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# On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling

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# On the Local Asymptotic Behavior of the Likelihood Function for Meixner Lévy Processes under High-Frequency Sampling

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**Abstract**

We discuss the local asymptotic behavior of the likelihood function associated with all the four characterizing parameters $(\alpha,\beta,\delta,\mu)$ of the Meixner Lévy process under high-frequency sampling scheme. We derive the optimal rate of convergence for each parameter and the Fisher information matrix in a closed form. The skewness parameter $\beta$ exhibits a slower rate alone, relative to the other three parameters free of sampling rate. An unusual aspect is that the Fisher information matrix is constantly singular for full joint estimation of the four parameters. This is a particular phenomenon in the regular high-frequency sampling setting and is of essentially different nature from low-frequency sampling. As soon as either $\alpha$ or $\delta$ is fixed, the Fisher information matrix becomes diagonal, implying that the corresponding maximum likelihood estimators are asymptotically orthogonal.

*Keywords:* High-frequency sampling, Lévy process, local asymptotic normality, Meixner process, Fisher information matrix.

*2010 Mathematics Subject Classification:* 60G51, 62E20.

## 1 Introduction and Preliminaries

The local asymptotic normality (LAN, for short) property is a vital concept in asymptotically optimal statistical analyses. In short, the LAN property is defined through the following locally asymptotically quadratic structure of a likelihood ratio

$$L_n(\theta+R_n u)-L_n(\theta)=\langle u,\mathscr{H}_n(\theta)\rangle-\frac{1}{2}\langle u,\mathscr{I}(\theta)u\rangle+o_{\mathbb{P}_\theta}(1) \tag{1.1}$$

for each $u$, where $\mathbb{P}_\theta$ is a probability measure associated with the parameter $\theta$, where $\{R_n\}_{n\in\mathbb{N}}$ is a sequence of nonrandom positive definite matrices tending to 0 in norm, where $\{\mathscr{H}_n(\theta)\}_{n\in\mathbb{N}}$ is a sequence of random vectors converging in law to $\mathscr{N}(0,\mathscr{I}(\theta))$ under $\mathbb{P}_\theta$, and where $\mathscr{I}(\theta)$ is a nonnegative definite deterministic matrix, called the Fisher information matrix. Once the identity (1.1) is confirmed with *nonsingular* $\mathscr{I}(\theta)$, one can formulate asymptotic optimality of estimation and testing hypothesis in terms of $\mathscr{H}_n(\theta)$. (See Le Cam [7], Le Cam and Yang [8], and van der Vaart [14] for a systematic account of the LAN theory.)

In this article, we discuss the local asymptotic behavior of the likelihood function associated with the four-parameter Meixner Lévy process observed under high-frequency sampling scheme. The Meixner process has been recognized as a successful class of Lévy processes for the purpose of practical modeling, such as mathematical finance and possibly turbulence, as well as of sufficient theoretical interest. We begin with some fundamental facts of the Meixner process with the most popular parametrization. (We refer the reader to [2, 13] for general details, and also [3, 6] for numerical aspects of the Meixner process.) The Meixner distribution, denoted by Meixner$(\alpha,\beta,\delta,\mu)$, is infinitely divisible and selfdecomposable, and admits a probability density

$$x\mapsto\frac{(2\cos(\beta/2))^{2\delta}}{2\pi\alpha\Gamma(2\delta)}\exp\left[\frac{\beta}{\alpha}(x-\mu)\right]\left|\Gamma\left(\delta+i\frac{x-\mu}{\alpha}\right)\right|^2,\quad x\in\mathbb{R}, \tag{1.2}$$

where $\alpha>0$, $|\beta|<\pi$, $\delta>0$, $\mu\in\mathbb{R}$. When $\beta=0$, the distribution is symmetric around $\mu$. We write

$$\theta:=(\alpha,\beta,\delta,\mu)\in\Theta,$$

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the parameter space $\Theta$ being a bounded convex domain satisfying

$$\Theta^- \subset \left\{(\alpha, \beta, \delta, \mu) \in \mathbb{R}^4 \middle| \alpha > 0, |\beta| < \pi, \delta > 0, \mu \in \mathbb{R}\right\}.$$

The Lévy measure of Meixner$(\alpha, \beta, \delta, \mu)$ admits the Lebesgue density

$$g(z;\theta) := \delta \frac{\exp(\beta z/\alpha)}{z\sinh(\pi z/\alpha)}, \quad z \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}.$$

Let $\{X_t : t \geq 0\}$ be a Lévy process satisfying

$$\mathscr{L}(X_1) = \text{Meixner}(\alpha, \beta, \delta, \mu),$$

which we call a *Meixner (Lévy) process*, which is of infinite variation. We denote by $\mathbb{P}_\theta$ the distribution of $X$ associated with the parameter $\theta \in \Theta$ and by $\mathbb{E}_\theta$ the expectation taken under the probability measure $\mathbb{P}_\theta$. In what follows, every stochastic asymptotics is taken under $\mathbb{P}_\theta$. The characteristic function of $\mathscr{L}(X_1)$ is given in closed form by

$$\mathbb{E}_\theta\left[e^{iyX_1}\right] = e^{iy\mu}\left(\frac{\cos(\beta/2)}{\cosh((\alpha y - i\beta)/2)}\right)^{2\delta}, \quad y \in \mathbb{R},$$

which implies that the Meixner distribution possesses the reproducing property, and that for each $c > 0$ and $t > 0$,

$$\mathscr{L}\left(c(X_t - t\mu)\right) = \text{Meixner}(c\alpha, \beta, t\delta, 0). \tag{1.3}$$

One of the remarkable properties of the Meixner process is its asymptotic behavior with respect to observation time, just like normal inverse Gaussian processes, tempered stable processes of Rosiński [11] and layered stable processes of Houdré and Kawai [4]. On the one hand, over short time intervals, it approximates a stable process; as $h \downarrow 0$, a scaled Meixner process

$$\left\{\frac{1}{h\alpha\delta}(X_{ht} - ht\mu) : t \geq 0\right\}$$

tends to a standard Cauchy (Lévy) process, where the convergence is in the weak sense of random processes in the space of càdlàg functions from $[0, +\infty)$ into $\mathbb{R}$ equipped with the Skorohod topology. (See also Lemma 3.1 below.) In a long time frame, on the other hand, it is close to a Brownian motion; as $h \uparrow +\infty$, another scaled Meixner process

$$\left\{\frac{\cos(\beta/2)}{\alpha}\sqrt{\frac{2}{h\delta}}\left(X_{ht} - ht\left(\mu + \alpha\delta\tan\frac{\beta}{2}\right)\right) : t \geq 0\right\}$$

approaches to the standard Brownian motion. (These can be proved in a similar manner to [4, 11].) The stable-type and Gaussian-type behaviors above have long been considered to be very appealing in various applications.

