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# Reweighting method for nuclear effective field theory on a lattice on the basis of renormalization group analysis 

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## Abstract

Recent development of experiments with radioactive isotope beams and observations of two-solar-mass neutron stars questioned existing knowledge of nuclear physics and have attracted extensive attention. Most ideal theoretical approach to figure out properties of these exotic nuclear systems is so-called lattice quantum chromodynamics (LQCD) which is the first-principle calculation of QCD. However, LQCD has difficulties in describing finite nuclei and nuclear matter. For finite nuclei, high computational costs are required. For nuclear matter, the notorious fermion sign problem caused by finite chemical potential prevents the probability interpretation of the integrand of path integrals in performing Monte Carlo simulations.

Since typical energy scale of nuclear physics is smaller than that of QCD, nuclear effective field theory (NEFT), i.e., low-energy effective theory of QCD, provides an alternative approach. By defining NEFT on a lattice and performing lattice simulations in the same manner as in LQCD, we can study finite nuclei and nuclear matter. However, when we try to increase the physical cutoff, the momentum scale below which effective field theory is valid, fermion sign problem occurs like LQCD with finite chemical potential.

In this dissertation, we propose a method to avoid the sign problem of NEFT on a lattice. The method is the same as the reweighting method which is often used in LQCD calculations with finite chemical potential except for how to determine the reference determinant. Unlike QCD, there is a hierarchy of importance of operators in NEFT. We perform renormalization group analysis to determine which operators are important for low-energy physics. On the basis of the analysis, we distinguish the operators which are relevant in low-energy physics and thus should be included in the reference determinant from those which may not be. To assess the method we propose, we perform lattice simulations and evaluate its effectiveness.

This dissertation is based on the following two papers:

- Numerical study of renormalization group flows of nuclear effective field theory without pions on a lattice,
K. Harada, S. Sasabe, and M. Yahiro, Phys. Rev. C 94, 024004 (2016).
- Reweighting method for nuclear effective field theory on a lattice: an application of renormalization group analysis,
S. Sasabe, K. Harada, and M. Yahiro (unpublished).


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## Contents

Abstract ..... i
Acknowledgement ..... iii
1 Introduction ..... 1
1.1 Current status of nuclear physics ..... 1
1.2 NEFT on a lattice ..... 2
1.3 Overview ..... 4
2 Renormalization group analysis ..... 5
2.1 NLO NEFT without pions in the continuum ..... 5
2.2 RG flows with the lattice-regularized integrals ..... 8
2.3 Ground-state wave function and the ANC ..... 12
2.4 NLO NEFT without pions on a lattice ..... 15
3 Reweighting Method ..... 27
3.1 Fermion sign problem ..... 27
3.2 Fermion matrix on a lattice ..... 30
3.3 Reweighting method based on RG analysis ..... 33
3.4 Computational strategy ..... 36
3.5 Numerical results ..... 37
4 Summary ..... 53

## List of Figures

2.1 The flow and the fixed points of the NLO NEFT in the $X-Y$ plane obtained by using a sharp momentum cutoff in the continuum formulation ..... 9
2.2 The flow and the fixed points of the NLO NEFT in the $X-Y$ plane obtained by using a lattice regularization with the three-point formula ..... 10
2.3 The flow and the fixed points of the NLO NEFT in the $X-Y$ plane obtained by using a lattice regularization with the five-point formula ..... 11
2.4 The cutoff dependence of the binding energy and the ANC ..... 14
2.5 The rotational symmetry breaking in the asymptotic behavior of the wave function ..... 19
2.6 The flow of the NLO NEFT in the strong coupling phase in the $X-Y$ plane obtained by numerical diagonalization of the Hamiltonian defined on a lattice with the five-point formula ..... 20
2.7 The difference of the calculated ground-state energies with $N_{s}=14$ and $N_{s}=16$ and that with $N_{s}=16$ and $N_{s}=18$ as functions of $X$ and $Y$ ..... 21
2.8 The flow calculated with the five-point formula with the ridge line, the nontrivial fixed point, and the relevant direction ..... 22
2.9 The flow calculated with the three-point formula with the ridge line, the nontrivial fixed point, and the relevant direction ..... 23
2.10 The flow line corresponding to deuteron calculated with the five-point formula ..... 24
2.11 The flow line corresponding to deuteron calculated with the three-point formula ..... 25
3.1 The Physical point corresponding to deuteron and the reference point of the reweighting method ..... 35
3.2 The reweighting factor for the point labeled as "Irr7" ..... 39
3.3 The reweighting factor for the point labeled as "Irr7" on the complex plane after tuning the chemical potential $\mu$ ..... 40
3.4 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr7" ..... 40
3.5 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr1" ..... 41
3.6 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr2" ..... 42
3.7 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr3" ..... 42
3.8 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr4" ..... 43
3.9 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr5" ..... 43
3.10 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr6" ..... 44
3.11 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Rel1" ..... 44
3.12 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Rel2" ..... 45
3.13 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Rel3" ..... 45
3.14 The standard deviation of the absolute value of the reweighting factor for the point labeled as "Rel4" ..... 46
3.15 The direction dependence of the standard deviation for $\nu \simeq-40 \mathrm{MeV}$ ..... 47
3.16 The direction dependence of the standard deviation for $\nu \simeq-36 \mathrm{MeV}$ ..... 48
3.17 The direction dependence of the standard deviation for $\nu \simeq-32 \mathrm{MeV}$ ..... 48
3.18 The direction dependence of the standard deviation for $\nu \simeq-28 \mathrm{MeV}$ ..... 49
3.19 The direction dependence of the standard deviation for $\nu \simeq-24 \mathrm{MeV}$ ..... 49
3.20 The direction dependence of the standard deviation for $\nu \simeq-20 \mathrm{MeV}$ ..... 50
3.21 The direction dependence of the standard deviation for $\nu \simeq-16 \mathrm{MeV}$ ..... 50
3.22 The standard deviations of the absolute value of the reweighting factor for the point labeled as "Irr7" obtained with renormalization group analysis and naive dimensional analysis ..... 51

## List of Tables

2.1 Locations of the nontrivial fixed point ..... 12
2.2 Constants for the RW equations ..... 12
3.1 Labels for various sets of $a_{0}$ and $r_{0}$. ..... 34

## Chapter 1

## Introduction

First, we review the current status of nuclear physics in Sec. 1.1. In Sec. 1.2, we introduce nuclear effective field theory (NEFT) and explain its application on a lattice. Finally, we show overview of this dissertation.

### 1.1 Current status of nuclear physics

Recently, main subject of nuclear physics research is switched over from stable nuclei to exotic nuclei with marked improvement in experimental equipment. A prominent example of exotic nuclei is halo nuclei which are unstable nuclei and locate in the vicinity of the drip line in the nuclear chart. Due to the fact that a few nucleons of halo nuclei spread out widely, halo nuclei have larger radii compared with stable nuclei with the same mass contrary to the relation $r \simeq 1.2 A^{1 / 3}$ between radius $r$ and mass number $A$ which reflects the saturation property. Typical example of halo nuclei is ${ }^{11} \mathrm{Li}$ whose radius is as large as that of ${ }^{208} \mathrm{~Pb}$ in spite of the fact that ${ }^{11} \mathrm{Li}$ has much smaller mass number than ${ }^{208} \mathrm{~Pb}$.

Another exotic nuclear system is neutron stars. Neutron stars are compact and dense stars formed in supernova explosions and composed mainly of neutrons. The typical values of their radii and the masses are about 10 km and 1.4 solar mass, respectively. At the center of neutron stars the density is expected to be several times normal nuclear density due to the neutron degeneracy pressure against gravitational collapse. Such high density nuclear matter does not realize in the usual circumstances since nuclear matter becomes stable at the saturation density; the strong external force, i.e., gravitation, makes it realized.

Physics of these exotic nuclear systems received extensive attention as new subjects which enrich our understanding of nuclear physics. The most popular theoretical approaches to investigate unstable nuclei and neutron star matter are quantum many body calculations starting from effective interactions. Since effective interactions are determined by systematic analysis of properties of stable nuclei, the application of effective interactions to such extreme systems is accompanied by uncertainties. Furthermore, the discovery of two-solar-mass neutron stars [12, 2] requires modification of the equation of state obtained with the existing effective interactions.

One of the ideal theoretical approaches which describe nuclear systems without uncertainties is so-called lattice quantum chromodynamics (LQCD). With LQCD, one can figure out properties of a system with strong interaction by defining quantum chromodynamics on a lattice and performing Monte Carlo simulation. Although LQCD
is a powerful tool for us to understand the physics of strongly interacting systems, its application to finite nuclei and nuclear matter is very difficult at present. For finite nuclei, high computational costs are required due to the following reasons. First of all, the number of quark contractions which should be taken in the correlators of nuclei increases rapidly as the number of nucleons of the nuclei increases. Some contraction algorithms [14, 13, 24] have been proposed, but solved the difficulty only partially. Second, a lattice with large spatial volume which is capable of accommodating finite nuclei is required. Third, a lattice with extremely large distance in temporal direction is necessary to separate the signal for the ground state from that for excited states. Recall that the typical energy difference between the ground state and excited states is $\mathcal{O}(1) \mathrm{MeV}$ for finite nuclei, whereas that for hadrons is $\mathcal{O}(100) \mathrm{MeV}$. For these reasons, LQCD simulations for finite nuclei is prohibitively expensive. For nuclear matter, the notorious fermion sign problem caused by finite chemical potential makes the probability interpretation of the integrand of path integrals impossible. To overcome these difficulties, a great deal of effort has been made for many years.

We therefore take an alternative approach to describe finite nuclei and nuclear matter; nuclear effective field theory on a lattice, which I will describe in the next section.

### 1.2 NEFT on a lattice

Since the seminal work on the low-energy effective field theory of nucleons, nuclear effective field theory (NEFT), by Weinberg [49, 50, 51], extensive investigation has been performed; see Refs. $[15,43]$ for the reviews. In contract to QCD in which the fundamental degrees of freedom are quarks and gluons, NEFT describes systems with strong interaction at low-energies in terms of low-lying hadrons, such as nucleons and pions. In NEFT, there is a certain momentum scale, the physical cutoff $\Lambda_{\text {phys }}$ below which NEFT is expected to be equivalent to QCD. The effects of heavier hadrons than $\Lambda_{\text {phys }}$, the processes with momenta higher than $\Lambda_{\text {phys }}$, and the internal structure of the hadrons are integrated out, represented by an infinite number of local operators which satisfy the same symmetries as QCD holds, and have been encoded in the coupling constants of the interactions, low-energy constants. For example, the effects of heavy-meson exchange processes between two nucleons are represented by local four-nucleon operator without derivatives and those with even number of derivatives. It is noteworthy that even if pions are included in NEFT, the exchange of the pion with momentum transfer higher than the cutoff is represented as local four-nucleon (and $2 n$-nucleon, in general) operators.

There is a hierarchy of importance among an infinite number of local operators in NEFT. Classically, the contribution of an operator with a canonical dimension $d$ is of $\left(Q / \Lambda_{\mathrm{phys}}\right)^{d-4}$ order, where $Q$ represents the typical momentum scale of interest. Thus, operators are classified through naive dimensional analysis into the leading order, the next-to-leading order, and so on. Based on the hierarchy of operators, the accuracy of NEFT can be improved systematically by introducing higher order operators. In most cases, the ordering of importance given by naive dimensional analysis is valid. However, quantum fluctuations change the situation drastically in some cases. In such cases, a "quantum" version of dimensional analysis is required, and it is renormalization group analysis that provides it. Renormalization group
analysis in which how the low-energy constants vary as a function of the cutoff is investigated reveals that the contribution of an operator with a canonical dimension $d$ changes from of $\left(Q / \Lambda_{\mathrm{phys}}\right)^{d-4}$ order to of $\left(Q / \Lambda_{\mathrm{phys}}\right)^{-\nu}$ order where $\nu$ is called scaling dimension. An operator with positive $\nu$ is called relevant, whereas that with negative $\nu$ is called irrelevant. Of course, as the typical momentum scale of interest decreases, relevant operators play important role and the effect of irrelevant operators becomes insignificant.

Although the early investigations exclusively employed continuous, semi-analytic approach based on the Lippmann-Schwinger (LS) equation, the Faddeev equation, etc., the methods of numerical simulation on a lattice have been developed recently [38, 4, $5,1,17,34,16,42,35,52,23,22,48,33]$; see Ref. [37] for the review. On a lattice, the cutoff in momentum is given by the lattice constant $a$ as $\pi / a$. Thus, unlike LQCD, the continuum limit in lattice NEFT should not be taken so that the cutoff does not exceed the physical cutoff.

Lattice simulation of NEFT has several advantages. First of all, in the framework of NEFT on a lattice, the investigation of many-nucleon systems can be performed without suffering from complications due to the increase of the number of nucleons. Recall that the Faddeev equation for three nucleons is more complex than the LS equation for two nucleons, and the Faddeev-Yakubovsky equation for four nucleons is even more complex. Lattice formulation does not have this kind of complication. However, there is a drawback; Construction of many-nucleon operators which are necessary when, for example, the correlators of nuclei are calculated becomes complicated corresponding to the increase of the number of nucleons. So far, considerably large nuclei to ${ }^{28}$ Si have already been investigated on a lattice [18, 34, 16]. Second, arbitrarily complicated pion interactions can be taken into account in lattice simulations, just like arbitrarily complicated interactions of gluons can be incorporated in LQCD. Therefore, it has potential of the calculations with the nonlinearly realized exactly chiral symmetric interactions of pions. Note that the truncation of pion interactions at a finite order which is often employed in semi-analytic calculations inevitably breaks chiral symmetry. Third, it is straightforward to make the system contact to a heat reservoir and/or to a particle reservoir. So, it allows us to study finite temperature, finite density system. It is noteworthy that finite chemical potential does not give rise to fermion sign problem in the case that nucleons are dealt with as non-relativistic particles unlike LQCD where relativistic quarks inhabit [7].

Meanwhile, pion interactions which should be taken into account with increasing the physical cutoff $\Lambda_{\text {phys }}$ cause the fermion sign problem [6]. Also, Including higher order contact interactions sometimes brings about the sign problem. This fact prevent us from increasing the cutoff. To increase the cutoff and to include higher order operators are important for NEFT to be more accurate and more applicable.

In this study, we propose a method to avoid the sign problem and assess its validity. Although we confine ourselves to considering the next-to-leading order (NLO) NEFT without pions in this study, in principle, the method we propose can be applied to the case where pion interactions and/or higher order operators are included. Note that the long-distance parts of pion interaction, corresponding to the pion exchanges with the momenta below the cutoff $\Lambda$, are irrelevant in low-energy physics, whereas the short-distance parts are included in the contact interactions in accordance with the general principle of renormalization [28]. The study is an important step toward the chirally symmetric NEFT with pions on a lattice.

### 1.3 Overview

In this dissertation, we develop a reweighting method based on renormalization group analysis. For this purpose, we consider the NLO NEFT without pions since it has both parameter regions where the sign problem occurs and does not.

This dissertation is organized as follows. In Chap. 2, we perform the renormalization group (RG) analysis of the NLO NEFT without pions defined on a lattice by diagonalizing the lattice Hamiltonian numerically. The obtained RG flows are compared with the flow in the continuum and the flows obtained analytically with latticeregularized integrals. Based on the RG analysis performed in Chap. 2, we develop the reweighting method and assess its effectiveness by executing Monte Carlo simulations in Chap. 3. The validity of the method is confirmed by comparing the resulting reweighting factor in the irrelevant direction with that in the relevant direction for various values of chemical potential. In addition to this, we compare the reweighting method based on the RG analysis with that based on the naive dimensional analysis. Finally, Chap. 4 is devoted to a summary.

