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Contents

Chapter 1

Introduction

The Siegel upper half space appears in the theory of algebraic functions as a period domain. And the consequential analytic theories on this domain, including modular forms and harmonic analysis, have been studied until today. In such analysis, the boundary structure of the domain plays an important role. In this thesis, we review the boundary structure of the Siegel upper half space. As its application, we explain a result on a statistical manifold stated below.

Let $\mathrm{Sym}_p^+(\mathbb{R})$ be the set of $p \times p$ positive-definite symmetric matrices. We consider a manifold $N_p = \left\{ (\Sigma, \mu) \in \mathrm{Sym}_p^+(\mathbb{R}) \times \mathbb{R}^p \right\}$ equipped with the metric

$$ds^2 = ({}^t d\mu)\Sigma^{-1}(d\mu) + \frac{1}{2}\mathrm{tr}((\Sigma^{-1}d\Sigma)^2). \tag{1.0.1}$$

This Riemannian manifold (N_p, ds^2) is known as the statistical manifold defined by p-dimensional normal distributions (cf. Shima [Shi01]). The positive affine motion group $\mathrm{GL}_p^+(\mathbb{R}) \ltimes \mathbb{R}^p$ transitively acts on N_p as

$$(A, b) : (\Sigma, \mu) \mapsto (A\Sigma {}^t A, A\mu + b) \quad (A \in \mathrm{GL}_p^+(\mathbb{R}),\ b \in \mathbb{R}^p),$$

and these actions are isometric with respect to ds^2. By Calvo and Oller [CO90], the manifold N_p has been realized as a Riemannian submanifold in the symmetric space $\mathrm{Sym}_{p+1}^+(\mathbb{R})$ as

$$\left\{ x = \begin{pmatrix} \Sigma + \mu {}^t\mu & \mu \\ {}^t\mu & 1 \end{pmatrix} \in \mathrm{Sym}_{p+1}^+(\mathbb{R}) \right\}. \tag{1.0.2}$$

Indeed, the metric ds^2 coincides with the restriction of the invariant metric $\frac{1}{2}\mathrm{tr}((x^{-1}dx)^2)$.

By the isometric and transitive group action, any geodesics is mapped to a geodesic through $(I_p, 0) \in N_p$ in the direction $(B, x) \in \mathrm{Sym}_p(\mathbb{R}) \times \mathbb{R}^p$. An explicit construction of geodesics on N_p was firstly obtained by Eriksen [Eri87] as follows.

Theorem 1.0.1 ([Eri87])**.** *For any $B \in \mathrm{Sym}_p(\mathbb{R})$ and $x \in \mathbb{R}^p$, define a square matrix $A \in M_{2p+1}(\mathbb{R})$ and a matrix exponential $\Lambda(t)$ $(t \in \mathbb{R})$ by*

$$A := \begin{pmatrix} B & x & 0 \\ {}^t x & 0 & -{}^t x \\ 0 & -x & -B \end{pmatrix}, \quad \Lambda(t) = \begin{pmatrix} \Delta & \delta & \Phi \\ {}^t\delta & \epsilon & {}^t\gamma \\ {}^t\Phi & \gamma & \Gamma \end{pmatrix} := \exp(tA). \tag{1.0.3}$$

Then, the curve $(\Sigma(t), \mu(t)) := (\Delta^{-1}, \Delta^{-1}\delta)$ is the geodesic on N_p satsfying the initial condition

$$(\Sigma(0), \mu(0)) = (I_n, 0), \quad (\dot\Sigma(0), \dot\mu(0)) = (B, x).$$

This theorem is well similar to the fact that the geodesics on a symmetric space are given by the exponential map (cf. [Hel62]). Indeed, the curve $\Lambda(t) = e^{tA}$ is a geodesic on the symmetric space $\mathrm{Sym}^+_{2p+1}(\mathbb{R})$. Eriksen [Eri87] suggests the relation with the Lie algebra $\mathfrak{so}(p+1,p)$, which may characterizes the matrix A as its element.

However, the relation between N_p and symmetric spaces remains unclear from the proof; Eriksen [Eri87] just showed that the curve $(\Sigma(t), \mu(t))$ satisfies the geodesic equation

$$\begin{cases} \ddot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\mu} = 0, \\ \ddot{\Sigma} + \dot{\mu}{}^t\dot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\Sigma} = 0 \end{cases} \tag{1.0.4}$$

on N_p by a direct computation from the equation $\dot{\Lambda}(t) = A\Lambda(t)$. The problem is that if we regard Theorem 1.0.1 as a statement on a Riemannian manifold N_p, the meaning of the construction of $(\Sigma(t), \mu(t))$ is not clear in the sense of manifolds. That is, this construction is not defined in a way invariant under the variable change. Then, we need to consider that there is a geometrical background that gives a proper meaning for this.

The purpose of this thesis is to give a geometrical explanation for Theorem 1.0.1 to answer the above question. As a natural background, the boundary structure of the Siegel domains is used. For instance, we will see that the extraction of the submatrix $\Lambda(t) \mapsto (\Delta, \delta)$ is regarded as the projection from the Siegel upper half space onto its boundary component, and the manifold N_p is realized as the 'imaginary part' of a fiber of this projection. Moreover, we restate Theorem 1.0.1 in terms of the property of this projection, which is called the Riemannian submersion. In this sense, we will conclude that Theorem 1.0.1 can be understood in the process of investigating the arrangement of a subdomain in the Siegel upper half space.

Let us see the content of the following chapters. In Chapter 2, we review the basic facts of the geometry of the Siegel upper half space, and see the embedding into a compact space. By the embedding, we see the definition of the boundary given by Pyatetskii-Shapiro [PS69], and review its basic structure. In Chapter 3, we introduce a symmetric subdomain of the Siegel upper half space, which is biholomorphic to the classical domain of type I. In Chapter 4, we describe Theorem 1.0.1 in terms of the boundary structure and the Riemannian submersion.

Remark 1.0.1. Calvo and Oller [CO91] gave an explicit formula for $(\Delta(t), \delta(t))$ with elementary functions. As a recent report, first author et al. [ITW11] emphasized to see Theorem 1.0.1 from the group theoretical point of view. □

Notation

Throughout this thesis, we denote the ring of rational integers by $\mathbb{Z}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$. Let $K = \mathbb{R}$ or $\mathbb{C}$. $K^{(n,m)}$ denotes the set of $n \times m$ matrices with each element in K. We simply write $K^{(n)} = K^{(n,n)}$ and $K^n = K^{(n,1)}$. For a matrix a, we write $a = a^{(n,m)}$ or $a^{(n)}$ to imply that $a \in K^{(n,m)}$ or $K^{(n)}$. For $a \in K^{(n,m)}$, we denote by ${}^ta$ or a' the transposed matrix, by $\overline{a}$ the complex conjugate matrix, and by a^* the conjugate-transposed matrix ${}^t\overline{a}$. For $a^{(n)}$, $a > 0$ implies that a is positive definite. $1_n \in K^{(n)}$ is the identity matrix and $0_n \in \mathbb{C}^{(n)}$ is the zero matrix. We write $\mathrm{GL}_n(K) = \mathrm{GL}(n, K)$ for the general linear group over K. For positive integers $n, m \in \mathbb{Z}$ and $H = H^{(n)}, a = a^{(n,m)}$, we write

$$H[a] := {}^taHa, \quad H\{a\} := a^*Ha$$

by the matrix multiplication.

For any domain in $\mathbb{C}^n$ or a general complex manifold, we consider the induced complex structure on it. Let D, D' are such domains. We say that a map $\Phi : D \to D'$ is biholomorphic if Φ is a bijective holomorphic map with holomorphic inverse. The group of all biholomorphic maps $D \to D$ is denoted by $\mathrm{Aut}(D)$.

Chapter 2

Siegel upper half space

Section 2.1 is a review of the geometry of the Siegel upper half space referring to Siegel [Sie64]. In section 2.2 – 2.4, 2.6, we will review a definition and the basic structure of the boundary referring to [PS69]. There, we will explicitly describe the projection onto the boundary. In section 2.5, we will add our consideration on the sequences of the boundary components.

2.1 Siegel's symplectic geometry

Let $n \in \mathbb{Z}$, $n > 0$. The Siegel upper half space $\mathfrak{H}_n$ of degree n is a complex domain defined as

$$\mathfrak{H}_n = \left\{ z \in \mathbb{C}^{(n)};\ {}^t z = z,\ \operatorname{Im} z > 0 \right\},$$

where $\operatorname{Im} z = y$ is the imaginary part of $z = x + iy$.

Let $G = \mathrm{Sp}(n, \mathbb{R})$ be the symplectic group of degree n defined as

$$\mathrm{Sp}(n, \mathbb{R}) = \{ g \in \mathrm{GL}_{2n}(\mathbb{R});\ J[g] = J \}, \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

We often decompose each element $g \in G$ into $n \times n$ blocks,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}^{(n)}.$$

Using this representation, the group G acts on $\mathfrak{H}_n$ by

$$g : z \mapsto g \cdot z = (az + b)(cz + d)^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ z \in \mathfrak{H}_n. \tag{2.1.1}$$

The action of G on $\mathfrak{H}_n$ is transitive; for any $z = x + iy \in \mathfrak{H}_n$, we can take a matrix $a = a^{(n)}$ such that $y = a^2$, ${}^t a = a, a > 0$, so that

$$z = \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} a & 0_n \\ 0_n & a^{-1} \end{pmatrix} \cdot i1_n.$$

The isotropy subgroup of G at $i1_n \in \mathfrak{H}_n$ is

$$K := \left\{ g \in G;\ {}^t g g = 1_{2n} \right\} = \left\{ g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix};\ {}^t a a + {}^t b b = 1_n \right\}.$$

This group is isomorphic to the unitary group $\mathrm{U}(n)$. Then we see the following isomorphism

$$\mathfrak{H}_n \cong \mathrm{Sp}(n, \mathbb{R}) / \mathrm{U}(n).$$

There is another realization of $\mathfrak{H}_n$ as a bounded domain. Let $\mathfrak{R}_n$ be the unit disk of degree n defined as

$$\mathfrak{R}_n = \left\{ w \in \mathbb{C}^{(n)};\ {}^t w = w,\ 1_n - w^* w > 0 \right\}.$$

It is known that $\mathfrak{H}_n$ is biholomorphic to $\mathfrak{R}_n$ by the Cayley transformation $\mathcal{C}$ defined as

$$\mathcal{C} : \mathfrak{H}_n \ni z \mapsto w = (z - i1_n)(iz - 1_n)^{-1} \in \mathfrak{R}_n. \tag{2.1.2}$$

Due to the boundedness of $\mathfrak{R}_n$, it is known that every biholomorphic map $\mathfrak{H}_n \to \mathfrak{H}_n$ is realized as an action of G (see [Sie64], or Faraut and Korányi [FK94] for general tube domains). The only elements of G that trivially act on $\mathfrak{H}_n$ are $\pm 1_{2n}$. To summarize, we have the following exact sequence

$$e \quad \to \quad \{\pm 1_{2n}\} \quad \to \quad G \quad \to \quad \mathrm{Aut}(\mathfrak{H}_n) \quad \to \quad e.$$

Cross ratio of $\mathfrak{H}_n$

Next, we consider the action of G on the product $\mathfrak{H}_n \times \mathfrak{H}_n$ by $g : (z_1, z_2) \mapsto g \cdot (z_1, z_2) := (g \cdot z_1, g \cdot z_2)$. Define the subset $\mathfrak{T}_n$ of $\mathfrak{H}_n$ by

$$\mathfrak{T}_n = \{ iT \in \mathfrak{H}_n;\ T = \mathrm{diag}[t_1, \ldots, t_n], 1 \le t_1 \le \cdots \le t_n \}.$$

Here $\mathrm{diag}[t_1, \ldots, t_n]$ is the diagonal matrix whose diagonal elements are $t_1, \ldots, t_n \in \mathbb{R}$ in this order.

Proposition 2.1.1. *The product $\mathfrak{H}_n \times \mathfrak{H}_n$ is decomposed into the G-orbits as*

$$\mathfrak{H}_n \times \mathfrak{H}_n = \bigsqcup_{iT \in \mathfrak{T}_n} G \cdot (iT, i1_n).$$

It follows from the above proposition that there is the following bijection

$$\mathcal{T} : G \backslash (\mathfrak{H}_n \times \mathfrak{H}_n) \xrightarrow{1:1} \mathfrak{T}_n. \tag{2.1.3}$$

The map $\mathcal{T}$ is constructed as follows. Define two maps $\{\cdot, \cdot\}_{H_1} : \mathfrak{H}_n \times \mathfrak{H}_n \to \mathbb{C}^{(n)}$ and $[\cdot, \cdot]_{J_1} : \mathfrak{H}_n \times \mathfrak{H}_n \to \mathbb{C}^{(n)}$ by

$$\{z_1, z_2\}_{H_1} = \begin{pmatrix} z_1 \\ 1_n \end{pmatrix}^* H_1 \begin{pmatrix} z_2 \\ 1_n \end{pmatrix} = i(\overline{z_1} - z_2), \quad H_1 = \begin{pmatrix} 0 & i1_n \\ -i1_n & 0 \end{pmatrix}, \tag{2.1.4}$$

$$[z_1, z_2]_{J_1} = \begin{pmatrix} z_1 \\ 1_n \end{pmatrix}' J_1 \begin{pmatrix} z_2 \\ 1_n \end{pmatrix} = z_1 - z_2, \quad J_1 = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \tag{2.1.5}$$

We observe that each $\{,\}_{H_1}, [,]_{J_1}$ satisfies a transformation law; Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, then

$$\begin{aligned} \{gz_1, gz_2\}_{H_1} &= (cz_1 + d)^{*-1} \{z_1, z_2\}_{H_1} (cz_2 + d)^{-1}, \\ [gz_1, gz_2]_{J_1} &= (cz_1 + d)'^{-1} [z_1, z_2]_{J_1} (cz_2 + d)^{-1}. \end{aligned} \tag{2.1.6}$$

We define the value of $\mathcal{T}$. Let z_1, z_2 be arbitrary two points in $\mathfrak{H}_n$. Define a matrix $R(z_1, z_2) \in \mathbb{C}^{(n)}$ by the quotient

$$R(z_1, z_2) = \{z_1, z_2\}_{H_1}^{-1} [z_1, z_2]_{J_1} \{z_1, z_2\}_{H_1}^{*-1} [z_1, z_2]_{J_1}^*.$$

R is called the cross ratio of $\mathfrak{H}_n$. Let $0 \le r_1 \le \cdots \le r_n < 1$ be the eigenvalues of $R(z_1, z_2)$. Then, we define

$$\mathcal{T}(z_1, z_2) = i\,\mathrm{diag}[t_1, \ldots, t_n], \quad t_k = \frac{1 + \sqrt{r_k}}{1 - \sqrt{r_k}}.$$

We notice that this definition does not depend on the choice of the representative of $G \cdot (z_1, z_2)$ due to the transformation law (2.1.6). From the point of view of invariants, the injectivity of (2.1.3) corresponds to the completeness of the invariant $\mathcal{T}$.

For a given $z = x + iy \in \mathfrak{H}_n$, we consider $R(z, z_1)$ as a function of $z_1 \in \mathfrak{H}_n$. Then, the second differential of R at $z_1 = z$ is

$$d^2R|_{z_1=z} = 2(\overline{z} - z)^{-1}dz(z - \overline{z})^{-1}d\overline{z} = \frac{1}{2}y^{-1}dz\, y^{-1}d\overline{z}.$$

The quadratic differential form

$$ds^2 = \operatorname{tr}(d^2R) = \frac{1}{2}\operatorname{tr}(y^{-1}dz\, y^{-1}d\overline{z}) \tag{2.1.7}$$

defines the Riemannian metric on $\mathfrak{H}_n$ invariant under G. It is known that for any two points z_1, z_2 in $\mathfrak{H}_n$, there is a unique geodesic on $\mathfrak{H}_n$ that connects z_1 and z_2. The geodesic distance $\rho(z_1, z_2)$ is regarded as a function $G\backslash(\mathfrak{H}_n \times \mathfrak{H}_n) \to \mathbb{R}$, and represented by $\mathcal{T}(z_1, z_2)$ as

$$\rho^2 = \frac{1}{2}\operatorname{tr}\left((\log|\mathcal{T}(z_1, z_2)|)^2\right).$$

2.2 Embedding of $\mathfrak{H}_n$ into a compact space

Let M^*_{2n} be a complex domain defined by $M^*_{2n,n} = \left\{u \in \mathbb{C}^{(2n,n)};\ \operatorname{rank} u = n\right\}$. On $M^*_{2n,n}$, we consider the action of $\mathrm{GL}_{2n}(\mathbb{C})$ from the left and the action of $\mathrm{GL}_n(\mathbb{C})$ from the right,

$$\mathrm{GL}_{2n}(\mathbb{C}) \curvearrowright M^*_{2n,n} \curvearrowleft \mathrm{GL}_n(\mathbb{C})$$

by the matrix multiplication. We consider a compact complex manifold D defined as the quotient space

$$D = M^*_{2n,n}/\,\mathrm{GL}_n(\mathbb{C}).$$

One can check the isomorphism $D \cong \mathrm{SU}(2n)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))$, the right hand side is the complex Grassmann manifold. We denote each element $z \in D$ by

$$z = [u] = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad u = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in M^*_{2n,n}, \quad z_1 = z_1^{(n)},\ z_2 = z_2^{(n)},$$

if it is the representative of the orbit including $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in M^*_{2n,n}$.

