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**Abstract**. In this article, linear operators satisfying anti-commutation relations are considered. It is proven that an anti-commutative type of the Glimm-Jaffe-Nelson commutator theorem follows.

## 1 Introduction and Main Theorem

In this article we consider the self-adjointness of linear operators satisfying anti-commutation relations. For criteria on the self-adjointness of the symmetric operator satisfying a commutation relation, there is the Glimm-Jaffe-Nelson commutator theorem. Refer to [2, 3, 4] of the original papers, and see also e.g. ([1]; Theorem 2.32, [6]; Theorem X.36). We investigate an anti-commutative type of the commutator theorem.

**Theorem 1**
*Let $H$ be a symmetric operator and $A$ be a strictly positive self-adjoint operator i.e. there exists a constant $\delta_A > 0$ such that $(\Psi, A\Psi) \geq \delta_A(\Psi,\Psi)$. We assume the following conditions :*
***(C.1)*** *There exists a core $\mathcal{D}_0$ of $A$ such that $\mathcal{D}_0 \subset \mathcal{D}(H)$ where $\mathcal{D}(H)$ denotes the domain of $H$.*
***(C.2)*** *There exists a constant $a > 0$ such that for all $\Psi \in \mathcal{D}_0$,*

$$\|H\Psi\| \ \leq \ a\|A\Psi\| \, . \tag{1}$$

***(C.3)*** *There exists a constant $b > 0$ such that for all $\Psi \in \mathcal{D}_0$,*

$$\Big| \, (H\Psi, A\Psi) + (A\Psi, H\Psi) \Big| \ \leq \ b \, \|A^{1/2}\Psi\|^2. \tag{2}$$

*Then $H$ is essentially self-adjoint on $\mathcal{D}_0$.*

**Remark 1** *Let $X$ and $Y$ be symmetric operators on a Hilbert space $\mathcal{X}$. Then the real part and the imaginary part of the inner product $(X\Psi, Y\Psi)$ for $\Psi \in \mathcal{D}(XY) \cap \mathcal{D}(YX)$ is expressed by using the commutator and the anti-commutator :*

$$Re\,(X\Psi, Y\Psi) \;=\; \frac{(\Psi,\, \{X,Y\}\Psi)}{2}\,, \tag{3}$$

*and*

$$Im\,(X\Psi, Y\Psi) \;=\; \frac{(\Psi,\, [X,Y]\Psi)}{2}\,, \tag{4}$$

*where $\{X,Y\} = XY + YX$ and $[X,Y] = XY - YX$. On the proof of the Glimm-Jaffe-Nelson commutator theorem, the estimate of a type of (4) is investigated. In this article we pply the estimate of a type of (3).*

**Remark 2** *Since $\|A^{1/2}\Phi\| \leq \|A\Phi\| + \|\Phi\|$ follows for $\Phi \in \mathcal{D}(A)$, we see that $D_0$ is a core of $A^{1/2}$. Then from **(C.2)** and **(C.3)**, it is seen that $\mathcal{D}(A) \subset \mathcal{D}(\overline{H_{\lceil \mathcal{D}_0}})$ and for $\Phi \in \mathcal{D}(A)$,*

$$\left| (\overline{H_{\lceil \mathcal{D}_0}}\Phi, A\Phi) + (A\Phi, \overline{H_{\lceil \mathcal{D}_0}}\Phi) \right| \;\leq\; b\,\|A^{1/2}\Phi\|^2 \tag{5}$$

*hold, where $\overline{X}$ denotes the closure of a operator $X$.*

**(Proof of Theorem 1)**

Let $c \in \mathbf{R}$ be a real number satisfying $c > \frac{b}{2}$, and let us set $z = c + i \;\in\; \mathbf{C}$. For a closable operator $X$, it is seen that $\overline{X} = (X^*)^*$ and $(\overline{X})^* = X^*$ follow. Then from the general theorem ([6], Theorem X.1) on the essential-self-adjointness of closed symmetric operator, it is enough to show that dim ker $\left( H^*_{\lceil \mathcal{D}_0} + z \right)$ = dim ker $\left( H^*_{\lceil \mathcal{D}_0} + z^* \right) = 0$. Let $\Psi \in \mathcal{D}_0$. It is noted that $A^{-\alpha}$, $\alpha > 0$ is bounded, since $A$ is a strictly positive. Then by using Re $(f,g) = \frac{(f,g)+(g,f)}{2}$ and $(H^*_{\lceil \mathcal{D}_0})^* = \overline{H_{\lceil \mathcal{D}_0}}$, we see that for $\Xi = A^{-1}\Psi \;\in \mathcal{D}(A)$,

$$\text{Re}\left(\Xi,\, (H^*_{\lceil \mathcal{D}_0} + z)\Psi\right) \;=\; \frac{1}{2}\left( (\overline{H_{\lceil \mathcal{D}_0}}\Xi, A\Xi) + (A\Xi, \overline{H_{\lceil \mathcal{D}_0}}\Xi) \right) \;+\; c\|A^{1/2}\Xi\|^2.$$

Then by using (5) and $\|A^{1/2}\Xi\| = \|A^{-1/2}\Psi\|$, we have

$$\left|\text{Re}\,\left(\Xi,\, (H^*_{\lceil \mathcal{D}_0} + z)\Psi\right)\right| \;\geq\; \left(c - \frac{b}{2}\right)\|A^{-1/2}\Psi\|. \tag{6}$$

Now let us assume that $\Psi \in$ ker $\left(H^*_{\lceil \mathcal{D}_0} + z\right)$. Then, we see from (6) that $A^{-1/2}\Psi = 0$. Then by acting $A^{1/2}$ the both side of this equation, we obtain $\Psi = 0$. Hence ker $\left(H^*_{\lceil \mathcal{D}_0} + z\right) = \{0\}$. Similarly, we can prove that ker $\left(H^*_{\lceil \mathcal{D}_0} + z^*\right) = \{0\}$. Thus the proof is obtained. ■

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