## Chapter 2

## Renormalization group analysis

### 2.1 NLO NEFT without pions in the continuum

We start with the following isospin $\mathrm{SU}(2)$ symmetric Lagrangian of the NLO NEFT without pions:

$$
\begin{align*}
\mathcal{L}= & N^{\dagger}\left(\mathrm{i} \partial_{t}+\frac{\nabla^{2}}{2 M}\right) N-C_{0}\left(N^{\mathrm{T}} P_{k} N\right)^{\dagger}\left(N^{\mathrm{T}} P_{k} N\right) \\
& +C_{2}\left[\left(N^{\mathrm{T}} P_{k} N\right)^{\dagger}\left(N^{\mathrm{T}} P_{k} \overleftrightarrow{\nabla}^{2} N\right)+\text { H.c. }\right] \tag{2.1}
\end{align*}
$$

where $N$ represents the nucleon field, $M$ is the nucleon mass, and $\overleftrightarrow{\nabla}^{2}=\overleftarrow{\nabla} \cdot \overleftarrow{\nabla}-2 \overleftarrow{\nabla}$ $\vec{\nabla}+\vec{\nabla} \cdot \vec{\nabla}$ corresponds to the momentum transfer squared in the center-of-mass frame of two incoming or outgoing nucleons. $P_{k}$ is a projection operator for a specific channel of the two-nucleon states; for the ${ }^{3} S_{1}$ (spin-triplet) channel, $P_{k}=\sigma^{2} \sigma^{k} \tau^{2} / \sqrt{8}$ where $\sigma^{a}$ and $\tau^{a}$ are spin and isospin Pauli matrices, respectively. The term represented by the momentum independent four-nucleon operator is the leading-order (LO) interaction, whereas the term represented by the four-nucleon operator with two spatial derivatives is the next-to-leading order (NLO) interaction. Thus, $C_{0}$ and $C_{2}$ are the low-energy constants for LO and NLO operators, respectively.

The $S$-wave Lippmann-Schwinger (LS) equation for the off-shell center-of-mass nucleon-nucleon ( $N N$ ) scattering amplitude derived form the Lagrangian, Eq. (2.1), is given by

$$
\begin{align*}
&-\mathrm{i} \mathcal{A}\left(p^{0}, \boldsymbol{p}_{1}, \boldsymbol{p}_{2}\right)=-\mathrm{i} V\left(\boldsymbol{p}_{1}, \boldsymbol{p}_{2}\right)+\int \frac{d^{3} k}{(2 \pi)^{3}} \\
&\left.\times-\mathrm{i} V\left(\boldsymbol{k}, \boldsymbol{p}_{2}\right)\right]  \tag{2.2}\\
& \times \mathrm{i} G\left(p^{0}, \boldsymbol{k}\right)\left[-\mathrm{i} \mathcal{A}\left(p^{0}, \boldsymbol{p}_{1}, \boldsymbol{k}\right)\right]
\end{align*}
$$

where $V$ is the vertex in momentum space,

$$
\begin{equation*}
V\left(\boldsymbol{p}_{1}, \boldsymbol{p}_{2}\right)=C_{0}+4 C_{2}\left(\boldsymbol{p}_{1}^{2}+\boldsymbol{p}_{2}^{2}\right) \tag{2.3}
\end{equation*}
$$

$G(\boldsymbol{k})$ represents the propagator,

$$
\begin{equation*}
G\left(p^{0}, \boldsymbol{k}\right)=\frac{1}{p^{0}-\boldsymbol{k}^{2} / M+\mathrm{i} \epsilon}, \tag{2.4}
\end{equation*}
$$

$p^{0}$ is the off-shell center-of-mass energy of the system, and $\boldsymbol{p}_{1}$ and $\boldsymbol{p}_{2}$ are half the relative momenta in the incoming and outgoing two-nucleon states, respectively.

The LS equation can be solved formally as

$$
\begin{equation*}
\mathcal{A}\left(\boldsymbol{p}_{1}, \boldsymbol{p}_{2}\right)=C_{0}+4 C_{2}\left(\boldsymbol{p}_{1}^{2}+\boldsymbol{p}_{2}^{2}\right)+\left(C_{0} \alpha\left(\boldsymbol{p}_{1}\right)+4 C_{2} \beta\left(\boldsymbol{p}_{1}\right)\right)+4 C_{2} \alpha\left(\boldsymbol{p}_{1}\right) \boldsymbol{p}_{2}^{2}, \tag{2.5}
\end{equation*}
$$

where we suppressed the argument $p^{0}$ and introduced functions $\alpha\left(\boldsymbol{p}_{1}\right)$ and $\beta\left(\boldsymbol{p}_{1}\right)$,

$$
\begin{align*}
\alpha\left(\boldsymbol{p}_{1}\right) & =\int \frac{d^{3} k}{(2 \pi)^{3}} G(\boldsymbol{k}) \mathcal{A}(\boldsymbol{p}, \boldsymbol{k})  \tag{2.6}\\
\beta\left(\boldsymbol{p}_{1}\right) & =\int \frac{d^{3} k}{(2 \pi)^{3}} \boldsymbol{k}^{2} G(\boldsymbol{k}) \mathcal{A}(\boldsymbol{p}, \boldsymbol{k}) . \tag{2.7}
\end{align*}
$$

By multiplying Eq. (2.5) by $G\left(\boldsymbol{p}_{2}\right)(2 \pi)^{3}$ and $\boldsymbol{p}_{2}^{2} G\left(\boldsymbol{p}_{2}\right)(2 \pi)^{3}$ and integrating over $\boldsymbol{p}_{2}$, we obtain the system of linear equations for $\alpha\left(\boldsymbol{p}_{1}\right)$ and $\beta\left(\boldsymbol{p}_{2}\right)$,

$$
\begin{array}{r}
\left(\begin{array}{cc}
1-C_{0} I_{0}-4 C_{2} I_{1} & -4 C_{2} I_{0} \\
-C_{0} I_{1}-4 C_{2} I_{2} & 1-4 C_{2} I_{1}
\end{array}\right)\binom{\alpha\left(\boldsymbol{p}_{1}\right)}{\beta\left(\boldsymbol{p}_{2}\right)} \\
=\binom{C_{0} I_{0}+4 C_{2} I_{1}+4 C_{2} I_{0} \boldsymbol{p}_{1}^{2}}{C_{0} I_{1}+4 C_{2} I_{2}+4 C_{2} I_{1} \boldsymbol{p}_{1}^{2}} \tag{2.8}
\end{array}
$$

where we have introduced the integrals,

$$
\begin{equation*}
I_{n}=-M \int \frac{d^{3} k}{(2 \pi)^{3}} \frac{|\boldsymbol{k}|^{2 n}}{|\boldsymbol{k}|^{2}+\mu^{2}}, \tag{2.9}
\end{equation*}
$$

with

$$
\begin{equation*}
\mu=\sqrt{-M p^{0}-\mathrm{i} \epsilon} \tag{2.10}
\end{equation*}
$$

The system of linear equations can be solved as

$$
\begin{align*}
\alpha\left(\boldsymbol{p}_{1}\right) & =D^{-1}\left[C_{0} I_{0}+4 C_{2}\left(I_{1}-4 C_{2} I_{1}^{2}+4 C_{2} I_{0} I_{2}\right)+4 C_{2} I_{0} \boldsymbol{p}_{1}^{2}\right]  \tag{2.11}\\
\beta\left(\boldsymbol{p}_{1}\right) & =D^{-1}\left[C_{0} I_{1}+4 C_{2} I_{2}+4 C_{2}\left(I_{1}-4 C_{2} I_{1}^{2}+4 C_{2} I_{0} I_{2}\right) \boldsymbol{p}_{1}^{2}\right] \tag{2.12}
\end{align*}
$$

where $D$ represents the determinant of the coefficient matrix,

$$
\begin{equation*}
D=1-C_{0} I_{0}-8 C_{2} I_{1}+16 C_{2}^{2} I_{1}^{2}-16 C_{2}^{2} I_{0} I_{2} \tag{2.13}
\end{equation*}
$$

By substituting Eqs. (2.11) and (2.12) into Eq. (2.5), we obtain the $N N$ scattering amplitude [21, 45, 26] as

$$
\begin{equation*}
\mathcal{A}\left(\boldsymbol{p}_{1}, \boldsymbol{p}_{2}\right)=x+y\left(\boldsymbol{p}_{1}^{2}+\boldsymbol{p}_{2}^{2}\right)+z \boldsymbol{p}_{1}^{2} \boldsymbol{p}_{2}^{2} \tag{2.14}
\end{equation*}
$$

with

$$
\begin{align*}
x & =\left(C_{0}+16 C_{2}^{2} I_{2}\right) / D  \tag{2.15}\\
y & =4 C_{2}\left(1-4 C_{2} I_{1}\right) / D  \tag{2.16}\\
z & =16 C_{2}^{2} I_{0} / D \tag{2.17}
\end{align*}
$$

Note that the integrals $I_{n}$ we introduced are divergent and thus we need to regularize them in some way. If we employ a sharp momentum cutoff $\Lambda$, they are expressed as

$$
\begin{equation*}
I_{n}=-\frac{M}{2 \pi^{2}} \int_{0}^{\Lambda} d k \frac{k^{2 n+2}}{k^{2}+\mu^{2}} \tag{2.18}
\end{equation*}
$$

The Wilsonian RG analysis of the off-shell $N N$ scattering amplitude can be performed elegantly by introducing the energy-dependent redundant operators, which can be eliminated by making use of equations of motion [3, 26, 27]. However, we consider the on-shell formulation since it is simpler and sufficient for our present purpose. See Ref. [26] for the relation between the two formulations. At low energies, the inverse of the on-shell amplitude can be written in powers of the momentum $p=\sqrt{M p^{0}}=\left|\boldsymbol{p}_{1}\right|=\left|\boldsymbol{p}_{2}\right|$ as

$$
\begin{equation*}
\left.\mathcal{A}^{-1}\right|_{\text {on-shell }}=-\frac{M}{4 \pi}\left[-\frac{1}{a_{0}}+\frac{1}{2} r_{0} p^{2}+\mathcal{O}\left(p^{4}\right)-\mathrm{i} p\right] \tag{2.19}
\end{equation*}
$$

This expansion is known as the effective range expansion and the low-energy parameters such as the scattering length $a_{0}$ and the effective range $r_{0}$ characterize the system at low energies.

By performing the effective range expansion for the obtained scattering amplitude, we obtain the scattering length and the effective range as

$$
\begin{align*}
& \frac{M}{4 \pi} \frac{1}{a_{0}}=\frac{M \Lambda}{2 \pi^{2}}\left[\theta_{1}+\frac{\left(1+\theta_{3} Y\right)^{2}}{X-\theta_{5} Y^{2}}\right]  \tag{2.20}\\
& \frac{M}{4 \pi} \frac{r_{0}}{2}=\frac{M}{2 \pi^{2} \Lambda}\left[-R(0)+\frac{Y\left(2+\theta_{3} Y\right)\left(1+\theta_{3} Y\right)^{2}}{\left(X-\theta_{5} Y^{2}\right)^{2}}\right] \tag{2.21}
\end{align*}
$$

where, according to Seki and van Kolck [46], we have introduced dimensionless coupling constants $X$ and $Y$ defined by

$$
\begin{equation*}
C_{0}=\frac{2 \pi^{2}}{M \Lambda} X, \quad 4 C_{2}=\frac{2 \pi^{2}}{M \Lambda^{3}} Y \tag{2.22}
\end{equation*}
$$

the constants $\theta_{n}(n=1,3,5)$ and the function $R(x)$ defined by

$$
\begin{align*}
I_{0} & =-\frac{M \lambda}{2 \pi^{2}}\left[\theta_{1}+\left(\frac{p^{2}}{\Lambda^{2}}\right) R\left(\frac{p^{2}}{\Lambda^{2}}\right)\right]-\frac{\mathrm{i} M}{4 \pi} p,  \tag{2.23}\\
L_{3} & \equiv-M \int \frac{d^{3} k}{(2 \pi)^{3}}=-\frac{M \Lambda^{3}}{2 \pi^{2}} \theta_{3},  \tag{2.24}\\
L_{5} & \equiv-M \int \frac{d^{3} k}{(2 \pi)^{3}}|\boldsymbol{k}|^{2}=-\frac{M \Lambda^{5}}{2 \pi^{2}} \theta_{5} . \tag{2.25}
\end{align*}
$$

If the regularization with the sharp momentum cutoff $\Lambda$ is employed,

$$
\begin{equation*}
\theta_{1}=1, \quad \theta_{3}=\frac{1}{3}, \quad \theta_{5}=\frac{1}{5}, \quad R(0)=-1 . \tag{2.26}
\end{equation*}
$$

It is noteworthy that the expressions Seki and van Kolck obtained do not contain the terms higher than linear in Y of Eqs. (2.20) and (2.21) as a result of their perturbative treatment of the NLO interaction.

We obtain the following RG equations by imposing the condition that $a_{0}$ and $r_{0}$
are independent of $\Lambda$ :

$$
\begin{align*}
\Lambda \frac{d X}{d \Lambda}= & X\left(1+6 \theta_{3} Y\right)+Y^{2}\left(5 \theta_{5}+3 \theta_{3}^{2} X+3 \theta_{3} \theta_{5} Y\right) \\
& +\frac{X-\theta_{5} Y^{2}}{\left(1+\theta_{3} Y\right)^{2}}\left[-R(0)\left(\theta_{3} X+\theta_{5} Y\right)\left(X-\theta_{5} Y^{2}\right)\right. \\
& \left.+\theta_{1}\left\{\theta_{5} Y^{2}\left(3+2 \theta_{3} Y\right)+X\left[1+2 \theta_{3} Y\left(2+\theta_{3} Y\right)\right]\right\}\right]  \tag{2.27}\\
\Lambda \frac{d Y}{d \Lambda}= & 3 Y\left(1+\frac{\theta_{3}}{2} Y\right)\left(1+\theta_{3} Y\right) \\
& +\frac{X-\theta_{5} Y^{2}}{2\left(1+\theta_{3} Y\right)}\left[-R(0) X+4 \theta_{1} Y+\left\{R(0) \theta_{5}+2 \theta_{1} \theta_{3}\right\} Y^{2}\right] \tag{2.28}
\end{align*}
$$

We find the fixed points which make both the right-hand side of Eq. (2.27) and that of Eq. (2.28) zero at the same time, and locate the nontrivial fixed point which is responsible for the "unnaturally" large scattering length in the ${ }^{3} S_{1}$ channel as

$$
\begin{align*}
\left(X_{\star}, Y_{\star}\right) & =\left[\frac{3}{5}(4-3 \sqrt{3}), \frac{3}{2}(-2+\sqrt{3})\right] \\
& =(-0.717691 \ldots,-0.401924 \ldots) \tag{2.29}
\end{align*}
$$

In Fig. 2.1, we show the obtained trivial and nontrivial fixed points together with the RG flow. The arrows indicate the directions in which $X$ and $Y$ evolve as the cutoff decreases.

Note that the location of the fixed point and the flow are not universal, while the existence of the nontrivial fixed point and the scaling dimensions, which are the eigenvalues of the linearized RG equations in the vicinity of the nontrivial fixed point, are universal. We see the fact that the location of the fixed point and the flow depend on the details of the regularization scheme and how they vary as the regularization is changed in the following sections.

