let $b = (b_1, b_2, b_3, \ldots) = V_s(a)$, $c = (c_1, c_2, c_3, \ldots) = F_r(b)$, $e = (e_1, e_2, e_3, \ldots) = F_r(a)$ and $f = (f_1, f_2, f_3, \ldots) = V_s(e)$. We prove $c_n = f_n$. If $n \mid s$, then we have $c_n = b_{nr} = sa_{nr/s} = sd_{n/s} = f_n$. If $n \nmid s$, we have $c_n = f_n = 0$. In any case, we have $c_n = f_n$ for any integer $n \geq 1$, that is, we have $F_r \circ V_s = V_s \circ F_r$.

(5) We have $F_r \circ V_s = F_{r/(r,s)} \circ F_{(r,s)} \circ V_{(r,s)} \circ V_{s/(r,s)} = (r, s)F_{r/(r,s)} \circ V_{s/(r,s)}$ by identities (1), (2) and (3).

□

**Proposition 2.3.** *The operation $F_r$ is a ring homomorphism for any integer $r \geq 1$.*

*Proof.* First, we prove this proposition in the case of $Nr(R)$. Obviously, the operation $F_r$ is an additive homomorphism. Thus, we consider the multiplication. Let $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ be elements of $Nr(R)$. Put $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = x \cdot_{Nr} y$, $u = (u_1, u_2, u_3, \ldots) = F_r(x)$, $v = (v_1, v_2, v_3, \ldots) = F_r(y)$ and $w = (w_1, w_2, w_3, \ldots) = F_r(\alpha)$. Then, have

$$\begin{aligned} w_n &= \sum_{[j,r]=nr} \frac{j}{n}\alpha_j \\ &= \sum_{[p,q]=j,[j,r]=nr} \frac{j}{n}(p,q)x_p y_q \end{aligned}$$

$$
\begin{aligned}
&= \sum_{[p,q,r]=nr} \frac{1}{n} p x_p q y_q \\
&= \sum_{[i,j]=n} \frac{1}{n} \Big( \sum_{[p,r]=ir} p x_p \Big) \Big( \sum_{[q,r]=jr} q y_q \Big) \\
&= \sum_{[i,j]=n} (i,j) \Big( \sum_{[p,r]=ir} \frac{p}{i} x_p \Big) \Big( \sum_{[q,r]=jr} \frac{q}{j} y_q \Big) \\
&= \sum_{[i,j]=n} (i,j) u_i v_j.
\end{aligned}
$$

In addition, we have $F_r(1) = 1$ by the definition of $F_r$ because if an integer $n \geq 1$ satisfies $[1, r] = nr$, then $n = 1$ holds. Hence, the operation $F_r$ is a ring homomorphism in the case of $Nr(R)$.

Next, we consider the case of $R^{\mathbb{N}}$. Let $a = (a_1, a_2, a_3, \ldots)$ and $b = (b_1, b_2, b_3, \ldots)$ be elements of $R^{\mathbb{N}}$. Put $e = (e_1, e_2, e_3, \ldots) = F_r(a)$ and $f = (f_1, f_2, f_3, \ldots) = F_r(b)$. Thus, we have $e_n f_n = a_{nr} b_{nr}$, which is the $n$-th element of $F_r(ab)$. In addition, we have $F_r(1) = 1$, and hence, the operation $F_r$ is a ring homomorphism. □

**Proposition 2.4.** *The map $\phi : Nr(R) \to R^{\mathbb{N}}$ preserves operations $V_r$ and $F_r$.*

*Proof.* First, we prove $V_r \circ \phi = \phi \circ V_r$ for any integer $r \geq 1$. Let $x = (x_1, x_2, x_3, \ldots)$ be an element of $Nr(R)$, and let $n \geq 1$ be an integer. We put $y = (y_1, y_2, y_3, \ldots) = V_r(x)$, $a = (a_1, a_2, a_3, \ldots) = \phi(y)$, $b = (b_1, b_2, b_3, \ldots) = \phi(x)$ and $c = (c_1, c_2, c_3, \ldots) = V_r(b)$. If $n \nmid r$, then all divisors of $n$ do not divide $r$. Thus we have $a_n = 0$ and $c_n = \sum_{d|n} d b_d = 0$. If $n \mid r$, then $c_n = \sum_{d|n/r} d r x_d = \sum_{d|n} d y_d = a_n$ holds. In any case we have $x_n = a_n$ for any integer $n \geq 1$, that is, we have $V_r \circ \phi = \phi \circ V_r$.

Next, we prove $F_r \circ \phi = \phi \circ F_r$. For any integer $n \geq 1$, we have

$$
f_n = \sum_{d|n} d \Big( \sum_{[j,r]=dr} \frac{j}{d} x_j \Big) = \sum_{[j,r]|nr} j x_j = e_n
$$

where $e = (e_1, e_2, e_3, \ldots) = F_r \circ \phi(x)$ and $f = (f_1, f_2, f_3, \ldots) = \phi \circ F_r(x)$. Note that $j$ satisfies $[j, r] \mid nr$ if and only if $j \mid nr$, and $e_n$ is the $nr$-th element of $\phi(x)$. Hence, we have $F_r \circ \phi = \phi \circ F_r$. □

In the remainder of §2.1, we suppose that a commutative ring $R$ is a $\mathbb{Q}$-algebra. For any integer $r \geq 1$, we define two maps

$$
\begin{aligned}
V'_r &: Nr(R) \to Nr(R), \\
V'_r &: R^{\mathbb{N}} \to R^{\mathbb{N}}
\end{aligned}
$$

by

$$
V'_r := \frac{1}{r} V_r.
$$

The operation $V'_r$ is said to be the $r$-th divided Verschiebung operation, which was defined as operations of aperiodic rings in [VW]. This definition comes from the definition in aperiodic rings.

**Proposition 2.5.** *The map $\phi$ preserves divided Verschiebung operations.*

*Proof.* By Proposition 2.4, we have $\phi \circ V'_r = \phi \circ \dfrac{1}{r} V_r = \dfrac{1}{r} V_r \circ \phi = V'_r \circ \phi$. Hence, the map $\phi$ preserves divided Verschiebung operations. □

**Proposition 2.6.** *For any integer $r \geq 1$, the operation $V'_r$ is a homomorphism of the additive and the multiplication.*

*Proof.* In both of cases, it is obvious that the operation $V'_r$ is an additive homomorphism. Hence, we consider the multiplication.

First, we consider the case of $R^{\mathbb{N}}$. Let $a = (a_1, a_2, a_3, \ldots)$ and $b = (b_1, b_2, b_3, \ldots)$ be elements of $R^{\mathbb{N}}$. Put $c = (c_1, c_2, c_3, \ldots) = ab$,
$e = (e_1, e_2, e_3, \ldots) = V'_r(a)$, $f = (f_1, f_2, f_3, \ldots) = V'_r(b)$ and $g = (g_1, g_2, g_3, \ldots) = V'_r(c)$. If $r \mid n$, then $g_n = c_{n/r} = a_{n/r} b_{n/r} = e_n f_n$ holds. If $r \nmid n$, then $g_n = e_n f_n = 0$ holds. Hence, the operation $V'_r$ satisfies $V'_r(ab) = V'_r(a) V'_r(b)$.

In case where $Nr(R)$, we use the map $\phi$ which is bijective by Proposition 2.1. The operation $V'_r$ is equal to $\phi^{-1} \circ V'_r \circ \phi$ by Proposition 2.5. Hence, the operation $V'_r$ satisfies $V'_r(x \cdot_{Nr} y) = V'_r(x) \cdot_{Nr} V'_r(y)$ for any $x, y \in Nr(R)$. □

The followings also hold by Proposition 2.2.

**Proposition 2.7.** *The operations $V'_r$ and $F_r$ satisfy the following relations.*

(1) $V'_r \circ V'_s = V'_{rs}$,

(2) $F_r \circ V'_r = \mathrm{id}_R$,

(3) $F_r \circ V'_s = V'_s \circ F_r$ if $(r, s) = 1$,

(4) $F_r \circ V'_s = F_{r/(r,s)} \circ V'_{s/(r,s)}$.

## 2.2 Pre-$\lambda$-rings and $\lambda$-rings

We discuss pre-$\lambda$-rings and $\lambda$-rings. Main references of §2.2 are [Hus, Chapter 13], [Knu] and [Yau].

A $\lambda$-semiring is a commutative semiring $R$ with operations $\lambda^n : R \to R$, $n = 0, 1, 2, \ldots$, such that

(1) $\lambda^0(r) = 1$,

(2) $\lambda^1(r) = r$,

(3) $\lambda^n(r + s) = \sum_{i+j=n} \lambda^i(r) \lambda^j(s)$

for any $r, s \in R$. A semi-$\lambda$-ring $R$ is said to be pre-$\lambda$-ring if $R$ has a commutative ring structure with the addition and the multiplication of $R$ as a semiring.

We define a $\lambda$-semiring homomorphism $f : R \to R'$ between two $\lambda$-semirings $R$ and $R'$ by a semiring homomorphism satisfying $f \circ \lambda^i = \lambda^i \circ f$ for any integer $i \geq 0$.

**Proposition 2.8** ([Hus, p.172])**.** *Let $(R^*, \theta)$ be the ring completion of the semiring $R$. There exists a pre-$\lambda$-ring structure on $R^*$ uniquely such that the map $\theta$ is a $\lambda$-semiring homomorphism.*

Let $R$ be a commutative ring. Next, we define $\Lambda(R)$ by

$$\Lambda(R) = \left\{ f = 1 + \sum_{i=1}^{\infty} r_i t^i \in R[[t]] \mid r_i \in R \right\}$$

where $t$ is an indeterminate variable. In [Knu] and [Yau], the set $\Lambda(R)$ is called universal $\lambda$-ring, and has a pre-$\lambda$-ring structure whose addition is defined by the multiplication of $R[[t]]$. For the multiplication and $\lambda$-operations of $\Lambda(R)$, see [Knu] and [Yau].

Let $R$ be a pre-$\lambda$-ring. For any $r \in R$, we define $\lambda_t(r) \in \Lambda(R)$ by

$$\lambda_t(r) = \sum_{i=0}^{\infty} \lambda^i(r) t^i.$$

By the definition of pre-$\lambda$-rings, we have $\lambda_t(r+s) = \lambda_t(r)\lambda_t(s)$ for any $r, s \in R$, which shows that the map $\lambda_t : R \to \Lambda(R)$ is an additive homomorphism.

We call a $\lambda$-semiring homomorphism between two pre-$\lambda$-rings a pre-$\lambda$-homomorphism.

A pre-$\lambda$-ring $R$ is said to be a $\lambda$-ring if the map $\lambda_t : R \to \Lambda(R)$ is a pre-$\lambda$-homomorphism. Knutson [Knu] showed that $\Lambda(R)$ is a $\lambda$-ring for any commutative ring $R$. A pre-$\lambda$-homomorphism between two $\lambda$-rings is called $\lambda$-homomorphism.

A $\lambda$-subring of a $\lambda$-ring $R$ is defined by a subring $R' \subset R$ such that $\lambda^i(x) \in R'$ for all integers $i \geq 0$ and $x \in R'$.

For a $\lambda$-homomorphism and a $\lambda$-subring, the following proposition holds.

**Proposition 2.9** ([Yau, Proposition 1.27 (2)])**.** *Let $R$ and $S$ be $\lambda$-rings and let $f : R \to S$ be a $\lambda$-homomorphism. The image of $f$ is a $\lambda$-subring of $S$.*

Let $R$ be a $\lambda$-ring. For each integer $n \geq 1$, we define the $n$-th Adams operation $\psi^n : R \to R$ by the following equation:

$$\sum_{n=1}^{\infty} \psi^n(r) t^n := -t \frac{d}{dt} \log \lambda_t(r) \quad (\text{for any } r \in R).$$

For Adams operations, the following propositions and theorem hold.

**Proposition 2.10.** *Let $R$ and $S$ be $\lambda$-rings and let $f : R \to S$ be a ring homomorphism. If $f$ is a $\lambda$-homomorphism, then $f \circ \psi^n = \psi^n \circ f$ holds for any integer $n \geq 1$.*

*Proof.* Assume that $f$ is a $\lambda$-homomorphism. By the definition of Adams operations, for any integer $n \geq 1$ and $r \in R$, an element $\psi^n(r)$ is a polynomial of $\lambda^i(r)$, $i = 0, 1, \ldots, n$. Hence $f \circ \psi^n = \psi^n \circ f$ holds. □

**Proposition 2.11** ([Yau, Corollary 3.16])**.** *Let $R$ and $S$ be $\lambda$-rings and let $f : R \to S$ be a ring homomorphism. The map $f$ is a $\lambda$-homomorphism if $f \circ \psi^n = \psi^n \circ f$ holds for any integer $n \geq 1$ and $S$ is $\mathbb{Z}$-torsion free.*

**Theorem 2.12** ([Yau, Theorem 3.6])**.** *On any $\lambda$-ring $R$ and integer $n \geq 1$, the $n$-th Adams operation $\psi^n$ is a $\lambda$-homomorphism.*

Let $R$ be a commutative ring. We define a map $z : \Lambda(R) \to R^{\mathbb{N}}$ by $z(f) := (a_1, a_2, a_3, \ldots)$ where

$$\sum_{i=1}^{\infty} a_i(-t)^i = -t\frac{d}{dt}\log(f(t)) \quad (\text{for any } f \in \Lambda(R)).$$

The map $z$ is a ring homomorphism. If $R$ is $\mathbb{Z}$-torsion free, then the map $z$ is injective ([Yau, p.133]).

A commutative ring $R$ equipped with ring homomorphisms $\psi^n$ for $n \geq 1$, such that $\psi^1 = \mathrm{id}_R$ and $\psi^n \circ \psi^m = \psi^{nm}$ hold for any integers $n, m \geq 1$, is said to be a $\psi$-ring.

For example, the set $R^{\mathbb{N}}$ is a $\psi$-ring with Frobenius operations. All $\lambda$-rings are $\psi$-rings with Adams operations. Conversely, the following theorem holds.

**Theorem 2.13** ([Knu, p.50])**.** *Suppose that $R$ is a $\mathbb{Q}$-algebra and a $\psi$-ring with operations $\psi^n, n \geq 1$. Then, there exists the $\lambda$-ring structure on $R$ uniquely such that $\psi^n$ is an $n$-th Adams operation.*

Next, we state the definition of binomial rings. It is a $\mathbb{Z}$-torsion free $\lambda$-ring such that all Adams operations are identity maps. There exists the following relation between the universal $\lambda$-ring $\Lambda(R)$ and the necklace ring $Nr(R)$.

**Lemma 2.14.** [Yau, Lemma 5.41] *Suppose that $R$ is a binomial ring. For each $f \in \Lambda(R)$, we can write uniquely as*

$$f(t) = \prod_{i=1}^{\infty}(1 - (-t)^i)^{a_i}$$

*for some elements $a_1, a_2, \ldots \in R$.*

By Lemma 2.14, we can state the following theorem.

**Theorem 2.15.** [Yau, Theorem 5.42] *Suppose that $R$ is a binomial ring. Then the map $E_{Nr} : \Lambda(R) \to Nr(R)$ defined by*

$$E_{Nr}\Big(\prod_{i=1}^{\infty}(1-(-t)^i)^{a_i}\Big) = (a_1, a_2, a_3, \ldots),$$

*is a ring isomorphism, and $z = \phi \circ E_{Nr}$ holds.*

We define three types of ring-homomorphisms. Let $R$ and $S$ be commutative rings and $F : R \to S$ be a ring homomorphism. We define maps $\Lambda(F) : \Lambda(R) \to \Lambda(S)$, $Nr(F) : Nr(R) \to Nr(S)$ and $F^{\mathbb{N}} : R^{\mathbb{N}} \to S^{\mathbb{N}}$ by

$$\begin{aligned}
\Lambda(F)\Big(1+\sum_{i=1}^{\infty} a_i t^i\Big) &:= 1+\sum_{i=1}^{\infty} F(a_i)t^i, \\
Nr(F)((x_1, x_2, x_3, \ldots)) &:= (F(x_1), F(x_2), F(x_3), \ldots) \\
F^{\mathbb{N}}((y_1, y_2, y_3, \ldots)) &:= (F(y_1), F(y_2), F(y_3), \ldots)
\end{aligned}$$

for any $a_1, a_2, \ldots \in R$, $(x_1, x_2, x_3, \ldots) \in Nr(R)$ and $(y_1, y_2, y_3, \ldots) \in R^{\mathbb{N}}$.

**Proposition 2.16.** *Let $F : R \to S$ be a ring homomorphism between two rings $R$ and $S$.*

(1) *Three maps $\Lambda(F), Nr(F)$ and $F^{\mathbb{N}}$ are ring homomorphisms.*

(2) *One has $z \circ \Lambda(F) = F^{\mathbb{N}} \circ z$ and $\phi \circ Nr(F) = F^{\mathbb{N}} \circ \phi$. If $R$ and $S$ are binomial rings, then $Nr(F) \circ E_{Nr} = E_{Nr} \circ \Lambda(F)$ holds.*

*Proof.* (1) For $\Lambda(F)$, see [NO, p.104], and for $Nr(F)$, see [VW, p.18]. For $F^{\mathbb{N}}$,

$$\begin{aligned}
& F^{\mathbb{N}}((a_1, a_2, a_3, \ldots)) + F^{\mathbb{N}}((b_1, b_2, b_3, \ldots)) \\
= \; & (F(a_1), F(a_2), F(a_3), \ldots) + (F(b_1), F(b_2), F(b_3), \ldots) \\
= \; & (F(a_1+b_1), F(a_2+b_2), F(a_3+b_3), \ldots) \\
= \; & F^{\mathbb{N}}(a_1+b_1, a_2+b_2, a_3+b_3, \ldots), \\
& F^{\mathbb{N}}((a_1, a_2, a_3, \ldots))F^{\mathbb{N}}((b_1, b_2, b_3, \ldots)) \\
= \; & (F(a_1), F(a_2), F(a_3), \ldots)(F(b_1), F(b_2), F(b_3), \ldots) \\
= \; & (F(a_1b_1), F(a_2b_2), F(a_3b_3), \ldots) = F^{\mathbb{N}}(a_1b_1, a_2b_2, a_3b_3, \ldots), \\
& F^{\mathbb{N}}(1, 1, 1, \ldots) = (1, 1, 1, \ldots)
\end{aligned}$$

holds for any $(a_1, a_2, a_3, \ldots), (b_1, b_2, b_3, \ldots) \in R^{\mathbb{N}}$. Then, the map $F^{\mathbb{N}}$ is a ring homomorphism.

(2) We prove $z \circ \Lambda(F) = F^{\mathbb{N}} \circ z$. For any integer $n \geq 1$, let $Z_n(x_1, x_2, x_3, \ldots)$ be a polynomial which satisfies that the $n$-th element of $z(1+\sum_{i=1}^{\infty} a_i t^i)$ is $Z_n(a_1, a_2, a_3, \ldots)$ where $a_1, a_2, \ldots \in R$. Then, the $n$-th element of $z \circ \Lambda(F)(f)$ and $F^{\mathbb{N}} \circ z(f)$ coincide with $Z_n(F(a_1), F(a_2), F(a_3), \ldots)$ for any $f = 1+\sum_{i=1}^{\infty} a_i t^i$. Then, we have $z \circ \Lambda(F) = F^{\mathbb{N}} \circ z$.

Similarly, we show that both $n$-th elements of $\phi \circ Nr(F)(x)$ and $F^{\mathbb{N}} \circ \phi(x)$ are equal in order to prove the second equation of (2), where $x = (x_1, x_2, x_3, \ldots) \in Nr(R)$. Clearly, it is $\sum_{d|n} dF(x_d)$. Then, we have $\phi \circ Nr(F) = F^{\mathbb{N}} \circ \phi$.

To prove the third equation of (2), we use Lemma 2.14. One has

$$\begin{aligned}\phi \circ Nr(F) \circ E_{Nr} &= F^{\mathbb{N}} \circ \phi \circ E_{Nr} = F^{\mathbb{N}} \circ z \\ &= z \circ \Lambda(F) = \phi \circ E_{Nr} \circ \Lambda(F) = \phi \circ E_{Nr} \circ \Lambda(F).\end{aligned}$$

By the assumption, the ring $S$ is $\mathbb{Z}$-torsion free. Hence $Nr(F) \circ E_{Nr} = E_{Nr} \circ \Lambda(F)$ holds by Proposition 2.1. □

By the definition of $\Lambda(F)$, $Nr(F)$ and $R^{\mathbb{N}}$, we have the following proposition.

**Proposition 2.17.** *For any ring homomorphisms $F : R \to S$ and $G : S \to T$, we have $\Lambda(G \circ F) = \Lambda(G) \circ \Lambda(F)$, $Nr(G \circ F) = Nr(G) \circ Nr(F)$ and $G \circ F^{\mathbb{N}} = G^{\mathbb{N}} \circ F^{\mathbb{N}}$.*

## 2.3 Exterior powers of representations of finite groups

In §2.3, we discuss representations of finite groups and, in particular, a character of exterior powers of representations of finite groups using $\lambda$-rings. Main references of §2.3 is [Knu].

The set of all representations of a finite group $G$ has an equivalent relation which is defined by isomorphisms of representations of $G$. Let $M(G)$ be the set of equivalence classes. For any representation $V$, we denote an equivalences class of $V$ by $[V] \in M(G)$. The set $M(G)$ has a $\lambda$-semiring structure whose addition is defined via the direct sum of two representations, multiplication is defined via the tensor product of two representations, and $\lambda$-operations are defined via exterior powers of representations.

We define the representation ring of $G$, written by $R(G)$, by the ring completion of the $\lambda$-semiring $M(G)$. By Proposition 2.8, the representation ring $R(G)$ has a pre-$\lambda$-ring structure, and the representation ring $R(G)$ is a free abelian group with equivalence classes of irreducible representations of $G$ as generators (see Appendix A.6).

Let $CF(G)$ be the set of all class functions. It has a $\mathbb{C}$-algebra $\psi$-ring structure with the following operations:

$$\begin{cases}(f_1 + f_2)(g) := f_1(g) + f_2(g), \\ (f_1 f_2)(g) := f_1(g) f_2(g), \\ (cf)(g) := cf(g), \\ \psi^n(f)(g) := f(g^n)\end{cases}$$

for any $f, f_1, f_2 \in CF(G), c \in \mathbb{C}$, integer $n \geq 1$ and $g \in G$. In particular, the set $CF(G)$ is a $\mathbb{Q}$-algebra. Thus, by Theorem 2.13 and the above operations, the set $CF(G)$ has a $\lambda$-ring structure.

For any representation $\rho : G \to GL(V)$, we denote the character of $V$ by $X(V)$, which satisfies that $X(V)(g) = \mathrm{Tr}(\rho(g))$ for any $g \in G$. The character of a representation satisfies the followings for any representations $V$ and $W$.

(1) $X(V) = X(W)$ whenever two representations $V$ and $W$ are isomorphic,

(2) $X(V \oplus W) = X(V) + X(W)$,

(3) $X(V \otimes W) = X(V)X(W)$.

Then, the mapping $V \mapsto X(V)$ defines the ring homomorphism $X : R(G) \to CF(G)$ by that $X([V]) = X(V)$ holds for any representation $V$.

**Theorem 2.18.** [Knu, p.84] *The map $X$ is an injective pre-$\lambda$-homomorphism. In particular, the representation ring $R(G)$ is a $\lambda$-ring.*

By Theorem 2.18, a character of exterior powers of representations can be written with $\lambda$-operations of $CF(G)$. For more detail, for any representation $V$ whose character is $\chi$, the character of the $i$-th exterior power of $\rho$ is $\lambda^i(\chi)$.

We define $Cl_{\mathbb{Z}}(G)$ by the image of the map $X$. It is a free abelian group with irreducible characters of $G$ as generators, and is a $\lambda$-ring. We call an element of $Cl_{\mathbb{Z}}(G)$ a virtual character.

For any $g \in G$, we define the map $\mathrm{E}_g^G : CF(G) \to \mathbb{C}$ by $\mathrm{E}_g^G(f) := f(g)$ for any $f \in CF(G)$. It is a ring homomorphism. By this define we can write $\lambda_t(\chi)(g)$ which is (1.1) by

$$\lambda_t(\chi)(g) = \Lambda(\mathrm{E}_g^G)(\lambda_t(\chi)) = \sum_{i=0}^{\infty} \lambda^i(\chi)(g)t^i$$

for any character $\chi$ of $G$ and $g \in G$.

## 2.4 Algebraic integers

In §2.4, we consider algebraic integers for §3.3. For more detail, see [CR, §17].

We define a monic polynomial $f(t) \in \mathbb{Z}[t]$ by $a_n = 1$ where $f(t) = \sum_{i=0}^{n} a_i t^i$. In this paper, we define an algebraic integer $\alpha$ by that there exists an irreducible monic polynomial $f(t) \in \mathbb{Z}[t]$ such that $f(\alpha) = 0$ holds.

**Theorem 2.19** ([CR, (17.2)])**.** *Any zero of a monic polynomial in $\mathbb{Z}[t]$ is an algebraic integer.*

By Theorem 2.19, we see that a primitive $e$-th roof of unity, written by $\omega$ is also an algebraic integer with $f(t) = t^e - 1$.

To state the main theorems, we consider the following lemmas.

**Lemma 2.20** ([CR, (17.7)])**.** *If $\alpha$ and $\beta$ are algebraic integers, then $\alpha + \beta$ and $\alpha\beta$ are also algebraic integers.*

**Lemma 2.21** ([CR, p.106, Exercise])**.** *A rational number is an algebraic integer if and only if it is an integer.*

*Proof.* Let $\alpha$ is a rational number. If $\alpha$ is an algebraic integers, then there exists an irreducible polynomial $f \in \mathbb{Z}[t]$ such that $f(\alpha) = 0$ holds. Hence, we have $t - \alpha \mid f(t)$. Moreover, a polynomial $f(t)$ is an irreducible as a polynomial of $\mathbb{Q}[t]$ by the Gauss Lemma (see Lemma A.8). Then, we have $f(t) = t-\alpha$. Hence, we have $\alpha \in \mathbb{Z}$.

Conversely, if $\alpha \in \mathbb{Z}$ then we put $f(t) = t - \alpha$. Then, an integer $\alpha$ is an algebraic integer. □

# 3 Main Result

In this section, we focus on integer-valued characters of finite groups as main results of this part with §2. First, we recall the definition of integer-valued characters.

**Definition 3.1.** A virtual character $\chi$ will be said to be an integer-valued character if $\chi(g)$ is an integer for any $g \in G$.

In §3.1, we introduce two examples of integer-valued characters. In §3.2, we define truncated operations, on $Nr(R)$ and $R^{\mathbb{N}}$. In §3.3, we characterize integer-valued characters in terms of $\lambda$-rings and necklace rings as a main result. In §3.4, we introduce the notion of support of elements of $Nr(R)$ and cycle of $R^{\mathbb{N}}$ for any commutative ring $R$, and state the relation between integer-valued characters and elements of necklace rings which have finite support. In §3.5, we discuss product groups and Frobenius operations to calculate $\lambda_t(\chi)(g)$ for a virtual character $\chi$ and an element $g$ of a finite group. Moreover, we discuss the multiplication of two elements of the image of truncation operations with Frobenius operations.

## 3.1 Examples of integer-valued characters

Now, we state two examples of integer-valued characters.

**Example 3.2.** Permutation characters are integer-valued characters. See Appendix A.4.

Next, we consider characters of symmetric groups.

**Lemma 3.3.** *Put $\sigma = (1\ 2 \cdots n) \in S_n$ and let $d \geq 1$ be an integer.*

(1) *If $d$ and $n$ are coprime, then $\sigma^d$ is also a cycle of length $n$.*

(2) *If $d \mid n$, then we have*

$$\sigma^d = (1\ \ d+1\ \ 2d+1 \cdots (d'-1)d+1) \cdots (d\ \ d+d \cdots (d'-1)d+d)$$

*where $d' = n/d$.*

*Proof.* For any integer $i = 1, \ldots, n$, put $a_{k,i} = \sigma^k(i)$. Then, we have $ki = b_{k,i}n + a_{k,i}$ for some integer $b_{k,i} \geq 0$.

First, we assume that $k$ and $n$ are coprime. If $a_{k,i} = a_{k,j}$, then $k(i-j) = (b_{k,i} - b_{k,j})n$ holds. Thus, the integer $n$ divides $k(i-j)$, that is, the integer $n$ divides $i-j$. Hence, we have $i = j$, that is, $\sigma^k = (a_{k,1} \cdots a_{k,n})$ holds.

Next, we assume $d \mid n$. If $i \leq (d'-1)d$, then $\sigma^d(i) = d+1$ holds. If $i = (d'-1)d + k$ where $k = 1, \ldots, d'$, then $\sigma^d(i) = k$ holds. Hence, we have $\sigma^d = (1\ d+1\ 2d+1 \cdots (d'-1)d+1) \cdots (d\ d+d \cdots (d'-1)d+d)$. □

**Example 3.4.** A finite group is called a Q-group if all characters are integer-valued [Kle, p.8]. A typical example of Q-group is a symmetric group, and more generally, Weyl groups.

Here we recall the proof [Kle, p.10] of the fact that a symmetric group is a Q-group.

Let $\sigma$ and $\tau$ be elements of $S_n$ satisfying $\langle\sigma\rangle = \langle\tau\rangle$. Then, there exists an integer $k \geq 1$ such that $\sigma^k = \tau$ holds and $k$ is coprime to $O(\sigma)$ by Lemma A.2. Let $\{\lambda_1, \ldots, \lambda_m\}$ be the cycle structure of $\sigma$. Then, the integer $k$ is coprime to $\lambda_i$ for any $i = 1, \ldots, m$ by Lemma A.4. Thus, the element $\tau$ has the same cycle structure by Lemma 3.3, that is, elements $\sigma$ and $\tau$ are conjugate by Lemma A.5. Hence, a character of $S_n$ is an integer-valued character by [Kle, Proposition 9].

## 3.2 Truncated operations

Let $R$ be a commutative ring. In §3.2, we define operations on $Nr(R)$ and $R^{\mathbb{N}}$ to state main theorems in §3.3, §3.4 and §3.5, and discuss relations among operations.

**Definition 3.5.** For any integer $r \geq 1$, we define

$$\begin{aligned} T_r &: Nr(R) \to Nr(R), \\ T_r &: R^{\mathbb{N}} \to R^{\mathbb{N}} \end{aligned}$$

by $y = (y_1, y_2, y_3, \ldots) = T_r(x)$ and $b = (b_1, b_2, b_3, \ldots) = T_r(a)$ where $x = (x_1, x_2, x_3, \ldots) \in Nr(R)$, $a = (a_1, a_2, a_3, \ldots) \in R^{\mathbb{N}}$, and

$$\begin{aligned} y_n &:= \begin{cases} x_n & \text{if } n \mid r, \\ 0 & \text{if } n \nmid r, \end{cases} \\ b_n &:= a_{(n,r)}. \end{aligned}$$

We call $T_r$ the $r$-th truncated operation.

In the following proposition, we consider the image of $T_r$.