## 2 Local Asymptotic Behavior of Likelihood Function

Consider the sample $(X_{t_{n,1}}, X_{t_{n,2}}, \ldots, X_{t_{n,n}})$ observed at equidistant observation points in time,

$$t_{n,k} := k\Delta_n, \quad k = 1, \ldots, n,$$

with a sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ of positive stepsizes satisfying

$$\Delta_n \downarrow 0 \quad \text{and} \quad n\Delta_n \uparrow \infty, \quad \text{as } n \uparrow +\infty. \tag{2.1}$$

Let us state the main claim of this article. To maintain the flow, we defer the proof to Section 3.1.

**Theorem 2.1.** *The LAN property (1.1) holds for each $\theta \in \Theta$, with*

$$R_n = \text{diag}\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n\Delta_n}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right),$$

$$\mathscr{I}(\theta) = \begin{bmatrix} \frac{1}{2\alpha^2} & 0 & \frac{1}{2\alpha\delta} & 0 \\ 0 & \frac{\delta}{2\cos^2(\beta/2)} & 0 & 0 \\ \frac{1}{2\alpha\delta} & 0 & \frac{1}{2\delta^2} & 0 \\ 0 & 0 & 0 & \frac{1}{2\alpha^2\delta^2} \end{bmatrix}. \tag{2.2}$$

Observe that the Fisher information matrix (2.2) is singular, that is, $|\mathscr{I}(\theta)| \equiv 0$ for each $\theta \in \Theta$. Due to this fact, on the one hand, the conventional asymptotic optimality theory is not applicable to the full joint estimation of the four parameters. On the other hand, it is clear that the singularity is caused solely by the off-diagonal elements between $\alpha$ and $\delta$. Nevertheless, as soon as $\alpha$ or $\delta$ is fixed, the Fisher information matrix then reduces to $\mathbb{R}^{3\times 3}$ and purely diagonal. This ensures that the maximum likelihood estimators are asymptotically independent.

Let us discuss the singularity issue for Meixner Lévy processes in terms of sampling scheme. First, it will shortly turn out (based on Lemma 3.2) that under low-frequency sampling with $\Delta_n \equiv \Delta > 0$, the Fisher information matrix remains involved with infinite sums. It seems difficult to judge in an analytical manner whether the matrix is singular. (See also [3].) Next, it is worthwhile to compare with the continuous sampling setting, based upon the following result.

**Proposition 2.2.** *Let $T > 0$ and let $\theta_k := (\alpha_k, \beta_k, \delta_k, \mu_k) \in \Theta$, $k = 1, 2$. The probability measures $\mathbb{P}_{\theta_1}|_{\mathscr{F}_T}$ and $\mathbb{P}_{\theta_2}|_{\mathscr{F}_T}$ are equivalent iff $\alpha_1\delta_1 = \alpha_2\delta_2$ and $\mu_1 = \mu_2$.*

This proposition implies that singularity in studying likelihood becomes more noteworthy in the continuously observed case than in the high-frequency sampling case: in the latter case, the likelihood itself does exist for every admissible parameter values, while the Fisher information may be singular; in the former case, the likelihood itself may not exist. Especially what is interesting is that the location parameter $\mu$ is required to be fixed.

Let us next discuss the singularity issue for different classes of Lévy processes under high-frequency sampling. It is well known that a similar phenomenon is observed in the case of non-Gaussian stable Lévy process. Precisely, the joint maximum-likelihood estimation of the stability index and the scale parameter leads to a constantly singular Fisher information matrix. (See [1, 10], for example.) Inferring from this, we suspect that the singularity arises from every Lévy process whose short-range behavior can be approximated in law by a stable process with unknown stability index and scale parameter. Typical examples are tempered stable processes [11] and layered stable processes [4]. In this direction, the present setting of Meixner processes is not directly relevant since its short-time stability index is necessarily 1 (see Lemma 3.1 later). As observed in the Fisher information matrix (2.2), the singularity issue in our framework comes instead from $(\alpha, \delta)$.

In principle, unlike the low-frequency sampling case, the high-frequency sampling scheme yields different optimal rates of convergence for different characterizing parameters. There exist several case studies in the literature that address the joint LAN property for univariate Lévy processes. The most well known case is the scaled Wiener process with drift, $X_t = t\mu + \sigma W_t$, where the LAN property holds true for $(\mu, \sigma)$ at rate $(\sqrt{n\Delta_n}, \sqrt{n})$ with a diagonal Fisher information matrix. In the case of inverse Gaussian subordinators or gamma processes, both of which are characterized by the two parameters $(\delta, \gamma)$, the LAN property holds true at rate $(\sqrt{n}, \sqrt{n\Delta_n})$ with a diagonal Fisher information matrix. (See Masuda [9] for details.) More recently, the authors derive in [5] the LAN property for the normal inverse Gaussian (NIG) process, which is characterized by the four parameters $(\alpha, \beta, \delta, \mu)$. Again, the LAN property holds true at rate $(\sqrt{n\Delta_n}, \sqrt{n\Delta_n}, \sqrt{n}, \sqrt{n})$ with a block-diagonal Fisher information matrix. Interestingly, the NIG process suffers no singularity issue, while, when suitably normalized, sharing the same short-range behavior of Cauchy type with the Meixner process.

