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Hirofumi OSADA

§ 0. Introduction

The purpose of this paper is to establish a rigorous derivation of the two dimensional vorticity equation associated with the Navier-Stokes equation from a many particle system as a propagation of chaos.

Let $v(t, z)$ $(z=(x, y) \in \boldsymbol{R}^2)$ be the vorticity of an incompressible and viscous two dimensional fluid, under the action of an external conservative field. Then v is described by the following evolution equation

(0.1) $$\partial_t v+(u\cdot \nabla)v-\nu\Delta v=0, \qquad u(t, z)=(\nabla^{\perp}G)*v(t, z),$$

where $G(z)=-(2\pi)^{-1}\log|z|$, $*$ denotes convolution, $\nabla=\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $\nabla^{\perp}=\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\right)$. Here $\nu>0$ denotes the viscosity constant. As far as strong solutions concerns, (0.1) is equivalent to the Navier-Stokes equation. In fact, $u(t, z)$ turns to be the velocity field described by the Navier-Stokes equation. Conversely we can get v from u as $v=\operatorname{curl} u$. Since the two dimensional Navier-Stokes equation is an equation of a vector valued function, a probabilistic treatment is not easy, while the vorticity equation (0.1) is nothing but a McKean's type non-linear equation (see [3]). Such an observation for the two dimensional Navier-Stokes equation was made by Marchioro-Pulvirenti in [2].

Let $\{Z_t\}$ denote the McKean process associated with (0.1);

(0.2) $$dZ_t=\sigma dB_t+u(t, Z_t)dt \qquad u(t, z)=(\nabla^{\perp}G)*(Z_t\circ P)(z)$$

where $\sigma^2=2\nu$, $\{B_t\}$ is a 2-dimensional Brownian motion. The precise definition of the McKean process associated with (0.1) will be presented in Section 2.

The n particle system associated with (0.1) are described by the following SDEs,

(0.3) $$\left\{dZ_t^i=\sigma dB_t^i+(n-1)^{-1}\sum_{\substack{j\neq i\\ j=1}}^{n}(\nabla^{\perp}G)(Z_t^i-Z_t^j)dt\right\}, \quad 1\leqq i\leqq n,$$

where $(B_t^1, \cdots, B_t^n)$ is a $2n$-dimensional Brownian motion. Since the coefficients of (0.3) have singularities at $N=\bigcup_{i\neq j}^n \{z=(z_1, \cdots, z_n) \in \boldsymbol{R}^{2n}, z_i \neq z_j\}$, it is not trivial to see that the solution of (0.3) defines a conservative diffusion process on $\boldsymbol{R}^{2n}$. However, if it starts out side of N, it can be shown that this diffusion process does not hit N (see Section 2 and Osada [6]).

Now we prepare some notations. For a separable metric space S, $\mathcal{M}(S)$ denotes the set of probalilities on S. For $m \in \mathcal{M}(S)$ and a measurable function f, we set $\langle m, f\rangle = \int_S f dm$. A sequence of symmetric probabilities m_n on S^n is said to be m-chaotic for a probability m on S, if for $f_1, \cdots, f_k \in C_b(S)$,

$$\lim_{n\to\infty} \langle m_n, f_1 \otimes \cdots \otimes f_k \otimes 1 \otimes \cdots \otimes 1\rangle = \prod_{i=1}^k \langle m, f_i\rangle,$$

holds. If we denote by $(X_1, \cdots, X_n) \in S^n$ a random variable with the distribution m_n, it can be shown (see Tanaka [12], Sznitman [10]) that being m-chaotic is equivalent to the convergence in law of $\bar{X}_n = (1/n) \sum_{i=1}^n \delta_{X_i}$ towards the non-random m.

In the following, C will denote $C([0, \infty) \to \boldsymbol{R}^2)$. Let $\{Z_t^n = (Z_t^1, \cdots, Z_t^n)\}$ (resp. $\{Z_t\}$) be the solution of (0.3) ((0.2)) with an initial distribution $\psi_n(z_1, \cdots, z_n) dz_1 \cdots dz_n$ ($\psi(z)dz$) and P_n (P) be the probability on C^n (C) induced by $\{Z_t^n\}$ ($\{Z_t\}$). Now we state our main result:

Theorem. *There exists a positive constant* ν_0 *such that, if* $\nu > \nu_0$, *then* P_n *is P-chaotic for all* $\{\psi_n dz_1 \cdots dz_n\}$ *which is* ψdz*-chaotic and satisfying*

$$\text{(0.4)} \qquad \varlimsup_{n\to\infty} \left\| \int_{\boldsymbol{R}^{2n-2i}} \psi_n dz_{i+1} \cdots dz_n \right\|_{L^\infty(\boldsymbol{R}^{2i})} < \infty$$

for $i=1, 2$.

Marchioro-Pulvilenti [2] presented first this propagation of chaos problem for the vorticity equation and Goodman [1] discussed also this problem. Their results are valid for all $\nu > 0$. However their arguments are not complete in two points. First they did not construct a diffusion process associated with (0.3). Second they discussed the propagation of chaos not from (0.3) but from another equation approximating (0.3). In the above theorem, we have proved the original problem without any modification.

As we shall remark in Section 2, $Z_t \circ P(dz)$ has a smooth density if $t > 0$. Hence $u(t, z) = \nabla^\perp G * (Z_t \circ P)(z)$ is a smooth solution of the Navier-Stokes equation.

It is convenient to state the theorem in another way. Let

$$\bar{Z}_n = C_n^{-1} \sum_{I_n} \delta_{(Z^{i_1}_\cdot, \dots, Z^{i_6}_\cdot)} \quad (I_n = \{(i_1, \cdots, i_6);\ 1 \leqq i_k \leqq n,\ i_k \neq i_j \text{ if } k \neq j\})$$

and $\bar{P}_n (\in \mathcal{M}(\mathcal{M}(C^6)))$ be the distribution of $(\bar{Z}_n, P_n)$, where C_n denotes the normalizing constant. Put $\bar{P} = \delta_{P \otimes \cdots \otimes P} \in \mathcal{M}(\mathcal{M}(C^6))$. Then as we explained above, Theorem is equivalent to

Theorem′. *Assume* $\{\psi_n dz_1 \cdots dz_n\}$ *and* ψdz *satisfy the same condition of Theorem. Let* $\nu > \nu_0$. *Then*

$$\lim_{n \to \infty} \bar{P}_n = \bar{P} \qquad \text{in } \mathcal{M}(\mathcal{M}(C^6)). \tag{0.5}$$

In Section 1, we prepare some uniform estimates for fundamental solution of parabolic equation to obtain a tightness result of $\{\bar{P}_n\}$. This tightness result is valid for all $\nu > 0$. In Section 2, we obtain a uniqueness result for a weak solution of (0.1) and give a precise meaning to (0.2). In Section 3, we identify the limit of $\{\bar{P}_n\}$ and complete the proof. In Section 4 we prove a certain uniform estimate of moments of fundamental solutions.