**Proposition 3.6.** *Let $r \geq 1$ be an integer.*

(1) *An element $x = (x_1, x_2, x_3, \ldots) \in Nr(R)$ belongs to the image of $T_r$ if and only if $x_n = 0$ holds for any integer $n \geq 1$ with $n \nmid r$.*

(2) *An element $a = (a_1, a_2, a_3, \ldots) \in R^{\mathbb{N}}$ belongs to the image of $T_r$ if and only if $a_n = a_{(n,r)}$ holds for any integer $n \geq 1$.*

*Proof.* We prove the statement (1). Let $x = (x_1, x_2, x_3, \ldots)$ be an element of the image of $T_r$. Then, there exists $y \in Nr(R)$ such that $x = T_r(y)$ holds. For any integer $n \geq 1$ with $n \nmid r$, the element $x_n$ is the $n$-th element of $T_r(y)$, which is equal to $0 \in R$.

Next, we assume that $x_n = 0$ holds for any integer $n \geq 1$ with $n \nmid r$. Put $u = (u_1, u_2, u_3, \ldots) = T_r(x)$. If an integer $n \geq 1$ satisfies $n \mid r$, then $x_n = u_n$ holds. If $n \nmid r$, then $x_n = u_n = 0$ holds. In any case, we have $x = u = T_r(x)$, that is, the element $x$ belongs to the image of $T_r$.

(2) Let $a = (a_1, a_2, a_3, \ldots)$ be the element of the image of $T_r$. Then, there exists $b = (b_1, b_2, b_3, \ldots) \in Nr(R)$ such that $a = T_r(b)$ holds. Hence we have $a_n = b_n = b_{(n,r)} = a_{(n,r)}$ for any integer $n \geq 1$.

Next, we assume that $a_n = a_{(n,r)}$ holds for any integer $n \geq 1$. Put $c = (c_1, c_2, c_3, \ldots) = T_r(a)$. Thus, we have $c_n = a_{(n,r)} = a_n$ for any integer $n \geq 1$, and hence, we have $a = c = T_r(a)$, that is, the element $a$ belongs to the image of $T_r$. □

**Proposition 3.7.** *The operation $T_r$ is a ring homomorphism for any integer $r \geq 1$, and $T_r \circ T_s = T_{(r,s)}$ holds for any integers $r, s \geq 1$.*

*Proof.* Obviously, the operation $T_r$ is an additive homomorphism and $T_r(1) = 1$ holds in both cases. Thus, we consider the multiplication.

First, we prove this proposition in the case of $Nr(R)$. Let $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ be elements of $Nr(R)$. We put $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = x \cdot_{Nr} y$, $u = (u_1, u_2, u_3, \ldots) = T_r(x)$, $v = (v_1, v_2, v_3, \ldots) = T_r(y)$ and $w = (w_1, w_2, w_3, \ldots) = T_r(\alpha)$. For any integers $r, n \geq 1$, the $n$-th elements of $T_r(x) \cdot_{Nr} T_r(y)$ is

$$\sum_{[i,j]=n} (i,j) u_i v_j \tag{3.1}$$

by the definition of multiplication of $Nr(R)$. If $n \mid r$, then integers $i$ and $j$ satisfying $[i, j] = n$ divide $r$. Thus, (3.1) coincides with

$$\sum_{[i,j]=n} (i,j) x_i y_j = \alpha_n = w_n.$$

If $n \nmid r$, then (3.1) is $0 \in R$ because either $i$ or $j$ with $[i, j] = n$ does not divide $r$ and $w_n = 0$ holds. Hence, we have $T_r(x) \cdot_{Nr} T_r(y) = w = T_r(\alpha) = T_r(x \cdots_{Nr} y)$.

Next, we prove this proposition in the case of $R^{\mathbb{N}}$. Let $a = (a_1, a_2, a_3, \ldots)$ and $b = (b_1, b_2, b_3, \ldots)$ be elements of $R^{\mathbb{N}}$. We put $c = (c_1, c_2, c_3, \ldots) = ab$, $e = (e_1, e_2, e_3, \ldots) = T_r(a)$, $f = (f_1, f_2, f_3, \ldots) = T_r(b)$ and $g = (g_1, g_2, g_3, \ldots) = T_r(c)$. For any integer $r, n \geq 1$, we have $e_n f_n = a_{(n,r)} b_{(n,r)} = c_{(n,r)} = g_n$, that is, the operation $T_r$ is a ring homomorphism. □

**Proposition 3.8.** *For any integers $r, s \geq 1$, we have $T_r \circ T_s = T_{(r,s)}$.*

*Proof.* First, we prove this proposition in the case of $Nr(R)$. Let $x = (x_1, x_2, x_3, \ldots)$ be an element of $Nr(R)$, and put $y = (y_1, y_2, y_3, \ldots) = T_s(x)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = T_r(y)$. For any integer $d \geq 1$, if $d \mid (r, s)$, which implies $d \mid r$ and $d \mid s$, then we have $\alpha_d = y_d = x_d$. If $d \mid r$ and $d \nmid s$, then we have $\alpha_d = y_d = 0$. If $d \nmid r$, then we have $\alpha_d = 0$. Hence, we have $T_r \circ T_s = T_{(r,s)}$.

Next, we prove this proposition in the case of $R^{\mathbb{N}}$. Let $x = (x_1, x_2, x_3, \ldots)$ be an element of $\in Nr(R)$, and put $y = (y_1, y_2, y_3, \ldots) = T_s(x)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = T_r(y)$. Then, we have $\alpha_n = y_{(n,r)} = x_{(n,r,s)}$ for any integer $n \geq 1$. Hence $T_r \circ T_s = T_{(r,s)}$ holds. $\square$

**Proposition 3.9.** *The map $\phi : Nr(R) \to R^{\mathbb{N}}$ preserves truncated operations.*

*Proof.* For any integer $r \geq 1$, we prove $T_r \circ \phi = \phi \circ T_r$. Let $x = (x_1, x_2, x_3, \ldots)$ be an element of $Nr(R)$. We put $a = (a_1, a_2, a_3, \ldots) = \phi(x)$, $b = (b_1, b_2, b_3, \ldots) = T_r(a)$, $y = (y_1, y_2, y_3, \ldots) = T_r(x)$ and $c = (c_1, c_2, c_3, \ldots) = \phi(y)$. Then, we have

$$c_n = \sum_{d|n} d y_d = \sum_{d|n, d|r} d x_d = \sum_{d|(n,r)} d x_d = a_{(n,r)} = b_n.$$

Hence $\phi \circ T_r = T_r \circ \phi$ holds. $\square$

By Proposition 3.9, we can state the following corollary.

**Corollary 3.10.** *Suppose that $R$ is $\mathbb{Z}$-torsion free. Let $x = (x_1, x_2, x_3, \ldots)$ be an element of $Nr(R)$, let $r \geq 1$ be an integer, and put $a = (a_1, a_2, a_3, \ldots) = \phi(x)$. Then, the followings are equivalent.*

(1) *For any integer $n \geq 1$ with $n \nmid r$, we have $x_n = 0$.*

(2) *For any integer $n \geq 1$, we have $a_n = a_{(n,r)}$.*

*Proof.* First, we assume the statement (1). We have $x = T_r(x)$ by definition of truncated operations. Thus, we have $a = \phi(x) = \phi \circ T_r(x) = T_r \circ \phi(x) = T_r(a)$. The $n$-th element of $a$ and $T_r(a)$ are $a_n$ and $a_{(n,r)}$, that is, the statement (2) holds by Proposition 3.6 (2).

Conversely, we assume the statement (2). Thus, we have $\phi(x) = a = T_r(a) = T_r \circ \phi(x) = \phi \circ T_r(x)$. Since $R$ is $\mathbb{Z}$-torsion free, we have $x = T_r(x)$. For any integer $n \geq 1$ with $n \nmid r$, the $n$-th element of $T_r(x)$ is $0 \in R$, and hence, $x_n = 0$ holds by Proposition 3.6(1). $\square$

Next, we state relations among truncated operations, Frobenius operations and Verschiebung operations.

**Proposition 3.11.** *For any integers $r, s \geq 1$, we have $F_r \circ T_s = T_{s/(r,s)} \circ F_{(r,s)}$.*

Note that Proposition 3.11 holds on both $Nr(R)$ and $R^{\mathbb{N}}$. To prove Proposition 3.11, we introduce the following notation.

**Definition 3.12.** For any $u \in R$ and integer $n \geq 1$, we define an element $u\delta_n$ of $Nr(R)$ by

$$u\delta_n = (x_1, x_2, x_3, \ldots), \quad x_n := \begin{cases} u & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

For elements $u\delta_n$, the following lemma holds.

**Lemma 3.13.** *For any $u \in R$ and integers $n, r \geq 1$, we have $F_r(u\delta_n) = (n,r)u\delta_{n/(n,r)}$.*

*Proof.* By the definition of Frobenius operations, we have

$$y_k = \sum_{[j,r]=kr} \frac{j}{k} x_j \tag{3.2}$$

where $x = (x_1, x_2, x_3, \ldots) = u\delta_n$ and $y = (y_1, y_2, y_3, \ldots) = F_r(x)$. If $[n,r] = kr$ does not hold, then the right side of (3.2) is equal to $0 \in R$. If $[n,r] = kr$ holds, which implies $k = n/(n,r)$, then the right side of (3.2) is equal to $(n,r)u$. Hence, this lemma holds. □

**Lemma 3.14.** *For any $x, y \in Nr(R)$, we have $x = y$ if $T_r(x) = T_r(y)$ holds for any integer $r \geq 1$.*

*Proof.* Put $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$. For any integer $n \geq 1$, both of $n$-th element of $T_n(x)$ and $T_n(y)$ are $x_n$ and $y_n$, and these are equal. Hence, we have $x = y$. □

*Proof of Proposition 3.11.* First, we prove $F_r \circ T_s(x) = T_{s/(r,s)} \circ F_{(r,s)}(x)$ when $x = u\delta_n$ for $u \in R$ and integer $n \geq 1$.

If $n \mid s$, then $F_r \circ T_s(u\delta_n) = (n,r)u\delta_{n/(n,r)}$ and

$$T_{s/(r,s)} \circ F_{(r,s)}(u\delta_n) = (n,r)T_{s/(r,s)}(u\delta_{n/(n,r)}) = (n,r)u\delta_{n/(n,r)}$$

hold by Lemma 3.13.

If $n \nmid s$, then $F_r \circ T_s(u\delta_n) = 0$ holds by the definition of truncated operations, and we have

$$T_{s/(r,s)} \circ F_{(r,s)}(u\delta_n) = (n,r)T_{s/(r,s)}(u\delta_{n/(n,r)}) = 0$$

by Lemma 3.13. Thus, we can show that this proposition holds when $x = u\delta_n$.

Next, we consider an arbitrary element $x = (x_1, x_2, x_3, \ldots) \in Nr(R)$. For any integer $n \geq 1$, we have

$$\begin{aligned} T_n(F_r \circ T_s(x)) &= \sum_{d|n} (F_r \circ T_s) x_d \delta_d \\ &= \sum_{d|n} (T_{s/(r,s)} \circ F_{(r,s)}) x_d \delta_d \end{aligned}$$

$$= \quad T_n(T_{s/(r,s)} \circ F_{(r,s)}(x)).$$

Thus, we have $T_n(F_r \circ T_s(x)) = T_n(T_{s/(r,s)} \circ F_{(r,s)}(x))$ for any integer $n \geq 1$. By Lemma 3.14, we have $F_r \circ T_s(x) = T_{s/(r,s)} \circ F_{(r,s)}(x)$, and hence, this proposition holds in the case of $Nr(R)$.

Finally, we prove this proposition in the case of $R^{\mathbb{N}}$. For any integers $r, s, n \geq 1$, we have

$$(s, (r, s)n) = (s, rn, sn) = (s, rn).$$

Thus, for any $a = (a_1, a_2, a_3, \dots) \in R^{\mathbb{N}}$ and integer $n \geq 1$, we have

$$b_n = a_{(s,rn)} = a_{(s,(r,s)n)} = a_{(s,rn)} = c_n$$

where $b = (b_1, b_2, b_3, \dots) = F_r \circ T_s(a)$ and $c = (c_1, c_2, c_3, \dots) = T_{s/(r,s)} \circ F_{(r,s)}(a)$. Hence, this proposition holds in the case of $R^{\mathbb{N}}$. □

**Proposition 3.15.** *For any integers $r, s \geq 1$ we have*

$$T_r \circ V_s = \begin{cases} V_s \circ T_{r/s} & \text{if } s \mid r, \\ 0 & \text{if } s \nmid r. \end{cases}$$

*Proof.* First, we consider the case of $Nr(R)$. Let $x = (x_1, x_2, x_3, \dots)$ be an element of $Nr(R)$ and let $n \geq 1$ be an integer. Put $y = (y_1, y_2, y_3, \dots) = T_r \circ V_s(x)$.

We assume that $s \mid r$ and put $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots) = V_s \circ T_{r/s}(x)$. If $s \mid n$ and $n \mid r$, then $y_n = x_{n/s}$ and $\alpha_n = x_{n/s}$ holds from $n/s \mid r/s$. If $s \nmid n$, then $y_n = 0$ and $\alpha_n = 0$ hold. If $n \nmid r$, then $y_n = 0$ holds and $\alpha_n = 0$ holds since $n/s \nmid r/s$ holds. Thus, we have $y = \alpha$, that is, one has $T_r \circ V_s = V_s \circ T_{r/s}$.

Assume that $s \nmid r$. If $n \nmid r$, then $y_n = 0$ holds. If $n \mid s$ then $y_n = 0$ holds and the $(r, s)$-element of $V_r(x)$ is $0 \in R$ from $r \nmid s$. Thus, this proposition holds for $Nr(R)$.

Next, we consider the case of $R^{\mathbb{N}}$. Let $a = (a_1, a_2, a_3, \dots)$ be an element of $R^{\mathbb{N}}$. Put $b = (b_1, b_2, b_3, \dots) = V_s(a)$ and $c = (c_1, c_2, c_3, \dots) = T_r(b)$. We assume $s \mid r$, then $s \mid (n, r)$ holds if and only if $s \mid n$. Thus, we have

$$c_n = b_{(n,r)} = a_{(n,r)/s} = a_{(n/s, r/s)} = e_n$$

where $e = (e_1, e_2, e_3, \dots) = V_s \circ T_{r/s}(a)$. If $s \nmid n$ then $c_n = 0$ holds by $s \nmid (n, r)$, and $e_n = 0$ holds. If $s \nmid r$, we have $c_n = 0$ by $s \nmid (n, r)$ for any $n \geq 1$.

As a result, this proposition holds for both cases. □

By Proposition 3.15, we have the following corollary.

**Corollary 3.16.** *Suppose that $R$ is a $\mathbb{Q}$-algebra. For any integers $r, s \geq 1$ we have*

$$T_r \circ V'_s = \begin{cases} V'_s \circ T_{r/s} & \text{if } s \mid r, \\ 0 & \text{if } s \nmid r. \end{cases}$$

## 3.3 Integer-valued characters

In §3.3, we discuss relations among integer-valued characters of a finite group $G$, $\lambda$-operations of $CF(G)$ and images of truncated operations.

First, we show the following result.

**Theorem 3.17.** *If a virtual character $\chi$ is an integer-valued character, then $\lambda^i(\chi)$ is also an integer-valued character for any integer $i \geq 0$.*

To prove Theorem 3.17, we characterize integer-valued characters in terms of a $\lambda$-ring. In the remainder of this section, we denote by $e$ the exponent of $G$ (see (viii) of the preface of this paper).

**Lemma 3.18.** *For any virtual character $\chi$ and $g \in G$, there exists a polynomial $f \in \mathbb{Z}[t]$ such that $\psi^n(\chi)(g) = f(\omega^n)$ holds for any integer $n \geq 1$ where $\omega$ is a primitive $e$-th root of unity. In particular, an element $\chi(g)$ is an algebraic integer.*

*Proof.* We may assume that $\chi$ is the character of a representation $\rho : G \to GL(V)$ by the definition of $R(G)$. All eigenvalue of the linear map $\rho(g) : V \to V$ are $e$-th roots of unity, so the set of eigenvalues of $\rho(g)$ is written as $\omega^{a_1}, \ldots, \omega^{a_m}$ with some non-negative integers $a_1, \ldots, a_m$.

We put $f(t) = t^{a_1} + \cdots + t^{a_m} \in \mathbb{Z}[t]$. Then

$$\psi^n(\chi)(g) = \chi(g^n) = \mathrm{Tr}(\rho(g)^n) = \omega^{na_1} + \cdots + \omega^{na_m} = f(\omega^n)$$

holds.

The last statements of this lemma holds using Lemmas 2.20 and 2.21 because elements $\omega^{a_1}, \ldots, \omega^{a_m}$ are algebraic integers. $\square$

By Lemma 3.18, we have the following Lemma 3.19.

**Lemma 3.19.** *A virtual character $\chi$ is an integer-valued character if and only if $\psi^n(\chi) = \psi^{(n,e)}(\chi)$ holds for any integer $n \geq 1$.*

*Proof.* In this proof, we denote a primitive $e$-th root of unity by $\omega$, and we denote the Galois group of the field extension $\mathbb{Q}(\omega^{(n,e)}) : \mathbb{Q}$ by $\Gamma_n$ for any integer $n \geq 1$. For more detail of Galois groups, see [Ste].

First, we assume that the character $\chi$ is an integer-valued character. Let $g$ be an element of $G$. By Lemma 3.18, there exists a polynomial $f \in \mathbb{Z}[t]$ such that $\psi^k(\chi)(g) = f(\omega^k)$ holds for any integer $k \geq 1$. By the assumption, $\psi^n(\chi)(g)$ and $\psi^{(n,e)}(\chi)(g)$ are integers.

For any integer $n \geq 1$, let $n'$ be $n/(n,e)$. Then we have $\psi^n(\chi)(g) = f(\omega^{(n,e)n'})$ and $\psi^{(n,e)}(\chi)(g) = f(\omega^{(n,e)})$, which belong to $\mathbb{Q}(\omega^{(n,e)})$. Thus, since $(n', e/(n,e)) = 1$ holds there exists an element $\sigma \in \Gamma_n$ such that $\sigma(\omega^{(n,e)}) = \omega^{(n,e)n'}$ holds. Hence we have

$$\psi^n(\chi)(g) = \sigma(\psi^{(n,e)}(\chi)(g)) = \psi^{(n,e)}(\chi)(g). \tag{3.3}$$

Conversely, we assume that $\psi^n(\chi) = \psi^{(n,e)}(\chi)$ holds for any integer $n \geq 1$. In particular, $\psi^n(\chi) = \chi$ holds for any integer $n \geq 1$ with $(n, e) = 1$. Use Lemma 3.18, there exists a polynomial $f \in \mathbb{Z}[t]$ such that $\psi^k(\chi)(g) = f(\omega^k)$ holds for any integer $k \geq 1$.

Let $\sigma$ be an element of $\Gamma_e$. Then, an integer $n$ satisfying $\sigma(\omega) = \omega^n$ is coprime to $e$. Thus, we have

$$\sigma(\chi(g)) = \sigma(f(\omega)) = f(\omega^n) = \psi^n(\chi)(g) = \chi(g).$$

Then, $\chi(g)$ is a rational number, which implies that $\chi(g)$ is an integer by Lemma 2.20 and Lemma 3.18. □

*Proof of Theorem 3.17.* The element $\chi$ is a virtual character. By Proposition 2.9 and Theorem 2.18, elements $\lambda^i(\chi)$ are also virtual characters for all integers $i \geq 0$. Since $CF(G)$ is a $\lambda$-ring, the $n$-th Adams operation $\psi^n$ is a $\lambda$-homomorphism by Theorem 2.12. Let $\chi$ be an integer-valued character. By Lemma 3.19, we have

$$\psi^n(\lambda^i(\chi)) = \lambda^i(\psi^n(\chi)) = \lambda^i(\psi^{(n,e)}(\chi)) = \psi^{(n,e)}(\lambda^i(\chi))$$

for any integers $i \geq 0$ and $n \geq 1$. We use Lemma 3.19 again, as a result, $\lambda^i(\chi)$ is an integer-valued character. □

Next, we characterize integer-valued characters with necklace rings and truncated operations.

**Theorem 3.20.** *A virtual character $\chi$ is an integer-valued character if and only if $E_{Nr}(\lambda_t(\chi))$ belongs to the image of the truncated operation $T_e$. In particular, if $\chi$ satisfies such conditions, we have the following finite product form of $\lambda_t(\chi)$,*

$$\lambda_t(\chi) = \prod_{d|e}(1 - (-t)^d)^{\alpha_d}$$

*where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = E_{Nr}(\lambda_t(\chi))$.*

*Proof.* By Lemma 3.19, a virtual character $\chi$ is an integer-valued character if and only if $\psi^n(\chi) = \psi^{(n,e)}(\chi)$ holds for any integer $n \geq 1$. The map $\phi : Nr(CF(G)) \to CF(G)^{\mathbb{N}}$ preserves the truncated operation $T_e$ by Proposition 3.9, and the $n$-th element of $\phi(\alpha) = \phi(E_{Nr}(\lambda_t(\chi)) = z(\lambda_t(\chi))$ is $\psi^n(\chi)$ for any $n \geq 1$. Thus, $\psi^n(\chi) = \psi^{(n,e)}(\chi)$ holds for any integer $n \geq 1$ if and only if $\alpha_n = 0$ holds for any integer $n \geq 1$ with $n \mid e$ by Corollary 3.10. Hence, the element $\alpha$ belongs to the image of the operation $T_e$ by Proposition 3.6. □

Let $\chi$ be a virtual character and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = E_{Nr}(\lambda_t(\chi))$. In Theorem 3.20 we saw that $\alpha_n = 0$ holds for any integer $n \geq 1$ with $n \nmid e$ if $\chi$ is an integer-valued character.

Next, we fix an element $g \in G$ satisfying that $\chi(g^k)$ is an integer for any integer $k \geq 1$, and study a power series $\lambda_t(\chi)(g)$. Put $(\beta_1, \beta_2, \beta_3, \ldots) = Nr(\mathrm{E}_g^G)(\alpha)$, and we show that the number of integers $n \geq 1$ such that $\beta_n \neq 0$ is equal to, or less than the number of divisors $d$ of $e$ by the following theorem.

**Theorem 3.21.** *Let $\chi$ be a virtual character and we denote $\alpha = E_{Nr}(\lambda_t(\chi))$. For any $g \in G$ satisfying that $\chi(g^k)$ is an integer for any integer $k \geq 1$, all elements of $Nr(\mathrm{E}_g^G)(\alpha)$ are integers and $Nr(\mathrm{E}_g^G)(\alpha)$ belongs to the image of $T_{O(g)}$. In particular, the power series $\lambda_t(\chi)(g)$ has the following form,*

$$\lambda_t(\chi)(g) = \prod_{d|O(g)} (1 - (-t)^d)^{\beta_d}$$

*where $(\beta_1, \beta_2, \beta_3, \ldots) = Nr(\mathrm{E}_g^G)(\alpha)$.*

To prove Theorem 3.21, we show the following lemma.

**Lemma 3.22.** *For any subgroups $H$ of $G$, the restriction map $\mathrm{Res}_H^G : CF(G) \to CF(H)$ is a $\lambda$-homomorphism.*

*Proof.* By Proposition 2.10 and that $CF(H)$ is $\mathbb{Z}$-torsion free, we show $\psi^n \circ \mathrm{Res}_H^G = \mathrm{Res}_H^G \circ \psi^n$ for any integer $n \geq 1$. For any $\chi \in CF(G)$ and $h \in H$, we have

$$\psi^n(\mathrm{Res}_H^G(\chi))(h) = \chi(h^n) = \mathrm{Res}_H^G(\psi^n(\chi))(h).$$

Hence, this lemma holds. □

*Proof of Theorem 3.21.* By the assumption, $\mathrm{Res}_{\langle g \rangle}^G(\chi)$ is an integer-valued character of $\langle g \rangle$. Thus, $\lambda_t(\chi)(g)$ belongs to $\Lambda(\mathbb{Z})$ by Lemma 3.22 and Theorem 3.17. The set of all integers $\mathbb{Z}$ is a binomial ring with the binomial coefficient ([Yau] §5.1). Hence, all elements of $Nr(\mathrm{E}_g^G)(\alpha)$ are integers.

Next, we consider $Nr(\mathrm{E}_g^G)(\alpha)$. One has

$$Nr(\mathrm{E}_g^G)(\alpha) = Nr(\mathrm{E}_g^G) \circ Nr(\mathrm{Res}_{\langle g \rangle}^G)(\alpha) = Nr(\mathrm{E}_g^G) \circ E_{Nr}(\lambda_t(\mathrm{Res}_{\langle g \rangle}^G(\chi))).$$

The exponent of a cyclic subgroup $\langle g \rangle$ is $O(g)$. Hence $Nr(\mathrm{E}_g^G)(\alpha)$ belongs to the image of $T_{O(g)}$. □

## 3.4 Finite support elements of a necklace ring

Let $R$ be a commutative ring. First, we define the following sets and properties associate with elements of $Nr(R)$ and $R^{\mathbb{N}}$.

**Definition 3.23.** For each $x = (x_1, x_2, x_3, \ldots) \in Nr(R)$, we define $\mathrm{supp}(x)$ by the set of integers $n \geq 1$ such that $x_n \neq 0$ holds. If $\mathrm{supp}(x)$ is finite, then we call that $x$ has the finite support.

**Lemma 3.24.** *An element of the image of $T_r$ has the finite support for any integer $r \geq 1$.*

*Proof.* Let $x$ be an element of the image of $T_r$. Then, the set $\mathrm{supp}(x)$ is contained the set of all divisor of $r$. In particular, the set $\mathrm{supp}(x)$ is finite. □

**Definition 3.25.** Let $c \geq 1$ be an integer. An element $a = (a_1, a_2, a_3, \ldots) \in R^{\mathbb{N}}$ has the $c$-cycle if $a_i = a_j$ holds for any integers $i \geq j \geq 1$ with $c \mid i - j$.

All elements of the image of the truncated operation $T_r$ has the $r$-cycle. As another example, all elements of $z \circ \lambda_t(CF(G))$ have the $e$-cycle by the definition of Adams operation of $CF(G)$ (see §2.3), where $e$ is the exponent of a finite group $G$.

Let $G$ be a finite group, let $\chi$ be a virtual character of $G$, and put $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = E_{Nr}(\lambda_t(\chi))$. In §3.3, we prove that if a virtual character $\chi$ is an integer-valued character, then $\alpha_n = 0$ holds for any integer $n \geq 1$ satisfying that $n$ does not divide the exponent of $G$. In particular, the element $E_{Nr}(\lambda_t(\chi))$ has the finite support.

In §3.4, we show that if $E_{Nr}(\lambda_t(\chi))$ has the finite support, then $\chi$ is an integer-valued character by the following theorem.

**Theorem 3.26.** *Suppose that $R$ is $\mathbb{Z}$-torsion free. For any element $x \in Nr(R)$ satisfying that $\phi(x)$ has the $c$-cycle, the element $x$ has the finite support if and only if $x$ belongs to the image of $T_c$.*

By Theorem 3.26 as $R = CF(G)$ and Theorem 3.20, we obtain the following corollary.

**Corollary 3.27.** *A virtual character $\chi$ is an integer-valued character if and only if $E_{Nr}(\lambda_t(\chi))$ has the finite support.*

First, we state the following lemma.

**Lemma 3.28.** *If an element of $x \in Nr(R)$ has the finite support, then $F_r(x)$ also has the finite support for any integer $r \geq 1$.*

*Proof.* Put $y = F_r(x)$. We prove that if $m \in \mathrm{supp}(y)$, then there exists integer $j$ such that $j \geq m$ and $j \in \mathrm{supp}(x)$ hold.

By the definition of Frobenius operation $F_r$, we have

$$y_m = \sum_{[j,r]=mr} \frac{j}{m} x_j.$$

Since $y_m \neq 0$ holds, there exists an integer $j$ such that $[j,r] = mr$ and $j \in \mathrm{supp}(x)$ hold. Hence, we have $mr = [j,r] \leq jr$, and then $m \leq j$. □

**Lemma 3.29.** *If an element $a \in R^{\mathbb{N}}$ has the $c$-cycle, then $F_r(a)$ has the $c/(c,r)$-cycle for any integer $r \geq 1$.*

*Proof.* Put $b = (b_1, b_2, b_3, \ldots) = F_r(a)$. For any integers $k_1, k_2 \geq 1$, we have

$$b_{(k_1 c/(c,r))+k_2} = a_{(k_1 rc/(c,r))+k_2 r} = a_{k_2 r} = b_{k_2}.$$

Hence, the element $b$ has the $c/(c,r)$-cycle. □

**Lemma 3.30.** *Let $r \geq 1$ be an integer with $(c,r) = 1$. If an element $a \in R^{\mathbb{N}}$ has the $c$-cycle, then there exists an integer $s \geq 1$ such that $r \mid s$ and $a = F_s(a)$ holds.*

*Proof.* By $(c, r) = 1$, there exist integers $p < 0$ and $q > 0$ such that $pc + br = 1$ holds.

We put $s = qr$. Then $r \mid s$. If $a = (a_1, a_2, a_3, \ldots)$ and $b = (b_1, b_2, b_3, \ldots) = F_s(a)$, then

$$b_n = a_{sn} = a_{(1-pc)n} = a_n$$

holds for any integer $n \geq 1$, that is, $a = F_s(a)$ holds. $\square$

We define the following maps.

**Definition 3.31.** For each $n \geq 1$, we define a map $d_n : \mathbb{N} \to \mathbb{N}$ as follows: The number $d_n(k)$ is the maximum number satisfying that $d_n(k) \mid k$ holds and $d_n(k)$ and $n$ are coprime, for any integer $k \geq 1$.

**Lemma 3.32.** *Suppose that $R$ is $\mathbb{Z}$-torsion free, and let $x$ be an element of $Nr(R)$ satisfying that $\phi(x)$ has the c-cycle. If the set $d_c(\mathrm{supp}(x))$ is finite, then the set $d_c(\mathrm{supp}(x))$ has the unique element $1 \in \mathbb{N}$.*

*Proof.* Let $n$ be an element of $\mathrm{supp}(x)$. We show $d_c(n) = 1$. By the assumption, we can put the least common multiple of all elements of $d_c(\mathrm{supp}(x))$, written by $l$. Two integers $l$ and $c$ are coprime. Thus, there exists an integer $s$ such that $l \mid s$ and $\phi(x) = F_s(\phi(x))$ holds by Lemma 3.30. The map $\phi$ preserves the Frobenius operation $F_s$, thus we have $\phi(x) = F_s(\phi(x)) = \phi(F_s(x))$. Since $R$ is $\mathbb{Z}$-torsion free, we have $x = F_s(x)$. Thus,

$$x_n = \sum_{[j,s]=ns} \frac{j}{n} x_j \tag{3.4}$$

holds where $x = (x_1, x_2, x_3, \ldots)$.

By $x_n \neq 0$, there exists an integer $j \in \mathrm{supp}(x)$ such that $[j, s] = ns$ holds. In particular, the integer $j$ satisfies $j/(j, s) = n$. By the definition of the map $d_c$, we have $d_c(j) \mid j$. On the other hand we have $d_c(j) \mid s$ by $j \in \mathrm{supp}(x)$. Thus we have $d_c(j) \mid (j, s)$. Hence $d_c(n) = d_c(j/(j, s)) = 1$ holds. $\square$

*Proof of Theorem 3.26.* We assume that $x$ belongs to the image of $T_c$. Then, the element $x$ has the finite support by Lemma 3.24. Conversely, we assume that $x$ has the finite support, and we show $x_n = 0$ for any integer $n \geq 1$ with $n \nmid c$ where $x = (x_1, x_2, x_3, \ldots)$.