Let $H = H^{(n)}$ be a Hermitian matrix and $J = J^{(n)}$ be a skew symmetric matrix. Here we write $H\{z\} > 0$ and $J[z] = 0$ to imply that $H\{u\} > 0$ and $J[u] = 0$ for $z = [u] \in D$. We notice that these conditions are well-defined. Let $D(J)$ be a complex manifold defined by

$$D(J) = \left\{z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in D;\ J[z] = 0\right\}.$$

We define a complex domain $D_H(J)$ of $D(J)$ by

$$D_H(J) = \left\{z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in D(J);\ H\{z\} > 0\right\}. \tag{2.2.1}$$

We also define a group $G_H(J)$ by

$$G_H(J) = \left\{g \in \mathrm{GL}_{2n}(\mathbb{C});\ H\{g\} = H,\ J[g] = J\right\}.$$

Then, the action $G_H(J) \curvearrowright D_H(J)$ is induced from $\mathrm{GL}_{2n}(\mathbb{C}) \curvearrowright M^*_{2n,n}$.

We observe the transformation law beween two domains $D_H(J)$ and $D_{\tilde{H}}(\tilde{J})$, where we assume that

$$\tilde{H} := H\{M^{-1}\}, \quad \tilde{J} := J[M^{-1}], \quad M \in \mathbb{C}^{(n)}.$$

Then, we obtain the isomorphism $D_H(J) \to D_{\tilde{H}}(\tilde{J})$ by

$$D_H(J) \ni z \mapsto Mz \in D_{\tilde{H}}(\tilde{J}),$$

and the isomorphism $G_H(J) \to G_{\tilde{H}}(\tilde{J})$ by

$$G_H(J) \ni g \mapsto MgM^{-1} \in G_{\tilde{H}}(\tilde{J}).$$

We note each transformation law as below:

H	H	$\to$	$H\{M^{-1}\}$
J	J	$\to$	$J[M^{-1}]$
D_H	z	$\mapsto$	Mz
$G_H(J)$	g	$\mapsto$	MgM^{-1}

Embedding $\mathfrak{H}_n \hookrightarrow D(J)$

Let $H_1 = H_1^{(n)}$ be a Hermitian matrix and $J_1 = J_1^{(n)}$ be a skew symmetric matrix defined by

$$H_1 = \begin{pmatrix} 0 & i1_n \\ -i1_n & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

It is shown that the domain $D_{H_1}(J_1)$ is represented as

$$D_{H_1}(J_1) = \left\{ \begin{bmatrix} z^{(n)} \\ 1_n \end{bmatrix} \in D;\ z = z',\ \mathrm{Im}\, z > 0 \right\}. \tag{2.2.2}$$

Then, we immediately obtain a biholomorphism $\mathfrak{H}_n \cong D_{H_1}(J_1)$ by the canonical map

$$\mathfrak{H}_n \ni z \mapsto \begin{bmatrix} z \\ 1_n \end{bmatrix} \in D_{H_1}(J_1).$$

This map also gives an embedding of $\mathfrak{H}_n$ into $D(J)$ by the including map. With this relation, domain $D(J_1)$ is called the compact dual of $\mathfrak{H}_n$. We see that $G_{H_1}(J_1) = \{g \in \mathrm{GL}_{2n}(\mathbb{R});\ J_1[g] = J_1\}$, and this is $G = \mathrm{Sp}(n, \mathbb{R})$. So we learn that the action of G on $\mathfrak{H}_n$ is extended to the space $D(J)$ by this embedding. We induce the metric on $D_{H_1}(J_1)$ from the metric (2.1.7) on $\mathfrak{H}_n$.

The inclusion relation of the domains and our embedding are described as follows.

$$\begin{array}{ccc} & & D \\ & & \cup \\ \mathrm{Sym}_n(\mathbb{C}) & \hookrightarrow & D(J_1) \\ \cup & & \cup \\ \mathfrak{H}_n & \xrightarrow{\sim} & D_{H_1}(J_1). \end{array}$$

Here $\mathrm{Sym}_n(\mathbb{C}) = \left\{a \in \mathbb{C}^{(n)};\ {}^t a = a\right\}$ is the set of $n \times n$ symmetric matrices over $\mathbb{C}$.

Throughout this thesis, we assign a map $\psi_1 : D_{H_1}(J_1) \xrightarrow{\sim} \mathfrak{H}_n$ by

$$\psi_1 : D_{H_1}(J_1) \ni \begin{bmatrix} z \\ 1_n \end{bmatrix} \mapsto z \in \mathfrak{H}_n, \tag{2.2.3}$$

and use it as a chart. Namely, we use $(\psi_1, \mathfrak{H}_n)$ as one of the canonical realizations of $D_{H_1}(J_1)$. On the other hand, our referring to geometrical concepts is considered to be on the manifold $D_{H_1}(J_1)$.

Realization $\mathfrak{R}_n$

Let $H_0 = H_0^{(n)}$ be a Hermitian matrix and $J_0 = J_0^{(n)}$ be a skew symmetric matrix defined by

$$H_0 = \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

It is shown that the domain $D_{H_0}(J_0)$ is represented as

$$D_{H_0}(J_0) = \left\{ \begin{bmatrix} w^{(n)} \\ 1_n \end{bmatrix} \in D;\ w = w',\ 1_n - w^* w > 0 \right\}.$$

This is biholomorphic to the unit disk $\mathfrak{R}_n$. The transformation matrix $M_{01} : D_{H_1}(J_1) \to D_{H_0}(J_0)$ is obtained as

$$M_{01} = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1_n & -i1_n \\ i1_n & -1_n \end{pmatrix},$$

which satisfies the relation $H_0 = H_1\{M_{01}^{-1}\},\ J_0 = J_1[M_{01}^{-1}]$.

The chart $\psi_0 : D_{H_1}(J_1) \xrightarrow{\sim} \mathfrak{R}_n$ is defined by

$$\psi_0 : D_{H_0}(J_0) \ni \begin{bmatrix} w \\ 1_n \end{bmatrix} \mapsto w \in \mathfrak{R}_n$$

similarly with (2.2.3). By ψ_0 and ψ_1, the transformation M_{01} is expressed as the Cayley transformation (2.1.2):

$$\begin{tikzcd} D_{H_1}(J_1) \arrow[r, "M_{01}"] \arrow[d, "\psi_1"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & D_{H_0}(J_0) \arrow[d, "\psi_0"', "\wr"] \\ \mathfrak{H}_n \arrow[r, "\mathcal{C}"'] & \mathfrak{R}_n. \end{tikzcd}$$

Realization as a Siegel domain S of genus 3

Let $H_2 = H_2^{(n)}$ be a Hermitian matrix and $J_2 = J_2^{(n)}$ be a skew symmetric matrix defined by

$$H_2 = \begin{pmatrix} 0 & 0 & 0 & i1_s \\ 0 & -1_r & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ -i1_s & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 1_s \\ 0 & 0 & 1_r & 0 \\ 0 & -1_r & 0 & 0 \\ -1_s & 0 & 0 & 0 \end{pmatrix}, \quad n = r + s.$$

It is shown that the domain $D_{H_2}(J_2)$ is represented as

$$D_{H_2}(J_2) = \left\{ \begin{bmatrix} \frac{1}{2}z^{(s)} & u \\ u' & t^{(r)} \\ 0 & 1_r \\ 1_s & 0 \end{bmatrix};\ \begin{array}{l} u = u^{(s,r)} \\ t \in \mathfrak{R}(r) \\ z' = z \\ \operatorname{Im} z - L_t(u,u) > 0 \end{array} \right\},$$

$$L_t(u,u) = \overline{u}u' + (\frac{1}{i}u - \overline{u}t)(1 - t^*t)^{-1}(\frac{1}{i}u - \overline{u}t)^*.$$

Define a complex domain S by

$$S = \left\{ (z,u,t) \in \mathfrak{H}_s \times \mathbb{C}^{(s,r)} \times \mathfrak{R}_r;\ \operatorname{Im} z - L_t(u,u) > 0 \right\}.$$

This is called a Siegel domain of genus 3. Then, we obtain the biholomorphism $\psi_2 : D_{H_2}(J_2) \xrightarrow{\sim} S$ by

$$\psi_2 : D_{H_2}(J_2) \ni \begin{bmatrix} \frac{1}{2}z^{(s)} & u \\ u' & t^{(r)} \\ 0 & 1_r \\ 1_s & 0 \end{bmatrix} \mapsto (z, u, t) \in S.$$

Below we compute the transformation between S and $\mathfrak{H}_n$. The transformation matrix $M_{02} : D_{H_2}(J_2) \to D_{H_0}(J_0)$ is obtained as

$$M_{02} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}i1_s & 0 & \frac{1}{\sqrt{2}}1_s \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & \frac{-1}{\sqrt{2}}1_s & 0 & \frac{-1}{\sqrt{2}}i1_s \end{pmatrix} \quad \left(\Leftrightarrow M_{02}^{-1} = \begin{pmatrix} 0 & 1_r & 0 & 0 \\ \frac{-1}{\sqrt{2}}i1_s & 0 & 0 & \frac{-1}{\sqrt{2}}1_s \\ 0 & 0 & 1_r & 0 \\ \frac{1}{\sqrt{2}}1_s & 0 & 0 & \frac{1}{\sqrt{2}}i1_s \end{pmatrix} \right),$$

which satisfies the relation $H_0 = H_2\{M_{02}^{-1}\},\ J_0 = J_2[M_{02}^{-1}]$.

Lemma 2.2.1. *The transformation $M_{02} : D_{H_2}(J_2) \xrightarrow{\sim} D_{H_0}(J_0)$ is expressed as*

$$\psi_0 \circ M_{02} \circ \psi_2^{-1} : S \ni (z, u, t) \mapsto \begin{pmatrix} t - iu'(\frac{i}{2}z - 1_r)^{-1}u & \sqrt{2}iu'(\frac{i}{2}z - 1_r)^{-1} \\ \sqrt{2}i(\frac{i}{2}z - 1_r)^{-1}u & (\frac{1}{2}z - i1_r)(\frac{i}{2}z - 1_r)^{-1} \end{pmatrix} \in \mathfrak{R}_n. \quad (2.2.4)$$

Proof. Let $z = (z_1, u, t)$ be any element of S. The transformation $M_{02}\psi_2^{-1}(z)$ is computed as

$$M_{02}\psi_2^{-1}(z) = \begin{pmatrix} 0 & 1_r & 0 & 0 \\ \frac{-1}{\sqrt{2}}i1_s & 0 & 0 & \frac{-1}{\sqrt{2}}1_s \\ 0 & 0 & 1_r & 0 \\ \frac{1}{\sqrt{2}}1_s & 0 & 0 & \frac{1}{\sqrt{2}}i1_s \end{pmatrix} \begin{bmatrix} \frac{1}{2}z_1^{(s)} & u \\ u' & t^{(r)} \\ 0 & 1_r \\ 1_s & 0 \end{bmatrix}$$

$$= \begin{bmatrix} u' & t \\ \frac{1}{\sqrt{2}}(-i\frac{1}{2}z - 1_s) & -\frac{1}{\sqrt{2}}iu \\ 0 & 1_r \\ \frac{1}{\sqrt{2}}(\frac{1}{2}z + i1_s) & \frac{1}{\sqrt{2}}u \end{bmatrix},$$

by the trivial action of $\mathrm{GL}_n(\mathbb{C})$ from the right,

$$= \begin{bmatrix} \sqrt{2}iu'(i\frac{1}{2}z - 1_s)^{-1} & t \\ (\frac{1}{2}z - i1_s)(\frac{1}{2}iz - 1_s)^{-1} & -\frac{1}{\sqrt{2}}iu \\ 0 & 1_r \\ 1_s & \frac{1}{\sqrt{2}}u \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}iu'(i\frac{1}{2}z - 1_s)^{-1} & t - iu'(i\frac{1}{2}z - 1_s)^{-1}u \\ (\frac{1}{2}z - i1_s)(\frac{1}{2}iz - 1_s)^{-1} & \sqrt{2}i(i\frac{1}{2}z - 1_s)^{-1}u \\ 0 & 1_r \\ 1_s & 0 \end{bmatrix}$$

$$= \begin{bmatrix} t - iu'(i\frac{1}{2}z - 1_s)^{-1}u & \sqrt{2}iu'(i\frac{1}{2}z - 1_s)^{-1} \\ \sqrt{2}i(i\frac{1}{2}z - 1_s)^{-1}u & (\frac{1}{2}z - i1_s)(\frac{1}{2}iz - 1_s)^{-1} \\ 1_r & 0 \\ 0 & 1_s \end{bmatrix}.$$

Therefore, we see that M_{02} is expressed as (2.2.4). $\square$

Lemma 2.2.2. *The transformation* $M_{01}^{-1}M_{02} : D_{H_2}(J_2) \xrightarrow{\sim} D_{H_1}(J_1)$ *is expressed as*

$$\psi_1 \circ (M_{01}^{-1}M_{02}) \circ \psi_2^{-1} : S \ni (z,u,t) \mapsto \begin{pmatrix} (t-i1_r)(it-1_r)^{-1} & -\sqrt{2}i(it-1_r)^{-1}u' \\ -\sqrt{2}iu(it-1_r)^{-1} & \frac{1}{2}z - iu(it-1_r)^{-1}u' \end{pmatrix} \in \mathfrak{H}_n. \tag{2.2.5}$$

Proof. Let $z = (z_1, u, t)$ be any element of S. Put $w \in \mathfrak{R}_n$ by $w := (\psi_0 M_{02}^{-1}\psi_2^{-1})(z)$. By the above lemma,

$$w = \begin{pmatrix} t - iu'(i\frac{1}{2}z - 1_s)^{-1}u & \sqrt{2}iu'(i\frac{1}{2}z - 1_s)^{-1} \\ \sqrt{2}i(i\frac{1}{2}z - 1_s)^{-1}u & (\frac{1}{2}z - i1_s)(\frac{1}{2}iz - 1_s)^{-1} \end{pmatrix} \in \mathfrak{R}_n.$$

To derive (2.2.5), we compute $(\psi_1 M_{01}^{-1}\psi_0^{-1})(w)$, which is the inverse of the Cayley transformation; $\mathcal{C}^{-1}(w) = (w - i1_n)(iw - 1_n)^{-1}$. For the matrix

$$iw - 1_n = \begin{pmatrix} (it - 1_r) + u'(i\frac{1}{2}z - 1_s)^{-1}u & -\sqrt{2}u'(\frac{1}{2}iz - 1_s)^{-1} \\ -\sqrt{2}(\frac{1}{2}iz - 1_s)^{-1}u & 2(\frac{1}{2}iz - 1_s)^{-1} \end{pmatrix},$$

use the formula of inverse of block matrix,

$$\begin{pmatrix} A^{-1} & A^{-1}B \\ B'A^{-1} & D \end{pmatrix} = \begin{pmatrix} A + BF^{-1}B' & -BF^{-1} \\ -F^{-1}B' & F^{-1} \end{pmatrix}^{-1}, \quad F = D - B'A^{-1}B,$$

to obtain

$$(iw - 1_n)^{-1} = \begin{pmatrix} (it - 1_r)^{-1} & \frac{1}{\sqrt{2}}(it - 1_r)^{-1}u' \\ \frac{1}{\sqrt{2}}u(it - 1_r)^{-1} & \frac{1}{2}(\frac{1}{2}iz - 1_s) + \frac{1}{2}u(it - 1_r)^{-1}u' \end{pmatrix}.$$

Therefore, $\mathcal{C}^{-1}(w)$ is computed as

$$\begin{aligned}(w - i1_n)(iw - 1_n)^{-1} &= \begin{pmatrix} (t - i1_r) - iu'(i\frac{1}{2}z - 1_s)^{-1}u & \sqrt{2}iu'(i\frac{1}{2}z - 1_s)^{-1} \\ \sqrt{2}i(i\frac{1}{2}z - 1_s)^{-1}u & z(\frac{1}{2}iz - 1_s)^{-1} \end{pmatrix} \\ &\quad \times \begin{pmatrix} (it - 1_r)^{-1} & \frac{1}{\sqrt{2}}(it - 1_r)^{-1}u' \\ \frac{1}{\sqrt{2}}u(it - 1_r)^{-1} & \frac{1}{2}(\frac{1}{2}iz - 1_s) + \frac{1}{2}u(it - 1_r)^{-1}u' \end{pmatrix} \\ &= \begin{pmatrix} (t - i1_r)(it - 1_r)^{-1} & -\sqrt{2}i(it - 1_r)^{-1}u' \\ -\sqrt{2}iu(it - 1_r)^{-1} & \frac{1}{2}z - iu(it - 1_r)^{-1}u' \end{pmatrix}.\end{aligned}$$

□

We describe the charts and transformations defined in this section as the diagram below.

$$\begin{tikzcd} D_{H_1}(J_1) \arrow[r, "M_{01}"] \arrow[d, "\psi_1", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & D_{H_0}(J_0) \arrow[d, "\psi_0", "\wr"'] & D_{H_2}(J_2) \arrow[l, "M_{02}"'] \arrow[d, "\psi_2", "\wr"'] \\ \mathfrak{H}_n \arrow[r, "\mathcal{C}"'] & \mathfrak{R}_n & S \end{tikzcd}$$

2.3 Boundary

Boundary component

We give a definition of the boundary of $\mathfrak{H}_n$ by using the embedding $\mathfrak{H}_n \xrightarrow{\sim} \mathcal{D} \subset D(J)$ and the complex structure of $D(J)$.