We explain the basic idea. Let L_n be the generator of (0.3);

$$L_n = \nu \Delta + (n-1)^{-1} \sum_{\substack{i \neq j \\ i,j=1}}^{n} (\nabla^\perp G)(z_i - z_j) \cdot \nabla_i,$$

where $\nabla_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right)$, and for fixed v let

$$L = \nu \Delta - u \cdot \nabla \qquad (u(t, z) = \nabla^\perp G * v(t, \cdot)(z)),$$

which is the generator of (0.1) if we regard it as linear equation for fixed v. The first point is to notice L_n and L are of G.D.F.. (see Section 1 for the definition of G.D.F.), which follows from

$$\nabla_x G = \nabla_x a_1 + \nabla_y a_2 \qquad \nabla_y G = \nabla_y a_1 + \nabla_x a_3. \tag{0.6}$$

Here $a_i(z)$ are bounded functions defined by

$$\begin{cases} a_1(z) = -x^2 y^2 / \pi |z|^4 \\ a_2(z) = -3xy / 2\pi |z|^2 + x^3 y / \pi |z|^4 \\ a_3(z) = -3xy / 2\pi |z|^2 + x y^3 / \pi |z|^4 \end{cases} \tag{0.7}$$

(G is unbounded; nevertheless the derivative of G is a sum of derivatives of *bounded* functions). Hence we can use the uniform estimates of fundamental solutions obtained in [8].

Now we have to identify a limit point $\bar{P}_\infty$ of $\{\bar{P}_n\}$. As we show in Section 3, for $\bar{P}_\infty$ a.e. $m \in \mathcal{M}(C^6)$, $m=\otimes\hat{m}$ ($\hat{m}$ is the distribution of $\{Z_t^1\}$), and $\hat{m}$ is a martingale solution of (0.1). (See Section 1 for the definition of martingale solution). If $Z_t^1 \circ \hat{m}(dz)$ has a density $v(t, z, m)dz$ such that

$$F_1(m)=\int_0^t\int_{R^2} v(s, z, m)^3 dzds<\infty, \tag{0.8}$$

then such an $\hat{m}$ is unique (Proposition 2.2). The second point is to see (0.8) follows from, for $\bar{P}_\infty$ a.e. m,

$$F_2(m)=\left\langle m, \int_0^t i_2(Z_s^1-Z_s^2, Z_s^1-Z_s^3, Z_s^i-Z_s^4)ds\right\rangle<\infty \qquad (i=1, 2, 3), \tag{0.9}$$

where $i_2(z_1, z_2, z_3)=(|z_1|^2+|z_2|^2)^{-3/2}|z_3|^{-1}$. Indeed, on account of $m=\hat{m}\otimes\cdots\otimes\hat{m}$, we have at least formally that

$$F_1(m)=\left\langle m, \int_0^t \delta(Z_s^1-Z_s^2, Z_s^1-Z_s^3)\right\rangle.$$

We can obtain by applying Ito's formula to $R(Z_2)=-1/(4\pi^2|Z_2|^2)$ that

$$\begin{aligned}4\nu F_1(m)=&\langle m, R(Z_\cdot^1-Z_\cdot^2, Z_\cdot^1-Z_\cdot^3)\rangle\Big|_0^t\\ &-2\left\langle m, \int_0^t (\nabla R)(Z_s^1-Z_s^2, Z_s^1-Z_s^3)(\nabla^\perp G)(Z_s^1-Z_s^4)ds\right\rangle\\ &+2\left\langle m, \int_0^t (\nabla R)(Z_s^1-Z_s^2, Z_s^1-Z_s^3)(\nabla^\perp G)(Z_s^2-Z_s^4)ds\right\rangle.\end{aligned}$$

The second term is bounded from above because $R<0$ and $\psi \in L^\infty(\boldsymbol{R}^2)$, and the last two terms are dominated by $F_2(m)$. Hence (0.8) follows from (0.9). Since F_2 is lower semicontinuous, (0.9) follows from

$$\overline{\lim_{n\to\infty}}\langle \bar{P}_n, F_2\rangle=\overline{\lim_{n\to\infty}} F_2(P_n)<\infty. \tag{0.10}$$

Here we extend the domain of F_2 over $\mathcal{M}(C^n)$ naturally. We obtain (0.10) by using Ito's formula for the function

$$k=\Delta^{-1}i_2.$$

The crucial points here are the singularity of $\nabla k\cdot\nabla^\perp G$ is similar to that of i_2. Hence we can obtain (0.10) if the viscosity ν is large enough. Thus we need the assumption that $\nu>\nu_0$ in order to control the singularity of the drift term.

We have to discuss the problem in $\mathcal{M}(C^6)$ instead of $\mathcal{M}(C)$ because we have been able to prove the uniqueness result only of the form Proposition 2.2. It is very hard to show the uniqueness under a weaker condition such that $\mu(t, \cdot)$ is only a probability measure.

In [10] Sznitman considered the propagation of chaos for the Burgers equation. There he had to treat singular drifts of δ-function type and he overcame this difficulty by introducing extra function spaces and by Tanaka's local time like arguments. Our method depends on Sznitman's ideas in the above two points.

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§ 1. Preliminaries from analysis and tightness results

In this section we prepare analytical estimates which will be used throughout this paper and obtain tightness results.

Let $a^1_{ij}(t)$, $a^2_{ij}(t, x)$, $c_{ij}(t, x)$ and $m(x)$ be measurable functions. Set

$$A=m^{-1}\left\{\sum_{i,j=1}^{n} \nabla_i a_{ij}\nabla_j+\sum_{j=1}^{n} b_j\nabla_j\right\} \quad \left(\nabla_i=\frac{\partial}{\partial x_i}\right), \tag{1.1}$$

where $a_{ij}(t, x)=a^1_{ij}(t)+a^2_{ij}(t, x)$ and $b_j=\sum_{i=1}^{n}\nabla_i c_{ij}$. Here b_j is not always a function. Without loss of generality, we can assume $a^1_{ij}=a^1_{ji}$. A is said to be of generalized divergence form (G.D.F. in abbreviation) if the coefficients satisfy the following conditions:

(G.1) $\quad \operatorname{div} b=0$ in distribution, $\quad (b=(b_j))$,

(G.2) $\quad \alpha^{-1}|\xi|^2\leqq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j\leqq\alpha|\xi|^2 \qquad$ for all $\xi=(\xi_i)\in \boldsymbol{R}^n$,

(G.3) $\quad |a^2_{ij}|,\ |c_{ij}|\leqq\beta/n, \qquad \sup_{1\leqq i\leqq n}\sum_{j=1}^{n}|a^1_{ij}|\leqq\beta$,

(G.4) $\quad \gamma^{-1}\leqq m\leqq\gamma$,

where α, β and γ are constants with $\alpha, \gamma\geqq 1$ and $\beta>0$.