Finally, let us note that the LAN property may be investigated, either singular or non-singular, only when the likelihood function are available in a sufficiently tractable form, such as (3.1) and (3.2). But, this is very rare. For example, the likelihood function for tempered stable processes is unknown in a closed form. Note also that the availability of an explicit likelihood function may not be enough. For example, without the reproducing property such as (1.3), even the explicit likelihood function is intractable in the high-frequency sampling framework. (A typical example is the generalized hyperbolic Lévy process.) This also ensures the importance of case studies.

# 3 Proofs

Throughout the proofs, we denote by $\Gamma(z)$ the Gamma function $\Gamma(z) := \int_0^{+\infty} x^{z-1}e^{-x}dx$, where $z$ is a complex number with a positive real part, denote by $\zeta$ the Riemann zeta function, that is, $\zeta(s) := \sum_{k=1}^{+\infty} k^{-s}$, $s > 1$, and write $\gamma := \lim_{n\uparrow+\infty}(\sum_{k=1}^{n} k^{-1} - \ln n) \approx 0.5772$ for the Euler-Mascheroni constant.

## 3.1 Proof of Theorem 2.1

It holds by (1.3) that for each $n \in \mathbb{N}$,

$$x_{n,k} := X_{t_{n,k}} - X_{t_{n,k-1}} \sim \text{Meixner}(\alpha, \beta, \Delta_n\delta, \Delta_n\mu), \quad k = 1, \ldots, n.$$

In view of this fact and the probability density function (1.2), define for each $n \in \mathbb{N}$ and $k = 1, \ldots, n$,

$$l_{n,k}(\theta) := 2\Delta_n\delta \ln\left(2\cos\frac{\beta}{2}\right) - \ln(2\pi\alpha) - \ln\Gamma(2\Delta_n\delta) + \frac{\beta}{\alpha}\left(x_{n,k} - \Delta_n\mu\right) + \ln\left|\Gamma\left(\Delta_n\delta + i\frac{x_{n,k} - \Delta_n\mu}{\alpha}\right)\right|^2. \quad (3.1)$$

Thanks to the stationarity and independence of increments of Lévy processes, the log-likelihood function to be maximized with discrete observations $\{X_{t_{n,k}}\}_{k=1,\ldots,n}$ is as simple as

$$L_n(\theta) := \sum_{k=1}^{n} l_{n,k}(\theta). \quad (3.2)$$

**1)** We begin with the local Cauchy approximation. We define a sequence $\{\varepsilon_{n,k}\}_{k=1,\ldots,n}$ of iid random variables by

$$\varepsilon_{n,k} := \varepsilon_{n,k}(\alpha,\delta,\mu,\Delta_n) := \frac{x_{n,k} - \Delta_n\mu}{\Delta_n\alpha\delta} \sim \text{Meixner}\left(\frac{1}{\Delta_n\delta}, \beta, \Delta_n\delta, 0\right). \quad (3.3)$$

In the pathwise sense, the random variables $\{\varepsilon_{n,k}\}_{k=1,\ldots,n}$ depend on $(\alpha,\delta,\mu,\Delta_n)$ and is independent of $\beta$. In contrast, the law $\mathscr{L}(\varepsilon_{n,1})$ depends on $(\beta,\delta,\Delta_n)$ and is independent of $(\alpha,\mu)$. We can show that the law $\mathscr{L}(\varepsilon_{n,1})$ has the mean, the variance, the skewness and the kurtosis, respectively,

$$\tan\frac{\beta}{2}, \quad \frac{1}{2\Delta_n\delta\cos^2(\beta/2)}, \quad \sin\frac{\beta}{2}\sqrt{\frac{2}{\Delta_n\delta}}, \quad 3 + \frac{2-\cos(\beta)}{\Delta_n\delta}.$$

(See, for example, Grigelionis [2].) It then follows that as $n \uparrow +\infty$, the first four moments are of order $O(1)$, $O(\Delta_n^{-1})$, $O(\Delta_n^{-3/2})$ and $O(\Delta_n^{-2})$ if $\beta \neq 0$, while 0, $O(\Delta_n^{-1})$, 0 and $O(\Delta_n^{-2})$ if $\beta = 0$.

We denote by *the standard Cauchy distribution* the infinitely divisible distribution with characteristic function $y \mapsto e^{-|y|}$ and the probability density function

$$\phi(x) := \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}. \quad (3.4)$$

The following lemma indicates that the random variables $\varepsilon_{n,k}$ act as suitably normalized increments.

**Lemma 3.1.** *The law $\mathscr{L}(\varepsilon_{n,1})$ converges to the standard Cauchy distribution, as $n \uparrow +\infty$.*

*Proof.* The claim can be deduced readily by observing that, for each $y \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}_\theta\left[e^{iy\varepsilon_{n,1}}\right] &= \left(\frac{\cos(\beta/2)}{\cosh\left((y/(\Delta_n\delta) - i\beta)/2\right)}\right)^{2\Delta_n\delta} \\
&= \left(\frac{2}{e^{y/(2\Delta_n\delta)}(1 - i\tan(\beta/2)) + e^{-y/(2\Delta_n\delta)}(1 + i\tan(\beta/2))}\right)^{2\Delta_n\delta} \\
&\sim \begin{cases} \left(\frac{2}{1 - i\tan(\beta/2)}\right)^{2\Delta_n\delta} e^{-y}, & \text{if } y > 0, \\ \left(\frac{2}{1 + i\tan(\beta/2)}\right)^{2\Delta_n\delta} e^{y}, & \text{if } y < 0, \end{cases} \\
&\to e^{-|y|},
\end{aligned}$$

as $n \uparrow +\infty$, with the help of the Lévy continuity theorem. □

**2)** The likelihood function (3.2) in question is smooth in $\theta$. We can rewrite (3.1) as

$$l_{n,k}(\theta) = 2\Delta_n\delta\ln\left[2\cos(\beta/2)\right] - \ln(2\pi\alpha) - \ln\Gamma(2\Delta_n\delta) + \Delta_n\beta\delta\varepsilon_{n,k} + \ln\left|\Gamma\left(\Delta_n\delta\left(1 + i\varepsilon_{n,k}\right)\right)\right|^2.$$