We denote the prime decomposition of $c$ and $n$ by $c = p_1^{e_1} \cdots p_r^{e_r}$ and $n = p_1^{f_1} \cdots p_r^{f_r}$ ($e_i, f_i \geq 0, i = 1, \ldots, r$). Let $i = 1, \ldots, r$ be an integer, and put $c_i = c/p_i^{e_i}$. Since the set $\mathrm{supp}(x)$ is finite by Lemma 3.28, the set $d_{c_i}(\mathrm{supp}(F_{p_i^{e_i}}(x)))$ is also finite. By Lemma 3.29, the element $\phi(F_{p_i^{e_i}}(x))$ has the $c_i$-cycle. Using Lemma 3.32, we have

$$d_{c_i}(\mathrm{supp}(F_{p_i^{e_i}}(x))) = \{1\}. \tag{3.5}$$

By $n \nmid c$, there exists an integer $k = 1, \ldots, r$ such that $p_k \mid n/p_k^{e_k}$ holds. Let $y = (y_1, y_2, y_3, \ldots) = F_{p_k^{e_k}}(x)$. Then, we have $p_k \mid d_{c_k}(n/p_k^{e_k})$ since $p_k \nmid c_k$ holds.

That is, one has $d_{c_k}(n/p_k^{e_k}) \neq 1$. Thus, we have $y_{n/p_k^{e_k}} = 0$ by (3.5). On the other hand, using the definition of Frobenius operations we have

$$y_{n/p_k^{e_k}} = \sum_{[j,p_k^{e_k}]=n} \frac{jp_k^{e_k}}{n} x_j = p_k^{e_k} x_n$$

because if the integer $j$ satisfying $[j, p_k^{e_k}] = n$, then $j = n$ holds. Thus we can obtain $x_n = 0$ since $R$ is $\mathbb{Z}$-torsion-free.

Hence, the element $x$ belongs to the image of $T_c$ by Proposition 3.6. □

## 3.5 Multiplication and Frobenius operation

In §3.5, we discuss product groups and Frobenius operations to calculate $\lambda_t(\chi)(g)$ for a virtual character $\chi$ and an element $g$ of a finite group. Moreover, we discuss the multiplication of two elements of the image of truncation operations with Frobenius operations. First, we state the following lemma.

**Lemma 3.33.** *Let $\chi$ be a character of a finite group $G$ and let $g$ be an element of $G$. Then, for any integer $k \geq 1$, the $k$-th element of $z(\Lambda(\mathrm{E}_g^G)(\lambda_t(\chi)))$ is $\chi(g^k)$.*

*Proof.* By Proposition 2.16, we have

$$z(\Lambda(\mathrm{E}_g^G)(\lambda_t(\chi))) = \mathrm{E}_g^{G^{\mathbb{N}}} \circ z \circ \lambda_t(\chi).$$

Hence, the $k$-th element of $z(\Lambda(\mathrm{E}_g^G)(\lambda_t(\chi)))$ is $\mathrm{E}_g^G(\psi^k(\chi))$, which coincides with $\chi(g^k)$ by the definition of Adams operations on $CF(G)$. □

We state the following theorem.

**Theorem 3.34.** *Let $G_1$ and $G_2$ be finite groups, and let $\chi_1$ (resp. $\chi_2$) be a virtual character of $G_1$ (resp. $G_2$). Then, the virtual character $\chi_1\chi_2$ of $G_1 \times G_2$ defined by $\chi_1\chi_2(g_1, g_2) := \chi_1(g_1)\chi_2(g_2)$ satisfies*

$$\lambda_t(\chi_1\chi_2)(g_1g_2) = \lambda_t(\chi_1)(g_1) \cdot_\Lambda \lambda_t(\chi_2)(g_2). \tag{3.6}$$

*Proof.* We consider Adams operations. Let $g_1 \in G_1$, $g_2 \in G_2$, and

$$\begin{aligned} a = (a_1, a_2, a_3, \ldots) &= z(\Lambda(\mathrm{E}_{(g_1,g_2)}^{G_1 \times G_2})(\lambda_t(\chi_1\chi_2))), \\ b = (b_1, b_2, b_3, \ldots) &= z(\Lambda(\mathrm{E}_{g_1}^{G_1})(\chi_1)), \\ c = (c_1, c_2, c_3, \ldots) &= z(\Lambda(\mathrm{E}_{g_2}^{G_2})(\chi_2)). \end{aligned}$$

Then, for any integer $k \geq 1$, we have $a_k = \chi_1\chi_2((g_1, g_2)^k) = \chi_1(g_1^k)\chi_1(g_2^k)$, $b_k = \chi_1(g_1^k)$ and $c_k = \chi_2(g_2^k)$ by Lemma 3.33. Thus, we have $a_k = b_k c_k$, and hence, we have $a = bc$. Since the complex field $\mathbb{C}$ is $\mathbb{Z}$-torsion free, the map $z : \Lambda(\mathbb{C}) \to \mathbb{C}^{\mathbb{N}}$ is an injective ring homomorphism. Hence, this theorem holds. □

Next, we prove the following theorem.

**Theorem 3.35.** *For any virtual character $\chi$ of a finite group $G$ and integer $k \geq 1$, we have $E_{Nr}(\lambda_t(\chi)(g^k)) = F_k \circ E_{Nr}(\lambda_t(\chi)(g))$.*

*Proof.* For any integer $k \geq 1$, we have

$$\phi \circ E_{Nr}(\lambda_t(\chi)(g^k)) \quad = \quad z(\Lambda(\mathrm{E}^G_{g^k})(\lambda_t(\chi))) \tag{3.7}$$

and

$$\phi \circ F_k \circ E_{Nr}(\lambda_t(\chi)(g)) \quad = \quad F_k(z(\Lambda(\mathrm{E}^G_g)(\lambda_t(\chi)))). \tag{3.8}$$

by Proposition 2.4. Then $n$-th elements of both (3.7) and (3.8) coincide with $\chi(g^{nk})$ by Lemma 3.33. Hence, we have

$$\phi \circ E_{Nr}(\lambda_t(\chi)(g^k)) = \phi \circ F_k \circ E_{Nr}(\lambda_t(\chi)(g)).$$

Since the map $\phi$ is injective, this proposition holds. □

Next, we discuss the problem to find another form of

$$T_r(x) \cdot_{Nr} T_s(y) \tag{3.9}$$

with Frobenius operation for any elements $x$ and $y$ of $Nr(R)$. For example, we consider when $r = 12$ and $s = 18$. Put $x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) = x \cdot_{Nr} y$. We have

$$\alpha_{36} = \sum_{[i,j]=36} (i, j)x_i y_j = x_4y_9 + 3x_{12}y_9 + 2x_4y_{18} + 6x_{12}y_{18}$$

by the definition of multiplication of $Nr(R)$. However, using Frobenius operation we have $\alpha_{36} = (x_4 + 3x_{12})(y_9 + 2y_{18}) = u_4v_9$ where $u = (u_1, u_2, u_3, \ldots) = F_3(x)$ and $v = (e_1, e_2, e_3, \ldots) = F_2(y)$, which gives a factorization. The calculation of (3.9) appears, for example, in the identities obtained from (3.6) using the map $E_{Nr}$ when virtual characters $\chi_1$ and $\chi_2$ are integer-valued characters. These results of this section will be used in the next section.

So, we calculate $T_r(x) \cdot_{Nr} T_s(y)$. However, we only need to calculate the $[r, s]$-th element since the following two lemmas hold.

**Lemma 3.36.** *Let $x$ and $y$ be elements of $Nr(R)$. For any integers $r, s \geq 1$, the element $T_r(x) \cdot_{Nr} T_s(y)$ belongs to the image of $T_{[r,s]}$.*

*Proof.* By Proposition 3.7, we have

$$\begin{aligned} T_{[r,s]}(T_r(x) \cdot_{Nr} T_s(y)) \quad &= \quad T_{[r,s]} \circ T_r(x) \cdot_{Nr} T_{[r,s]} \circ T_s(y) \\ &= \quad T_r(x) \cdot_{Nr} T_s(y). \end{aligned}$$

Hence, the element $T_r(x) \cdot_{Nr} T_s(y)$ belongs to the image of $T_{[r,s]}$. □

**Lemma 3.37.** *For any integers $r, s, u \geq 1$ with $u \mid [r, s]$, we have $[(r, u), (s, u)] = u$.*

*Proof.* We denote the prime decomposition of $r, s, u$ and $[(r, u), (s, u)]$ by $r = p_1^{r_1} \cdots p_q^{r_q}$, $s = p_1^{s_1} \cdots p_q^{s_q}$, $u = p_1^{u_1} \cdots p_q^{u_q}$ and $[(r, u), (s, u)] = p_1^{e_1} \cdots p_q^{e_q}$. Then, we have $u_i \leq \max(r_i, s_i)$ since $u \mid [r, s]$ holds.

By the definition of $e_i$, we have

$$e_i = \max(\min(r_i, u_i), \min(s_i, u_i)).$$

If $r_i \leq u_i$ and $s_i > u_i$, then one has $e_i = \max(r_i, u_i) = u_i$. If $r_i > u_i$ and $s_i \leq u_i$. Thus one has $e_i = \max(u_i, s_i) = u_i$. If $r_i > u_i$ and $s_i > u_i$, then we have $e_i = u_i$.

In any case, we have $e_i = u_i$ for any $i = 1, \ldots, q$, that is, $[(r, u), (s, u)] = u$ holds. $\square$

By Lemma 3.36 and Lemma 3.37, the $n$-th element of $T_r(x) \cdot_{Nr} T_s(y)$ is equal to the $r$-th element of $T_{(a,r)}(x) \cdot_{Nr} T_{(b,r)}(y)$ for any divisor $n$ of $[r, s]$. Hence, we calculate the $[r, s]$-element of $T_r(x) \cdot_{Nr} T_s(y)$.

In the remainder of this section, we denote the $n$-th element of $x \in Nr(R)$ by $x_n$ and the $n$-th element for $a \in R^{\mathbb{N}}$ by $a_n$ for simplicity.

**Theorem 3.38.** *Let $r, s \geq 1$ be integers, and we denote the prime decomposition of $r$ and $s$ by $r = p_1^{r_1} \cdots p_q^{r_q}$ and $s = p_1^{s_1} \cdots p_q^{s_q}$. We define integers $a_1, a_2, a_3, b_1, b_2, b_3 \geq 1$ by*

$$a_1 = \prod_{i, r_i > s_i} p_i^{r_i}, \quad a_2 = \prod_{i, r_i = s_i} p_i^{r_i}, \quad a_3 = \prod_{i, r_i < s_i} p_i^{r_i},$$
$$b_1 = \prod_{i, r_i > s_i} p_i^{s_i}, \quad b_2 = \prod_{i, r_i = s_i} p_i^{s_i}, \quad b_3 = \prod_{i, r_i < s_i} p_i^{s_i}.$$

*For integers $a_1, a_2, a_3, b_1, b_2$ and $b_3$, the followings hold,*

$$\begin{aligned}
&r = a_1 a_2 a_3, \quad s = b_1 b_2 b_3, \\
&(a_1, a_2) = (a_2, a_3) = (a_3, a_1) = 1, \\
&(b_1, b_2) = (b_2, b_3) = (b_3, b_1) = 1, \\
&b_1 \mid a_1, \quad D^{\mathrm{pr}}(a_1) = D^{\mathrm{pr}}(a_1/b_1), \\
&a_2 = b_2, \\
&a_3 \mid b_3, \quad D^{\mathrm{pr}}(b_3) = D^{\mathrm{pr}}(b_3/a_3)
\end{aligned}$$

*where $D^{\mathrm{pr}}(n)$ is the set of all prime divisors of $n$ for $n \geq 1$.*

*Suppose that $R$ is a $\mathbb{Q}$-algebra. Let $x$ and $y$ be elements of $Nr(R)$. Then, we have*

$$(T_r(x) \cdot_{Nr} T_s(y))_{[r,s]} = \frac{1}{a_2} \sum_{d_2 \mid a_2} \mu\Big(\frac{a_2}{d_2}\Big) (F_{a_1 d_2}(x))_{a_3} (F_{d_2 b_3}(y))_{b_1}.$$

*In particular, we have*

$$(T_r(x) \cdot_{Nr} T_s(y))_{[r,s]} = (F_{a_1}(x))_{a_3}(F_{b_3}(y))_{b_1}$$

*when* $a_2 = b_2 = 1$.

*Proof.* We use the fact that the map $\phi : Nr(R) \to R^{\mathbb{N}}$ is bijective and preserves truncated operations $T_r$ and Frobenius operations $F_r$. We modify the left side. Then, we have

$$\begin{aligned}
& (T_{a_1a_2a_3}(x) \cdot_{Nr} T_{b_1b_2b_3}(y))_{[a_1a_2a_3, b_1b_2b_3]} \\
= \;& (\phi^{-1}(T_{a_1a_2a_3} \circ \phi(x))(T_{b_1b_2b_3} \circ \phi(y)))_{a_1a_2b_3} \\
= \;& \frac{1}{a_1a_2b_3} \sum_{d|a_1a_2b_3} \mu\Big(\frac{a_1a_2b_3}{d}\Big)(T_{a_1a_2a_3} \circ \phi(x))_d(T_{b_1b_2b_3} \circ \phi(y))_d \\
= \;& \frac{1}{a_1a_2b_3} \sum_{d_1|a_1} \sum_{d_2|a_2} \sum_{d_3|b_3} \mu\Big(\frac{a_1}{d_1}\Big)\mu\Big(\frac{a_2}{d_2}\Big)\mu\Big(\frac{b_3}{d_3}\Big)(\phi(x))_{d_1d_2a_3}(\phi(y))_{b_1d_2d_3} \\
= \;& \frac{1}{a_2} \sum_{d_2|a_2} \mu\Big(\frac{a_2}{d_2}\Big)\Big\{\frac{1}{a_1} \sum_{d_1|a_1} \mu\Big(\frac{a_1}{d_1}\Big)(F_{d_2a_3} \circ \phi(x))_{d_1}\Big\} \\
& \Big\{\frac{1}{b_3} \sum_{d_3|b_3} \mu\Big(\frac{b_3}{d_3}\Big)(F_{b_1d_2} \circ \phi(x))_{d_3}\Big\} \\
= \;& \frac{1}{a_2} \sum_{d_2|a_2} \mu\Big(\frac{a_2}{d_2}\Big)\Big\{\frac{1}{a_1} \sum_{d_1|a_1} \mu\Big(\frac{a_1}{d_1}\Big)(\phi \circ F_{b_1d_2}(x))_{d_1}\Big\} \\
& \Big\{\frac{1}{b_3} \sum_{d_3|b_3} \mu\Big(\frac{b_3}{d_3}\Big)(\phi \circ F_{d_2a_3}(x))_{d_3}\Big\} \\
= \;& \frac{1}{a_2} \sum_{d_2|a_2} \mu\Big(\frac{a_2}{d_2}\Big)(F_{b_1d_2}(x))_{a_1}(F_{d_2a_3}(y))_{b_3}
\end{aligned}$$

where the second equality follows from Propositions 2.1 and 3.9, the third and seventh equality follow from the definition of the map $\phi$ and the Möbius inversion formula, the fourth and fifth equality follows from the definition of truncated operations and Frobenius operations, respectively, and the sixth equality follows from Proposition 2.4. ☐

# 4 A representation of a symmetric group with a multiplicative anti-symmetric matrix

In this section, we calculate $E_{Nr}(\lambda_t(\chi))$ where a character $\chi$ is defined by the following as an example. We fix an integer $k$, and let $\mathbf{q} = (q_{i,j})_{i,j}$ be a square matrix whose size is $k$, and $\mathbf{q}$ satisfies $q_{i,j}q_{j,i} = 1$ for any integers $i, j = 1, \ldots, k$. This matrix $\mathbf{q}$ was introduced in [BG, Appendix I.10], and called a multiplicative anti-symmetric matrix.

In §4.1, we will define maps with divided Verchiebung operation and Frobenius operations, written by $W_r$, and state properties. In §4.2, we define a representation of a symmetric group $S_n$ associated with $\mathbf{q}$, and will be written by $\rho_{\mathbf{q},n}$. We will calculate the character $\chi_{\mathbf{q},n}$ of $\rho_{\mathbf{q},n}$ in §4.3 and $E_{Nr}(\lambda_t(\chi_{\mathbf{q},n}))((1\,2 \cdots n))$ in §4.4. and we will discuss the calculation method of $E_{Nr}(\lambda_t(\chi_{\mathbf{q},n}))(\sigma)$ for all $\sigma \in S_n$ with Theorem 3.38.

## 4.1 The map $W_r$

In §4.1, we suppose that $R$ is a $\mathbb{Q}$-algebra. First, we introduce new operations on $Nr(R)$ and $R^{\mathbb{N}}$ for §4.4.

**Definition 4.1.** For any integer $r \geq 1$, we define two maps

$$\begin{aligned} W_r &: Nr(R) \to Nr(R), \\ W_r &: R^{\mathbb{N}} \to R^{\mathbb{N}} \end{aligned}$$

by $W_r := V'_r \circ F_r$.

By this definition, the following holds.

**Proposition 4.2.** *The map $\phi$ preserves the map $W_r$ for any integer $r \geq 1$.*

*Proof.* By Proposition 2.4, we have

$$\phi \circ W_r = \phi \circ \frac{1}{r} V'_r \circ F_r = \frac{1}{r} V'_r \circ \phi \circ F_r = \frac{1}{r} V'_r \circ F_r \circ \phi = W_r \circ \phi.$$

Thus, the map $\phi$ preserves $W_r$. $\square$

By Propositions 2.2 and 3.16, we have the following proposition.

**Proposition 4.3.** *The followings hold for any integers $r, s \geq 1$.*

(1) $F_r \circ W_s = W_{s/(r,s)} \circ F_r$,

(2) $T_r \circ W_s = 0$ if $s \nmid r$.

*Proof.* (1) By Proposition 2.2, we have $F_r \circ W_s = F_r \circ V'_s \circ F_s = V'_{s/(r,s)} \circ F_{r/(r,s)} \circ F_s = W_{s/(r,s)} \circ F_r$.

(2) By Proposition 3.16, we have $T_r \circ W_s = T_r \circ V'_s \circ F_s = 0$. $\square$

We will use the following lemma in §4.4.

**Lemma 4.4.** *For any element $a = (a_1, a_2, a_3, \dots) \in R^{\mathbb{N}}$ and integer $r \geq 1$, we have*

$$b_n = \begin{cases} a_n & \text{if } r \mid n, \\ 0 & \text{if } r \nmid n, \end{cases}$$

*where $(b_1, b_2, b_3, \dots) = W_r(a)$.*

*Proof.* Put $c = (c_1, c_2, c_3, \dots) = F_r(a)$. If an integer $n \geq 1$ satisfies $r \mid n$ then $b_n = c_{n/r} = a_n$ holds. If $r \nmid n$, then $b_n = 0$ holds since $b = V'_r(c)$ holds. $\square$

## 4.2 Definition

In §4.2, we define a representation $\rho_{\mathbf{q},n}$. First, we consider a set of generators of $S_n$.

**Lemma 4.5** ([Sag, p.88])**.** *The followings hold for a symmetric group $S_n$.*

(1) *The group $S_n$ is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$ where $\sigma_i = (i\ i+1)$ for any integer $i = 1, \ldots, n-1$.*

(2) *All $\sigma_i$'s satisfy the following relations.*

$$\begin{aligned} \sigma_i^2 &= 1, \qquad i = 1, \ldots, n-1, \\ (\sigma_i \sigma_{i+1})^3 &= 1, \qquad i = 1, \ldots, n-2, \\ (\sigma_i \sigma_j)^2 &= 1, \qquad |i-j| \geq 2. \end{aligned}$$

(3) *Suppose that $n \geq 2$. The group $S_n$ is isomorphic to a group $G_n$ which is generated by $n-1$ elements $\tau_1, \ldots, \tau_{n-1}$ satisfying $\tau_i^2 = 1$ for any integer $i = 1, \ldots, n-1$, $(\tau_i \tau_{i+1})^3 = 1$ for any integer $i = 1, \ldots, n-2$ and $(\tau_i \tau_j)^2 = 1$ for any integer $i, j = 1, \ldots, n-1$ with $|i-j| \geq 2$.*

*Proof.* First, we prove (1). All elements of $S_n$ can be written as some product of cycles. For $(i_1\ i_2 \cdots i_q)$, we have

$$(i_1\ i_2 \cdots i_q) = (i_1\ i_2)(i_2\ i_3) \cdots (i_{q-1}\ i_q).$$

For an element $(i\ j)$ where $i, j = 1, \ldots, n$ with $i < j$, we have $(i\ j) = \sigma \sigma_j \sigma^{-1}$ where $\sigma = (i\ i+1 \cdots j) = \sigma_i \cdots \sigma_{j-1}$. Then, all elements of $S_n$ are generated by $\sigma_1, \ldots, \sigma_{n-1}$.

It is obvious that the statement (2) holds. So, we prove (3). By the definition of $G_n$ and (2), there exists a surjective group homomorphism $\iota : G_n \to S_n$ such that $\iota(\tau_i) = \sigma_i$ holds for any integer $i = 1, \ldots, n$.

We prove $|G_n| = n!$ by induction on $n \geq 2$. If $n = 2$, we have $G_2 = \{1, \tau_1\}$. In particular, the cardinality of $G_2$ is 2, that is, the statement (3) holds when $n = 2$.

Next, let $n \geq 3$ be an integer and we show $|G_n| = n!$ with the assumption that $G_{n-1}$, which is the subgroup generated by $\tau_1, \ldots, \tau_{n-2}$, satisfies $|G_{n-1}| = (n-1)!$.

Put

$$H_i = \begin{cases} G_{n-1} \tau_{n-1} \tau_{n-2} \cdots \tau_i & \text{if } i = 1, \ldots, n-1, \\ G_{n-1} & \text{if } i = n, \end{cases}$$

and $H = \bigcup_{i=1}^{n-1} H_i$. Now, we consider $H_i \tau_j$. If $i = n$, then $H_1 \tau_j \in H$. We assume $i < n$. If $j > i+1$ then $H_i \tau_j = H_i \subset H$ holds. If $j = i+1$, then $H_i \tau_j = H_{i+1} \subset H$. If $j = i$ then $H_i \tau_j = H_{i-1}$. If $j < i$, then

$$\begin{aligned} H_i \tau_j &= (G_{n-1} \tau_{n-1} \tau_{n-2} \cdots \tau_i) \tau_j \\ &= G_{n-1} \tau_{n-1} \tau_{n-2} \cdots \tau_{j+1} \tau_j \tau_{j-1} \tau_j \tau_{j-2} \cdots \tau_i \end{aligned}$$

$$
\begin{aligned}
&= G_{n-1}\tau_{n-1}\tau_{n-2}\cdots\tau_{j+1}\tau_{j-1}\tau_j\tau_{j-1}\tau_{j-2}\cdots\tau_i \\
&= G_{n-1}\tau_{n-1}\tau_{n-2}\cdots\tau_{j+1}\tau_j\tau_{j-1}\tau_{j-2}\cdots\tau_i \subset H.
\end{aligned}
$$

In any case, we have $H_i\tau_j \in H$. Thus, all generators $\tau_1, \dots, \tau_{n-1}$ belong to $H$. Then, we have $G_n = H$. For each $H_i$ and $H_j$, the set $H_i \cap H_j$ has no elements if $i \neq j$. Hence, we have $|G_n| = |H| = n|G_{n-1}| = n!$. $\square$

In §4, we denote by $V$ a vector space over $\mathbb{C}$ with a basis $\{v_1, \dots, v_k\}$.

**Definition 4.6.** We define a bijective linear map $F : V \otimes V \to V \otimes V$ by

$$F(v_i \otimes v_j) := q_{i,j}(v_j \otimes v_i)$$

for any integers $i, j = 1, \dots, k$.

**Proposition 4.7.** *The map $F$ satisfies $F \circ F = \mathrm{id}_V$ and the Yang Baxter equation. That is,*

$$(\mathrm{id}_V \otimes F) \circ (F \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes F) = (F \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes F) \circ (F \otimes \mathrm{id}_V)$$

*holds as a linear map on $V \otimes V \otimes V$.*

*Proof.* First, we show $F \circ F = \mathrm{id}_V$. For any integers $i, j = 1, \dots, n$, we have $F \circ F(v_i \otimes v_j) = q_{i,j}F(v_j \otimes v_i) = q_{i,j}q_{j,i}(v_i \otimes v_j) = v_i \otimes v_j$ by the definition of $\mathbf{q}$. Then, we have $F \circ F = \mathrm{id}_V$.

Next, we show that the map $F$ satisfies the Yang Baxter equation. For any integers $i_1, i_2, i_3 = 1, \dots, k$, we have

$$
\begin{aligned}
& (\mathrm{id}_V \otimes F) \circ (F \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes F)(v_{i_1} \otimes v_{i_2} \otimes v_{i_3}) \\
= \; & q_{i_2,i_3}(\mathrm{id}_V \otimes F) \circ (F \otimes \mathrm{id}_V)(v_{i_1} \otimes v_{i_3} \otimes v_{i_2}) \\
= \; & q_{i_1,i_3}q_{i_2,i_3}(\mathrm{id}_V \otimes F)(v_{i_3} \otimes v_{i_1} \otimes v_{i_2}) \\
= \; & q_{i_1,i_2}q_{i_1,i_3}q_{i_2,i_3}(v_{i_3} \otimes v_{i_2} \otimes v_{i_1}),
\end{aligned}
$$

and

$$
\begin{aligned}
& (F \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes F) \circ (F \otimes \mathrm{id}_V)(v_{i_1} \otimes v_{i_2} \otimes v_{i_3}) \\
= \; & q_{i_1,i_2}(F \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes F)(v_{i_2} \otimes v_{i_1} \otimes v_{i_3}) \\
= \; & q_{i_1,i_3}q_{i_1,i_2}(\mathrm{id}_V \otimes F)(v_{i_2} \otimes v_{i_3} \otimes v_{i_1}) \\
= \; & q_{i_2,i_3}q_{i_1,i_3}q_{i_1,i_2}(v_{i_3} \otimes v_{i_2} \otimes v_{i_1}).
\end{aligned}
$$

Hence, we have $(\mathrm{id}_V \otimes F) \circ (F \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes F) = (F \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes F) \circ (F \otimes \mathrm{id}_V)$. $\square$

**Definition 4.8.** For each integer $n \geq 1$, we define a representation $\rho_{\mathbf{q},n} : S_n \to GL(V^{\otimes n})$ by

$$\rho_{\mathbf{q},n}(\sigma_i) := \mathrm{id}_V^{\otimes(i-1)} \otimes F \otimes \mathrm{id}_V^{\otimes(n-i-1)}$$

where $\sigma_i = (i\ i+1)$.

By Lemma 4.5 and Proposition 4.7, this representation is well-defined.

## 4.3 A representation matrix of $\rho_{\mathbf{q},n}(\sigma)$

In §4.3, for any $\sigma \in S_n$, we consider a representation matrix of the linear map $\rho_{\mathbf{q},n}(\sigma)$ with respect to the basis $B_n$ on $V^{\otimes n}$, which is defined by

$$\begin{aligned} \{ \quad & v_1 \otimes \cdots \otimes v_1, \dots, v_1 \otimes \cdots \otimes v_k, \\ & \dots, \\ & v_k \otimes \cdots \otimes v_1, \dots, v_k \otimes \cdots \otimes v_k\}. \end{aligned}$$

We define the following two serieses of matrices.

**Definition 4.9.** For any integers $i, j = 1, \dots, k$ and $l \geq 0$, we define matrices $F_{l,j,i} \in M(k^l, \mathbb{C})$ by induction on $l \geq 0$, and

$$\begin{aligned} F_{0,j,i} &:= \delta_{j,i}, \\ F_{l,j,i} &:= q_{j,i} \begin{pmatrix} & 0 & \\ F_{l-1,1,i} & \dots & F_{l-1,k,i} \\ & 0 & \end{pmatrix} \quad (l \geq 1) \end{aligned}$$

where matrices $F_{l-1,1,i}, \dots, F_{l-1,k,i}$ lies on $j$-th row of $F_{l,j,i}$ for $l \geq 1$.

**Definition 4.10.** For any integer $l \geq 1$, we define a matrix $P(\mathbf{q}, l) \in M(k^l, \mathbb{C})$ by $P(\mathbf{q}, l) := (F_{l-1,j,i})_{i,j}$.

First, we consider $\rho_{\mathbf{q},n}(\sigma)$ when $\sigma$ is a cycle of length $p$.

**Proposition 4.11.** *Let $m$ and $p$ be integers with $m + (p-1) < n$. Then the matrix of the linear map $\rho_{\mathbf{q},n}((m\ m+1 \cdots m+(p-1)))$ with respect to the basis $B_n$ is equal to $I_k^{\otimes(m-1)} \otimes P(\mathbf{q}, p) \otimes I_k^{(n-m+(p-1))}$.*

To prove Proposition 4.11, we state the following lemmas.

**Lemma 4.12.** *The matrix of the linear map $F : V \otimes V \to V \otimes V$ with respect to the basis $B_2$ is $P(\mathbf{q}, 2)$.*

*Proof.* For any integer $j = 1, \dots, k$, we have

$$\begin{aligned} & (F(v_j \otimes v_1), \dots, F(v_j \otimes v_k)) \\ = \ & (q_{j,i}(v_1 \otimes v_j), \dots, q_{j,n}(v_k \otimes v_j)) \\ = \ & \sum_{p=1}^{k} (v_p \otimes v_1, \dots, v_p \otimes v_k) F_{1,j,p} \\ = \ & (v_1 \otimes v_1, \dots, v_1 \otimes v_k, \dots, v_k \otimes v_1, \dots, v_k \otimes v_k) \begin{pmatrix} F_{1,j,1} \\ \vdots \\ F_{1,j,k} \end{pmatrix}. \end{aligned}$$

Hence, this lemma holds. $\square$

**Lemma 4.13.** *For any integer $l \geq 2$,*

$$F_{l,j,i} = (F_{l-1,j,i} \otimes I_k)(I_k^{\otimes(l-2)} \otimes P(\mathbf{q}, 2)) \tag{4.1}$$

*holds.*

*Proof.* We show this lemma by induction on $l \geq 2$. If $l = 2$, then the identity (4.1) holds by

$$\begin{aligned}
(F_{1,j,i} \otimes I_k)P(\mathbf{q}, 2) &= q_{j,i} \begin{pmatrix} 0 & & 0 \\ & I_k & \\ 0 & & 0 \end{pmatrix} \begin{pmatrix} F_{1,1,1} & \dots & F_{1,k,1} \\ \vdots & \ddots & \vdots \\ F_{1,1,k} & \dots & F_{1,k,k} \end{pmatrix} \\
&= \begin{pmatrix} & 0 & \\ F_{1,1,i} & \dots & F_{1,k,i} \\ & 0 & \end{pmatrix} = F_{2,j,i}
\end{aligned}$$

where the matrix $I_k$ is the $(j, i)$ block in $(F_{1,j,i} \otimes I_k)$ and matrices $F_{1,1,i}, \dots, F_{1,k,i}$ lies on $j$-th row.