We shall denote by $G(n; \alpha, \beta, \gamma)$ the totality of G.D.F.s satisfying the above conditions, and by $G_0(n; \alpha, \beta, \gamma)$ the subcollection of $G(n; \alpha, \beta, \gamma)$ with smooth coefficients.

Let us consider the norm $\|\cdot\|_{[a,b]}$ defined by

$$\|v\|_{[a,b]}=\sup_{a\leqq t\leqq b}\|u(t, \cdot)\|_{L^2(\boldsymbol{R}^n)}+\left(\int_a^b\int_{\boldsymbol{R}^n}|\nabla u|^2 dxdt\right)^{1/2}.$$

We denote the function space equiped the norm $\|\cdot\|_{[a,b]}$ by $H_{[a,b]}$. $u(t, x)\in H_{[a,b]}$ is said to be a solution of the Cauchy problem of ∂_t-A on $[a, b]\times\boldsymbol{R}^n$ $(a<b<\infty)$ with initial condition $u(a, \cdot)\in L^2(\boldsymbol{R}^n)\cap L^1(\boldsymbol{R}^n)$ if

$$(1.2)\quad \begin{aligned}&\int_{R^n} u\phi\Big|_a^t mdx-\int_a^t\int_{R^n} u\partial_s\phi mdxds\\ &+\int_a^t\int_{R^n}\sum_{i,j=1}^n (a_{ij}\nabla_j u\nabla_i\phi-c_{ij}\nabla_i u\nabla_j\phi-c_{ij}u\nabla_i\nabla_j\phi)dxds=0\end{aligned}$$

for all $t\in[a, b]$ and all $\phi\in C_0^2([a, b]\times \boldsymbol{R}^n)$.

A continuous function $p(s, x, t, y)$ $(s<t,\ x, y\in \boldsymbol{R}^n)$ is said to be a fundamental solution of ∂_t-A, if it satisfies the following conditions:

$$\int_{R^n} p(s, x, t, y)m(y)dy=1,\qquad p(s, x, t, y)\geqq 0.$$

Moreover, set $u(t, y)=\int_{R^n} p(a, x, t, y)u(a, x)m(x)dx$ for $u(a, x)\in L^2(\boldsymbol{R}^n)\cap L^1(\boldsymbol{R}^n)$. Then u is a solution of the Cauchy problem of ∂_t-A on $[a, b]\times \boldsymbol{R}^n$ with an initial condition $u(a, x)$.

We call p a regular fundamental solution of ∂_t-A $(A\in G(n;\alpha,\beta,\gamma))$ if p is a fundamental solution of ∂_t-A and there exists a sequence $\{A_k\}_{k=1,2,\dots}$ of G.D.F.s satisfying

$$A_k\in G_0(n; 2\alpha, 2\beta, 2\gamma)\qquad \text{for all } k$$

and

$$\lim_{k\to\infty} p^k(s, x, t, y)=p(s, x, t, y)\quad \text{compact uniformly,}$$

where p^k is a fundamental solution of ∂_t-A_k.

Lemma 1.1. *Let $A\in G(n;\alpha,\beta,\gamma)$. Then there exists a regular fundamental solution of ∂_t-A. Moreover, an arbitrary regular fundamental solution $p(s, x, t, y)$ satisfies*

$$(1.3)\qquad p(s, x, t, y)\leqq C_1(t-s)^{-n/2}\exp[-C_2|x-y|^2/(t-s)]$$

for all $s<t$ and $x, y\in \boldsymbol{R}^n$ with positive constants C_1 and C_2 depending only on α, β, γ and n, that

$$(1.4)\quad \begin{aligned}&|P(s, x, t, y)-p(s', x', t', y')|\\ &\leqq C_3(|s-s'|^{\theta/2}+|x-x'|^{\theta}+|t-t'|^{\theta/2}+|y-y'|^{\theta})\end{aligned}$$

for all $T<t-s,\ t'-s'<\infty$ and $x, y, x', y'\in \boldsymbol{R}^n$, where C_3 depends only on α, β, γ, n and $T(T>0)$, and θ $(0<\theta<1)$ depends only on α, β, γ and n.

See Theorem 1 and 2 in [8] for a proof.

Lemma 1.2. *Let A be G.D.F. defined by* (1.1). *Let $u(t, x)$ is a solu-*

tion of the Cauchy problem of $\partial_t - A$ *on* $[a, b]\times \boldsymbol{R}^n$ *with* $u(a, \cdot) \in L^2(\boldsymbol{R}^n) \cap L^1(\boldsymbol{R}^n)$. *Suppose* $b_i \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ *and that*

$$\int_a^b \int_{\boldsymbol{R}^n} |b_i u|^2 \, dx dt < \infty \qquad \text{for } i=1, \cdots, n. \tag{1.5}$$

Then $u(t, x)$ *is unique.*

See Proposition 4.1 in [8] for proof.

Lemma 1.3. *Let* $p(s, x, t, y)$ *be a regular fundamental solution of* $\partial_t - A$ $(A \in G(n; \alpha, \beta, \gamma))$. *Then*

$$\sum_{i=1}^{n} \int_{\boldsymbol{R}^n} p(s, x, t, y) |x_i - y_i|^k \, dy \leqq C_4 (t-s)^{k/2} n \tag{1.6}$$

for all $x=(x_i) \in \boldsymbol{R}^n$ *and* $0<t-s<\infty$, *where* C_4 *is a positive constant depending only on* α, β, γ *and* k $(k=1, 2, \cdots)$. *Especially* C_4 *is independent of the dimension* n.

We shall prove Lemma 1.3 in Section 4.

Now let L_n be the generator of the n-particle system (0.3). Then

$$L_n = \nu\Delta + (n-1)^{-1} \sum_{\substack{i \neq j \\ i,j=1}}^{n} (\nabla^{\perp} G)(z_i - z_j) \cdot \nabla_i. \tag{1.7}$$

Let L'_n be the formal adjoint of L_n with respect to Lebesgue measure. Then

$$L'_n = \nu\Delta - (n-1)^{-1} \sum_{\substack{i \neq j \\ i,j=1}}^{n} (\nabla^{\perp} G)(z_i - z_j) \cdot \nabla_i. \tag{1.8}$$

Lemma 1.4. $L'_n \in G(2n; \nu, 2, 1)$. *Moreover there exists a unique regular fundamental solution* $p_n(s, x, t, y)$ *of* $\partial_t - L'_n$ *satisfying*

$$P(\boldsymbol{Z}^n_t \in dy \,|\, \boldsymbol{Z}^n_s = x) = p_n(s, x, t, y) dy,$$

where $\{\boldsymbol{Z}^n_t\}$ *is the solution of* (0.3).

See [6] and example 2 in Section 1 in [8] for proof.

As a corollary of Lemma 1.1 and Lemma 1.4, we have the following estimate, which will be used frequently in Section 3.

$$p_n(s, x, t, y) \leqq C_5 (t-s)^{-n/2} \exp\left[-C_6 |x-y|^2/(t-s)\right] \tag{1.9}$$

for all $x, y \in \boldsymbol{R}^n$, $0<t-s<\infty$.