Let $\mathcal{D} = D_H(J)$ be any domain defined as (2.2.1). Since $\mathcal{D}$ is a subdomain of the compact dual $D(J)$, we can consider the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in $D(J)$. We put $\partial\mathcal{D} := \overline{\mathcal{D}} \backslash \mathcal{D}$ and call it the boundary of $\mathcal{D}$.

Let A be a subset of a complex domain. It is said to be analytic if for each point $p \in A$, there are a neighborhood U of p and analytic functions $f_i : U \to \mathbb{C}$ $(i = 1, \ldots, m)$ such that $A \cap U = \{p \in U;\ f_1(p) = 0, \ldots, f_m(p) = 0\}$.

Definition 2.3.1. An analytic set $\phi \neq \mathcal{F} \subset \partial\mathcal{D}$ is called a boundary component of $\mathcal{D}$ if any analytic curve $\phi : \{t \in \mathbb{C};\ |t| < \varepsilon\} \to \partial\mathcal{D}$ that intersects $\mathcal{F}$ is completely contained in $\mathcal{F}$.

The closure $\overline{\mathcal{D}}$ and the boundary $\partial\mathcal{D}$ are described as

$$\overline{\mathcal{D}} = \{z \in D(J);\ H\{z\} \geq 0\}, \quad \partial\mathcal{D} = \left\{z \in \overline{\mathcal{D}};\ \det(H\{z\}) = 0\right\}.$$

Then, the group $G_H(J)$ acts on $\partial\mathcal{D}$. If another domain $\tilde{\mathcal{D}} = D_{\tilde{H}}(\tilde{J})$ and an isomorphic are given by $M : \mathcal{D} \to \tilde{\mathcal{D}}$, then their boundary and boundary components correspond to each other.

We fix $\mathcal{D} = D_{H_1}(J_1)$. For $0 \leq r \leq n$, we define the subsets $\mathcal{F}_r \subset \partial\mathcal{D}$ by

$$\mathcal{F}_0 = \left\{\begin{bmatrix} i1_n \\ 0_n \end{bmatrix} \in F\right\}, \quad \mathcal{F}_r = \left\{\begin{bmatrix} z_1^{(r)} & 0 \\ 0 & i1_s \\ 1_r & 0 \\ 0 & 0_s \end{bmatrix} \in \partial\mathcal{D};\ z_1 \in \mathfrak{H}_r\right\} \ (1 \leq r \leq n-1), \quad \mathcal{F}_n = \mathcal{D}. \tag{2.3.1}$$

We note that

$$\overline{\mathcal{F}_0} \subset \overline{\mathcal{F}_1} \subset \cdots \subset \overline{\mathcal{F}_{n-1}} \subset \overline{\mathcal{F}_n}, \quad \mathcal{F}_r \cong \mathfrak{H}_r,$$

where $\overline{\mathcal{F}_r}$ is the closure of $\mathcal{F}_r$ in $D(J_1)$.

Proposition 2.3.1. *$\mathcal{F}_r$ is a boundary component of $\mathcal{D}$. For any boundary component $\mathcal{F}$ of $\mathcal{D}$, there exist $0 \leq r \leq n$ and $g \in G$ such that $\mathcal{F} = g\mathcal{F}_r$. The boundary $\partial\mathcal{D}$ is covered by the boundary components.*

Example 2.3.1. When $n = 1$, we have $\mathcal{D} = D_{H_1}(J_1) = \left\{\begin{bmatrix} z \\ 1 \end{bmatrix};\ z \in \mathbb{C},\ \operatorname{Im} z > 0\right\}$. The boundary component $\mathcal{F}_0$ of $\mathcal{D}$ consists of one point; $\mathcal{F}_0 = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. By the action of $G = \mathrm{Sp}(1, \mathbb{R})$, this point is mapped to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}.$$

All the boundary components of $\mathcal{D}$ are exhausted by

$$\mathcal{F}_0 = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \quad \mathcal{F}(x) := \left\{\begin{bmatrix} x \\ 1 \end{bmatrix}\right\} \quad (x \in \mathbb{R}).$$

$\square$

We describe a relationship of two points in $\overline{\mathcal{D}}$ by an invariant. We denote $\mathcal{B} := D_{H_0}(J_0)$, the realization of the unit disk, and recall that $M_{01}(\mathcal{D}) = \mathcal{B}$. The closure $\overline{\mathcal{B}}$ and the boundary $\partial\mathcal{B}$ are described as

$$\overline{\mathcal{B}} = \left\{\begin{bmatrix} w^{(n)} \\ 1_n \end{bmatrix} \in D(J);\ 1_n - w^* w \geq 0\right\}, \quad \partial\mathcal{B} = \left\{\begin{bmatrix} w^{(n)} \\ 1_n \end{bmatrix} \in \overline{\mathcal{B}};\ \det(1_n - w^* w) = 0\right\}.$$

We can define a map $\{\cdot,\cdot\}_{H_0} : \mathcal{B}\times\mathcal{B} \to \mathbb{C}^{(n)}$ as

$$\{w_1, w_2\}_{H_0} = \begin{pmatrix} \psi_0(w_1) \\ 1_n \end{pmatrix}^* H_0 \begin{pmatrix} \psi_0(w_2) \\ 1_n \end{pmatrix} = 1_n - \psi_0(w_1)^*\psi_0(w_2),$$

where $\psi_0 : \mathcal{B} \xrightarrow{\sim} \mathfrak{R}_n$. Next, we continuously extend $\{,\}_{H_0}$ to $\overline{\mathcal{B}}\times\overline{\mathcal{B}}$. Then, we define an invariant $r : \overline{\mathcal{B}}\times\overline{\mathcal{B}} \to \{0,\dots,n\}$ by

$$r(w_1, w_2) = \operatorname{rank}\left(\{w_1, w_2\}_{H_0}\right).$$

We see the transformation law for $\{,\}_{H_0}$ similarly as (2.1.6), so that r depends only on the representatives of the orbit $G\cdot(w_1, w_2)$, with $G = G_{H_0}(J_0)$.

Proposition 2.3.2. *For two points $w_1, w_2 \in \overline{\mathcal{B}}$, the following conditions are equivalent.*

1. *w_1 and w_2 belong to the same boundary component $\mathcal{F}$, which is isomorphic to $\mathcal{F}_s$ with $0 \le s \le n$.*
2. *$r(w_1, w_1) = r(w_1, w_2) = r(w_2, w_2) = s$.*

Connectedness by a geodesic

Definition 2.3.2. Consider a domain $\mathcal{D} = D_H(J)$. Let $z \in \mathcal{D}$ and $a \in \partial\mathcal{D}$. The two points z, a are said to be connected by a geodesic if there is a geodesic $c(t)$ $(t \in \mathbb{R})$ of $\mathcal{D}$ such that $c(0) = z$ and $\lim_{t\to\infty} c(t) = a$ with respect to the topology of $D(J)$.

We notice that the relation of the connectedness is preserved by the action of $G = G_H(J)$. So we describe the connectedness in $\mathcal{B}$ by an invariant. Let $\mathfrak{S}_n$ be the set defined by

$$\mathfrak{S}_n = \{S = (s_1,\dots,s_n) \in \mathbb{R}^n;\ 0 \le s_1 \le \cdots \le s_n \le 1\}.$$

Let $w_1 \in \mathcal{B}$ and $w_2 \in \overline{\mathcal{B}}$. We can define a matrix $R = R^{(n)}$ by the quotient

$$R(w_1, w_2) = \{w_1, w_2\}_{H_0}^{-1}\{w_1, w_1\}_{H_0}\{w_2, w_1\}_{H_0}^{-1}\{w_2, w_2\}_{H_0}.$$

Let $0 \le s_1 \le \cdots \le s_n \le 1$ be the eigenvalues of $R(w_1, w_2)$, and put $\mathcal{R}(w_1, w_2) := (s_1,\dots,s_n)$. Then, we defined an invariant $\mathcal{R} : G\backslash(\mathcal{B}\times\overline{\mathcal{B}}) \to \mathfrak{S}_n$.

Proposition 2.3.3. *For $w_1 \in \mathcal{B}$ and $w_2 \in \partial\mathcal{B}$, the following conditions are equivalent.*

1. *w_1 and w_2 are connected by a geodesic, and $w_2 \in \mathcal{F} \cong \mathcal{F}_r$.*
2. *$\mathcal{R}(w_1, w_2) = (\overbrace{0,\dots,0}^{n-r}, \overbrace{1,\dots,1}^{r})$.*

2.4 Projection onto a boundary component

For the rest of this chapter, we fix $\mathcal{D} = D_{H_1}(J_1)$, $G = \mathrm{Sp}(n,\mathbb{R})$.

Proposition 2.4.1. *Let $\mathcal{F}$ be any boundary component of $\mathcal{D}$. Then, for any $z \in \mathcal{D}$, there uniquely exists $a \in \mathcal{F}$ that is connected with z by a geodesic.*

Definition 2.4.1. With the notation of Proposition 2.4.1, we define a map $\pi : \mathcal{D} \to \mathcal{F}$ so that $\pi(z) = a$. We call π the projection onto $\mathcal{F}$.

For $\mathcal{F} = \mathcal{F}_r$, we write $\pi = \pi_r$. In this section, we write the map $\pi_r : \mathcal{D} \to \mathcal{F}_r$ explicitly. Define an isomorphism $\varphi_r : \mathcal{F}_r \to \mathfrak{H}_r$ by

$$\varphi_r : \begin{bmatrix} z_1^{(r)} & 0 \\ 0 & i1_s \\ 1_r & 0 \\ 0 & 0_s \end{bmatrix} \mapsto z_1.$$

Then our purpose here is to calculate the map $\varphi_r \circ \pi_r \circ \psi_1^{-1} : \mathfrak{H}_n \to \mathfrak{H}_r$.

We define a subgroup of G by $G_1(\mathcal{F}_r) := \{g \in G;\ g(\mathcal{F}_r) = \mathcal{F}_r\}$. $G_1(\mathcal{F}_r)$ is written in matrices as

$$G_1(\mathcal{F}_r) = \left\{ g = \begin{pmatrix} a_1^{(r)} & 0 & b_1 & b_{12} \\ a_{21} & a_2^{(s)} & b_{21} & b_2 \\ c_1 & 0 & d_1^{(r)} & d_{12} \\ 0 & 0 & 0 & d_2^{(s)} \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R}) \right\}.$$

Remark 2.4.1. A general element g of $G_1(\mathcal{F}_r)$ is uniquely written in the form of

$$g = \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & a_2^{(s)} & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & d_2^{(s)} \end{pmatrix} \begin{pmatrix} a_1^{(r)} & 0 & b_1 & 0 \\ 0 & 1_s & 0 & 0 \\ c_1 & 0 & d_1^{(r)} & 0 \\ 0 & 0 & 0 & 1_s \end{pmatrix} \begin{pmatrix} 1_r & 0 & 0 & b_{12} \\ a_{21} & 1_s & b_{21} & b_2 \\ 0 & 0 & 1_r & d_{12} \\ 0 & 0 & 0 & 1_s \end{pmatrix}.$$

The last two components generate the Jacobi group $G^J(\mathbb{R})$ (ref. [Zie89]). The above decomposition implies the semidirect product $G_1(\mathcal{F}_r) \cong \mathrm{GL}_s(\mathbb{R}) \ltimes G^J(\mathbb{R})$. □

Because the connectedness by a geodesic is invariant under G, the map $\pi_r : \mathcal{D} \to \mathcal{F}_r$ is $G_1(\mathcal{F}_r)$-invariant, that is, $\pi_r(gz) = g\pi_r(z)$ for $g \in G, z \in \mathcal{D}$. We can define a group homomorphism $\pi_r^* : G_1(\mathcal{F}_r) \to \mathrm{Sp}(r, \mathbb{R})$ by

$$\pi_r^* : g = \begin{pmatrix} a_1^{(r)} & 0 & b_1 & b_{12} \\ a_{21} & a_2^{(s)} & b_{21} & b_2 \\ c_1 & 0 & d_1^{(r)} & d_{12} \\ 0 & 0 & 0 & d_2^{(s)} \end{pmatrix} \mapsto \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}. \tag{2.4.1}$$

It can be shown that π_r^* is surjective, more precisely,

$$\pi_r^* \left(\cap_{i=r}^n G_1(\mathcal{F}_i)\right) = \mathrm{Sp}(r, \mathbb{R}).$$

The action of $G_1(\mathcal{F}_r)$ on $\mathcal{F}_r$ is expressed through $\varphi_r : \mathcal{F}_r \cong \mathfrak{H}_r$ as

$$\varphi_r(gz) = \pi_r^*(g)\varphi_r(z) \quad (g \in G_1(\mathcal{F}_r),\ z \in \mathcal{F}_r). \tag{2.4.2}$$

Here the action in the right hand side is the same as (2.1.1). Thus, the action of $G_1(\mathcal{F}_r) \curvearrowright \mathcal{F}_r$ contains all elements of $\mathrm{Aut}(\mathcal{F}_r)$.

$$\begin{array}{ccc} G_1(\mathcal{F}_r) & \curvearrowright & \mathcal{D} \\ \downarrow{\scriptstyle \pi_r^*} & & \downarrow{\scriptstyle \pi_r} \\ \mathrm{Sp}(r, \mathbb{R}) & \curvearrowright & \mathcal{F}_r \end{array}$$

Put the subgroups of $G = \mathrm{Sp}(n,\mathbb{R})$ by

$$N := \left\{ \begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix};\ \begin{array}{l} a \in \mathrm{GL}_n(\mathbb{R}) : \text{lower triangular, with all diagonal elements equal to } 1 \\ b = b^{(n)} : b\,{}^t a \in \mathrm{Sym}_n(\mathbb{R}) \end{array} \right\},$$

$$A := \left\{ \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix};\ d \in \mathrm{GL}_n(\mathbb{R}) : \text{diagonal}, d > 0 \right\},$$

$$T := NA.$$

We have the decomposition $G = NAK$, where K is the isotropy subgroup at $i1_n \in \mathfrak{H}_n$. By the matrix expression, we verify the following lemma.

