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Zeros of certain weakly holomorphic modular forms and their transcendence

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Contents

1 Introduction

Modular forms and zeta functions often appear in number theory. It is important to study locations of zeros for such complex-valued functions. For the Eisenstein series $E_k(z)$, which is perhaps the easiest example of modular forms, a great deal is known about the locations of the zeros. In the 1960s, Wohlfahrt [14] proved that all zeros of the Eisenstein series $E_k(z)$ in the standard fundamental domain for $SL_2(\mathbb{Z})$ lie on the arc for even $4 \leq k \leq 26$. In 1970, F.K.C. Rankin and Swinnerton-Dyer [11] showed this result for all weight $k \geq 4$. In 2007, Miezaki, Nozaki and Shigezumi [10] proved similar results for Eisenstein series for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$. Shigezumi showed similar results for Eisenstein series for $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$ in 2007 [12], and for Poincaré series for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$ in 2010 [13].

In 2008, Duke and Jenkins [2] constructed a canonical basis $f_{k,m}$ for the space of weakly holomorphic modular forms for level 1. Let Δ be the Ramanujan Δ function and j be weight 0 modular function which is known simply as the j-function. The basis $f_{k,m}$ is defined by

$$f_{k,m} = \Delta^\ell E_{k'} F_{k,D}(j)$$

where $k = 12\ell + k'$, $k' \in \{0, 4, 6, 8, 10, 14\}$ and $F_{k,D}(x)$ is a monic polynomial in x of degree $D = \ell + m$. The basis $f_{k,m}$ have the following form

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}).$$

They considered $f_{k,m}$ as a two-parameter family of weakly holomorphic modular forms that is a canonical basis for the space and proved almost all of the basis elements have all of their zeros on a lower boundary of the standard fundamental domain for $SL_2(\mathbb{Z})$. Similar results have been obtained for level 2 and 3 by Garthwaite and Jenkins in 2013 [4], and for level 4 by Haddock and Jenkins in 2014 [6].

In 2004, J. Getz [5] generalized Rankin's theorem [11], providing conditions under which the zeros of other modular forms lie only on the arc. For weakly holomorphic modular forms of level 1, we have the information of the location of the zeros for the canonical basis $f_{k,m}$ by Jenkins's theorem [2]. However, we do not know where the zeros of general weakly holomorphic modular forms exist. In this paper, we consider the locations of the zeros for weakly holomorphic modular forms $g_{k,m}$ of level 1 defined by

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j,$$

where $a_j \in \mathbb{R}$. The result is given by the following theorem.

Main Theorem. *(Theorem 2.1)*
Let $a_j \in \mathbb{R}, m \geq 0$ and $\ell + m \geq 1$. We define a weakly holomorphic form $g_{k,m}(z)$ as above. If $\{a_j\}_{j=1}^{\ell+m}$ satisfy a certain assumption (see p.8), then all of the zeros of $g_{k,m}$ in the fundamental domain for $SL_2(\mathbb{Z})$ lie on the circle $|z| = 1$.

Besides, we prove the transcendence of their zeros when the coefficients a_j are rational numbers and satisfy the assumption (Theorem 3.1 and Corollary 3.1). Finally, we show similar results for the locations of the zeros of weakly holomorphic modular forms of level 2 (Theorem 4.2, Theorem 4.3 and Theorem 4.4).

This paper is organized as follows. In Section 2, we recall the definition of weakly holomorphic modular forms and prove the location of the zeros for level 1 case. In Subsection 2.3, we show the main theorem without detailed computations of the numerical bounds for weakly holomorphic modular forms. The proof of numerical bounds is given in Subsection 2.4. In Section 3, we recall some basic facts for imaginary quadratic fields and consider the transcendence of zeros for level 1. In Section 4, similar results are obtained for the locations of the zeros of weakly holomorphic modular forms of level 2. In Subsection 4.1, we construct a canonical basis for level 2 and give the statements for several cases. We prove the theorems in Subsection 4.2, 4.3 and 4.4 for each case.

2 Zeros of weakly holomorphic modular forms of level 1

We recall the definition of weakly holomorphic modular forms and study the location of their zeros in this section.

2.1 Definitions and statement of a result

Let $k \in 2\mathbb{Z}$, N be a prime number or 1, and $\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod{N} \right\}$. Put $\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}$, and $q = e^{2\pi i z}$ for $z \in \mathbb{H}$.

A holomorphic function f on $\mathbb{H}$ is a weakly holomorphic modular form of weight k with respect to $\Gamma_0(N)$ if f satisfies the following two conditions:

- $f\Big(\dfrac{az+b}{cz+d}\Big) = (cz+d)^k f(z) \quad \text{for any } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N).$

- $f(z) = \displaystyle\sum_{n \geq n_0} a(n) q^n \text{ and } \frac{1}{z^k} f\Big(-\frac{1}{z}\Big) = \sum_{n \geq n_1} b(n) q^{\frac{n}{N}}$

  with $a(n_0) \neq 0$ and $b(n_1) \neq 0$.

We define f is holomorphic if $n_0 \geq 0$ and $n_1 \geq 0$, a cusp form if $n_0 \geq 1$ and $n_1 \geq 1$. We denote the space of holomorphic modular form of weight k on $\Gamma_0(N)$ by $M_k(N)$, the space of weakly holomorphic modular forms by $M_k^!(N)$. Put $M_k = M_k(1)$ and $M_k^! = M_k^!(1)$ in this paper.

Duke and Jenkins considered an explicit basis of $M_k^!$ which is indexed by the order of the pole at ∞ in [2]. Let $k = 12\ell + k'$ where $\ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. For any integer $m \geq -\ell$, there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!$ which has an expansion

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}). \tag{1}$$

For any $f = \sum a(n) q^n \in M_k^!$, we can write

$$f = \sum_{n_0 \leq n \leq \ell} a(n) f_{k,-n}$$

when we know first few Fourier coefficients of f. Therefore we see that $\{f_{k,m}\}_{m \geq -\ell}$ form a natural basis of $M_k^!$.

We define three modular forms to construct the basis $\{f_{k,m}\}_{m\geq -\ell}$. Bernoulli numbers B_k and $\sigma_{k-1}(n)$ are each defined by

$$\frac{x}{e^x-1}=\sum_{j=0}^{\infty}B_j\frac{x^j}{j!},\qquad \sigma_{k-1}(n)=\sum_{d|n}d^{k-1}.$$

Then Ramanujan Δ function, Eisenstein series E_k and j function are each defined by

$$\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n,$$

$$E_k(z)=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\quad (k\geq 4),\ E_0=1,$$

$$j(z)=\frac{E_4(z)^3}{\Delta(z)}=q^{-1}+744+\sum_{n\geq 1}c(n)q^n.$$

Their weights are each 12, k, 0 and orders at ∞ are each 1, 0, -1. The function $f_{k,m}$ is constructed by

$$f_{k,m}=\Delta^{\ell}E_{k'}F_{k,D}(j)$$

where $F_{k,D}(x)$ is a monic polynomial in x of degree $D=\ell+m$.

For the group $SL_2(\mathbb{Z})$, we use a fundamental domain in the upper half-plane bounded by the lines $\Re(z)=-\frac{1}{2}$ and $\Re(z)=\frac{1}{2}$, the circles of radius 1 centered at $z=0$. We include the boundary on the left half of this fundamental domain. The cusps of this fundamental domain can be taken to be at ∞.

For any $f\in M_k^!(f\neq 0)$, the following valence formula holds.

$$\frac{k}{12}=\mathrm{ord}_\infty(f)+\frac{1}{2}\mathrm{ord}_i(f)+\frac{1}{3}\mathrm{ord}_\rho(f)+\sum_{\tau\in\mathcal{F}\setminus\{i,\rho\}}\mathrm{ord}_\tau(f).$$

Here $\rho=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ and $\mathcal{F}$ is the fundamental domain for $SL_2(\mathbb{Z})$. We know that for any $f\in M_k^!$ $(f\neq 0)$, the inequality

$$\mathrm{ord}_\infty(f)\leq \ell$$

holds. Hence we note that there exists a unique $f_{k,m}\in M_k^!$ with the expansion (1).

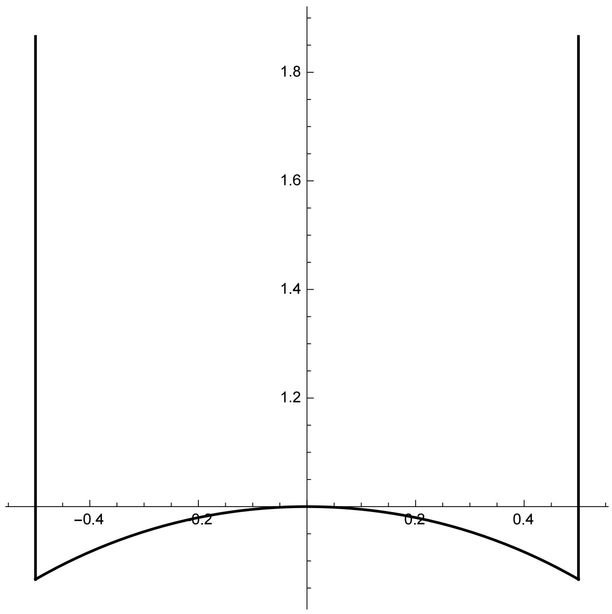

Figure 1: A fundamental domain for $SL_2(\mathbb{Z})$.

The description of the zeros of a weakly holomorphic modular form $f \in M_k^!$ on $\mathbb{H}$ is clearly equivalent to the description of the zeros of f on $\mathcal{F}$. Thus, for the remainder of this paper, when we speak of a zero z_0 of $f \in M_k^!$, we assume $z_0 \in \mathcal{F}$.

We define four constants by $\delta_1 = 0.432207, \delta_2 = 0.024975, \delta_3 = 0.004807$ and $\delta_4 = 0.257348$. Then we define $\gamma(j)$ and $A_{k'}$ by

$$\gamma(j) = \begin{cases} \delta_3^j \delta_1^{\ell-j} & \text{if } 1 \leq j \leq \ell, \\ \delta_2^j \delta_3^\ell & \text{if } \ell+1 \leq j \leq \ell+m. \end{cases} \quad A_{k'} = \begin{cases} 2.76009 & \text{if } k' = 0, \\ 0.684214 & \text{if } k' = 4, \\ 0.950549 & \text{if } k' = 6, \\ 0.184724 & \text{if } k' = 8, \\ 0.258108 & \text{if } k' = 10, \\ 0.075404 & \text{if } k' = 14. \end{cases}$$

We note here, and will prove later in Subsection 2.4, that

$$\begin{aligned}
\left|\frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)}\right| &\leq \delta_1,\\
\left|\Delta\left(x+0.65i\right)\right| &\leq \delta_2,\\
\left|\Delta\left(e^{i\theta}\right)\right| &\leq \delta_3,\\
e^{-2\pi m(\sin\theta-0.65)} &\leq \delta_4
\end{aligned}$$

$$\text{and}\quad \int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{1}{\Delta\left(x+0.65i\right)}\frac{E_{k'}\left(e^{i\theta}\right)E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right)-j\left(e^{i\theta}\right)}\right|dx \leq A_{k'},$$

for $\theta \in [1.9, 2\pi/3]$ and $x \in [-1/2, 1/2]$. Then we have the following theorem.

Theorem 2.1. *Let $k = 12\ell + k'$, where $\ell \in \mathbb{Z}_{\geq 0}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. Let*

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j,$$

where $a_j \in \mathbb{R}, m \geq 0$ and $\ell + m \geq 1$. If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j|(3\delta_3^j + \delta_4^m\gamma(j)^j A_{k'}) < 1 - \delta_4^m\delta_1^\ell A_{k'},$$

then all of the zeros of $g_{k,m}$ in the fundamental domain for $SL_2(\mathbb{Z})$ lie on the circle $|z| = 1$.

2.2 Generating functions and integration

We use the following generating function for $f_{k,m}$ to obtain the integral formula as with [2].

Theorem 2.2. *([2, Theorem 2])*
For any even integer k we have

$$\sum_{m\geq -\ell} f_{k,m}(z)q^m = \frac{f_k(z)f_{2-k}(\tau)}{j(\tau)-j(z)},$$

where $f_k = \Delta^\ell E_{k'}$ with $k = 12\ell + k'$.

This is equivalent to the following lemma.

Lemma 2.1. *([2, Lemma 2])*
We have

$$f_{k,m}(z) = \frac{1}{2\pi i} \oint_C \frac{\Delta^\ell(z) E_{k'}(z) E_{14-k'}(\tau)}{\Delta^{1+\ell}(\tau)\,(j(\tau) - j(z))} q^{-m-1} dq,$$

for C a (counterclockwise) circle centered at 0 in the q-plane with a sufficiently small radius.