We show finally tightness results for $\{Z_t^n\}$ appearing in Theorem.

Proposition 1.1. *Let $\{Z_t^{n,m}\}$ be the first m components of $\{Z_t^n\}$. Then*
(i) *$\{Z_t^{n,m}\}$ is tight in $C([0,\infty)\to \mathbf{R}^{2m})$.*
(ii) *$\{\bar{P}_n\}$ is tight.*

Proof. By Lemma 1.3 and Lemma 1.4, we have

$$\sum_{i=1}^{m} \langle P_n, |Z_t^i - Z_s^i|^4\rangle = (m/n) \sum_{i=1}^{n} \langle P_n, |Z_t^i - Z_s^i|^4\rangle \leqq Cm(t-s)^2.$$

Here we used the symmetry of $(Z_\cdot^1, \cdots, Z_\cdot^n)$. This proves (i). It is well known that (ii) follows from (i).

§ 2. Uniqueness results for the non-linear process

Let $\{Z_t\}$ be a $\mathbf{R}^2$ valued measurable process and $\{\mu_t\}$ be the distribution of $\{Z_t\}$. $\{\mu_t\}$ is said to be a weak solution of (0.1) with an initial distribution μ_0, if $\{\mu_t\}$ satisfies the following conditions;

$$\text{(W.1)} \qquad \int_0^t \int_{\mathbf{R}^4} |z_1 - z_2|^{-1} d\mu_s(dz_1) d\mu_s(dz_2) ds < \infty,$$

$$\text{(W.2)} \qquad \begin{aligned} &\langle \mu_s, \phi(s,\cdot)\rangle\Big|_{s=0}^{s=t} + \int_0^t \langle \mu_s, -\partial_s\phi(s,\cdot) - \nu\Delta\phi(s,\cdot)\rangle ds \\ &- \int_0^t \langle \mu_s \otimes \mu_s, (\nabla^\perp G)(z_1 - z_2)\cdot(\nabla\phi)(s, z_1)\rangle ds = 0 \end{aligned}$$

for all $\phi \in C_0^2([0,\infty)\times \mathbf{R}^2)$ and $0 < t < \infty$. It should be noted that

$$\text{(2.1)} \qquad \lim_{t\to 0} \langle \mu_t, \psi\rangle = \langle \mu_0, \psi\rangle \qquad \text{for all } \psi \in C_b(\mathbf{R}^2)$$

follows from the definition immediately.

We present a uniqueness result for weak solutions of (0.1).

Proposition 2.1. *Assume $\{\mu_t\}$ is a weak solution of (0.1) with an initial condition μ_0. Suppose that $\mu_t(dz)$ has a density $\mu(t,z)dz$ for a.e. $t \geqq 0$ and that $\mu_0(dz) = \mu_0(z)dz$ with $\mu_0 \in L^\infty(\mathbf{R}^2)$. Assume further that there exists a $\delta > 0$ satisfying*

$$\text{(2.2)} \qquad \int_0^t \left(\int_{\mathbf{R}^2} \mu(s,z)^2 dz\right)^{1/2} \left(\int_{\mathbf{R}^2} \mu(s,z)^{2+\delta} dz\right)^{1/(2+\delta)} ds < \infty$$

for all $t > 0$. Then $\{\mu_t\}$ is unique and has a continuous density $\mu(t,z)$ on $(0,\infty)\times \mathbf{R}^2$ satisfying

$$\mu(t,z) \leqq \|\mu_0\|_{L^\infty(\mathbf{R}^2)}.$$

Remark. One can show (see [4]) that there exists a weak solution $\mu(t, z)dz$ such that $\mu(t, z)$ is smooth if $t>0$ for an arbitrary initial condition $\mu_0 \in \mathscr{M}(\boldsymbol{R}^2)$. Moreover $\mu(t, z)$ satisfies (2.2) if μ_0 has a density $\mu_0(z)$ with $\mu_0 \in L^\infty(\boldsymbol{R}^2)$.

We reduce Proposition 1.1 to the following lemma, due to Marchioro-Pulvilenti [2].

Lemma 2.1. *Let $\{\mu_t\}$ be a weak solution of* (0.1) *with an initial condition $\mu_0(z)dz$. Suppose $\mu_0 \in L^\infty(\boldsymbol{R}^2)$ and that $\mu_t(dz)$ has a density $\mu(t, z)dz$ for all $t>0$ satisfying*

$$\sup_{0\leq s\leq t} \|\mu(s, \cdot)\|_{L^\infty(\boldsymbol{R}^2)}<\infty$$

for all $0<t<\infty$. Then $\{\mu_t\}$ is unique.

Proof of Proposition 2.1. Let L_μ be the differential operator defined by

$$L_\mu=\nu\Delta-u\cdot\nabla \quad (u=(\nabla^\perp G)*\mu_t). \tag{2.3}$$

The key point of the proof is to notice that

$$L_\mu \in G(2; \nu, 1/2, 1) \tag{2.4}$$

and that μ is a solution of the Cauchy problem for L_μ in $[s, t]\times\boldsymbol{R}^2$ which satisfies the uniqueness condition (1.5) in Lemma 1.2.

We show first (2.4). By (0.6) and the definition of u, we have

$$u_1(t, z)=\nabla_x a_1*\mu_t(z)+\nabla_y a_2*\mu_t(z)$$
$$u_2(t, z)=-\nabla_x a_3*\mu_t(z)-\nabla_y a_1*\mu_t(z).$$

Since $\|a_i(z)\|_{L^\infty(\boldsymbol{R}^2)}<1/4$ $(i=1, 2, 3)$, we have

$$\sup_t \|a_i*\mu_t(\cdot)\|_{L^\infty(\boldsymbol{R}^2)}<1/4.$$

Then L_μ satisfies (G.2). (G.1) follows from

$$\operatorname{div} u=\nabla\cdot\nabla^\perp G*\mu_t=0.$$

Hence we obtain (2.4). Next we shall show that μ is a solution of the Cauchy problem for L_μ in $[s, t]\times\boldsymbol{R}^2$ $(0<s<t<\infty)$ which satisfies the uniqueness condition (1.5) in Lemma 1.2. For this purpose it is enough to check that

$$\sup_{s\leq\tau\leq t}\left\{\int_{\boldsymbol{R}^2}\mu(\tau, z)^2dz\right\}<\infty, \qquad \int_s^t\int_{\boldsymbol{R}^2}|\nabla\mu(\tau, z)|^2dzd\tau<\infty, \tag{2.5}$$

and that

$$\int_0^t \int_{R^2} |u(s, z)|^2 \mu(s. z)^2 dz ds < \infty. \tag{2.6}$$

By $|\nabla^\perp G(z)| \leqq 1/2\pi |z|$ and $u = \nabla^\perp G * \mu$, we have

$$|u(t, z)| \leqq C_1 \left\{ \int_{R^2} \mu(t, z)^{2+\delta} dz \right\}^{1/2+\delta} + 1, \tag{2.7}$$

where C_1 is a positive constant depending on δ. Hence by (2.2)

$$\int_0^t \int_{R^2} |u(s, z)|^2 \mu(s, z)^2 dz ds < \infty,$$

which implies that L_μ satisfies (2.6).