Lemma 2.4.2. *Put*

$$K_W := \left\{ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \in G;\ d : \textit{diagonal, with all diagonal elements equal to } \pm 1 \right\}.$$

Then, $TK_W = \cap_{i=0}^{n} G_1(\mathcal{F}_i)$. *The group* T *acts on each* $\mathcal{F}_r$ $(0 \le r \le n)$ *transitively.*

Proposition 2.4.3. *The projection* $\pi_r : \mathcal{D} \to \mathcal{F}_r$ *is expressed by* $\psi_1 : \mathcal{D} \cong \mathfrak{H}_n$ *and* $\varphi_r : \mathcal{F}_r \cong \mathfrak{H}_r$ *as*

$$\varphi_r \circ \pi_r \circ \psi_1^{-1} : \mathfrak{H}_n \ni \begin{pmatrix} z_1^{(r)} & * \\ * & * \end{pmatrix} \mapsto z_1 \in \mathfrak{H}_r. \tag{2.4.3}$$

Proof. Let $z_0 \in \mathcal{D}$ and $a_0 \in \mathcal{F}_r$ be defined by

$$z_0 = \begin{bmatrix} i1_r & 0 \\ 0 & i1_s \\ 1_r & 0 \\ 0 & 1_s \end{bmatrix}, \quad a_0 = \begin{bmatrix} i1_r & 0 \\ 0 & i1_s \\ 1_r & 0 \\ 0 & 0_s \end{bmatrix}.$$

We claim that $\pi_r(z_0) = a_0$. By the isomorphism $M_{01} : \overline{\mathcal{D}} \to \overline{\mathcal{B}}$, they are mapped to

$$M_{01} z_0 = \begin{bmatrix} 0_r & 0 \\ 0 & 0_s \\ 1_r & 0 \\ 0 & 1_s \end{bmatrix} \in \mathcal{B}, \quad M_{01} a_0 = \begin{bmatrix} 0_r & 0 \\ 0 & -i1_s \\ 1_r & 0 \\ 0 & 1_s \end{bmatrix} \in \partial\mathcal{B}.$$

By Proposition 2.3.3, we find that these two points are connected by a geodesic. Pulling back to $\overline{\mathcal{D}}$ by M_{01}^{-1}, the claim is verified. Next, let

$$z = \begin{bmatrix} z_1^{(r)} & z_{12} \\ z_{21} & z_2^{(s)} \\ 1_r & 0 \\ 0 & 1_s \end{bmatrix} \in \mathcal{D}$$

be any point in $\mathcal{D}$. We recall that the group $G_1(\mathcal{F}_r)$ transitively acts on $\mathcal{D}$ and $\pi_r : \mathcal{D} \to \mathcal{F}_r$ is $G_1(\mathcal{F}_r)$-invariant. Then, we can take

$$g = \begin{pmatrix} a_1^{(r)} & 0 & b_1 & b_{12} \\ a_{21} & a_2^{(s)} & b_{21} & b_2 \\ c_1 & 0 & d_1^{(r)} & d_{12} \\ 0 & 0 & 0 & d_2^{(s)} \end{pmatrix} \in G_1(\mathcal{F}_r)$$

such that $z = g \cdot z_0$, so that $\pi_r(z) = g \cdot a_0$. By the expression (2.4.2), we have

$$g \cdot a_0 = \begin{bmatrix} (ia_1 + b_1)(ic_1 + d_1)^{-1} & 0 \\ 0 & i1_s \\ 1_r & 0 \\ 0 & 0_s \end{bmatrix}.$$

To prove the lemma, we show that $z_1 = (ia_1+b_1)(ic_1+d_1)^{-1}$, which follows from the computation of $g \cdot z_0$. We compute it in $M^*_{2n,n}/\operatorname{GL}_n(\mathbb{C})$ as

$$\begin{aligned} g \cdot z_0 &= \begin{pmatrix} a_1^{(r)} & 0 & b_1 & b_{12} \\ a_{21} & a_2^{(s)} & b_{21} & b_2 \\ c_1 & 0 & d_1^{(r)} & d_{12} \\ 0 & 0 & 0 & d_2^{(s)} \end{pmatrix} \begin{bmatrix} i1_r & 0 \\ 0 & i1_s \\ 1_r & 0 \\ 0 & 1_s \end{bmatrix} \\ &= \begin{bmatrix} ia_1 + b_1 & b_{12} \\ ia_{21} + b_{21} & ia_2 + b_2 \\ ic_1 + d_1 & d_{12} \\ 0 & d_2 \end{bmatrix} \\ &= \begin{bmatrix} (ia_1 + b_1)(ic_1 + d_1)^{-1} & b_{12}d_2^{-1} \\ (ia_{21} + b_{21})(ic_1 + d_1)^{-1} & (ia_2 + b_2)d_2^{-1} \\ 1_r & d_{12}d_2^{-1} \\ 0 & 1_s \end{bmatrix} \\ &= \begin{bmatrix} (ia_1 + b_1)(ic_1 + d_1)^{-1} & b_{12}d_2^{-1} - (ia_1 + b_1)(ic_1 + d_1)^{-1}d_{12}d_2^{-1} \\ (ia_{21} + b_{21})(ic_1 + d_1)^{-1} & (ia_2 + b_2)d_2^{-1} - (ia_{21} + b_{21})(ic_1 + d_1)^{-1}d_{12}d_2^{-1} \\ 1_r & 0 \\ 0 & 1_s \end{bmatrix}. \end{aligned}$$

By expressing the equation $\pi_r(g \cdot z_0) = g \cdot a_0$, we obtain (2.4.3). $\square$

2.5 Sequence of boundary components

Let $1 \le r \le n-1$. Since $\mathcal{F}_r \cong \mathfrak{H}_r$ as a complex domain, we can consider the boundary components of $\mathcal{F}_r$ as we did for $\mathcal{D}$. Here, we see that they are already realized in the complex dual $D(J_1)$.

Let D' be a closed domain in $D(J_1)$ defined by

$$D' := \left\{ \begin{bmatrix} z_1^{(r)} & 0 \\ 0 & 1_s \\ z_2^{(r)} & 0 \\ 0 & 0_s \end{bmatrix} \in D(J_1);\ z_1, z_2 \in \mathbb{C}^{(r)} \right\}.$$

D' contains $\mathcal{F}_r$ as a subdomain. Then, the pair $\mathcal{F}_r, D'$ is a biholomorphic realization of the pair: $\mathfrak{H}_r$ and its compact dual $D(J^{(r)})$;

$$\begin{array}{ccccc} D(J^{(r)}) & \cong & D' & \subset & D(J_1) \\ & & \cup & & \cup \\ \mathfrak{H}_r & \cong & \mathcal{F}_r & \subset & \overline{\mathcal{D}}. \end{array}$$

We can see that the boundary components of $\mathfrak{H}_r$ in $D(J^{(r)})$ is embedded into $D(J_1)$ and they coincide with the boundary components of $\mathcal{D}$ in $D(J_1)$, that is, the set

$$\left\{\mathcal{F} \subset \overline{\mathcal{F}_r};\ \mathcal{F} \text{ is a boundary component of } \mathcal{D}.\ \right\}.$$

We define the Riemannian metric on $\mathcal{F}_r$ inducing the metric on $\mathfrak{H}_r$ defined as (2.1.7) through the isomorphism $\varphi_r : \mathcal{F}_r \cong \mathfrak{H}_r$. We notice that this metric is invariant under the action of $G_1(\mathcal{F}_r)$.

We put the set $\mathfrak{X}$ of all sequences of boundary components of $(\mathcal{G}_0, \ldots, \mathcal{G}_{n-1})$ of $\mathcal{D}$ such that

$$\overline{\mathcal{G}_0} \subset \overline{\mathcal{G}_1} \subset \cdots \subset \overline{\mathcal{G}_{n-1}} \subset \overline{\mathcal{D}}, \quad \mathcal{G}_i \cong \mathcal{F}_i \ (i = 0, \ldots, n-1).$$

Proposition 2.5.1. *1. For any boundary component $\mathcal{F}$ of $\mathcal{D}$, there exists $(\mathcal{G}_i)_{i=0}^{n-1} \in \mathfrak{X}$ such that $\mathcal{F} = \mathcal{G}_r$ for some r.*

2. The group G acts on $\mathfrak{X}$ transitively. The isotropy subgroup at $(\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}) \in \mathfrak{X}$ is TK_W. Thus, we have the bijection $\mathfrak{X} \cong G/TK_W$.

Proof. *1* follows from Proposition 2.3.1 and the surjectivity of π_r^*.
2: To prove the transitivity, we show that any element of $\mathfrak{X}$ is mapped to $(\mathcal{F}_i)_{i=0}^{n-1}$ by the action of G. Let $(\mathcal{G}_i)_{i=0}^{n-1} \in \mathfrak{X}$. Because $g\mathcal{G}_{n-1} = \mathcal{F}_{n-1}$ for some $g \in G$, we can assume that $\mathcal{G}_{n-1} = \mathcal{F}_{n-1}$. Now, assume that

$$(\mathcal{G}_0, \ldots, \mathcal{G}_{r-1}, \mathcal{G}_r, \ldots, \mathcal{G}_{n-1}) = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1}, \mathcal{F}_r, \ldots, \mathcal{F}_{n-1}),$$

for $1 \leq r \leq n-1$. Because $\pi_r^*\left(\cap_{i=r}^{n-1} G_1(\mathcal{F}_i)\right) = \mathrm{Sp}(r, \mathbb{R})$, we can take $g \in \cap_{i=r}^{n-1} G_1(\mathcal{F}_i)$ such that $g\mathcal{G}_{r-1} = \mathcal{F}_{r-1}$. Therefore,

$$g(\mathcal{G}_i)_{i=0}^{n-1} = (g\mathcal{G}_0, \ldots, g\mathcal{G}_{r-2}, \mathcal{F}_{r-1}, \mathcal{F}_r, \ldots, \mathcal{F}_{n-1}).$$

Repeating this argument, we obtain $(\mathcal{F}_i)_i$ from $(\mathcal{G}_i)_i$ by the action of G.
The isotropy subgroup at $(\mathcal{F}_i)_i$ is $\cap_{i=0}^{n} G_1(\mathcal{F}_i) = TK_W$. □

Proposition 2.5.2. *Take an arbitrary $(\mathcal{G}_i)_{i=0}^{n-1} \in \mathfrak{X}$. For $0 \leq s < r \leq n-1$, let $\pi_s : \mathcal{D} \to \mathcal{G}_s$, $\pi_r : \mathcal{D} \to \mathcal{G}_r$ be the projections and $\pi_{s,r} : \mathcal{G}_r \to \mathcal{G}_s$ be the projection regarding $\mathcal{G}_s$ as a boundary component of $\mathcal{G}_r$. Then, it holds that*

$$\pi_s = \pi_{s,r} \circ \pi_r$$

on $\mathcal{D}$.

Proof. It suffices to show for $(\mathcal{G}_i)_i = (\mathcal{F}_i)_i$. In this case, $\pi_s = \pi_{s,r} \circ \pi_r$ is obvious from the expression (2.4.3). □

2.6 Fiber of π_r

We recall the Siegel domain of genus 3

$$S = \left\{(z, u, t) \in \mathfrak{H}_s \times \mathbb{C}^{(s,r)} \times \mathfrak{R}_r;\ \mathrm{Im}\, z - L_t(u, u) > 0\right\},$$

$$L_t(u, u) = \overline{u}u' + (\frac{1}{i}u - \overline{u}t)(1 - t^*t)^{-1}(\frac{1}{i}u - \overline{u}t)^*$$

and the transformation (2.2.5). We define a chart $\varphi_S := \psi_2 \circ (M_{02}^{-1}M_{01}) : \mathcal{D} \xrightarrow{\sim} S$. Then, the projection $\pi_r : \mathcal{D} \to \mathcal{F}_r$ is expressed by $\varphi_S : \mathcal{D} \cong S$ and $\varphi_r : \mathcal{F}_r \cong \mathfrak{H}_r$ as

$$\varphi_r \circ \pi_r \circ \varphi_S^{-1} : S \ni (z, u, t) \mapsto \mathcal{C}^{-1}(t) \in \mathfrak{H}_r.$$

Definition 2.6.1. For $a \in \mathcal{F}_r$, we define the subdomain $\mathcal{D}_a$ of $\mathcal{D}$ by

$$\mathcal{D}_a = \pi_r^{-1}(a).$$

This is the set of all points in $\mathcal{D}$ that is connected with a by a geodesic. We call $\mathcal{D}_a$ the fiber of π_r at a.

From the expression (2.4.3), $\mathcal{D}_a$ is given by the image

$$\psi_1(\mathcal{D}_a) = \left\{ \begin{pmatrix} z_1 & z_{12} \\ z_{21} & z_2 \end{pmatrix} \in \mathfrak{H}_n;\ z_1 = \varphi_r(a) \right\}.$$

Proposition 2.6.1. *For any $a \in \mathcal{F}_r$, the fiber $\mathcal{D}_a = \pi_r^{-1}(a)$ is biholomorphic to the domain*

$$\left\{ (z, u) \in \mathfrak{H}_s \times \mathbb{C}^{(s,r)};\ \operatorname{Im} z - 2\operatorname{Re}(uu^*) > 0 \right\}.$$

The group $G_a = \{g \in G_1(\mathcal{F}_r);\ ga = a\}$ acts on $\mathcal{D}_a$ transitively.

Proof. Let a, b be any points in $\mathcal{F}_r$ and g be the element of $G_1(\mathcal{F}_r)$ such that $g \cdot a = b$. Since $\pi_r : \mathcal{D} \to \mathcal{F}_r$ is $G_1(\mathcal{F}_r)$-invariant, we have $g\mathcal{D}_a = \mathcal{D}_b$; the two fibers $\mathcal{D}_a$ and $\mathcal{D}_b$ are biholomorphic. If we take $a = a_0 := \varphi_r^{-1}(i1_r) \in \mathcal{F}_r$, then the fiber $\mathcal{D}_a$ is mapped to $\varphi_S(\mathcal{D}_a) = \{(z, u, 0) \in S\}$. This is biholomorphic to the domain

$$\left\{ (z, u) \in \mathfrak{H}_s \times \mathbb{C}^{(s,r)};\ \operatorname{Im} z - 2\operatorname{Re}(uu^*) > 0 \right\}. \tag{2.6.1}$$

The isotropy subgroup $G_a := \{g \in G_1(\mathcal{F}_r);\ ga = a\}$ acts on $\mathcal{D}_a$. We show that this action is transitive. Let $z_1, z_2 \in \mathcal{D}_a$. We can take $g \in G_1(\mathcal{F}_r)$ such that $gz_1 = z_2$. Then $\pi_r(gz_1) = \pi_r(z_2) = a$ implies that $g \in G_a$. □

Remark 2.6.1. The domain given in (2.6.1) is one of those called the Siegel domains of genus 2, ref. [PS69]. □

Chapter 3

Symmetric subdomain $\mathcal{D}^l$

Let $n = 2p + q$ with positive integers $p, q \in \mathbb{Z}$. Fix $\mathcal{D} = D_{H_1}(J_1) \cong \mathfrak{H}_n$ as the previous chapter.

3.1 Involution l

We take an element l of the isotropy subgroup $K = \mathrm{U}(n)$ by

$$l := \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 1_p \\ 0 & 1_q & 0 \\ 1_p & 0 & 0 \end{pmatrix}$$

Definition 3.1.1. Define a subdomain $\mathcal{D}^l$ of $\mathcal{D}$ by

$$\mathcal{D}^l = \{z \in \mathcal{D};\ l \cdot z = z\}.$$

Let $\mathcal{D}^{l'}$ be a subdomain of $\mathcal{D}$ defined by

$$\mathcal{D}^{l'} = \left\{z \in \mathcal{D};\ l'z = z\right\}, \quad l' = \begin{pmatrix} 0 & 1_{p+q,p} \\ -1_{p+q,p} & 0 \end{pmatrix}, \quad 1_{p+q,p} = \begin{pmatrix} 1_{p+q} & 0 \\ 0 & -1_p \end{pmatrix}.$$

The element $m \in K$ defined by

$$m := \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} \frac{1}{\sqrt{2}}1_p & 0 & \frac{1}{\sqrt{2}}1_p \\ 0 & 1_q & 0 \\ \frac{1}{\sqrt{2}}1_p & 0 & -\frac{1}{\sqrt{2}}1_p \end{pmatrix},$$

satisfies $l' = mlm^{-1}$, so that it gives the biholomorphism $m : \mathcal{D}^l \xrightarrow{\sim} \mathcal{D}^{l'} : z \mapsto mz$.

Let $\mathcal{B}^{l_0}$ be a subdomain of $\mathcal{B} = D_{H_0}(J_0)$ defined by

$$\mathcal{B}^{l_0} := \{w \in \mathcal{B};\ l_0 w = w\}, \quad l_0 = \begin{pmatrix} 1_{p+q,p} & 0 \\ 0 & -1_{p+q,p} \end{pmatrix}.$$

The Cayley transformation $M_{01} = \begin{pmatrix} 1_n & -i1_n \\ i1_n & -1_n \end{pmatrix} : \mathcal{D} \to \mathcal{B}$ satisfies that $l_0 = M_{01} l' M_{01}^{-1}$, so that it gives the biholomorphism $M_{01} : \mathcal{D}^{l'} \xrightarrow{\sim} \mathcal{B}^{l_0}$. $\mathcal{B}^{l_0}$ is realized in the unit disk by $\psi_0 : \mathcal{B} \to \mathfrak{R}_n$ as

$$\begin{aligned} \psi_0(\mathcal{B}^{l_0}) &= \{w \in \mathfrak{R}_n;\ 1_{p+q,p} w 1_{p+q,p} = -w\} \\ &= \left\{ w = \begin{pmatrix} 0 & w_2' \\ w_2 & 0 \end{pmatrix} \in \mathbb{C}^{(n)};\ \begin{matrix} w_2 \in \mathbb{C}^{(p,p+q)}, \\ 1_p - w_2 w_2^* > 0 \end{matrix} \right\}. \end{aligned}$$

This is the irreducible bounded symmetric domain (or the classical domain) of type I (ref. [FKK+00], Part IV). It is given by

$$\mathrm{SU}(p+q,p)/\mathrm{S}(\mathrm{U}(p+q)\times\mathrm{U}(p)).$$

We define a subgroup G^l of $G = \mathrm{Sp}(n,\mathbb{R})$ by

$$G^l = \left\{g \in G;\ lgl^{-1} = g\right\}.$$

G^l obviously acts on $\mathcal{D}^l$.