We assume that the identity (4.1) holds for any integer $m \geq l - 1$. By this assumption, we have

$$\begin{aligned}
&(F_{l-1,j,i} \otimes I_k)(I_k^{\otimes(l-2)} \otimes P(\mathbf{q}, 2)) \\
= &\begin{pmatrix} & 0 & \\ q_{j,i}(F_{l-2,1,i} \otimes I_k) & \dots & q_{j,i}(F_{l-2,k,i} \otimes I_k) \\ & 0 & \end{pmatrix} \\
&\begin{pmatrix} I_k^{\otimes(l-3)} \otimes P(\mathbf{q}, 2) & & 0 \\ & \ddots & \\ 0 & & I_k^{\otimes(l-3)} \otimes P(\mathbf{q}, 2) \end{pmatrix} \\
= &\ q_{j,i} \begin{pmatrix} & 0 & \\ F_{l-1,1,i} & \dots & F_{l-1,k,i} \\ & 0 & \end{pmatrix} = F_{l,j,i}
\end{aligned}$$

where matrices $q_{j,i}(F_{l-2,1,i} \otimes I_k), \dots, q_{j,i}(F_{l-2,k,i} \otimes I_k)$ and $F_{l-1,1,i}, \dots, F_{l-1,k,i}$ lie on $j$-th row. Hence, the identity (4.1) holds for any $l \geq 2$. $\square$

**Lemma 4.14.** *For any integer $l \geq 2$,*

$$P(\mathbf{q}, l) = (P(\mathbf{q}, l - 1) \otimes I_k)(I_k^{\otimes(l-2)} \otimes P(\mathbf{q}, 2))$$

*holds.*

*Proof.* We use (4.1) to prove this lemma. If $l = 2$, then we have

$$(P(\mathbf{q}, 1) \otimes I_k)P(\mathbf{q}, 2) = I_{k^2} P(\mathbf{q}, 2) = P(\mathbf{q}, 2).$$

If $l \geq 3$, then we have

$$(P(\mathbf{q}, l - 1) \otimes I_k)(I_k^{\otimes(l-2)} \otimes P(\mathbf{q}, 2))$$

$$
\begin{aligned}
&= (F_{l-2,j,i} \otimes I_k)_{i,j} \begin{pmatrix} I_k^{\otimes(l-3)} \otimes P(\mathbf{q},2) & & 0 \\ & \ddots & \\ 0 & & I_k^{\otimes(l-3)} \otimes P(\mathbf{q},2) \end{pmatrix} \\
&= ((F_{l-2,j,i} \otimes I_k)(I_k^{\otimes(l-3)} \otimes P(\mathbf{q},2)))_{i,j} = (F_{l-1,j,i})_{i,j} = P(\mathbf{q},l).
\end{aligned}
$$

Hence, this lemma holds. $\square$

*Proof of Proposition 4.11.* Let $\rho'_{\mathbf{q},n}(\sigma)$ be a matrix of the linear map $\rho_{\mathbf{q},n}(\sigma)$ with respect to the basis $B_n$. We prove Proposition 4.11 by induction on $p \geq 2$.

We assume $p = 2$. In this case, we have $\rho'_{\mathbf{q},n}((m\ m+1)) = \mathrm{id}_V^{\otimes i-1} \otimes P(\mathbf{q},2) \otimes \mathrm{id}_V^{\otimes(n-i+1)}$ by Lemma 4.12. Thus, this proposition holds if $p = 2$.

For any integer $p \geq 2$, we suppose that this proposition holds when $p-1$. For any integer $m \geq 1$,

$$(m\ m+1 \cdots m+(p-1)) = (m\ m+1 \cdots m+(p-2))(m+(p-2)\ m+(p-1))$$

holds. Thus,

$$
\begin{aligned}
& \rho'_{\mathbf{q},n}((m\ m+1 \cdots m+(p-1))) \\
= \ & \rho'_{\mathbf{q},n}((m\ m+1 \cdots m+(p-2)))(\rho'_{\mathbf{q},n}((m+(p-2)\ m+(p-1))) \\
= \ & (I_k^{\otimes(m-1)} \otimes (P(\mathbf{q},p-1) \otimes I_k) \otimes I_k^{\otimes(n-m+(p-1))}) \\
& (I_k^{\otimes(m-1)} \otimes (I^{\otimes(p-2)} \otimes P(\mathbf{q},2)) \otimes I_k^{\otimes(n-m+(p-1))}) \\
= \ & I_k^{\otimes(m-1)} \otimes P(\mathbf{q},p) \otimes I_k^{\otimes(n-m+(p-1))}
\end{aligned}
$$

holds. Hence, this proposition holds. $\square$

By Proposition 4.11, the following corollary holds.

**Corollary 4.15.** *Put* $\pi_1 = (1\ 2 \cdots \lambda_1), \pi_2 = (\lambda_1+1\ \cdots\ \lambda_1+\lambda_2), \ldots, \pi_q = (\lambda_1+\cdots+\lambda_{q-1}+1\ \cdots\ n) \in S_n$. *Then, a matrix of* $\rho_{\mathbf{q},n}(\pi_1 \cdots \pi_q)$ *with respect to the basis* $B_n$ *is* $P(\mathbf{q},\lambda_1) \otimes \cdots \otimes P(\mathbf{q},\lambda_q)$.

## 4.4 Calculation of $E_{Nr}(\lambda_t(\chi))$

In §4.4, we show the calculation method of $E_{Nr}(\lambda_t(\chi_{\mathbf{q},n}))$, where $\chi_{\mathbf{q},n}$ is the character of the representation $\rho_{\mathbf{q},n}$.

One has $\phi(E_{Nr}(\lambda_t(\chi_{\mathbf{q},n}))) = z \circ \lambda_t(\chi_{\mathbf{q},n})$, hence its $n$-th element is $\psi^n(\chi_{\mathbf{q},n})$. Then, we investigate $\chi_{\mathbf{q},n}$ in order to calculate $E_{Nr}(\lambda_t(\chi))$. We investigate $\chi_{\mathbf{q},n}$ by the following propositions.

**Proposition 4.16.** *For any integer* $l \geq 2$, *we have*

$$\mathrm{Tr}(P(\mathbf{q},l)) = \begin{cases} \mathrm{Tr}(\mathbf{q}) & \text{if } 2 \mid l, \\ k & \text{if } 2 \nmid l. \end{cases}$$

*In particular, the number* $\mathrm{Tr}(P(\mathbf{q},l))$ *is an integer.*

*Proof.* First, we prove $\mathrm{Tr}(F_{l,j,i}) = \delta_{j,i} q_{j,i}^l$ by induction on $l \geq 0$. If $l = 0$, then we have $\mathrm{Tr}(F_{0,j,i}) = \delta_{j,i} = \delta_{j,i} q_{j,i}^0$. Let $l \geq 1$ be an integer and we assume that $\mathrm{Tr}(F_{l-1,j,i}) = \delta_{j,i} q_{j,i}^{l-1}$. Then, we have

$$\mathrm{Tr}(F_{l,j,i}) = \mathrm{Tr}(q_{j,i} F_{l-1,j,i}) = q_{j,i} \delta_{j,i} q_{j,i}^{l-1} = \delta_{j,i} q_{j,i}^l.$$

For $P(\mathbf{q}, l)$, we have

$$\mathrm{Tr}(P(\mathbf{q}, l)) = \sum_{i=1}^{k} \mathrm{Tr}(F_{l-1,i,i}) = \sum_{i=1}^{k} q_{i,i}^{l-1}$$

Thus, for any integer $i = 1, \ldots, k$ we have

$$q_{i,i}^{l-1} = \begin{cases} q_{i,i} & \text{if } 2 \mid l, \\ 1 & \text{if } 2 \nmid l. \end{cases}$$

Hence, we have

$$\mathrm{Tr}(P(\mathbf{q}, l)) = \sum_{i=1}^{k} q_{i,i}^{l-1} = \begin{cases} \mathrm{Tr}(\mathbf{q}) & \text{if } 2 \mid l, \\ k & \text{if } 2 \nmid l. \end{cases}$$

By the definition of $\mathbf{q}$, we have $q_{i,i} q_{i,i} = 1$, which means that $q_{i,i} = 1$ or $q_{i,i} = -1$ holds for any integer $i = 1, \ldots, k$. Thus, the number $\mathrm{Tr}(\mathbf{q})$ is an integer. Hence, the number $\mathrm{Tr}(P(\mathbf{q}, l))$ is an integer. □

By Corollary 4.15, Proposition 4.16 and Lemma A.1, we have the following proposition.

**Proposition 4.17.** *Let $\sigma$ be an element of $S_n$, and let $\{\lambda_1, \ldots, \lambda_m\}$ be the cycle structure of $\sigma$. Then, we have*

$$\chi_{\mathbf{q},n}(\sigma) = k^{s_1(\sigma)} \mathrm{Tr}(\mathbf{q})^{s_2(\sigma)}$$

*where $s_1(\sigma)$ is the number of $l = 1, \ldots, m$ such that $2 \nmid \lambda_l$ holds, and $s_2(\sigma)$ is the number of $l = 1, \ldots, m$ such that $2 \mid \lambda_l$ holds. In particular, the character $\chi_{\mathbf{q},n}$ is an integer-valued character.*

**Remark 4.18.** Remark that if matrices $\mathbf{q} = (q_{i,j})_{i,j}, \mathbf{q}' = (q'_{i,j})_{i,j} \in M(k, \mathbb{C})$ satisfying $q_{i,j} q_{j,i} = 1$ and $q'_{i,j} q'_{j,i} = 1$ for any integers $i, j = 1, \ldots k$, if $\mathrm{Tr}(\mathbf{q}) = \mathrm{Tr}(\mathbf{q}')$ then two representation $\rho_{\mathbf{q},n}$ and $\rho_{\mathbf{q}',n}$ are isomorphic.

First, we calculate $E_{Nr}(\lambda_t(\chi_{\mathbf{q},n})(\sigma))$ when $\sigma = (1\,2 \cdots n)$. Next, we calculate when $\sigma = (1\ 2 \cdots n)^r$ for any integer $r \geq 1$. Finally, we consider all $\sigma \in S_n$ by using Theorem 3.38.

To calculate $E_{Nr}(\lambda_t(\chi))$, we use the following identity.

**Definition 4.19.** For any integer $l$, we define $M(l) = (m_1, m_2, m_3, \ldots) \in Nr(\mathbb{C})$ by

$$m_n := \frac{1}{n}\sum_{d|n}\mu\Big(\frac{n}{d}\Big)l^n$$

for any integer $n \geq 1$.

Note that if $l \geq 1$, then $m_n$ is said to be the necklace polynomial in [MR]. We will define necklace polynomials again in (5.1).

For $M(l)$, the following proposition holds.

**Lemma 4.20.** *The followings hold.*

(1) *The $n$-th element of $\phi(M(l))$ is $l^n$ for any integer $l$ and $n \geq 1$.*

(2) *We have $F_r(M(l)) = M(l^r)$ for any integer $l$ and $r \geq 1$.*

*Proof.* For the statement (1), the $n$-th element of $\phi(M(l))$ is

$$\sum_{d|n} dm_d = \sum_{d|n}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)l^{d'} = l^n$$

by the Möbius inversion formula, where $(m_1, m_2, m_3, \ldots) = M(l)$. The statement (2) was proved by Metropolis and Rota [MR, p.104, Proposition 2]. $\square$

First, we consider $E_{Nr}(\lambda_t(\chi_{\mathbf{q},n})(\sigma))$ when $\sigma = (1\ 2 \cdots\ n)$.

**Proposition 4.21.** *Let $n \geq 1$ be an integer and put $\sigma = (1\ 2 \cdots n)$. Then,*

$$E_{Nr}(\lambda_t(\chi_{\mathbf{q},n})(\sigma)) = T_n(M(\mathrm{Tr}(\mathbf{q})) + W_{2^c}(M(k) - M(\mathrm{Tr}(\mathbf{q}))))$$

*holds where $c \geq 1$ is the maximum integer satisfying $2^c \mid n$.*

*Proof.* We show that two elements

$$a = (a_1, a_2, a_3, \ldots) = \phi(E_{Nr}(\lambda_t(\chi_{\mathbf{q},n})(\sigma)))$$

and

$$b = (b_1, b_2, b_3, \ldots) = \phi \circ T_n(M(\mathrm{Tr}(\mathbf{q})) + W_{2^c}(M(k) - M(\mathrm{Tr}(\mathbf{q}))))$$

are equal. To show it, we prove three identities $a_d = a_{(n,d)}$ for any integer $d \geq 1$, $b_d = b_{(n,d)}$ for any integer $d \geq 1$ and $a_d = b_d$ if $d \mid n$.

First, we consider the element $a$. By Theorem 2.15, we have $\phi(E_{Nr}(\lambda_t(\chi_{\mathbf{q},n})(\sigma))) = z(\lambda_t(\chi_{\mathbf{q},n})(\sigma))$. Thus,

$$a_d = \psi^d(\chi_{\mathbf{q},n})(\sigma) = \chi_{\mathbf{q},n}(\sigma^d).$$

By Proposition 4.17, $\chi_{\mathbf{q},n}$ is an integer-valued character. Thus we have $a_d = a_{(n,d)}$ by Lemma 3.19. If $d \mid n$, then $\sigma^d$ is a $d$-times multiple of cycles of length $n/d$ by Lemma 3.3. Hence, we have

$$\chi_{\mathbf{q},n}(\sigma^d) = \begin{cases} k^d & \text{if } 2 \nmid n/d, \\ (\mathrm{Tr}(\mathbf{q}))^d & \text{if } 2 \mid n/d. \end{cases} \tag{4.2}$$

Next, we consider the element $b$. By Propositions 3.9 and 4.2,

$$(b_1, b_2, b_3, \ldots) = T_n(\phi(M(\mathrm{Tr}(\mathbf{q}))) + W_{2^c}(\phi(M(k)) - \phi(M(\mathrm{Tr}(\mathbf{q})))))$$

holds. Thus, the element $b$ belongs to the image of $T_n$, which means that $b_d = b_{(n,d)}$ holds by Proposition 3.6 (2). If $d \mid n$, then

$$b_d = \begin{cases} k^d & \text{if } 2^c \mid d, \\ (\mathrm{Tr}(\mathbf{q}))^d & \text{if } 2^c \nmid d \end{cases} \tag{4.3}$$

by Lemmas 4.4 and 4.20.

A divisor $d$ of $n$ satisfies $2 \nmid n/d$ if and only if $2^c \mid d$ holds. Thus, $a_d = b_d$ holds for any divisor $d$ of $n$ by (4.2) and (4.3). Hence, we have $a = b$. Since $\mathbb{C}$ is $\mathbb{Z}$-torsion free, this proposition holds. □

Next, we have the following proposition.

**Proposition 4.22.** *For any divisor $r$ of $n$ we have*

$$F_r(\alpha) = T_{n/r}(M(\mathrm{Tr}(\mathbf{q})^r) + W_{2^{c-f}}(M(k^r) - M(\mathrm{Tr}(\mathbf{q})^r)))$$

*where $f$ is the maximum number satisfying $2^f \mid r$ and*

$$\alpha = T_n(M(\mathrm{Tr}(\mathbf{q})) + W_{2^c}(M(k) - M(\mathrm{Tr}(\mathbf{q})))).$$

*Proof.* One has

$$\begin{aligned} F_r(\alpha) &= F_r \circ T_n(M(\mathrm{Tr}(\mathbf{q})) + W_{2^c}(M(k) - M(\mathrm{Tr}(\mathbf{q})))) \\ &= T_{n/(n,r)} \circ F_r(M(\mathrm{Tr}(\mathbf{q})) + W_{2^c}(M(k) - M(\mathrm{Tr}(\mathbf{q})))) \\ &= T_{n/r}(F_r(M(\mathrm{Tr}(\mathbf{q}))) + F_r \circ W_{2^c}(M(k) - M(\mathrm{Tr}(\mathbf{q})))) \\ &= T_{n/r}(F_r(M(\mathrm{Tr}(\mathbf{q}))) + W_{2^{c-f}}(F_r(M(k)) - F_r(M(\mathrm{Tr}(\mathbf{q}))))) \\ &= T_{n/r}(M(\mathrm{Tr}(\mathbf{q})^r) + W_{2^{c-f}}(M(k^r) - M(\mathrm{Tr}(\mathbf{q})^r))). \end{aligned}$$

where the second, fourth or fifth equality follows from Proposition 3.11, Proposition 4.3 or Lemma 4.20 (2), respectively. □

By Proposition 4.22, we have the following corollary.

**Corollary 4.23.** *For any integers $n \geq 1$ and $r \geq 1$,*

$$E_{Nr}(\lambda_t(\chi_{\mathbf{q},n})(\sigma^r)) = T_{n/r}(M(\mathrm{Tr}(\mathbf{q})^r) + W_{2^{a-b}}(M(k^r) - M(\mathrm{Tr}(\mathbf{q})^r)))$$

*holds where $\sigma = (1\ 2 \cdots n)$, and integer $a$ or $b$ is the maximum number satisfying $2^a \mid n$ holds or $2^b \mid n/r$ holds, respectively.*

*Proof.* Put $\alpha = T_n(M(\mathrm{Tr}(\mathbf{q})) + W_{2^a}(M(k) - M(\mathrm{Tr}(\mathbf{q}))))$. Then

$$F_r(\alpha) = T_{n/r}(M(\mathrm{Tr}(\mathbf{q})^r) + W_{2^{a-b}}(M(k^r) - M(\mathrm{Tr}(\mathbf{q})^r)))$$

by Proposition 4.22. On the other hand, we have $F_r(\alpha) = E_{Nr}(\lambda_t(\chi_{\mathbf{q},n})(\sigma^r))$. Hence, this corollary holds. □

We calculated $\lambda_t(\chi_{\mathbf{q},n}))(\sigma)$ where $\sigma = (1\ 2 \cdots n)$ in Proposition 4.21 and where the cycle structure $\{\lambda_1, \ldots, \lambda_m\}$ of an element $\sigma \in S_n$ satisfies $\lambda_i = \lambda_j$ for any integer $i, j = 1, \ldots m$ in Corollary 4.23.

Finally, we state the calculation method when $\sigma$ is the product of two disjoint cycles by the following proposition. Hence, we can calculate $E_{Nr}(\lambda_t(\chi_{\mathbf{q},n}))(\sigma)$ for any $\sigma \in S_n$, because all elements of $S_n$ can be written as a product of cycles.

**Proposition 4.24.** *For any $\sigma \in S_m$ and $\tau \in S_n$, we have*

$$\lambda_t(\chi_{m+n,\mathbf{q}})(\sigma\tau) = \lambda_t(\chi_{m,\mathbf{q}})(\sigma) \cdot_\Lambda \lambda_t(\chi_{n,\mathbf{q}})(\tau).$$

*where we identify the subgroup $\{\sigma \in S_{m+n} \mid \sigma(i) = i,\ i = m+1, \ldots, n\}$ or $\{\tau \in S_{m+n} \mid \tau(i) = i,\ i = 1, \ldots, m\}$ of $S_{m+n}$ with $S_m$ or $S_n$, respectively.*

*Proof.* First, we show $\mathrm{Res}^{S_{m+n}}_{S_m \times S_n}(\chi_{m+n,\mathbf{q}}) = \chi_{n,\mathbf{q}}\chi_{m,\mathbf{q}}$ as a character of the product group $S_m \times S_n$. For any $\sigma' \in S_m$ and let $\tau' \in S_n$, we have $s_1(\sigma'\tau') = s_1(\sigma') + s_1(\tau')$ and $s_2(\sigma'\tau') = s_1(\sigma') + s_2(\tau')$. Thus, we have $\chi_{m+n,\mathbf{q}}(\sigma'\tau') = \chi_{n,\mathbf{q}}(\sigma)\chi_{m,\mathbf{q}}(\tau)$. Hence, we have $\mathrm{Res}^{S_{m+n}}_{S_m \times S_n}(\chi_{m+n,\mathbf{q}}) = \chi_{n,\mathbf{q}}\chi_{m,\mathbf{q}}$.

We use this fact. By Lemma 3.33, the $d$-th element of $z(\lambda_t(\chi_{m+n,\mathbf{q}})(\sigma\tau))$ is

$$\begin{aligned} \psi^d(\chi_{m+n,\mathbf{q}})(\sigma\tau) &= \chi_{m+n,\mathbf{q}}(\sigma^d\tau^d) \\ &= \mathrm{Res}^{S_{m+n}}_{S_m \times S_n}(\chi_{m+n,\mathbf{q}})((\sigma^d, \tau^d)) \\ &= \chi_{m,\mathbf{q}}(\sigma^d)\chi_{n,\mathbf{q}}(\tau^d), \end{aligned}$$

which is the product of $d$-th elements of $z(\lambda_t(\chi_{m,\mathbf{q}})(\sigma))$ and $z(\lambda_t(\chi_{n,\mathbf{q}})(\tau))$. Thus, we have $z(\lambda_t(\chi_{m+n,\mathbf{q}})(\sigma\tau)) = z(\lambda_t(\chi_{m,\mathbf{q}})(\sigma) \cdot_\Lambda \lambda_t(\chi_{n,\mathbf{q}})(\tau))$.

Since $\mathbb{C}$ is $\mathbb{Z}$-torsion free, the map $z$ is injective. It follows that we have $\lambda_t(\chi_{m+n,\mathbf{q}})(\sigma\tau) = \lambda_t(\chi_{m,\mathbf{q}})(\sigma) \cdot_\Lambda \lambda_t(\chi_{n,\mathbf{q}})(\tau)$. $\square$

By Proposition 4.24, we can calculate $\lambda_t(\chi_{\mathbf{q},n})(\sigma)$ for all $\sigma \in S_n$, and we can apply Theorem 3.38.

# Part II

# Symmetric powers of permutation representations of finite groups and primitive colorings on polyhedrons

## 5 Introduction

Let $G$ be a finite group. In this part, we will discuss actions of $G$, which will be prepared in §6. In §7.1, we will define the set "$|A|$-colored $N$-nested $G$-set", written by $J_{N,A,a}(X)$. Moreover, we will define the degree of an element of $J_{N,A,a}(X)$ and decompose $J_{N,A,a}(X)$ into the disjoint union of $G$-set $J^n_{N,A,a}(X)$, which is the set of $f \in J_{N,A,a}(X)$ such that $\deg(f) = n$ holds.

We will also define a power series $\varphi_{H,t}(J_{N,A,a}(X))$ by

$$\varphi_{H,t}(J_{N,A,a}(X)) = \sum_{n=0}^{\infty} \varphi_H(J^n_{N,A,a}(X))t^n$$

for any subgroup $H$ of $G$.

Note that the definition of $J_{N,A,a}(X)$ depends on a finite set $A$, an element $a \in A$, a non-empty subset $N \subset \mathbb{N} \cup \{0\}$ and a $G$-set $X$. In §7.2 and §7.3, we will discuss the following two problems with $J_{N,A,a}(X)$ and the power series $\varphi_{H,t}(J_{N,A,a}(X))$:

(1) Calculation of characters of exterior powers of representations with a character of symmetric powers.

(2) Calculation of the number of primitive colorings on some objects of polyhedrons.

In §5.1 and §5.2, we outline problems (1) and (2).

### 5.1 Symmetric powers of representations

Let $n \geq 1$ be an integer. For a representation $\rho : G \to GL(V)$ whose dimension over the complex field $\mathbb{C}$ is finite, we can define the $n$-th symmetric power of the representation $\rho$ by

$$S^n\rho : G \to GL(S^n(V)),$$
$$S^n\rho(g)(v_1 \cdots v_n) := (\rho(g)v_1) \cdots (\rho(g)v_n)$$

where $g \in G$ and $v_1, \ldots, v_n \in V$.

In §6.3, we will introduce symmetric powers operations $S^n$, $n = 0, 1, 2, \dots$ on $\lambda$-rings, and prove that $S^n(\chi)$ is the character of the $n$-th symmetric powers of the representation whose character is $\chi$. By the definition of symmetric powers operations, the calculating of the generating function character $\lambda^n(\chi)$ is equivalent to the calculating of the generating function of $S^n(\chi)$.

In §7.2, we will discuss the case of $N = \mathbb{N} \cup \{0\}$ and $|A| = 2$. We will show that the $G$-set $J_{N,A,a}(X)$ is isomorphic to the symmetric algebra of $X$ as a $G$-set. The symmetric algebra of $X$ was defined in [DS, §2.13]. Furthermore, for any $g \in G$ we will show that the generating function $\varphi_{\langle g \rangle, t}(J_{N,A,a}(X))$ is equal to the generating function of $S^n(\chi)(g)$ where $\chi$ is the permutation character associated with $X$. .

## 5.2 Primitive colorings on polyhedrons

In this part, we denote the group with one element by $C_1$. For any subgroup $V$ of $G$, we denote by $\mu_H(X)$ the number of orbits in $X$ which is isomorphic to $G/H$ (This notation will be defined again in §6).

In §7.3, we will discuss the case of $N = \{0, 1\}$. We will show that the set $J_{N,A,a}(X)$ is identified with the set of maps $\iota : X \to A$, and the set $J^n_{N,A,a}(X)$ is identified with the set of maps $\iota : X \to A$ such that $|\{x \in X \mid \iota(x) \neq a\}| = n$ holds. We will introduce the method for calculating $\mu_{C_1}(J_{N,A,a}(X))$ and will introduce the method for calculating the generating function of $\mu_{C_1}(J^n_{N,A,a}(X))$.

Now, we discuss the meaning to calculate $\mu_{C_1}(J_{N,A,a}(X))$. We assume that $X$ is the set of objects on polyhedrons (vertices, edges), and $G$ is a rotation group of $X$ and $A$ is a finite color set. A primitive coloring on $X$ of $G$ with $A$ is a coloring on $X$ with $A$ which is asymmetric under rotation of $G$. More detail, see Definition 7.15.

We can regard the set $J^n_{N,A,a}(X)$ as the set of colorings on $X$ which has $(|X| - n)$ times of $a$ colored, and the set $J_{N,A,a}(X)$ as the set of all colorings on $X$ with color of $A$. Moreover, $\mu_{C_1}(J^n_{N,A,a}(X))$ is the number of primitive colorings which has $(|X| - n)$ times of $a$ colored, and $\mu_{C_1}(J^n_{N,A,a}(X))$ provides the number of primitive colorings.

For the these colorings, Metropolis and Rota [MR] considered the number of primitive necklaces. Given a set of colors $A$, a necklace is a result of placing $n$ colored beads around a circle. A necklace which is asymmetric under rotation is said to be primitive.

For example, we consider when $n = 6$ and $A = \{0, 1\}$. Figure 1 is an example of primitive. However, Figure 2 is not primitive because it has a symmetry under $\dfrac{2\pi}{3}$ rotation.

The number of primitive necklaces $M(k, n)$ is computed by the following formula,

$$M(k, n) = \frac{1}{n} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) k^d, \tag{5.1}$$

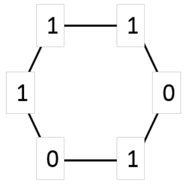


Figure 1: primitive necklace

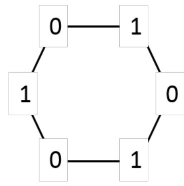


Figure 2: non-primitive necklace

where $k$ is the cardinality of $A$. Note that $M(k,n)$ is a polynomial with an indeterminate variable $k$.

For each integer $n \geq 1$, let $X_n$ be the set of all vertices of a regular $n$-gon and let $C_n$ be the cyclic group whose cardinality is $n$. Then, we can regard the set $X_n$ as a $C_n$ set, and hence, we will show that $\mu_{C_1}(J_{N,A,a}(X_n))$ is the number of asymmetric $k$-colorings of vertices of the $n$-regular polygons, which also coincides with the necklace polynomial $M(|A|,n)$.

Metropolis and Rota [MR] showed the following formula:

$$\frac{1}{1-kx} = \prod_{n=1}^{\infty} \Big(\frac{1}{1-x^n}\Big)^{M(k,n)} \tag{5.2}$$

for any integer $k \geq 1$, as combinatorial theory. The formula (5.2) is called a cyclotomic identity.

In addition, Metropolis and Rota showed the following identities for necklace polynomials:

$$M(k_1k_2,n) = \sum_{[i,j]=n} (i,j)M(k_1,i)M(k_2,j), \tag{5.3}$$

$$M(k^r,n) = \sum_{[j,r]=nr} \frac{j}{n} M(k,j) \tag{5.4}$$

for any integers $k_1, k_2, n \geq 1$ and $r \geq 1$. Identities (5.3) and (5.4) are origin of the multiplication of necklace rings and Frobenius operations. In §7.4, we generalize identities (5.3) and (5.4) to (7.20) and (7.22) with $\mu_V(J_{N,A,a}(X))$ for all subgroups $V$ of $G$. We consider a regular $n$-gon in (7.20) or (7.22), and hence, we will obtain the same result of (5.3) or (5.4), respectively.

In §8, we consider $J_{N,A,a}(X)$ when

(i) The set $X$ is the set of all vertices of a regular $n$-prism and $G$ is a dihedral group $D_n$,

(ii) The set $X$ is the set of all vertices of a regular $n$-gon and $G$ is a dihedral group $D_n$,

as examples.

**Remark 5.1.** Suppose that $X$ is isomorphic to $G/C_1$. Note that for any subgroup $V$ of $G$, $\mu_V(J_{N,A,a}(X))$ is equal to the polynomial $M_G(k,V)$, which was introduced in [Oh1], [Oh2], where $k$ is the cardinality of $A$. In [Oh1], $k$ belongs to some $\lambda$-rings, and Oh used the polynomial $M_G(k, V)$ to define the ring homomorphism, called exponential map. In [Oh2], $k$ belongs to the set of all integers $\mathbb{Z}$, and Oh generalized identities (5.3) and (5.4) for $M_G(k, V)$.

# 6 Preliminaries

In this section, let $G$ be a finite group. First, we define notations on group actions to state main results of this part. Main references of this section are [MR] and [Sch].

## 6.1 $G$-set

First, we define $G$-sets. A $G$-set $X$ is defined as a set equipped with a map

$$\iota : G \times X \to X, \quad \iota(g, x) = gx$$

which satisfies

$$g_1(g_2x) = (g_1g_2)x, \quad 1x = x$$

for any $g_1, g_2 \in G$ and $x \in X$.

Note that the empty set $\emptyset$ has a $G$-set structure.

For any $G$-sets $S$ and $T$, the set $T^S$, which is the set of all maps from $S$ to $T$, is also has the $G$-set structure which is defined as

$$(gf)(s) := gf(g^{-1}s) \tag{6.1}$$

for any $g \in G$, $f : S \to T$ and $s \in S$.

For any $G$-sets $X$ and $Y$, the disjoint union and the cartesian product of $X$ and $Y$ are again $G$-sets.

Next, we define the notion of the isomorphism as $G$-sets. We call that two $G$-sets $X_1$ and $X_2$ are $G$-isomorphic if there exists a bijective map $f : X_1 \to X_2$ such that $f(gx) = gf(x)$ holds for any $g \in G$ and $x \in X_1$. We call the map $f$ a $G$-isomorphism.

For any $x \in X$, we denote an orbit of $x$ by $Gx$. A $G$-set $X$ is said to be transitive if $Gx = X$ for some $x \in X$. For example, the left quotient set $G/H$ is a transitive $G$-set for any subgroup $H \subset G$. Moreover, a transitive $G$-sets $X$ is $G$-isomorphic to $G/H$ for some subgroup $H \subset G$.

For transitive $G$-sets, the following proposition holds.

**Proposition 6.1** ([Knu, p.111])**.** *For any subgroups $H_1$ and $H_2$, two $G$-sets $G/H_1$ and $G/H_2$ are $G$-isomorphism if and only if there exists an element $g \in G$ such that $H_1 = gH_2g^{-1}$ holds.*

Let $\Phi(G)$ be the set of conjugacy subgroups of $G$. By Proposition 6.1, a $G$-set $X$ is isomorphic to some disjoint union of $G$-sets $G/H$ where $H \in \Phi(G)$.