Secondly we shall check (2.5). Let $f(t, z)$ be a non-negative smooth function defined on $(-\infty, \infty) \times \boldsymbol{R}^2$ with support in $t^2 + |z|^2 \leqq 1/2$ and $\iint_{R^3} f(t, z) dz dt = 1$. Define

$$\mu_\rho(t, z) = \int_0^\infty \int_{R^2} f_\rho(t-s, z-z') \mu(s, z') dz' ds,$$

where $f_\rho(t, z) = \rho^{-3} f(t/\rho, z/\rho)$. Then in $[\rho, \infty) \times \boldsymbol{R}^2$, μ_ρ satisfies the following equality;

$$\partial_t \mu_\rho = \nu \Delta \mu_\rho - \nabla \cdot b_\rho, \tag{2.8}$$

where $b_\rho(t, z) = \int_0^\infty \int_{R^2} f_\rho(t-s, z-z') \mu(s, z') u(s, z') dz' ds$. Multiplying μ_ρ to the both sides of (2.8) and integrate over $[s, t] \times \boldsymbol{R}^2$ $(\rho < s)$, we have

$$F_\rho(t) - F_\rho(s) + \nu H_\rho(s, t) = I_\rho(s, t). \tag{2.9}$$

Here we define

$$F_\rho(t) = (1/2) \int_{R^2} \mu_\rho(t, z)^2 dz, \tag{2.10}$$

$$H_\rho(s, t) = \int_s^t \int_{R^2} |\nabla \mu_\rho(\tau, z)|^2 d\tau dz, \tag{2.11}$$

$$I_\rho(s, t) = \int_s^t \int_{R^2} b_\rho(\tau, z) \nabla \mu_\rho(\tau, z) d\tau dz. \tag{2.12}$$

On account of (2.6), we have

$$\lim_{\rho \to 0} b_\rho = u\mu \qquad \text{in } L^2([0, t] \times \boldsymbol{R}^2 \to \boldsymbol{R}^2) \tag{2.13}$$

and

$$\overline{\lim_{\rho\to 0}} |I_\rho(s, t)| \leqq C_2 \overline{\lim_{\rho\to 0}} H_\rho(s, t)^{1/2}, \tag{2.14}$$

where C_2 is the L^2 norm of $u\mu$ over $[0, t]\times \boldsymbol{R}^2$. Let $F(t)=\int \mu(t, z)^2 dz$. Then by (2.2), we have $F(t)<\infty$ a.e. t and that

$$\lim_{\rho\to 0} F_\rho(t)=F(t) \qquad \text{for a.e. t.} \tag{2.15}$$

Combining these results with (2.9) yields

$$F(t)-F(s)+\nu \overline{\lim_{\rho\to 0}} H_\rho(s, t) \leqq C_2 \{\overline{\lim_{\rho\to 0}} H_\rho(s, t)\}^{1/2}$$

for a.e. $t-s>0$, which shows

$$\overline{\lim_{\rho\to 0}} H_\rho(s, t)<\infty.$$

Then we have

$$H(s, t)=\int_s^t \int_{\boldsymbol{R}^2} |\nabla \mu(\tau, z)|^2 d\tau dz<\infty \qquad \text{for all } t-s>0. \tag{2.16}$$

Hence we obtain, for all $t-s>0$,

$$\lim_{\rho\to 0} \mu_\rho=\mu \qquad \text{in } H^1([s, t]\times \boldsymbol{R}^n), \tag{2.17}$$

$$\lim_{\rho\to 0} I_\rho(s, t)=\int_s^t \int_{\boldsymbol{R}^2} u\mu\nabla \mu\, d\tau dz=0. \tag{2.18}$$

Here we used (2.13) and $\operatorname{div} u=0$. Combining (2.16), (2.17) and (2.18) with (2.9) yields

$$F(t)-F(s)+\nu H(s, t)=0 \qquad \text{for a.e. } t-s>0. \tag{2.19}$$

A moment reflection tells that (2.19) holds for all s and t with $0<s<t<\infty$, which implies (2.5).

Thus μ is a solution of the Cauchy problem for L_μ in $[s, t]\times \boldsymbol{R}^2$ with the initial condition $\mu(s, z)$. Moreover we can apply Lemma 1.2 to see that μ is unique. Consequently

$$\mu(t, z)=\int_{\boldsymbol{R}^2} \mu(s, \xi) p(s, \xi, t, z) d\xi, \tag{2.20}$$

where p is a regular fundamental solution of $\partial_t - L_\mu$ defined on $\{(s, \xi, t, z);\ 0\leqq s\leqq t<\infty,\ \xi, z \in \boldsymbol{R}^2\}$. By means of Lemma 1.1, we see, for fixed (t, ξ), (2.21) $$\lim_{s\to 0} p(s, z, t, \xi)=p(0, z, t, \xi) \qquad \text{uniformly in } z.$$

This together with (2.1) shows

$$\mu(t, z)=\lim_{s\to 0}\int_{R^2} \mu(s, \xi)p(s, \xi, t, z)d\xi$$
$$=\int_{R^2} \mu(0, \xi)p(0, \xi, t, z)d\xi.$$

Since $\int_{R^2} p(s, z, t, \xi)dz=1$ for all $0\leqq s<t$ and ξ, we conclude

(2.22) $$\mu(t, z)\leqq\|\mu(0, z)\|_{L^\infty(R^2)},$$

which completes the proof of Proposition 2.1.

We say $P \in \mathcal{M}(C)$ is a martingale solution of (0.1) if

(2.24) $$\left\langle P\otimes P, \int_0^t |Z_s^1-Z_s^2|^{-1}ds\right\rangle<\infty$$

and, for all $f \in C_0^2(R^2)$

(2.25) $$f(Z_t)-\int_0^t L'_\mu f(Z_s)ds$$

is P martingale with respect to the canonical filtration. Here μ is the distribution of (Z_t, P) and L'_μ is the operator defined by

(2.26) $$L'_\mu=\nu\Delta+u\nabla\cdot \qquad (u(t, z)=(\nabla^\perp G)*\mu_t(z)).$$

The expression for L'_μ is formal, however owing to (2.24), the second term of (2.25) has always a finite expectation.

We call this $P \in \mathcal{M}(C)$ to be a McKean process associated with (0.1) because we can show

Proposition 2.2. *Let P be a martingale solution of* (0.1) *such that $Z_t \circ P(dz)$ has a density $\mu(t, z)dz$ for a.e. $t\geqq 0$ satisfying* (2.2). *Suppose that $\mu(0, \cdot) \in L^\infty(R^2)$. Then P is unique.*

Proof. It is clear that $\{\mu(t, z)dz\}$ is a weak solution of (0.1). Then it follows from Proposition 2.2 that $\{\mu(t, z)dz\}$ is unique. Hence the generator L_μ is determined uniquely. Since $u(t, z)$ is bounded measurable, the martingale problem for L'_μ starting from $(0, z)$ is well posed (see [9]). Then we obtain the uniqueness of martingale solutions.