Changing variables $q \mapsto \tau$ in the formula of Lemma 2.1 and deforming the resulting contour by Cauchy's theorem gives that for $\alpha > 1$,

$$f_{k,m}(z) = \int_{-\frac{1}{2}+i\alpha}^{\frac{1}{2}+i\alpha} \frac{\Delta^\ell(z)}{\Delta^{1+\ell}(\tau)} \frac{E_{k'}(z) E_{14-k'}(\tau)}{j(\tau) - j(z)} e^{-2\pi i m \tau} d\tau.$$

For brevity, we write

$$G(\tau, z) = \frac{\Delta^\ell(z)}{\Delta^{1+\ell}(\tau)} \frac{E_{k'}(z) E_{14-k'}(\tau)}{j(\tau) - j(z)} e^{-2\pi i m \tau},$$

so that

$$f_{k,m}(z) = \int_{-\frac{1}{2}+i\alpha}^{\frac{1}{2}+i\alpha} G(\tau, z) d\tau.$$

We now assume that $z = e^{i\theta}$ for some $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$, and move the contour of integration downward to a height α'. As we do so, each pole τ_0 of $G(\tau, z)$ in the region defined by

$$-\frac{1}{2} \leq \Re(\tau) < \frac{1}{2} \quad \text{and} \quad \alpha' < \Im(\tau) < \alpha$$

will contribute a term $2\pi i \cdot \mathrm{Res}_{\tau=\tau_0} G(\tau, z)$ to the equation. The poles of $G(\tau, z)$ occur only when $\tau = z$ or when τ is equivalent to z under the action of $SL_2(\mathbb{Z})$. In moving the contour, then, the first nonzero contributions occur at $\tau = z = e^{i\theta}$ and $\tau = -1/z = e^{i(\pi-\theta)}$, and these are the only poles for $\sqrt{3}/2 < \alpha' < \alpha$. The residues can be easily calculated using the alternative formula

$$G(\tau, z) = \frac{e^{-2\pi i m \tau}}{-2\pi i} \frac{\Delta^\ell(z) E_{k'}(z)}{\Delta^\ell(\tau) E_{k'}(\tau)} \frac{\frac{d}{d\tau}\,(j(\tau) - j(z))}{j(\tau) - j(z)}.$$

If $\sqrt{3}/2 < \alpha' < \sin\theta$, the result is the equation

$$\int_{-\frac{1}{2}+i\alpha'}^{\frac{1}{2}+i\alpha'} G(\tau, z)d\tau = f_{k,m}(z) - e^{-2\pi imz} - z^{-k}e^{-2\pi im(-1/z)}.$$

We replace z with $e^{i\theta}$ and multiply by $e^{ik\theta/2}e^{-2\pi m\sin\theta}$; simplifying, we find that

$$e^{ik\theta/2}e^{-2\pi m\sin\theta}f_{k,m}\left(e^{i\theta}\right) - 2\cos\left(\frac{k\theta}{2} - 2\pi m\cos\theta\right),$$

which is the quantity we are trying to bound, is equal to

$$e^{ik\theta/2}e^{-2\pi m\sin\theta}\int_{-\frac{1}{2}+i\alpha'}^{\frac{1}{2}+i\alpha'} G(\tau, e^{i\theta})d\tau.$$

As α' decreases, the next nonzero contribution occurs when $\tau = -\frac{1}{z+1}$ or $\tau = \frac{z}{z+1}$. Since these points have real part $-1/2$ and $1/2$, respectively, we add a small circular arc to each of the vertical contours of integration in the usual way. The result is a contribution of

$$\frac{e^{-\pi im}}{(2\cos(\theta/2))^k}e^{-\pi m(2\sin\theta - \tan(\theta/2))}$$

from this pole. However, if θ is close to $\pi/2$, the pole at $-\frac{z}{z-1}$ will be nearby. To avoid this, we choose α' so that the contribution from this pole appears only if θ is not close to $\pi/2$. Specifically, if $1.9 \le \theta < 2\pi/3$, we choose

$$\alpha' = 0.65 < \Im\left(-\frac{1}{e^{i\theta}+1}\right),$$

so that the quantity we are bounding equals

$$\frac{e^{-\pi im}}{(2\cos(\theta/2))^k}e^{-\pi m(2\sin\theta - \tan(\theta/2))} + e^{ik\theta/2}e^{-2\pi m\sin\theta}\int_{-\frac{1}{2}}^{\frac{1}{2}} G(x + 0.65i, e^{i\theta})dx.$$

Alternatively, if $\pi/2 < \theta < 1.9$, we choose

$$\alpha' = 0.75 > \Im\left(-\frac{1}{e^{i\theta}+1}\right),$$

and the quantity we are bounding will equal

$$e^{ik\theta/2}e^{-2\pi m\sin\theta}\int_{-\frac{1}{2}}^{\frac{1}{2}} G(x + 0.75i, e^{i\theta})dx.$$

We deal with these cases separately.

2.3 Proof of Theorem 2.1

Writing $k = 12\ell + k'$, note that k' determines the residue class of k modulo 12. Bearing in mind the valence formula, an examination of the possible values of k' implies that

$$\mathrm{ord}_i(f) \geq \begin{cases} 1 & \text{if } k \equiv 2 \pmod 4, \\ 0 & \text{if } k \equiv 0 \pmod 4, \end{cases}$$

and

$$\mathrm{ord}_\rho(f) \geq \begin{cases} 2 & \text{if } k \equiv 2 \pmod 6, \\ 1 & \text{if } k \equiv 4 \pmod 6, \\ 0 & \text{if } k \equiv 0 \pmod 6. \end{cases}$$

Again applying the valence formula for $k = 12\ell + k'$, there are at most $\ell + m$ zeros on $\mathcal{F} - \{\rho, i\}$. Thus if $g_{k,m} \in M_k^!$ satisfies the hypotheses of Theorem 1.1, then to prove Theorem 1.1 it suffices to demonstrate that $g_{k,m}$ has $\ell + m$ simple zeros in $\{e^{i\theta} : \frac{\pi}{2} < \theta < \frac{2\pi}{3}\}$.

An easy argument [5, Proposition 2.1] shows that for any weakly holomorphic modular form f of weight k with real coefficients, the quantity $e^{ik\theta/2} f(e^{i\theta})$ is real for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. Thus, we approximate $e^{ik\theta/2} g_{k,m}(e^{i\theta})$ by an elementary function having the required number of zeros on the arc.

Suppose $\ell \geq 1$ and $m \geq 1$. Then we set

$$H(\theta) = e^{ik\theta/2} e^{-2\pi m \sin\theta} g_{k,m}\left(e^{i\theta}\right) = H_{0,m}(\theta) + \sum_{j=1}^{\ell+m} a_j e^{12ji\theta/2} \Delta\left(e^{i\theta}\right)^j H_{j,m}(\theta),$$

where $H_{j,m}(\theta) = e^{(k-12j)i\theta/2} e^{-2\pi m \sin\theta} f_{k-12j,m}\left(e^{i\theta}\right)$. Since $\ell \geq 1$ and $m \geq 1$, we write

$$\begin{aligned} H(\theta) = H_{0,m}(\theta) \quad &+ \quad \sum_{j=1}^{\ell} a_j e^{12ji\theta/2} \Delta\left(e^{i\theta}\right)^j H_{j,m}(\theta) \\ &+ \quad \sum_{j=1}^{m} a_{j+\ell} e^{12(j+\ell)i\theta/2} \Delta\left(e^{i\theta}\right)^{j+\ell} H_{j+\ell,m}(\theta). \end{aligned}$$

We define the function $R_{j,m}(\theta)$ for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ by

$$H_{j,m}(\theta) = 2\cos\left(\frac{(k-12j)\theta}{2} - 2\pi m \cos\theta\right) + R_{j,m}(\theta).$$

We seek a bound for the function $R_{j,m}(\theta)$. Details for the computation of the numerical bounds that appear in this subsection are provided in the next subsection. By the argument in Subsection 2.2,

$$\begin{aligned}
|R_{j,m}(\theta)| &= \left| e^{(k-12j)i\theta/2} e^{-2\pi m \sin\theta} f_{k-12j,m}\left(e^{i\theta}\right) - 2\cos\left(\frac{(k-12j)\theta}{2} - 2\pi m\cos\theta\right)\right| \\
&= \left| e^{(k-12j)i\theta/2} e^{-2\pi m \sin\theta} \int_{-\frac{1}{2}+\alpha'}^{\frac{1}{2}+\alpha'} \frac{\Delta^{\ell-j}(z)}{\Delta^{1+\ell-j}(\tau)} \frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)} e^{-2\pi i m\tau} d\tau \right| \\
&= \left| e^{-2\pi m \sin\theta} \int_{-\frac{1}{2}+\alpha'}^{\frac{1}{2}+\alpha'} \frac{\Delta^{\ell-j}(z)}{\Delta^{1+\ell-j}(\tau)} \frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)} e^{-2\pi i m\tau} d\tau \right|.
\end{aligned}$$

When $1.9 \le \theta \le 2\pi/3$, we have

$$|R_{j,m}(\theta)| = \left| \frac{e^{-i\pi m}}{(2\cos(\theta/2))^k} e^{-\pi m(2\sin\theta - \tan(\theta/2))} \right.$$

$$\left. + e^{ik\theta/2} e^{-2\pi m\sin\theta} \int_{-\frac{1}{2}}^{\frac{1}{2}} G_j(x+0.65i, e^{i\theta}) e^{-2\pi i m\tau} dx \right|,$$

where

$$G_j(\tau, z) = \frac{\Delta^{\ell-j}(z)}{\Delta^{1+\ell-j}(\tau)} \frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)}.$$

Similarly, when $\pi/2 \le \theta < 1.9$, we have

$$|R_{j,m}(\theta)| = \left| e^{ik\theta/2} e^{-2\pi m\sin\theta} \int_{-\frac{1}{2}}^{\frac{1}{2}} G_j(x+0.75i, e^{i\theta}) e^{-2\pi i m\tau} dx \right|.$$

Suppose $1.9 \le \theta \le 2\pi/3$. It holds that

$$|R_{j,m}(\theta)| \le \frac{e^{-\pi m(2\sin\theta - \tan(\theta/2))}}{(2\cos(\theta/2))^k} + e^{-2\pi m(\sin\theta - 0.65)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| G_j(x+0.65i, e^{i\theta}) \right| dx.$$

Looking at the first term,

$$1 \le 2\cos(\theta/2) \le \sqrt{2}$$

for $\theta \in [1.9, 2\pi/3]$, and

$$-\pi(2\sin\theta - \tan(\theta/2)) < 0$$

for these θ. Thus for $m \geq 0$, we have

$$\left|\frac{e^{-\pi m(2\sin\theta-\tan(\theta/2))}}{(2\cos(\theta/2))^k}\right| \leq 1.$$

Considering the exponential term $e^{-2\pi m(\sin\theta-0.65)}$, it is bounded above by 0.257348 for $\theta \in [1.9, 2\pi/3]$. We set $\delta_4 = 0.257348$.

We next seek a bound for $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|G_j(x+0.65i, e^{i\theta})\right| dx$. This integral is equal to

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)}\right|^{\ell-j} \left|\frac{1}{\Delta\left(x+0.65i\right)}\right| \left|\frac{E_{k'}\left(e^{i\theta}\right)E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right)-j\left(e^{i\theta}\right)}\right| dx.$$

First, we consider

$$\left|\frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)}\right|^{\ell-j}.$$

From computations in the next subsection, we have

$$0.002691 \leq \left|\Delta\left(e^{i\theta}\right)\right| \leq 0.004807.$$

We set $\delta_3 = 0.004807$. We compute that

$$0.011122 \leq \left|\Delta\left(x+0.65i\right)\right| \leq 0.024975.$$

We set $\delta_2 = 0.024975$. Putting this together, we have, for $\ell \geq j$,

$$\left|\frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)}\right|^{\ell-j} \leq \left|0.432207\right|^{\ell-j}.$$

We set $\delta_1 = 0.432207$.

Next, we consider

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{\Delta\left(x+0.65i\right)}\frac{E_{k'}\left(e^{i\theta}\right)E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right)-j\left(e^{i\theta}\right)}\right| dx.$$

We will break our path of integration into small pieces, and consider $j(\tau)$ in relation to $j(z)$ on each. We can bound the quotient by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{\Delta\left(x+0.65i\right)}\frac{E_{k'}\left(e^{i\theta}\right)E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right)-j\left(e^{i\theta}\right)}\right| dx \leq A_{k'},$$

where

$$A_{k'} = \begin{cases} 2.76009 & \text{if } k' = 0, \\ 0.684214 & \text{if } k' = 4, \\ 0.950549 & \text{if } k' = 6, \\ 0.184724 & \text{if } k' = 8, \\ 0.258108 & \text{if } k' = 10, \\ 0.075404 & \text{if } k' = 14. \end{cases}$$

Putting all of these pieces together, we see that

$$|R_{j,m}(\theta)| \leq 1 + \delta_4^m \delta_1^{\ell-j} A_{k'}$$

for $1 \leq j \leq \ell$ and

$$|R_{j+\ell,m}(\theta)| \leq 1 + \delta_4^m \left| \frac{\delta_2}{\Delta\left(e^{i\theta}\right)} \right|^j A_{k'}$$

for $1 \leq j \leq m$.