Theorem 3.1.1. *It holds the group isomorphism $G^l \cong \mathrm{SU}(p+q,p)$. Moreover, G^l acts on $\mathcal{D}^l$ transitively.*

Proof. By the definition, we have

$$\begin{aligned} G^l &= \{g \in \mathrm{Sp}(n,\mathbb{R});\ lg = gl\} \\ &= \{g \in \mathrm{SL}_{2n}(\mathbb{R});\ J_1[g] = J_1,\ lg = gl\}, \quad J_1 = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \end{aligned}$$

The pair (J_1, l) is mapped by the adjoint transformations as

$$\begin{aligned} (J_1, l) &\mapsto (mJ_1m^{-1}, mlm^{-1}) = (J_1, l') \\ &\mapsto (m_2J_1m_2^{-1}, m_2l'm_2^{-1}) = (l', J_1), \quad m_2 = \begin{pmatrix} 1_{p+q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_p \\ 0 & 0 & 1_{p+q} & 0 \\ 0 & 1_p & 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, we have the following group isomorphism

$$\begin{aligned} G^l &\overset{\mathrm{Ad}(m_2m)}{\cong} \{g \in \mathrm{SL}_{2n}(\mathbb{R});\ l'[g] = l',\ J_1g = gJ_1\} \\ &\cong \{g \in \mathrm{SL}_n(\mathbb{C});\ 1_{p+q,p}\{g\} = 1_{p+q,p}\} = \mathrm{SU}(p+q,p). \end{aligned}$$

For the second equivalence, we used the correspondence

$$\begin{array}{ccc} \{g \in \mathrm{SL}_{2n}(\mathbb{R});\ J_1g = gJ_1\} & \xrightarrow{\sim} & \mathrm{SL}_n(\mathbb{C}) \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} & \mapsto & x + iy. \end{array}$$

To prove the transitivity, we consider the orbit $G^l \cdot e,\ e = \psi_1^{-1}(i1_n)$ in $\mathcal{D}^l$. We see that

$$G^l \cdot e \cong G^l/(G^l \cap K) \cong \mathrm{SU}(p+q,p)/\mathrm{S}(\mathrm{U}(p+q)\times\mathrm{U}(p)).$$

Thus, we obtain that $G^l \cdot e = \mathcal{D}^l$. □

3.2 Fiber $\mathcal{D}_e$

Boundary components

Let $\mathcal{F}_{p+q}$ be the boundary component of $\mathcal{D}$ defined as (2.3.1);

$$\mathcal{F}_{p+q} = \left\{ z = \begin{bmatrix} z_1^{(p+q)} & 0 \\ 0 & i1_p \\ 1_{p+q} & 0 \\ 0 & 0_p \end{bmatrix} \in \partial\mathcal{D};\ z_1 \in \mathfrak{H}_{p+q} \right\}.$$

We consider the image $l\mathcal{F}_{p+q}$ by $l \in K$ defined in the previous section. $l\mathcal{F}_{p+q}$ is a boundary component of $\mathcal{D}$, and represented as

$$l\mathcal{F}_{p+q} = \left\{ z = \begin{bmatrix} 0_p & 0 \\ 0 & z_1^{(p+q)} \\ 1_p & 0 \\ 0 & 1_{p+q} \end{bmatrix} \in \partial\mathcal{D};\ z_1 \in \mathfrak{H}_{p+q} \right\}.$$

Definition 3.2.1. We define a boundary component $\mathcal{G}_q$ of $\mathcal{D}$ by

$$\mathcal{G}_q = \left\{ z = \begin{bmatrix} 0_p & 0 & 0 \\ 0 & z_2^{(q)} & 0 \\ 0 & 0 & i1_p \\ 1_p & 0 & 0 \\ 0 & 1_q & 0 \\ 0 & 0 & 0_p \end{bmatrix} \in \partial\mathcal{D};\ z_2 \in \mathfrak{H}_q \right\}.$$

It is obvious that $\mathcal{G}_q$ is biholomorphic to $\mathcal{F}_q$.

Proposition 3.2.1. *$\overline{\mathcal{G}_q} = \overline{\mathcal{F}_{p+q}} \cap \overline{l\mathcal{F}_{p+q}}$. Moreover, $l(\mathcal{G}_q) = \mathcal{G}_q$.*

Proof. By the biholomorphism $M_{01} : \mathcal{D} \to \mathcal{B}$, the image of $\overline{\mathcal{F}_{p+q}}$ is

$$M_{01}(\overline{\mathcal{F}_{p+q}}) = \left\{ \begin{bmatrix} w^{(n)} \\ 1_n \end{bmatrix} \in \partial\mathcal{B};\ w = \begin{pmatrix} w_1^{(p+q)} & 0 \\ 0 & -i1_p \end{pmatrix},\ 1_{p+q} - w_1^* w_1 \geq 0 \right\},$$

and the image of $\overline{l\mathcal{F}_{p+q}}$ is

$$M_{01}(\overline{l\mathcal{F}_{p+q}}) = \left\{ \begin{bmatrix} w^{(n)} \\ 1_n \end{bmatrix} \in \partial\mathcal{B};\ w = \begin{pmatrix} i1_p & 0 \\ 0 & w_2^{(p+q)} \end{pmatrix},\ 1_{p+q} - w_2^* w_2 \geq 0 \right\}.$$

Then, their intersection is

$$M_{01}(\overline{\mathcal{F}_{p+q}}) \cap M_{01}(\overline{l\mathcal{F}_{p+q}}) = \left\{ \begin{bmatrix} w^{(n)} \\ 1_n \end{bmatrix} \in \partial\mathcal{B};\ w = \begin{pmatrix} i1_p & 0 & 0 \\ 0 & w_3^{(q)} & 0 \\ 0 & 0 & -i1_p \end{pmatrix},\ 1_q - w_3^* w_3 \geq 0 \right\}.$$

This coincides with the image $M_{01}(\overline{\mathcal{G}_q})$. Pulling back the equation $M_{01}(\overline{\mathcal{F}_{p+q}}) \cap M_{01}(\overline{l\mathcal{F}_{p+q}}) = M_{01}(\overline{\mathcal{G}_q})$ by M_{01}^{-1}, we obtain the proposition. □

Let $k \in K$ be defined by

$$k = \begin{pmatrix} 0 & 0 & A_{p,q} & 0 \\ 0 & 1_p & 0 & 0 \\ -A_{p,q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_p \end{pmatrix}, \quad A_{p,q} = \begin{pmatrix} 0 & 1_p \\ 1_q & 0 \end{pmatrix}.$$

Then, we can verify that $k\mathcal{F}_q = \mathcal{G}_q$, $k\mathcal{F}_{n+q} = \mathcal{F}_{n+q}$. If we put $G_1(\mathcal{G}_q) := \{g \in G;\ g(\mathcal{G}_q) = \mathcal{G}_q\}$, then $G_1(\mathcal{G}_q) = kG_1(\mathcal{F}_q)k^{-1}$.

We define a chart $\phi_q : \mathcal{G}_q \xrightarrow{\sim} \mathfrak{H}_q$ by

$$\phi_q = \sigma \circ \varphi_q \circ k^{-1},$$

where $\sigma : \mathfrak{H}_q \to \mathfrak{H}_q : z \mapsto -z^{-1}$. ϕ_q is explicitly written as

$$\phi_q : \begin{bmatrix} 0_p & 0 & 0 \\ 0 & z_1^{(q)} & 0 \\ 0 & 0 & i1_p \\ 1_p & 0 & 0 \\ 0 & 1_q & 0 \\ 0 & 0 & 0_p \end{bmatrix} \mapsto z_1.$$

We define a group homomorphism $\varpi_q^* : G_1(\mathcal{G}_q) \to \mathrm{Sp}(q, \mathbb{R})$ by

$$\varpi_q^* = \sigma^* \circ \pi_q^* \circ \mathrm{Ad}(k^{-1}),$$

where $\sigma^* : \mathrm{Sp}(q, \mathbb{R}) \to \mathrm{Sp}(q, \mathbb{R}) : g \mapsto {}^t g^{-1}$ and $\pi_q^* : G_1(\mathcal{F}_q) \to \mathrm{Sp}(q, \mathbb{R})$ is defined as (2.4.1). Similarly with (2.4.2), the action $G_1(\mathcal{G}_q) \curvearrowright \mathcal{G}_q$ is expressed through ϕ_q as

$$\phi_q(gz) = \varpi_q^*(g)\phi_q(z) \quad (g \in G_1(\mathcal{G}_q),\ z \in \mathcal{G}_q).$$

Let $\pi_{p+q,n} : \mathcal{D} \to \mathcal{F}_{p+q}$ and $\pi_{q,p+q} : \mathcal{F}_{p+q} \to \mathcal{G}_q$ be the projection onto each boundary component as in Definition 2.4.1. We notice that $\pi_{q,n} := \pi_{q,p+q} \circ \pi_{p+q,n}$ is the projection from $\mathcal{D}$ onto $\mathcal{G}_q$. We have the following sequence.

$$\mathcal{D} \xrightarrow{\pi_{p+q,n}} \mathcal{F}_{p+q} \xrightarrow{\pi_{q,p+q}} \mathcal{G}_q.$$

$\pi_{q,p+q}$ is also given by $\pi_{q,p+q} = k \circ \tilde{\pi}_{q,p+q}$, where $\tilde{\pi}_{q,p+q} : \mathcal{F}_{p+q} \to \mathcal{F}_q$ is the projection defined as Proposition 2.5.2. Then, $\pi_{q,p+q}$ is expressed through φ_{p+q} and ϕ_q as

$$\phi_q \circ \pi_{q,p+q} \circ \varphi_{p+q}^{-1} : \mathfrak{H}_{p+q} \ni -\begin{pmatrix} * & * \\ * & z^{(q)} \end{pmatrix}^{-1} \mapsto -z^{-1} \in \mathfrak{H}_q. \tag{3.2.1}$$

As we saw in Proposition 2.5.2, the composition $\pi_{q,n} = k\pi_q$ is the projection from $\mathcal{D}$ onto $\mathcal{G}_q$.

Put $T_0 := kTk^{-1}$. Then $\pi_{p+q,n}, \pi_{q,p+q}$ are T_0-invariant and T_0 transitively acts on each $\mathcal{D}$, $\mathcal{F}_{n+q}$, $\mathcal{G}_q$ by Lemma 2.4.2. A general element of T_0 is an element $g \in G$ of the form

$$g = \begin{pmatrix} {}^t a_1^{(p)} & 0 & 0 & 0 & 0 & b_{13} \\ a_{21} & {}^t a_2^{(q)} & 0 & 0 & 0 & b_{23} \\ a_{31} & a_{32} & a_3^{(p)} & b_{31} & b_{32} & b_3 \\ c_1 & c_{12} & 0 & d_1^{(p)} & d_{12} & d_{13} \\ c_{21} & c_2 & 0 & 0 & d_2^{(q)} & d_{23} \\ 0 & 0 & 0 & 0 & 0 & {}^t d_3^{(p)} \end{pmatrix}, \quad \begin{array}{l} a_i, d_i\ (i = 1, 2, 3) \text{ are lower triangular,} \\ \text{with all diagonal elements positive.} \end{array}$$

By the definition, the image $\varpi_q^*(g)$ is

$$\varpi_q^*(g) = \begin{pmatrix} {}^t a_2 & 0 \\ c_2 & d_2 \end{pmatrix}.$$

Let $e_n := \psi_1^{-1}(i1_n) \in \mathcal{D}$ and

$$e_{p+q} := \pi_{p+q,n}(e_n), \quad e_q := \pi_{q,n}(e_n).$$

If there is no confusion, we omit the subscripts and just write e. We consider the fibers

$$\mathcal{D}_e := \pi_{q,n}^{-1}(e_q), \quad \mathcal{D}_e^{(p+q)} := \pi_{q,p+q}^{-1}(e_q).$$

Each of them is a subdomain of $\mathcal{D}$ and $\mathcal{F}_{p+q}$.

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{\pi_{p+q,n}} & \mathcal{F}_{p+q} & \xrightarrow{\pi_{q,p+q}} & \mathcal{G}_q \\ \cup & & \cup & & \cup \\ \mathcal{D}_e & \longrightarrow & \mathcal{D}_e^{(p+q)} & \longrightarrow & \{e_q\}. \end{array}$$

Relation between $\mathcal{D}^l$ and $\mathcal{D}_e$

Lemma 3.2.2.

$$G^l \subset (T_0 \cap G_2(\mathcal{G}_q))K.$$

Here $G^l = \{g \in G;\ lgl^{-1} = g\}$, $T_0 = kTk^{-1}$, $G_2(\mathcal{G}_q) = \{g \in G_1(\mathcal{G}_q);\ gz = z\ (z \in \mathcal{G}_q)\}$.

Proof. Let $g \in G^l$ be any element. Since $G = T_0K$, we assume that $g = g_1k$ with $g_1 \in T_0$ and $k \in K$. $g \in G^l$ implies that $g_1k = (lg_1l^{-1})(lkl^{-1})$. Then, we have

$$(lg_1l^{-1})^{-1}g_1 \in K. \tag{3.2.2}$$

Now we put the homomorphism $\varpi := \varpi_q^* : G_1(\mathcal{G}_q) \to \mathrm{Sp}(q, \mathbb{R})$. Because $g_1 \in T_0$, we see that

$$\varpi(g_1) \in \pi_q^*(T).$$

On the other hand, we saw that $l \in G_1(\mathcal{G}_q)$. l is mapped to

$$\varpi(l) = J = \begin{pmatrix} 0 & 1_q \\ -1_q & 0 \end{pmatrix}.$$

Applying ϖ to (3.2.2), we get

$${}^t\varpi(g_1)\varpi(g_1) = (J\varpi(g_1)J^{-1})^{-1}\varpi(g_1) \in \mathrm{U}(q).$$

Therefore, we get ${}^t\varpi(g_1)\varpi(g_1) \in \mathrm{U}(q) \cap \mathrm{Sym}_q^+(\mathbb{R}) = \{1_q\}$, so that $\varpi(g_1) = 1_q$. This means that $g_1 \in G_2(\mathcal{G}_q)$. □

Remark 3.2.1. A similar situation to Lemma 3.2.2 is treated for orthogonal groups of a quadratic form instead of $G = \mathrm{Sp}(n, \mathbb{R})$ in [Sat03], Chap.21. □

Theorem 3.2.3. $\mathcal{D}^l \subset \mathcal{D}_e$.

Proof. In the previous section, we saw that $\mathcal{D}^l = G^l \cdot e_n$. By Lemma 3.2.2, we have

$$\mathcal{D}^l \subset G_2(\mathcal{G}_q) \cdot e_n.$$

Then, we get $\pi_{q,n}(\mathcal{D}^l) = e_q$. This implies that $\mathcal{D}^l \subset \pi_{q,n}^{-1}(e_q) = \mathcal{D}_e$. □

3.3 Imaginary part $\mathcal{S}$

We define the imaginary part of the domain $\mathcal{D}$.

Definition 3.3.1. We define a real domain $\mathcal{S}$ in $\mathcal{D}$ by the image

$$\psi_1(\mathcal{S}) = \left\{ iy \in \mathfrak{H}_n;\ y \in \mathbb{R}^{(n)} \right\}.$$

We notice that $\mathcal{S}$ is also given by

$$\mathcal{S} = \{z \in \mathcal{D};\ R\{z\} = 0\}, \quad R := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

With this representation, it is clear that the subgroup G_R of G defined by

$$G_R = \{g \in G;\ R\{g\} = R\},$$

acts on $\mathcal{S}$.

Lemma 3.3.1. *1. The group G_R is written as*

$$G_R = \left\{ g = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix};\ a \in \mathrm{GL}_n(\mathbb{R}) \right\}.$$

2. G_R acts on S transitively.

3. The group $\{g \in G;\ g(\mathcal{S}) = \mathcal{S}\}$ is generated by G_R and $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$.