In our proof of Theorem 2.1, we will need to specify the partial derivatives up to the second order. To this end, we define an array $\{g_{n,k}(\theta)\}_{n\in\mathbb{N};k=1,\ldots,n}$ of random vectors in $\mathbb{R}^4$ by $g_{n,k}(\theta) := \nabla_\theta\left(l_{n,k}(\theta)\right)$, where

$$g_{n,k}(\theta) := \begin{bmatrix} g_{n,k}^{(1)} \\ g_{n,k}^{(2)} \\ g_{n,k}^{(3)} \\ g_{n,k}^{(4)} \end{bmatrix} := \begin{bmatrix} -\frac{1}{\alpha} - \frac{\Delta_n\beta\delta}{\alpha}\varepsilon_{n,k} + \frac{2\Delta_n\delta}{\alpha}\varepsilon_{n,k}\mathrm{Im}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,k}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,k}))}\right) \\ \Delta_n\delta\left(\varepsilon_{n,k} - \tan(\beta/2)\right) \\ 2\Delta_n\ln\left[2\cos(\beta/2)\right] - 2\Delta_n\frac{\Gamma'(2\Delta_n\delta)}{\Gamma(2\Delta_n\delta)} + 2\Delta_n\mathrm{Re}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,k}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,k}))}\right) \\ -\Delta_n\frac{\beta}{\alpha} + \Delta_n\frac{2}{\alpha}\mathrm{Im}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,k}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,k}))}\right) \end{bmatrix}. \quad (3.5)$$

(See Grigoletto and Provasi [3, Appendix A] for derivation of the gradient.) We also prepare the following asymptotics.

**Lemma 3.2.** *It holds almost surely that*

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\mathrm{Im}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right) &= -\frac{\Delta_n\delta\varepsilon_{n,1}}{\alpha}\sum_{k=0}^{+\infty}\frac{(k+\Delta_n\delta)^2-(\Delta_n\delta\varepsilon_{n,1})^2}{((k+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2)^2}\\
&\sim -\frac{\Delta_n\delta\varepsilon_{n,1}}{\alpha}\left(\frac{1}{(\Delta_n\delta)^2}\frac{1-\varepsilon_{n,1}^2}{(1+\varepsilon_{n,1}^2)^2}+\zeta(2)\right),\\
\frac{\partial}{\partial\delta}\mathrm{Im}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right) &= -2\Delta_n^2\delta\varepsilon_{n,1}\sum_{k=0}^{+\infty}\frac{k+\Delta_n\delta}{((k+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2)^2}\\
&\sim -2\Delta_n^2\delta\varepsilon_{n,1}\left(\frac{1}{(\Delta_n\delta)^3(1+\varepsilon_{n,1}^2)^2}+\zeta(3)\right),\\
\frac{\partial}{\partial\mu}\mathrm{Im}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right) &= -\frac{\Delta_n}{\alpha}\sum_{k=0}^{+\infty}\frac{(k+\Delta_n\delta)^2-(\Delta_n\delta\varepsilon_{n,1})^2}{((k+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2)^2}\\
&\sim -\frac{\Delta_n}{\alpha}\left(\frac{1}{(\Delta_n\delta)^2}\frac{1-\varepsilon_{n,1}^2}{(1+\varepsilon_{n,1}^2)^2}+\zeta(2)\right),\\
\frac{\partial}{\partial\alpha}\mathrm{Re}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right) &= -\frac{2(\Delta_n\delta\varepsilon_{n,1})^2}{\alpha}\sum_{k=0}^{+\infty}\frac{k+\Delta_n\delta}{((k+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2)^2}\\
&\sim -\frac{2(\Delta_n\delta\varepsilon_{n,1})^2}{\alpha}\left(\frac{1}{(\Delta_n\delta)^3}\frac{1}{(1+\varepsilon_{n,1}^2)^2}+\zeta(3)\right),\\
\frac{\partial}{\partial\delta}\mathrm{Re}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right) &= \Delta_n\sum_{k=0}^{+\infty}\frac{(k+\Delta_n\delta)^2-(\Delta_n\delta\varepsilon_{n,1})^2}{((k+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2)^2}\\
&\sim \Delta_n\left(\frac{1}{(\Delta_n\delta)^2}\frac{1-\varepsilon_{n,1}^2}{(1+\varepsilon_{n,1}^2)^2}+\zeta(2)\right),\\
\frac{\partial}{\partial\mu}\mathrm{Re}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right) &= -\frac{2\Delta_n}{\alpha}\sum_{k=0}^{+\infty}\frac{k+\Delta_n\delta}{((k+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2)^2}\\
&\sim -\frac{2\Delta_n}{\alpha}\left(\frac{1}{(\Delta_n\delta)^3}\frac{1}{(1+\varepsilon_{n,1}^2)^2}+\zeta(3)\right),
\end{aligned}
$$

*where all the asymptotics hold when* $n\uparrow+\infty$.

*Proof.* The proof entails rather lengthy algebra of somewhat routine nature. To avoid overloading this proof, we only consider the first claim and omit all the rest. To this end, we have only to justify the interchange of the differentiation and the infinite sum as

$$\frac{\partial}{\partial\alpha}\sum_{k=1}^{+\infty}\frac{1}{(k+\Delta_n\delta)^2+((x_{n,1}-\Delta_n\mu)/\alpha)^2}=\sum_{k=1}^{+\infty}\frac{\partial}{\partial\alpha}\frac{1}{(k+\Delta_n\delta)^2+((x_{n,1}-\Delta_n\mu)/\alpha)^2}.$$

For convenience, we use the notation

$$H(k;\alpha):=\frac{1}{(k+\Delta_n\delta)^2+((x-\Delta_n\mu)/\alpha)^2},$$

with $x \in \mathbb{R}$, $\delta$, $\mu$, $\Delta_n$ fixed. It holds by the Taylor theorem that for $\lambda > 0$ and for each $k \in \mathbb{N}$,

$$
\begin{aligned}
\left| \frac{H(k;\alpha+\lambda) - H(k;\alpha)}{\lambda} \right| &= \left| \int_0^1 \frac{\partial}{\partial \alpha} H(k;\alpha+\lambda s) ds \right| \\
&\leq \int_0^1 \frac{2}{\alpha+\lambda s} \left( \frac{(x-\Delta_n\mu)/(\alpha+\lambda s)}{(k+\Delta_n\delta)^2 + ((x-\Delta_n\mu)/(\alpha+\lambda s))^2} \right)^2 ds \\
&\leq \frac{2}{\alpha^3} \frac{(x-\Delta_n\mu)^2}{(k+\Delta_n\delta)^4}.
\end{aligned}
$$