Similarly, for $\theta \in [\pi/2, 1.9)$, we have

$$|R_{j,m}(\theta)| \leq e^{-2\pi m(\sin\theta - 0.75)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| G_j(x + 0.75i, e^{i\theta}) \right| dx.$$

$e^{-2\pi m(\sin\theta - 0.75)}$ is bounded above by 0.29131.

It holds that

$$0.00178 \leq \left|\Delta\left(e^{i\theta}\right)\right| \leq 0.00270,$$

and

$$0.00721 \leq |\Delta\left(x + 0.75i\right)| \leq 0.01112.$$

Putting this together, we have, for $\ell \geq j$,

$$\left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x + 0.75i\right)} \right|^{\ell-j} \leq |0.3745|^{\ell-j}.$$

We can bound the quotient by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{1}{\Delta\left(x + 0.75i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x + 0.75i\right)}{j\left(x + 0.75i\right) - j\left(e^{i\theta}\right)} \right| dx \leq A'_{k'},$$

where

$$A'_{k'} = \begin{cases} 2.8039 & \text{if } k' = 0, \\ 1.97763 & \text{if } k' = 4, \\ 1.1423 & \text{if } k' = 6, \\ 1.63148 & \text{if } k' = 8, \\ 0.82393 & \text{if } k' = 10, \\ 0.696154 & \text{if } k' = 14. \end{cases}$$

Thus we have that

$$|R_{j,m}(\theta)| \leq 0.29131^m 0.3745^{\ell-j} A'_{k'}$$

for $1 \leq j \leq \ell$ and

$$|R_{j+\ell,m}(\theta)| \leq 0.29131^m \left| \frac{0.01112}{\Delta\left(e^{i\theta}\right)} \right|^j A'_{k'}$$

for $1 \leq j \leq m$.

We note that the bound of $|R_{j+\ell,m}(\theta)|$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$ is larger than for $\frac{\pi}{2} \leq \theta < 1.9$ since $|R_{j+\ell,m}(\theta)| > 1$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$. Therefore we also use the bound of $|R_{j+\ell,m}(\theta)|$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$ when $\frac{\pi}{2} \leq \theta < 1.9$.

We prove Theorem 2.1 using the bound for $|R_{j,m}(\theta)|$. $H(\theta)$ is written by

$$\begin{aligned} H(\theta) &= H_{0,m}(\theta) + \sum_{j=1}^{\ell} a_j e^{12ji\theta/2} \Delta\left(e^{i\theta}\right)^j H_{j,m}(\theta) \\ &\qquad + \sum_{j=1}^{m} a_{j+\ell} e^{12(j+\ell)i\theta/2} \Delta\left(e^{i\theta}\right)^{j+\ell} H_{j+\ell,m}(\theta) \\ &= 2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right) + R_{0,m}(\theta) \\ &+ \sum_{j=1}^{\ell} a_j e^{12ji\theta/2} \Delta\left(e^{i\theta}\right)^j \left(2\cos\left(\frac{(k-12j)\theta}{2} - 2\pi m\cos\theta\right) + R_{j,m}(\theta)\right) \\ &+ \sum_{j=1}^{m} a_{j+\ell} e^{12(j+\ell)i\theta/2} \Delta\left(e^{i\theta}\right)^{j+\ell} \left(2\cos\left(\frac{(k-12(j+\ell))\theta}{2} - 2\pi m\cos\theta\right)\right. \\ &\qquad\qquad \left. + R_{j+\ell,m}(\theta)\right). \end{aligned}$$

Thus $\left|H(\theta) - 2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)\right|$ is bounded above by

$$\begin{aligned} & |R_{0,m}(\theta)| + \sum_{j=1}^{\ell} |a_j| \left(2 + |R_{j,m}(\theta)|\right) \left|\Delta\left(e^{i\theta}\right)\right|^j \\ & \qquad\qquad + \sum_{j=1}^{m} |a_{j+l}| \left(2 + |R_{j+\ell,m}(\theta)|\right) \left|\Delta\left(e^{i\theta}\right)\right|^{j+\ell} \\ \leq & \quad 1 + \delta_4^m \delta_1^\ell A_{k'} + \sum_{j=1}^{\ell} |a_j| \left(2 + 1 + \delta_4^m \delta_1^{\ell-j} A_{k'}\right) \delta_3^j \\ + & \quad \sum_{j=1}^{m} |a_{j+l}| \left(2 + 1 + \delta_4^m \left|\frac{\delta_2}{\Delta\left(e^{i\theta}\right)}\right|^j A_{k'}\right) \left|\Delta\left(e^{i\theta}\right)\right|^{j+\ell} \\ = & \quad 1 + \delta_4^m \delta_1^\ell A_{k'} + \sum_{j=1}^{\ell+m} |a_j| \left(3\delta_3^j + \delta_4^m \gamma^j(j) A_{k'}\right). \end{aligned}$$

Now suppose

$$\sum_{j=1}^{\ell+m} |a_j| (3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'}.$$

Then we have

$$\left|H(\theta) - 2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)\right| < 2.$$

This inequality is enough to prove the theorem. To see this, note that as θ increases from $\pi/2$ to $2\pi/3$, the quantity

$$\frac{k\theta}{2} - 2\pi m \cos\theta$$

increases from $\pi\left(3\ell + k'/4\right)$ to $\pi\left(3\ell + k'/3 + D\right)$, where $D = \ell + m$, hitting $D + 1$ distinct consecutive integer multiples of π (this is independent of the choice of k'). A short computation shows that if $D \geq |\ell|$, then the quantity $\frac{k\theta}{2} - 2\pi m \cos\theta$ is strictly increasing on this interval. Thus, there are exactly $D + 1$ values of θ in the interval $[\pi/2, 2\pi/3]$ where the function

$$2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)$$

has absolute value 2, alternating between $+2$ and -2 as θ increases. Then real-valued function $H(\theta)$ must have at least D distinct zeros as θ moves through the interval $(\pi/2, 2\pi/3)$. This accounts for all D nontrivial zeros of $g_{k,m}$.

2.4 Details of computing upper and lower bounds

We seek a bound for $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|G_j(x+0.65i, e^{i\theta})\right| dx$. Firstly, we bound

$$\left|\frac{\Delta^{\ell-j}(z)}{\Delta^{\ell-j}(\tau)}\right|.$$

To do this we consider the upper and lower bounds for $\Delta(e^{i\theta})$ and $\Delta(x+0.65i)$. We can write the cusp form Δ in term of Eisenstein series E_4 and E_6 such that

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728}.$$

Here for a modular form f with Fourier series $f = \sum a_f(n)q^n$, we will choose a positive integer N and let $\widetilde{f}$ be the truncation of the Fourier series of f up to and including the q^N term, and we let $R_N f = f - \widetilde{f}$ be the remaining tail of the series. By the definition of Eisenstein series, we have

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n.$$

For $k \geq 1$, we can generously bound $\sigma_k(n) = \sum_{d|n} d^k$ by $n \cdot n^k = n^{k+1}$. If $|e^{2\pi iz}| \leq t$, then we can bound $R_{20}E_4(z)$ by

$$\begin{aligned} |R_{20}E_4(z)| &\leq 240\sum_{n=21}^{\infty} \sigma_3(n)t^n \\ &\leq 240\sum_{n=21}^{30} \sigma_3(n)t^n + 240\sum_{n=31}^{\infty} n^4t^n. \end{aligned}$$

Standard Taylor series methods involving derivatives of the geometric series $(1-x)^{-1} = \sum x^n$ taken at $x = t$ allows us to bound the infinite series. Since we have

$$\sum_{n=31}^{\infty} n^4t^n = \frac{t^4 + 11t^3 + 11t^2 + t}{(1-t)^5} - \sum_{n=1}^{30} n^4t^n,$$

it holds that

$$|R_{20}E_4(z)| \le 240 \sum_{n=21}^{30} \sigma_3(n)t^n + 240 \left(\frac{t^4+11t^3+11t^2+t}{(1-t)^5} - \sum_{n=1}^{30} n^4 t^n \right).$$

The tail $|R_{20}E_4^3(z)|$ is bounded by

$$\begin{aligned} \left|R_{20}E_4^3(z)\right| &= \left| R_{20} \left(\widetilde{E_4}(z) + R_{20}E_4(z) \right)^3 \right| \\ &\le \left| R_{20}\widetilde{E_4}^3(z) \right| + 3\left| \widetilde{E_4}(z) \right|^2 |R_{20}E_4(z)| + 3 \left| \widetilde{E_4}(z) \right| |R_{20}E_4(z)|^2 \\ &\qquad + |R_{20}E_4(z)|^3 . \end{aligned}$$

Therefore we compute explicit bounds on all of these terms for $|q| = e^{-\sqrt{3}\pi}$ and we find $|R_{20}E_4(z)| < 5.491887 \times 10^{-44}$, $\left| R_{20}\widetilde{E_4}^3(z) \right| < 7.905146 \times 10^{-34}$, and $\left| \widetilde{E_4}(z) \right| < 2.081136$. Thus it holds that

$$\left|R_{20}E_4^3(z)\right| < 7.905147 \times 10^{-34}.$$

Similarly we compute the bound for $R_{20}E_6^2(z)$. By the definition of Eisenstein series we have

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n.$$

We can bound $R_{50}E_6(z)$ by

$$\begin{aligned} |R_{20}E_6(z)| &\le 504 \sum_{n=21}^{\infty} \sigma_5(n)t^n \\ &\le 504 \sum_{n=21}^{30} \sigma_5(n)t^n + 504 \sum_{n=31}^{\infty} n^6 t^n \\ &\le 504 \sum_{n=21}^{30} \sigma_5(n)t^n \\ &\qquad +504 \left(\frac{t^6+57t^5+302t^4+302t^3+57t^2+t}{(1-t)^7} - \sum_{n=1}^{30} n^6 t^n \right). \end{aligned}$$

We have

$$\begin{aligned}\left|R_{20}E_6^2(z)\right| &= \left|R_{20}\left(\widetilde{E_6}(z)+R_{20}E_6(z)\right)^2\right| \\ &\leq \left|R_{20}\widetilde{E_6}^2(z)\right| + 2\left|\widetilde{E_6}(z)\right|\left|R_{20}E_6(z)\right| + \left|R_{20}E_6(z)\right|^2.\end{aligned}$$

Therefore we compute explicit bounds on all of these terms for $|e^{2\pi iz}| = e^{-\sqrt{3}\pi}$ and we find $|R_{20}E_6(z)| < 4.911666 \times 10^{-41}$, $\left|R_{20}\widetilde{E_6}^2(z)\right| < 7.905146 \times 10^{-34}$, and $\left|\widetilde{E_6}(z)\right| < 3.506567$. Thus it holds that

$$\left|R_{20}E_6^2(z)\right| < 7.905150 \times 10^{-34}.$$

We can bound $R_{20}\Delta(z)$ by

$$|R_{20}\Delta(z)| \leq \frac{|R_{20}E_4^3(z)| + |R_{20}E_6^2(z)|}{1728} < 9.149478 \times 10^{-37}.$$

We can do similar upper bound calculations for $R_{20}E_4^3(\tau)$, $R_{20}E_6^2(\tau)$ and $R_{20}\Delta(\tau)$, where $\tau = x + 0.65i$ and $-1/2 \leq x \leq 1/2$, using the additional fact that $|e^{2\pi i\tau}| = e^{-1.3\pi}$. Since we have $|R_{20}E_4(\tau)| < 1.335417 \times 10^{-31}$, $\left|R_{20}\widetilde{E_4}^3(\tau)\right| < 1.932892 \times 10^{-21}$, $\left|\widetilde{E_4}(\tau)\right| < 5.687301$, $|R_{20}E_6(\tau)| < 1.195172 \times 10^{-28}$, $\left|R_{20}\widetilde{E_6}^2(\tau)\right| < 1.932889 \times 10^{-21}$, and $\left|\widetilde{E_6}(\tau)\right| < 14.83488$, it holds that

$$\left|R_{20}E_4^3(\tau)\right| < 1.932893 \times 10^{-21},$$

and

$$\left|R_{20}E_6^2(\tau)\right| < 1.932893 \times 10^{-21}.$$

Thus it follows that

$$|R_{20}\Delta(\tau)| \leq \frac{|R_{20}E_4^3(\tau)| + |R_{20}E_6^2(\tau)|}{1728} < 2.237145 \times 10^{-24}.$$

To compute upper and lower bounds for $|\Delta(z)|$, we trivially bound the derivative of $\widetilde{\Delta}(z)$ with respect to θ for $\theta \in [1.9, 2\pi/3]$ by

$$\begin{aligned}\left|\frac{d}{d\theta}\widetilde{\Delta}(z)\right| &= \left|\frac{d}{d\theta}\left(\sum_{n=1}^{20} a(n)q^n\right)\right| = \left|-\sum_{n=1}^{20} 2\pi n e^{i\theta}\cdot a(n)q^n\right| \\ &\leq \sum_{n=1}^{20} 2\pi n\cdot a(n)t^n \leq 0.021938.\end{aligned}$$