§ 3. Identification of a limit

We have already known by Proposition 1.1 that $\{\bar{P}_n\}$ is precompact in $\mathscr{M}(\mathscr{M}(C^6))$. Throughout this section, we shall denote an arbitrary convergent subsequence of $\{\bar{P}_n\}$, also, by $\{\bar{P}_n\}$, and its limit by $\bar{P}_\infty$. The purpose of this section is to show $\bar{P}_\infty$ concentrates on $P\otimes\cdots\otimes P$ to complete the proof of Theorem.

For $m\in\mathscr{M}(C^6)$, $\tilde{m}$ ($\tilde{m}\in\mathscr{M}(C)$) denotes the distribution of the first coordinate and $\tilde{m}_t$ denotes $Z_t^1\circ\tilde{m}$.

Lemma 3.1.

(3.1) $\bar{P}_\infty(\{m\in\mathscr{M}(C^6)$: *m is independent copies of* $\tilde{m}\})=1$.

The proof of Lemma 3.1 is elementary. Hence we omit it. See, for example, Proposition 4.2 of [10].

Proposition 3.1. *Suppose* $\nu>1/2\pi$. *Then, for* $\bar{P}_\infty$ *a.e.m*, $\tilde{m}$ *is a martingale solution of* (0.1).

We prepare first the following two lemmas to show the above proposition.

Lemma 3.2. *Let* $0\leqq\alpha<1$ *and suppose* $\nu>1/2\pi(1-\alpha)$. *Then*

$$\varlimsup_{n\to\infty}\left\langle P_n, \int_0^t |Z_s^1-Z_s^2|^{-1-\alpha}ds\right\rangle<\infty \tag{3.2}$$

for all t.

Proof. Let $f(z)=|z|^{-1-\alpha}$ and $g(z)=(1-\alpha)^{-2}|z|^{1-\alpha}$. Define for $0<\iota<1$,

$$f_l(z)=\min\{f(z), l^{-1-\alpha}\} \quad \text{and} \quad g_l(z)=-G*f_l(z).$$

Here $*$ denotes convolution. Then it is clear that $\Delta g=f$ and $\Delta g_l=f_l$. By Ito's formula and symmetry of $(Z_s^1, \cdots, Z_s^n)$, we have

$$\begin{aligned}\langle P_n, g_l(Z_s^1-Z_s^2)\rangle\Big|_{s=\varepsilon}^{s=t} &= \frac{2}{n-1}\sum_{i=2}^n\left\langle P_n, \int_\varepsilon^t (\nabla g_l)(Z_s^1-Z_s^2)\cdot(\nabla^\perp G)(Z_s^1-Z_s^i)ds\right\rangle \\ &\quad +2\nu\left\langle P_n, \int_\varepsilon^t f_l(Z_s^1-Z_s^2)ds\right\rangle.\end{aligned} \tag{3.3}$$

By (1.9), we can apply Lebesgue's convergence theorem to the both sides of (3.3). Hence

$$
\begin{aligned}
&\langle P_n, g(Z_s^1-Z_s^2)\rangle\Big|_{s=\varepsilon}^{s=t} \\
&\quad=\frac{2}{n-1}\sum_{i=2}^{n}\left\langle P_n, \int_\varepsilon^t (\nabla g)(Z_s^1-Z_s^2)\cdot(\nabla^\perp G)(Z_s^1-Z_s^i)ds\right\rangle \\
&\qquad+2\nu\left\langle P_n, \int_\varepsilon^t f_l(Z_s^1-Z_s^2)ds\right\rangle \\
&\quad\geqq\left(2\nu-\frac{1}{\pi(1-\alpha)}\right)\left\langle P_n, \int_\varepsilon^t f_l(Z_s^1-Z_s^2)ds\right\rangle.
\end{aligned}
\tag{3.4}
$$

Here we used the symmetry of $(Z_t^1, \cdots, Z_t^n)$ and

$$
\begin{aligned}
|\nabla g(z_1)\cdot\nabla^\perp G(z_2)| &\leqq \{2\pi(1-\alpha)|z_1|^\alpha|z_2|\}^{-1} \\
&\leqq \{2\pi(1-\alpha)\}^{-1}\left(\frac{\alpha}{\alpha+1}f(z_1)+\frac{1}{\alpha+1}f(z_2)\right).
\end{aligned}
\tag{3.5}
$$

By Lemma 1.3, we have

$$
\langle P_n, g(Z_s^1-Z_s^2)\rangle\Big|_{s=\varepsilon}^{s=t} < C(t-\varepsilon)^{1/2}+2. \tag{3.6}
$$

Here C is a positive constant independent of t, ε and n. Lemma 3.2 follows from (3.4) and (3.6) immediately.

For $\phi(t, z) \in C_0^2([0, \infty)\times \boldsymbol{R}^2)$ set

$$
H(t, z_1, z_2)=(\nabla^\perp G)(z_1-z_2)\cdot(\nabla\phi)(t, z_1).
$$

Let $H_\varepsilon(t, z_1, z_2) \in C_b^2([0, \infty)\times\boldsymbol{R}^2\times\boldsymbol{R}^2)$ such that

$$
H_\varepsilon(t, z_1, z_2)=H(t, z_1, z_2) \qquad \text{if } |z_1-z_2|\geqq\varepsilon. \tag{3.7}
$$

We can do this in such a way that

$$
|H_\varepsilon(t, z_1, z_2)|\leqq 2|H(t, z_1, z_2)| \tag{3.8}
$$

for all $(t, z_1, z_2) \in [0, \infty)\times\boldsymbol{R}^2\times\boldsymbol{R}^2$. For $m \in \mathcal{M}(C^4)$ satisfying

$$
\left\langle m, \int_0^t |Z_s^1-Z_s^2|^{-1}ds\right\rangle < \infty
$$

for all $t\geqq 0$, define

$$
\begin{aligned}
F(m)=\Big\langle m, \Big\{&\phi(t, Z_t^1)-\phi(s, Z_s^1) \\
&-\int_s^t (\partial_t\phi+\nu\Delta\phi)(\tau, Z_\tau^1)+H(\tau, Z_\tau^1, Z_\tau^2)d\tau\Big\}\Psi\Big\rangle.
\end{aligned}
$$

Otherwise we set $F(m)=+\infty$. Here $\Psi=\prod_{i=1}^{k}\psi_i(Z_{t_i}^1)$ with $\psi_i \in C_0(\boldsymbol{R}^2)$ and $0\leqq t_i\leqq s$. Let

$$F_\varepsilon(m)=\Big\langle m, \Big\{\phi(t, Z_t^1)-\phi(s, Z_s^1) - \int_s^t (\partial_t\phi+\nu\Delta\phi)(\tau, Z_\tau^1)+H_\varepsilon(\tau, Z_\tau^1, Z_\tau^2)d\tau\Big\}\Psi\Big\rangle.$$

Lemma 3.3.