### 2.2 RG flows with the lattice-regularized integrals

In this section, we consider to regularize the integrals defined by Eq. (2.9) with a lattice following Seki and van Kolck [46] to obtain RG flows which approximate those on a lattice. We suppose the lattice whose volume is infinite and lattice constant is $a$. We restrict the interval of integration to be the first Brillouin zone,

$$
\begin{equation*}
-\frac{\pi}{a} \leq k_{i} \leq \frac{\pi}{a} \quad(i=1,2,3) \tag{2.30}
\end{equation*}
$$

and replace the momentum square $|\boldsymbol{k}|^{2}$ coming from the Laplacian $\nabla^{2}$ in the continuum with the corresponding discretized one obtained with the three-point formula,

$$
\begin{equation*}
|\boldsymbol{k}|^{2} \rightarrow \frac{4}{a^{2}} \sum_{i=1}^{3} \sin ^{2}\left(\frac{k_{i} a}{2}\right) . \tag{2.31}
\end{equation*}
$$

With this prescription, for example, the integral $I_{0}$ is given as

$$
\begin{equation*}
I_{0}=\frac{M}{a} \prod_{i=1}^{3}\left[\int_{\pi}^{\pi} \frac{d k_{i}}{2 \pi}\right] \frac{1}{p^{2}-4 \sum_{i=1}^{3} \sin ^{2}\left(k_{i} / 2\right)+\mathrm{i} \epsilon} \tag{2.32}
\end{equation*}
$$



Figure 2.1: The flow and the trivial and nontrivial fixed points of the NLO NEFT in the $X-Y$ plane obtained by using a sharp momentum cutoff in the continuum formulation. The arrows indicate the directions in which $X$ and $Y$ evolve as the cutoff decreases.
where we have introduced a dimensionless quantity $p=\sqrt{(M a)\left(p^{0} a\right)}$ and performed the change of variables $k_{i} \rightarrow k_{i} / a$ so that the integration variables are dimensionless.

By identifying $\Lambda$ with $\pi / a$, Seki and van Kolck [46] obtained the values of the constants,

$$
\begin{equation*}
\theta_{1}=1.58796 \ldots, \quad \theta_{3}=\frac{2}{\pi}, \quad R(0)=0.754330 \ldots, \tag{2.33}
\end{equation*}
$$

and $\theta_{5}$ is easily evaluated as $\theta_{5}=12 / \pi^{3}$. (The integral $I_{0}$ in (2.32) can be calculated in a closed form; see Refs. [11, 20].) With these parameters, we find that the nontrivial fixed point is now located at $\left(X_{\star}, Y_{\star}\right)=(-0.76602 \ldots,+0.17501 \ldots)$. Fig. 2.2 shows the obtained fixed points and the RG flow. The flow is very different from the one in the continuum, especially in the strong-coupling phase, i.e., the left-hand part of the figure. It is noteworthy that the sign of $Y_{\star}$ is changed depending on the regularization. These result show a non-universal feature of the RG flow, as one might expect.


Figure 2.2: The same as in Fig. 2.1, but obtained with the lattice regularization with the three-point formula.

In addition to the three-point formula, we also consider the five-point formula,

$$
\begin{equation*}
|\boldsymbol{k}|^{2} \rightarrow \frac{4}{a^{2}} \sum_{i=1}^{3}\left[\sin ^{2}\left(\frac{k_{i} a}{2}\right)+\frac{1}{3} \sin ^{4}\left(\frac{k_{i} a}{2}\right)\right] . \tag{2.34}
\end{equation*}
$$

This formula has higher-order discretization errors than the three-point formula.
As we will show later, effects of the rotational symmetry breaking caused by the discretization with the three-point formula are large. Thus, we perform the same RG analysis with the five-point formula as that with the three-point formula. With this prescription, we obtain values of the constants as

$$
\begin{equation*}
\theta_{1}=1.37619 \ldots, \quad \theta_{3}=\frac{2}{\pi}, \quad \theta_{5}=\frac{15}{\pi^{3}}, \quad R(0)=-0.41278 \ldots . \tag{2.35}
\end{equation*}
$$

Here we have calculated the constants $\theta_{1}$ and $R(0)$ by reference to the method of Appendix of Ref. [46]; see the Appendix of Ref. [29] for more details. With these parameters, we find that the nontrivial fixed point is now located at $\left(X_{\star}, Y_{\star}\right)=$ $(-0.63338 \ldots,-0.098805 \ldots)$. In Fig. 2.3, we show the fixed points and the RG flow.


Figure 2.3: The same as in Fig. 2.1, but obtained with the lattice regularization with the five-point formula.

The flow changes considerably from the case of the three-point formula, especially in the strong-coupling phase and becomes more similar to the flow in the continuum, as one might expect. We summarize the locations of the nontrivial fixed point in Table 2.1 as well as the constants $\theta_{n}(n=1,3,5)$ and $R(0)$ for the RW equations in Table 2.2.

Note that the prescription discussed in this section does not produce a genuine lattice result. On a lattice, the rotational invariance is explicitly broken so that the notion of "partial waves" is not good. However, we just substituted the integrals evaluated with the lattice regularization into the RG equations derived from the LS equation for the $S$ waves. Although the procedure is not fully consistent, the analytic results obtained here are a very useful guide for the genuine lattice study, as shown later.

Table 2.1: Locations of the nontrivial fixed point

| Regularization scheme | $\left(X_{\star}, Y_{\star}\right)$ |
| :---: | :---: |
| Sharp momentum cutoff $\Lambda$ | $(-0.717691 \ldots,-0.401924 \ldots)$ |
| Lattice regularization with |  |
| the three-point formula | $(-0.76602 \ldots,+0.17501 \ldots)$ |
| Lattice regularization with |  |
| the five-point formula | $(-0.63338 \ldots,-0.098805 \ldots)$ |

Table 2.2: Constants for the RW equations

| Regularization scheme | $\theta_{1}$ | $\theta_{3}$ | $\theta_{5}$ | $R(0)$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Sharp momentum cutoff $\Lambda$ | 1 | $\frac{1}{3}$ | $\frac{1}{5}$ | -1 |  |
| Lattice regularization with |  |  |  |  |  |
| the three-point formula | $1.58796 \ldots$ | $\frac{2}{\pi}$ | $\frac{12}{\pi^{3}}$ | $0.754330 \ldots$ |  |
| Lattice regularization with |  |  | $\frac{2}{\pi}$ | $\frac{15}{\pi^{3}}$ | $-0.41278 \ldots$ |
| the five-point formula | $1.37619 \ldots$ | $\frac{15}{\pi}$ |  |  |  |

### 2.3 Ground-state wave function and the ANC

In this section, we consider the stationary Schrödinger equation in the continuum, before we proceed to that defined on a lattice to obtain the genuine lattice result. In momentum space, the stationary Schrödinger equation for the relative motion of the two-nucleon state is given by

$$
\begin{equation*}
E \psi(\boldsymbol{p})=\frac{\boldsymbol{p}^{2}}{M} \psi(\boldsymbol{p})+\int^{\Lambda} \frac{d^{3} q}{(2 \pi)^{3}}\left[C_{0}+4 C_{2}\left(\boldsymbol{p}^{2}+\boldsymbol{q}^{2}\right)\right] \psi(\boldsymbol{q}), \tag{2.36}
\end{equation*}
$$

where we suppose that the wave function satisfies $\psi(\boldsymbol{p})=0$ for $|\boldsymbol{p}|>\Lambda$ and restrict the interval of integration to be the region $|\boldsymbol{q}| \leq \Lambda$. Hereafter, we concentrate on the case with $E<0$, i.e., the bound state.

The Schrödinger equation can be solved in the same matter as the LS equation in Sec. 2.1. By introducing constants $\alpha$ and $\beta$,

$$
\begin{equation*}
\alpha=\int^{\Lambda} \frac{d^{3} q}{(2 \pi)^{3}} \psi(\boldsymbol{q}), \quad \beta=\int^{\Lambda} \frac{d^{3} q}{(2 \pi)^{3}} \boldsymbol{q}^{2} \psi(\boldsymbol{q}), \tag{2.37}
\end{equation*}
$$

we can formally solve the Schrödinger equation as

$$
\begin{equation*}
\psi(\boldsymbol{p})=\frac{-M}{\boldsymbol{p}^{2}+\mu^{2}}\left[\left(C_{0}+4 C_{2} \boldsymbol{p}^{2}\right) \alpha+4 C_{2} \beta\right] \tag{2.38}
\end{equation*}
$$

where $\mu=\sqrt{M|E|}$. By multiplying Eq. (2.38) by $(2 \pi)^{-3}$ and $\boldsymbol{p}^{2}(2 \pi)^{-3}$ and integrating
over $\boldsymbol{p}$, we obtain the system of linear equations for $\alpha$ and $\beta$,

$$
\left(\begin{array}{cc}
1-C_{0} I_{0}-4 C_{2} I_{1} & -4 C_{2} I_{0}  \tag{2.39}\\
-C_{0} I_{1}-4 C_{2} I_{2} & 1-4 C_{2} I_{1}
\end{array}\right)\binom{\alpha}{\beta}=\binom{0}{0} .
$$

Here, we use the integrals $I_{n}$ defined by Eq. (2.9) but with $\mu=\sqrt{M|E|}$. Note that the coefficient matrix is the same as that appears in Eq. (2.8). The system of linear equations has a nonzero solution if the determinant $D$ defined by Eq. (2.13) is equal to 0 . This condition which corresponds to the vanishing of the denominator of the scattering amplitude determines $\mu$, hence the energy eigenvalue $E$.

When the condition is satisfied, by substituting the ratio $\beta / \alpha$,

$$
\begin{equation*}
\frac{\beta}{\alpha}=\frac{1-C_{0} I_{0}-4 C_{2} I_{1}}{4 C_{2} I_{0}} \tag{2.40}
\end{equation*}
$$

into Eq. (2.38), the wave function is written as

$$
\begin{equation*}
\psi(\boldsymbol{p})=-4 M \alpha C_{2}+\frac{M \alpha\left[4 C_{2} \mu^{2}-\left(1-4 C_{2} I_{1}\right) / I_{0}\right]}{\boldsymbol{p}^{2}+\mu^{2}} \tag{2.41}
\end{equation*}
$$

We determine the overall normalization by the usual condition $\int^{\Lambda} d^{3} p /(2 \pi)^{3}|\psi(\boldsymbol{p})|^{2}=$ 1. In the coordinate space, this function is expressed by a sum of the regularized $\delta$ function and the regularized Yukawa function. Therefore, it is natural to define the asymptotic normalization constants (ANC) as the numerator of the second term divided by $4 \pi$ since the Yukawa function governs the asymptotic behavior of the coordinate-space wave function in the limit of $\Lambda \rightarrow \infty$.

By utilizing formulations explained in Sec. 2.1 and this section, We can relate the scattering length and the effective range to the binding energy and the ANC as follows. First, we determine the coupling constants $X$ and $Y$ correspond to the given values of the scattering length and the effective range as a function of the cutoff $\Lambda$ by solving Eqs. (2.20) and (2.21) for each value of the cutoff $\Lambda$. We then obtain the binding energy and the ANC by solving the equation $D=0$ numerically.

In Fig. 2.4, we show the results for the physical set of the scattering length and the effective range for the spin-triplet isospin-singlet S -wave channel, $\left(a_{0}, r_{0}\right)=$ $(5.42,1.75) \mathrm{fm}[10]$, corresponding to deuteron, as a function of the cutoff $\Lambda$. We see that the binding energy and the ANC are constant for a wide range of the cutoff and approximately equal to 2.19 MeV and $0.244 \mathrm{fm}^{-1 / 2}$, respectively. It is noteworthy that the obtained value of the ANC is very close to the recommended value in Ref. [10], $0.8845(8) / \sqrt{4 \pi} \mathrm{fm}^{-1 / 2}=0.2495(2) \mathrm{fm}^{-1 / 2}$, obtained with a completely different $N N$ potential. (The factor $\sqrt{4 \pi}$ comes from the normalization of the spherical harmonics.)

Note that both the binding energy and the ANC vanish at $\Lambda \approx 57.2 \mathrm{MeV}$, corresponding to $\Lambda^{2} / M \approx 3.5 \mathrm{MeV}$ in energy scale or $\pi / \Lambda \approx 10.8 \mathrm{fm}$ in length scale. Recalling that binding energy and the mean-square radius of deuteron are 2.22 MeV and 1.97 fm , we guess the behavior comes from the resolution there is too low to detect the deuteron.

The result we numerically obtained supports that we can take the binding energy and the ANC instead of the scattering length and the effective range as the parameters which characterize the system at low energies. Thus, to obtain the RG flow on a lattice, we use the binding energy and the ANC as low-energy physical quantities to be fixed in the next section.


Figure 2.4: The binding energy (upper) and the ANC (lower) as a function of the cutoff $\Lambda$, calculated for the physical set of the scattering length and the effective range, $\left(a_{0}, r_{0}\right)=(5.42,1.75) \mathrm{fm}$, corresponding to deuteron [10]. The binding energy and the ANC are constant for a wide range of the cutoff and approximately equal to 2.19 MeV and $0.244 \mathrm{fm}^{-1 / 2}$, respectively. The obtained value of the ANC is very close to the recommended value in Ref. [10], $0.8845(8) / \sqrt{4 \pi} \mathrm{fm}^{-1 / 2}=0.2495(2) \mathrm{fm}^{-1 / 2}$, obtained with a completely different $N N$ potential.

### 2.4 NLO NEFT without pions on a lattice

In this section, we consider the Hamiltonian of the NLO NEFT without pions defined on a spatial cubic lattice to obtain a genuine lattice result. We suppose the lattice has $N_{\mathrm{s}}$ sites in each directions, a finite lattice constant $a$, and a finite size $L=N_{\mathrm{s}} a$ and is imposed the periodic boundary condition. On the lattice, the three-dimensional position vector $\boldsymbol{x}$ is replaced with $\boldsymbol{n} a$, where $\boldsymbol{n}$ is a three-dimensional vector with integer components $\boldsymbol{n}=\left(n_{1}, n_{2}, n_{3}\right)$. The periodic boundary condition identifies $\boldsymbol{n}$ with $\boldsymbol{n}+N_{\mathrm{s}} \boldsymbol{e}_{i}$, where $\boldsymbol{e}_{i}(i=1,2,3)$ is the unit vector in the $i$ th direction.