If we evaluate $\widetilde{\Delta}(z)$ at the points $\theta = 1.9 + \frac{n}{10000}$ for $0 \le n \le 10000 \cdot (2\pi/3 - 1.9)$, the spacing between the points is small enough that on the entire interval, $\widetilde{\Delta}(z)$ cannot be below $0.021938 \times \frac{1}{20000} = 1.0969 \times 10^{-6}$ less than its minimum value on these points. The minimum value of $\widetilde{\Delta}(z)$ on these points is at least 0.0026913 and the maximum value is at most 0.0048052. Since we have $|R_{20}\Delta(z)| \le 9.149478 \times 10^{-37}$, for $\theta \in [1.9, 2\pi/3]$, it follows that

$$0.002691 \le |\Delta(z)| \le 0.004807.$$

Similarly, we seek upper and lower bounds for $|\Delta(\tau)|$ where $\tau = x + 0.65i$ and $-1/2 \le x \le 1/2$. We trivially bound the derivative of $\widetilde{\Delta}(\tau)$ with respect to x for $x \in [-1/2, 1/2]$ by

$$\begin{aligned} \left|\frac{d}{dx}\widetilde{\Delta}(\tau)\right| &= \left|\frac{d}{dx}\left(\sum_{n=1}^{20} a(n)q^n\right)\right| = \left|-\sum_{n=1}^{20} 2\pi i n \cdot a(n)q^n\right| \\ &\le \sum_{n=1}^{20} 2\pi n \cdot a(n)t^n \le 0.040192. \end{aligned}$$

We consider the minimum value of $\widetilde{\Delta}(\tau)$ at the points $x = -1/2 + \frac{n}{10000}$ for $0 \le n \le 10000$. $\widetilde{\Delta}(\tau)$ cannot be below $0.040192 \times \frac{1}{20000} = 2.0096 \times 10^{-6}$ less than its minimum value on these points. The minimum value of $\widetilde{\Delta}(\tau)$ on these points is at least 0.0111249 and the maximum value is at most 0.0249721. Since we have $|R_{20}\Delta(\tau)| \le 2.237145 \times 10^{-24}$, for $x \in [-1/2, 1/2]$, it follows that

$$0.011122 \le |\Delta(\tau)| \le 0.024975.$$

Therefore we can bound $\Delta(z)/\Delta(\tau)$ by

$$\left|\frac{\Delta(z)}{\Delta(\tau)}\right| < \frac{0.004807}{0.011122}$$

and it holds that

$$\left|\frac{\Delta(z)}{\Delta(\tau)}\right| < 0.432207,$$

for $\theta \in [1.9, 2\pi/3]$ and $\tau = x + 0.65i$ where $-1/2 \le x \le 1/2$.

We will also need to find upper bounds for $|E_4(z)|$ and $|E_6(z)|$ for $\theta \in [1.9, 2\pi/3]$. We can calculate them in the same way as the bounds for $|\Delta(z)|$.

The derivative of $\widetilde{E_4}(z)$ with respect to θ for $\theta \in [1.9, 2\pi/3]$ is bounded above by

$$\begin{aligned}\left|\frac{d}{d\theta}\widetilde{E_4}(z)\right| &= \left|\frac{d}{d\theta}\left(\sum_{n=1}^{20}\sigma_3(n)q^n\right)\right| = \left|-\sum_{n=1}^{20}2\pi n e^{i\theta}\cdot\sigma_3(n)q^n\right| \\ &\le \sum_{n=1}^{20}2\pi n\cdot\sigma_3(n)t^n \le 7.054822.\end{aligned}$$

The function $\widetilde{E_4}(z)$ cannot be above $7.054822 \times \frac{1}{2000000} = 3.527411 \times 10^{-6}$ more than its maximum value on the points $\theta = 1.9 + \frac{n}{1000000}$ for $0 \le n \le 1000000 \cdot (2\pi/3 - 1.9)$. The maximum value of $\widetilde{E_4}(z)$ on these points is at most 0.900254. Since we have $|R_{20}E_4(z)| \le 5.491887 \times 10^{-44}$, for $\theta \in [1.9, 2\pi/3]$, it follows that

$$|E_4(z)| < 0.900258.$$

Similarly, the derivative of $\widetilde{E_6}(z)$ with respect to θ for $\theta \in [1.9, 2\pi/3]$ is bounded above by

$$\begin{aligned}\left|\frac{d}{d\theta}\widetilde{E_6}(z)\right| &= \left|\frac{d}{d\theta}\left(\sum_{n=1}^{20}\sigma_5(n)q^n\right)\right| = \left|-\sum_{n=1}^{20}2\pi n e^{i\theta}\cdot\sigma_5(n)q^n\right| \\ &\le \sum_{n=1}^{20}2\pi n\cdot\sigma_5(n)t^n \le 17.8410.\end{aligned}$$

The function $\widetilde{E_6}(z)$ cannot be above $17.8410 \times \frac{1}{2000000} = 8.9205 \times 10^{-6}$ more than its maximum value on the points $\theta = 1.9 + \frac{n}{1000000}$ for $0 \le n \le 1000000 \cdot (2\pi/3 - 1.9)$. The maximum value of $\widetilde{E_6}(z)$ on these points is at most 2.881542. Since we have $|R_{20}E_6(z)| \le 4.911666 \times 10^{-41}$, for $\theta \in [1.9, 2\pi/3]$, it follows that

$$|E_6(z)| < 2.881551.$$

We now consider

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\Delta(\tau)}\frac{E_{14-k'}(\tau)}{j(\tau)-j(z)}dx. \tag{2}$$

We note $j(z)$ is real valued by [5, Proposition 2.1] and

$$\frac{1}{j(\tau) - j(z)} = \frac{1}{(\Re(j(\tau)) - j(z)) + i\Im(j(\tau))}.$$

We need information about the size of the real and imaginary parts of $j(\tau)$ for values of x on each of several subintervals of $[-1/2, 1/2]$ and the value of $j(z)$. For these computations, we work with the truncations $\widetilde{j}(z)$ and $\widetilde{j}(\tau)$, taking into account the growth of the real and imaginary parts of the truncations and the error caused by ignoring the tail. We can express $j(z)$ in terms of Eisenstein series of weight 4 and the Ramanujan Δ function as

$$j(z) = \frac{E_4^3(z)}{\Delta(z)}.$$

We use this representation to bound $R_{20}j(z)$. Observe that if we truncate $j(z)$, then the tail satisfies

$$\begin{aligned}
|R_{20}j(z)| &= \left| j(z) - \widetilde{j}(z) \right| \\
&= \left| \frac{E_4^3(z)}{\Delta(z)} - \widetilde{j}(z) \right| \\
&= \left| \left(\widetilde{j}(z) + \frac{E_4^3(z) - \widetilde{j}(z)\widetilde{\Delta}(z)}{\widetilde{\Delta}(z)} \right) \frac{\widetilde{\Delta}(z)}{\Delta(z)} - \widetilde{j}(z) \right| \\
&\leq \left| \widetilde{j}(z) \right| \left| \frac{\widetilde{\Delta}(z) - \Delta(z)}{\Delta(z)} \right| + \frac{\left| \widetilde{E_4^3}(z) - \widetilde{j}(z)\widetilde{\Delta}(z) \right| + |R_{20}E_4^3(z)|}{|\Delta(z)|} \\
&= \left| \widetilde{j}(z) \right| \left| \frac{R_{20}\Delta(z)}{\Delta(z)} \right| + \frac{\left| \widetilde{E_4^3}(z) - \widetilde{j}(z)\widetilde{\Delta}(z) \right| + |R_{20}E_4^3(z)|}{|\Delta(z)|}.
\end{aligned}$$

We compute explicit bounds on all of these terms for $\theta \in [1.9, 2\pi/3]$, and have

$$\begin{aligned}
|R_{20}j(z)| &< 10505.2 \cdot \frac{9.149478 \times 10^{-37}}{0.002691} + \frac{6.859820 \times 10^{-29} + 7.905147 \times 10^{-34}}{0.002691} \\
&< 2.549558 \times 10^{-26}.
\end{aligned}$$

Similarly, for $x \in [-1/2, 1/2]$, it holds that

$$\begin{aligned} |R_{20}j(\tau)| &< 16535.7 \cdot \frac{2.237145 \times 10^{-24}}{0.011122} + \frac{5.684342 \times 10^{-14} + 1.932893 \times 10^{-21}}{0.011122} \\ &< 5.110903 \times 10^{-12}. \end{aligned}$$

We also bound the derivatives of the truncations of $j(z)$ and the real and imaginary parts of $j(\tau)$. Doing so allows us to evaluate the functions at equally spaced points as before to get maximum and minimum values for $j(z)$ and the real and imaginary parts of $j(\tau)$.

We take the derivative of $\widetilde{j}(\tau)$ with respect to x, and for both the real and imaginary parts we achieve a bound of

$$\begin{aligned} \left|\frac{d}{dx}\Re\left(\widetilde{j}(\tau)\right)\right|, \left|\frac{d}{dx}\Im\left(\widetilde{j}(\tau)\right)\right| &\leq \left|\frac{d}{dx}\left(\sum_{n=-1}^{20} a(n)q^n\right)\right| = \left|-\sum_{n=-1}^{20} 2\pi i n \cdot a(n)q^n\right| \\ &\leq \sum_{n=-1}^{20} 2\pi n \cdot a(n)t^n \leq 234470, \end{aligned}$$

for $x \in [-1/2, 1/2]$. The bound on the derivative of $\widetilde{j}(z)$ with respect to θ is quite manageable as well. We have

$$\begin{aligned} \left|\frac{d}{d\theta}\widetilde{j}(z)\right| &= \left|\frac{d}{d\theta}\left(\sum_{n=-1}^{20} a(n)q^n\right)\right| = \left|-\sum_{n=-1}^{20} 2\pi n e^{i\theta} \cdot a(n)q^n\right| \\ &\leq \sum_{n=-1}^{20} 2\pi n \cdot a(n)t^n \leq 10505.2, \end{aligned}$$

for $\theta \in [1.9, 2\pi/3]$. We again compute values of $\widetilde{j}(z)$ at a sampling of points and use these bounds to find upper and lower bounds of $j(z)$ for $\theta \in [1.9, 2\pi/3]$. We have

$$0 \leq j(z) \leq 271.1,$$

for $\theta \in [1.9, 2\pi/3]$.

We also need upper bounds for the derivatives of $\widetilde{E_4}(\tau)$ and $\widetilde{E_6}(\tau)$ with respect to x for $x \in [-1/2, 1/2]$. They are bounded above by

$$\begin{aligned} \left|\frac{d}{dx}\widetilde{E_4}(\tau)\right| &= \left|\frac{d}{dx}\left(\sum_{n=1}^{20} \sigma_3(n)q^n\right)\right| = \left|-\sum_{n=1}^{20} 2\pi i n \cdot \sigma_3(n)q^n\right| \\ &\leq \sum_{n=1}^{20} 2\pi n \cdot \sigma_3(n)t^n \leq 33.7302, \end{aligned}$$

and

$$\begin{aligned}\left|\frac{d}{dx}\widetilde{E_6}(\tau)\right| &= \left|\frac{d}{dx}\left(\sum_{n=1}^{20}\sigma_5(n)q^n\right)\right| = \left|-\sum_{n=1}^{20}2\pi in\cdot\sigma_5(n)q^n\right| \\ &\le \sum_{n=1}^{20}2\pi n\cdot\sigma_5(n)t^n \le 124.801,\end{aligned}$$

for $x \in [-1/2, 1/2]$.

We break our path of integration (2) into pieces, and consider $j(\tau)$ in relation to $j(z)$ on each. Since it is clear that $e^{2\pi i\tau} = e^{-1.3\pi}(\cos(2\pi x) + i\sin(2\pi x))$, we have $\Re(j(x+0.65i)) = \Re(j(-x+0.65i))$ while $\Im(j(x+0.65i)) = -\Im(j(-x+0.65i))$. Similar equations hold for $\Delta(\tau)$ and $E_{14-k'}(\tau)$. Thus we have

$$\left|\frac{1}{\Delta(x+0.65i)}\frac{E_{14-k'}(x+0.65i)}{j(x+0.65i)-j(z)}\right| = \left|\frac{1}{\Delta(-x+0.65i)}\frac{E_{14-k'}(-x+0.65i)}{j(-x+0.65i)-j(z)}\right|,$$

for $x \in [-1/2, 1/2]$. With this in mind, we restrict our calculations to $x \in [-1/2, 0]$ and use symmetry for $x \in [0, 1/2]$. Using the upper bounds for the derivatives, the values at sampling points and the upper bounds for the tails, we can obtain bounds for $\Re(j(\tau))$, $\Im(j(\tau))$, $|E_4(\tau)|$, $|E_6(\tau)|$ and $|\Delta(\tau)|$ for x on each of several subintervals of $[-1/2, 1/2]$. Their bounds are showed by the following tables.