(i) $$\langle \bar{P}_n, |F|\rangle=0 \qquad \textit{for all } n\geqq 4.$$

Moreover, assume $\nu>1/2\pi$. *Then for any* $0<\alpha<1$ *there exists* C_α *such that*

(ii) $$\langle \bar{P}_n, |F_\varepsilon-F|\rangle\leqq C_\alpha\varepsilon^\alpha\Big\langle P_n, \int_0^t |Z_s^1-Z_s^2|^{-1-\alpha}ds\Big\rangle$$

holds for all t, $n\geqq 4$ *and* $\varepsilon>0$.

Proof. On account of symmetry of $(Z_t^1, \cdots, Z_t^n)$, we have

$$\langle \bar{P}_n, |F|\rangle=\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\ i,j=1}}^{n}\Big|\Big\langle P_n, \Big\{\phi(t, Z_t^i)-\phi(s, Z_s^i) - \int_s^t \Big((\nu\Delta\phi+\partial_t\phi)(\tau, Z_\tau^i)+\frac{1}{n-1}\sum_{\substack{k\neq i\\ k=1}}^{n} H(\tau, Z_\tau^i, Z_\tau^k)\Big)d\tau\Big\}\Psi\Big\rangle\Big|=0,$$

which shows (i). Now we have

$$\begin{aligned}\langle \bar{P}_n, |F_\varepsilon-F|\rangle &\leqq C_1\Big\langle P_n, \int_s^t |(H-H_\varepsilon)(\tau, Z_\tau^1, Z_\tau^2)|d\tau\Big\rangle \\ &\leqq C_1C_2\Big\langle P_n, \int_s^t |Z_\tau^1-Z_\tau^2|^{-1}1_{[-\varepsilon,\varepsilon]}(|Z_\tau^1-Z_\tau^2|)d\tau\Big\rangle,\end{aligned} \tag{3.9}$$

where $C_1=\max_{i,z}|\psi_i(z)|$ and $C_2=\max_{(t,z)}|\phi(t, z)|$. Here we used (3.8). (ii) follows from this immediately.

Proof of Proposition 3.1. We observe first

$$\Big\langle P_n, \int_0^t |Z_s^1-Z_s^2|^{-1-\alpha}ds\Big\rangle=\Big\langle \bar{P}_n, \Big\langle m, \int_0^t |Z_s^1-Z_s^2|^{-1-\alpha}ds\Big\rangle\Big\rangle.$$

Then, it follows from Lemma 3.2 that

$$\Big\langle \bar{P}_\infty, \Big\langle m, \int_0^t |Z_s^1-Z_s^2|^{-1}ds\Big\rangle\Big\rangle<\infty, \tag{3.10}$$

which implies that $\tilde{m}$ satisfies (2.24).

Since $|F(m)|\leqq C_1+C_2\left\langle m, \int_0^t |Z_s^1-Z_s^2|^{-1}ds\right\rangle$ with constants C_1 and C_2, we can apply Lebesgue's convergence theorem to $\langle \bar{P}_\infty, |F_\varepsilon|\rangle$. Then

$$\lim_{\varepsilon\to 0} \langle \bar{P}_\infty, |F_\varepsilon|\rangle=\langle \bar{P}_\infty, |F|\rangle. \tag{3.11}$$

Since F_ε is a bounded continuous function on $\mathcal{M}(\mathcal{M}(C^4))$, we have

$$\lim_{n\to\infty} \langle \bar{P}_n, |F_\varepsilon|\rangle=\langle \bar{P}_\infty, |F_\varepsilon|\rangle. \tag{3.12}$$

Combining (3.11) and (3.12) with Lemma 3.3 yields

$$\langle \bar{P}_\infty, |F|\rangle=0,$$

which implies (2.25). The proof is thus completed.

Proposition 3.2. *There exists a positive constant* ν_2 *such that, if* $\nu\geqq\nu_2$, *then, for* $\bar{P}_\infty$ a.e. m, $\tilde{m}_t$ *has a density* $v(t, z, m)dz=v_t(z, m)dz$ *for* a.e. t *satisfying*

$$\int_0^t \left\{\int_{R^2} v_s^2dz\right\}^{1/2}\left\{\int_{R^2} v_s^3dz\right\}^{1/3}ds<\infty \qquad \textit{for all } t<\infty. \tag{3.13}$$

We prepare first several notations. Let r_p be the function defined by

$$r_p(z_p)\equiv r_p(z_1, \cdots, z_p)=(|z_1|^2+\cdots+|z_p|^2)^{-1/2},$$

where $z_p=(z_1, \cdots, z_p)\in R^{2p}$. We set $r(z_1)=r_1(z_1)$. Let $*$ denote convolution as before and $\otimes$ denote the following operation;

$$f\otimes g=f(z_1, \cdots, z_p)g(z_{p+1}, \cdots, z_{p+q})$$

for functions f defined on R^{2p} and g defined on R^{2q}. Define

$$h_{\alpha,\beta}=r_2^3*(r^\alpha\otimes r^\beta) \qquad (0<\alpha, \beta<2, \alpha+\beta>1)$$

and

$$h=r_3^5*(r^{3/2}\otimes r^{3/2}\otimes r).$$

It is easy to see that

$$h_{\alpha,\beta}(\theta z_2)=|\theta|^{1-\alpha-\beta}h_{\alpha,\beta}(z_2), \tag{3.14}$$

$$h(\theta z_3)=|\theta|^{-3}h(z_3) \tag{3.15}$$

and that

(3.16) $$h_{\alpha,\beta}(z_1, z_2)=h_{\alpha,\beta}(|z_1|, |z_2|),$$

(3.17) $$h(z_1, z_2, z_3)=h(|z_1|, |z_2|, |z_3|).$$

Let ρ_α ($\alpha \in \boldsymbol{R}$) be the function on $\boldsymbol{R}^2$ defined by $\rho_\alpha(z)=|z|^{-\alpha}$ ($\alpha>0$), $\rho_\alpha(z)=\max\{-\log|z|, 1\}$ ($\alpha=0$) and $\rho_\alpha(z)=1$ ($\alpha<0$). Also we set

$$\rho_{\alpha,\beta}(z_1, z_2)=\rho_\alpha(z_1/|z_2|)+\rho_\beta(z_2/|z_1|)$$

and

$$\xi_{\alpha,\beta,\gamma}(z_1, z_2, z_3)=r_2^\alpha(z_1/|z_3|, z_2/|z_3|)\rho_{\beta,\gamma}(z_1, z_2).$$