The Hamiltonian of the NLO NEFT without pions in the continuum is given as

$$
\begin{align*}
H=\int d^{3} x\left[N^{\dagger}\left(-\frac{\nabla^{2}}{2 M}\right) N\right. & +C_{0}\left(N^{\mathrm{T}} P_{k} N\right)^{\dagger}\left(N^{\mathrm{T}} P_{k} N\right) \\
& \left.-C_{2}\left\{\left(N^{\mathrm{T}} P_{k} N\right)^{\dagger}\left(N^{\mathrm{T}} P_{k} \stackrel{\nabla}{ }^{2} N\right)\right\}\right] . \tag{2.42}
\end{align*}
$$

By performing the substitutions, $H \rightarrow H_{\mathrm{L}} a^{-1}, \boldsymbol{x} \rightarrow \boldsymbol{n} a, \int d^{3} x \rightarrow a^{3} \sum_{\boldsymbol{n}}, N(\boldsymbol{x}) \rightarrow$ $N_{n} a^{-3 / 2}, M \rightarrow M_{\mathrm{L}} a^{-1}, C_{0} \rightarrow C_{0}^{\mathrm{L}} a^{2}$, and $C_{2} \rightarrow C_{2}^{\mathrm{L}} a^{4}$, we obtain the dimensionless Hamiltonian on a lattice, $H_{\mathrm{L}}$, in terms of dimensionless quantities as

$$
\begin{align*}
H_{\mathrm{L}}= & \sum_{n}\left[-\frac{1}{2 M_{\mathrm{L}}} N_{\boldsymbol{n}}^{\dagger} \nabla_{\mathrm{L}}^{2} N_{\boldsymbol{n}}+C_{0}^{\mathrm{L}}\left(N_{\boldsymbol{n}}^{\mathrm{T}} P_{k} N_{\boldsymbol{n}}\right)^{\dagger}\left(N_{\boldsymbol{n}}^{\mathrm{T}} P_{k} N_{\boldsymbol{n}}\right)\right. \\
& \left.-C_{2}^{\mathrm{L}}\left\{\left(N_{n}^{\mathrm{T}} P_{k} N_{n}\right)^{\dagger}\left(N_{n}^{\mathrm{T}} P_{k} \overleftrightarrow{\nabla}_{\mathrm{L}}^{2} N_{\boldsymbol{n}}\right)++ \text { H.c. }\right\}\right], \tag{2.43}
\end{align*}
$$

where we introduced the dimensionless discretized Laplacian $\nabla_{\mathrm{L}}^{2}$ and $\overleftrightarrow{\nabla}_{\mathrm{L}}^{2}$. For the three-point formula, we define

$$
\begin{equation*}
\nabla_{\mathrm{L}}^{2} N_{\boldsymbol{n}}=\sum_{i=1}^{3}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}-2 N_{\boldsymbol{n}}+N_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right), \tag{2.44}
\end{equation*}
$$

and

$$
\begin{align*}
N_{\boldsymbol{n}}^{\mathrm{T}} \overleftrightarrow{\nabla}_{\mathrm{L}}^{2} N_{\boldsymbol{n}}= & \sum_{i=1}^{3}\left[\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\mathrm{T}}-2 N_{\boldsymbol{n}}^{\mathrm{T}}+N_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\mathrm{T}}\right) P_{k} N_{\boldsymbol{n}}-\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\mathrm{T}}-N_{\boldsymbol{n}}^{\mathrm{T}}\right) P_{k}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}-N_{\boldsymbol{n}}\right)\right. \\
& \left.-\left(N_{\boldsymbol{n}}^{\mathrm{T}}-N_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\mathrm{T}}\right) P_{k}\left(N_{\boldsymbol{n}}-N_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right)+N_{\boldsymbol{n}}^{\mathrm{T}} P_{k}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}-2 N_{\boldsymbol{n}}+N_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right)\right], \tag{2.45}
\end{align*}
$$

so that these derivatives give the same discretized Laplacian in momentum space as we employed in Eq. (2.31) as shown shortly. Similarly, for the five-point formula, we define

$$
\begin{equation*}
\nabla_{\mathrm{L}}^{2} N_{\boldsymbol{n}}=\sum_{i=1}^{3}\left(-\frac{1}{12} N_{\boldsymbol{n}+2 e_{i}}+\frac{4}{3} N_{n+e_{i}}-\frac{5}{2} N_{\boldsymbol{n}}+\frac{4}{3} N_{\boldsymbol{n}-\boldsymbol{e}_{i}}-\frac{1}{12} N_{n-2 e_{i}}\right), \tag{2.46}
\end{equation*}
$$

and

$$
\begin{align*}
N_{\boldsymbol{n}}^{\mathrm{T}} \overleftrightarrow{\nabla}_{\mathrm{L}}^{2} N_{\boldsymbol{n}}= & -\frac{1}{12} \sum_{i=1}^{3}\left[\left(N_{\boldsymbol{n}+2 e_{i}}^{\mathrm{T}}-16 N_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\mathrm{T}}+30 N_{\boldsymbol{n}}^{\mathrm{T}}-16 N_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\mathrm{T}}+N_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\mathrm{T}}\right) P_{k} N_{\boldsymbol{n}}\right. \\
& -\frac{1}{2}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\mathrm{T}}-N_{\boldsymbol{n}}^{\mathrm{T}}\right) P_{k}\left(N_{\boldsymbol{n}+2 e_{i}}-15 N_{\boldsymbol{n}+\boldsymbol{e}_{i}}+15 N_{\boldsymbol{n}}-N_{\boldsymbol{n}-e_{i}}\right) \\
& -\frac{1}{2}\left(N_{\boldsymbol{n}}^{\mathrm{T}}-N_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\mathrm{T}}\right) P_{k}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}-15 N_{\boldsymbol{n}}+15 N_{\boldsymbol{n}-\boldsymbol{e}_{i}}-N_{\boldsymbol{n}-2 e_{i}}\right) \\
& -\frac{1}{2}\left(N_{\boldsymbol{n}+2 \boldsymbol{e}_{i}}^{\mathrm{T}}-15 N_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\mathrm{T}}+15 N_{\boldsymbol{n}}^{\mathrm{T}}-N_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\mathrm{T}}\right) P_{k}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}-N_{\boldsymbol{n}}\right) \\
& -\frac{1}{2}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\mathrm{T}}-15 N_{\boldsymbol{n}}^{\mathrm{T}}+15 N_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\mathrm{T}}-N_{\boldsymbol{n}-2 \boldsymbol{e}_{i}}^{\mathrm{T}}\right) P_{k}\left(N_{\boldsymbol{n}}-N_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right) \\
& \left.+N_{\boldsymbol{n}}^{\mathrm{T}} P_{k}\left(N_{\boldsymbol{n}+2 \boldsymbol{e}_{i}}-16 N_{\boldsymbol{n}+\boldsymbol{e}_{i}}+30 N_{\boldsymbol{n}}-16 N_{\boldsymbol{n}-\boldsymbol{e}_{i}}+N_{\boldsymbol{n}-2 \boldsymbol{e}_{i}}\right)\right] . \tag{2.47}
\end{align*}
$$

We perform Fourier transform for the nucleon operators,

$$
\begin{equation*}
N_{\boldsymbol{n}}=\frac{1}{N_{\mathrm{s}}^{3 / 2}} \sum_{\boldsymbol{p}} e^{\mathrm{i} \boldsymbol{p} \cdot \boldsymbol{n}} a_{\boldsymbol{p}} \tag{2.48}
\end{equation*}
$$

where we suppress the spin and isospin indices, and work in momentum space. Due to the periodic boundary condition, the momentum $\boldsymbol{p}=\left(p_{1}, p_{2}, p_{3}\right)$ takes the discrete values

$$
\begin{equation*}
p_{i}=\frac{2 \pi}{N_{\mathrm{s}}} \hat{p}_{i}, \tag{2.49}
\end{equation*}
$$

where $\hat{p}_{i}(i=1,2,3)$ are integers and satisfy

$$
\begin{equation*}
-\frac{N_{\mathrm{s}}}{2}<\hat{p}_{i} \leq \frac{N_{\mathrm{s}}}{2} . \tag{2.50}
\end{equation*}
$$

The creation and annihilation operators satisfy the canonical anticommutation relations

$$
\begin{equation*}
\left\{a_{\boldsymbol{p}}, a_{\boldsymbol{p}^{\prime}}\right\}=\left\{a_{\boldsymbol{p}}^{\dagger}, a_{\boldsymbol{p}^{\prime}}^{\dagger}\right\}=0, \quad\left\{a_{\boldsymbol{p}}, a_{\boldsymbol{p}^{\prime}}^{\dagger}\right\}=\delta_{\boldsymbol{p}, \boldsymbol{p}^{\prime}} \tag{2.51}
\end{equation*}
$$

By substituting Eq. (2.48) into Eq. (2.43), we obtain the Hamiltonian in terms of creation and annihilation operators,

$$
\begin{align*}
H_{\mathrm{L}}= & \sum_{\boldsymbol{p}} \frac{\Delta_{\boldsymbol{p}}}{2 M_{\mathrm{L}}} a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}}+\frac{1}{N_{\mathrm{s}}^{3}} \sum_{\left\{\boldsymbol{p}_{i}\right\}} \delta_{\boldsymbol{p}_{1}+\boldsymbol{p}_{2}-\boldsymbol{p}_{3}-\boldsymbol{p}_{4}, \mathbf{0}} \\
& \times\left[C_{0}^{\mathrm{L}}+C_{2}^{\mathrm{L}}\left(\Delta_{\boldsymbol{p}_{1}, \boldsymbol{p}_{2}}+\Delta_{\boldsymbol{p}_{3}, \boldsymbol{p}_{4}}\right)\right]\left(a_{\boldsymbol{p}_{1}}^{\dagger} P_{k} a_{\boldsymbol{p}_{2}}^{\dagger}\right)\left(a_{\boldsymbol{p}_{3}} P_{k} a_{\boldsymbol{p}_{4}}\right), \tag{2.52}
\end{align*}
$$

where

$$
\begin{equation*}
\Delta_{p}=4 \sum_{i=1}^{3} \sin ^{2}\left(\frac{p_{i}}{2}\right) \tag{2.53}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta_{p, \boldsymbol{q}}=4 \sum_{i=1}^{3}\left[\sin ^{2}\left(\frac{p_{i}}{2}\right)+\sin ^{2}\left(\frac{q_{i}}{2}\right)-2 \cos \left(\frac{p_{i}+q_{i}}{2}\right) \sin \left(\frac{p_{i}}{2}\right) \sin \left(\frac{q_{i}}{2}\right)\right] \tag{2.54}
\end{equation*}
$$

for the three-point formula, whereas

$$
\begin{equation*}
\Delta_{p}=4 \sum_{i=1}^{3}\left[\sin ^{2}\left(\frac{p_{i}}{2}\right)+\frac{1}{3} \sin ^{4}\left(\frac{p_{i}}{2}\right)\right], \tag{2.55}
\end{equation*}
$$

and

$$
\begin{array}{r}
\Delta_{\boldsymbol{p}, \boldsymbol{q}}=4 \sum_{i=1}^{3}\left[\sin ^{2}\left(\frac{p_{i}}{2}\right)+\frac{1}{3} \sin ^{4}\left(\frac{p_{i}}{2}\right)+\sin ^{2}\left(\frac{q_{i}}{2}\right)+\frac{1}{3} \sin ^{4}\left(\frac{q_{i}}{2}\right)\right. \\
-\cos \left(\frac{p_{i}+q_{i}}{2}\right)\left\{\sin \left(\frac{p_{i}}{2}\right)\left[\sin \left(\frac{q_{i}}{2}\right)+\frac{1}{3} \sin ^{3}\left(\frac{q_{i}}{2}\right)\right]\right. \\
\left.\left.+\left[\sin \left(\frac{p_{i}}{2}\right)+\frac{1}{3} \sin ^{3}\left(\frac{p_{i}}{2}\right)\right] \sin \left(\frac{q_{i}}{2}\right)\right\}\right] \tag{2.56}
\end{array}
$$

for the five-point formula. Note that, for both formulas, $\Delta_{p}$ has the same form as the discretized version of the momentum square we employed in Sec. 2.2 and the relations $\Delta_{p, q}=\Delta_{q, p}$ and $\Delta_{p,-p}=4 \Delta_{p}$ are satisfied.

### 2.4.1 Schrödinger equation for two-nucleon states

We proceed to consider the lattice version of the stationary Schrödinger equation. Let $\left|\Psi^{k}\right\rangle$ be the spin-triplet isospin-singlet two-nucleon state with zero total momentum,

$$
\begin{equation*}
\left|\Psi^{k}\right\rangle=\sum_{\boldsymbol{p}} \psi(\boldsymbol{p}) a_{\boldsymbol{p}}^{\dagger} P_{k}^{\dagger} a_{-\boldsymbol{p}}^{\dagger}|0\rangle . \tag{2.57}
\end{equation*}
$$

The Schrödinger equation is given by

$$
\begin{equation*}
H_{\mathrm{L}}\left|\Psi^{k}\right\rangle=E_{\mathrm{L}}\left|\Psi^{k}\right\rangle, \tag{2.58}
\end{equation*}
$$

where $E_{\mathrm{L}}=E a$ is the dimensionless energy eigenvalue.
In terms of the discretized wave function $\psi(\boldsymbol{p})$ of relative motion in momentum space, we can write the equations as

$$
\begin{equation*}
E_{\mathrm{L}} \psi(\boldsymbol{p})=\frac{\Delta_{\boldsymbol{p}}}{M_{\mathrm{L}}} \psi(\boldsymbol{p})+\frac{1}{N_{\mathrm{s}}^{3}} \sum_{\boldsymbol{q}}\left[C_{0}^{\mathrm{L}}+4 C_{2}^{\mathrm{L}}\left(\Delta_{p}+\Delta_{\boldsymbol{q}}\right)\right] \psi(\boldsymbol{q}) . \tag{2.59}
\end{equation*}
$$

Note that this equation is nothing but the discretized version of Eq. (2.36).
To obtain the binding energy and the ANC, we numerically diagonalize the eigenvalue equation (2.59). Since there is no self-energy contribution in our theory, the lattice constant $a$ in physical unit can be assigned by giving $M_{\mathrm{L}}$ through the relation $M_{\mathrm{L}}=M a$, where we set the nucleon mass, $M=938.9 \mathrm{MeV}$. In the following analysis, we typically consider the case $a=5 \mathrm{fm}$, which corresponds to the momentum cutoff $\Lambda=\pi / a \approx 124 \mathrm{MeV}$. For most of the calculations, we use the lattice whose size is $N_{s}=16$, which corresponds to a cube with the edge of length $L=80 \mathrm{fm}$

Note that although there is no bound state in the weak-coupling phase physically, the periodic boundary condition makes the ground state have negative energy [41].

### 2.4.2 Determination of the ANC

The wave function obtained by diagonalizing the lattice Hamiltonian (2.43) is not rotationally symmetric due to the fact that the lattice discretization breaks the rotational symmetry. To demonstrate the broken rotational symmetry, in Figs. 2.5, we show the coordinate-space wave function normalized by its asymptotic form, $\psi(\boldsymbol{r}) /\left(e^{-\sqrt{M_{\mathrm{L}}\left|E_{\mathrm{L}}\right| r}} / r\right)$, in the $(1,0,0),(1,1,0)$, and $(1,1,1)$ directions where we obtain $\psi(\boldsymbol{r})$ by performing Fourier transform for $\psi(\boldsymbol{p})$. Precisely, for the asymptotic form, we took into account the periodicity. To incorporate the effect of the potentials within the distance $L=N_{\mathrm{s}} a$ is sufficient to obtain converged results. For example, we plot the function in the $(1,0,0)$ direction,

$$
\begin{equation*}
\psi(n a, 0,0) /\left[\frac{e^{-\sqrt{M_{\mathrm{L}}\left|E_{\mathrm{L}}\right|} \mid n a}}{n a}+\frac{e^{-\sqrt{M_{\mathrm{L}}\left|E_{\mathrm{L}}\right|}\left(N_{\mathrm{s}} a-n a\right)}}{N_{\mathrm{s}} a-n a}\right], \tag{2.60}
\end{equation*}
$$

where integers $n$ satisfy $0 \leq n \leq N_{\mathrm{s}}$. Note that the function is symmetric with respect to $N_{\mathrm{s}} / 2$. If the system holds the rotational symmetry, at sufficiently long distances, the wave functions would coincide with each other and show a plateau with the value of ANC. However, the wave function obtained with the three-point formula (upper) shows large direction-dependence. Also, with the five-point formula, we can reduce it largely, but there remains.

Thus, there is an ambiguity in determining the ANC by the asymptotic form of the coordinate space wave function. As an alternative way, we employ the method which utilize the momentum-space wave function. By fitting the numerically obtained the normalized momentum-space wave function $\psi(\boldsymbol{p})$ to the expression implied by the Schrödinger equation (2.59),

$$
\begin{equation*}
\psi(\boldsymbol{p})=A+\frac{B}{\Delta_{p}+M_{\mathrm{L}}\left|E_{\mathrm{L}}\right|}, \tag{2.61}
\end{equation*}
$$

we determine the constants $A$ and $B$. Note that this form of the wave function corresponds to the continuum wave function, Eq. (2.41). As same as in the continuum, we identify $B / 4 \pi$ with the ANC. In this method, the effect of the explicit rotationalsymmetry breaking is absorbed by the discretized Laplacian in momentum space $\Delta_{p}$, so we can determine the ANC without suffering from ambiguity.

### 2.4.3 RG analysis

It has already validated in Sec. 2.3 that the binding energy and the ANC can be taken as the parameters characterize the system at low energies instead of the scattering length and the effective range. In the following analysis, we thus use the binding energy and the ANC as low-energy physical quantities and require them to be independent of the lattice constant, corresponding to the cutoff. To obtain the RG flow, we first calculate the binding energy and the ANC for a set of $(X, Y)$ where $X$ and $Y$ are the dimensionless coupling constants defined in Eq. (2.22), but with $\Lambda=\pi / a$. Then, we change the lattice constant a bit from $a$ to $a+\delta a$, and numerically search the new set of $(X+\delta X, Y+\delta Y)$ which reproduces the same binding energy and the ANC.