Let $H$ be a subgroup of $G$. We denote a $G$-set $X$ by $\mathrm{Res}^G_H(X)$ if we regard $X$ as an $H$-set. For the restriction map $\mathrm{Res}^G_H$, the following proposition holds.

**Proposition 6.2.** *Let $X$ and $Y$ be $G$-set. Then, two $H$-sets $\mathrm{Res}^G_H(X)$ and $\mathrm{Res}^G_H(Y)$ are $H$-isomorphic if $X$ and $Y$ are $G$-isomorphic. Moreover, two $H$-sets $\mathrm{Res}^G_H(X \cup Y)$ and $\mathrm{Res}^G_H(X) \cup \mathrm{Res}^G_H(Y)$ are $H$-isomorphic, and two $H$-sets $\mathrm{Res}^G_H(X \times Y)$ and $\mathrm{Res}^G_H(X) \times \mathrm{Res}^G_H(Y)$ are $H$-isomorphic*

*Proof.* If $X$ and $Y$ are $G$-isomorphic, then there exists $G$-isomorphism $\iota : X \to Y$. This map is also $H$-isomorphism, and hence, two $H$-sets $\mathrm{Res}^G_H(X)$ and $\mathrm{Res}^G_H(Y)$ are $H$-isomorphic.

Next, we consider the set $X \cup Y$ and $X \times Y$. The identity map on $X \cup Y$ and $X \times Y$ are $H$-isomorphic, then two $H$-sets $\mathrm{Res}^G_H(X \cup Y)$ and $\mathrm{Res}^G_H(X) \cup \mathrm{Res}^G_H(Y)$ are $H$-isomorphic and two $H$-sets $\mathrm{Res}^G_H(X)$ and $\mathrm{Res}^G_H(Y)$ are $H$-isomorphic. □

The following propositions will be used in §7.4 and §8.1.

**Proposition 6.3** ([Oh2, p.398])**.** *For any subgroups $K_1$ and $K_2$ of $G$, the cartesian product of two transitive $G$-sets $G/K_1$ and $G/K_2$ is $G$-isomorphic to*

$$\bigcup_{K_1 g K_2} G/K_1 \cap gK_2g^{-1}$$

*where $g$ ranges over a set of double coset representatives of $K_1$ and $K_2$ in $G$.*

**Proposition 6.4** ([Oh2, p.400])**.** *For any subgroups $H$ and $K$ of $G$, the $H$-set $\mathrm{Res}^G_H(G/K)$ is $H$-isomorphic to*

$$\bigcup_{HgK} H/H \cap gKg^{-1}$$

*where $g$ ranges over a set of double coset representatives of $H$ and $K$ in $G$.*

**Remark 6.5.** Note that Proposition 6.3 and 6.4 were discussed when $G$ is not a finite group, however subgroups $H$ and $K$ satisfy that $|G/H|$ and $|G/K|$ is finite in [Oh2]. For all subgroups $H$ of finite groups $G$ satisfies $|G/H| < \infty$.

## 6.2 Calculation of $\mu_H(X)$

We denote by $\mu_H(X)$ the number of orbits in the $G$-set $X$ which is $G$-isomorphic to $G/H$. We discuss a method of calculating $\mu_H(X)$ for any finite $G$-set $X$ and subgroup $H$ of $G$ with super characters, which was introduced by Knutson [Knu].

The set of all $G$-sets has an equivalent relation which is defined by $G$-isomorphisms. We denote an equivalent class of a finite $G$-set $X$ by $[X]$.

We define $M'(G)$ by a set of all equivalence classes $[X]$. It has a semiring structure whose addition is defined via disjoint union and whose multiplication is defined via cartesian product. The zero element of $M'(G)$ is $[\emptyset]$, and the unit element 1 of $M'(G)$ is the set whose cardinality is 1.

We define the Burnside ring by the ring completion of $M'(G)$, written by $B(G)$. The Burnside ring $B(G)$ has a commutative ring structure (see Appendix A.6). The Burnside ring $B(G)$ has a $\mathbb{Z}$-basis $\{[G/H] \mid H \in \Phi(G)\}$.

For any $H \in \Phi(G)$, there exists an additive homomorphism $\mu_H : B(G) \to \mathbb{Z}$ such that $\mu_H([X]) = \mu_H(X)$ holds for any finite $G$-set $X$. Then, we have

$$[X] = \sum_{H \in \Phi(G)} \mu_H([X])[G/H]$$

for any finite $G$-set $X$.

For any finite $G$-set $X$ and subgroup $H$ of $G$, we define a set $X_H$ by the set of $x \in X$ such that $hx = x$ holds for any $h \in H$. In addition, we define $\varphi_H(X)$ by the cardinality of $X_H$. Note that we have $\varphi_H(\emptyset) = 0$ and $\varphi_H(1) = 1$ for any subgroup $H$.

**Remark 6.6.** Let $\chi$ be the permutation character associate with a finite $G$-set $X$. Then, we have $\varphi_{\langle g \rangle}(X) = \chi(g)$.

**Proposition 6.7.** *The followings hold for any finite $G$-sets $S, T, X, Y$ and subgroups $H, H_1, H_2$.*

(1) $\varphi_H(S) = \varphi_H(T)$ *if $S$ and $T$ are $G$-isomorphism.*

(2) $\varphi_{H_1}(X) = \varphi_{H_2}(X)$ *if $H_1$ and $H_2$ are conjugate.*

(3) $\varphi_H(X \cup Y) = \varphi_H(X) + \varphi_H(Y)$.

(4) $\varphi_H(X \times Y) = \varphi_H(X)\varphi_H(Y)$.

*Proof.* Identities (3) and (4) were proved in [Knu, p.111]. First, we prove the identity (1).

Let $f : S \to T$ be a $G$-isomorphism. We show $f(S_H) = T_H$. For any $s \in S_H$, we have $hf(s) = f(hs) = f(s)$. Thus, we have $f(s) \in T_H$. Conversely, for any $t \in T_H$ we put $s = f^{-1}(t)$. Then $hs = hf^{-1}(t) = f^{-1}(ht) = f^{-1}(t) = s$ holds for any $h \in H$. Thus the element $s$ belongs to $S_H$. Hence, we have $f(S_H) = T_H$, that is, $\varphi_H(S) = \varphi_H(T)$ holds.

Next, we prove the identity (2). Take $g \in G$ which satisfies $H_1 = g^{-1}H_2g$. Let $\iota : X \to X$ be the map defined by $\iota(x) = gx$ for any $x \in X$. The map $\iota$ is bijective. We show $\iota(X_{H_1}) = X_{H_2}$. For any $x \in X_{H_1}$, we have $h_2\iota(x) = h_2gx = gh_1x = gx = \iota(x)$ for any $h_2 \in H_2$ where $h_1 \in H_1$ satisfies $gh_1 = h_2g$. Thus, we have $\iota(x) \in X_{H_2}$. Conversely, for any $y \in X_{H_2}$, we put $x = \iota^{-1}(y)$. Then, $\iota(h_1x) = gh_1x = h_2gx = h_2\iota(x) = \iota(x)$ for any $h_1 \in H_1$ where $h_2 \in H_2$ satisfies $gh_1 = h_2g$. Thus, we have $x \in X_{H_1}$, and hence, we have $\varphi_{H_1}(X) = \varphi_{H_2}(X)$. $\square$

Let $SCF(G)$ be the set of all maps from $\Phi(G)$ to $\mathbb{C}$. The set $SCF(G)$ has a commutative ring structure with the following operations.

$$\begin{aligned} (f_1 + f_2)(H) &:= f_1(H) + f_2(H), \\ (f_1f_2)(H) &:= f_1(H)f_2(H) \end{aligned}$$

where $f_1, f_2 \in SCF(G)$ and $H \in \Phi(G)$. In [Knu, p.110], an element in $SCF(G)$ is called a super central function on $G$. The zero element $0_{SFC(G)}$ is the map defined by $0_{SCF(G)}(H) = 0$ for any $H \in \Phi(G)$ and the unit element of $1_{SFC(G)}$ is the map defined by $1_{SCF(G)}(H) = 1$ for any $H \in \Phi(G)$.

By Proposition 6.7, there exists a ring homomorphism $\varphi : B(G) \to SCF(G)$ such that $\varphi([X])(H) = \varphi_H(X)$ holds for any finite $G$-set $X$. If $X$ is a finite $G$-set, then the map $\varphi([X])$ is called the super character of $G$-set $X$. In addition, the following theorem holds.

**Theorem 6.8.** *For any $H, V \in \Phi(G)$ satisfying $gHg^{-1} \subset V$ for some $g \in G$, there exists a unique rational integer $a_{H,V}$ such that*

$$\mu_H(\alpha) = \sum_{H \lesssim V} a_{H,V}\varphi(\alpha)(V)$$

*holds for any $\alpha \in B(G)$ where the notation $H \lesssim V$ means that there exists $g \in G$ such that $g^{-1}Hg \subset V$ holds. In particular, the map $\varphi$ is injective.*

Hence, if we know $\varphi_H(X)$ for all $H \in \Phi(G)$ and rational numbers $a_{H,V}$ for all $H, V \in \Phi(G)$, then we can calculate $\mu_H(X)$ for all $H \in \Phi(G)$.

To prove this theorem, we use the following lemma.

**Lemma 6.9.** [Knu, p.111] *Let $H_1$ and $H_2$ be subgroups of $G$. Then, one has $\varphi_{H_1}(G/H_2) \neq 0$ holds if and only if there exists an element $g \in G$ such that $g^{-1}H_1g \subset H_2$ holds.*

*Proof of Theorem 6.8.* We prove this theorem by the induction on $|G/H|$. Recall that we have

$$\varphi(\alpha)(H) = \sum_{V \in \Phi(G)} \mu_V(\alpha)\varphi_H(G/V) \tag{6.2}$$

for any $\alpha \in B(G)$.

First, we consider $H = G$. By (6.2) and Lemma 6.9, we have $\mu_G(\alpha) = \varphi(\alpha)(G) = 0$.

Next, we assume that this theorem holds for any $H \in \Phi(G)$ such that $|G/H| < n$ holds. Let $\alpha$ be an element of $B(G)$. By (6.2) and Lemma 6.9, we have

$$\varphi(\alpha)(H) = \sum_{V : H \lesssim V} \mu_V(\alpha)\varphi_H(G/V).$$

Thus,

$$\mu_H(\alpha) = \frac{1}{\varphi_H(G/H)}\Big(\varphi(\alpha)(H) - \sum_{V : H \lesssim V, H \not\cong V} \mu_V(\alpha)\varphi_H(G/V)\Big) \tag{6.3}$$

holds where $H \not\cong V$ means that $H$ and $V$ are not conjugate. By (6.3) and the induction hypothesis, we can write

$$\mu_H(\alpha) = \sum_{H \lesssim V} a_{H,V}\varphi(\alpha)(V)$$

for some rational numbers $a_{H,V}$'s.

Next, for any $V \in \Phi(G)$ satisfying $H \lesssim V$, let $a_{H,V}$ and $a'_{H,V}$ be rational numbers satisfying

$$\mu_H(\alpha) = \sum_{H \lesssim V} a_{H,V}\varphi(\alpha)(V) = \sum_{H \lesssim V} a'_{H,V}\varphi(\alpha)(V). \tag{6.4}$$

We show $a_{H,V} = a'_{H,V}$ by the induction of $|V/H|$. We substitute $\alpha = [G/H]$ in (6.4), then $1 = a_{H,H} = a'_{H,H}$ holds. If $a_{H,V_1} = a'_{H,V_1}$ holds for any $V_1 \in \Phi(G)$ such that $|V_1/H| < |V/H|$, then we substitute $\alpha = [G/V]$ in (6.4), thus we have

$$0 = \sum_{H \lesssim V_1 \lesssim V} a_{H,V_1}\varphi([G/V])(V_1) = \sum_{H \lesssim V_1 \lesssim V} a'_{H,V_1}\varphi([G/V])(V_1).$$

Hence, we have $a_{H,V} = a'_{H,V}$ by the induction hypothesis.

That is, this theorem holds.

□

From the proof of Theorem 6.8, we can calculate rational numbers $a_{H,V}$'s inductively. Moreover, there exists cases where calculation can be performed more effectively. For example, if $G$ is a cyclic group, we can use Möbius inversion formula to calculate $a_{H,V}$. In the following example, we consider the case where $G$ is a cyclic group.

**Example 6.10.** Let $n \geq 1$ be an integer and let $C_n$ be the cyclic group whose cardinality is $n$. We consider when $G = C_n$.

First, we investigate all elements of $\Phi(C_n)$. Note that the set of all subgroups of $C_n$ is $\{\langle g^d \rangle \mid d \text{ divides } n\}$ where $g \in C_n$ is a generator of $C_n$. For any two distinct divisors $d_1$ and $d_2$ of $n$, we have $\langle g^{d_1} \rangle \neq \langle g^{d_2} \rangle$. Thus, for each divisor $d$ of $n$, there exists the subgroup uniquely whose cardinality is $n/d$.

Moreover, the set of all subgroups of $C_n$ is $\Phi(C_n)$ since $C_n$ is an abelian group.

Now, we denote $\langle g^d \rangle$ by $C_{n/d}$. For any subgroups $C_{n/d}$ and $C_{n/d'}$ in $\Phi(C_n)$, we have

$$[\mathrm{Res}^{C_n}_{C_{n/d'}}(C_n/C_{n/d})] = (d, d')[C_{n/d'}/C_{n/[d,d']}]. \tag{6.5}$$

In particular, we have

$$\varphi_{C_{n/d'}}(C_n/C_d) = \begin{cases} d' & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d. \end{cases} \tag{6.6}$$

Hence, for any $C_n$-set $X$ and divisor $d$ of $r$, we have

$$\varphi_{C_{n/d}}(X) = \sum_{d' \mid n} \mu_{C_{n/d'}}(X)\varphi_{C_{n/d}}(C_n/C_{n/d'}) = \sum_{d' \mid d} d'\mu_{C_{n/d'}}(X). \tag{6.7}$$

By Theorem 6.8, all $\mu_{C_{n/d}}(X)$'s are determined uniquely from all $\varphi_{C_{n/d}}(X)$'s. In fact, by the Möbius inversion formula with the identity (6.7), we have

$$\mu_{C_{n/d}}(X) = \frac{1}{d} \sum_{d'|d} \mu\Big(\frac{d}{d'}\Big) \varphi_{C_{n/d'}}(X). \tag{6.8}$$

Next, we consider the cartesian product of two finite $G$-sets $X$ and $Y$.

**Proposition 6.11.** *Let $V_1, V_2, W_1, W_2, H_1, H_2$ be subgroups of $G$ and $V_1$, $V_2$ or $H_1$ is conjugate to $W_1$, $W_2$ or $H_2$, respectively. Then, we have $\mu_{H_1}(G/V_1 \times G/V_2) = \mu_{H_2}(G/W_1 \times G/W_2)$.*

*Proof.* Since $V_1$ is conjugate to $W_1$ and $V_2$ is conjugate $W_2$, the $G$-set $G/V_1 \times G/V_2$ is $G$-isomorphic to $G/W_1 \times G/W_2$ by Proposition 6.1. Moreover, since $H_1$ is conjugate to $H_2$, two maps $\mu_{H_1}$ is equal to $\mu_{H_2}$. Thus, we have $\mu_{H_1}(G/V_1 \times G/V_2) = \mu_{H_2}(G/W_1 \times G/W_2)$. □

**Definition 6.12.** For any $V_1, V_2, H \in \Phi(G)$. we define the number $b_{V_1,V_2}(H)$ by $\mu_H(G/V_1 \times G/V_2)$. By Proposition 6.11, this definition is well-defined.

**Proposition 6.13.** *For any finite $G$-sets $X$ and $Y$ and $H \in \Phi(G)$, we have*

$$\mu_H(X \times Y) = \sum_{V_1, V_2 \in \Phi(G)} b_{V_1,V_2}(H) \mu_{V_1}(X) \mu_{V_2}(Y)$$

*Proof.* Two $G$-sets $X$ and $Y$ have the following form,

$$[X] = \sum_{V_1 \in \Phi(G)} \mu_{V_1}(X)[G/V_1], \quad [Y] = \sum_{V_2 \in \Phi(G)} \mu_{V_2}(X)[G/V_2].$$

Thus, we have

$$\begin{aligned} [X \times Y] &= \sum_{V_1, V_2 \in \Phi(G)} \mu_{V_1}(X) \mu_{V_2}(Y) [G/V_1 \times G/V_2] \\ &= \sum_{H \in \Phi(G)} \Big( \sum_{V_1, V_2 \in \Phi(G)} b_{V_1,V_2}(H) \mu_{V_1}(X) \mu_{V_2}(Y) \Big) [G/H]. \end{aligned}$$

Hence, this proposition holds. □

In the remainder of §6.2, we fix a subgroup $H$ of $G$. By Proposition 6.2, there are exist a ring homomorphism $\mathrm{Res}^G_H : R(G) \to R(H)$ such that $\mathrm{Res}^G_H([X]) = [\mathrm{Res}^G_H(X)]$ holds for any finite $G$-set $X$.

**Proposition 6.14.** *Let $K_1$ and $K_2$ be subgroups of $G$ and let $V_1, V_2$ be subgroup $H$. Suppose that $K_1$ and $K_2$ are conjugate as subgroups of $G$ and $V_1$ and $V_2$ are conjugate as subgroups of $H$. Then, we have $\mu_{V_1}(\mathrm{Res}^G_H(G/K_1)) = \mu_{V_2}(\mathrm{Res}^G_H(G/K_2)$.*

*Proof.* Since $K_1$ and $K_2$ are conjugate, then $G/K_1$ and $G/K_2$ are $G$-isomorphic by Proposition 6.1, that is, two $H$-sets $\mathrm{Res}^G_H(G/K_1)$ and $\mathrm{Res}^G_H(G/K_2)$ are $H$-isomorphic. Moreover, since $V_1$ and $V_2$ are conjugate, then two maps $\mu_{H_1}$ is equal to $\mu_{H_2}$. Hence, this proposition holds. □

**Definition 6.15.** Let $V$ be an element of $\Phi(G)$ and let $K$ be an element of $\Phi(H)$. We define the number $c_V(K)$ by $\mu_K(\mathrm{Res}^G_H(G/V))$. By Proposition 6.14, this definition is well-defined.

**Proposition 6.16.** *For any finite $G$-set $X$ and $K \in \Phi(H)$, we have*

$$\mu_K(\mathrm{Res}^G_H(X)) = \sum_{V \in \Phi(G)} c_V(K)\mu_V(X).$$

*Proof.* A finite $G$-set $X$ has the following form,

$$[X] = \sum_{V \in \Phi(G)} \mu_V(X)[G/V].$$

Then, we have

$$\begin{aligned} [\mathrm{Res}^G_H(X)] &= \sum_{V \in \Phi(G)} \mu_V(X)[\mathrm{Res}^G_H(G/V)] \\ &= \sum_{V \in \Phi(G)} \mu_V(X)\Big(\sum_{K \in \Phi(H)} c_V(K)[H/K]\Big) \\ &= \sum_{K \in \Phi(H)} \Big(\sum_{V \in \Phi(G)} c_V(K)\mu_V(X)\Big)[H/K]. \end{aligned}$$

Hence, this proposition holds. □

## 6.3 Symmetric powers operations on $\lambda$-rings.

In §6.3, we discuss symmetric powers operations on $\lambda$-rings and a character of symmetric powers of representations of finite groups which will be used in §7.2. We will obtain the generating function of the character of the $i$-th symmetric powers of a representation in §7.2.

For a relation between representations of finite groups and $\lambda$-rings, see §2.2 and §2.3.

We show that calculating the generating function of the character of the $i$-th symmetric powers of a representation $\rho$ is equivalent to calculating $\lambda_t(\chi)$ where $\chi$ is the character of $\rho$.

**Definition 6.17.** Let $R$ be a $\lambda$-ring (which was defined in §2.2). We define operations $S^n : R \to R, n = 0, 1, 2, \ldots$ by the following equation,

$$S_t(r) = \frac{1}{\lambda_{-t}(r)}$$

for any $r \in R$, where $S_t(r) := \sum_{n=0}^{\infty} S^n(r)t^n$. The operation $S^n$ is said to be the $n$-th symmetric powers operation.

Symmetric powers operations were defined in [Mar]. For each finite group $G$, symmetric powers operations of $CF(G)$ give the character of symmetric powers representations of $G$ by the following proposition.

**Proposition 6.18.** *Let $\chi$ be the character of a representation $\rho$. Then, the character of the $n$-th symmetric powers of representation $\rho$ is $S^n(\chi)$ for any integer $n \geq 0$.*

*Proof.* Let $m$ be the degree of $\rho$. For each $g \in G$, let $\alpha_1, \ldots, \alpha_m$ be all eigenvalues of the linear map $\rho(g)$. It is known that

$$\lambda^n(\chi)(g) = \sum_{1 \leq i_1 < \cdots < i_n \leq m} \alpha_{i_1} \cdots \alpha_{i_n}, \quad X(S^n\rho)(g) = \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq m} \alpha_{i_1} \cdots \alpha_{i_n},$$

where $X(S^n\rho)$ is the character of $S^n\rho$. Then, we have

$$S_t(\chi)(g) = \frac{1}{\lambda_t(\chi)(g)} = \prod_{i=1}^{m} \frac{1}{1 - \alpha_i t} = \sum_{n=0}^{\infty} X(S^n\rho)(g)t^n$$

where $S_t(\chi)(g) := \sum_{n=0}^{\infty} S^n(\chi)(g)$. Hence, $S^n(\chi) = X(S^n\rho)$ holds for any integer $n \geq 0$. $\square$

# 7 $|A|$-colored $N$-nested $G$-set

Let $G$ be a finite group. In this section, we define $|A|$-colored $N$-nested $G$-set, written by $J_{N,A,a}(X)$, for a given finite color set $A$, an element $a \in A$, a non-empty subset $N \subset \mathbb{N} \cup \{0\}$ and a finite $G$-set $X$.

In §7.1, we will define $|A|$-colored $N$-nested $G$-set and state its properties. In §7.2, we will put $|A| = 2$ and $N = \mathbb{N} \cup \{0\}$, and discuss generating functions of the character of $n$-th symmetric power of representations of finite groups with $|A|$-colored $N$-nested $G$-set. Note that the discussion in §7.2 is also an idea of the definition and properties of $|A|$-colored $N$-nested $G$-set. In §7.3, we will put $N = \{0, 1\}$ and establish the method of calculating the number of primitive colorings. In §7.4, we will generalize identities (5.3) and (5.4) for all $\mu_V(J_{N,A,a}(X))$ where $V \in \Phi(G)$.

In this section, we denote the set $\{1, \ldots, n\}$ by $[n]$ for any integer $n \geq 1$.

## 7.1 Definition

In §7.1, let $N$ be a non-empty subset of $\mathbb{N} \cup \{0\}$, let $A$ be a finite set with $|A| \geq 2$, and let $a$ be an element of $A$. First, we define $|A|$-colored $N$-nested $G$-set.

**Definition 7.1.** Let $X$ be a finite $G$-set. We define the $|A|$-colored $N$-nested $G$-set of $X$ with $a \in A$ by

$$J_{N,A,a}(X) := \Big( \bigcup_{n \in N} A'^{[n]} \Big)^X,$$

where $A' := A \setminus \{a\}$ and $A'^{[0]} = \{a\}$.

We regard the set $\left(\bigcup_{n\in N} A'^{[n]}\right)$ as the trivial $G$-set. We define a $G$-set structure on $J_{N,A,a}(X)$ by (6.1).

Next, we state the decomposition of the disjoint union with the degree of $J_{N,A,a}(X)$.

**Definition 7.2.** For any elements of $a \in A'^{[n]}$, the symbol $\deg(a)$ is defined by $n$. We define $\deg(f)$ by

$$\deg(f) := \sum_{x\in X} \deg(f(x))$$

for any $f \in J_{N,A,a}(X)$. Next, we define the set $J^n_{N,A,a}(X)$ by

$$J^n_{N,A,a}(X) := \{f \in J_{N,A,a}(X) \mid \deg(f) = n\}$$

for any integer $n \geq 0$.

By this definition, we have

$$J_{N,A,a}(X) = \bigcup_{n=0}^{\infty} J^n_{N,A,a}(X).$$

**Proposition 7.3.** *Let $X$ be a finite $G$-set. For any integer $n \geq 0$, the set $J^n_{N,A,a}(X)$ is finite, and closed under the $G$-action of $J_{N,A,a}(X)$.*

*Proof.* For any integer $n \geq 0$, the set $A'^{[n]}$ is finite. Since an element $f \in J^n_{N,A,a}(X)$ satisfies $f(x) \in A^{[0]} \cup A^{[1]} \cup \cdots \cup A^{[n]}$ for any $x \in X$. Then, the set $J^n_{N,A,a}(X)$ is finite.

Next, for any $f \in J^n_{N,A,a}(X)$ and $g \in G$, we have

$$\deg(gf) = \sum_{x\in X} \deg(gf(x)) = \sum_{x\in X} \deg(f(g^{-1}x)) = \deg(f) = n.$$

Hence, the element $gf$ belongs to $J^n_{N,A,a}(X)$. $\square$

**Proposition 7.4.** *For given finite $G$-sets $X$ and $Y$ which are $G$-isomorphic, the $G$-set $J_{N,A,a}(X)$ is $G$-isomorphic to $J_{N,A,a}(Y)$.*

*Proof.* By the assumption, there exists a $G$-isomorphism $\iota : X \to Y$. We define a map $\iota_1 : J_{N,A,a}(Y) \to J_{N,A,a}(X)$ by $\iota_1(f)(x) := f(\iota(x))$ for any $f \in J_{N,A,a}(Y)$ and $x \in X$. First, we prove that the map $\iota_1$ is injective. Let $f_1$ and $f_2$ be elements of $J_{N,A,a}(Y)$, and we assume $\iota_1(f_1) = \iota_1(f_2)$ holds. Then, we have $f_1(\iota(y)) = f_2(\iota(y))$ holds for any $y \in Y$. Thus $f_1 = f_2$ holds, that is, the map $\iota_1$ is injective.

Next we prove that the map $\iota_1$ is surjective. Let $f$ be an element of $J_{N,A,a}(X)$. We put $g \in J_{N,A,a}(X)$ which satisfies $g(x) = f(\iota^{-1}(x))$ for any

$x \in X$. Thus, we have $\iota_1(g)(y) = g(\iota(y)) = f(y)$, that is, the map $\iota_1$ is surjective.

Thus $\iota_1$ is bijective, and

$$\iota_1(gf)(x) = gf(\iota(x)) = f(g^{-1}\iota(x)) = f(\iota(g^{-1}x)) = \iota_1(f)(g^{-1}x) = g(\iota_1(f))(x)$$

holds for any $f \in J_{N,A,a}(Y)$, $g \in G$ and $x \in X$.

Hence, the map $\iota_1$ is a $G$-isomorphism, that is, this proposition holds. $\square$

**Lemma 7.5.** *Let $S, T$ and $U$ be $G$-sets. Then, the map $\iota_2 : U^S \times U^T \to U^{S\cup T}$, which is defined by*

$$\iota_2(f,g)(u) = \begin{cases} f(u) & \text{if } u \in S, \\ g(u) & \text{if } u \in T, \end{cases}$$

*where $f \in U^S, g \in U^T$ and $u \in S \cup T$, is a $G$-isomorphism.*

*Proof.* First, we prove that the map $\iota_2$ is injective. Let $f_1$ and $f_2$ be elements of $U^S$, and let $g_1$ and $g_2$ be elements of $U^T$. We assume that $\iota_2(f_1, g_1) = \iota_2(f_2, g_2)$ holds. If $u \in S$, then we have $f_1(u) = f_2(u)$. If $z \in T$, then we have $g_1(u) = g_2(u)$. Hence, $f_1 = f_2$ and $g_1 = g_2$ hold, that is, the map $\iota_2$ is injective.

Next we prove that the map $\iota_2$ is surjective. Let $f$ be an element of $U^{S\cup T}$. We put $f_1 \in U^S$ and $g_1 \in U^T$ which satisfies $f_1(x) = f(x)$ for any $x \in S$ and $g_1(x) = f(x)$ for any $y \in T$. Then, we have $\iota_2((f_1, g_1)) = f$, that is, the map $\iota_2$ is surjective.

Finally, for any $g \in G$ and $(f_1, g_1) \in U^S \times U^T$, if $u \in S$ we have

$$g\iota_2((f_1, g_1))(u) = \iota_2((f_1, g_1))(g^{-1}u) = f_1(g^{-1}u) = (gf_1)(u) = \iota_2(g(f_1, g_1))(z).$$

Similarly, we have $g\iota_2((f_1, g_1))(u) = \iota_2(g(f_1, g_1))(u)$ if $u \in T$. In any case, we have $g\iota_2((f_1, g_1)) = \iota_2(g(f_1, g_2))$.

As a result, the map $\iota$ is a $G$-isomorphism. $\square$

**Proposition 7.6.** *Let $X$ and $Y$ be finite $G$-sets. Then, two $G$-sets $J_{N,A,a}(X \cup Y)$ and $J_{N,A,a}(X) \times J_{N,A,a}(Y)$ are $G$-isomorphic. In particular, for any integer $n \geq 1$ the following two $G$-sets are $G$-isomorphic:*

$$\bigcup_{i+j=n} J^i_{N,A,a}(X) \times J^j_{N,A,a}(Y), \quad J^n_{N,A,a}(X \cup Y). \tag{7.1}$$

*Proof.* The first statement of this proposition holds by Lemma 7.5. Then, we show the second statement of this proposition.

Let $n \geq 1$ be an integer and let $\iota_3 : J_{N,A,a}(X) \times J_{N,A,a}(Y) \to J_{N,A,a}(X \cup Y)$ be a map defined by Lemma 7.5. For any $f \in J^i_{N,A,a}(X)$ and $g \in J^j_{N,A,a}(Y)$ with $i + j = n$,

$$\deg(\iota_3((f,g))) = \sum_{x\in X} \deg(f(x)) + \sum_{y\in Y} \deg(g(y)) = i + j = n$$

holds. Conversely, for any $h \in J^n_{N,A,a}(X \cup Y)$, $f \in J_{N,A,a}(X)$ and $g \in J_{N,A,a}(Y)$ with $\iota_3(f,g) = h$, we put $i = \deg(f)$ and $j = \deg(f)$. Then, $f \in J^i_{N,A,a}(X)$ and $g \in J^j_{N,A,a}(Y)$ hold. Hence,

$$\iota_3\Big(\bigcup_{i+j=n} J^i_{N,A,a}(X) \times J^j_{N,A,a}(Y)\Big) = J^n_{N,A,a}(X \cup Y)$$

holds. That is, two $G$-sets (7.1) are $G$-isomorphic. $\square$

Next, we consider the super character and orbits of $J^n_{N,A,a}(X)$.

**Definition 7.7.** Let $X$ be a finite $G$-set and let $H$ be a subgroup of $G$. By Proposition 7.3, the set $J^n_{N,A,a}(X)$ is a finite $G$-set. We define two powerserieses $\varphi_{H,t}(J_{N,A,a}(X))$ and $\mu_{H,t}(J_{N,A,a}(X))$ by

$$\varphi_{H,t}(J_{N,A,a}(X)) := \sum_{n=0}^{\infty} \varphi_H(J^n_{N,A,a}(X))t^n$$

and

$$\mu_{H,t}(J_{N,A,a}(X)) := \sum_{n=0}^{\infty} \mu_H(J^n_{N,A,a}(X))t^n$$

with an indeterminate variable $t$.