	$\Re(j(\tau))$	$\Im(j(\tau))$	$\lvert E_4(\tau)\rvert$	$\lvert E_6(\tau)\rvert$	$\lvert\Delta(\tau)\rvert$
$-0.50 \le x \le -0.49$	> 593	≥ 0	< 2.462	< 5.329	> 0.0249
$-0.49 \le x \le -0.48$	> 584.4	> 81.01	< 2.473	< 5.363	> 0.0249
$-0.48 \le x \le -0.47$	> 569.9	> 162.5	< 2.488	< 5.420	> 0.0248
$-0.47 \le x \le -0.46$	> 548.9	> 244.9	< 2.510	< 5.498	> 0.0246
$-0.46 \le x \le -0.45$	> 520.7	> 328.7	< 2.537	< 5.596	> 0.0245
$-0.45 \le x \le -0.44$	> 484.4	> 414.2	< 2.571	< 5.713	> 0.0243
$-0.44 \le x \le -0.43$	> 439.1	> 501.7	< 2.610	< 5.848	> 0.0241
$-0.43 \le x \le -0.42$	> 383.2	> 591.3	< 2.654	< 6.000	> 0.0238
$-0.42 \le x \le -0.41$	> 315.4	> 683.0	< 2.704	< 6.166	> 0.0235
$-0.41 \le x \le -0.40$	> 233.9	> 776.6	< 2.760	< 6.346	> 0.0232
$-0.40 \le x \le -0.39$	$> 21, < 234$	> 871.5	< 2.820	< 6.537	> 0.0229
$-0.39 \le x \le -0.38$	$> 21, < 234$	> 966.9	< 2.885	< 6.738	> 0.0225
$-0.38 \le x \le -0.37$	< 21.0	> 1061	< 2.954	< 6.948	> 0.0222

	$\Re(j(\tau))$	$\Im(j(\tau))$	$\lvert E_4(\tau)\rvert$	$\lvert E_6(\tau)\rvert$	$\lvert\Delta(\tau)\rvert$
$-0.37 \le x \le -0.36$	< -112.8	> 1153	< 3.027	< 7.166	> 0.0218
$-0.36 \le x \le -0.35$	< -270.0	> 1241	< 3.104	< 7.389	> 0.0214
$-0.35 \le x \le -0.34$	< -451.9	> 1321	< 3.185	< 7.618	> 0.0209
$-0.34 \le x \le -0.33$	< -660.7	> 1389	< 3.270	< 7.851	> 0.0205
$-0.33 \le x \le -0.32$	< -898.4	> 1442	< 3.357	< 8.086	> 0.0201
$-0.32 \le x \le -0.31$	< -1166	> 1473	< 3.447	< 8.323	> 0.0196
$-0.31 \le x \le -0.30$	< -1466	> 1444	< 3.540	< 8.561	> 0.0192
$-0.30 \le x \le -0.29$	< -1797	> 1368	< 3.635	< 8.798	> 0.0187
$-0.29 \le x \le -0.28$	< -2158	> 1239	< 3.731	< 9.034	> 0.0183
$-0.28 \le x \le -0.27$	< -2546	> 1048	< 3.829	< 9.269	> 0.0178
$-0.27 \le x \le -0.26$	< -2957	> 783.7	< 3.929	< 9.500	> 0.0174
$-0.26 \le x \le -0.25$	< -3384	> 436.4	< 4.029	< 9.728	> 0.0170
$-0.25 \le x \le -0.24$	< -3817	$> -3.3, < 437$	< 4.129	< 9.951	> 0.0165
$-0.24 \le x \le -0.23$	< -4242	< -3.212	< 4.229	< 10.17	> 0.0161
$-0.23 \le x \le -0.22$	< -4645	< -543.1	< 4.329	< 10.39	> 0.0157
$-0.22 \le x \le -0.21$	< -5007	< -1189	< 4.428	< 10.59	> 0.0153
$-0.21 \le x \le -0.20$	< -5304	< -1943	< 4.527	< 10.79	> 0.0149
$-0.20 \le x \le -0.19$	< -5513	< -2805	< 4.623	< 10.99	> 0.0146
$-0.19 \le x \le -0.18$	< -5559	< -3767	< 4.718	< 11.17	> 0.0142
$-0.18 \le x \le -0.17$	< -5341	< -4816	< 4.810	< 11.35	> 0.0139
$-0.17 \le x \le -0.16$	< -4929	< -5932	< 4.900	< 11.51	> 0.0136
$-0.16 \le x \le -0.15$	< -4301	< -7087	< 4.986	< 11.67	> 0.0133
$-0.15 \le x \le -0.14$	< -3445	< -8248	< 5.069	< 11.82	> 0.0130
$-0.14 \le x \le -0.13$	< -2352	< -9373	< 5.149	< 11.96	> 0.0127
$-0.13 \le x \le -0.12$	< -1209	< -10413	< 5.223	< 12.09	> 0.0125
$-0.12 \le x \le -0.11$	$> -1030, < 508$	< -11319	< 5.294	< 12.21	> 0.0123
$-0.11 \le x \le -0.10$	> 508.2	< -12037	< 5.359	< 12.32	> 0.0121
$-0.10 \le x \le -0.09$	> 2231	< -12515	< 5.420	< 12.42	> 0.0119
$-0.09 \le x \le -0.08$	> 4098	< -12574	< 5.474	< 12.51	> 0.0117
$-0.08 \le x \le -0.07$	> 6054	< -12089	< 5.523	< 12.58	> 0.0116
$-0.07 \le x \le -0.06$	> 8033	< -11241	< 5.566	< 12.65	> 0.0114
$-0.06 \le x \le -0.05$	> 9964	< -10033	< 5.603	< 12.71	> 0.0113
$-0.05 \le x \le -0.04$	> 11770	< -8490	< 5.633	< 12.76	> 0.0112
$-0.04 \le x \le -0.03$	> 13374	< -6651	< 5.657	< 12.79	> 0.0112
$-0.03 \le x \le -0.02$	> 14706	< -4574	< 5.674	< 12.82	> 0.0111
$-0.02 \le x \le -0.01$	> 15706	< -2330	< 5.684	< 12.83	> 0.0111
$-0.01 \le x \le -0.00$	> 16325	≤ 0	< 5.688	< 12.84	> 0.0111

We consider

$$\frac{1}{j(\tau)-j(z)} = \frac{1}{(\Re\left(j(\tau)\right)-j(z))+i\Im\left(j(\tau)\right)}.$$

We note that since $j(z)$ is real, if we have the bound $|\Im(j(z))| > b > 0$ for a subinterval, it follows that

$$\left|\frac{1}{j(\tau)-j(z)}\right| < \left|\frac{1}{(\Re\left(j(\tau)\right)-j(z))+ib}\right|.$$

If $\Re(j(\tau)) > a_1 > 271.1$, then this is bounded by

$$\left|\frac{1}{j(\tau)-j(z)}\right| < \left|\frac{1}{(a_1-271.1)+ib}\right|.$$

If $\Re(j(\tau)) < -a_2 < 0$, then this is bounded by

$$\left|\frac{1}{j(\tau)-j(z)}\right| < \left|\frac{1}{(a_2-0)+ib}\right|.$$

If $\Re(j(\tau)) \in [0, 271.1]$ for some τ, then this is bounded by

$$\left|\frac{1}{j(\tau)-j(z)}\right| < \left|\frac{1}{(0-0)+ib}\right|.$$

Since $\Im(j(z))$ may be equal to 0, $\left|\frac{1}{j(\tau)-j(z)}\right|$ is bounded by

$$\left|\frac{1}{(593-271.1)+0}\right|, \quad \left|\frac{1}{(3817-0)+0}\right| \quad \text{and} \quad \left|\frac{1}{(16325-271.1)+0}\right|,$$

for $-0.50 \leq x \leq -0.49$, $-0.25 \leq x \leq -0.24$ and $-0.01 \leq x \leq 0$, respectively. Moreover, we note that $E_{k'}$ is written in term of E_4 and E_6 for $k' \in \{0, 4, 6, 8, 10, 14\}$. Thus we can bound $E_{k'}$ using the bounds of E_4 and E_6. Therefore we can bound the integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{\Delta\left(x+0.65i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right)-j\left(e^{i\theta}\right)}\right| dx. \tag{3}$$

For example, when $k' = 0$, (3) is bounded above by

$$
\begin{aligned}
2 \cdot 0.01 \cdot \Bigg(& \frac{2.462^2 \cdot 5.329}{0.0249 \cdot \sqrt{(593 - 271.1)^2 + 0^2}} + \frac{2.473^2 \cdot 5.363}{0.0249 \cdot \sqrt{(584.4 - 271.1)^2 + 81.01^2}} \\
& + \frac{2.488^2 \cdot 5.420}{0.0248 \cdot \sqrt{(569.9 - 271.1)^2 + 162.5^2}} + \frac{2.510^2 \cdot 5.498}{0.0246 \cdot \sqrt{(548.9 - 271.1)^2 + 244.9^2}} \\
& + \frac{2.537^2 \cdot 5.596}{0.0245 \cdot \sqrt{(520.7 - 271.1)^2 + 328.7^2}} + \frac{2.571^2 \cdot 5.713}{0.0243 \cdot \sqrt{(484.4 - 271.1)^2 + 414.2^2}} \\
& + \frac{2.610^2 \cdot 5.848}{0.0241 \cdot \sqrt{(439.1 - 271.1)^2 + 501.7^2}} + \frac{2.654^2 \cdot 6.000}{0.0238 \cdot \sqrt{(383.2 - 271.1)^2 + 591.3^2}} \\
& + \frac{2.704^2 \cdot 6.166}{0.0235 \cdot \sqrt{(315.4 - 271.1)^2 + 683.0^2}} + \frac{2.760^2 \cdot 6.346}{0.0232 \cdot \sqrt{(0 - 0)^2 + 776.6^2}} \\
& + \frac{2.820^2 \cdot 6.537}{0.0229 \cdot \sqrt{(0 - 0)^2 + 871.5^2}} + \frac{2.885^2 \cdot 6.738}{0.0225 \cdot \sqrt{(0 - 0)^2 + 966.9^2}} \\
& + \frac{2.954^2 \cdot 6.948}{0.0222 \cdot \sqrt{(0 - 0)^2 + 1061^2}} + \frac{3.027^2 \cdot 7.166}{0.0218 \cdot \sqrt{(112.8 - 0)^2 + 1153^2}} \\
& + \frac{3.104^2 \cdot 7.389}{0.0214 \cdot \sqrt{(270.0 - 0)^2 + 1241^2}} + \frac{3.185^2 \cdot 7.618}{0.0209 \cdot \sqrt{(451.9 - 0)^2 + 1321^2}} \\
& + \frac{3.270^2 \cdot 7.851}{0.0205 \cdot \sqrt{(660.7 - 0)^2 + 1389^2}} + \frac{3.357^2 \cdot 8.086}{0.0201 \cdot \sqrt{(898.4 - 0)^2 + 1442^2}} \\
& + \frac{3.447^2 \cdot 8.323}{0.0196 \cdot \sqrt{(1166 - 0)^2 + 1473^2}} + \frac{3.540^2 \cdot 8.561}{0.0192 \cdot \sqrt{(1466 - 0)^2 + 1444^2}} \\
& + \frac{3.635^2 \cdot 8.798}{0.0187 \cdot \sqrt{(1797 - 0)^2 + 1368^2}} + \frac{3.731^2 \cdot 9.034}{0.0183 \cdot \sqrt{(2158 - 0)^2 + 1239^2}} \\
& + \frac{3.829^2 \cdot 9.269}{0.0178 \cdot \sqrt{2546 - 0)^2 + 1048^2}} + \frac{3.929^2 \cdot 9.500}{0.0174 \cdot \sqrt{2957 - 0)^2 + 783.7^2}} \\
& + \frac{4.029^2 \cdot 9.728}{0.0170 \cdot \sqrt{3384 - 0)^2 + 436.4^2}} + \frac{4.129^2 \cdot 9.951}{0.0165 \cdot \sqrt{(3817 - 0)^2 + 0^2}} \\
& + \frac{4.229^2 \cdot 10.17}{0.0161 \cdot \sqrt{(4242 - 0)^2 + 3.212^2}} + \frac{4.329^2 \cdot 10.39}{0.0157 \cdot \sqrt{(4645 - 0)^2 + 543.1^2}} \\
& + \frac{4.428^2 \cdot 10.59}{0.0153 \cdot \sqrt{(5007 - 0)^2 + 1189^2}} + \frac{4.527^2 \cdot 10.79}{0.0149 \cdot \sqrt{(5304 - 0)^2 + 1943^2}}
\end{aligned}
$$