Figure 2.5: The rotational symmetry breaking in the asymptotic behavior of the wave function. The wave function in the $(1,0,0),(1,1,0)$, and $(1,1,1)$ directions are obtained with the three-point formula (upper) and with the five-point formula (lower). The gray line indicates the ANC defined as $B / 4 \pi$ from the coefficient $B$ of the regularized Yukawa term in Eq. (2.61). The calculation corresponds to the deuteron state, so that ANC is $0.244 \mathrm{fm}^{-1 / 2}$.

Fig. 2.6 shows the RG flow calculated with the five-point formula. The arrows indicate the change $(\delta X, \delta Y)$ for $a=5 \mathrm{fm}$ and $\delta a=0.25 \mathrm{fm}$. Due to the fictitious feature of the ground-state energy in the weak-coupling phase, we do not calculate the flow in the right upper part of the figure corresponds to the weak coupling phase.


Figure 2.6: The flow of the NLO NEFT in the strong coupling phase in the $X-Y$ plane obtained by numerical diagonalization of the Hamiltonian defined on a lattice with the five-point formula. The allows indicate the directions of the smaller values of the cutoff. The right upper part of the figure corresponds to the weak coupling phase where the periodic boundary condition makes the unphysical ground state with negative energy.

Near the phase boundary, it is difficult to calculate the flow since the groundstate energy becomes small and the effects of the periodic boundary condition become noticeable [40]. The boundary condition affects the wave function with the radius $\sim L / 2=40 \mathrm{fm}$ which corresponds to the binding energy $\sim 0.03 \mathrm{MeV}$. However, we find that the finite-size effect brings about useful information, as shown below.

In Fig. 2.7, we show the difference of the ground-state energies calculated with $N_{\mathrm{s}}=14$ and $N_{\mathrm{s}}=16$, and the difference with $N_{\mathrm{s}}=16$ and $N_{\mathrm{s}}=18$. As one naturally expects, the difference is larger in the $N_{\mathrm{s}}=14 \mathrm{v} . \mathrm{s} . N_{\mathrm{s}}=16$ case than in the $N_{\mathrm{s}}=16$ v.s. $N_{\mathrm{s}}=18$ case.

It is numerically shown that the ridge line is independent of the lattice size $L$. We argue that this ridge line corresponds to the phase boundary as follows. First of all, as we discussed above, the $L$ dependence of the ground-state energy in the strong-coupling phase comes from the spreading of the wave function and thus the energy difference becomes larger as we approach the phase boundary. Second, the $L$ dependence in the weak-coupling phase arises for a different reason; In the weakcoupling phase, the wave function spreads out over the whole space and feels an infinite number of potentials placed periodically, where the lattice size $L$ corresponds to the period of the potentials. As the period is smaller, the "density" of the attractive potential becomes higher and the ground-state gains more negative energy. Finally, for the both sides of the ridge line, $L$ dependence of calculated ground-state energies fit well with the known $L$ dependence of Refs. [40, 41].


Figure 2.7: The difference of the calculated ground-state energies with $N_{s}=14$ and $N_{s}=16$ (upper surface), and that with $N_{s}=16$ and $N_{s}=18$ (lower surface) are shown as functions of $X$ and $Y$. This side of the mountain range corresponds to the weak coupling phase, whereas the other side is the strong coupling phase.

Once we establish that the ridge line represents the phase boundary, it is easy to locate the nontrivial fixed point. In Fig. 2.8, we show the ridge line together with the RG flow shown in Fig. 2.6. Recalling that, around the nontrivial fixed point, the RG flow falls into along the phase boundary and springs from there, we identify where the nontrivial fixed point is. It is $\left(X_{\star}, Y_{\star}\right)=(-0.65$ to $-0.63,-0.13$ to -0.11$)$, which is surprisingly close to the one obtained analytically with the five-point formula in Sec. 2.2.

The direction in which the RG flow springs from the nontrivial fixed point corresponds to the relevant operator. The unit vector for the direction is obtained as $(-1 / \sqrt{2},-1 / \sqrt{2})$ within the accuracy of the present analysis. This is very different from ( $-0.933 \ldots,-0.359 \ldots$ ) obtained from the linearized RGEs derived from Eqs. (2.27) and (2.28) with the five-point formula. We guess that we obtain more similar results as we approach the nontrivial fixed point.


Figure 2.8: The ridge line (red) together with the RG flow in Fig. 2.6 calculated with the five-point formula. From the flow, we infer that the nontrivial fixed point is on the dashed line (green). The nontrivial fixed point is also on the ridge line, it is at the crossing point (blue bullet). The small point (magenta) just above the crossing point is the location of the nontrivial fixed point obtained by the analytic calculation, $\left(X_{\star}, Y_{\star}\right)=(-0.63338 \ldots,-0.098805 \ldots)$.

We perform similar analysis with the three-point formula and show the ridge line together with the RG flow in Fig. 2.9. Note that the RG flow is considerably different from that with the five-point formula. The location of the nontrivial fixed point is obtained as $\left(X_{\star}, Y_{\star}\right)=(-0.75$ to $-0.77,0.12$ to 0.14$)$. It is again very close to the one analytically obtained. The relevant direction is now represented by a unit vector $(-1 / 2,-\sqrt{3} / 2)$ within the accuracy of the present analysis. It should be compared with ( $-0.935 \ldots, 0.353 \ldots$ ) obtained by linearizing RGEs Eqs. (2.27) and (2.28) with the three-point formula around the nontrivial fixed point.


Figure 2.9: The same as in Fig. 2.8, but with the three-point formula. The small point (magenta) indicates the location of the nontrivial fixed point obtained by the analytic calculation, $\left(X_{\star}, Y_{\star}\right)=(-0.76602 \ldots, 0.17501 \ldots)$.

To summarize, the analytic results with the lattice regularization, which are not obtained on a lattice, are very accurate for the location of the nontrivial fixed point, but the direction of the relevant operator is considerably different from the one on a lattice within the accuracy of the present analysis.

### 2.4.4 The flow line corresponding to deuteron

Finally, we draw a flow line that corresponds to deuteron. As input parameters, we use the binding energy $E=2.19 \mathrm{MeV}$ and the $\mathrm{ANC}=0.244 \mathrm{fm}^{-1 / 2}$, which are obtained from the physical set of the scattering length and the effective range in Sec. 2.3. The flow line is obtained as follows. First, we search a point ( $X_{0}, Y_{0}$ ) for which the binding energy and the ANC take the values given above for a certain value of the lattice constant $a_{0}$. Then, we change the lattice constant $5 \%$ larger, $a_{1}=1.05 a_{0}$, and numerically search a new set of coupling constants ( $X_{1}, Y_{1}$ ) which reproduces the same binding energy and ANC. We repeat the procedure iteratively. Recall that the lattice constant is assigned by the dimensionless nucleon mass $M_{\mathrm{L}}=M a$ due to the absence of the self-energy contribution.

We show the flow line together with the RG flow, the nontrivial fixed point, the phase boundary, and the relevant direction in Fig. 2.10 for the five-point formula and in Fig. 2.11 for the three-point formula, respectively.


Figure 2.10: The flow line corresponding to deuteron calculated with the five-point formula is shown as a dotted line (magenta) against the RG flow given in Fig. 2.8. The nontrivial fixed point (blue bullet) phase boundary (red solid line), and the relevant direction (green dashed line) are also shown.

When we draw Figs. 2.10 and 2.11, we change $M_{\mathrm{L}}$ in the region $9 \lesssim M_{\mathrm{L}} \lesssim 80$ which corresponds to $2 \mathrm{fm} \lesssim a \lesssim 17 \mathrm{fm}$. Of course, the lattice with $a \sim 2 \mathrm{fm}$ is too fine for the present EFT; the calculation there should not be taken too seriously.

The part of the flow closest to the nontrivial fixed point corresponds to the lattice constant $a$ in the range from 5 fm to 10 fm , corresponding to the momentum scale from 62 MeV to 124 MeV . This is just the region where the EFT without pions is valid.


Figure 2.11: The same as in Fig. 2.10, but calculated with the three-point formula.

## Chapter 3

## Reweighting Method

### 3.1 Fermion sign problem

We start with the isospin $\operatorname{SU}(2)$ symmetric Lagrangian of the NLO NEFT without pions in Euclidean space:

$$
\begin{align*}
\mathcal{L}_{\mathrm{E}}= & N^{\dagger}\left(-\partial_{4}+\frac{\nabla^{2}}{2 M}+\mu\right) N-C_{0}\left(N^{\mathrm{T}} P_{k} N\right)^{\dagger}\left(N^{\mathrm{T}} P_{k} N\right) \\
& +C_{2}\left[\left(N^{\mathrm{T}} P_{k} N\right)^{\dagger}\left(N^{\mathrm{T}} P_{k} \ddot{\nabla}^{2} N\right)+\text { H.c. }\right], \tag{3.1}
\end{align*}
$$

where we have introduced the chemical potential $\mu$ in the usual manner. The grand canonical partition function is given by

$$
\begin{equation*}
Z \propto \int \mathcal{D} \chi \mathcal{D} \chi^{*} \exp \left(-S_{\mathrm{E}}\right)=\int \mathcal{D} \chi \mathcal{D} \chi^{*} \exp \left[\int d^{4} x \mathcal{L}_{\mathrm{E}}\right] \tag{3.2}
\end{equation*}
$$

where $\chi, \chi^{*}$ are the Grassmann variables corresponding to the nucleon fields $N, N^{\dagger}$, respectively. In the following, we rewrite the Lagrangian density applying HubbardStratonovich transformations [32, 47], perform path-integral for the Grassmann variables analytically, and obtain the partition function in terms of the auxiliary fields and the fermion determinant.

By using identities

$$
\begin{align*}
\left(\sigma^{i} \sigma^{2}\right)_{\alpha \beta}\left(\sigma^{2} \sigma^{i}\right)_{\dot{\alpha} \dot{\beta}} & =\frac{3}{2}\left(\sigma^{0}\right)_{\alpha \dot{\alpha}}\left(\sigma^{0}\right)_{\beta \dot{\beta}}+\frac{1}{2}\left(\sigma^{i}\right)_{\alpha \dot{\alpha}}\left(\sigma^{i}\right)_{\beta \dot{\beta}}  \tag{3.3}\\
\left(\tau^{2}\right)_{\dot{\gamma} \dot{\delta}}\left(\tau^{2}\right)_{\gamma \dot{\delta}} & =-\frac{1}{2}\left(\tau^{0}\right)_{\gamma \dot{\gamma}}\left(\tau^{0}\right)_{\delta \dot{\delta}}+\frac{1}{2}\left(\tau^{i}\right)_{\gamma \dot{\gamma}}\left(\tau^{i}\right)_{\delta \dot{\delta}} \tag{3.4}
\end{align*}
$$

and Fierz transformations, we rewrite the leading order contact interaction term as

$$
\begin{equation*}
-\frac{1}{8} C_{0}\left[\left(N^{\dagger} N\right)^{2}-\left(N^{\dagger} \boldsymbol{\tau} N\right)^{2}\right]=-\frac{1}{8} C_{0}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)^{2} \tag{3.5}
\end{equation*}
$$

where we have introduced

$$
\begin{equation*}
\tilde{\boldsymbol{\tau}}=\left(\tilde{\tau}^{0}, \tilde{\tau}^{1}, \tilde{\tau}^{2}, \tilde{\tau}^{3}\right)=\left(1_{2 \times 2}, \mathrm{i} \tau^{1}, \mathrm{i} \tau^{2}, \mathrm{i} \tau^{3}\right) \tag{3.6}
\end{equation*}
$$

Similarly, the next-to-leading order contact interaction term can be written as

$$
\begin{align*}
& \frac{1}{8} C_{2}\left[\left\{N^{\dagger}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\}\left(N^{\dagger} N\right)+\left(N^{\dagger} N\right)\left\{N^{\dagger}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\}\right. \\
& -\left\{N^{\dagger} \boldsymbol{\tau}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\}\left(N^{\dagger} \boldsymbol{\tau} N\right)-\left(N^{\dagger} \boldsymbol{\tau} N\right)\left\{N^{\dagger} \boldsymbol{\tau}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\} \\
& \left.-2\left\{\left(N^{\dagger} \bar{\nabla} N\right)^{2}+\left(N^{\dagger} \vec{\nabla} N\right)^{2}-\left(N^{\dagger} \boldsymbol{\tau} \overleftarrow{\nabla} N\right)^{2}-\left(N^{\dagger} \boldsymbol{\tau} \vec{\nabla} N\right)^{2}\right\}\right] \\
= & \frac{1}{8} C_{2}\left[\left\{N^{\dagger} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\} \cdot\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)+\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \cdot\left\{N^{\dagger} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\}\right. \\
& \left.-2\left\{\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \overleftarrow{\nabla} N\right)^{2}+\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \vec{\nabla} N\right)^{2}\right\}\right] . \tag{3.7}
\end{align*}
$$

Furthermore, we can rewrite terms in the last line of the above expression as

$$
\begin{align*}
\left(N^{\dagger} \tilde{\boldsymbol{\tau}}\right. & \overleftarrow{\nabla} N)^{2}+\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \vec{\nabla} N\right)^{2} \\
& \rightarrow \frac{1}{2}\left\{\left[\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \overleftarrow{\nabla} N\right)-\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \vec{\nabla} N\right)\right]^{2}-\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \cdot \nabla^{2}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)\right\} \tag{3.8}
\end{align*}
$$

where we have omitted the following total derivative terms:

$$
\begin{equation*}
\nabla\left[\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \cdot \nabla\left(N^{\dagger} \boldsymbol{\tau} N\right)\right] \tag{3.9}
\end{equation*}
$$

Then, the Lagrangian density is given by

$$
\begin{align*}
& \mathcal{L}_{\mathrm{E}}= N^{\dagger}( \\
&\left.-\partial_{4}+\frac{\nabla^{2}}{2 M}+\mu\right) N-c_{0}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)^{2} \\
&+c_{2}[ {\left[N^{\dagger} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\} \cdot\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)+\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \cdot\left\{N^{\dagger} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\} }  \tag{3.10}\\
&\left.-\left[\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \overleftarrow{\nabla} N\right)-\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \vec{\nabla} N\right)\right]^{2}+\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \cdot \nabla^{2}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)\right],
\end{align*}
$$

where we have defined the coupling constants $c_{0}$ and $c_{2}$ by

$$
\begin{equation*}
c_{0} \equiv \frac{C_{0}}{8}, \quad c_{2} \equiv \frac{C_{2}}{8} \tag{3.11}
\end{equation*}
$$