By Proposition 7.6, we have

$$\varphi_{H,t}(J_{N,A,a}(X \cup Y)) = \varphi_{H,t}(J_{N,A,a}(X))\varphi_{H,t}(J_{N,A,a}(Y)) \tag{7.2}$$

for any two finite $G$-sets $X$ and $Y$.

To calculate $\varphi_{H,t}(J_{N,A,a}(X))$, we show the following theorem.

**Theorem 7.8.** *For any subgroup $H$ of $G$ and finite $G$-set $X$, we have*

$$\varphi_{H,t}(J_{N,A,a}(X)) = \prod_{i=1}^{\infty}\Big(\sum_{n \in N}(|A'|t^i)^n\Big)^{O_{X,H,i}}$$

*where $O_{X,H,i}$ is the number of orbits $Hx \subset \mathrm{Res}^G_H(X)$ such that $|Hx| = i$ holds for any integer $i \geq 1$.*

We prepare the following proposition.

**Proposition 7.9.** *Let $X$ be a finite $G$-set, and let $H$ be a subgroup of $G$. Then, two $G$-sets $\mathrm{Res}^G_H(J_{N,A,a}(X))$ and $J_{N,A,a}(\mathrm{Res}^G_H(X))$ are $H$-isomorphic.*

*Proof.* We show that the identity map on $J_{N,A,a}(X)$, written by $F$ in this proof, is a $G$-isomorphism. For any $f \in J_{N,A,a}(X)$, $x \in X$ and $h \in H$,

$$hF(f)(x) = F(f)(h^{-1}x) = f(h^{-1}x) = hf(x) = F(hf)(x)$$

holds. Then, $\mathrm{Res}^G_H(J_{N,A,a}(X))$ and $J_{N,A,a}(\mathrm{Res}^G_H(X))$ are $H$-isomorphic. $\square$

*Proof of Theorem 7.8.* First, we consider the case of $H = G$ and $X = G/K$. we prove

$$\varphi_G(J^n_{N,A,a}(G/K)) = \begin{cases} |A'|^{n/|G/K|} & \text{if } \frac{n}{|G/K|} \in N, \\ 0 & \text{otherwise} \end{cases} \tag{7.3}$$

for any subgroup $K$ of $G$. Any element $f \in J^n_{N,A,a}(G/K)$ satisfies $gf = f$ for all $g \in G$, so that $f$ is a constant map since $G/K$ is a transitive $G$-set. Thus, there does not exist an element $f \in J^n_{N,A,a}(G/K)$ such that $|G/K| \mid n$ holds.

We assume $n \mid |G/K|$. Let $\alpha$ be an element of $G/K$ with $n' = \deg(f(\alpha))$. Then we have $n' \in N$ and $n'|G/K| = n$. The number of $f$ such that $f(\alpha)(m) \in A'$ holds for any $1 \leq m \leq n'$ is $|A'|^{n'}$. Hence, we have (7.3).

Next, we show when $H = G$. By (7.3), we have

$$\varphi_{G,t}(J_{N,A,a}(G/H)) = \sum_{n \in N} (|A'|t^{|G/H|})^n.$$

Next, we prove this theorem for any finite $G$-set $X$. We use the orbit decomposition of $X$ and (7.2). Thus, we have

$$\varphi_{G,t}(J_{N,A,a}(X)) = \prod_{i=1}^{\infty} \Big( \sum_{n \in N} (|A'|t^i)^n \Big)^{O_{X,G,i}}. \tag{7.4}$$

Finally, we calculate the power series $\varphi_{H,t}(J_{N,A,a}(X))$ for any subgroup $H$ of $G$ with (7.4). By Proposition 7.9, we have

$$\begin{aligned} \varphi_{H,t}(J_{N,A,a}(X)) &= \sum_{n=0}^{\infty} \varphi_H(J_{N,A,a}(X))t^n \\ &= \sum_{n=0}^{\infty} \varphi_H(\mathrm{Res}^G_H(J_{N,A,a}(X)))t^n \\ &= \sum_{n=0}^{\infty} \varphi_H(J_{N,A,a}(\mathrm{Res}^G_H(X)))t^n \\ &= \varphi_{H,t}(J_{N,A,a}(\mathrm{Res}^G_H(X))) = \prod_{i=1}^{\infty} \Big( \sum_{n \in N} (|A'|t^i)^n \Big)^{O_{X,H,i}}. \end{aligned}$$

Hence, this theorem holds. $\square$

To calculate $\mu_{H,t}(J_{N,A,a}(X))$, we use the following proposition.

**Proposition 7.10.** *For any $V \in \Phi(G)$ and finite $G$-set $X$,*

$$\mu_{H,t}(J_{N,A,a}(X)) = \sum_{H \lesssim V} a_{H,V} \varphi_{V,t}(J_{N,A,a}(X))$$

*holds where rational numbers $a_{H,V}$'s satisfy the identity of Theorem 6.8.*

*Proof.* By Theorem 6.8, we have

$$\begin{aligned}
\mu_{H,t}(J_{N,A,a}(X)) &= \sum_{i=0}^{\infty} \mu_H(J^i_{N,A,a}(X))t^i \\
&= \sum_{i=0}^{\infty} \sum_{H \lesssim V} a_{H,V}\varphi_V(J^i_{N,A,a}(X))t^i \\
&= \sum_{H \lesssim V} a_{H,V}\varphi_{V,t}(J_{N,A,a}(X)).
\end{aligned}$$

□

## 7.2 A relation with symmetric powers of representations of finite group

We discuss a generating function of the character of the $n$-th symmetric powers of representation with §7.1.

In §7.2, we denote $\mathbb{N} \cup \{0\}$ by $N_s$, and we assume $|A| = 2$. We consider $J_{N_s,A,a}(X)$ for any finite $G$-set $X$.

Dress and Siebeneicher [DS, §2.13] defined the $n$-th symmetric power of a finite $G$-set $X$, written by $S^n(X)$, by the set of $f : X \to \mathbb{N} \cup \{0\}$ satisfying $\sum_{x \in X} f(x) = n$. The set $S^n(X)$ has a $G$-set structure by (6.1). In this condition, an element $f \in S^n(X)$ can be written

$$f = x_1 \cdots x_1 \cdots x_m \cdots x_m$$

where $X = \{x_1, \dots, x_m\}$ and the element $x_i$ appears $f(x_i)$ times. Moreover, for any $g \in G$ we have $gf = gx_1 \cdots gx_1 \cdots gx_m \cdots gx_m$. Hence, the permutation representation associated with $S^n(X)$ is the $n$-th symmetric power of the permutation representation associated with $X$.

For $S^n(X)$, we have the following proposition.

**Proposition 7.11.** *The $G$-set $S^n(X)$ is $G$-isomorphic to $J^n_{N_s,A,a}(X)$ for any integer $n \geq 1$.*

*Proof.* From $|A| = 2$, the set $A'^{[n]}$ has the unique element $f$ such that $f(k) = b$ holds for any $k = 1, \dots, n$, where $b$ is the unique element of $A \setminus \{a\}$. Thus, if $x, y \in \bigcup_{n \in N_s}^{\infty} A'^{[n]}$ satisfy $\deg(x) = \deg(y)$, then $x = y$ holds.

We define a map $\iota_s : J^n_{N_s,A,a}(X) \to S^n(X)$ by $\iota_s(f)(x) := \deg(f(x))$ for any $x \in X$.

First, we show that the map $\iota_s$ is injective. Let $f$ and $g$ be elements of $J^n_{N_s,A,a}(X)$ satisfying $\iota_s(f) = \iota_s(g)$. For any $x \in X$, we have $\deg(f(x)) = \deg(g(x))$. Hence, we have $f(x) = g(x)$. That is, the map $\iota_s$ is injective.

Next, for any $g \in S^n(X)$, we put $f \in J^n_{N_s,A,a}(X)$ satisfying $f(x) \in A'^{[\deg(f(x))]}$. Then, we have $\iota_s(f)(x) = g(x)$. That is, the map $\iota_s$ is surjective.

Finally, we prove that $\iota_s(gf) = g\iota_s(f)$ holds for any $g \in G$ and $f \in J^n_{N_s,A,a}(X)$. One has

$$\iota_s(gf)(x) = \deg(gf(x)) = \deg(f(g^{-1}x)) = \iota_s(f(g^{-1}x) = g\iota_s(f)(x).$$

Then, the map $\iota_s$ is $G$-isomorphism. $\square$

Substituting $N = N_s$ in the identities of Theorem 7.8, we have

$$\varphi_{\langle g\rangle,t}(J_{N_s,A,a}(X)) = \prod_{i=1}^{\infty}\Big(\frac{1}{1-t^i}\Big)^{O_{X,\langle g\rangle,i}}. \tag{7.5}$$

For any $g \in G$, the right side of (7.5) coincide with the generating function of the $S^n(\chi)(g)$ where $\chi$ is the character of the permutation representation associated with finite $G$-set $X$, and $S^n(\chi)$ is the character of the $n$-th symmetric power of permutation representation associated with a finite $G$-set $X$.

**Example 7.12.** Let $S_n$ be the symmetric group on $n$-letters. We consider the natural action of $S_n$ on $[n]$. Then, $S_n$-set $[n]$ is $S_n$-isomorphic to $S_n/S_{n-1}$ where we identify $S_{n-1}$ with the isotropy subgroup $\{\sigma \in S_n \mid \sigma(1) = 1\}$ of $S_n$.

Let $\{\lambda_1, \dots, \lambda_m\}$ be the cycle structure of $\sigma \in S_n$. Then, the number of orbits of $\mathrm{Res}^{S_n}_{\langle\sigma\rangle}([n])$ whose cardinality is $i$, which is $O_{[n],\langle\sigma\rangle,i}$, is equal to the number of $\lambda_k$ such that $\lambda_k = i$. Hence, we have

$$\varphi_{\langle\sigma\rangle,t}(J_{N_s,A,a}([n])) = \prod_{i=1}^{\infty}\Big(\frac{1}{1-t^i}\Big)^{O_{[n],\langle\sigma\rangle,i}}$$

**Remark 7.13.** By the identity (7.5), we have

$$\lambda_t(\chi)(g) = \prod_{i=1}^{\infty}(1-(-t)^i)^{O_{X,\langle g\rangle,i}}$$

where $\chi$ is the permutation character associated with a finite $G$-set $X$. Then, the element $E_{Nr}(\lambda_t(\chi))$ belongs to the image of operation $T_{O(g)}$ (For the map $E_{Nr}$ and the operation $T_{O(g)}$, see §2.2 and §3.2).

Note that we can not prove Theorem 3.20 with the discussion of this section, because there exists an integer-valued character $\chi$ such that $\chi$ is not generated by permutation characters.

For example, we consider the dihedral group $D_5$ which is a group generated by two elements $a$ and $b$ satisfying $a^5 = b^2 = 1$ and $bab^{-1} = a^{-1}$. The group $D_5$ has two irreducible characters, and the sum of two irreducible characters is an integer-valued character. However, this character is not generated by permutation characters of $D_5$. For more detail, see §8.2.

## 7.3 Calculation of the number of primitive colorings

In §7.3, we discuss $|A|$-colored $N$-nested $G$-set $X$ when $N = N_c := \{0, 1\}$ in order to calculate the number of primitive colorings.

**Theorem 7.14.** *For any finite $G$-set $X$, the map $\iota_c : J_{N_c,A,a}(X) \to A^X$ defined by*

$$\iota_c(f)(x) = \begin{cases} a & \text{if } f(x) \in {A'}^{[0]}, \\ f(x)(1) & \text{if } f(x) \in {A'}^{[1]}, \end{cases}$$

*is a $G$-isomorphism. In particular, two $G$-sets $J_{N_c,A,a}(X)$ and $A^X$ are $G$-isomorphic, and the set $J_{N_c,A,a}(X)$ is finite. Moreover, for any integer $n \geq 0$ and $f \in J^n_{N_c,A,a}(X)$,*

$$|\{x \in X \mid \iota_c(f)(x) \neq a\}| = n$$

*holds.*

*Proof.* First, we prove that the map $\iota_c$ is injective. Let $f$ and $g$ be elements of $J_{N_c,A,a}(X)$ with $\iota_c(f) = \iota_c(g)$. For any $x \in X$, if $f(x) \in {A'}^{[0]}$ then we have $f(x) = g(x) = a$. If $f(x) \in {A'}^{[1]}$, then $g(x)$ also belongs to ${A'}^{[1]}$ and $f(x)(1) = g(x)(1)$ holds. In any cases, we have $f(x) = g(x)$, thus, the map $\iota_c$ is injective.

Next, we prove that the map $\iota_c$ is surjective. For any $g \in A^X$, we put $f \in J_{N_c,A,a}(X)$ which satisfies

$$f(x) = \begin{cases} a & \text{if } g(x) = a, \\ 1 \mapsto g(x) & \text{if } g(x) \neq a. \end{cases}$$

Thus, we have $\iota_c(f)(x) = g(x)$, that is, the map $\iota_c$ is surjective.

Finally, we prove $g\iota_c(f) = \iota_c(gf)$ for any $g \in G$ and $f \in J_{N_c,A,a}(X)$. Let $x$ be an element of $X$. If $f(g^{-1}x) \in {A'}^{[0]}$, then we have $g\iota_c(f)(x) = \iota_c(f)(g^{-1}x) = a$ and $\iota_c(gf)(x) = a$. If $f(g^{-1}x) \in {A'}^{[1]}$, then $(gf)(x)$ also belongs to ${A'}^{[1]}$, and $\iota_c(f)(g^{-1})(x) = f(g^{-1}x)(1) = (gf)(x)(1) = \iota_c(gf)(x)$ holds. In any case, we have $g\iota_c(f) = \iota_c(gf)$. $\square$

We discuss a relation between the $G$-set $J_{N_c,A,a}(X)$ and the number of primitive colorings. First, we give the set of objects $X$ on polyhedrons, for example all vertices or all edges, and a rotation group $G$ of $X$. We assume that $X$ is a $G$-set.

**Definition 7.15.** We define a coloring on $X$ with $A$ by an element of $A^X$. We define a primitive coloring on $X$ of $G$ with $A$ by a coloring on $X$ which is asymmetric under rotations of $G$. In other words, a primitive coloring is defined by an element $f \in A^X$ such that $|Gf| = |G|$ holds.

By the definition, the number of primitive colorings up to symmetry is $\mu_{C_1}(A^X)$.

An element $f \in \iota_c(J^n_{N_c,A,a}(X))$, which satisfies that the number of $x \in X$ such that $f(x) \neq a$ is equal to $n$, is a colorings that there exists $(|X| - n)$ times of $a$ colored place. The number of primitive colorings which has $(|X| - n)$ times of $a$ colored place is equal to $\mu_{C_1}(\iota_c(J^n_{N_c,A,a}(X)))$.

To calculate the number of primitive colorings, we calculate $\mu_{C_1}(J_{N_c,A,a}(X))$ and $\mu_{C_1}(J^n_{N_c,A,a}(X))$. Calculation of $\mu_{C_1}(J^n_{N_c,A,a}(X))$ is equivalent to calculating the generating function $\mu_{C_1,t}(J_{N_c,A,a}(X))$, and we obtain $\mu_{C_1,t}(J_{N_c,A,a}(X))$ by substituting $H = C_1$ in the identity of Proposition 7.10.

Thus, we investigate rational numbers $a_{V,W}$'s and $\varphi_{H,t}(J^n_{N_c,A,a}(X))$ for any subgroup $H$ of $G$. To calculate $\varphi_{H,t}(J^n_{N_c,A,a}(X))$, we use the following theorem.

**Theorem 7.16.** *For any finite $G$-set $X$ and subgroup $H$ of $G$,*

$$\varphi_{H,t}(J_{N_c,A,a}(X)) = \prod_{i=1}^{\infty}(1 + |A'|t^i)^{O_{X,H,i}} \tag{7.6}$$

*holds. In particular,*

$$\varphi_H(J_{N_c,A,a}(X)) = |A|^{O_H(X)} \tag{7.7}$$

*holds where $O_H(X)$ is the number of orbits of $\mathrm{Res}^G_H(X)$ as an $H$-set.*

To prove this theorem, we state the following lemmas.

**Lemma 7.17.** *Let $X$ be a finite $G$-set. Then, an element $f \in J_{N_c,A,a}(X)$ satisfies $\deg(f) \leq |X|$. In particular, the set $J^n_{N_c,A,a}(X)$ is the empty set for any integer $n > |X|$.*

*Proof.* From $N_c = \{0, 1\}$, the degree of $f(x)$ is 0 or 1 for any $x \in X$. Then,

$$\deg(f) = \sum_{x \in X} \deg(f(x)) \leq |X|$$

holds. In particular, there does not exists an element $f \in J_{N_c,A,a}(X)$ such that $\deg(f) > |X|$. Hence, the set $J^n_{N_c,A,a}(X)$ is the empty set for any integer $n > |X|$. □

The following lemma shows that $\varphi_{H,t}$ is a refinement of $\varphi_H$ and that $\mu_{H,t}$ is that of $\mu_H$.

**Lemma 7.18.** *Let $X$ be a finite $G$-set and let $H$ be a subgroup of $G$. Consider that we substitute $t = 1$ in $\mu_{H,t}(J_{N_c,A,a}(X))$ and $\varphi_{H,t}(J_{N_c,A,a}(X))$. Then,*

$$\mu_{H,1}(J_{N_c,A,a}(X)) = \mu_H(J_{N_c,A,a}(X)), \quad \varphi_{H,1}(J_{N_c,A,a}(X)) = \varphi_H(J_{N_c,A,a}(X))$$

*hold.*

*Proof.* By Definition 7.7, we recall

$$\mu_{H,t}(J_{N_c,A,a}(X)) = \sum_{n=0}^{\infty} \mu_H(J^n_{N_c,A,a}(X))t^n.$$

By Lemma 7.17, we have $\mu_H(J^n_{N_c,A,a}(X)) = 0$ for any integer $n > |X|$. Substituting $t = 1$ in $\mu_{H,t}(J_{N_c,A,a}(X))$, we have

$$\begin{aligned}
\mu_{H,1}(J_{N_c,A,a}(X)) &= \sum_{n=0}^{\infty} \mu_H(J^n_{N_c,A,a}(X)) \\
&= \sum_{n=0}^{|X|} \mu_H(J^n_{N_c,A,a}(X)) \\
&= \mu_H\Big(\bigcup_{n=0}^{|X|} J^n_{N_c,A,a}(X)\Big) = \mu_H(J_{N_c,A,a}(X)).
\end{aligned}$$

Similarly, we have $\varphi_{H,1}(J_{N_c,A,a}(X)) = \varphi_H(J_{N_c,A,a}(X))$ since Proposition 6.7 (3) and

$$\varphi_{H,t}(J_{N_c,A,a}(X)) = \sum_{n=0}^{\infty} \mu_H(J^n_{N_c,A,a}(X))t^n$$

hold. □

*Proof of Theorem 7.16.* For (7.6), we substitute $N = N_c$ in the identities of Theorem 7.8. Thus, we have (7.6).

For (7.7), we substitute $t = 1$ in the identity (7.6). The left side of (7.6) is equal to $\varphi_H(J_{N_c,A,a}(X))$ since Lemma 7.18 holds and the right side of (7.6) is

$$|A|^{O_{X,H,1}+O_{X,H,2}+\cdots+O_{X,H,|X|}} = |A|^{O_H(X)}$$

by the definition of $O_H(X)$ and $O_{X,H,i}$'s. □

**Example 7.19.** Let $n \geq 1$ be an integer. We denote the set of all vertices of a regular $n$-gon by $X_n$, and we denote by $C_n$ the cyclic group of order $n$, which is the group of rotation symmetry of the regular $n$-gon. Then, we can regard $X_n$ as a $C_n$-set, and the set $X_n$ is isomorphic to $C_n/C_1$.

We show $\mu_{C_1}(J_{N_c,A,a}(X)) = M(|A|, n)$ which is the same result as the number of primitive necklaces [MR]. First, we calculate $\varphi_{C_{n/d},t}(J_{N_c,A,a}(X_n))$ for any divisor $d$ of $n$. By (6.5), we have

$$O_{X_n,C_{n/d},i} = \begin{cases} d & \text{if } i = \frac{n}{d}, \\ 0 & \text{otherwise.} \end{cases}$$

By (7.6), we have

$$\varphi_{C_{n/d},t}(J_{N_c,A,a}(X_n)) = (1 + |A'|t^{n/d})^d.$$

Thus, for any divisor $d$ of $n$, we have

$$
\begin{aligned}
\mu_{C_{n/d},t}(J_{N_c,A,a}(X_n)) =& \sum_{i=0}^{\infty} \mu_{C_{n/d}}(J^i_{N_c,A,a}(X_n))t^i \\
=& \frac{1}{d}\sum_{i=0}^{\infty}\sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'}}(J^i_{N_c,A,a}(X_n))t^i \\
=& \frac{1}{d}\sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'},t}(J_{N_c,A,a}(X_n)) \\
=& \frac{1}{d}\sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)(1+|A'|t^{n/d'})^{d'}
\end{aligned}
\tag{7.8}
$$

where the third equation follows from (6.8) in the case of $J^i_{N_c,A,a}(X)$. Substituting $t = 1$, we have

$$
\mu_{C_{n/d}}(J_{N_c,A,a}(X_n)) = \frac{1}{d}\sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)|A|^{d'} = M(|A|, d) \tag{7.9}
$$

by Lemma 7.18 and (5.1). We substitute $d = n$ in (7.9), and we obtain $\mu_{C_1}(J_{N_c,A,a}(X_n)) = M(|A|, n)$.

For example, we consider when $n = 6$, $A = \{0, 1\}$ and $a = 2$. Substituting these condition and $d = n = 6$ in (7.8). Then we have

$$
\begin{aligned}
\mu_{C_1,t}(J_{N_c,A,a}(X_6)) &= \frac{1}{6}\Big((1+t)^6 - (1+t^2)^3 - (1+t^3)^2 + (1+t^6)\Big) \\
&= t + 2t^2 + 3t^3 + 2t^4 + t^5.
\end{aligned}
$$

Substituting $t = 1$, we have $\mu_{C_1}(J_{N_c,A,a}(X_6)) = 9$ by Lemma 7.18. Figure 3 is all primitive colorings on 6-gon of $C_6$ with $A = \{0, 1\}$.

Recall that the coefficient of $t^n$ in $\mu_{C_1,t}(J_{N_c,A,a}(X_6))$ is the number of primitive colorings on $X_6$ of $C_6$ which has $(6-n)$ times of 0-colored places. In Figure 3, the image (a) is the unique primitive coloring which has five 0-colored places, images (b) and (c), or (g) and (h) are primitive colorings which have four or two 0-colored places, respectively, images (c), (d) and (e) are primitive colorings which have three 0-colored places, the image (i) is the unique primitive coloring which has one 0-colored place.

## 7.4 Generalization of identities of necklace polynomials

In §7.4, we generalize identities (5.3) and (5.4) for $\mu_V(J_{N_c,A,a}(X))$ where $V$ is an element of $\Phi(G)$, $A$ is a finite set with $|A| \geq 2$ and $a \in A$ and $X$ is a finite $G$-set $X$. We show that if $X = X_n$, then we obtain identities (5.3) and (5.4).

**Theorem 7.20.** *Let $X$ be a finite $G$-set. Take finite sets $A$ and $B$ with $|A|, |B| \geq 2$, and take elements $a \in A$ and $b \in B$. Then, two $G$-sets*

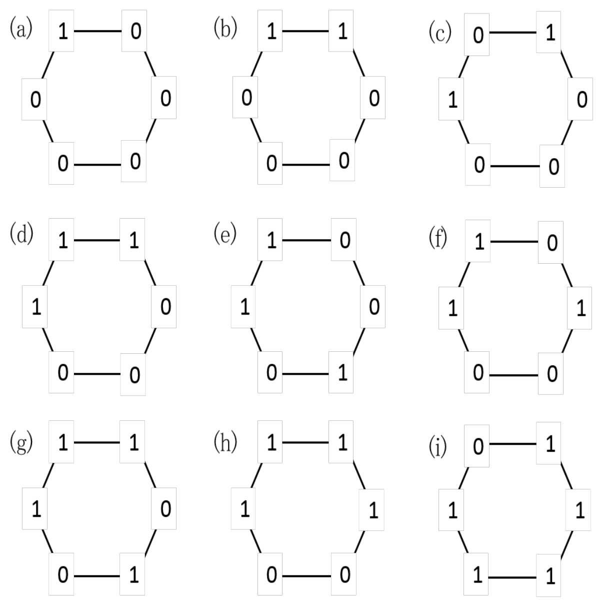


Figure 3: Primitive colorings on vertices of a regular 6-gon with 2 colors $\{0,1\}$.

*$J_{A\times B,N_c,(a,b)}(X)$ and $J_{A,N_c,a}(X)\times J_{B,N_c,b}(X)$ are $G$-isomorphic. In particular, one has*

$$\begin{aligned}&\mu_H(J_{N_c,A\times B,(a,b)}(X))\\ =&\sum_{V_1,V_2\in\Phi(G)} b_{V_1,V_2}(H)\mu_{V_1}(J_{N_c,A,a}(X))\mu_{V_2}(J_{N_c,B,b}(X))\end{aligned} \tag{7.10}$$

*for any $H\in\Phi(G)$.*

For the number $b_{V_1,V_2}(H)$, recall Definition 6.12.

*Proof.* We calculate both super characters of two $G$-sets $J_{N_c,A\times B,(a,b)}(X)$ and $J_{N_c,A,a}(X)\times J_{N_c,B,b}(X)$. By (7.7), we have

$$\begin{aligned}\varphi_H(J_{N_c,A\times B,(a,b)}(X)) &= (|A||B|)^{O_H(X)}\\ &= |A|^{O_H(X)}|B|^{O_H(X)} = \varphi_H(J_{N_c,A,a}(X)\times J_{N_c,B,b}(X)).\end{aligned}$$

for any subgroup $H\subset G$.

For (7.10), we calculate two integers $\mu_H(J_{A\times B,N_c,(a,b)}(X))$ and $\mu_H(J_{A,N_c,a}(X)\times J_{B,N_c,b}(X))$. Hence, we obtain the identity (7.10) by Proposition 6.13. □

**Example 7.21.** Under the assumption of Example 7.19, we show that the identity (7.20) is a generalization of the identity (5.3).

Substituting $X = X_n$ in (7.10), we have

$$\begin{aligned} &\mu_{C_1}(J_{N_c,A\times B,(a,b)}(X_n)) \\ &= \sum_{i,j|n} b_{C_{n/i},C_{n/j}}(C_n)\mu_{C_{n/i}}(J_{N_c,A,a}(X_n))\mu_{C_{n/j}}(J_{N_c,B,b}(X_n)). \end{aligned} \tag{7.11}$$

We use Proposition 6.3 to calculate numbers $b_{C_{n/i},C_{n/j}}$. For any divisors $i$ and $j$ of $n$, if $[i,j] = n$ then the number of double cosets $C_{n/i}gC_{n/j}$ in $C_n$ such that $C_{n/i} \cap C_{n/j} = C_n$ is $(i,j)$. If $[i,j] \neq n$, there does not exist a double coset satisfying $C_{n/i} \cap C_{n/j} = C_n$. With this fact and (7.9), we have

$$M(k_1k_2, n) = \sum_{[i,j]=n} (i,j)M(k_1, i)M(k_2, j)$$

where $k_1 = |A|$ and $k_2 = |B|$. It is the same as (5.3).

Next, we consider a generalization of (5.4)

**Theorem 7.22.** *Let $X$ be a finite $G$-set and let $r \geq 1$ be an integer. Take a finite set $|A|$ with $|A| \geq 2$, and an element $a \in A$. We put $P = A \times \cdots \times A$ ($r$-times of cartesian product) and $p = (a, \ldots, a) \in P$.*

*Suppose that there exists a finite group $G'$ and a $G'$-set $Y$ such that $G$ is embedded to $G'$ and $[\mathrm{Res}_G^{G'}(Y)] = r[X]$ holds. Then the $G$-set $J_{N_c,P,p}(X)$ is $G$-isomorphic to $\mathrm{Res}_G^{G'}(J_{N_c,A,a}(Y))$. In particular, one has*

$$\mu_H(J_{N_c,P,p}(X)) = \sum_{V\in\Phi(G')} c_V(H)\mu_V(J_{N_c,A,a}(Y)) \tag{7.12}$$

*for any $H \in \Phi(G)$.*

For the number $c_V(H)$, recall Definition 6.15.

*Proof.* We calculate both super characters of two $G$-sets $J_{N_c,P,p}(X)$ and $\mathrm{Res}_G^{G'}(J_{N_c,A,a}(Y))$. For any subgroup $H$ of $G$, we have

$$\begin{aligned} \varphi_H(J_{N_c,P,p}(X)) &= |P|^{O_H(X)} \\ &= |A|^{rO_H(X)} \\ &= |A|^{O_H(\mathrm{Res}_G^{G'}(Y))} \\ &= \varphi_H(J_{N_c,A,a}(\mathrm{Res}_G^{G'}(Y))) = \varphi_H(\mathrm{Res}_G^{G'}(J_{N_c,A,a}(Y))). \end{aligned}$$

where the first and fourth equality follows from (7.7) and fifth equality follows from Proposition 7.9.

Calculating $\mu_H(J_{N_c,P,p}(X))$ and $\mu_H(\mathrm{Res}_G^{G'}(J_{N_c,A,a}(Y)))$ we obtain the identity (7.12) by Proposition 6.16. □

**Example 7.23.** Under the assumption of Example 7.19, we show that the identity (7.22) is a generalization of the identity (5.4). Let $r \geq 1$ be an integer.

We can identify $C_n$ with $C_{nr/r} \subset C_{nr}$, and we have $[\mathrm{Res}^{C_{nr}}_{C_{nr/r}}(X_{nr})] = r[X_n]$.

Then, two $C_n$-sets $J_{N_c,P,p}(X_n)$ and $\mathrm{Res}^{C_{nr}}_{C_{nr/r}}(J_{N_c,A,a}(X_{nr}))$ are $C_n$-isomorphic. In particular, we have

$$\mu_{C_1}(J_{N_c,P,p}(X_n)) = \sum_{d|nr} c_{C_{nr/d}}(C_1)\mu_{C_{nr/d}}(J_{N_c,A,a}(X_{nr}))$$

We use Proposition 6.4 to calculate numbers $c_{C_{nr/d}}(C_1)$. The number of double cosets $C_{nr/d}gC_{nr/r}$ in $C_{nr}$ such that $C_{nr/r} \cap C_{nr/d} = C_1$ is $(d,r)$ when $[d,r] = nr$. In this case, we have $(d,r) = d/n$. Ir $[d,r] \neq nr$, then there does not exists a double coset $C_{nr/d}gC_{nr/r}$ in $C_{nr}$ such that $C_{nr/r} \cap C_{nr/d} = C_1$. Thus, we have

$$M(k^r, n) = \sum_{[d,r]=nr} \frac{d}{n} M(k,d)$$

which is the same as (5.4).

We assume that the subgroup $C_1 \times G \subset C_r \times G$ is identified with $G$. Substituting $Y = C_r \times X$ in Theorem 7.22, we have the following corollary.