Hence, we get

$$
\sum_{k=1}^{+\infty} \left| \frac{H(k;\alpha+\lambda) - H(k;\alpha)}{\lambda} \right| \leq \frac{2(x-\Delta_n\mu)^2}{\alpha^3} \sum_{k=1}^{+\infty} \frac{1}{(k+\Delta_n\delta)^4} \leq \frac{2(x-\Delta_n\mu)^2}{\alpha^3} \zeta(4),
$$

which justifies the interchange with the help of the dominated convergence theorem. The asymptotics are straightforward by splitting the sum into two parts $k = 0$ and $k \geq 1$ with the help of the definition of the Riemann zeta function. □

**3)** Note that using the expressions (3.5), we can show that

$$
\mathscr{I}(\theta) = \lim_{n\uparrow+\infty} \sum_{k=1}^{n} R_n \mathbb{E}_\theta \left[ g_{n,k}(\theta) g_{n,k}(\theta)^\top \right] R_n.
$$

In order to complete the proof of Theorem 2.1, it suffices to prove the following two lemmas (see Section 4.1 in Kawai and Masuda [5] and the references therein for details).

**Lemma 3.3.** *The symmetric matrix $\mathscr{I}(\theta)$ is well defined and is given by (2.2).*

**Lemma 3.4.** *(i) It holds that*

$$
\lim_{n\uparrow+\infty} R_n \left( \sum_{k=1}^{n} \mathbb{E}_\theta \left[ g_{n,k}(\theta) \right] \mathbb{E}_\theta \left[ g_{n,k}(\theta) \right]^\top \right) R_n = 0,
$$

*where the right hand side indicates the zero matrix in $\mathbb{R}^{4\times 4}$.*
*(ii) It holds that*

$$
\lim_{n\uparrow+\infty} \sup_{\theta\in\Theta} \sum_{k=1}^{n} \left( \mathbb{E}_\theta \left[ \left| R_n g_{n,k}(\theta) \right|^4 \right] + \mathbb{E}_\theta \left[ \left| R_n \mathrm{Hess}_\theta(l_{n,k}(\theta)) R_n \right|^2 \right] \right) = 0.
$$

Note that (ii) in particular verifies the Lindeberg condition: for every $a > 0$,

$$
\lim_{n\uparrow+\infty} \sum_{k=1}^{n} \mathbb{E}_\theta \left[ \left| R_n g_{n,k}(\theta) \right|^2 1\left( \left| R_n g_{n,k}(\theta) \right| \geq a \right) \right] = 0.
$$

**4)** We here prove Lemma 3.3. Recall the definition of the digamma function

$$
\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{l=1}^{+\infty} \left( \frac{1}{l+z} - \frac{1}{l} \right), \quad z \in \mathbb{C},
$$

and also that for each $x > 0$ and $y \in \mathbb{R}$,

$$
\begin{aligned}
\mathrm{Re}\left( \frac{\Gamma'(x+iy)}{\Gamma(x+iy)} \right) &= -\frac{x}{x^2+y^2} - \gamma - \sum_{l=1}^{+\infty} \left( \frac{l+x}{(l+x)^2+y^2} - \frac{1}{l} \right), \\
\mathrm{Im}\left( \frac{\Gamma'(x+iy)}{\Gamma(x+iy)} \right) &= \sum_{l=0}^{+\infty} \frac{y}{(l+x)^2+y^2},
\end{aligned}
$$

where the both infinite sums are well defined. Note that with respect to the variable $y$, the former is even, while the latter is odd. It is straightforward that as $n\uparrow+\infty$,

$$-2\Delta_n\frac{\Gamma'(2\Delta_n\delta)}{\Gamma(2\Delta_n\delta)}\to\frac{1}{\delta},$$

and that almost surely as $n\uparrow+\infty$,

$$\mathrm{Re}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right)=-\frac{1}{\Delta_n\delta}\frac{1}{1+\varepsilon_{n,1}^2}-\gamma-\sum_{l=1}^{+\infty}\left(\frac{l+\Delta_n\delta}{(l+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2}-\frac{1}{l}\right)\sim-\frac{1}{\Delta_n\delta}\frac{1}{1+\varepsilon_{n,1}^2},$$

$$\mathrm{Im}\left(\frac{\Gamma'(\Delta_n\delta(1+i\varepsilon_{n,1}))}{\Gamma(\Delta_n\delta(1+i\varepsilon_{n,1}))}\right)=\frac{1}{\Delta_n\delta}\frac{\varepsilon_{n,1}}{1+\varepsilon_{n,1}^2}+\sum_{l=1}^{+\infty}\frac{\Delta_n\delta\varepsilon_{n,1}}{(l+\Delta_n\delta)^2+(\Delta_n\delta\varepsilon_{n,1})^2}\sim\frac{1}{\Delta_n\delta}\frac{\varepsilon_{n,1}}{1+\varepsilon_{n,1}^2}.$$