By multiplying the partition function, Eq. (3.2), by

$$
\begin{equation*}
(\text { const. })=\int \mathcal{D} \phi_{0}^{\prime} \exp \left[-\phi_{0}^{\prime 2}\right] \tag{3.12}
\end{equation*}
$$

and making a change of variable from $\phi_{0}^{\prime}$ to $\phi_{0}$ which satisfies

$$
\begin{equation*}
\boldsymbol{\phi}_{0}^{\prime}=\boldsymbol{\phi}_{0}+\mathrm{i} \sqrt{c_{0}}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)-\mathrm{i} \frac{c_{2}}{\sqrt{c_{0}}}\left\{N^{\dagger} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\}-\mathrm{i} \frac{c_{2}}{2 \sqrt{c_{0}}} \nabla^{2}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \tag{3.13}
\end{equation*}
$$

we obtain the terms which eliminate

$$
\begin{align*}
& -c_{0}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)^{2}+c_{2}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \cdot \nabla^{2}\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \\
& +c_{2}\left[\left\{N^{\dagger} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\} \cdot\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right)+\left(N^{\dagger} \tilde{\boldsymbol{\tau}} N\right) \cdot\left\{N^{\dagger} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) N\right\}\right] \tag{3.14}
\end{align*}
$$

in the Lagrangian, Eq. (3.10). Here, we have omitted the terms with four derivatives in accordance with the fact that we have truncated the Lagrangian up to including NLO. Similarly, by introducing another auxiliary field $\vec{\phi}_{2}^{\prime}$ and making a change of variable from $\vec{\phi}_{2}^{\prime}$ to $\vec{\phi}_{2}$ which satisfies

$$
\begin{equation*}
\vec{\phi}_{2}^{\prime}=\vec{\phi}_{2}+\mathrm{i} \sqrt{c_{2}}\left[\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \tilde{\nabla} N\right)-\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \vec{\nabla} N\right)\right] \tag{3.15}
\end{equation*}
$$

we obtain the term which cancels out

$$
\begin{equation*}
-c_{2}\left[\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \overleftarrow{\nabla} N\right)-\left(N^{\dagger} \tilde{\boldsymbol{\tau}} \vec{\nabla} N\right)\right]^{2}, \tag{3.16}
\end{equation*}
$$

in Eq. (3.10).
Thus, we can write the partition function, Eq. (3.2), in terms of auxiliary fields as

$$
\begin{equation*}
Z \propto \int \mathcal{D} \chi \mathcal{D} \chi^{*} \mathcal{D} \phi_{0} \mathcal{D} \vec{\phi}_{2} \exp \left[\int d^{4} x \mathcal{L}_{\mathrm{E}}\left(\chi, \chi^{*}, \boldsymbol{\phi}_{0}, \vec{\phi}_{2}\right)\right] \tag{3.17}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{L}_{\mathrm{E}}= & \chi^{*}\left(-\partial_{4}+\frac{\nabla^{2}}{2 M}+\mu\right) \chi-2 \mathrm{i} \sqrt{c_{0}} \boldsymbol{\phi}_{0} \cdot\left(\chi^{*} \tilde{\boldsymbol{\tau}} \chi\right) \\
& +2 \mathrm{i} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0} \cdot\left\{\chi^{*} \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) \chi\right\}+\mathrm{i} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0} \cdot \nabla^{2}\left(\chi^{*} \tilde{\boldsymbol{\tau}} \chi\right) \\
& -2 \mathrm{i} \sqrt{c_{2}} \overrightarrow{\boldsymbol{\phi}}_{2} \cdot\left[\left(\chi^{*} \tilde{\boldsymbol{\tau}} \overleftarrow{\nabla} \chi\right)-\left(\chi^{*} \tilde{\boldsymbol{\tau}} \vec{\nabla} \chi\right)\right]-\boldsymbol{\phi}_{0}^{2}-\overrightarrow{\boldsymbol{\phi}}_{2}^{2} . \tag{3.18}
\end{align*}
$$

Here derivatives in coordinate space do not act on the auxiliary fields $\boldsymbol{\phi}_{0}$ and $\overrightarrow{\boldsymbol{\phi}}_{2}$. Now, we can perform the path-integral for the Grassmann variables $\chi$ and $\chi^{*}$, then we obtain

$$
\begin{equation*}
Z \propto \int \mathcal{D} \boldsymbol{\phi}_{0} \mathcal{D} \overrightarrow{\boldsymbol{\phi}}_{2} \operatorname{det} M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2}\right) \exp \left[\int d^{4} x\left(-\boldsymbol{\phi}_{0}^{2}-\vec{\phi}_{2}^{2}\right)\right], \tag{3.19}
\end{equation*}
$$

where

$$
\begin{align*}
\operatorname{det} M\left(\phi_{0}, \vec{\phi}_{2}\right)=\operatorname{det}[ & \left(-\partial_{4}+\frac{\nabla^{2}}{2 M}+\mu\right)-2 \mathrm{i} \sqrt{c_{0}} \boldsymbol{\phi}_{0} \cdot \tilde{\boldsymbol{\tau}}+2 \mathrm{i} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0} \cdot \tilde{\boldsymbol{\tau}}\left(\overleftarrow{\nabla}^{2}+\vec{\nabla}^{2}\right) \\
& \left.+\mathrm{i} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0} \cdot \nabla^{2} \tilde{\boldsymbol{\tau}}-2 \mathrm{i} \sqrt{c_{2}} \overrightarrow{\boldsymbol{\phi}}_{2} \cdot\{\tilde{\boldsymbol{\tau}} \overleftarrow{\nabla}-\tilde{\boldsymbol{\tau}} \vec{\nabla}\}\right] . \tag{3.20}
\end{align*}
$$

In order to use Monte Carlo method for integrating the auxiliary fields, we would like to interpret the determinant multiplied by the exponent as a Gibbs factor, i.e., a probability distribution. For the interpretation, it is necessary for the determinant to be positive semidefinite. In the case of $c_{0}, c_{2}<0, \sqrt{c_{i}}=\mathrm{i} \sqrt{\left|c_{i}\right|}(i=0,2)$, the fermion determinant $\operatorname{det} M$ is positive semidefinite, because the fermion matrix $M$ satisfies

$$
\begin{equation*}
\tau^{2} M \tau^{2}=M^{*} \tag{3.21}
\end{equation*}
$$

so that for an eigenvalue $\lambda$ of the matrix $M$, the value $\lambda^{*}$ is also an eigenvalue and if $\lambda \in \boldsymbol{R}, \lambda$ is doubly degenerate [25,36]. In the case of either $c_{0}>0$ or $c_{2}>$ 0 , $\operatorname{det} M\left(\phi_{0}, \vec{\phi}_{2}\right)$ takes a complex value. It is important to note that the chemical potential does not cause fermion sign problem unlike the case of lattice QCD [7].

### 3.2 Fermion matrix on a lattice

In this section, we will obtain the discretized version of the fermion matrix Eq. (3.20). For the spatial directions, the lattice has the same lattice constant, size and boundary condition as defined in Sec. 2.4. For the temporal direction, we introduce a lattice constant $a_{\mathrm{t}}=\alpha_{\mathrm{t}} a$ and a size $L_{\mathrm{t}}=N_{\mathrm{t}} a_{\mathrm{t}}$ and impose the anti-periodic boundary condition.

Although, we consider the discretization with the five-point formula as well as that with the three-point formula in Chap. 2, we adopt only the three-point formula in this chapter to reduce computational costs.

In contrast to the improvement of lattice discretization in spatial directions, that in temporal direction can be done by utilizing the relation between a trace of a product of several normal ordered operators and path integrals without increase of computational costs. Following Lee et al. [38, 39], let us consider the free nucleon case where contact interaction terms are absent. In this case, the simplest lattice action is given by

$$
\begin{equation*}
S_{\mathrm{E}}^{\text {simple }}=\sum_{\boldsymbol{n}}\left[\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{4}}-(1+\mu-6 h) \chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}}-h \sum_{i=1}^{3}\left\{\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}+\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right\}\right], \tag{3.22}
\end{equation*}
$$

where $h=\alpha_{t} / 2 M_{\mathrm{L}}$ and $\boldsymbol{e}_{4}$ means the unit vector in the temporal direction. This action has temporal discretization errors of $\mathcal{O}\left(\alpha_{t}\right)$. To reduce the discretization error, let us start with the operator formalism. The free lattice Hamiltonian is

$$
\begin{equation*}
H^{\mathrm{free}}=\sum_{n}\left[\left(\frac{3}{M_{\mathrm{L}}}-\mu\right) N_{\boldsymbol{n}}^{\dagger} N_{\boldsymbol{n}}-\frac{1}{2 M_{\mathrm{L}}} \sum_{i=1}^{3}\left(N_{\boldsymbol{n}}^{\dagger} N_{\boldsymbol{n}+\boldsymbol{e}_{i}}+N_{\boldsymbol{n}}^{\dagger} N_{n-e_{i}}\right)\right] \tag{3.23}
\end{equation*}
$$

and the grand canonical partition function can be written as

$$
\begin{equation*}
Z=\operatorname{Tr} \exp \left(-\beta H^{\text {free }}\right)=\operatorname{Tr} \exp \left(-\alpha_{\mathrm{t}} H^{\text {free }}\right)^{N_{\mathrm{t}}} \tag{3.24}
\end{equation*}
$$

where we have divided into $N_{\mathrm{t}}$ pieces. By using the identity [8, 9], one can get

$$
\begin{equation*}
\exp \left(-\alpha_{\mathrm{t}} H^{\text {free }}\right)=: \exp \left(-h^{\text {free }}\right):+\mathcal{O}(h) \tag{3.25}
\end{equation*}
$$

where

$$
\begin{equation*}
h^{\mathrm{free}}=\sum_{n}\left[\left\{1-e^{\mu \alpha_{t}}(1-6 h)\right\} N_{\boldsymbol{n}}^{\dagger} N_{\boldsymbol{n}}-h e^{\mu \alpha_{t}} \sum_{i=1}^{3}\left\{N_{\boldsymbol{n}}^{\dagger} N_{\boldsymbol{n}+\boldsymbol{e}_{i}}+N_{\boldsymbol{n}}^{\dagger} N_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right\}\right] . \tag{3.26}
\end{equation*}
$$

Following Refs. [38, 39], we have introduced the additional $e^{\mu \alpha_{t}}$ factor multiplying $h$ so that in the resultant action the chemical potential appears in the usual manner [30, 31]. Then, we use the relation between a trace of a product of several normal ordered operators and path integrals,

$$
\begin{align*}
& \operatorname{Tr}: f_{n}\left(N^{\dagger}, N\right):: \ldots:: f_{1}\left(N^{\dagger} N\right): \\
& =\int d \chi_{n} d \chi_{n}^{*} \ldots d \chi_{1} d \chi_{1}^{*} \exp \left[\sum_{j=1}^{n} \chi_{j}^{*}\left(\chi_{j}-\chi_{j+1}\right)\right] \prod_{j=1}^{n} f_{j}\left(\chi_{j}^{*}, \chi_{j}\right), \tag{3.27}
\end{align*}
$$

with $\chi_{n+1}=-\chi_{1}$ and obtain

$$
\begin{equation*}
S_{\mathrm{E}}^{\text {free }}=\sum_{n}\left[\chi_{n}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{4}}-e^{\mu \alpha_{\mathrm{t}}}(1-6 h) \chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}}-h e^{\mu \alpha_{\mathrm{t}}} \sum_{i=1}^{3}\left\{\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}+\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right\}\right] . \tag{3.28}
\end{equation*}
$$

It is conventional to redefine the Grassmann field $\chi_{n}$ as

$$
\begin{equation*}
\chi_{n}^{\prime}=e^{\mu \alpha_{t}} \chi_{n} . \tag{3.29}
\end{equation*}
$$

Thus, the partition function is given by

$$
\begin{equation*}
Z \simeq e^{g \mu \alpha_{\mathrm{t}} N_{\mathrm{t}} N_{\mathbf{s}}^{3}} \int \mathcal{D} \chi^{\prime} \mathcal{D} \chi^{*} \exp \left(-\tilde{S}_{\mathrm{E}}^{\text {free }}\right) \tag{3.30}
\end{equation*}
$$

where $g$ represents the number of internal degrees of freedom (in the present case of $g=2 \times 2=4$, taking into account the spin and isospin degrees of freedom) and

$$
\begin{equation*}
S_{\mathrm{E}}^{\mathrm{free}}=\sum_{\boldsymbol{n}}\left[e^{-\mu \alpha_{\mathrm{t}}} \chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{4}}^{\prime}-(1-6 h) \chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}}^{\prime}-h \sum_{i=1}^{3}\left\{\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\prime}+\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\prime}\right\}\right] \tag{3.31}
\end{equation*}
$$

This action has temporal discretization errors of $\mathcal{O}(h)$. In the region where the NEFT without pions is valid, $M_{\mathrm{L}}=M a / \hbar c$ is much larger than 1 ; for example, for $a=5 \mathrm{fm}$, $M_{\mathrm{L}} \sim 25$. Also, this action differs from the simple one of Eq (3.22) just by coefficients. Thus, the discretization errors are reduced without increase of computational costs and this action is superior to the simple one.

Next, we consider to take into account the contact interaction terms. When we write down the free lattice Hamiltonian, Eq (3.23), we have performed the substitution,

$$
\begin{equation*}
N^{\dagger} \frac{\nabla^{2}}{2 M} N \rightarrow \frac{1}{2 M_{\mathrm{L}}} \sum_{i=1}^{3} N_{n}^{\dagger}\left(N_{\boldsymbol{n}+\boldsymbol{e}_{i}}-2 N_{\boldsymbol{n}}+N_{\boldsymbol{n}-\boldsymbol{e}_{i}}\right) \tag{3.32}
\end{equation*}
$$

Corresponding to this substitution, we also perform the following substitutions,

$$
\begin{align*}
N^{\dagger} \overleftarrow{\nabla}^{2} N \rightarrow & \sum_{i=1}^{3}\left(N_{n+e_{i}}^{\dagger}-2 N_{n}^{\dagger}+N_{n-e_{i}}^{\dagger}\right) N_{n}  \tag{3.33}\\
2 N^{\dagger} \overleftarrow{\nabla} \cdot \vec{\nabla} N \rightarrow & \sum_{i=1}^{3}\left[\left(N_{n+e_{i}}^{\dagger}-N_{n}^{\dagger}\right)\left(N_{n+e_{i}}-N_{n}\right)\right. \\
& \left.+\left(N_{n}^{\dagger}-N_{n-e_{i}}^{\dagger}\right)\left(N_{n}-N_{n-e_{i}}\right)\right] . \tag{3.34}
\end{align*}
$$

This definition is determined so as to the relation,

$$
\begin{equation*}
\int d^{3} x \nabla^{2}\left(N^{\dagger} N\right)=\int d^{3} x N^{\dagger}\left(\overleftarrow{\nabla}^{2}+2 \overleftarrow{\nabla} \cdot \vec{\nabla}+\vec{\nabla}^{2}\right) N=0 \tag{3.35}
\end{equation*}
$$

holds on a lattice. In addition to this, the substitution for the first derivative,

$$
\begin{align*}
& 2 \nabla_{i}\left(N^{\dagger} N\right) \rightarrow\left(N_{n+e_{i}}^{\dagger}-N_{n}^{\dagger}\right)\left(N_{n+e_{i}}+N_{n}\right) \\
& \quad+\left(N_{n+e_{i}}^{\dagger}+N_{n}^{\dagger}\right)\left(N_{n+e_{i}}-N_{n}\right), \tag{3.36}
\end{align*}
$$

is determined so that the relation,

$$
\begin{equation*}
\int d^{3} x \nabla\left[N^{\dagger} N \nabla\left(N^{\dagger} N\right)\right]=\int d^{3} x\left[\left\{\nabla\left(N^{\dagger} N\right)\right\}^{2}+N^{\dagger} N \nabla^{2}\left(N^{\dagger} N\right)\right]=0 \tag{3.37}
\end{equation*}
$$

also holds on a lattice by combining the substitution rule for the second derivative defined above.