$$+\frac{4.623^2\cdot 10.99}{0.0146\cdot\sqrt{(5513-0)^2+2805^2}}+\frac{4.718^2\cdot 11.17}{0.0142\cdot\sqrt{(5559-0)^2+3767^2}}$$
$$+\frac{4.810^2\cdot 11.35}{0.0139\cdot\sqrt{(5341-0)^2+4816^2}}+\frac{4.900^2\cdot 11.51}{0.0136\cdot\sqrt{(4929-0)^2+5932^2}}$$
$$+\frac{4.986^2\cdot 11.67}{0.0133\cdot\sqrt{(4301-0)^2+7087^2}}+\frac{5.069^2\cdot 11.82}{0.0130\cdot\sqrt{(3445-0)^2+8248^2}}$$
$$+\frac{5.149^2\cdot 11.96}{0.0127\cdot\sqrt{(2352-0)^2+9373^2}}+\frac{5.223^2\cdot 12.09}{0.0125\cdot\sqrt{(1209-0)^2+10413^2}}$$
$$+\frac{5.294^2\cdot 12.21}{0.0123\cdot\sqrt{(0-0)^2+11319^2}}+\frac{5.359^2\cdot 12.32}{0.0121\cdot\sqrt{(508.2-271.1)^2+12037^2}}$$
$$+\frac{5.420^2\cdot 12.42}{0.0119\cdot\sqrt{(2231-271.1)^2+12515^2}}+\frac{5.474^2\cdot 12.51}{0.0117\cdot\sqrt{(4098-271.1)^2+12574^2}}$$
$$+\frac{5.523^2\cdot 12.58}{0.0116\cdot\sqrt{(6054-271.1)^2+12089^2}}+\frac{5.566^2\cdot 12.65}{0.0114\cdot\sqrt{(8033-271.1)^2+11241^2}}$$
$$+\frac{5.603^2\cdot 12.71}{0.0113\cdot\sqrt{(9964-271.1)^2+10033^2}}+\frac{5.633^2\cdot 12.76}{0.0112\cdot\sqrt{(11770-271.1)^2+8490^2}}$$
$$+\frac{5.657^2\cdot 12.79}{0.0112\cdot\sqrt{(13374-271.1)^2+6651^2}}+\frac{5.674^2\cdot 12.82}{0.0111\cdot\sqrt{(14706-271.1)^2+4574^2}}$$
$$+\frac{5.684^2\cdot 12.83}{0.0111\cdot\sqrt{(15706-271.1)^2+2330^2}}+\left.\frac{5.688^2\cdot 12.84}{0.0111\cdot\sqrt{(16325-271.1)^2+0^2}}\right)$$
$$<2.76009.$$

We set $A_0 = 2.76009$. For $k' \in \{4, 6, 8, 10, 14\}$, constants $A_{k'}$ which bound (3) is obtained by similar calculation.

Similarly, when $\frac{\pi}{2} \leq \theta < 1.9$ and $\tau = x + 0.75i$, we have the bounds $A'_{k'}$ in Subsection 2.3.

We completed our proof of Theorem 2.1.

3 Transcendence of zeros

In this section, we consider transcendence of zeros of $g_{k,m}$. We have the following theorem.

Theorem 3.1. *Let z_0 be a zero of $g_{k,m}(z) = f_{k,m}(z)+\sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j$ in the fundamental domain for $SL_2(\mathbb{Z})$ lying on the circle $|z| = 1$. Let $a_j \in \mathbb{Q}$. Then z_0 is transcendental if it is not equal to i or $\rho = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$.*

For the proof of Theorem 3.1, we use the following lemma of Schneider.

Lemma 3.1. *[9, Corollary 3.4] If $z \in \mathbb{H}$ and $j(z)$ is algebraic, then either z is transcendental or z is imaginary quadratic, i.e. $\mathbb{Q}(z)$ is a degree 2 extension of $\mathbb{Q}$, with $z \notin \mathbb{R}$.*

We can prove Theorem 3.1 as with [7]. We now consider some properties from class field theory and complex multiplication discussed in [1]. Let D be a negative integer so that $K = \mathbb{Q}(\sqrt{D})$ is an imaginary quadratic field. An oder $\mathfrak{O}$ of K is a subring of K containing 1 that is a free $\mathbb{Z}$-module of rank 2. A proper fractional of $\mathfrak{O}$ is a nonzero fractional ideal $\mathfrak{A}$ of $\mathfrak{O}$ such that

$$\mathfrak{O} = \{\alpha \in K : \alpha\mathfrak{A} \subset \mathfrak{A}\}.$$

We consider many nice properties for the set of all proper fractional ideals of K forms a multiplicative group.

We set a polynomial $P(x) = ax^2 + bx + c$ of negative discriminant $D = b^2 - 4ac$ with integer coefficients such that $a > 0$ and $\gcd(a, b, c) = 1$. If $z \in \mathbb{H}$ is a root of $P(x)$, as seen in [1, Lemma 7.5], $\mathfrak{O} = [1, az]$ is an order of K and $\Lambda = [1, z]$ is a proper fractional ideal of $\mathfrak{O}$.

To see the structure of $\mathfrak{O}$, we note that since $z \in \mathbb{H}$ is a root of the polynomial $ax^2 + bx + c$, by the quadratic formula, $z = \frac{-b+\sqrt{D}}{2a}$. Therefore

$$[1, az] = \left[1, \frac{-b + \sqrt{D}}{2}\right] = \begin{cases} \frac{i\sqrt{D}}{2} & \text{if } b \equiv 0 \pmod 2, \\ \frac{1+\sqrt{D}}{2} & \text{if } b \equiv 1 \pmod 2. \end{cases}$$

Since $D = b^2 - 4ac$, we have that b is even if and only if $D \equiv 0 \pmod 4$. Similarly, b is odd if and only if $D \equiv 1 \pmod 4$. Thus the following lemma holds as with [7].

Lemma 3.2. *[7, Lemma 2.2] Let $a, b, c \in \mathbb{Z}$ such that $a > 0$, $\gcd(a, b, c) = 1$, and $D = b^2 - 4ac < 0$. If $z \in \mathbb{H}$ is a root of the polynomial $ax^2 + bx + c$, then the lattice $[1, z]$ is a proper fractional ideal of the order $\mathfrak{O} = [1, az]$ of $K = \mathbb{Q}(\sqrt{D})$. Moreover,*

$$\mathfrak{O} = \begin{cases} \frac{i\sqrt{D}}{2} & \text{if } D \equiv 0 \pmod 4, \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

We find that the order $\mathfrak{O}$ does not depend on z, but instead on the discriminant D of the reduced integer polynomial that has z as a root. Recall, if Λ is a lattice of C we define $j(\Lambda) = j(z)$, where $z \in \mathbb{H}$ and $\Lambda = [1, z]$. The choice of $z \in \mathbb{H}$ is well defined. By Lemma 3.2, we see that we can map a point $z \in \mathbb{H}$ to the proper fractional ideal $\Lambda = [1, z]$ of $\mathfrak{O}$, where $j([1, z]) = j(z)$.

The following lemma follows from [1, Theorem 11.1 and Proposition 13.2], and is the last result we need before the proof of Theorem 3.1 as with [7].

Lemma 3.3. *[7, Lemma 2.3] If $\mathfrak{A}$ is a proper fractional ideal of an order $\mathfrak{O}$ of an imaginary quadratic field K, then $j(\mathfrak{A})$ is an algebraic over $\mathbb{Q}$. If $\mathfrak{B}$ is any other proper fractional ideal of $\mathfrak{O}$, then $K(j(\mathfrak{A})) = K(j(\mathfrak{B}))$ and $j(\mathfrak{A})$ and $j(\mathfrak{B})$ are conjugate over K. Furthermore, the degree of $j(\mathfrak{A})$ is the class number of $\mathfrak{O}$.*

Let $g_{k,m}(z)$ satisfy the assumption of Theorem 3.1. Then we can write

$$\begin{aligned} g_{k,m}(z) &= f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j \\ &= \Delta(z)^\ell E_{k'}(z) F_{k,L}(j(z)), \end{aligned}$$

where $F_{k,L}(j(z))$ is a monic polynomial in $j(z)$ of degree $L = \ell + m$ with rational number coefficients. By Kohnen [8], the only possible zeros of $E_{k'}(z)$ are i and ρ. Also, we see from the valence formula that $\Delta(z)$ is never zero on $\mathbb{H}$. Thus, the only zeros of $g_{k,m}(z)$ in $\mathcal{F}$ other than i, ρ are the zeros of $F_{k,L}(j(z))$.

Suppose $z_0 \in \mathcal{F}$ such that $F_{k,L}(j(z_0)) = 0$. Since $F_{k,L}(x)$ is a polynomial with rational number coefficients, $j(z_0)$ is algebraic. Thus from Lemma 3.1, z_0 is either transcendental or imaginary quadratic.

If z_0 is imaginary quadratic, then z_0 is a root of a polynomial $P(x) = ax^2 + bx + c$, where $\gcd(a, b, c) = 1, a > 0$, and the discriminant $D_0 = b^2 - 4ac < 0$. Let $K = \mathbb{Q}(\sqrt{D_0})$.

We consider the order $\mathfrak{D} = [1, az_0]$ of K. From Lemma 3.2, the lattice $[1, z_0]$ is a proper fractional ideal of $\mathfrak{D}$, and the order $\mathfrak{D}$ has the form

$$\mathfrak{D} = \begin{cases} \left[1, \frac{i\sqrt{D_0}}{2}\right] & \text{if } D_0 \equiv 0 \pmod 4, \\ \left[1, \frac{1+\sqrt{D_0}}{2}\right] & \text{if } D_0 \equiv 1 \pmod 4. \end{cases}$$

Thus by Lemma 6.3, if $\mathfrak{A}$ is any other proper fractional ideal of $\mathfrak{D}$, $j(z_0) = j([1, z_0])$ and $j(\mathfrak{A})$ are conjugate.

We consider the point $z_1 \in \mathbb{C}$ defined by

$$z_1 = \begin{cases} \frac{i\sqrt{|D_0|}}{2} & \text{if } D_0 \equiv 0 \pmod 4, \\ \frac{1+i\sqrt{|D_0|}}{2} & \text{if } D_0 \equiv 1 \pmod 4. \end{cases}$$

Then $z_1 \in \mathcal{F}$ and we have $[1, z_1] = \mathfrak{D}$. Thus by definition $[1, z_1]$ is a proper fractional ideal of $\mathfrak{D}$, and so $j(z_0)$ and $j(z_1)$ are conjugate.

We take an automorphism σ of $K(j(\mathfrak{D}))$ such that $\sigma(j(z_0)) = j(z_1)$. Since σ acts as the identity on $\mathbb{Q}$ and $F_{k,L}$ is a polynomial with rational number coefficients, we have that

$$\begin{aligned} 0 &= \sigma(0) \\ &= \sigma(F_{k,L}(j(z_0))) \\ &= F_{k,L}(\sigma(j(z_0))) \\ &= F_{k,L}(j(z_1)). \end{aligned}$$

Thus z_1 is also a zero of $F_{k,L}$ and hence a zero of $g_{k,m}$. Since $z_1 \in \mathcal{F}$, by Theorem 2.1 we have that z_1 must lie on the arc of the unit circle given by

$$\left\{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3} \right\}.$$

Suppose $D_0 \equiv 0 \pmod 4$, so that $D_0 = -4n$ for some positive integer n. Then $z_1 = i\sqrt{n}$, but since z_1 must lie on the unit circle we must have $n = 1$. Thus, $D_0 = -4$. Since $z_0 \in \mathbb{H}$, we have by the quadratic formula that

$$z_0 = \frac{-b + 2i}{2a}.$$

But $z_0 \in \mathcal{F}$, and so $\Im(z_0) \geq \frac{\sqrt{3}}{2}$. Thus $a = 1$, and so

$$z_0 = -\frac{b}{2} + i.$$

But again by Theorem 2.1 we have that z_0 must lie on the unit circle, so $b = 0$ and $z_0 = i$.

.If $D_0 \equiv 1 \pmod 4$, then $D_0 = -4n + 1$ for some positive integer n. Hence,

$$z_1 = \frac{-1 + i\sqrt{4n-1}}{2},$$

and thus $|z_1|^2 = n$. Again, since z_1 must lie on the unit circle we must have $n = 1$. Therefore $D_0 = -3$. Since $z_0 \in \mathbb{H}$, we have that

$$z_0 = \frac{-b + i\sqrt{3}}{2a}$$

by the quadratic formula. And again since $z_0 \in \mathcal{F}$, we have $a = 1$ so that

$$z_0 = -\frac{b}{2} + i\frac{\sqrt{3}}{2}.$$

But again by Theorem 2.1 we have that z_0 must lie on the unit circle, so $b = 1$ and $z_0 = \rho$. Thus, we completed Theorem 3.1.

We have the following corollary by Theorem 2.1 and Theorem 3.1.

Corollary 3.1. *Let $k = 12\ell + k'$, where $\ell \in \mathbb{Z}_{\geq 0}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. Let*

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j,$$

where $a_j \in \mathbb{Q}, m \geq 0$ and $\ell + m \geq 1$. If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j|(3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'},$$

then all of zeros of $g_{k,m}$ in the fundamental domain for $SL_2(\mathbb{Z})$ are transcendental or equal to i or $\rho = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$.

4 Zeros of weakly holomorphic modular form of level 2

In this section, we consider zeros of weakly holomorphic modular form of level 2.