**Corollary 7.24.** *For any finite $G$-set $X$, two $G$-sets $\mathrm{Res}^{C_r \times G}_G(J_{N_c,A,a}(C_r \times X))$ and $J_{N_c,P,p}(X)$ are $G$-isomorphic. In particular, one has*

$$\mu_H(J_{N_c,P,p}(X)) = \sum_{V \in \Phi(C_r \times G)} c_V(H)\mu_V(J_{N_c,A,a}(C_r \times X))$$

*for any subgroup $H$.*

# 8 The number of primitive colorings on some objects of polyhedrons.

In this section, we assume $N = N_c = \{0,1\}$ and calculate the number of primitive colorings on a given the set of object $X$ and a rotation group $G$ of $X$ with a color set $A$. For more detail, we calculate $\mu_{C_1}(J_{N_c,A,a}(X))$ and $\mu_{C_1,t}(J_{N_c,A,a}(X))$ in the following cases as examples where $a \in A$.

(i) We assume that $X$ is the set of all vertices of $n$-prism and $G$ is the dihedral group $D_n$.

(ii) We assume that $X$ is the set of all vetrices of $n$-gon and $G$ is the dihedral group.

In Example 7.19, we considered the set of all vertices of $n$-gon $X_n$ and the cyclic group $C_n$. The difference of the condition of Example 7.19 and (ii) is that we consider not only rotations but also reverses in (ii). In §8.1.1, we investigate the cardinality of orbits of $\mathrm{Res}^{D_n}_K(D_n/K)$ and $\varphi_K(D_n/K')$ for any $K, K' \in \Phi(D_n)$. In addition, we represent $\mu_K(X)$ as all $\varphi_K(X)$ for any $K \in \Phi(D_n)$ (Corollary 8.4). In §8.1.2, we consider the case (ii), and in §8.1.3 we consider the case (ii).

## 8.1 A dihedral group

We consider the case of $G = D_n$ which is a finite group generated by two elements $a$ and $b$ satisfying $a^n = b^2 = 1$ and $bab = a^{-1}$. In §8.1, we redefine $C_{n/d}$ by the subgroup $\langle a^d \rangle$ for any divisor $d$ of $n$.

### 8.1.1 Calculation of orbits

First, we investigate all elements of $D_n$.

**Proposition 8.1.** *Conjugacy subgroups of $D_n$ are the following subgroups.*

$$\begin{aligned} C_{n/d} &:= \{1, a^d, \ldots, a^{(n/d-1)d}\}, \\ D_{n/d} &:= C_{n/d} \cup bC_{n/d}, \\ D'_{n/d} &:= C_{n/d} \cup baC_{n/d}, \end{aligned}$$

*for each divisor $d$ of $n$ where two subgroups $D_{n/d}$ and $D'_{n/d}$ are conjugate if $2 \nmid d$.*

*Proof.* Put $D_n = \{1, a, a^2, \ldots, a^{n-1}, b, ba, ba^2, \ldots, ba^{n-1}\}$. First, we investigate all subgroups $H$ of $D_n$. Let $d$ be the minimum number such that $a^d$ is an element of $H$. Thus, $C_{n/d}$ is contained to $H$. If there does not exist an element which has the form $ba^k$ in $H$, then $H = C_{n/d}$ holds. If $H$ has an element which has the form $ba^k$, put $k = 0, \ldots, n-1$ which is the minimum number such that $ba^k$ is an element of $H$. Hence, we have $H = C_{n/d} \cup ba^k C_{n/d}$.

Next, we consider all conjugacy subgroups of $D_n$. Let $d$ be a divisor of $n$. For elements $a, b \in D_n$, we have $aC_{n/d}a^{-1} = C_{n/d}$ and $bC_{n/d}b^{-1} = C_{n/d}$ by the definition of $D_n$. Thus, $C_{n/d}$ is one of conjugacy subgroup of $D_n$. In addition, we have $a(C_{n/d} \cap ba^k C_{n/d})a^{-1} = C_{n/d} \cap ba^{k-2}$ and $b(C_{n/d} \cap ba^k C_{n/d})b^{-1} = C_{n/d} \cap ba^{-k}$. Now, put

$$D_{n/d} := C_{n/d} \cup bC_{n/d}, \quad D'_{n/d} := C_{n/d} \cup baC_{n/d}.$$

Thus, all subgroups of $D_n$ which has the form $C_{n/d} \cup ba^k C_{n/d}$ are conjugate to either of the above subgroups. Moreover, if $2 \nmid d$ then the above two subgroups are conjugate. If $2 \mid d$ then the above two subgroups are not conjugate. $\square$

We investigate orbits of $K$-set $\mathrm{Res}^{D_n}_K(D_n/K')$ for any $K, K' \in \Phi(D_n)$. For the calculation, we recall Proposition 6.4, and we focus on the cardinality of orbits of $\mathrm{Res}^{D_n}_K(D_n/K')$ from (7.6).

(i) First, we consider the case of $K' = C_{n/d'}$. Recall that $C_{n/d'}$ is the normal subgroup of $D_n$. Then, we have

$$[\mathrm{Res}^{D_n}_{C_{n/d'}}(D_n/K)] = \sum_{C_{n/d'}gK} [C_{n/d'}/(C_{n/d'} \cap gKg^{-1})]$$

where $g$ ranges over a set of double coset representatives of $C_{n/d'}$ and $K$ in $D_n$. For any $g \in D_n$, we have $|(C_{n/d'} \cap gKg^{-1})| = |C_{n/d'} \cap K|$. In addition, the number of double cosets of $C_{n/d'}$ and $K$ is $|D_n/C_{n/d'}K|$. Thus, we have

$$O_{D_n/K,C_{n/d'},i} = \begin{cases} |D_n/C_{n/d'}K| & \text{if } i = |C_{n/d'}/(C_{n/d'} \cap K)|, \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

Hence, we have

$$\varphi_{C_{n/d'}}(D_n/K) = \begin{cases} |D_n/C_{n/d'}K| & \text{if } C_{n/d'} \subset K, \\ 0 & \text{otherwise,} \end{cases}$$

and by (7.6) and (8.1), we have

$$\varphi_{C_{n/d'},t}(J_{N_c,A,a}(D_n/K)) = (1 + |A'|t^{|C_{n/d'}/(C_{n/d'} \cap K)|})^{|D_n/C_{n/d'}K|}.$$

In particular,

$$\varphi_{C_{n/d'}}(D_n/C_{n/d}) = \begin{cases} 2d' & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.2}$$

$$\varphi_{C_{n/d'}}(D_n/D_{n/d}) = \begin{cases} d' & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.3}$$

$$\varphi_{C_{n/d'}}(D_n/D'_{n/d}) = \begin{cases} d' & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.4}$$

and

$$\varphi_{C_{n/d'},t}(J_{N_c,A,a}(D_n/C_{n/d})) = (1 + |A'|t^{[d,d']/d'})^{2(d,d')}, \tag{8.5}$$

$$\varphi_{C_{n/d'},t}(J_{N_c,A,a}(D_n/D_{n/d})) = (1 + |A'|t^{[d,d']/d'})^{(d,d')}, \tag{8.6}$$

$$\varphi_{C_{n/d'},t}(J_{N_c,A,a}(D_n/D'_{n/d})) = (1 + |A'|t^{[d,d']/d'})^{(d,d')} \tag{8.7}$$

hold. Identities (8.6) and (8.7) are equal.

(ii) We consider the case of $K = C_{n/d}$. We discuss the method similar to (i). We have

$$\mathrm{Res}^{D_n}_K(D_n/C_{n/d})] = \sum_{KgC_{n/d}} [K/K \cap gC_{n/d}g^{-1}]$$

where $g$ ranges over a set of double coset representatives of $K$ and $C_{n/d}$ in $D_n$. For any $g \in D_n$, we have $|K \cap gC_{n/d}g^{-1})| = |K \cap C_{n/d}|$. In addition, the number of double cosets of $C_{n/d}$ and $K$ is $|D_n/C_{n/d}K|$. Thus, we have

$$O_{D_n/C_{n/d},K,i} = \begin{cases} |D_n/C_{n/d}K| & \text{if } i = |K/(K \cap C_{n/d})|, \\ 0 & \text{otherwise.} \end{cases} \tag{8.8}$$

Hence, we have

$$\varphi_K(D_n/C_{n/d}) = \begin{cases} |D_n/C_{n/d}K| & \text{if } K \subset C_{n/d}, \\ 0 & \text{otherwise,} \end{cases}$$

and by (7.6) and (8.8), we have

$$\varphi_{K,t}(J_{N_c,A,a}(D_n/C_{n/d})) = (1 + |A'|t^{|K/K\cap C_{n/d'})|})^{|D_n/C_{n/d}K|}.$$

In particular,

$$\begin{aligned} \varphi_{C_{n/d'}}(D_n/C_{n/d}) &= \begin{cases} 2d' & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} && (8.9) \\ \varphi_{D_{n/d'}}(D_n/C_{n/d}) &= 0, && (8.10) \\ \varphi_{D'_{n/d'}}(D_n/C_{n/d}) &= 0, && (8.11) \end{aligned}$$

and

$$\begin{aligned} \varphi_{C_{n/d'},t}(J_{N_c,A,a}(D_n/C_{n/d})) &= (1 + |A'|t^{[d,d']/d'})^{2(d,d')}, && (8.12) \\ \varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/C_{n/d})) &= (1 + |A'|t^{2[d,d']/d'})^{(d,d')}, && (8.13) \\ \varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/C_{n/d})) &= (1 + |A'|t^{2[d,d']/d'})^{(d,d')} && (8.14) \end{aligned}$$

hold. We remark that identities (8.2) and (8.9), or (8.5) and (8.12) are same calculation results respectively, and identities (8.13) and (8.14) are equal.

(iii) We consider the case of $K = D_{n/d}$ and $K' = D_{n/d'}$. For any integer $i \geq 0$, we have

$$D_{n/d'}a^iD_{n/d} = a^iC_{n/(d,d')} \cup a^{-i}C_{n/(d,d')} \cup ba^iC_{n/(d,d')} \cup ba^{-i}C_{n/(d,d')}$$

which gives the following double cosets representation,

$$D_n = \begin{cases} \bigcup_{i=0}^{(d,d')/2} D_{n/d'}a^iD_{n/d} & \text{if } 2 \mid (d,d'), \\ \bigcup_{i=0}^{((d,d')-1)/2} D_{n/d'}a^iD_{n/d} & \text{if } 2 \nmid (d,d'). \end{cases}$$

Next, we investigate a subgroup $D_{n/d'} \cap a^iD_{n/d}a^{-i}$ for any integer $i \geq 0$ to consider orbits of $\mathrm{Res}^{D_n}_{D_{n/d'}}(D_n/D_{n/d})$. A subgroup $D_{n/d'} \cap a^iD_{n/d}a^{-i}$ is conjugate to $C_{n/[d,d']}, D_{n/[d,d']}$ or $D'_{n/[d,d']}$. An integer $i \geq 0$ satisfies $2i \mid (d,d')$ if and only if $D_{n/d'} \cap a^iD_{n/d}a^{-i}$ has an element which has the form $ba^k$.

If $2 \mid (d,d')$, then $D_{n/d'} \cap a^iD_{n/d}a^{-i}$ is conjugate to $D_{n/[d,d']}$ when $i = 0$ or $(d,d')/2$, and is conjugate to $C_{n/[d,d']}$ when $i \neq 0$ and $(d,d')/2$. Then, we have

$$O_{D_n/D_{n/d},D_{n/d'},i} = \begin{cases} 2 & \text{if } i = \frac{[d,d']}{d'}, \\ \frac{(d,d')}{2} - 1 & \text{if } i = \frac{2[d,d']}{d'}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.15}$$

Hence, we have

$$\varphi_{D_{n/d'}}(D_n/D_{n/d}) = \begin{cases} 2 & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.16}$$

and by (7.6) and (8.15), we have

$$\begin{aligned} &\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D_{n/d})) \\ =&(1+|A'|t^{[d,d']/d'})^2(1+|A'|t^{2[d,d']/d'})^{((d,d')/2)-1}. \end{aligned} \tag{8.17}$$

If $2 \nmid (d,d')$, then $D_{n/d'} \cap a^i D_{n/d} a^{-i}$ is conjugate to $D_{n/[d,d']}$ when $i=0$ and is conjugate to $C_{n/[d,d']}$ when $i \neq 0$. Then, we have

$$O_{D_n/D_{n/d},D_{n/d'},i} = \begin{cases} 1 & \text{if } i = \frac{[d,d']}{d'}, \\ \frac{(d,d')-1}{2} & \text{if } i = \frac{2[d,d']}{d'}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.18}$$

Hence, we have

$$\varphi_{D_{n/d'}}(D_n/D_{n/d}) = \begin{cases} 1 & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.19}$$

and by (7.6) and (8.18), we have

$$\begin{aligned} &\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D_{n/d})) \\ =&(1+|A'|t^{[d,d']/d'})(1+|A'|t^{2[d,d']/d'})^{((d,d')-1)/2}. \end{aligned} \tag{8.20}$$

(iv) We consider the case of $K = D_{n/d}$ and $K' = D'_{n/d'}$. For any integer $i \geq 0$ we have

$$D'_{n/d'} a^i D_{n/d} = a^i C_{n/(d,d')} \cup a^{-(i+1)} C_{n/(d,d')} \cup b a^{i+1} C_{n/(d,d')} \cup b a^{-i} C_{n/(d,d')}$$

which gives the following double cosets representation.

$$D_n = \begin{cases} \bigcup_{i=0}^{((d,d')/2)-1} D'_{n/d'} a^i D_{n/d} & \text{if } 2 \mid (d,d'), \\ \bigcup_{i=0}^{((d,d')-1)/2} D'_{n/d'} a^i D_{n/d} & \text{if } 2 \nmid (d,d'). \end{cases} \tag{8.21}$$

A subgroup $D'_{n/d'} \cap a^i D_{n/d} a^{-i}$ is conjugate to either $C_{n/[d,d']}$, $D_{n/[d,d']}$ or $D'_{n/[d,d']}$. An integer $i \geq 0$ satisfies $2i+1 \mid (d,d')$ if and only if $D_{n/d'} \cap a^i D_{n/d} a^{-i}$ has an element which has the form $ba^k$.

If $2 \mid (d,d')$, then there exists no integer $i \geq 0$ such that $D'_{n/d'} \cap a^i D_{n/d} a^{-i}$ is conjugate to $D_{n/[d,d']}$ or $D'_{n/[d,d']}$. Then, we have

$$O_{D_n/D_{n/d},D'_{n/d'},i} = \begin{cases} \frac{(d,d')}{2} & \text{if } i = \frac{2[d,d']}{d'}, \\ 0 & \text{otherwise} \end{cases} \tag{8.22}$$

which is equal to (8.18). Hence, we have

$$\varphi_{D'_{n/d'}}(D_n/D_{n/d}) = 0, \tag{8.23}$$

and by (7.6) and (8.22), we have

$$\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/D_{n/d})) = (1+|A'|t^{2[d,d']/d'})^{(d,d')/2}. \tag{8.24}$$

If $2 \nmid (d,d')$, then $D_{n/d'} \cap a^i D_{n/d} a^{-i}$ is conjugate to $D_{n/[d,d']}$ when $i=0$ and is conjugate to $C_{n/[d,d']}$ when $i \neq 0$. Then, we have

$$O_{D_n/D_{n/d},D'_{n/d'},i} = \begin{cases} 1 & \text{if } i = \frac{[d,d']}{d'}, \\ \frac{(d,d')-1}{2} & \text{if } i = \frac{2[d,d']}{d'}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.25}$$

Hence, we have

$$\varphi_{D'_{n/d'}}(D_n/D_{n/d}) = \begin{cases} 1 & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.26}$$

which is equal to (8.19), and by (7.6) and (8.25), we have

$$\begin{aligned} &\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/D_{n/d})) \\ =&(1+|A'|t^{[d,d']/d'})(1+|A'|t^{2[d,d']/d'})^{((d,d')-1)/2} \end{aligned} \tag{8.27}$$

which is equal to (8.20).

(v) We consider the case of $K = D'_{n/d}$ and $K' = D_{n/d'}$ with the method similar to (iv). By (8.28), we have

$$D_n = \begin{cases} \bigcup_{i=0}^{((d,d')/2)-1} D_{n/d'} a^{-i} D'_{n/d} & \text{if } 2 \mid (d,d'), \\ \bigcup_{i=0}^{((d,d')-1)/2} D_{n/d'} a^{-i} D'_{n/d} & \text{if } 2 \nmid (d,d'). \end{cases} \tag{8.28}$$

If $2 \mid (d,d')$, then there exists no integer $i \geq 0$ such that $D'_{n/d} \cap a^{-i} D_{n/d'} a^i$ is conjugate to $D_{n/[d,d']}$ or $D'_{n/[d,d']}$. Then, we have

$$O_{D_n/D'_{n/d},D_{n/d'},i} = \begin{cases} \frac{(d,d')}{2} & \text{if } i = \frac{2[d,d']}{d'}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.29}$$

Hence, we have

$$\varphi_{D_{n/d'}}(D_n/D'_{n/d}) = 0, \tag{8.30}$$

and by (7.6) and (8.29), we have

$$\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D'_{n/d})) = (1+|A'|t^{2[d,d']/d'})^{(d,d')/2}. \tag{8.31}$$

If $2 \nmid (d,d')$, then $D_{n/d'} \cap a^i D_{n/d} a^{-i}$ is conjugate to $D_{n/[d,d']}$ when $i = 0$ and is conjugate to $C_{n/[d,d']}$ when $i \neq 0$. Thus, we have

$$O_{D_n/D'_{n/d}, D_{n/d'}, i} = \begin{cases} 1 & \text{if } i = \frac{[d,d']}{d'}, \\ \frac{(d,d')-1}{2} & \text{if } i = \frac{2[d,d']}{d'}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.32}$$

Hence, we have

$$\varphi_{D_{n/d'}}(D_n/D_{n/d}) = \begin{cases} 1 & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.33}$$

and by (7.6) and (8.32), we have

$$\begin{aligned} &\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D'_{n/d})) \\ =&(1+|A'|t^{[d,d']/d'})(1+|A'|t^{2[d,d']/d'})^{((d,d')-1)/2}. \end{aligned} \tag{8.34}$$

(vi) We assume $2 \mid n$, and we consider when $K = D'_{n/d}$ and $K' = D'_{n/d'}$. For any integer $i \geq 0$ we have

$$D'_{n/d'} a^i D'_{n/d} = a^i C_{n/(d,d')} \cup a^{-i} C_{n/(d,d')} \cup ba^{i+1} C_{n/(d,d')} \cup ba^{-i+1} C_{n/(d,d')}$$

which gives the following double cosets representation,

$$D_n = \begin{cases} \bigcup_{i=0}^{(d,d')/2} D'_{n/d'} a^i D'_{n/d} & \text{if } 2 \mid (d,d'), \\ \bigcup_{i=0}^{((d,d')-1)/2} D'_{n/d'} a^i D'_{n/d} & \text{if } 2 \nmid (d,d'). \end{cases} \tag{8.35}$$

A subgroup $D'_{n/d'} \cap a^i D'_{n/d} a^{-i}$ is conjugate to $C_{n/[d,d']}$, $D_{n/[d,d']}$ or $D'_{n/[d,d']}$. An integer $i \geq 0$ satisfies $2i \mid (d,d')$ if and only if $D'_{n/d'} \cap a^i D'_{n/d} a^{-i}$ has an element which has a form $ba^k$. Then we have

$$O_{D_n/D'_{n/d}, D_{n/d'}, i} = \begin{cases} 2 & \text{if } i = \frac{[d,d']}{d'}, \\ \frac{(d,d')}{2} - 1 & \text{if } i = \frac{2[d,d']}{d'}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.36}$$

Hence, we have

$$\varphi_{D'_{n/d'}}(D'_n/D'_{n/d}) = \begin{cases} 2 & \text{if } d' \mid d, \\ 0 & \text{if } d' \nmid d, \end{cases} \tag{8.37}$$

which is equal to (8.16), and by (7.6) and (8.36), we have

$$\begin{aligned} &\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/D'_{n/d})) \\ =&(1+|A'|t^{[d,d']/d'})^2(1+|A'|t^{2[d,d']/d'})^{((d,d')/2)-1} \end{aligned} \tag{8.38}$$

which is equal to (8.17).

Next, we represent $\mu_H(X)$ with $\varphi_V(X)$ for any $D_n$-set $X$.

**Proposition 8.2.** *For any $D_n$-set $X$ and divisor $d$ of $n$, the followings hold,*
(1)

$$\begin{aligned}&\frac{1}{d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'}}(X)\\ =&\begin{cases}2\mu_{C_{n/d}}(X)+\mu_{D_{n/d}}(X) & \text{if } 2\nmid d,\\ 2\mu_{C_{n/d}}(X)+\mu_{D_{n/d}}(X)+\mu_{D'_{n/d}}(X) & \text{if } 2\mid d.\end{cases}\end{aligned}$$

(2)

$$\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'}}(X)=\begin{cases}\mu_{D_{n/d}}(X) & \text{if } 2\nmid d,\\ 2\mu_{D_{n/d}}(X) & \text{if } 2\mid d.\end{cases}$$

(3) *If $2\mid d$, then*

$$\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D'_{n/d'}}(X)=2\mu_{D'_{n/d}}(X)$$

*holds.*

*Proof.* By Proposition 8.1, an arbitrary $D_n$-set $X$ has the following form,

$$\begin{aligned}[X]=&\sum_{d'|n,2\nmid d'}\Big(\mu_{C_{n/d'}}(X)[D_n/C_{n/d'}]+\mu_{D_{n/d'}}(X)[D_n/D_{n/d'}]\Big)\\ &+\sum_{d'|n,2|d'}\Big(\mu_{C_{n/d'}}(X)[D_n/C_{n/d'}]+\mu_{D_{n/d'}}(X)[D_n/D_{n/d'}]\\ &+\mu_{D_{n/d'}}(X)[D_n/D'_{n/d'}]\Big).\end{aligned}\tag{8.39}$$

By (8.39), (8.2), (8.3) and (8.4), we have

$$\begin{aligned}\varphi_{C_{n/d}}(X)=&\sum_{d'|d,2\nmid d'}(2d'\mu_{C_{n/d'}}(X)+d'\mu_{D_{n/d'}}(X))\\ &+\sum_{d'|d,2|d'}(2d'\mu_{C_{n/d'}}(X)+d'\mu_{D_{n/d'}}(X)+d'\mu_{D_{n/d'}}(X)).\end{aligned}\tag{8.40}$$

Thus, we have the identity (1) by (8.40) and the Möbius inversion formula.
By (8.39), (8.10), (8.16), (8.19), (8.30) and (8.33) we have

$$\varphi_{D_{n/d}}(X)\quad=\quad\sum_{d'|d,2\nmid d'}\mu_{D_{n/d'}}(X)+\sum_{d'|d,2|d'}2\mu_{D_{n/d'}}(X).\tag{8.41}$$

Thus, we have the identity (2) by (8.41) and the Möbius inversion formula.

By (8.39), (8.11), (8.30), (8.33) and (8.37) we have

$$\varphi_{D_{n/d'}}(X) = \sum_{d'|d, 2\nmid d'} \mu_{D_{n/d'}}(X) + \sum_{d'|d, 2|d'} 2\mu_{D'_{n/d'}}(X). \tag{8.42}$$

Thus, we have the identity (3) by (8.42) and the Möbius inversion formula. □

By Proposition 8.2, we have the following proposition.

**Proposition 8.3.** *For any $D_n$-set $X$ and divisor $d$ of $n$, the followings hold,*
(1)

$$\begin{aligned}
&\frac{1}{d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'},t}(J_{N_c,A,a}(X)) \\
&= \begin{cases} 2\mu_{C_{n/d},t}(J_{N_c,A,a}(X)) + \mu_{D_{n/d},t}(J_{N_c,A,a}(X)) & \text{if } 2\nmid d, \\ 2\mu_{C_{n/d},t}(J_{N_c,A,a}(X)) & \\ +\mu_{D_{n/d},t}(J_{N_c,A,a}(X)) + \mu_{D'_{n/d},t}(J_{N_c,A,a}(X)) & \text{if } 2\mid d. \end{cases}
\end{aligned}$$

(2)

$$\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'},t}(J_{N_c,A,a}(X)) = \begin{cases} \mu_{D_{n/d},t}(J_{N_c,A,a}(X)) & \text{if } 2\nmid d, \\ 2\mu_{D_{n/d},t}(J_{N_c,A,a}(X)) & \text{if } 2\mid d. \end{cases}$$

(3) *If $2\mid d$, then*

$$\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(X)) = 2\mu_{D'_{n/d},t}(J_{N_c,A,a}(X))$$

*holds.*

Note that (2) and (3) are equal if $2\mid d$ holds.

*Proof.* To proof this Proposition 8.3, we consider the coefficient of $t^n$ in these elements and use Proposition 8.2.

(1) If $2\nmid d$, we have

$$\begin{aligned}
&\frac{1}{d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'},t}(J_{N_c,A,a}(X)) \\
&= \frac{1}{d}\sum_{d'|d}\sum_{i=0}^{\infty}\mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'}}(J^i_{N_c,A,a}(X))t^i \\
&= \sum_{i=0}^{\infty}(2\mu_{C_{n/d}}(J^i_{N_c,A,a}(X)) + \mu_{D_{n/d}}(J^i_{N_c,A,a}(X)))t^i \\
&= 2\mu_{C_{n/d},t}(J_{N_c,A,a}(X)) + \mu_{D_{n/d},t}(J_{N_c,A,a}(X)).
\end{aligned}$$

Similarly, if $2 \mid d$, we have

$$\begin{aligned}
& \frac{1}{d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'},t}(J_{N_c,A,a}(X)) \\
= \ & \frac{1}{d}\sum_{d'|d}\sum_{i=0}^{\infty}\mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'}}(J^i_{N_c,A,a}(X))t^i \\
= \ & \sum_{i=0}^{\infty}(2\mu_{C_{n/d'}}(J^i_{N_c,A,a}(X)) + \mu_{D_{n/d}}(J^i_{N_c,A,a}(X)) + \mu_{D'_{n/d}}(J^i_{N_c,A,a}(X)))t^i \\
= \ & 2\mu_{C_{n/d},t}(J_{N_c,A,a}(X)) + \mu_{D_{n/d},t}(J_{N_c,A,a}(X)) + \mu_{D_{n/d'},t}(J_{N_c,A,a}(X)).
\end{aligned}$$

(2) If $2 \nmid d$, we have

$$\begin{aligned}
& \frac{1}{d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'},t}(J_{N_c,A,a}(X)) \\
= \ & \frac{1}{d}\sum_{d'|d}\sum_{i=0}^{\infty}\mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'}}(J^i_{N_c,A,a}(X))t^i \\
= \ & \sum_{i=0}^{\infty}\mu_{D_{n/d}}(J^i_{N_c,A,a}(X))t^i = \mu_{D_{n/d},t}(J_{N_c,A,a}(X)).
\end{aligned}$$

Similarly, if $2 \mid d$, we have

$$\begin{aligned}
& \frac{1}{d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'},t}(J_{N_c,A,a}(X)) \\
= \ & \frac{1}{d}\sum_{d'|d}\sum_{i=0}^{\infty}\mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'}}(J^i_{N_c,A,a}(X))t^i \\
= \ & \sum_{i=0}^{\infty}2\mu_{D_{n/d}}(J^i_{N_c,A,a}(X))t^i = 2\mu_{D_{n/d},t}(J_{N_c,A,a}(X)).
\end{aligned}$$

(3) We assume $2 \mid d$. Thus, we have

$$\begin{aligned}
& \frac{1}{d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(X)) \\
= \ & \frac{1}{d}\sum_{d'|d}\sum_{i=0}^{\infty}\mu\Big(\frac{d}{d'}\Big)\varphi_{D'_{n/d'}}(J^i_{N_c,A,a}(X))t^i \\
= \ & \sum_{i=0}^{\infty}2\mu_{D'_{n/d}}(J^i_{N_c,A,a}(X))t^i = 2\mu_{D'_{n/d},t}(J_{N_c,A,a}(X)).
\end{aligned}$$

$\square$

By Proposition 8.3, we have the following corollary.

**Corollary 8.4.** *For any $D_n$-set $X$ and divisor $d$ of $n$, the followings hold,*
(1) *If $2 \nmid d$, the following identities hold,*

$$\mu_{D_{n/d},t}(J_{N_c,A,a}(X)) = \sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'},t}(J_{N_c,A,a}(X)), \tag{8.43}$$

$$\begin{aligned} & \mu_{C_{n/d},t}(J_{N_c,A,a}(X)) \quad (8.44)\\ = & \sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}\varphi_{C_{n/d'},t}(J_{N_c,A,a}(X)) - \frac{1}{2}\varphi_{D_{n/d'},t}(J_{N_c,A,a}(X))\Big). \end{aligned}$$

(2) *If $2 \mid d$, the following identities hold,*

$$\mu_{D_{n/d},t}(J_{N_c,A,a}(X)) = \frac{1}{2}\sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'},t}(J_{N_c,A,a}(X)), \tag{8.45}$$

$$\mu_{D'_{n/d},t}(J_{N_c,A,a}(X)) = \frac{1}{2}\sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(X)), \tag{8.46}$$

$$\begin{aligned} & \mu_{C_{n/d},t}(J_{N_c,A,a}(X)) \\ = & \frac{1}{2d}\sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\varphi_{C_{n/d'},t}(J_{N_c,A,a}(X)) - \frac{1}{2}\mu_{D_{n/d},t}(J_{N_c,A,a}(X)) \quad (8.47)\\ & -\frac{1}{2}\mu_{D_{n/d},t}(J_{N_c,A,a}(X)) \\ = & \sum_{d'|d} \mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}\varphi_{C_{n/d'},t}(J_{N_c,A,a}(X)) - \frac{1}{4}\varphi_{D_{n/d'},t}(J_{N_c,A,a}(X)) \quad (8.48)\\ & -\frac{1}{4}\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(X))\Big). \end{aligned}$$

*Proof.* We assume $2 \nmid d$. The identity about $\mu_{D_{n/d},t}(J_{N_c,A,a}(X))$ holds since Proposition 8.3 (2) holds. For $\mu_{C_{n/d},t}(J_{N_c,A,a}(X))$, substitute the identity of Proposition 8.3 (2) in (1). Then, we have the identity (8.44).