By using Eqs. (3.32), (3.33), (3.34), and (3.36), the lattice action for the NLO NEFT without pions is given by

$$
\begin{align*}
S_{\mathrm{E}}^{\mathrm{NLO}}=\sum_{n}[ & e^{-\mu \alpha_{\mathrm{t}}} \chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{4}}^{\prime}-(1-6 h) \chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}}^{\prime}-h \sum_{i=1}^{3}\left(\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\prime}+\chi_{\boldsymbol{n}}^{*} \chi_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\prime}\right) \\
& +2 \mathrm{i} \alpha_{\mathrm{t}} \sqrt{c_{0}} \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot\left(\chi_{n}^{*} \tilde{\boldsymbol{\tau}} \chi_{\boldsymbol{n}}^{\prime}\right) \\
& -2 \mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot \sum_{i=1}^{3}\left(\chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{*}-2 \chi_{\boldsymbol{n}}^{*}+\chi_{\boldsymbol{n}}^{*}\right) \tilde{\boldsymbol{\tau}} \chi_{\boldsymbol{n}}^{\prime} \\
& -2 \mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot \sum_{i=1}^{3} \chi_{\boldsymbol{n}}^{*} \tilde{\boldsymbol{\tau}}\left(\chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\prime}-2 \chi_{\boldsymbol{n}}^{\prime}+\chi_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\prime}\right) \\
& -\mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot \sum_{i=1}^{3}\left(\chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{*} \tilde{\boldsymbol{\tau}} \chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\prime}-2 \chi_{\boldsymbol{n}}^{*} \tilde{\boldsymbol{\tau}} \chi_{\boldsymbol{n}}^{\prime}+\chi_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{*} \tilde{\boldsymbol{\tau}} \chi_{\boldsymbol{n}-\boldsymbol{e}_{i}}^{\prime}\right) \\
& -2 \mathrm{i} \alpha_{\mathrm{t}} \sqrt{c_{2}} \chi_{n}^{*} \sum_{i=1}^{3}\left\{\left(\vec{\phi}_{2, \boldsymbol{n}}\right)_{i} \cdot \tilde{\boldsymbol{\tau}} \chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{\prime}\right\}+2 \mathrm{i} \alpha_{\mathrm{t}} \sqrt{c_{2}} \sum_{i=1}^{3}\left\{\left(\overrightarrow{\boldsymbol{\phi}}_{2, \boldsymbol{n}}\right)_{i} \chi_{\boldsymbol{n}+\boldsymbol{e}_{i}}^{*}\right\} \cdot \tilde{\boldsymbol{\tau}} \chi_{\boldsymbol{n}}^{\prime} \\
& \left.+\boldsymbol{\phi}_{0, \boldsymbol{n}}^{2}+\vec{\phi}_{2, \boldsymbol{n}}^{2}\right] . \tag{3.38}
\end{align*}
$$

Thus, the fermion matrix on the lattice is written as

$$
\begin{aligned}
M_{\boldsymbol{n}, \boldsymbol{m}}= & e^{-\mu \alpha_{\mathrm{t}}} \delta_{\boldsymbol{n}+\boldsymbol{e}_{4}, \boldsymbol{m}}-(1-6 h) \delta_{\boldsymbol{n}, \boldsymbol{m}}-h \sum_{i=1}^{3}\left(\delta_{\boldsymbol{n}+\boldsymbol{e}_{i}, \boldsymbol{m}}+\delta_{\boldsymbol{n}+\boldsymbol{e}_{\boldsymbol{i}}, \boldsymbol{m}}\right) \\
& +2 \mathrm{i} \alpha_{\mathrm{t}} \sqrt{c_{0}} \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot \tilde{\boldsymbol{\tau}} \delta_{\boldsymbol{n}, \boldsymbol{m}} \\
& -2 \mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot \tilde{\boldsymbol{\tau}} \sum_{i=1}^{3}\left(\delta_{\boldsymbol{n}, \boldsymbol{m}+\boldsymbol{e}_{i}}-2 \delta_{\boldsymbol{n}, \boldsymbol{m}}+\delta_{\boldsymbol{n}, \boldsymbol{m}-\boldsymbol{e}_{i}}\right) \\
& -2 \mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{\sqrt{c_{0}}} \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot \tilde{\boldsymbol{\tau}} \sum_{i=1}^{3}\left(\delta_{\boldsymbol{n}+\boldsymbol{e}_{i}, \boldsymbol{m}}-2 \delta_{\boldsymbol{n}, \boldsymbol{m}}+\delta_{\boldsymbol{n}-\boldsymbol{e}_{i}, \boldsymbol{m}}\right) \\
& -\mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{\sqrt{c_{0}}} \sum_{i=1}^{3}\left(\boldsymbol{\phi}_{0, \boldsymbol{n}+\boldsymbol{e}_{i}}-2 \boldsymbol{\phi}_{0, \boldsymbol{n}}+\boldsymbol{\phi}_{0, \boldsymbol{n}-\boldsymbol{e}_{i}}\right) \cdot \tilde{\boldsymbol{\tau}} \delta_{\boldsymbol{n}, \boldsymbol{m}} \\
& -2 \mathrm{i} \alpha_{\mathrm{t}} \sqrt{c_{2}} \sum_{i=1}^{3}\left\{\left(\overrightarrow{\boldsymbol{\phi}}_{2, \boldsymbol{n}}\right)_{i} \cdot \tilde{\boldsymbol{\tau}} \delta_{\boldsymbol{n}+\boldsymbol{e}_{i}, \boldsymbol{m}}-\left(\overrightarrow{\boldsymbol{\phi}}_{2, \boldsymbol{m}}\right)_{i} \cdot \tilde{\boldsymbol{\tau}} \delta_{\boldsymbol{n}, \boldsymbol{m}+\boldsymbol{e}_{i}}\right\}
\end{aligned}
$$

$$
\begin{align*}
= & e^{-\mu \alpha_{t}} \delta_{\boldsymbol{n}+\boldsymbol{e}_{4}, \boldsymbol{m}}+\left[-(1-6 h)+2 \mathrm{i} \alpha_{t} \sqrt{c_{0}}\left(1+15 \frac{c_{2}}{c_{0}}\right) \boldsymbol{\phi}_{0, \boldsymbol{n}} \cdot \tilde{\boldsymbol{\tau}}\right. \\
& \left.-\mathrm{i} \alpha_{t} \frac{c_{2}}{\sqrt{c_{0}}} \sum_{i=1}^{3}\left(\boldsymbol{\phi}_{0, \boldsymbol{n}+\boldsymbol{e}_{i}}+\boldsymbol{\phi}_{0, \boldsymbol{n}-\boldsymbol{e}_{i}}\right) \cdot \tilde{\boldsymbol{\tau}}\right] \delta_{\boldsymbol{n}, \boldsymbol{m}} \\
& +\sum_{i=1}^{3}\left[-h-2 \mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{\sqrt{c_{0}}}\left(\boldsymbol{\phi}_{0, \boldsymbol{n}+\boldsymbol{e}_{i}}+\boldsymbol{\phi}_{0, \boldsymbol{n}}\right) \cdot \tilde{\boldsymbol{\tau}}-2 \mathrm{i} \alpha_{\mathrm{t}} \sqrt{c_{2}}\left(\overrightarrow{\boldsymbol{\phi}}_{2, \boldsymbol{n}}\right)_{i} \cdot \tilde{\boldsymbol{\tau}}\right] \delta_{\boldsymbol{n}+\boldsymbol{e}_{i}, \boldsymbol{m}} \\
& +\sum_{i=1}^{3}\left[-h-2 \mathrm{i} \alpha_{\mathrm{t}} \frac{c_{2}}{c_{0}} \sqrt{c_{0}}\left(\boldsymbol{\phi}_{0, \boldsymbol{n}-\boldsymbol{e}_{i}}+\boldsymbol{\phi}_{0, \boldsymbol{n}}\right) \cdot \tilde{\boldsymbol{\tau}}+2 \mathrm{i} \alpha_{\mathrm{t}} \sqrt{c_{2}}\left(\overrightarrow{\boldsymbol{\phi}}_{2, \boldsymbol{n}-\boldsymbol{e}_{i}}\right)_{i} \cdot \tilde{\boldsymbol{\tau}}\right] \delta_{\boldsymbol{n}-\boldsymbol{e}_{i}, \boldsymbol{m}} \tag{3.39}
\end{align*}
$$

We suppress the effect of the boundary condition in this expression.
Since the lattice discretization does not affect internal symmetry, in the case of $c_{0}, c_{2}<0$, the matrix $M_{n, m}$ satisfies the relation Eq. (3.21) in the continuum, so there is no sign problem. On the other hand, in the case of either $c_{0}>0$ or $c_{2}>0$, the sign problem occurs and we cannot perform path integrals by using a Monte Carlo method.

In the most part of the RG flow which corresponds to the deuteron, $Y$ takes a positive value as shown in Fig. 2.11, so, if we would like to perform the simulation with such a value of $Y$, we would suffer from the sign problem.

### 3.3 Reweighting method based on RG analysis

In this section, we will propose the reweighting method based on the RG analysis to avoid the sign problem. The reweighting method [19] is one of the methods often employed to avoid the sign problem in LQCD simulations. The basic idea of the reweighting method is shown schematically as follows: First, we rewrite the partition function as

$$
\begin{equation*}
Z=\int \mathcal{D} \phi \operatorname{det} M(\phi) e^{-S(\phi)}=\int \mathcal{D} \phi \frac{\operatorname{det} M(\phi)}{\operatorname{det} M^{\prime}(\phi)} \operatorname{det} M^{\prime}(\phi) e^{-S(\phi)}, \tag{3.40}
\end{equation*}
$$

where $\operatorname{det} M^{\prime}(\phi)$ is assumed to be positive semidefinite. Second, we perform path integral based on Monte Carlo method regarding $\operatorname{det} M^{\prime}(\phi) e^{-S(\phi)}$ as the probability function. Finally, we deal with $\operatorname{det} M(\phi) / \operatorname{det} M^{\prime}(\phi)$ as the reweighting factor which corrects the difference between the original "probability" function, $\operatorname{det} M(\phi) e^{-S(\phi)}$, and the probability function which we employ to sample configurations, $\operatorname{det} M^{\prime}(\phi) e^{-S(\phi)}$. In this method, the expectation value of an operator $\mathcal{O}$ is evaluated as

$$
\begin{equation*}
\langle\mathcal{O}\rangle=\frac{\int \mathcal{D} \phi \mathcal{O} \frac{\operatorname{det} M(\phi)}{\operatorname{det} M^{\prime}(\phi)} \operatorname{det} M^{\prime}(\phi) e^{-S(\phi)}}{\int \mathcal{D} \phi \frac{\operatorname{det} M^{\prime}(\phi)}{\operatorname{det} M^{\prime}(\phi)} \operatorname{det} M^{\prime}(\phi) e^{-S(\phi)}} \tag{3.41}
\end{equation*}
$$

Although det $M^{\prime}(\phi)$ can be chosen arbitrarily, it is preferable in practice that the reweighting factor does not strongly depend on $\phi$. From the viewpoint of the RG analysis, we propose to omit irrelevant operators from the fermion matrix, since irrelevant operators are insignificant at low energies, whereas relevant operators play an

Table 3.1: Labels for various sets of $a_{0}$ and $r_{0}$.

| Label | $a_{0}[\mathrm{fm}]$ | $r_{0}[\mathrm{fm}]$ |
| :---: | :---: | :---: |
| Ref | 5.42 | 0.00 |
| Irr1 | 5.42 | 0.25 |
| Irr2 | 5.42 | 0.55 |
| Irr3 | 5.42 | 0.85 |
| Irr4 | 5.42 | 1.05 |
| Irr5 | 5.42 | 1.25 |
| Irr6 | 5.42 | 1.55 |
| Irr7 | 5.42 | 1.75 |
| Rel1 | 4.42 | 0.00 |
| Rel2 | 3.42 | 0.00 |
| Rel3 | 2.42 | 0.00 |
| Rel4 | 2.00 | 0.00 |

important role. We expect that the fermion determinant without irrelevant operators mimics the original one well and that the reweighting factor depends on $\phi$ weakly.

Since the relevant operator corresponds to the scattering length and the leading irrelevant operator to the effective range, we exclude the irrelevant operator by fixing the effective range calculated by the diagonalization to be 0.00 fm , while making the strength of the relevant operator by requiring the scattering length to be the physical value, 5.42 fm . In Fig. 3.1, we show the reference point, $\left(a_{0}, r_{0}\right)=(5.42,0.00) \mathrm{fm}$, of the reweighting method and the physical point, $\left(a_{0}, r_{0}\right)=(5.42,1.75) \mathrm{fm}$, in the case of the lattice constant $a=5 \mathrm{fm}$ together with the RG flow, the nontrivial fixed point, the phase boundary, and the relevant direction. We also show in Fig. 3.1 the points with the physical values of scattering length and various values of effective range, as well as points with the vanishing value of the effective range and various values of scattering length. The former points are considered to be along the irrelevant direction, while the latter along the relevant direction. In Table 3.1, we summarize the relation between $\left(a_{0}, r_{0}\right)$ and labels shown in Fig. 3.1. Note that the nontrivial fixed point corresponds to $\left(a_{0}, r_{0}\right)=(\infty, 0.00)$. In the following analysis, we will study the dependence of the reweighting factor on these directions.

The reweighting method we propose can be expressed as

$$
\begin{align*}
Z \simeq & e^{g \mu \alpha_{\mathrm{t}} N_{\mathrm{t}} N_{\mathrm{s}}^{3}} \int \mathcal{D} \phi_{0} \mathcal{D} \overrightarrow{\boldsymbol{\phi}}_{2} \operatorname{det} M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}, c_{2} ; \mu\right) \exp \left[\sum_{n}\left(-\boldsymbol{\phi}_{0, n}^{2}-\overrightarrow{\boldsymbol{\phi}}_{2, n}^{2}\right)\right], \\
= & e^{g \mu \alpha_{\mathrm{t}} N_{\mathrm{t}} N_{\mathrm{s}}^{3}} \int \mathcal{D} \boldsymbol{\phi}_{0} \mathcal{D} \overrightarrow{\boldsymbol{\phi}}_{2} \frac{\operatorname{det} M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}, c_{2} ; \mu\right)}{\operatorname{det} M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}^{\text {Ref }}, c_{2}^{\mathrm{Ref}} ; \nu\right)} \\
& \quad \times \operatorname{det} M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}^{\mathrm{Ref}}, c_{2}^{\text {Ref }} ; \nu\right) \exp \left[\sum_{n}\left(-\boldsymbol{\phi}_{0, n}^{2}-\overrightarrow{\boldsymbol{\phi}}_{2, n}^{2}\right)\right], \tag{3.42}
\end{align*}
$$

where the form of the fermion matrix $M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}, c_{2} ; \mu\right)$ is given in Eq. (3.39). Here, we have introduced a different value of the chemical potential for the reference deter-


Figure 3.1: The Physical point corresponding to deuteron and the reference point of the reweighting method where $\left(a_{0}, r_{0}\right)=(5.42,0.00) \mathrm{fm}$ at $a=5 \mathrm{fm}$ are shown by magenta and red points, respectively. The cyan-colored points are in the irrelevant direction, corresponding to $a_{0}=5.42 \mathrm{fm}$ and $0.00 \mathrm{fm}<r_{0}<1.75 \mathrm{fm}$, while the orange-colored points are in the relevant direction, corresponding to $a_{0}<5.42 \mathrm{fm}$ and $r_{0}=0.00 \mathrm{fm}$.
minant. The purpose of it will be explained shortly.
The expectation value of the reweighting factor we can evaluate on the lattice is

$$
\begin{align*}
& \left\langle\frac{\operatorname{det} M\left(\phi_{0}, \vec{\phi}_{2} ; c_{0}, c_{2} ; \mu\right)}{\operatorname{det} M\left(\phi_{0}, \vec{\phi}_{2} ; c_{0}^{\text {Ref }}, c_{2}^{\text {Ref }} ; \nu\right)}\right\rangle \\
= & e^{g \nu \alpha_{t} N_{t} N_{\mathbf{s}}^{3}} \int \mathcal{D} \phi_{0} \mathcal{D} \vec{\phi}_{2} \frac{\operatorname{det} M\left(\phi_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}, c_{2} ; \mu\right)}{\operatorname{det} M\left(\phi_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}^{\text {Ref }}, c_{2}^{\text {Ref }} ; \nu\right)} \\
& \quad \times \frac{1}{Z^{\text {Ref }}} \operatorname{det} M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}^{\text {Ref }}, c_{2}^{\text {Ref }} ; \nu\right) \exp \left[\sum_{n}\left(-\boldsymbol{\phi}_{0, n}^{2}-\overrightarrow{\boldsymbol{\phi}}_{2, n}^{2}\right)\right], \tag{3.43}
\end{align*}
$$

where
$Z^{\text {Ref }} \simeq e^{g \nu \alpha_{t} N_{t} N_{\mathbf{s}}^{3}} \int \mathcal{D} \boldsymbol{\phi}_{0} \mathcal{D} \overrightarrow{\boldsymbol{\phi}}_{2} \operatorname{det} M\left(\boldsymbol{\phi}_{0}, \overrightarrow{\boldsymbol{\phi}}_{2} ; c_{0}^{\mathrm{Ref}}, c_{2}^{\mathrm{Ref}} ; \nu\right) \exp \left[\sum_{n}\left(-\boldsymbol{\phi}_{0, n}^{2}-\overrightarrow{\boldsymbol{\phi}}_{2, n}^{2}\right)\right]$.