4.1 Definitions and statement of results

Let $M_k^\sharp(2)$ be the subspace of $M_k^!(2)$ consisting of forms which are holomorphic away from the cusp at ∞. Garthwaite and Jenkins considered a canonical basis of $M_k^\sharp(2)$ in [4]. Let $k = 4\ell + k'$ where $\ell \in \mathbb{Z}$ and $k' \in \{0, 2\}$. For any integer $m \geq -\ell$, there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^\sharp(2)$ which has an expansion

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}).$$

We define three modular forms of level 2 to construct $f_{k,m}(z)$. Let

$$\psi(z) = \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} = q^{-1} - 24 + 276q + \cdots \in M_0^\sharp(2)$$

be the Hauptmodul for $\Gamma_0(2)$. This form has integer coefficients, has a pole at ∞, and vanishes at 0. Moreover, by the above argument $\psi(z)$ is real-valued on the lower boundary of the fundamental domain.

Next, let

$$F_2(z) = 2E_2(2z) - E_2(z) = 1 + 24\sum_n \left(\sum_{d|n, d \text{ odd}} d\right) q^n$$

be the unique normalized holomorphic modular form of weight 2 and level 2. Here $E_2(z)$ is the weight 2 Eisenstein series $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n$. The form $F_2(z)$ has integer coefficients and a single zero at the elliptic point $-\frac{1}{2} + \frac{i}{2}$.

Additionally, we define the Eisenstein series $S_4(z) \in M_4(2)$ as

$$S_4(z) = \frac{E_4(z) - E_4(2z)}{240} = q + 8q^2 + 28q^3 + \cdots .$$

It is easily checked that S_4 has integral Fourier coefficients and vanishes at ∞. It does not vanish at the cusp at 0, as there are no cusp forms of weight 4 and level 2.

We now use these forms to construct a basis for $M_k^\sharp(2)$. We can write

$$f_{k,m}(z) = S_4^\ell(z)F_{k'}(z)F(\psi(z)),$$

where $F(x)$ is a polynomial with integer coefficients of degree $n+\ell = n+\lfloor \frac{k}{4} \rfloor$. Similar sequences of modular forms for many levels appear in [3].

For the group $\Gamma_0(2)$, we use a fundamental domain in the upper half-plane bounded by the lines $\Re(z) = -\frac{1}{2}$ and $\Re(z) = \frac{1}{2}$, the circles of radius $\frac{1}{2}$ centered at $z = -\frac{1}{2}$ and $z = \frac{1}{2}$. We include the boundary on the left half of this fundamental domain. The cusps of this fundamental domain can be taken to be at ∞ and at 0.

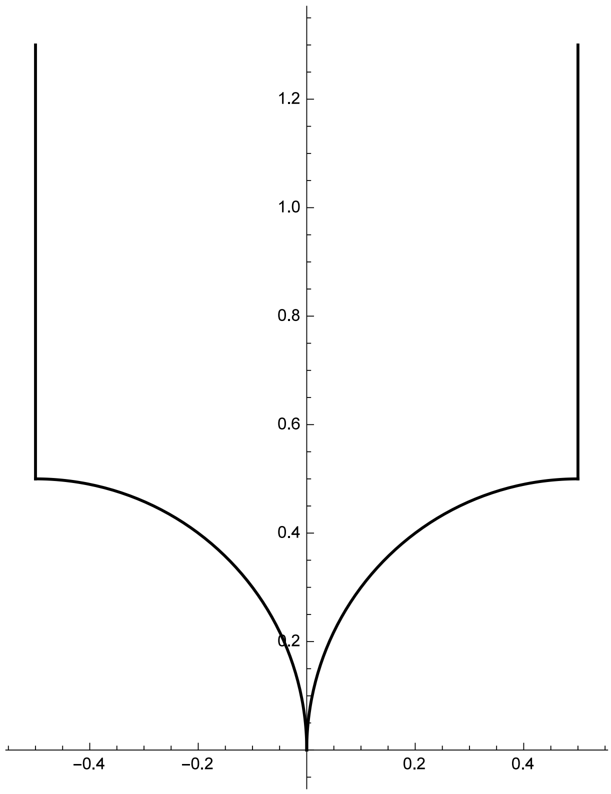

Figure 2: A fundamental domain for $\Gamma_0(2)$.

Garthwaite and Jenkins proved the following theorem [4].

Theorem 4.1. *[4, Theorem 1] Let $f_{k,m}(z)$ be as above. If $\ell \geq 0$ and $m \geq 14\ell + 8$, or if $\ell < 0$ and $n \geq 15|\ell| + 8$, then at least $\lfloor \frac{\sqrt{3}}{2}n + \frac{k}{6} \rfloor$ of the $n + \lfloor \frac{k}{4} \rfloor$ nontrivial zeros of $f_{k,m}(z)$ in the fundamental domain for $\Gamma_0(2)$ lie on the lower boundary of the fundamental domain.*

We define $g_{k,m}(z)$ to generalize Theorem 4.1. It is defined by

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-4j,m}(z) S_4(z)^j,$$

where $a_j \in \mathbb{R}, m \geq 1$ and $\ell \geq 1$. The main results for level 2 of this paper are the following three theorems.

Theorem 4.2. *Let $k = 4\ell + k'$, where $\ell \geq 1$ and $k' \in \{0, 2\}$. Let $g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-4j,m}(z) S_4(z)^j$, where $a_j \in \mathbb{R}$ and $m \geq 1$. Suppose*

$$\begin{aligned}
\delta_5 &= 62.574, \\
\delta_6 &= 2.444141, \\
\delta_7 &= 0.87063, \\
\delta_8 &= 0.73041, \\
A &= 21.8151 \\
\text{and} \quad \gamma_2(j) &= \begin{cases} \delta_7^j \delta_5^{\ell-j} & \text{if } 1 \leq j \leq \ell, \\ \delta_6^j \delta_7^{\ell} & \text{if } \ell + 1 \leq j \leq \ell + m. \end{cases}
\end{aligned}$$

If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j| (2\delta_7^j + \delta_8^m \gamma_2(j)^j A) < 2 - \delta_8^m \delta_5^\ell A,$$

then at least $\lfloor \frac{\sqrt{3}}{2}n + \frac{k}{6} \rfloor$ of the $n + \lfloor \frac{k}{4} \rfloor$ nontrivial zeros of $f_{k,m}(z)$ in the fundamental domain for $\Gamma_0(2)$ lie on the lower boundary of the fundamental domain.

Theorem 4.3. *Let $f_{k,m}(z)$ be as above. If $\ell \geq 0$ and $m \geq 3\ell + 5$, or if $\ell < 0$ and $n \geq 8|\ell| + 5$, then at least $\lfloor \frac{n}{2} + \frac{k}{12} \rfloor$ of the $n + \lfloor \frac{k}{4} \rfloor$ nontrivial zeros of $f_{k,m}(z)$ in the fundamental domain for $\Gamma_0(2)$ lie on the lower boundary of the fundamental domain.*

Theorem 4.4. *Let $k = 4\ell + k'$, where $\ell \geq 1$ and $k' \in \{0, 2\}$. Let $g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-4j,m}(z) S_4(z)^j$, where $a_j \in \mathbb{R}$ and $m \geq 1$. Suppose*

$$\begin{aligned}
\delta_9 &= 1.6326, \\
\delta_{10} &= 0.15165, \\
\delta_{11} &= 0.066968, \\
\delta_{12} &= 0.81268, \\
B &= 5.50471 \\
\textit{and} \quad \gamma_3(j) &= \begin{cases} \delta_{11}^j \delta_9^{\ell-j} & \text{if } 1 \leq j \leq \ell, \\ \delta_{10}^j \delta_{11}^{\ell} & \text{if } \ell+1 \leq j \leq \ell+m. \end{cases}
\end{aligned}$$

If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j| (2\delta_{11}^j + \delta_{12}^m \gamma_3(j)^j B) < 2 - \delta_{12}^m \delta_9^{\ell} B,$$

then at least $\lfloor \frac{n}{2} + \frac{k}{12} \rfloor$ of the $n + \lfloor \frac{k}{4} \rfloor$ nontrivial zeros of $f_{k,m}(z)$ in the fundamental domain for $\Gamma_0(2)$ lie on the lower boundary of the fundamental domain.

Proofs of these theorems is given by Subsection 4.2, 4.3 and 4.4, respectively.

4.2 Generalization of a theorem of Garthwaite and Jenkins

In this subsection, we prove Theorem 4.2. An easy argument [4] shows that for any weakly holomorphic modular form f of weight k and level 2 with real coefficients, the quantity $e^{ik\theta/2} f(-\frac{1}{2} + \frac{1}{2} e^{i\theta})$ is real for $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. In [4], Garthwaite and Jenkins proved that

$$\left| e^{ik\theta/2} e^{-\pi m \sin\theta} f\left(-\frac{1}{2} + \frac{1}{2} e^{i\theta}\right) - (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m \cos\theta\right) \right| < 2,$$

for $\ell \geq 0$ and $m \geq 14\ell + 8$, or $\ell < 0$ and $n \geq 15|\ell| + 8$. They showed the inequality in term of several bounds for weakly holomorphic modular forms of level 2. In Subsection 2.4, we computed bounds for weakly holomorphic modular forms of level 1. Now we can improve the bounds for level 2 by the similar technique.

Suppose $\ell \geq 1$ and $m \geq 1$. Then we set

$$\begin{aligned} H(\theta) &= e^{ik\theta/2} e^{-\pi m \sin\theta} g_{k,m}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) \\ &= H_{0,m}(\theta) + \sum_{j=1}^{\ell+m} a_j e^{4ji\theta/2} S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)^j H_{j,m}(\theta), \end{aligned}$$

where $H_{j,m}(\theta) = e^{(k-4j)i\theta/2} e^{-\pi m \sin\theta} f_{k-4j,m}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)$. Since $\ell \geq 1$ and $m \geq 1$, we write

$$\begin{aligned} H(\theta) = H_{0,m}(\theta) &+ \sum_{j=1}^{\ell} a_j e^{4ji\theta/2} S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)^j H_{j,m}(\theta) \\ &+ \sum_{j=1}^{m} a_{j+\ell} e^{4(j+\ell)i\theta/2} S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)^{j+\ell} H_{j+\ell,m}(\theta). \end{aligned}$$

We define the function $R_{j,m}(\theta)$ for $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ by

$$H_{j,m}(\theta) = (-1)^m 2\cos\left(\frac{(k-4j)\theta}{2} - \pi m \cos\theta\right) + R_{j,m}(\theta).$$

We seek a bound for the function $R_{j,m}(\theta)$. Details for the computation of the numerical bounds that appear in this subsection are provided by the similar technique of Subsection 2.4. By [4], we find that

$$\begin{aligned} |R_{j,m}(\theta)| &= \Bigg| e^{(k-4j)i\theta/2} e^{-\pi m \sin\theta} f_{k-4j,m}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) \\ &\qquad - (-1)^m 2\cos\left(\frac{(k-4j)\theta}{2} - \pi m \cos\theta\right) \Bigg| \\ &= \Bigg| e^{(k-4j)i\theta/2} e^{-\pi m \sin\theta} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_4^{\ell-j}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4^{\ell-j}(x+\frac{i}{5})} \frac{F_{k'}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) F_2\left(x+\frac{i}{5}\right)}{F_{k'}\left(x+\frac{i}{5}\right)} \\ &\qquad \times \frac{\psi\left(x+\frac{i}{5}\right)}{\psi\left(x+\frac{i}{5}\right) - \psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)} e^{-2\pi i m\tau} dx \Bigg| \end{aligned}$$

$$= \quad e^{-\pi m\left(\sin\theta - \frac{2}{5}\right)}\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_4^{\ell-j}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4^{\ell-j}(x+\frac{i}{5})} \frac{F_{k'}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) F_2\left(x+\frac{i}{5}\right)}{F_{k'}\left(x+\frac{i}{5}\right)} \times \frac{\psi\left(x+\frac{i}{5}\right)}{\psi\left(x+\frac{i}{5}\right) - \psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)} dx \right|$$

We consider the exponential term $e^{-\pi m\left(\sin\theta - \frac{2}{5}\right)}$. It holds that $e^{-\pi m\left(\sin\theta - \frac{2}{5}\right)} < 0.73041$ for $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. We set $\delta_8 = 0.73041$.

First, we consider

$$\left| \frac{S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4\left(x+\frac{i}{5}\right)} \right|^{\ell-j}.$$

We compute

$$0.03 \leq \left| S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) \right| \leq 0.87063.$$

We set $\delta_7 = 0.87063$. From [4], we have that

$$0.014 \leq \left| S_4\left(x+\frac{i}{5}\right) \right| \leq 2.44141.$$

We set $\delta_6 = 2.44141$. Putting this together, we have, for $\ell - j \geq 0$,

$$\left| \frac{S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4\left(x+\frac{i}{5}\right)} \right|^{\ell-j} \leq |62.574|^{\ell-j},$$

and for $\ell - j < 0$,

$$\left| \frac{S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4\left(x+\frac{i}{5}\right)} \right|^{\ell-j} \leq \left(\frac{\delta_6}{\left|S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|} \right)^{|\ell-j|}.$$

We set $\delta_5 = 62.574$.