By using the above results, we get

$$\begin{aligned}
g_{n,1}^{(1)}(\theta)&\sim-\frac{1}{\alpha}\frac{1-\varepsilon_{n,1}^2}{1+\varepsilon_{n,1}^2},\\
g_{n,1}^{(2)}(\theta)&=\Delta_n\delta\left(\varepsilon_{n,1}-\tan\frac{\beta}{2}\right),\\
g_{n,1}^{(3)}(\theta)&\sim-\frac{1}{\delta}\frac{1-\varepsilon_{n,1}^2}{1+\varepsilon_{n,1}^2},\\
g_{n,1}^{(4)}(\theta)&\sim\frac{2}{\alpha\delta}\frac{\varepsilon_{n,1}}{1+\varepsilon_{n,1}^2},
\end{aligned}$$

as $n\uparrow+\infty$. By denoting by $\mathscr{I}_{l_1,l_2}(\theta)$ the $(l_1,l_2)$-entry of $\mathscr{I}(\theta)$, we readily deduce by means of the LLN that

$$\begin{aligned}
\mathscr{I}_{1,1}(\theta)&=\frac{1}{\alpha^2}\int_{\mathbb{R}}\left(\frac{1-x^2}{1+x^2}\right)^2\phi(x)dx=\frac{1}{2\alpha^2},\\
\mathscr{I}_{2,2}(\theta)&=\frac{\delta}{2(\cos(\beta/2))^2},\\
\mathscr{I}_{3,3}(\theta)&=\frac{1}{\delta^2}\int_{\mathbb{R}}\left(\frac{1-x^2}{1+x^2}\right)^2\phi(x)dx=\frac{1}{2\delta^2},\\
\mathscr{I}_{4,4}(\theta)&=\frac{4}{\alpha^2\delta^2}\int_{\mathbb{R}}\left(\frac{x}{1+x^2}\right)^2\phi(x)dx=\frac{1}{2\alpha^2\delta^2},\\
\mathscr{I}_{1,2}(\theta)&=\mathscr{I}_{2,1}(\theta)=0,\\
\mathscr{I}_{2,3}(\theta)&=\mathscr{I}_{3,2}(\theta)=0,\\
\mathscr{I}_{3,4}(\theta)&=\mathscr{I}_{4,3}(\theta)=-\frac{2}{\alpha\delta^2}\int_{\mathbb{R}}\frac{1-x^2}{1+x^2}\frac{x}{1+x^2}\phi(x)dx=0,\\
\mathscr{I}_{1,3}(\theta)&=\mathscr{I}_{3,1}(\theta)=\frac{1}{\alpha\delta}\int_{\mathbb{R}}\frac{1-x^2}{1+x^2}\frac{1-x^2}{1+x^2}\phi(x)dx=\frac{1}{2\alpha\delta},\\
\mathscr{I}_{2,4}(\theta)&=\mathscr{I}_{4,2}(\theta)=0,\\
\mathscr{I}_{1,4}(\theta)&=\mathscr{I}_{4,1}(\theta)=-\frac{2}{\alpha^2\delta}\int_{\mathbb{R}}\frac{1-x^2}{1+x^2}\frac{x}{1+x^2}\phi(x)dx=0.
\end{aligned}$$

This completes the proof of Lemma 3.3.

**5)** It remains to prove Lemma 3.4.

(i) With the help of Lemma 3.2 and the asymptotic behaviors of $g_{n,1}(\theta)$ given in the proof of Lemma 3.3 together with the bounded convergence theorem, it holds that as $n\uparrow+\infty$,

$$\sqrt{n}R_n\mathbb{E}_\theta\left[g_{n,1}(\theta)\right]\sim\begin{bmatrix}-\frac{1}{\alpha}\int_{\mathbb{R}}\frac{1-x^2}{1+x^2}\phi(x)dx\\ 0\\ -\frac{1}{\delta}\int_{\mathbb{R}}\frac{1-x^2}{1+x^2}\phi(x)dx\\ \frac{2}{\alpha\delta}\int_{\mathbb{R}}\frac{x}{1+x^2}\phi(x)dx\end{bmatrix}=0,$$

which is enough to prove the claim.

(ii) With the help of the asymptotic behaviors of $g_{n,1}(\theta)$ given in the proof of Lemma 3.3 and notation $r_n^{(2)} := 1/\sqrt{n\Delta_n}$ and $r_n^{(k)} := 1/\sqrt{n}$ for $k = 1,3,4$, it suffices to check that

$$\sum_{k=1}^{n} \mathbb{E}_\theta\left[\left|r_n^{(1)} g_{n,k}^{(1)}(\theta)\right|^4\right] \sim \frac{1}{n\alpha^4}\int_{\mathbb{R}}\left(\frac{1-x^2}{1+x^2}\right)^4 \phi(x)dx = \frac{3}{8n\alpha^4},$$

$$\sum_{k=1}^{n} \mathbb{E}_\theta\left[\left|r_n^{(2)} g_{n,1}^{(2)}(\theta)\right|^4\right] = \frac{1}{n\Delta_n}\frac{3\delta\Delta_n + 2 - \cos\beta}{4\delta^3(\cos(\beta/2))^4},$$

$$\sum_{k=1}^{n} \mathbb{E}_\theta\left[\left|r_n^{(3)} g_{n,1}^{(3)}(\theta)\right|^4\right] \sim \frac{1}{n^2\delta^4}\int_{\mathbb{R}}\left(\frac{1-x^2}{1+x^2}\right)^4 \phi(x)dx = \frac{3}{8n\delta^4},$$

$$\sum_{k=1}^{n} \mathbb{E}_\theta\left[\left|r_n^{(4)} g_{n,1}^{(4)}(\theta)\right|^4\right] \sim \frac{1}{n^2}\left(\frac{2}{\alpha\delta}\right)^4\int_{\mathbb{R}}\left(\frac{x}{1+x^2}\right)^4 \phi(x)dx = \frac{3}{8n\alpha^4\delta^4},$$

each of which tends to zero as $n \uparrow +\infty$. Thanks to the compactness of the set $\Theta$, it follows that

$$\lim_{n\uparrow+\infty}\sup_{\theta\in\Theta}\sum_{k=1}^{n}\mathbb{E}_\theta\left[\left|R_n g_{n,k}(\theta)\right|^4\right] = 0.$$