This corresponds to the ratio of the partition function $Z$ to $Z^{\text {Ref }}$ up to the factor $e^{g(\mu-\nu) \alpha_{t} N_{\mathrm{t}} N_{s}^{3}}$. Thus, it varies as a function of the difference of the thermodynamic potentials or the pressure as follows:

$$
\begin{equation*}
\frac{Z}{Z^{\mathrm{Ref}}}=e^{-\beta\left(\Omega-\Omega^{\mathrm{Ref}}\right)}=e^{+\beta N_{\mathrm{s}}^{3}\left(p-p^{\mathrm{Ref}}\right)} . \tag{3.45}
\end{equation*}
$$

This value just shows the normalization of the probability function. To examine how close the two probability functions are to each other, we tune the expectation value of the reweighting factor to be one within errors by changing the chemical potential for the reference determinant from that for the original determinant and then evaluate the standard deviation of the reweighting factor. This tuning is equivalent to choosing the pressure of the reference system equal to the pressure of the target system, as is easily seen from Eq. (3.45). This process is realized by utilizing the relation between the pressure of the system and the chemical potential:

$$
\begin{equation*}
p(\mu)=\int_{-\infty}^{\mu} d \mu^{\prime} \rho\left(\mu^{\prime}\right) \tag{3.46}
\end{equation*}
$$

where $\rho(\mu)$ is the nucleon density.
Since to generate configurations is more time-consuming than to evaluate the reweighting factor, in actual calculations, we use configurations obtained with fixed $\nu$ common to all reweighting points along the relevant and the irrelevant directions and change the chemical potential for the original determinant, $\mu$, instead of changing $\nu$. For the calculations shown below, the difference between $\mu$ and $\nu$ is only of a few percent order.

### 3.4 Computational strategy

In this section, we will describe how to generate auxiliary field configurations. We sample the partition function,

$$
\begin{equation*}
Z \propto \int \prod_{n} \mathcal{D} \boldsymbol{\phi}_{0, n} \mathcal{D} \boldsymbol{\phi}_{2, \boldsymbol{n}} \operatorname{det} M\left(\boldsymbol{\phi}_{0}, \boldsymbol{\phi}_{2} ; c_{0}^{\mathrm{Ref}} c_{2}^{\mathrm{Ref}} ; \nu\right) \exp \left[\sum_{n}\left(-\boldsymbol{\phi}_{0, \boldsymbol{n}}^{2}-\boldsymbol{\phi}_{2, \boldsymbol{n}}^{2}\right)\right], \tag{3.47}
\end{equation*}
$$

where the fermion matrix, $M_{n, m}$ is given in Eq. (3.39), by using the Metropolis algorithm. In this process, we utilize the locality of the auxiliary fields, $\phi_{0, n}$ and $\phi_{2, n}$, in the fermion matrix. Let us consider to replace the old auxiliary fields on the site $\boldsymbol{n}, \boldsymbol{\Phi}_{\boldsymbol{n}}=\left(\boldsymbol{\phi}_{0, \boldsymbol{n}}, \boldsymbol{\phi}_{2, \boldsymbol{n}}\right)$, with the new one $\boldsymbol{\Phi}_{\boldsymbol{n}}+\Delta \boldsymbol{\Phi}_{\boldsymbol{n}}$. The fermion matrix for the new configuration, $M\left(\boldsymbol{\Phi}_{n}+\Delta \boldsymbol{\Phi}_{\boldsymbol{n}}\right)$, differs from that for the old configuration, $M\left(\boldsymbol{\Phi}_{\boldsymbol{n}}\right)$, only by elements which correspond to the site and the nearest neighbor sites in spatial directions. These non-zero elements of $\Delta M=M\left(\boldsymbol{\Phi}_{\boldsymbol{n}}+\Delta \boldsymbol{\Phi}_{\boldsymbol{n}}\right)-M\left(\boldsymbol{\Phi}_{\boldsymbol{n}}\right)$ can be collected into one square $I \times I$ block, $(\Delta M)_{I \times I}$, by permuting lines and columns so that the
other blocks of $\Delta M$ are empty. Therefore, the ratio of two fermion determinants can be expressed by

$$
\begin{equation*}
\frac{\operatorname{det}(M+\Delta M)}{\operatorname{det} M}=\operatorname{det}\left(1+\Delta M M^{-1}\right)=\operatorname{det}\left(1_{I \times I}+(\Delta M)_{I \times I}\left(M^{-1}\right)_{I \times I}\right) \tag{3.48}
\end{equation*}
$$

due to the fact that the matrix $1+\Delta M M^{-1}$ is a block triangular matrix [44]. Note that only we have to do is to evaluate the determinant of a $I \times I$ matrix. The size of the matrix does not depend on the lattice volume and is related to which discretization formula is employed. If we employ the discretization with the five-point formula, $\Delta M$ has the non-zero elements which correspond not only to the site and the nearest neighbor sites but also to the next-nearest neighbor sites in each spatial direction. This is why we adopt the three-point formula to reduce computational costs. In addition to this, we have to calculate only the $I \times I$ block of $M^{-1}$ to evaluate the ratio. We calculate the $I \times I$ block of $M^{-1}$ column by column by using the conjugate gradient method.

For auxiliary fields on a site, we try to update $n_{\text {hit }}$ times successively. The ratio between the new matrix, $M^{(k+1)}$, and the lastly accepted matrix, $M^{(k)}$, can be evaluated recursively by using the inverse matrix of the original matrix $M^{(0)}$ as follows [44]:

$$
\begin{align*}
\rho^{(0)} & =1, \\
\rho^{(k+1)} & =\frac{\operatorname{det} M^{(k+1)}}{\operatorname{det} M^{(k)}} \\
& =\frac{\operatorname{det} M^{(k+1)}}{\operatorname{det} M^{(0)}} \frac{\operatorname{det} M^{(0)}}{\operatorname{det} M^{(k)}} \\
& =\frac{\operatorname{det} M^{(k+1)}}{\operatorname{det} M^{(0)}} \frac{\operatorname{det} M^{(k-1)}}{\operatorname{det} M^{(k)}} \cdots \frac{\operatorname{det} M^{(0)}}{\operatorname{det} M^{(1)}} \\
& =\operatorname{det}\left(1_{I \times I}+\left(M^{(k+1)}-M^{(0)}\right)_{I \times I}\left(M^{(0)}\right)_{I \times I}^{-1}\right) \frac{1}{\rho^{(k)}} \frac{1}{\rho^{(k-1)}} \cdots \frac{1}{\rho^{(2)}} \frac{1}{\rho^{(1)}} \tag{3.49}
\end{align*}
$$

Note that the index $1, \ldots, k$ runs only for the matrices which corresponds to the accepted fields.

### 3.5 Numerical results

In this section, we discuss the reweighting factor obtained by the Monte Carlo simulations. All simulations are performed on $N_{\mathrm{s}}^{3} \times N_{\mathrm{t}}=4^{3} \times 4$ lattice with $a=a_{\mathrm{t}}=5 \mathrm{fm}$. Thus, the temperature of the system is about 10 MeV . For almost all the parameter set, we generate about 250,000 configurations and discard the first 500 configurations as thermalization. We perform the measurement at every 100 configurations so the sample size is 2,500 .

### 3.5.1 Reweighting factor

First, we show the result of the reweighting factor. Fig. 3.2 (a) shows the reweighting factor for the point labeled as "Irr7" corresponding to the physical point for various values of chemical potential in the case of $\mu=\nu$ on the complex plane and Fig. 3.2 (b) shows the absolute value of it as a function of the chemical potential. For all values
of $\mu$, the real part of the reweighting factor takes larger value than the imaginary part. The absolute value of the reweighting factor increase as the chemical potential increases. This behavior can be understood as the increase of the difference of the partition functions up to the factor $e^{g(\mu-\nu) \alpha_{\mathrm{t}} N_{\mathrm{t}} N_{\mathrm{s}}^{3}}$ as already mentioned in Sec. 3.3.

By changing the chemical potential for the original determinant from that for the reference determinant, we have tuned the expectation value of the reweighting factor to be one within errors. Fig. 3.3 shows the resulting reweighting factor for the point labeled as "Irr7" on the complex plane.

The tuning requires the difference between $\mu$ and $\nu$ to be only of a few percent order. Since the sample size is common to all results, the larger error bar corresponds to the larger standard deviation. Hereafter, we are going to argue that the standard deviation represents the similarity of the probability functions.

Fig. 3.4 shows the standard deviation of the absolute value of the reweighting factor as a function of the chemical potential of the reference determinant $\nu$. The standard deviation increases as the chemical potential increases. This behavior is natural because the renormalization group analysis on which the reweighting method is based have been performed at zero density and the larger chemical potential corresponds to the larger density.



Figure 3.2: The reweighting factor for the point labeled as "Irr7" corresponding to the physical point for various values of chemical potential in the case of $\mu=\nu$ on the complex plane (a), and the absolute value of it as a function of the chemical potential (b).


Figure 3.3: The reweighting factor for the point labeled as "Irr7" on the complex plane after tuning the chemical potential $\mu$ so that the real part of the expectation value to be one within errors.


Figure 3.4: The standard deviation of the absolute value of the reweighting factor for the point labeled as "Irr7" as a function of the chemical potential of the reference determinant $\nu$.

Fig. 3.5-3.14 shows the same as in Fig. 3.4, but for the points labeled as "Irr1""Irr6" and "Rel1"-"Rel4", respectively. For all results, the monotonic increase of the standard deviation can be seen.


Figure 3.5: The same as in Fig. 3.4, but for the point labeled as "Irr1".


Figure 3.6: The same as in Fig. 3.4, but for the point labeled as "Irr2".


Figure 3.7: The same as in Fig. 3.4, but for the point labeled as "Irr3".


Figure 3.8: The same as in Fig. 3.4, but for the point labeled as "Irr4".


Figure 3.9: The same as in Fig. 3.4, but for the point labeled as "Irr5".


Figure 3.10: The same as in Fig. 3.4, but for the point labeled as "Irr6".


Figure 3.11: The same as in Fig. 3.4, but for the point labeled as "Rel1".


Figure 3.12: The same as in Fig. 3.4, but for the point labeled as "Rel2".


Figure 3.13: The same as in Fig. 3.4, but for the point labeled as "Rel3".


Figure 3.14: The same as in Fig. 3.4, but for the point labeled as "Rel4".

To examine the direction dependence of the standard deviation, we plot the standard deviation for $\nu \simeq-40 \mathrm{MeV}$ as a function of the distance in the $X-Y$ plane in Fig. 3.15. The standard deviation in the irrelevant direction is significantly smaller than that in the relevant direction at the same distance: Compare label "Irr1" with label "Rel1" and label "Irr6" with label "Rel4", respectively.


Figure 3.15: The direction dependence of the standard deviation for $\nu \simeq-40 \mathrm{MeV}$.
Fig. 3.16-3.21 shows the same as in Fig. 3.15, but for various values of chemical potential. The tendency of the direction dependence is preserved for all values of the chemical potential.


Figure 3.16: The same as in Fig. 3.15, but for $\nu \simeq-36 \mathrm{MeV}$.


Figure 3.17: The same as in Fig. 3.15, but for $\nu \simeq-32 \mathrm{MeV}$.


Figure 3.18: The same as in Fig. 3.15, but for $\nu \simeq-28 \mathrm{MeV}$.


Figure 3.19: The same as in Fig. 3.15, but for $\nu \simeq-24 \mathrm{MeV}$.


Figure 3.20: The same as in Fig. 3.15, but for $\nu \simeq-20 \mathrm{MeV}$.


Figure 3.21: The same as in Fig. 3.15, but for $\nu \simeq-16 \mathrm{MeV}$.

### 3.5.2 Scaling dimension vs. canonical dimension

In the reweighting method we have concerned, we perform the power counting based on the scaling dimension through renormalization group analysis. On the other hand, we can also perform the power counting based on the canonical dimension through naive dimensional analysis. In this subsection, we compare the reweighting method based on renormalization group analysis with that based on naive dimensional analysis.

Although both the leading order and the next-to-leading order contact interaction terms are irrelevant operators in the power counting based on the canonical dimension, we deal with the next-to-leading order term as the irrelevant operator which is omitted from the reference determinant. Fig. 3.22 shows the the standard deviations of the absolute value of the reweighting factor for the point labeled as "Irr7" obtained with renormalization group analysis and naive dimensional analysis as a function of the chemical potential of the reference determinant $\nu$. For the result with naive dimensional analysis, the sample size is 500 . The standard deviation obtained with renormalization group analysis is significantly smaller than that obtained with naive dimensional analysis for all values of the chemical potential. This fact indicates that the power counting based on the scaling dimension is superior to that based on the canonical dimension even if the nucleon density of the system is finite.


Figure 3.22: The standard deviations of the absolute value of the reweighting factor for the point labeled as "Irr7" obtained with renormalization group analysis and naive dimensional analysis as a function of the chemical potential of the reference determinant $\nu$.

## Chapter 4

## Summary

In this dissertation, we developed the reweighting method on the basis of RG analysis by considering the NLO NEFT without pions.

First, we performed the RG analysis of the NLO NEFT without pions defined on a spatial lattice by diagonalizing the lattice Hamiltonian numerically. To obtain the RG flows, we change the lattice constant with the binding energy and the ANC fixed. We showed the validity of using the binding energy and the ANC as a low-energy physical quantity to fix the effective field theory couplings for a wide range of the cutoff. Through this analysis, we not only obtained the RG flows, but also inferred the relevant operator, the phase boundary, and the location of the nontrivial fixed point. We compared the obtained RG flows with the flow in the continuum and the flows obtained analytically with lattice-regularized integrals. It became clear that the location of the nontrivial fixed point is close to that obtained by the corresponding analytic calculation with lattice-regularized integrals.

Then, we proceeded to lattice simulations. We determined the reference point of the reweighting method by fixing the effective range to be 0.00 fm and the scattering length to be the physical value, 5.42 fm . We also considered the points that are along with the irrelevant direction with the physical scattering length and the points that are along with the relevant direction with the effective range 0.00 fm . We generated configurations which obey the probability function with the reference determinant by performing Monte Carlo simulations. Since the expectation value of the reweighting factor just represents the normalization of the probability function, in evaluating the reweighting factor, we tuned it to be one within errors by utilizing the relation between the pressure of the system and the chemical potential. We compared the standard deviations of the tuned reweighting factor in the relevant direction with those in the irrelevant direction and concluded that the reweighting in the irrelevant direction is significantly superior to that in the relevant direction. We also compared the reweighting method based on RG analysis with that based on naive dimensional analysis and confirmed the former is better than the latter.

By performing RG analysis and lattice simulations, we have established the reweighting method for nuclear effective field theory on a lattice on the basis of renormalization group analysis. It is noteworthy that, although we confine ourselves to considering the NLO NEFT without pions in this study, the method we developed can be applied to the case where pion interactions and/or higher order operators are included. Recall that NEFT with pions has the sign problem even if the chemical potential is absent and the long-distance parts of pion interaction, corresponding to the pion exchanges
with the momenta below the cutoff $\Lambda$, are irrelevant. It is worth applying the method to the NEFT with pions in which (nonlinearly realized) chiral symmetry is exactly implemented.

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