Next, we consider the term

$$\left| \frac{F_{k'}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) F_2\left(x+\frac{i}{5}\right)}{F_{k'}\left(x+\frac{i}{5}\right)} \right|.$$

If $k' = 2$, this is $\left|F_2\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|$, which is bounded above by 8.00067. If $k' = 0$, this is $\left|F_2\left(x+\frac{i}{5}\right)\right|$, which is bounded above by 12.50005. Therefore the contribution is bounded above by 12.50005.

Finally, we consider

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\psi\left(x+\frac{i}{5}\right)}{\psi\left(x+\frac{i}{5}\right)-\psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)} \right| dx.$$

From [4], We can bound the Hauptmodul quotient by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\psi\left(x+\frac{i}{5}\right)}{\psi\left(x+\frac{i}{5}\right)-\psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)} \right| dx \leq 1.74520.$$

Thus

$$\left| \frac{F_{k'}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) F_2\left(x+\frac{i}{5}\right)}{F_{k'}\left(x+\frac{i}{5}\right)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\psi\left(x+\frac{i}{5}\right)}{\psi\left(x+\frac{i}{5}\right)-\psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)} du \right|$$

is bounded above by

$$12.50005 \cdot 1.74520 = 21.8151.$$

We set $A = 21.8151$.

Putting all of these pieces together, we see that, for $1 \leq j \leq \ell$,

$$|R_{j,m}(\theta)| < \delta_8^m \delta_5^{\ell-j} A,$$

for $1 \leq j \leq m$,

$$|R_{j+\ell,m}(\theta)| < \delta_8^m \left(\frac{\delta_6}{\left|S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|} \right)^j A.$$

We prove Theorem 4.2 using the bound for $|R_{j,m}(\theta)|$. When $\ell \geq 1$ and

$m \geq 1$, we can write

$$\begin{aligned}
H(\theta) &= H_{0,m}(\theta) + \sum_{j=1}^{\ell} a_j e^{4ji\theta/2} S_4 \left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)^j H_{j,m}(\theta) \\
&\quad + \sum_{j=1}^{m} a_{j+\ell} e^{4(j+\ell)i\theta/2} S_4 \left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)^{j+\ell} H_{j+\ell,m}(\theta) \\
&= (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m \cos\theta\right) + R_{0,m}(\theta) \\
&\quad + \sum_{j=1}^{\ell} a_j e^{4ji\theta/2} S_4 \left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)^j \Bigg((-1)^m 2\cos\left(\frac{(k-4j)\,\theta}{2} - \pi m\cos\theta\right) \\
&\qquad\qquad + R_{j,m}(\theta)\Bigg) \\
&\quad + \sum_{j=1}^{m} a_{j+\ell} e^{4(j+\ell)i\theta/2} S_4 \left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)^{j+\ell} \\
&\qquad \times \left((-1)^m 2\cos\left(\frac{(k-4\,(j+\ell))\,\theta}{2} - \pi m\cos\theta\right) + R_{j+\ell,m}(\theta)\right).
\end{aligned}$$

Thus $\left|H(\theta) - (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m\cos\theta\right)\right|$ is bounded above by

$$\begin{aligned}
&|R_{0,m}(\theta)| + \sum_{j=1}^{\ell} |a_j| \left(2 + |R_{j,m}(\theta)|\right) \left|S_4\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)\right|^j \\
&\quad + \sum_{j=1}^{m} |a_{j+l}| \left(2 + |R_{j+\ell,m}(\theta)|\right) \left|S_4\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)\right|^{j+\ell} \\
\leq\ & \delta_8^m \delta_5^\ell A + \sum_{j=1}^{\ell} |a_j| \left(2 + \delta_8^m \delta_5^{\ell-j} A\right) \delta_7^j \\
&\quad + \sum_{j=1}^{m} |a_{j+l}| \left(2 + \delta_8^m \left|\frac{\delta_6}{S_4\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)}\right|^j A\right) \left|S_4\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)\right|^{j+\ell} \\
=\ & \delta_8^m \delta_5^\ell A + \sum_{j=1}^{\ell+m} |a_j| \left(2\delta_7^j + \delta_8^m \gamma_2^j(j) A\right).
\end{aligned}$$

Now suppose

$$\sum_{j=1}^{\ell+m} |a_j| \left(2\delta_7^j + \delta_8^m \gamma_2^j(j) A\right) < 2 - \delta_8^m \delta_5^\ell A.$$

Then we have

$$\left| H(\theta) - (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m \cos\theta\right)\right| < 2.$$

This inequality is enough to prove Theorem 4.2 by the same argument as Theorem 2.1. We completed the proof of Theorem 4.2.

4.3 Improving assumption for a theorem of Garthwaite and Jenkins

The assumption for coefficients a_j of Theorem 4.2 is very strict. In this subsection, we consider more manageable assumption for a_j and the proof of Theorem 4.3 is given. For $-\frac{1}{2} + \frac{1}{2}e^{i\theta}$, if θ is close to 0, $-\frac{1}{2} + \frac{1}{2}e^{i\theta}$ is also close to 0. Then $S_4(z)$ and $S_4(\tau)$ have more large values. To avoid this, we restrict the interval $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ to $\theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and change the integral contour from $\tau = x + \frac{i}{5}$ to $\tau = x + \frac{2}{5}i$.

Suppose $\theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$. We consider to bound

$$\begin{aligned}
&\left| e^{ki\theta/2} e^{-\pi m \sin\theta} f_{k,m}\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right) - (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m\cos\theta\right)\right| \\
= \;& e^{-\pi m\left(\sin\theta - \frac{2}{5}\right)} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_4^\ell\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)}{S_4^\ell(x + \frac{2}{5}i)} \frac{F_{k'}\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right) F_2\left(x + \frac{2}{5}i\right)}{F_{k'}\left(x + \frac{2}{5}i\right)} \right. \\
& \qquad\qquad \left. \times \frac{\psi\left(x + \frac{2}{5}i\right)}{\psi\left(x + \frac{2}{5}i\right) - \psi\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)} dx \right|
\end{aligned}$$

We consider the exponential term $e^{-\pi m\left(\sin\theta - \frac{2}{5}\right)}$. It holds that $e^{-\pi m\left(\sin\theta - \frac{2}{5}\right)} < 0.81268$ for $\theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$. We set $\delta_{12} = 0.81268$.

First, we consider

$$\left| \frac{S_4\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)}{S_4\left(x + \frac{2}{5}i\right)} \right|^\ell.$$

We compute
$$0.03 \leq \left| S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right| \leq 0.066968.$$
We set $\delta_{11} = 0.066968$. It holds that
$$0.014 \leq \left| S_4\left(x+\frac{2}{5}i\right)\right| \leq 0.15165.$$
We set $\delta_{10} = 0.15165$. Putting this together, we have, for $\ell \geq 0$,
$$\left|\frac{S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4\left(x+\frac{2}{5}i\right)}\right|^{\ell} \leq |1.6326|^{\ell},$$
and for $\ell < 0$,
$$\left|\frac{S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4\left(x+\frac{2}{5}i\right)}\right|^{\ell} \leq \left(\frac{\delta_{11}}{\left|S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|}\right)^{|\ell|}.$$
We set $\delta_9 = 1.6326$.

Next, we consider the term
$$\left|\frac{F_{k'}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)F_2\left(x+\frac{2}{5}i\right)}{F_{k'}\left(x+\frac{2}{5}i\right)}\right|.$$
If $k' = 2$, this is $\left|F_2\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|$, which is bounded above by 1.7929. If $k' = 0$, this is $\left|F_2\left(x+\frac{2}{5}i\right)\right|$, which is bounded above by 3.1542. Therefore the contribution is bounded above by 3.1542.

Finally, we consider
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\psi\left(x+\frac{2}{5}i\right)}{\psi\left(x+\frac{2}{5}i\right)-\psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}\right| dx.$$
and it is bounded above by 1.74520.

Thus
$$\left|\frac{F_{k'}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)F_2\left(x+\frac{2}{5}i\right)}{F_{k'}\left(x+\frac{2}{5}i\right)}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\psi\left(x+\frac{2}{5}i\right)}{\psi\left(x+\frac{2}{5}i\right)-\psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)} du\right|$$

is bounded above by

$$3.1542 \cdot 1.74520 = 5.50471.$$

We set $B = 5.50471$.

Putting all of these pieces together, we see that for $\ell \geq 0$,

$$e^{-\pi m\left(\sin\theta - \frac{2}{5}\right)} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_4^\ell\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}{S_4^\ell(x+\frac{2}{5}i)} \frac{F_{k'}\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right) F_2\left(x+\frac{2}{5}i\right)}{F_{k'}\left(x+\frac{2}{5}i\right)} \right.$$
$$\left. \times \frac{\psi\left(x+\frac{2}{5}i\right)}{\psi\left(x+\frac{2}{5}i\right) - \psi\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)} dx \right|$$
$$< 0.81268^m \left|1.6326\right|^\ell (5.50471).$$

Note that $(0.81268^m)(5.50471) < 2$ if $m \geq 5$, and $(0.81268^m)(1.6326) < 1$ if $m \geq 3$. Hence, the integral is less than our desired bound 2 if $\ell \geq 0$ and $m \geq 3\ell + 5$. Similarly, for $\ell < 0$, we find that our integral is bounded by 2 if $m \geq 8\left|\ell\right| + 5$. Therefore, we completed the proof of Theorem 4.3.

4.4 Generalization for improved assumption

Finally, we prove Theorem 4.4 in this subsection. By the bounds of Subsection 4.3, we see that

$$\left|R_{j,m}(\theta)\right| < \delta_{12}^m \delta_9^{\ell - j} B$$

for $1 \leq j \leq \ell$ and that

$$\left|R_{j+\ell,m}(\theta)\right| < \delta_{12}^m \left(\frac{\delta_{10}}{\left|S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|} \right)^j B$$

for $1 \leq j \leq m$.

We prove Theorem 4.4 using the bound for $\left|R_{j,m}(\theta)\right|$. When $\ell \geq 1$ and

$m \geq 1$, we can write

$$\begin{aligned}
H(\theta) &= H_{0,m}(\theta) + \sum_{j=1}^{\ell} a_j e^{4ji\theta/2} S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)^j H_{j,m}(\theta) \\
&\quad + \sum_{j=1}^{m} a_{j+\ell} e^{4(j+\ell)i\theta/2} S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)^{j+\ell} H_{j+\ell,m}(\theta) \\
&= (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m\cos\theta\right) + R_{0,m}(\theta) \\
&\quad + \sum_{j=1}^{\ell} a_j e^{4ji\theta/2} S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)^j \Big((-1)^m 2\cos\left(\frac{(k-4j)\,\theta}{2} - \pi m\cos\theta\right) \\
&\qquad + R_{j,m}(\theta)\Big) \\
&\quad + \sum_{j=1}^{m} a_{j+\ell} e^{4(j+\ell)i\theta/2} S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)^{j+\ell} \\
&\qquad \times \left((-1)^m 2\cos\left(\frac{(k-4\,(j+\ell))\,\theta}{2} - \pi m\cos\theta\right) + R_{j+\ell,m}(\theta)\right).
\end{aligned}$$

Thus $\left|H(\theta) - (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m\cos\theta\right)\right|$ is bounded above by

$$\begin{aligned}
&|R_{0,m}(\theta)| + \sum_{j=1}^{\ell} |a_j| \left(2 + |R_{j,m}(\theta)|\right) \left|S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|^j \\
&\quad + \sum_{j=1}^{m} |a_{j+l}| \left(2 + |R_{j+\ell,m}(\theta)|\right) \left|S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|^{j+\ell} \\
\leq\ & \delta_{12}^m \delta_9^\ell B + \sum_{j=1}^{\ell} |a_j| \left(2 + \delta_{12}^m \delta_9^{\ell-j} B\right) \delta_{11}^j \\
&\quad + \sum_{j=1}^{m} |a_{j+l}| \left(2 + \delta_{12}^m \left|\frac{\delta_{10}}{S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)}\right|^j A\right) \left|S_4\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right|^{j+\ell} \\
=\ & \delta_{12}^m \delta_9^\ell B + \sum_{j=1}^{\ell+m} |a_j| \left(2\delta_{11}^j + \delta_{12}^m \gamma_2^j(j) B\right).
\end{aligned}$$

Now suppose

$$\sum_{j=1}^{\ell+m} |a_j| \left(2\delta_{11}^{j} + \delta_{12}^{m}\gamma_3^{j}(j)B\right) < 2 - \delta_{12}^{m}\delta_9^{\ell}B.$$

Then we have

$$\left| H(\theta) - (-1)^m 2\cos\left(\frac{k\theta}{2} - \pi m \cos\theta\right) \right| < 2.$$

This inequality is enough to prove Theorem 4.4 by the same argument as Theorem 2.1 and Theorem 4.2. We completed the proof of Theorem 4.4.

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