We can derive each entry of the Hessian matrix $\mathrm{Hess}_\theta(l_{n,k}(\theta))$ of the likelihood as

$$\begin{aligned}
\partial_\alpha^2 l_{n,k}(\theta) &= \frac{1}{\alpha^2} + \frac{2\beta\Delta_n\delta\varepsilon_{n,k}}{\alpha} - \frac{2(\Delta_n\delta\varepsilon_{n,k})^2}{\alpha^2}\sum_{l=0}^{+\infty}\frac{3(l+\Delta_n\delta)^2 + (\Delta_n\delta\varepsilon_{n,k})^2}{((l+\Delta_n\delta)^2 + (\Delta_n\delta\varepsilon_{n,k})^2)^2}\\
&\sim \frac{1}{\alpha^2}\frac{1-4\varepsilon_{n,k}^2 - \varepsilon_{n,k}^4}{(1+\varepsilon_{n,k}^2)^2},\\
\partial_\beta^2 l_{n,k}(\theta) &= -\Delta_n\frac{\delta}{2(\cos(\beta/2))^2},\\
\partial_\delta^2 l_{n,k}(\theta) &= -4\Delta_n^2\sum_{l=0}^{+\infty}\frac{1}{(l+2\Delta_n\delta)^2} + 2\Delta_n^2\sum_{l=0}^{+\infty}\frac{(l+\Delta_n\delta)^2 - (\Delta_n\delta\varepsilon_{n,k})^2}{((l+\Delta_n\delta)^2 + (\Delta_n\delta\varepsilon_{n,k})^2)^2}\\
&\sim -\frac{1}{\delta^2}\frac{1-4\varepsilon_{n,k}^2 - \varepsilon_{n,k}^4}{(1+\varepsilon_{n,k}^2)^2},\\
\partial_\mu^2 l_{n,k}(\theta) &= \frac{2\Delta_n^2}{\alpha^2}\sum_{l=0}^{+\infty}\frac{-(l+\Delta_n\delta)^2 + (\Delta_n\delta\varepsilon_{n,k})^2}{((l+\Delta_n\delta)^2 + (\Delta_n\delta\varepsilon_{n,k})^2)^2} \sim \frac{2}{\alpha^2\delta^2}\frac{-1+\varepsilon_{n,k}^2}{(1+\varepsilon_{n,k}^2)^2},
\end{aligned}$$

and

$$\begin{aligned}
\partial_\alpha\partial_\beta l_{n,k}(\theta) &= -\frac{\Delta_n\delta}{\alpha}\varepsilon_{n,k},\\
\partial_\alpha\partial_\delta l_{n,k}(\theta) &= -\frac{4\Delta_n}{\alpha}\sum_{l=0}^{+\infty}\frac{(l+\Delta\delta)(\Delta_n\delta\varepsilon_{n,k})^2}{((l+\Delta\delta)^2 + (\Delta\delta\varepsilon_{n,k})^2)^2} \sim -\frac{4}{\alpha\delta}\left(\frac{\varepsilon_{n,k}}{1+\varepsilon_{n,k}^2}\right)^2,\\
\partial_\alpha\partial_\mu l_{n,k}(\theta) &= \frac{\beta\Delta_n}{\alpha^2} - \frac{4\Delta_n}{\alpha^2}\sum_{l=0}^{+\infty}\frac{\Delta_n\delta\varepsilon_{n,k}(l+\Delta_n\delta)^2}{((l+\Delta_n\delta)^2 + (\Delta_n\delta\varepsilon_{n,k})^2)^2} \sim -\frac{4}{\alpha^2\delta}\frac{\varepsilon_{n,k}}{(1+\varepsilon_{n,k}^2)^2},\\
\partial_\beta\partial_\delta l_{n,k}(\theta) &= -\Delta_n\frac{\tan(\beta/2)}{2},\\
\partial_\beta\partial_\mu l_{n,k}(\theta) &= -\Delta_n\frac{1}{\alpha},\\
\partial_\delta\partial_\mu l_{n,k}(\theta) &= -\frac{4\Delta_n^2}{\alpha}\sum_{l=0}^{+\infty}\frac{(l+\Delta_n\delta)\Delta_n\delta\varepsilon_{n,k}}{((l+\Delta_n\delta)^2 + (\Delta_n\delta\varepsilon_{n,k})^2)^2} \sim -\frac{4}{\alpha\delta^2}\frac{\varepsilon_{n,k}}{(1+\varepsilon_{n,k}^2)^2},
\end{aligned}$$

where all the asymptotics hold almost surely as $n\uparrow+\infty$. It is straightforward to deduce that as $n\uparrow+\infty$,

$$
\sup_{\theta\in\Theta}\sum_{k=1}^{n}\mathbb{E}_{\theta}\left[\left|R_n\mathrm{Hess}_{\theta}(l_{n,k}(\theta))R_n\right|^2\right]=\left[\begin{array}{cccc} O(1/n) & O(\Delta_n/n) & O(1/n) & O(1/n) \\ O(\Delta_n/n) & O(1/n) & O(\Delta_n/n) & O(\Delta_n/n) \\ O(1/n) & O(\Delta_n/n) & O(1/n) & O(1/n) \\ O(1/n) & O(\Delta_n/n) & O(1/n) & O(1/n) \end{array}\right],
$$

where the squared norm inside the expectation are understood to be componentwise. The proof of Lemma 3.4 is complete.

## 3.2 Proof of Proposition 2.2

The mean of $X_1$ is given by $\mu_0(\theta):=\mu+\alpha\delta\tan(\beta/2)$, so that we may write

$$
\mathbb{E}_{\theta}\left[e^{iyX_1}\right]=\exp\left[iy\mu_0(\theta)+\int_{\mathbb{R}_0}\left(e^{iyz}-1-iyz\right)g(z;\theta)dz\right],\quad y\in\mathbb{R}.
$$

According to Sato [12, Theorem 33.1], for each $T>0$, the measures $\mathbb{P}_{\theta_1}|_{\mathscr{F}_T}$ and $\mathbb{P}_{\theta_2}|_{\mathscr{F}_T}$ are equivalent iff the following conditions are fulfilled:

(a) $g(z;\theta_2)=\gamma(z;\theta_1,\theta_2)g(z;\theta_1)$ for some Borel function $\gamma(\cdot;\theta_1,\theta_2):\mathbb{R}\to(0,\infty)$;
(b) $\mu_0(\theta_2)=\mu_0(\theta_1)+\int_{\mathbb{R}}z(\gamma(z;\theta_1,\theta_2)-1)g(z;\theta_1)dz$;
(c) $\int_{\mathbb{R}}(1-\sqrt{\gamma(z;\theta_1,\theta_2)})^2g(z;\theta_1)dz<+\infty$.

Hence, it suffices to show that these three conditions hold true iff $\alpha_1\delta_1=\alpha_2\delta_2$ and $\mu_1=\mu_2$.