Next, we assume $2 \mid d$. The identities (8.46) or (8.46) holds by Proposition 8.3 (2) or (3), respectively. Then, by Proposition 8.3 (1), (8.45)and (8.46), we have the identity (8.47) and (8.48). □

### 8.1.2 The set of all vertices of a regular $n$-prism

Let $X'_n$ be the set of all vertices of a regular $n$-prism. We can regard the set $X'_n$ as a $D_n$-set, and $X'_n$ is $D_n$-isomorphic to $D_n/C_1$. Substituting $d = n$ in (8.5), (8.13) and (8.14), we have $n/n = 1$ and

$$\begin{aligned} \varphi_{C_{n/d'},t}(J_{N_c,A,a}(D_n/C_1)) &= (1 + |A'|t^{n/d'})^{2d'}, \\ \varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/C_1)) &= (1 + |A'|t^{2n/d'})^{d'}, \end{aligned}$$

$$\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/C_1)) \quad = \quad (1+|A'|t^{2n/d'})^{d'}.$$

Let $d$ be a divisor of $n$. If $2 \nmid d$, then we substitute these identities in (8.44). Then, we have

$$\begin{aligned}
&\mu_{C_{n/d},t}(J_{N_c,A,a}(D_n/C_1)) \\
=&\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}(1+|A'|t^{n/d'})^{2d'}-\frac{1}{2}(1+|A'|t^{2n/d'})^{d'}\Big).
\end{aligned} \tag{8.49}$$

If $2 \mid d$, them we substitute in (8.47). Then, we have

$$\begin{aligned}
& \mu_{C_{n/d},t}(J_{N_c,A,a}(D_n/C_1)) \\
= & \sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}(1+|A'|t^{n/d'})^{2d'}-\frac{1}{4}(1+|A'|t^{2n/d'})^{d'} \\
& -\frac{1}{4}(1+|A'|t^{2n/d'})^{d'}\Big) \\
= & \sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}(1+|A'|t^{n/d'})^{2d'}-\frac{1}{2}(1+|A'|t^{2n/d'})^{d'}\Big).
\end{aligned}$$

In any case, we have (8.49) for any divisor $d$ of $n$. Substituting $d = n$ in (8.49), we have

$$\begin{aligned}
&\mu_{C_1,t}(J_{N_c,A,a}(D_n/C_1)) \\
=&\sum_{d|n}\mu\Big(\frac{n}{d}\Big)\Big(\frac{1}{2n}(1+|A'|t^{n/d})^{2d}-\frac{1}{2}(1+|A'|t^{2n/d})^{d}\Big)
\end{aligned} \tag{8.50}$$

and substituting $t = 1$ in (8.50), we have

$$\begin{aligned}
& \mu_{C_1}(J_{N_c,A,a}(D_n/C_1)) \\
= & \frac{1}{2}\sum_{d|n}\mu\Big(\frac{n}{d}\Big)\Big(\frac{1}{n}|A|^{2d}-|A|^{d}\Big)=\frac{1}{2}(M(|A|^2,n)-nM(|A|,n))
\end{aligned}$$

by Lemma 7.18, which is the number of primitive colorings on the set of all vertices of the $n$-prism.

**Example 8.5.** We consider the case of $A = \{0, 1\}$, $a = 0$ and $n = 3$. Substitute these condition and $d = n$ in (8.50), we have

$$\begin{aligned}
& \mu_{C_1,t}(J_{N_c,A,a}(D_3/C_1)) \\
= & \frac{1}{2}\Big(-\frac{1}{3}(1+t^3)^2+(1+t^6)+\frac{1}{3}(1+t)^6-(1+t^2)^3\Big) \\
= & t+t^2+3t^3+t^4+t^5.
\end{aligned}$$

In particular, we have $\mu_{C_1}(J_{N_c,A,a}(D_3/C_1)) = 7$. All results of placing 2 colored on the vertices of a regular 3-prism are described in Figure 4. In Figure 4, the

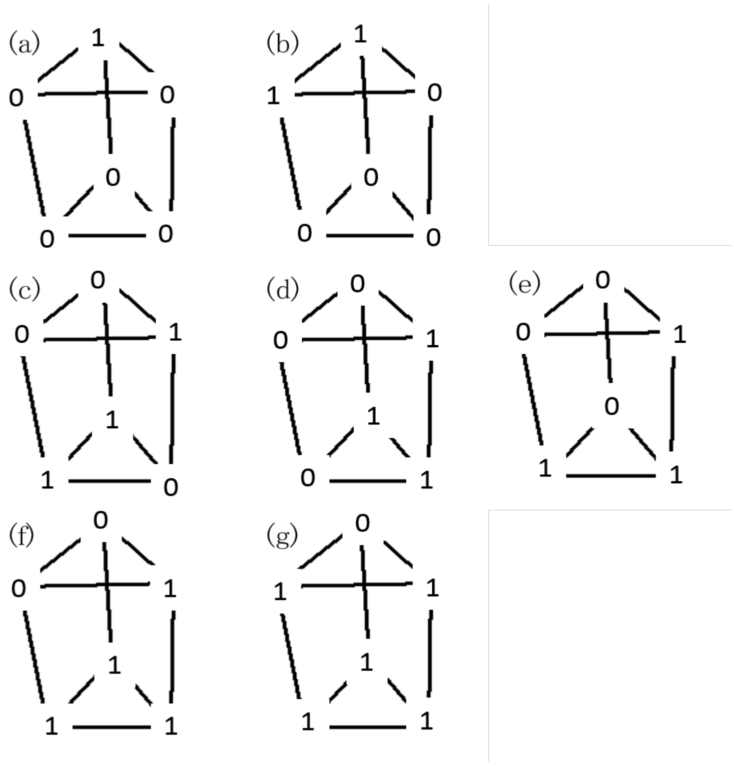


Figure 4: Primitive colorings on vertices of a regular 3-prism of $D_3$ with 2-colors $\{0, 1\}$.

image (a), (b), or (f) is the unique primitive coloring which has five, four or two 0-colored places, respectively, images (c), (d) and (e) are primitive colorings which have 3-times 0-colored places, and the image (g) is the unique primitive coloring which has one 0-colored place.

### 8.1.3 The set of all vertices of a regular $n$-gon with the dihedral group

Let $X_n$ be the set of all vertices of a $n$-gon. In §8.1, we considered cycle rotations. In addition, we consider cycle rotations and reverses of a regular $n$-gon. We assume that the set $X_n$ is a $D_n$-set which is $D_n$-isomorphic to $D_n/D_1$, and we calculate $\mu_{C_1,t}(J_{N_c,A,a}(D_n/D_1))$ and $\mu_{C_1}(J_{N_c,A,a}(D_n/D_1))$.

In the remainder of this section, we define a polynomial $M(k,n) \in \mathbb{C}[k]$ by (5.1). There are cases that we substitute $k = \sqrt{A}$ in $M(k,n)$.

Substituting $d = n$ in (8.6), we have

$$\varphi_{C_{n/d'},t}(J_{N_c,A,a}(D_n/D_1)) = (1 + |A'|t^{n/d'})^{d'}.$$

A divisor $d'$ of $n$ satisfies $2 \mid d'$ if and only if $2 \mid (n, d')$. Thus, if $2 \mid d'$, then we have

$$\begin{aligned}
\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D_1)) &= (1+|A'|t^{n/d'})^2(1+|A'|t^{2n/d'})^{(d'/2)-1},\\
\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/D_1)) &= (1+|A'|t^{2n/d'})^{d'/2},
\end{aligned}$$

since we substitute $d=n$ in (8.17) and (8.24).

If $2\nmid d'$, then we have

$$\begin{aligned}
\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D_1)) &= (1+|A'|t^{n/d'})(1+|A'|t^{2n/d'})^{(d'-1)/2},\\
\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/D_1)) &= (1+|A'|t^{n/d'})(1+|A'|t^{2n/d'})^{(d'-1)/2},
\end{aligned}$$

since we substitute $d=n$ in (8.20) and (8.27).

Substituting these five identities in identities of Corollary 8.4, we calculate $\mu_{C_{n/d}}(J_{N_c,A,a}(D_n/D_1))$ for any divisor $d$ of $n$.

(I) If $2\nmid d$, then we have

$$\begin{aligned}
&\mu_{C_{n/d},t}(J_{N_c,A,a}(D_n/D_1))\\
=&\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}(1+|A'|t^{n/d'})^{d'}\\
&-\frac{1}{2}(1+|A'|t^{n/d'})(1+|A'|t^{2n/d'})^{(d'-1)/2}\Big)
\end{aligned}\tag{8.51}$$

by (8.44). Substituting $t=1$, we have

$$\begin{aligned}
&\mu_{C_{n/d}}(J_{N_c,A,a}(D_n/D_1))\\
=&\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}|A|^{d'}-\frac{1}{2}|A|^{(d'+1)/2}\Big)\\
=&\frac{1}{2}\Big(M(|A|,d)-d\sqrt{|A|}M(\sqrt{|A|},d)\Big)
\end{aligned}\tag{8.52}$$

by Lemma 7.18.

(II) If $2\mid d$ and $4\nmid d$, then we have

$$\begin{aligned}
&\mu_{D_{n/d},t}(J_{N_c,A,a}(D_n/D_1))\\
=&\ \frac{1}{2}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D_1))\\
=&\ \frac{1}{2}\sum_{d'|d/2}\mu\Big(\frac{d}{2d'}\Big)\Big(-\varphi_{D_{n/d'},t}(J_{N_c,A,a}(D_n/D_1))\\
&+\varphi_{D_{n/2d'},t}(J_{N_c,A,a}(D_n/D_1))\Big)\\
=&\ \frac{1}{2}\sum_{d'|d/2}\mu\Big(\frac{d}{2d'}\Big)\Big(-(1+|A'|t^{n/d'})(1+|A'|t^{2n/d'})^{(d'-1)/2}
\end{aligned}$$

$$+(1+|A'|t^{n/2d'})^2(1+|A'|t^{n/d'})^{d'-1}\Big)$$

and

$$\begin{aligned}
&\mu_{D'_{n/d},t}(J_{N_c,A,a}(D_n/D_1))\\
=\ &\frac{1}{2}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/D_1))\\
=\ &\frac{1}{2}\sum_{d'|d/2}\mu\Big(\frac{d}{2d'}\Big)\Big((-\varphi_{D'_{n/d'},t}(J_{N_c,A,a}(D_n/D_1))\\
&+\varphi_{D'_{n/2d'},t}(J_{N_c,A,a}(D_n/D_1))\Big)\\
=\ &\frac{1}{2}\sum_{d'|d/2}\mu\Big(\frac{d}{2d'}\Big)\Big(-(1+|A'|t^{n/d'})(1+|A'|t^{2n/d'})^{(d'-1)/2}\\
&+(1+|A'|t^{n/d'})^{d'}\Big)
\end{aligned}$$

by (8.45). Thus, we have

$$\begin{aligned}
&\mu_{C_{n/d},t}(J_{N_c,A,a}(D_n/D_1))\\
=&\frac{1}{2d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)(1+|A'|t^{n/d'})^{d'}\\
&-\frac{1}{4}\sum_{d'|d/2}\mu\Big(\frac{d}{2d'}\Big)\Big(-2(1+|A'|t^{n/d'})(1+|A'|t^{2n/d'})^{(d'-1)/2}\\
&+(1+|A|t^{n/2d'})^2(1+|A'|t^{n/d'})^{d'-1}+(1+|A'|t^{n/d'})^{d'}\Big)
\end{aligned}\tag{8.53}$$

by (8.47). Substituting $t=1$, we have

$$\begin{aligned}
&\mu_{C_{n/d}}(J_{N_c,A,a}(D_n/D_1))\\
=&\frac{1}{2d}\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)|A|^{d'}\\
&-\frac{1}{4}\sum_{d'|d/2}\mu\Big(\frac{d}{2d'}\Big)(-2|A|^{(d'+1)/2}+|A|^{d'+1}+|A|^{d'})\\
=&\frac{1}{2}M(|A|,d)-\frac{d}{8}(|A|+1)M(|A|,\frac{d}{2})+\frac{d}{4}\sqrt{|A|}M(\sqrt{|A|},\frac{d}{2})
\end{aligned}\tag{8.54}$$

by Lemma 7.18.

(III) If $4\mid d$, then all divisor $d'$ of $d$ such that $\mu(d/d')\neq 0$ holds satisfy $2\mid d'$.

Then, we have

$$\begin{aligned}
&\mu_{C_{n/d},t}(J_{N_c,A,a}(D_n/D_1))\\
=&\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}(1+|A'|t^{n/d'})^{d'}\\
&-\frac{1}{4}(1+|A'|t^{n/d'})^2(1+|A'|t^{2n/d'})^{(d'/2)-1}\\
&-\frac{1}{4}(1+|A'|t^{2n/d'})^{d'/2}\Big)
\end{aligned} \tag{8.55}$$

by (8.48). Substituting $t=1$, we have

$$\begin{aligned}
&\mu_{C_{n/d}}(J_{N_c,A,a}(D_n/D_1))\\
=&\sum_{d'|d}\mu\Big(\frac{d}{d'}\Big)\Big(\frac{1}{2d}|A|^{d'}-\frac{1}{4}|A|^{(d'/2)+1}-\frac{1}{4}|A|^{d'/2}\Big)\\
=&\frac{1}{2}M(|A|,d)-\frac{d}{4}(|A|+1)M(\sqrt{|A|},d)
\end{aligned} \tag{8.56}$$

by Lemma 7.18.

Substituting $d=n$ in (8.52), (8.54) and (8.56), we have the following results.

(I) If $2\nmid n$, we have

$$\mu_{C_1}(J_{N_c,A,a}(D_n/D_1))=\frac{1}{2}\Big(M(|A|,n)-n\sqrt{|A|}M(\sqrt{|A|},n)\Big).$$

(II) If $2\mid n$ and $4\nmid n$, we have

$$\begin{aligned}
&\mu_{C_1}(J_{N_c,A,a}(D_n/D_1))\\
=&\frac{1}{2}M(|A|,n)-\frac{n}{8}(|A|+1)M(|A|,\frac{n}{2})+\frac{n}{4}\sqrt{|A|}M(\sqrt{|A|},\frac{n}{2}).
\end{aligned}$$

(III) If $4\mid n$, we have

$$\mu_{C_1}(J_{N_c,A,a}(D_n/D_1))=\frac{1}{2}M(|A|,n)-\frac{n}{4}(|A|+1)M(\sqrt{|A|},n).$$

The number $\mu_{C_1}(J_{N_c,A,a}(D_n/D_1))$ is the number of all primitive colorings on a regular $n$-gon of $D_n$ with $A$.

**Example 8.6.** We consider when $n=4$, $A=\{0,1,2\}$ and $a=0$. Substituting these condition and $d=n$ in (8.55), we have

$$\begin{aligned}
&\mu_{C_1,t}(J_{N_c,A,a}(D_4/D_1))\\
=&-\frac{1}{8}(1+2t^2)^2+\frac{1}{4}(1+2t^2)^2+\frac{1}{4}(1+2t^4)
\end{aligned}$$

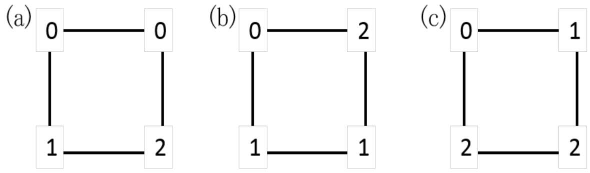


Figure 5: Primitive colorings on vertices of a regular 4-gon, of $D_4$, with 3-colors $\{0, 1, 2\}$.

$$\begin{aligned} & +\frac{1}{8}(1+2t)^4 - \frac{1}{4}(1+2t)^2(1+2t^2) - \frac{1}{4}(1+2t^2)^2 \\ = \ & t^2 + 2t^3. \end{aligned}$$

In particular, we have $\mu_{C_1}(J_{N_c,A,a}(D_4/D_1)) = 3$. All results of placing 3 colored on the vertices of a regular 4-gon are described in Figure 5. In Figure 5, the image (a) is the primitive coloring which has two 0-colored places, and images (b) and (c) are primitive colorings which have one 0-colored place.

**Example 8.7.** We consider when $n = 5$, $A = \{0, 1, 2\}$ and $a = 0$. Substituting these condition and $d = n$ in (8.51), we have

$$\begin{aligned} & \mu_{C_1,t}(J_{N_c,A,a}(D_5/D_1)) \\ = \ & -\frac{1}{10}(1+2t)^5 + \frac{1}{2}(1+2t^5) + \frac{1}{10}(1+2t)^5 - \frac{1}{2}(1+2t)(1+2t^2)^2 \\ = \ & 2t^2 + 4t^3 + 6t^4. \end{aligned}$$

In particular, we have $\mu_{C_1}(J_{N_c,A,a}(D_5/D_1)) = 12$. All results of placing 3 colored on the vertices of a regular 5-gon are described in Figure 6.

In Figure 6, images (a) and (b) are primitive colorings which have three 0-colored places, and images (c), (d), (e) and (f) are primitive colorings which have two 0-colored places, and (g), (h), (i), (j), (k) and (l) are primitive colorings which have one 0-colored place.

**Example 8.8.** We consider when $n = 6$, $A = \{0, 1\}$ and $a = 0$. Substituting these condition and $d = n$ in (8.53), we have

$$\begin{aligned} & \mu_{C_1,t}(J_{N_c,A,a}(D_6/D_1)) \\ = \ & \frac{1}{12}\Big((1+t)^6 - (1+t^2)^3 - (1+t^3)^2 + (1+t^6)\Big) \\ & +\frac{1}{4}\Big(-2(1+t^2)(1+t^4) + (1+t)^2(1+t^2)^2 + (1+t^2)^3 \\ & +2(1+t^6) - (1+t^3)^2 - (1+t^6)\Big) = t^3. \end{aligned}$$

In particular, we have $\mu_{C_1}(J_{N_c,A,a}(D_6/D_1)) = 1$. Figure 7 is the unique result which has three 0-colored places.

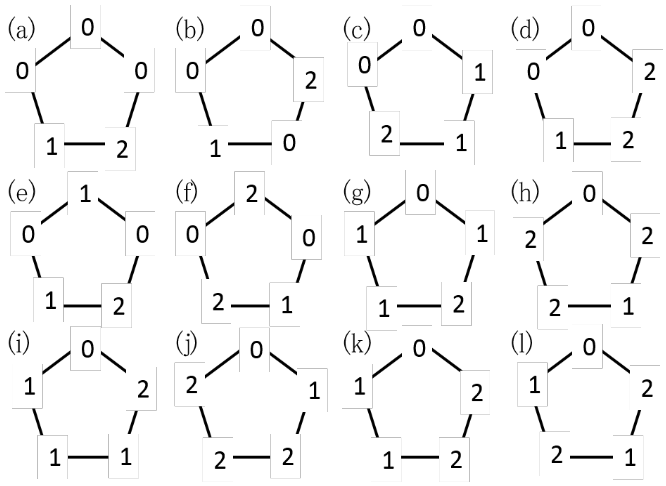


Figure 6: Primitive colorings on vertices of a regular 5-gon, of $D_5$, with 3-colors $\{0, 1, 2\}$.

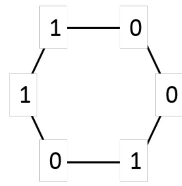

Figure 7: Primitive colorings on vertices of a regular 6-gon, of $D_6$, with 2-colors $\{0, 1\}$.

## 8.2 Remark: the sum of two distinct irreducible characters of $D_5$

In Remark 7.13, we told that there exists an integer-valued character which is not generated by permutation characters. We show it by using the results in §8.1 and ones in [CR].

Note that every $f \in CF(D_5)$ are determined by $f(1), f(a)$ and $f(b)$.

By [CR, p.339], the character $\chi$, which is the sum of two irreducible characters $T_1^G$ and $T_2^G$, satisfies

$$\chi(1) = 4, \quad \chi(a) = -1, \quad \chi(b) = 0.$$

Next, we calculate all permutation characters of $D_5$. By Proposition 8.1, all conjugacy subgroups of $D_5$ is $C_1, C_5, D_1$ and $D_5$. Let $\chi_{D_5/K}$ be the permutation character associated with the $D_5$-set $D_5/K$ for any $K \in \Phi(D_5)$ to $\mathbb{C}$ via $\chi$. Note that $\chi_{D_5/K}(a) = \varphi_{C_5}(D_5/K)$ and $\chi_{D_5/K}(b) = \varphi_{D_1}(D_5/K)$. Then, we have

$$\chi_{D_5/C_1}(1) = 10, \quad \chi_{D_5/C_1}(a) = 0, \quad \chi_{D_5/C_1}(b) = 0,$$

$$\chi_{D_5/C_5}(1) = 2, \quad \chi_{D_5/C_5}(a) = 2, \quad \chi_{D_5/C_5}(b) = 0,$$
$$\chi_{D_5/D_1}(1) = 5, \quad \chi_{D_5/D_1}(a) = 0, \quad \chi_{D_5/D_1}(b) = 2,$$
$$\chi_{D_5/D_5}(1) = 1, \quad \chi_{D_5/D_5}(a) = 1, \quad \chi_{D_5/D_5}(b) = 1.$$

Assume that there exist integers $a, b, c$ and $d$ such that $\chi = p\chi_{D_5/C_1} + q\chi_{D_5/C_5} + r\chi_{D_5/D_1} + s\chi_{D_5/D_5}$ holds. By mapping $a$ and $b \in D_5$, we have $1 = 2q + s$ and $0 = 2r + s$, however, there does not exists an integer $s$ satisfying these two identities, and hence, it is a contradiction. So, there exists an integer-valued character which is not generated by permutation characters.

# A Appendix

In this section, we state some notions and propositions which are used in this paper.

## A.1 Vector spaces

For any $A \in M(k, \mathbb{C})$, we denote the trace of $A$ by $\mathrm{Tr}(A)$.

For any $A, B \in M(k, \mathbb{C})$, we denote the Kronecker product of $A$ and $B$ by $A \otimes B$.

For any complex vector space $V$, we denote the set of all bijective linear maps on $V$ by $GL(V)$.

Let $F : V \to V$ be a linear map of a vector space $V$. We define $\mathrm{Tr}(F)$ by the trace of the matrix of $F$ with respect to a basis $\{x_1, \ldots, x_n\}$. Note that $\mathrm{Tr}(F)$ does not depend on the basis $\{x_1, \ldots, x_n\}$.

For any vector spaces $V$ and $W$, we denote the direct sum of $V$ and $W$ by $V \oplus W$, and the tensor product of $V$ and $W$ by $V \otimes W$.

Let $V$ and $W$ be vector spaces and let $F : V \to W$ be a linear map. For any integer $n \geq 1$, we denote $n$-times tensor product of $V$ by $V^{\otimes n}$, and we define a linear map $F^{\otimes n} : V^{\otimes n} \to W^{\otimes}$ by $F^{\otimes n}(v_1 \otimes \cdots \otimes v_n) := F(v_1) \otimes \cdots \otimes F(v_n)$ for any $v_1, \ldots, v_n \in V$.

Let $V_1, V_2, W_1$ and $W_2$ be vector spaces and let $F_1 : V_1 \to W_1$ and $F_2 : V_2 \to W_2$ be linear maps. Then, we define a linear map $F_1 \otimes F_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ by $F_1 \otimes F_2(v_1 \otimes v_2) := F_1(v_1) \otimes F_2(v_2)$ for any $v_1 \in V_1$ and $v_2 \in V_2$.

For the Kronecker product of two matrices, the following lemma holds from Lemma (v) in [Knu, p.83].

**Lemma A.1.** *For any $A \in M(k_1, \mathbb{C})$ and $B \in M(k_2, \mathbb{C})$, we have* $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)$.

## A.2 Finite groups

Let $G$ be a finite group.

**Lemma A.2.** *Two elements $g_1, g_2 \in G$ generate the same cyclic group $\langle g_1 \rangle = \langle g_2 \rangle$ if and only if there exists an integer $k \geq 1$ such that $g_1^k = g_2$ and $(k, O(g_1)) = 1$ hold.*

**Lemma A.3.** *Let $g_1$ and $g_2$ be elements of $G$ with $\langle g_1 \rangle \cap \langle g_2 \rangle = \{1\}$ and $g_1 g_2 = g_2 g_1$. Then, we have $O(g_1 g_2) = [O(g_1), O(g_2)]$.*

*Proof.* First, we show $(g_1 g_2)^{[O(g_1), O(g_2)]} = g_1^{[O(g_1), O(g_2)]} g_2^{[O(g_1), O(g_2)]} = 1$. In addition, if an arbitrary integer $k$ satisfies $(g_1 g_2)^k = 1$, then $O(g_1) \mid k$ and $O(g_2) \mid k$ hold by $\langle g_1 \rangle \cap \langle g_2 \rangle = \{1\}$. Hence, we have $[O(g_1), O(g_2)] \mid k$. This means $O(g_1 g_2) \mid [O(g_1), O(g_2)]$.

Conversely, $(g_1 g_2)^{O(g_1 g_2)} = 1$ implies that $g_1^{O(g_1 g_2)} = g_2^{-O(g_1 g_2)} \in \langle g_1 \rangle \cap \langle g_2 \rangle = \{1\}$ which means $O(g_1) \mid O(g_1 g_2)$ and $O(g_2) \mid O(g_1 g_2)$. Hence we have $[O(g_1), O(g_2)] \mid O(g_1 g_2)$. Hence, $O(g_1 g_2) = [O(g_1), O(g_2)]$ holds. $\square$

## A.3 For symmetric groups

In this paper, we denote the symmetric group on $n$-letters by $S_n$. For more detail, see [Knu, p.124].

An element $\sigma$ of $S_n$ is a cycle of length $q$, there exists a subset $\{i_1, \dots, i_q\} \subset \{1, \dots, n\}$ such that $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \dots, \sigma(i_q) = i_1$ and $\sigma(m) = m$ hold where $m$ is not the form $i_j$. In the case, we write $\sigma = (i_1\ i_2 \cdots i_n)$.

An element $\sigma \in S_n$ can be written as a product of disjoint cycles: $\sigma = (i_1\ i_2 \cdots\ i_q)(j_1\ j_2 \cdots\ j_r) \cdots (k_1\ k_2 \cdots\ k_s)$ where all integers which have the form $i_p, j_q, \dots, k_s$ are distinct. The cycle structure of $\sigma$ is a partition $(\lambda_1, \dots, \lambda_m)$ of $n$ where $\lambda_i$ is the size of the cycles in the decomposition.

By Lemma A.3, we have the following lemma.

**Lemma A.4.** *An element of $\sigma \in S_n$ which has the cycle structure $(\lambda_1, \dots, \lambda_m)$ satisfies that $O(\sigma)$ is the least common multiple of $\lambda_1, \dots, \lambda_m$*

For the cycle structure of $\sigma \in S_n$, it is known the following lemma.

**Lemma A.5.** [Knu, p.125] *Two elements of $S_n$ are conjugate if and only if they have the same cycle structure.*

## A.4 For representations of finite groups

Let $G$ be a finite group.

We define a group homomorphism $\rho : G \to GL(V)$ by a representation of $G$, where the vector space $V$ is a complex vector space whose dimension is finite. We define the degree of $\rho$ by $\dim_{\mathbb{C}}(V)$.

The character of $\rho$, which is denoted by $\chi$, is the map $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\rho(g))$ for any $g \in G$.

Next, we define a $G$-set. A $G$-set $X$ is a set equipped with a map

$$\iota : G \times X \to X, \quad \iota(g, x) = gx$$

which satisfies

$$g_1(g_2 x) = (g_1 g_2)x, \quad 1x = x$$

for any $g_1, g_2 \in G$ and $x \in X$ (Note that we have mentioned it in §6.1).

Put $X = \{x_1, \dots, x_n\}$. We define the permutation representation associated with $X$ by the representation $\rho_X : G \to GL(V)$ where $V$ is the $n$-dimensional vector space which is formally generated by $\{x_1, \dots, x_n\}$, and $\rho_X(g)(x_i) := gx_i$ holds for any $g \in G$. Finally, we define the permutation character associated with $X$ by the character of $\rho_X$.

For any $g \in G$, the matrix of $\rho_X(g)$ with respect to the basis $\{x_1, \dots, x_n\}$ is written by $(x_{i,j})_{1 \le i,j \le n}$, satisfies

$$x_{i,j} = \begin{cases} 1 & \text{if } x_j = gx_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\chi$ be the permutation character associated with $X$. Then, $\chi(g)$ is the number of $x \in X$ such that $gx = x$ holds.

## A.5 For commutative rings

We consider the following two properties on a commutative ring $R$.

**Definition A.6.** A commutative ring $R$ is $\mathbb{Z}$-torsion free if the map $r \mapsto nr$ is injective for any integer $n \geq 1$. A commutative ring $R$ is a $\mathbb{Q}$-algebra if $R$ contains a subring which is isomorphic to $\mathbb{Q}$ as rings.

With the Möbius function, the following proposition holds.

**Proposition A.7** (Möbius inversion formula)**.** *Let $(a_1, a_2, a_3, \ldots)$ and $(b_1, b_2, b_3, \ldots)$ be infinite vectors of a commutative ring $R$. Two elements $a$ and $b$ satisfy*

$$b_n = \sum_{d|n} a_d$$

*for any integer $n \geq 1$ if and only if*

$$a_n = \sum_{d|n} \mu\Big(\frac{n}{d}\Big) b_d$$

*holds for any integer $n \geq 1$.*

We state the Gauss Lemma.

**Lemma A.8** ([Ste, Lemma 3.17])**.** *Let $f$ be a polynomial over $\mathbb{Z}$ that is irreducible over $\mathbb{Z}$. Then $f$, considered as a polynomial over $\mathbb{Q}$, is also irreducible over $\mathbb{Q}$.*

## A.6 For semirings and their ring completion

In this section, we tell the notion of the ring completion. For more detail, see [Hus, §9.3].

We define a semiring by a triple $(S, \alpha, \mu)$ where $S$ is a set, the map $\alpha : S \times S \to S$ is the addition function usually denoted by $\alpha(a, b) = a + b$, and the map $\mu : S \times S \to S$ is the multiplication function usually denoted by $\mu(a, b) = ab$, and satisfies all axioms of a ring except the existence of negative or additive inverse. For simplicity, we denote a semiring $(S, \alpha, \mu)$ by $S$.

We define a semiring homomorphism from a semiring $S$ to a semiring $S'$ by a map $f : S \to S'$ such that $f(a + b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$ and $f(0) = 0$ hold.

**Proposition A.9.** *For any semiring $S$, there exists a pair $(S^*, \theta)$, where $S^*$ is a ring and $\theta : S \to S^*$ is a semiring homomorphism such that if for any ring $R$ and a semiring homomorphism $f : S \to R$, there exists a ring homomorphism $g : S^* \to R$ such that $g \circ \theta = f$ holds.*

In this paper, we call $(S^*, \theta)$ a ring completion, and we construct the ring completion $(S^*, \theta)$ of a semiring $S$ as follows: We define an equivalence relation on $S \times S$ by that $(a_1, b_1)$ and $(a_2, b_2)$ are equivalent if there exists $c \in S$ such that

$a_1+b_2+c=a_2+b_1+c$ holds. We denote the equivalence class of $(a,b)$ by $\langle a,b\rangle$, and let $S^*$ be the set of all equivalence classes $\langle a,b\rangle$. The set $S^*$ has the following addition and multiplication which are defined by $\langle a,b\rangle+\langle c,d\rangle := \langle a+c, b+d\rangle$ and $\langle a,b\rangle\langle c,d\rangle := \langle ac+bd, ad+bc\rangle$. Next, we define a map $\theta : S \to S^*$ by $\theta(s)=\langle s,0\rangle$ for any $s \in S$.

**Proposition A.10.** *Let $\{x_1,\dots,x_n\}$ be a subset of a semiring $S$. We assume that every elements $\alpha$ of $S$ can be write uniquely*

$$\alpha = m_1x_1+\cdots+m_nx_n$$

*for some integers $m_1,\dots,m_n \geq 0$. Let $(S^*,\theta)$ be the ring completion of $S$. Then $S^*$ has a $\mathbb{Z}$-basis $\{\theta(x_1),\dots,\theta(x_n)\}$.*

*Proof.* First, we show that the semiring homomorphism $\theta$ is injective. Let $s_1, s_2 \in S$ satisfying $\theta(s_1)=\theta(s_2)$. Then there exists $c \in S$ such that $s_1+c = s_2+c$ holds. By the assumption, we have $s_1=s_2$, that is, the map $\theta$ is injective. From this, elements $\{\theta(x_1),\dots,\theta(x_n)\}$ are linearly independent. For any $\langle a,b\rangle \in S^*$, $\langle a,b\rangle = \theta(a)-\theta(b)$ holds. Hence, $S^*$ has a $\mathbb{Z}$-basis $\{\theta(x_1),\dots,\theta(x_n)\}$. □

For simplicity, we write the ring completion of a semiring $S$ as $S^*$, and we write $\theta(a)$ as $a \in S^*$ for any $a \in S$.

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