Proof. 1. We recall the definition $G = \{g \in \mathrm{GL}_{2n}(\mathbb{R});\ J[g] = g\}$. Now for $g \in \mathrm{GL}_{2n}(\mathbb{R})$, we have

$$\begin{cases} J[g] = J \\ R\{g\} = R \end{cases} \Leftrightarrow \begin{cases} J[g] = J \\ R[g] = R \end{cases} \Leftrightarrow \begin{cases} J[g] = J \\ (J^{-1}R)[g] = (J^{-1}R). \end{cases}$$

As $J^{-1}R = \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix}$, $(J^{-1}R)[g] = (J^{-1}R)$ implies that

$$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad a, d \in \mathrm{GL}_n(\mathbb{R}).$$

This is contained in G if and only if $d = {}^t a^{-1}$.
2. It follows from the transitivity of the action of $\mathrm{GL}_n(\mathbb{R})$ on the set $\{A \in \mathrm{Sym}_n(\mathbb{R});\ A > 0\}$ defined by $g \cdot A = gA{}^t g$.
3. Let $\langle G_R, J\rangle$ be the subgroup generated by G_R and J. It is obvious that $\langle G_R, J\rangle \subset \{g \in G;\ g(\mathcal{S}) = \mathcal{S}\}$. We show the converse $\supset$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,\ g(\mathcal{S}) = \mathcal{S}$. Then,

$$g(iy) = (aiy + b)(ciy + d)^{-1} \in i\mathbb{R}^{(n)}, \quad {}^\forall y \in \mathbb{R}^{(n)}, y > 0.$$

Now we assume that $d \neq 0$. Considering a limit $y \to i0_n$, we see that $g(iy) \to bd^{-1} \in \mathbb{R}^{(n)} \cap i\mathbb{R}^{(n)}$. So we must have $b = 0$, and consequently $c = 0$. Thus $g \in G_R$. Next, we assume that $d = 0$. In this case, $g(iy) = ac^{-1} + b(ciy)^{-1}$. Then $g(iy) \in i\mathbb{R}^{(n)}$ implies that $a = 0$. So g is of the form

$$g = \begin{pmatrix} 0 & b \\ -{}^t b^{-1} & 0 \end{pmatrix} = J \begin{pmatrix} {}^t b^{-1} & 0 \\ 0 & b \end{pmatrix}.$$

Thus $g \in JG_R$. In both cases, we have $g \in \langle G_R, J\rangle$. $\square$

Restriction to the imaginary part

We put subdomains of $\mathcal{D}$ as

$$\mathcal{S}^l := \mathcal{D}^l \cap \mathcal{S}, \quad \mathcal{S}_e := \mathcal{D}_e \cap \mathcal{S},$$

and a subdomains of $\mathcal{F}_{p+q}$ as

$$\mathcal{S}^{(p+q)} := \pi_{p+q,n}(\mathcal{S}), \quad \mathcal{S}_e^{(p+q)} := \mathcal{D}_e^{(p+q)} \cap \mathcal{S}^{(p+q)}.$$

Proposition 3.3.2. $\pi_{p+q,n}(\mathcal{S}_e) = \mathcal{S}_e^{(p+q)}$.

Proof. The inclusion $\subset$ is obvious by the definition $\mathcal{S}_e = \mathcal{D}_e \cap \mathcal{S}$. To prove $\supset$, let $z \in \mathcal{D}_e^{(p+q)} \cap \mathcal{S}^{(p+q)}$. Suppose that

$$z = \pi_{p+q,n}(z_1), \quad z_1 \in \mathcal{S}.$$

Because $\mathcal{D}_e^{(p+q)} = \pi_{q,p+q}^{-1}(e_q)$, we have $\pi_{q,n}(z_1) = \pi_{q,p+q}(z) = e_q$. Thus, we obtain that $z_1 \in \mathcal{D}_e \cap \mathcal{S}$. It means that $z \in \pi_{p+q,n}(\mathcal{S}_e)$. $\square$

We review these domains as follows.

$$\begin{array}{ccccccc}
 & & \mathcal{D} & \xrightarrow{\pi_{p+q,n}} & \mathcal{F}_{p+q} & \xrightarrow{\pi_{q,p+q}} & \mathcal{G}_q \\
 & & \cup & & \cup & & \cup \\
\mathcal{D}^l & \subset & \mathcal{D}_e & \longrightarrow & \mathcal{D}_e^{(p+q)} & \longrightarrow & \{e_q\} \\
\cup & & \cup & & \cup & & \\
\mathcal{S}^l & \subset & \mathcal{S}_e & \longrightarrow & \mathcal{S}_e^{(p+q)}. & &
\end{array}$$

By the biholomorphism $M_{01} \circ m : \mathcal{D}^l \xrightarrow{\sim} \mathcal{B}^{l_0}$ in Section 3.1, the real domain $\mathcal{S}^l$ is mapped to

$$\left\{ w = \begin{pmatrix} 0 & iw_2' \\ iw_2 & 0 \end{pmatrix} \in \mathbb{C}^{(n)};\ \begin{array}{l} w_2 \in \mathbb{R}^{(p,p+q)}, \\ 1_p - w_2 w_2' > 0 \end{array} \right\}.$$

This is a realization of the non compact symmetric space of type BDI (ref. [FKK+00], part IV). It is given by

$$\mathrm{SO}(p+q,p)/\mathrm{S}(\mathrm{O}(p+q) \times \mathrm{O}(p)).$$

We define a subgroup G_R^l of G_R by

$$G_R^l = G^l \cap G_R.$$

G_R^l obviously acts on $\mathcal{S}^l$.

Theorem 3.3.3. *It holds the group isomorphism $G_R^l \cong \mathrm{SO}(p+q,p)$. Moreover, G_R^l acts on $\mathcal{S}^l$ transitively.*

Proof. By the definition, we have

$$G_R^l = \left\{ g = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix};\ \begin{array}{l} a \in \mathrm{GL}_n(\mathbb{R}), \\ LaL = {}^t a^{-1} \end{array} \right\}, \quad L = \begin{pmatrix} 0 & 0 & 1_p \\ 0 & 1_q & 0 \\ 1_p & 0 & 0 \end{pmatrix}.$$

Then the isomorphism $G_R^l \cong \mathrm{SO}(p+q,p)$ is given by $g \mapsto a$.

To prove the transitivity, we consider the orbit $G_R^l \cdot e_n$ in $\mathcal{S}^l$. We see that

$$G_R^l \cdot e \cong G_R^l/(G_R^l \cap K) \cong \mathrm{SO}(p+q,p)/\mathrm{S}(\mathrm{O}(p+q) \times \mathrm{O}(p)).$$

Thus, we obtain that $G_R^l \cdot e = \mathcal{S}^l$. $\square$

We put subgroups of G_R as

$$T_R := T_0 \cap G_R,$$

$$K_R := K \cap G_R.$$

A general element of T_R is a $g \in G(\mathcal{S})$ of the form

$$g = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \quad a = \begin{pmatrix} {}^t a_1^{(p)} & 0 & 0 \\ a_{21} & {}^t a_2^{(q)} & 0 \\ a_{31} & a_{32} & a_3^{(p)} \end{pmatrix}, \quad \begin{array}{l} a_i \ (i = 1, 2, 3) \text{ are lower triangular,} \\ \text{with all diagonal elements positive.} \end{array}$$

Lemma 3.3.4. *$G_R = T_R K_R$. Moreover, $G_R^l = T_R^l K_R^l$. Here $T_R^l = T_R \cap G^l$, $K_R^l = K_R \cap G^l$.*

Proof. The equation $G_R = T_R K_R$ immediately follows from the Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$. By the uniqueness of the decomposition $G = TK$ and Lemma 3.2.2, we have

$$G_R^l \subset (T_R \cap G_2(\mathcal{G}_2))K_R.$$

Each element of $T_R \cap G_2(\mathcal{G}_2)$ is of the form

$$g = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \quad a = \begin{pmatrix} {}^t a_1^{(p)} & 0 & 0 \\ a_{21} & 1_q & 0 \\ a_{31} & a_{32} & a_3^{(p)} \end{pmatrix}.$$

Then, we notice that $T_R \cap G_2(\mathcal{G}_2)$ is invariant under the action of $\mathrm{Ad}(l)$.

Let $g \in G_R^l$ be any element. Now we can write

$$g = tk, \quad t \in T_R \cap G_2(\mathcal{G}_2), \ k \in K_R.$$

$lgl^{-1} = g$ implies that

$$(ltl^{-1})(lkl^{-1}) = tk.$$

By the uniqueness of the decomposition $g = tk$, it follows that $ltl^{-1} = t$, $lkl^{-1} = k$. Therefore, $t \in T_R^l$, $k \in K_R^l$. We showed that $G_R^l \subset T_R^l K_R^l$. The converse $\supset$ is obvious. □

We see that

$$\mathcal{S}^{(p+q)} = \left\{ z \in \mathcal{F}_{p+q};\ \varphi_{p+q}(z) = iy \in i\mathbb{R}^{(p+q)} \right\},$$

$$\mathcal{S}_e^{(p+q)} = \left\{ z \in \mathcal{F}_{p+q};\ \varphi_{p+q}(z) = iy \in i\mathbb{R}^{(p+q)},\ -(iy)^{-1} = \begin{pmatrix} * & * \\ * & i1_q \end{pmatrix} \right\}.$$

Lemma 3.3.5. *Define a subgroup $(T_R)_e$ by $(T_R)_e := \{t \in T_R;\ t \cdot e_q = e_q\}$. Then, $(T_R)_e$ acts on $\mathcal{S}_e^{(p+q)}$ transitively.*

The proof is almost the same as the proof of Proposition 2.6.1.

Proof. Let $z \in \mathcal{S}_e^{(p+q)}$ be any point. By Lemma 3.3.4, T_R transitively acts on $\mathcal{S}$, so on $\mathcal{S}^{(p+q)}$ as well. Then, there exists $t \in T_R$ such that $t \cdot e_{p+q} = z$. We see that

$$e_q = \pi_{q,p+q}(z) = \pi_{q,p+q}(te_{p+q}) = te_q.$$

This implies that $t \in (T_R)_e$. □

Chapter 4

Arrangement of $\mathcal{D}^l$ in $\mathcal{D}$

4.1 Riemannian submersion

Definition and general theory

In this section, we review the definition and the property of the Riemannian submersion, referring to Urakawa [Ura90].

Let $(M, g), (N, h)$ be Riemannian manifolds and $f : M \to N$ be a smooth map. We consider the differential $df_x : T_xM \to T_{f(x)}N$. For each $x \in M$, define subspaces in T_xM by

$$V_x := \ker df_x, \quad H_x := (V_x)^\perp,$$

where $(V_x)^\perp$ is the orthogonal complement of V_x with respect to the metric g_x.

Definition 4.1.1. $f : M \to N$ is said to be a Riemannian submersion if the restriction $df_x|_{H_x}$ gives the isometry

$$df_x|_{H_x} : H_x \cong T_{f(x)}N \tag{4.1.1}$$

for each $x \in M$.

For a Riemannian submersion f, a vector $v \in T_xM$ is said to be horizontal with respect to f if $v \in H_x$.

Remark 4.1.1. The term 'horizontal' is independent of the same term of connections as in [KN93]. □

Lemma 4.1.1 (O'Neill's formula). *Let ∇, ∇' be the Riemannian connection of $(M, g), (N, h)$ respectively. If $f : M \to N$ is a Riemannian submersion, then it holds the equation*

$$df_x(\nabla_X Y) = \nabla'_{df_x X} df_x Y$$

for any point $x \in M$ and any horizontal vector fields X, Y on M.

By the above lemma, a Riemannian submersion leaves the Riemannian connection invariant for horizontal vectors. We recall that the differential equation for each geodesic $c(t)$ on M is written as

$$\nabla_{\dot{c}(t)}\dot{c}(t) = 0. \tag{4.1.2}$$

Here $\dot{c}(t) \in T_{c(t)}M$ is the tangent vector of $c(t)$ at t.

Proposition 4.1.2. *Let $f : M \to N$ be a Riemannian submersion. Suppose that $c(t)$ $(t \in (-\varepsilon, \varepsilon),\ \varepsilon > 0)$ is a geodesic on M and the vector $\dot{c}(0) \in T_{c(0)}M$ is horizontal. Then, the following conditions are equivalent.*

1. *The vector $\dot{c}(t) \in T_{c(t)}M$ is horizontal at each $t \in (-\varepsilon, \varepsilon)$.*
2. *The image $f(c(t))$ $(t \in (-\varepsilon, \varepsilon))$ is a geodesic on N.*

Remark 4.1.2. The statement *1* $\Rightarrow$ *2* is a special case of a proposition in [Ura90], Section 2.3. $\square$

Proof. *1* $\Rightarrow$ *2* is obvious from Lemma 4.1.1 and (4.1.2).
Now, suppose *2*. Take an arbitrary $t_0 \in (-\varepsilon, \varepsilon)$. We claim that the vector $\dot{c}(t_0)$ is horizontal. We write

$$\dot{c}(t_0) = h(t_0) + v(t_0), \quad h(t_0) \in H_{c(t_0)},\ v(t_0) \in V_{c(t_0)}.$$

We denote by $\|\dot{c}(t)\|$ the norm of $\dot{c}(t)$ at each t. Since the norm of tangent vector of any geodesic is constant, we see that

$$\|\dot{c}(t_0)\|^2 = \|\dot{c}(0)\|^2,$$

$$\|df(\dot{c}(t_0))\|^2 = \|\frac{d}{dt}(f(c(t)))|_{t=t_0}\|^2 = \|\frac{d}{dt}(f(c(t)))|_{t=0}\|^2 = \|df(\dot{c}(0))\|^2.$$

Since f is a Riamannian submersion and $\dot{c}(0)$ is horizontal, we have

$$\|df(\dot{c}(t_0))\|^2 = \|h(t_0)\|^2, \quad \|df(\dot{c}(0))\|^2 = \|\dot{c}(0)\|^2.$$

Combining these equations, we obtain that

$$\|h(t_0)\|^2 + \|v(t_0)\|^2 = \|\dot{c}(t_0)\|^2 = \|h(t_0)\|^2.$$

Then, $v(t_0) = 0$. This means that $\dot{c}(t_0)$ is horizontal. $\square$

In the next section, we will show that the projection $\pi_r : \mathcal{D} \to \mathcal{F}_r$ and its restriction on $\mathcal{S}^l$ are Riemannian submersions. Here, we prepare a lemma to reduce the condition for a Riemannian submersion $f : M \to N$ to the condition on a point $e \in M$ for a homogeneous space M.

Lemma 4.1.3. *Let M, N be Riemannian submanifolds. Let G be a Lie group acting on M and N. Suppose that the actions of G are isometric on M, N and transitive on M. Let $f : M \to N$ be a smooth map which is G-invariant; $f(gx) = gf(x)$ $(g \in G, x \in M)$. Let $e \in M$. Suppose that we have the isometry*

$$df_e|_{H_e} : H_e \xrightarrow{\sim} T_{f(e)}N$$

where

$$V_e = \ker df_e, \quad H_e = (V_e)^{\perp}.$$

Then, $f : M \to N$ is a Riemannian submersion.

Proof. We must verify the isometry (4.1.1) for any point $x \in M$. Let $x \in M$. We can take $g \in G$ such that $ge = x$. Since f is G-invariant, it holds that $f = g \circ f \circ g^{-1}$, so we have

$$df_x = dg_{f(e)} \circ df_e \circ d(g^{-1})_x,$$

where $d(g^{-1})_x : T_xM \to T_eM$ is the differential of $g^{-1} : M \to M$ and $dg_{f(e)}$ is as well. Then, we have

$$V_x := \ker df_x = \ker \left(dg_{f(e)} \circ df_e \circ d(g^{-1})_x\right) = dg_e(V_e).$$

Since the action of G is isometric, we have

$$H_x := (V_x)^{\perp} = dg_e(H_e).$$

Now we obtain the following commutative diagram:

$$\begin{tikzcd} H_e \arrow[r, "df_e"] \arrow[d, "dg_e"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{f(e)}N \arrow[d, "dg_{f(e)}"] \\ H_x \arrow[r, "df_x"'] & T_{f(x)}N. \end{tikzcd}$$

Thus, the isometry (4.1.1) is verified. □

Riemannian submersion $\pi_{p+q,n}$

Recall that we defined the metric on $\mathcal{F}_r$ through the isomorphism $\varphi_r : \mathcal{F}_r \cong \mathfrak{H}_r$ in Section 2.5. The tangent space $T_{i1_n}\mathfrak{H}_n$ is identified with the space $\mathrm{Sym}_n(\mathbb{C})$ equipped with the metric

$$(Z_1, Z_2) := \mathrm{tr}(Z_1\overline{Z_2}) \quad (Z_1, Z_2 \in \mathrm{Sym}_n(\mathbb{C})),$$

and it is the same for $T_{i1_r}\mathfrak{H}_r$.