Concerning the behaviors of the Lévy density $g(z;\theta)$ near the origin and at infinity, in view of the series expansion $z/\sinh(z)=1-z^2/6+O(z^4)$ as $|z|\to0$, it is easy to see that the Lévy density $g(z;\theta)$ admits the following expansion

$$
g(z;\theta)=\frac{\alpha\delta}{\pi z^2}\left(1+\frac{\beta}{\alpha}z+O(z^2)\right), \tag{3.6}
$$

as $|z|\to0$. Note that $\sinh(x)$ behaves like $e^x/2$ as $x\uparrow+\infty$, while it behaves like $-e^{-x}/2$ as $x\downarrow-\infty$. We thus get

$$
g(z;\theta)\sim\begin{cases} 2\delta z^{-1}\exp\{-(\pi-\beta)z/\alpha\}, & z\uparrow+\infty,\\ 2\delta|z|^{-1}\exp\{-(\pi+\beta)|z|/\alpha\}, & z\downarrow-\infty.\end{cases} \tag{3.7}
$$

In particular, (3.6) as well as the fact that $g(z;\theta)>0$ for every $z\neq0$ ensures (a); more specifically, the singularities of $g(z;\theta_1)$ and $g(z;\theta_2)$ at the origin are canceled out since $g(z;\theta_2)/g(z;\theta_1)\sim\alpha_2\delta_2/(\alpha_2\delta_2)$ as $|z|\to0$.

We turn to (c) with $\gamma(z;\theta_1,\theta_2)=g(z;\theta_2)/g(z;\theta_1)$. Due to (3.6) and (3.7), it holds that

$$
\begin{aligned}
&\left(1-\sqrt{\gamma(z;\theta_1,\theta_2)}\right)^2g(z;\theta_1)=\left(\sqrt{g(z;\theta_1)}-\sqrt{g(z;\theta_2)}\right)^2\\
&\sim\begin{cases}\frac{1}{\pi|z|^2}\left(\left(\sqrt{\alpha_1\delta_1}-\sqrt{\alpha_2\delta_2}\right)^2+\left(\sqrt{\alpha_1\delta_1}-\sqrt{\alpha_2\delta_2}\right)\left(\beta_1\sqrt{\frac{\delta_1}{\alpha_1}}-\beta_2\sqrt{\frac{\delta_2}{\alpha_2}}\right)z+O(z^2)\right), & |z|\to0,\\ C_+z^{-1}\exp(-q_+z), & z\uparrow+\infty,\\ C_-|z|^{-1}\exp(-q_-|z|), & z\downarrow-\infty,\end{cases}
\end{aligned}
$$

for some positive constants $C_\pm$ and $q_\pm$, depending on $(\theta_1,\theta_2)$. Hence, (c) holds true iff $\alpha_1\delta_1=\alpha_2\delta_2$, which we will impose in the rest of this proof.

The remaining (b) is equivalent to

$$
\mu_1+\alpha_1\delta_1\tan\frac{\beta_1}{2}-\mu_2-\alpha_2\delta_2\tan\frac{\beta_2}{2}=\int_{\mathbb{R}_0}\left(\delta_1\frac{\exp(\beta_1z/\alpha_1)}{\sinh(\pi z/\alpha_1)}-\delta_2\frac{\exp(\beta_2z/\alpha_2)}{\sinh(\pi z/\alpha_2)}\right)dz.
$$

In the case $\alpha_1\delta_1=\alpha_2\delta_2=:C>0$, the last display can be rewritten as

$$
\mu_1-\mu_2+C\left(\tan\frac{\beta_1}{2}-\tan\frac{\beta_2}{2}\int_{\mathbb{R}_0}\left(\frac{\exp(\beta_1z/\alpha_1)}{\alpha_1\sinh(\pi z/\alpha_1)}-\frac{\exp(\beta_2z/\alpha_2)}{\alpha_2\sinh(\pi z/\alpha_2)}\right)dz\right)=0.
$$

We now show that the function

$$f(\alpha_1,\beta_1;\alpha_1,\beta_2) := \tan\frac{\beta_1}{2} - \tan\frac{\beta_2}{2} - \int_{\mathbb{R}_0}\left(\frac{\exp(\beta_1 z/\alpha_1)}{\alpha_1\sinh(\pi z/\alpha_1)} - \frac{\exp(\beta_2 z/\alpha_2)}{\alpha_2\sinh(\pi z/\alpha_2)}\right)dz \equiv 0,$$

rendering that (b) holds true iff $\mu_1 = \mu_2$, which completes the proof of the proposition. First, we observe that

$$f(\alpha_1,0;\alpha_2,0) = \int_{\mathbb{R}_0}\left(\frac{1}{\alpha_2\sinh(\pi z/\alpha_2)} - \frac{1}{\alpha_1\sinh(\pi z/\alpha_1)}\right)dz \equiv 0,$$

since the integrand is odd, continuous in $\mathbb{R}$, and exponentially decreasing as $|z| \uparrow +\infty$. Next, using the fact that the variance of Meixner$(\alpha_k,\beta_k,\delta_k,\mu_k)$ equals $\int_{\mathbb{R}} z^2 g(z;\theta_k)dz$, we derive

$$\frac{1}{\alpha_k^2}\int_{\mathbb{R}_0} z\frac{\exp(\beta_k z/\alpha_k)}{\sinh(\pi z/\alpha_k)}dz = \frac{1}{2\cos^2(\beta_k/2)}.$$

Hence, we get

$$\frac{\partial}{\partial\beta_1}f(\alpha_1,\beta_1;\alpha_1,\beta_2) = \frac{1}{2(\cos(\beta_1/2))^2} - \frac{1}{\alpha_1^2}\int_{\mathbb{R}_0} z\frac{\exp(\beta_1 z/\alpha_1)}{\sinh(\pi z/\alpha_1)}dz \equiv 0,$$

and $(\partial/\partial\beta_2)f(\alpha_1,\beta_1;\alpha_1,\beta_2) \equiv 0$ in a similar manner. These imply that $f(\alpha_1,\beta_1;\alpha_1,\beta_2) \equiv 0$. The proof is complete.


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