Theorem 4.1.4. *The projection $\pi_r : \mathcal{D} \to \mathcal{F}_r$ $(1 \le r \le n-1)$ is a Riemannian submersion.*

Proof. We denote by π the expression of π_r in Proposition 2.4.3;

$$\pi : \mathfrak{H}_n \ni \begin{pmatrix} z_1^{(r)} & * \\ * & * \end{pmatrix} \mapsto z_1 \in \mathfrak{H}_r.$$

At the point $i1_n \in \mathfrak{H}_n$, the differential $d\pi : T_{i1_n}\mathfrak{H}_n \to T_{i1_r}\mathfrak{H}_r$ is expressed as

$$d\pi : \mathrm{Sym}_n(\mathbb{C}) \ni \begin{pmatrix} Z_1^{(r)} & Z_{12} \\ Z_{21} & Z_2 \end{pmatrix} \mapsto Z_1 \in \mathrm{Sym}_r(\mathbb{C}). \tag{4.1.3}$$

Then we have

$$V_e := \ker d\pi = \left\{ \begin{pmatrix} 0_r & Z_{12} \\ Z_{21} & Z_2 \end{pmatrix} \in \mathrm{Sym}_n(\mathbb{C}) \right\}.$$

Therefore,

$$H_e := (V_e)^{\perp} = \left\{ \begin{pmatrix} Z_1^{(r)} & 0 \\ 0 & 0_{n-r} \end{pmatrix} \in \mathrm{Sym}_n(\mathbb{C}) \right\}.$$

It is obvious that the restriction $d\pi|_{H_e} : H_e \to \mathrm{Sym}_r(\mathbb{C})$ gives an isometry. Applying Lemma 4.1.3 with $f = \pi : \mathfrak{H}_n \to \mathfrak{H}_r$ and $G = G_1(\mathcal{F}_r)$, we obtain that π is a Riemannian submersion and so is π_r. □

We return to the sequence

$$\mathcal{D} \xrightarrow{\pi_{p+q,n}} \mathcal{F}_{p+q} \xrightarrow{\pi_{q,p+q}} \mathcal{G}_q.$$

We consider the tangent space $T_e\mathcal{D}$ at $e = e_n = \psi_1^{-1}(i1_n) \in \mathcal{D}$. For the projection π to each boundary component, we put the subspace $V(\pi)$ of $T_e\mathcal{D}$ by

$$V(\pi) := \ker d\pi_e,$$

and put

$$H(\pi) := (V(\pi))^{\perp},$$

the orthogonal complement of $V(\pi)$ in $T_e\mathcal{D}$. In addition, we define subspaces V_1, V_2 of $T_e\mathcal{D}$ by

$$V_1 = V(\pi_{q,n}) \cap H(\pi_{p+q,n}), \quad V_2 = V(\pi_{p+q,n}) \cap V(l\pi_{p+q,n}l^{-1}).$$

Here $l \in K$ is the element defined in Section 3.1.

Now we have the refinement of the decompositions

$$T_e\mathcal{D} = H(\pi_{q,n}) \oplus V(\pi_{q,n}), \quad T_e\mathcal{D} = H(\pi_{p+q,n}) \oplus V(\pi_{p+q,n}) \tag{4.1.4}$$

as follows.

Lemma 4.1.5. *$T_e\mathcal{D}$ admits the orthogonal decomposition*

$$T_e\mathcal{D} = H(\pi_{q,n}) \oplus V_1 \oplus V(\pi_{p+q,n}). \tag{4.1.5}$$

Proof. $V(\pi_{p+q,n})$ is obviously a subspace of $V(\pi_{q,n})$. Then we have the decomposition

$$V(\pi_{q,n}) = \Big(V(\pi_{q,n}) \cap V(\pi_{p+q,n})^{\perp}\Big) \oplus V(\pi_{p+q,n}) = V_1 \oplus V(\pi_{p+q,n}).$$

The lemma is obtained from this and (4.1.4). $\square$

We notice that the isotoropy group K acts on the vector space $T_e\mathcal{D}$ as the differentials. For any $l_0 \in K$, we denote this action by $l_0 : T_e\mathcal{D} \to T_e\mathcal{D}$, instead of dl_0.

Lemma 4.1.6. *Let $l \in K$ be the element defined in Section 3.1.*

1. *l acts on $H(\pi_{q,n})$ by $l : v \mapsto -v$ ($v \in H(\pi_{q,n})$).*
2. *$lV_1 \subset V(\pi_{p+q,n})$.*
3. *$T_e\mathcal{D}$ admits the orthogonal decomposition*

$$T_e\mathcal{D} = H(\pi_{q,n}) \oplus V_1 \oplus lV_1 \oplus V_2. \tag{4.1.6}$$

Proof. *1.* In the proof of Lemma 3.2.2, we saw that the action of l on $\mathcal{G}_q$ is expressed as

$$\varpi(l)z = Jz = -z^{-1} \quad (z \in \mathfrak{H}_q).$$

This implies that l acts on $T_{e_q}\mathcal{G}_q \cong H(\pi_{q,n})$ by $l : v \mapsto -v$.
2. Here we consider the matrix expression through $d\psi_1 : T_e\mathcal{D} \cong \mathrm{Sym}_n(\mathbb{C})$. By the proof of Theorem 4.1.4 and (3.2.1), the subspaces $H(\pi_{p+q,n}), H(\pi_{q,n})$ are expressed by ψ_1 as

$$H(\pi_{p+q,n}) \cong \left\{ \begin{pmatrix} Z_1^{(p)} & Z_{12} & 0 \\ Z_{21} & Z_2^{(q)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_n(\mathbb{C}) \right\},$$

$$H(\pi_{q,n}) \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & Z_2^{(q)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_n(\mathbb{C}) \right\}.$$

Then we have

$$V_1 = V(\pi_{q,n}) \cap H(\pi_{p+q,n}) \cong \left\{ \begin{pmatrix} Z_1^{(p)} & Z_{12} & 0 \\ Z_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_n(\mathbb{C}) \right\}.$$

The action of l on $T_e\mathcal{D}$ is expressed as $l : v \mapsto L(-v)L$ ($v \in T_{i1_n}\mathfrak{H}_n$). Therefore, we have

$$lV_1 \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Z_{21} \\ 0 & Z_{12} & Z_1^{(p)} \end{pmatrix} \in \mathrm{Sym}_n(\mathbb{C}) \right\}.$$

It follows that $lV_1 \subset H(\pi_{p+q,n})^\perp = V(\pi_{p+q,n})$.
3. Let $\tilde{V}_2$ be a subspace of $V(\pi_{p+q,n})$ such that

$$V(\pi_{p+q,n}) = lV_1 \oplus \tilde{V}_2, \quad \tilde{V}_2 \subset (lV_1)^\perp. \tag{4.1.7}$$

Then we have the orthogonal decomposition

$$T_e\mathcal{D} = H(\pi_{q,n}) \oplus V_1 \oplus lV_1 \oplus \tilde{V}_2.$$

$\tilde{V}_2$ is the orthogonal complement of the subspace $H(\pi_{q,n}) \oplus V_1 \oplus lV_1$, so $\tilde{V}_2$ is invariant under the action of l. Now we see that

$$V(l\pi_{p+q,n}l^{-1}) = lV(\pi_{p+q,n}) = l(lV_1 \oplus \tilde{V}_2) = V_1 \oplus \tilde{V}_2. \tag{4.1.8}$$

Combining (4.1.7) and (4.1.8), we get $\tilde{V}_2 = V(\pi_{p+q,n}) \cap V(l\pi_{p+q,n}l^{-1}) = V_2$. $\square$

Lemma 4.1.7. *1. $T_e\mathcal{D}^l$ admits the orthogonal decomposition*

$$T_e\mathcal{D}^l = (l + 1_n)V_1 \oplus (l + 1_n)V_2.$$

Here $(l + 1_n) : T_e\mathcal{D} \to T_e\mathcal{D} : v \mapsto lv + v$.

2. $\ker d\pi_{p+q,n}|_{\mathcal{D}^l} = (l + 1_n)V_2$.

3. The differential $d\pi_{p+q,n}|_{\mathcal{D}^l} : T_e\mathcal{D}^l \to T_{e_{p+q}}\mathcal{D}_e^{(p+q)}$ is surjective.

4. The linear isomorphism $\frac{1}{2}d\pi_{p+q,n} : (l + 1_n)V_1 \cong T_{e_{p+q}}\mathcal{D}_e^{(p+q)}$ is an isometry.

Proof. 1. We see that

$$T_e\mathcal{D}^l = \{v \in T_e\mathcal{D};\ lv = v\} = (l + 1_n)T_e\mathcal{D}.$$

By Lemma 4.1.6, we have

$$T_e\mathcal{D}^l = (l + 1_n)V_1 \oplus (l + 1_n)V_2.$$

This is obviously an orthogonal decomposition.
2. It follows from $V_1 \subset H(\pi_{p+q,n})$, $lV_1 \subset V(\pi_{p+q,n})$ and $(l + 1_n)V_2 \subset V_2 \subset V(\pi_{p+q,n})$.
3. The rank of the linear map $d\pi_{p+q,n}|_{\mathcal{D}_e} : T_e\mathcal{D}^l \to T_{e_{p+q}}\mathcal{D}_e^{(p+q)}$ is

$$\dim(T_e\mathcal{D}^l / \ker d\pi_{p+q,n}|_{\mathcal{D}^l}) = \dim(l + 1_n)V_1 = \dim V_1.$$

Since $H(\pi_{p+q,n}) = H(\pi_{q,n}) \oplus V_1$, we see that

$$\begin{aligned} \dim V_1 &= \dim H(\pi_{p+q,n}) - \dim H(\pi_{q,n}) \\ &= \dim \mathcal{F}_{p+q} - \dim \mathcal{G}_q \\ &= \dim \pi_{q,p+q}^{-1}(e) = \dim \mathcal{D}_e^{(p+q)}. \end{aligned}$$

Then the rank of $d\pi_{p+q,n}|_{\mathcal{D}_e}$ is equal to $\dim T_{e_{p+q}}\mathcal{D}_e^{(p+q)}$. This implies that $d\pi_{p+q,n}|_{\mathcal{D}_e}$ is surjective.

4. It follows from $V_1 \subset H(\pi_{p+q,n}), lV_1 \subset V(\pi_{p+q,n})$. $\square$

We derive the similar decomposition of $T_e\mathcal{S}^l$ as Lemma 4.1.7. Let us define a map $\iota_n : T_e\mathcal{D} \to T_e\mathcal{D}$ by inducing the complex conjugation

$$z = x + iy \mapsto \overline{z} = x - iy \quad (z \in \mathrm{Sym}_n(\mathbb{C}))$$

through $d\psi_1 : T_e\mathcal{D} \cong \mathrm{Sym}_n(\mathbb{C})$. We also define $\iota_{p+q} : T_e\mathcal{F}_{p+q} \to T_e\mathcal{F}_{p+q}$ and $\iota_q : T_e\mathcal{G}_q \to T_e\mathcal{G}_q$ by inducing the complex conjugation through $d\varphi_{p+q}$ and $d\phi_q$.

Lemma 4.1.8. *ι_n is isometric on $T_e\mathcal{D}$. In addition, we have the following equations of differentials on $T_e\mathcal{D}$.*

1. $\iota_n^2 = \mathrm{id}$.
2. $\iota_{p+q} \circ d\pi_{p+q,n} = d\pi_{p+q,n} \circ \iota_n, \quad \iota_q \circ d\pi_{q,n} = d\pi_{q,n} \circ \iota_n$.
3. $\iota_n \circ l = l \circ \iota_n$.

Proof. By the definition, it is obvious that ι_n is isometric and $\iota_n^2 = \mathrm{id}$. Since $d\pi_{p+q,n} = d\pi_{p+q}$ is expressed as (4.1.3), we verify the first equation of *2*. By (3.2.1), we see that $d\pi_{q,n}$ is expressed as

$$\mathrm{Sym}_n(\mathbb{C}) \ni \begin{pmatrix} Z_1^{(p)} & Z_{12} & Z_{13} \\ Z_{21} & Z_2^{(q)} & Z_{23} \\ Z_{31} & Z_{32} & Z_3^{(p)} \end{pmatrix} \mapsto Z_2 \in \mathrm{Sym}_q(\mathbb{C}).$$

Then we verify the second equation of *2*. For *3*, we see that the map $l : v \mapsto L(-v)L$ ($v \in \mathrm{Sym}_n(\mathbb{C})$) is obviously commutative with ι_n. $\square$

Lemma 4.1.9. *ι_n preserves the subspaces V_1, V_2. ι_{p+q} preserves the subspace $T_e\mathcal{D}_e^{(p+q)}$.*

Proof. By Lemma 4.1.8, we see that ι_n preserves the subspaces $H(\pi), V(\pi)$ for $\pi = \pi_{p+q,n}, \pi_{q,n}$. Then ι_n also preserves V_1 and V_2. By the isomorphism $d\pi_{p+q,n} : (l + 1_n)V_1 \cong T_e\mathcal{D}_e^{(p+q)}$ as in Lemma 4.1.7, we see that ι_{p+q} preserves the subspace $T_e\mathcal{D}_e^{(p+q)}$. $\square$

Lemma 4.1.10.

$$T_e\mathcal{S}^l = (1_n - \iota_n)T_e\mathcal{D}^l,$$
$$T_e\mathcal{S}_e^{(p+q)} = (1_n - \iota_n)T_e\mathcal{D}_e^{(p+q)}.$$

Proof. We see that

$$T_e\mathcal{S} = (1_n - \iota_n)T_e\mathcal{D}, \quad T_e\mathcal{S}^{(p+q)} = (1_n - \iota_n)T_e\mathcal{F}_{p+q}.$$

By the definition $\mathcal{S}^l = \mathcal{D}^l \cap \mathcal{S}$, we have

$$T_e\mathcal{S}^l = T_e\mathcal{D}^l \cap T_e\mathcal{S} = (l + 1_n)T_e\mathcal{D} \cap (1_n - \iota_n)T_e\mathcal{D}.$$

By the commutativity between l and ι_n, we have

$$T_e\mathcal{S}^l = (1_n - \iota_n)(l + 1_n)T_e\mathcal{D} = (1_n - \iota_n)T_e\mathcal{D}^l.$$

Since ι_n preserves $T_e\mathcal{D}_e^{(p+q)}$, we see that $T_e\mathcal{S}_e^{(p+q)} = (1_n - \iota_{p+q})T_e\mathcal{D}_e^{(p+q)}$. $\square$

Lemma 4.1.11. *1. $T_e\mathcal{S}^l$ admits the orthogonal decomposition*

$$T_e\mathcal{S}^l = U_1 \oplus U_2, \quad U_i = (1_n - \iota_n)(l + 1_n)V_i \quad (i = 1, 2).$$

2. $\ker d\pi_{p+q,n}|_{\mathcal{S}^l} = U_2$.

3. $\frac{1}{2} d\pi_{p+q,n}|_{U_1} : U_1 \cong T_{e_{p+q}} \mathcal{S}_e^{(p+q)}$ *is an isometry.*

Proof. By applying $(1_n - \iota_n)$ to both sides of the decomposition in Lemma 4.1.7, we obtain

$$T_e \mathcal{S}^l = U_1 + U_2.$$

Since ι_n commutes with l and preserves V_1, V_2, we see that $U_i \subset V_i$ $(i = 1, 2)$. This implies that $U_1 + U_2$ is orthogonal decomposition and $\ker d\pi_{p+q,n}|_{\mathcal{S}^l} = U_2$. By the commutative diagram

$$\begin{tikzcd} T_e\mathcal{D}^l \arrow[r, two heads, "d\pi_{p+q,n}"] \arrow[d, two heads, "(1-\iota_n)"'] \arrow[dr, phantom, "\circlearrowleft"] & T_e\mathcal{D}_e^{(p+q)} \arrow[d, two heads, "(1-\iota_{p+q})"] \\ T_e\mathcal{S}^l \arrow[r, "d\pi_{p+q,n}"'] & T_e\mathcal{S}_e^{(p+q)}, \end{tikzcd}$$

we see that $d\pi_{p+q,n}(T_e\mathcal{S}^l) = T_e\mathcal{S}_e^{(p+q)}$. Then, $d\pi_{p+q,n}$ gives the linear isomorphism $U_1 \cong T_e\mathcal{S}_e^{(p+q)}$. By Lemma 4.1.7, this is isometric. ☐

Theorem 4.1.12. *We denote by g (resp. h) the induced metric on $\mathcal{S}^l$ (resp. $\mathcal{S}_e^{(p+q)}$) from the metric on $\mathcal{D}$ (resp. $\mathcal{F}_{p+q}$). Then, the map $\pi_{p+q,n}|_{\mathcal{S}^l} : (\mathcal{S}^l, \frac{1}{2}g) \to (\mathcal{S}_e^{(p+q)}, h)$ is a Riemannian submersion.*

Proof. We can apply Lemma 4.1.3 to $\pi_{p+q,n}|_{\mathcal{S}^l} : \mathcal{S}^l \to \mathcal{S}_e^{(p+q)}$ as $f : M \to N$ and $G = T_R^l$. ☐

4.2 Description of Theorem 1.0.1

Let $J = \begin{pmatrix} 0 & 1_{p+q} \\ -1_{p+q} & 0 \end{pmatrix} \in \mathrm{Sp}(p+q, \mathbb{R})$. We consider the sequence

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{\pi_{p+q,n}} & \mathcal{S}^{(p+q)} & \xrightarrow{J} & \mathcal{S}^{(p+q)} \\ \cup & & \cup & & \cup \\ \mathcal{S}^l & \longrightarrow & \mathcal{S}_e^{(p+q)} & \longrightarrow & J\mathcal{S}_e^{(p+q)}. \end{array}$$

For these manifolds, we use the charts $i^{-1}\psi_1$ and $i^{-1}\varphi_{p+q}$, that give the diffeomorphisms

$$i^{-1}\psi_1 : \mathcal{S} \xrightarrow{\sim} \mathrm{Sym}_n^+(\mathbb{R}), \quad i^{-1}\varphi_{p+q} : \mathcal{S}^{(p+q)} \xrightarrow{\sim} \mathrm{Sym}_{p+q}^+(\mathbb{R}).$$

Each manifold is expressed as follows.

$$i^{-1}\psi_1 : \mathcal{S}^l \xrightarrow{\sim} \left\{ y \in \mathrm{Sym}_n^+(\mathbb{R});\ LyL = y^{-1} \right\} =: N,$$

$$i^{-1}\varphi_{p+q} : \mathcal{S}_e^{(p+q)} \xrightarrow{\sim} \left\{ y \in \mathrm{Sym}_{p+q}^+(\mathbb{R});\ y^{-1} = \begin{pmatrix} y_1^{(p)} & y_{12} \\ y_{21} & 1_q \end{pmatrix} \right\} =: N_{p,q},$$

$$i^{-1}\varphi_{p+q} : J\mathcal{S}_e^{(p+q)} \xrightarrow{\sim} \left\{ y \in \mathrm{Sym}_{p+q}^+(\mathbb{R});\ y = \begin{pmatrix} y_1^{(p)} & y_{12} \\ y_{21} & 1_q \end{pmatrix} \right\} =: JN_{p,q}.$$

We induce the metrics on these manifolds from $\mathcal{D}$ and $\mathcal{F}_{p+q}$. In particular, the metric on $\mathcal{S}^{(p+q)}$ is expressed on $\mathrm{Sym}^+_{p+q}(\mathbb{R})$ as

$$g_y(dy, dy) = \frac{1}{2}\operatorname{tr}((y^{-1}dy)^2) \quad (y \in \mathrm{Sym}^+_{p+q}(\mathbb{R})).$$

The metric on $\mathcal{S}_e^{(p+q)} \cong N_{p,q}$ is the further restriction of this metric.

Here we recall the statistical manifold N_p mentioned in Chapter 1. Let N_p be defined by (1.0.2).

Proposition 4.2.1. *$N_{p,1}$, $JN_{p,1}$ and N_p are isomorhic as Riemannian manifolds.*

Proof. We see that $JN_{p,1}$ coincides with N_p as a subset of $\mathrm{Sym}^+_{p+1}(\mathbb{R})$. They are equipped with the same induced metric from $\mathrm{Sym}^+_{p+1}(\mathbb{R})$. Then, we have $JN_{p,1} \cong N_p$ as Riemannian manifolds. Since the action of J is isomorphic, we also have $N_{p,1} \cong JN_{p,1}$. □

Remark 4.2.1. $\mathcal{S}_e^{(p+q)}$ is regarded as the imaginary part of the Siegel domain $\mathcal{D}_e$. Indeed, the manifold N_p is realized in the form of so-called a real Siegel domain in [Shi01]. □

We put $\pi := (i^{-1}\varphi_{p+q}) \circ \pi_{p+q,n} \circ (i^{-1}\psi_1)^{-1}$. Then $\pi : N \to N_{p,q}$ is given by

$$\pi : N \ni \begin{pmatrix} y_1^{(p)} & y_{12} & y_{13} \\ y_{21} & y_2^{(q)} & y_{23} \\ y_{31} & y_{32} & y_3^{(q)} \end{pmatrix} \mapsto \begin{pmatrix} y_1 & y_{12} \\ y_{21} & y_2 \end{pmatrix} \in N_{p,q}. \tag{4.2.1}$$

The action of J is expressed as

$$J : N_{p,q} \ni y = \begin{pmatrix} y_1 & -y_1 y_{12} \\ -y_{21}y_1 & 1_q + y_{21}y_1y_{12} \end{pmatrix} \mapsto y^{-1} = \begin{pmatrix} y_1^{-1} + y_{12}y_{21} & y_{12} \\ y_{21} & 1_q \end{pmatrix} \in JN_{p,q}. \tag{4.2.2}$$

The decomposition $T_e\mathcal{S}^l = U_1 \oplus U_2$ in Lemma 4.1.11 is expressed through $d(i^{-1}\psi_1)$ as

$$\begin{aligned} T_{1_n}N &= d(i^{-1}\psi_1)(U_1) \oplus d(i^{-1}\psi_1)(U_2) \\ &= \left\{ \begin{pmatrix} B & x & 0 \\ {}^tx & 0_q & -{}^tx \\ 0 & -x & -B \end{pmatrix} ;\ B = B^{(p)} = {}^tB \right\} \oplus \left\{ \begin{pmatrix} 0_p & 0 & D \\ 0 & 0_q & 0 \\ {}^tD & 0 & 0_p \end{pmatrix} ;\ D = D^{(p)} = -{}^tD \right\}. \end{aligned}$$

Now we see the following correspondence.

Theorem 1.0.1	our setting
N_p	$JN_{p,1}$
A	a vector in $d(i^{-1}\psi_1)(U_1)$
$\Lambda(t)$	a curve on N
$\Lambda \mapsto (\Delta, \delta)$	the map $\pi : N \to N_{p,1}$
$(\Delta, \delta) \mapsto (\Sigma, -\mu)$	the map $J : N_{p,1} \to JN_{p,1}$

Proposition 4.2.2. *For any $A \in d(i^{-1}\psi_1)(U_1)$, define a curve $\Lambda(t)$ on N by*

$$\Lambda(t) = \begin{pmatrix} \Delta & \delta & \Phi \\ {}^t\delta & \epsilon & {}^t\gamma \\ {}^t\Phi & \gamma & \Gamma \end{pmatrix} := \exp(tA) \quad (t \in \mathbb{R}).$$

Then, $\Lambda(t)$ is a geodesic on N, and the curve $c(t) := \pi(\Lambda(t))$ is a geodesic on $N_{p,q}$. Moreover, any geodesic on $N_{p,q}$ through 1_{p+q} is given in this way.

Theorem 1.0.1 follows from this proposition in the case of $q = 1$.

Lemma 4.2.3. *The geodesic equation on $N_{p,q}$ is written as*

$$\ddot{y}(t) - \dot{y}(t)y(t)^{-1}\dot{y}(t) - \begin{pmatrix} 0_p & 0 \\ 0 & \lambda(t) \end{pmatrix} = 0, \tag{4.2.3}$$

for a curve $y(t)$ on $N_{p,q}$. Here $\lambda(t) = \lambda^{(q)}(t)$ is any curve on $\mathrm{Sym}_q(\mathbb{R})$.

Proof. The Lagrangian for the geodesic equation on $\mathrm{Sym}^+(\mathbb{R})$ is given by $L_0(y, \dot{y}) = \frac{1}{2}\operatorname{tr}((y^{-1}\dot{y})^2)$. We consider the constraint of $N_{p,q}$. We see that

$$N_{p,q} = \left\{ y \in \mathrm{Sym}^+_{p+q}(\mathbb{R});\ (y^{-1})_{ij} = \delta_{ij}\ (p+1 \le i, j \le p+q) \right\},$$

where $\delta_{ij} = \pm 1$ is Kroneccker's delta. Then the Lagrangian for the geodesic equation on $N_{p,q}$ is given by

$$\begin{aligned} L(y, \dot{y}) &= L_0(y, \dot{y}) - \sum_{p+1 \le i,j \le p+q} \lambda_{ij}(t)(y^{-1})_{ij} \\ &= \frac{1}{2}\operatorname{tr}((y^{-1}\dot{y})^2) - \operatorname{tr}\left(\begin{pmatrix} 0_p & 0 \\ 0 & \lambda(t) \end{pmatrix} y^{-1} \right). \end{aligned}$$

Here $\lambda(t) = (\lambda_{ij}(t))_{ij} \in \mathrm{Sym}_q(\mathbb{R})$ is the Lagrange multiplier. Using the formula of differential $d(y^{-1}) = -y^{-1}dy\, y^{-1}$, we compute as

$$\begin{aligned} &\frac{\partial}{\partial \dot{y}} L = y^{-1}\dot{y}y^{-1}, \\ &\frac{d}{dt}\frac{\partial}{\partial \dot{y}} L = y^{-1}\ddot{y}y^{-1} - 2y^{-1}\dot{y}y^{-1}\dot{y}y^{-1}, \\ &\frac{\partial}{\partial y} L = -y^{-1}\dot{y}y^{-1}\dot{y}y^{-1} + y^{-1}\begin{pmatrix} 0_p & 0 \\ 0 & \lambda(t) \end{pmatrix} y^{-1}. \end{aligned}$$

Then, the Euler-Lagrange equation $\frac{d}{dt}\frac{\partial}{\partial \dot{y}} L - \frac{\partial}{\partial y} L = 0$ is written as

$$\ddot{y} - \dot{y}y^{-1}\dot{y} - \begin{pmatrix} 0_p & 0 \\ 0 & \lambda(t) \end{pmatrix} = 0.$$

This is the geodesic equation on $N_{p,q}$. $\square$

Proof of Proposition 4.2.2. Here we prove Proposition 4.2.2. Let $A \in d(i^{-1}\psi_1)U_1$ be an element given by

$$A = \begin{pmatrix} B & x & 0 \\ {}^t x & 0_q & -{}^t x \\ 0 & -x & -B \end{pmatrix}, \quad B \in \mathrm{Sym}_p(\mathbb{R}).$$

We can write the matrix $\Lambda(t) = e^{tA}$ in the form

$$\Lambda(t) = \begin{pmatrix} 1_{p+q} & 0 \\ {}^t b(t) & 1_q \end{pmatrix} \begin{pmatrix} c(t) & 0 \\ 0 & d^{(q)}(t) \end{pmatrix} \begin{pmatrix} 1_{p+q} & b(t) \\ 0 & 1_q \end{pmatrix}.$$

We decompose it as

$$\Lambda(t) = \begin{pmatrix} 1_{p+q} & 0 \\ {}^t b(t) & 1_q \end{pmatrix} \begin{pmatrix} c(t) & 0 \\ 0 & 0_q \end{pmatrix} \begin{pmatrix} 1_{p+q} & b(t) \\ 0 & 1_q \end{pmatrix} + \begin{pmatrix} 0_{p+q} & 0 \\ 0 & d(t) \end{pmatrix}.$$

Then, we have

$$\Lambda(t)^{-1} = \Lambda(t)^{-1}\Lambda(t)\Lambda(t)^{-1} = \begin{pmatrix} c(t)^{-1} & 0 \\ 0 & 0_q \end{pmatrix} + \Lambda^{-1} \begin{pmatrix} 0_{p+q} & 0 \\ 0 & d(t) \end{pmatrix} \Lambda^{-1}. \tag{4.2.4}$$

Now, one can derive the geodesic equation on N,

$$\ddot{\Lambda} - \dot{\Lambda}\Lambda^{-1}\dot{\Lambda} = 0,$$

from the Lagrangian $L_0(\Lambda, \dot{\Lambda}) = \mathrm{tr}((\Lambda^{-1}\dot{\Lambda})^2)$. $\Lambda(t) = e^{tA}$ obviously satisfies this equation, then it is a geodesic on N. Combining this equation, $\dot{\Lambda}\Lambda^{-1} = A$ and (4.2.4), we have

$$\ddot{\Lambda} - \dot{\Lambda} \begin{pmatrix} c(t)^{-1} & 0 \\ 0 & 0_q \end{pmatrix} \dot{\Lambda} - A \begin{pmatrix} 0_{p+q} & 0 \\ 0 & d(t) \end{pmatrix} A = 0_n. \tag{4.2.5}$$

Applying the map (4.2.1) to both sides, we obtain

$$\ddot{c}(t) - \dot{c}(t)c(t)^{-1}\dot{c}(t) - \begin{pmatrix} 0_p & 0 \\ 0 & \lambda(t) \end{pmatrix} = 0_{p+q}, \quad \lambda(t) = {}^t\!x d(t) x.$$

Therefore, $c(t)$ satisfies the geodesic equation (4.2.3).

For the last assertion, let $c(t)$ be any geodesic on $N_{p,q}$. We can assume that $c(0) = 1_{p+q}$. Put $v := \dot{c}(0)$. Since $\pi_{p+q,n}$, or equivalently π, is a Riemannian submersion, we can find $A \in d(i^{-1}\psi_1)(U_1)$ such that

$$v = d\pi(A).$$

As we saw, the curve $c_0(t) := \pi(e^{tA})$ is a geodesic on $N_{p,q}$. We see that

$$c_0(0) = 1_{p+q}, \quad \dot{c}_0(0) = d\pi(A) = v.$$

Then $c_0(t)$ satisfies the same initial condition as $c(t)$. By the uniqueness of geodesic, it holds that $c_0(t) = c(t)$. □

Theorem 4.2.4. *Any geodesic $c(t)$ on $\mathcal{S}_e^{(p+q)}$ is given by $c(t) = \pi_{p+q,n}(\tilde{c}(t))$ with a geodesic $\tilde{c}(t)$ on $\mathcal{S}^l$.*

Proof. Since the group T_R^l acts on $\mathcal{S}_e^{(p+q)}$ transitively and $\pi_{p+q,n}$ is T_R^l-invariant, it is sufficient to consider the geodesic $c(t)$ such that $c(0) = e_{p+q}$. Then the assertion follows from Proposition 4.2.2. □

Theorem 4.2.5. $\pi_{p+q,n}(\mathcal{S}^l) = \mathcal{S}_e^{(p+q)}$.

Proof. We claim that $\mathcal{S}_e^{(p+q)}$ is a complete Riemannian manifold. Let $c_0(t)$ $(t \in (-\varepsilon, \varepsilon))$ be any geodesic on $\mathcal{S}_e^{(p+q)}$. By Theorem 4.2.4, $c_0(t)$ is an image by $\pi_{p+q,n}$ of a geodesic on $\mathcal{S}^l$, and it is defined for all $t \in \mathbb{R}$. Then, $c_0(t)$ is extended to all $t \in \mathbb{R}$. This means that $\mathcal{S}_e^{(p+q)}$ is complete. We have already seen that $\pi_{p+q,n}(\mathcal{S}^l) \subset \mathcal{S}_e^{(p+q)}$ in Section 3.2, 3.3. So we need to show $\supset$. Fix a point $x \in \mathcal{S}^l$. Let $y \in \mathcal{S}_e^{(p+q)}$ be any point. Since $\mathcal{S}_e^{(p+q)}$ is complete, y is connected with $\pi_{p+q,n}(x) \in \mathcal{S}_e^{(p+q)}$ by a geodesic $c(t)$. We may suppose that $c(0) = \pi_{p+q,n}(x)$, $c(t_0) = y$ with $t_0 \in \mathbb{R}$. By Theorem 4.2.4, there is a geodesic $\tilde{c}(t)$ such that $\pi_{p+q,n}(\tilde{c}(t)) = c(t)$ $(t \in \mathbb{R})$. Then, $\pi_{p+q,n}(\tilde{c}(t_0)) = c(t_0) = y$. This implies that $\pi_{p+q,n}(\mathcal{S}^l) = \mathcal{S}_e^{(p+q)}$. □

We can restate Theorem 1.0.1, Proposition 4.2.2 and Theorem 4.2.4 as follows.

Theorem 4.2.6. *For any geodesic $c(t)$ on $\mathcal{S}^l$, if the tangent vector $\dot{c}(t_0)$ is horizontal at a $t_0 \in \mathbb{R}$, then it is horizontal at every $t \in \mathbb{R}$.*

Proof. Suppose that $c(t)$ is a geodesic on $\mathcal{S}^l$ and $\dot{c}(t_0)$ is horizontal at a $t_0 \in \mathbb{R}$. We can assume that $t_0 = 0$. By the action of T_R^l, we can assume that $c(0) = e_n$. By Proposition 4.2.2, $\pi_{p+q,n}(c(t))$ is a geodesic on $\mathcal{S}_e^{(p+q)}$. By Proposition 4.1.2, $\dot{c}(t)$ is horizontal at every $t \in \mathbb{R}$. □

4.3 Conclusion

In the Siegel upper half space $\mathcal{D} \cong \mathfrak{H}_n$, we defined a symmetric subdomain $\mathcal{D}^l$, and investigated $\mathcal{D}^l$ in the relation with the boundary of $\mathcal{D}$. As a result, we found that

$$\mathcal{D}^l \subset \mathcal{D}_e, \quad \pi_{p+q,n}(\mathcal{S}^l) = \mathcal{S}_e^{(p+q)}.$$

To obtain the second relation, we used Proposition 4.2.2, which is a generalization of Theorem 1.0.1. Proposition 4.2.2 is restated as a property of geodesics on $\mathcal{S}^l$ as Theorem 4.2.6. Therefore, we conclude that the significance of Theorem 1.0.1 is naturally understood in the process of investigating the arrangement of the subdomain $\mathcal{D}^l$.

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