Lemma 3.4. *There exist positive constants C_1 and C_2 satisfying*

(i) $$h_{\alpha,\beta}(z_2)\leqq C_1 r_2^{\alpha+\beta-1}(z_2)\rho_{\alpha-1,\beta-1}(z_2)$$

for all z_2 with $|z_1||z_2|\neq 0$, and

(ii) $h(z_3)\leqq C_2 r_3^3(z_3)\{\xi_{2,1/2,1/2}(z_1, z_2, z_3)+\xi_{3/2,1/2,0}(z_2, z_3, z_1)+\xi_{3/2,0,1/2}(z_3, z_1, z_2)\}$

for all z_3 with $|z_1||z_2||z_3|\neq 0$. Here C_1 and *C_2 depends on α and β.*

Proof. Let $e \in \boldsymbol{R}^2$ with $|e|=1$ and $\tau=|z_1-\zeta_1|^{-1}$. Then

(3.18) $$h_{\alpha,\beta}(z_1, e)=\int_{R^2}\tau^{1+\beta}r^\alpha(\zeta_1)\int_{R^2}r_2^3(e, \tau e-\zeta_2)r^\beta(\zeta_2)d\zeta_1 d\zeta_2.$$

Since $\int_{R^2} r_2^3(e, \tau e-\zeta_2)r^\beta(\zeta_2)d\zeta_2\leqq C\min\{\tau^{-\beta}, 1\}$, we have

(3.19) $$h_{\alpha,\beta}(z_1, e)\leqq C\left\{\int_{S_1} r(z_1-\zeta_1)r^\alpha(\zeta_1)d\zeta_1+\int_{S_2} r^{1+\beta}(z_1-\zeta_1)r^\alpha(\zeta_1)d\zeta_1\right\},$$

where $S_1=\{\zeta_1; |z_1-\zeta_1|\leqq 1\}$ and $S_2=\boldsymbol{R}^2-S_1$. Here we used the assumption $\alpha+\beta+1>2$ to show the second term of the right hand side is finite. Hence we obtain

(3.20) $$h_{\alpha,\beta}(z_1, e)\leqq C_{\alpha\beta}\{\rho_{\alpha-1}(z_1)+1\}$$

with a positive constant $C_{\alpha\beta}$ depending only on α and β. Similarly we have

(3.21) $$h_{\alpha,\beta}(e, z_2)\leqq C_{\alpha\beta}\{\rho_{\beta-1}(z_2)+1\}.$$

Combining these results with (3.14) yields (i).

Now we proceed the proof of (ii). Let $\eta=r_2(z_1-\zeta_1, z_2-\zeta_2)$. Then

$$h(z_1, z_2, e)=\int_{R^4} \eta^4 r^{3/2}(\zeta_1) r^{3/2}(\zeta_2) d\zeta_1 d\zeta_2 \int_{R^2} r_2^5(e, \eta e-\zeta_3) r(\zeta_3) d\zeta_3$$
$$\leqq C\int_{R^4} \eta^3 r^{3/2}(\zeta_1) r^{3/2}(\zeta_2) d\zeta_1 d\zeta_2.$$

Here we used $\int_{R^2} r_2^5(e, \eta e-\zeta_3) r(\zeta_3) d\zeta_3 \leqq C\eta^{-1}$. By (i), we have

$$h(z_1, z_2, e)\leqq Cr_2^2(z_1, z_2)\rho_{1/2,1/2}(z_1, z_2).$$

Similarly we have

$$h(z_1, e, z_3)\leqq Cr_2^{3/2}(z_1, z_3)\rho_{1/2,0}(z_1, z_3)$$

and

$$h(e, z_2, z_3)\leqq Cr_2^{3/2}(z_2, z_3)\rho_{1/2,0}(z_2, z_3).$$

Combining these results with (3.15) yields (ii).

Let $i_1=r\otimes r$, $i_2=r^{3/2}\otimes r^{1/2}$ and $j_1=r^{3/2}\otimes r^{3/2}\otimes r$. Then, as a corollary of Lemma 3.4, we have the following

Lemma 3.5. *Let C_1 and C_2 be positive constants in Lemma* 3.4. *Then*

(i) $$\begin{cases} h_{3/2,1/2}(z_2) r(z_1)\leqq C_1\{i_1(z_1, z_2)+i_2(z_1, z_2)\} \\ h_{3/2,1/2}(z_2) r(z_q)\leqq 2C_1 i_1(z_1, z_q) \qquad (q=2, 3), \end{cases}$$

(ii) $$\begin{cases} h_{1,1}(z_2) r(z_1)\leqq C_1\{i_2(z_1, z_2)+i_2(z_2, z_1)\} \\ h_{1,1}(z_2) r(z_3)\leqq C_1\{i_1(z_1, z_3)+i_1(z_2, z_3)\}, \end{cases}$$

(iii) $$\begin{cases} h(z_3) r(z_1)\leqq 6C_2 j_1(z_1, z_2, z_3) \\ h(z_3) r(z_3)\leqq C_2\{4j_1(z_1, z_2, z_3)+2j_1(z_1, z_3, z_2)\} \\ h(z_3) r(z_4)\leqq C_2\{4j_1(z_1, z_2, z_3)+j_1(z_1, z_3, z_2)+j_1(z_2, z_3, z_4)\}. \end{cases}$$

We now have

Lemma 3.6. *Let $j_2=r_2^3\otimes r$ and set $\nu_0=\max\{1/2\pi, C_1/\pi^2, 48C_2/\pi^3\}$ (C_1 and C_2 are constants appearing in Lemma* 3.4). *Suppose $\nu>\nu_0$. Then*

(i) $$\left\langle \bar{P}_\infty, \left\langle m, \int_0^t i_q(Z_s^1-Z_s^2, Z_s^1-Z_s^3) ds\right\rangle\right\rangle<\infty$$

(ii) $$\left\langle \bar{P}_\infty, \left\langle m, \int_0^t j_q(Z_s^1-Z_s^2, Z_s^1-Z_s^3, Z_s^p-Z_s^4) ds\right\rangle\right\rangle<\infty \quad (p=1, 2),$$

for $q=1, 2$ and all t.

Proof. Let $i_q^*(z_2)=i_q(z_2)$ if $|z_1|\leqq 1$ and $|z_2|\leqq 1$, and $i_q^*(z_2)=0$ if $|z_1|\geqq 1$ or $|z_2|\geqq 1$ $(q=1, 2)$. Define

$$k_q=\Delta^{-1}i_q^*=-(1/4\pi^2)r_2^2 * i_q^*.$$

k_q is not of C^2-class. However by the device similar to the proof of Lemma 3.2, we can apply Ito's formula for k_q to obtain

$$\begin{aligned}
&\langle P_n, k_q(Z_\cdot^1-Z_\cdot^2, Z_\cdot^1-Z_\cdot^3)\rangle\Big|_0^t \\
&=(n-1)^{-1}\sum_{i=2}^{n}\Big\langle P_n, \int_0^t (\nabla_1 k_q)(Z_s^1-Z_s^2, Z_s^1-Z_s^3)\cdot(\nabla^\perp G)(Z_s^1-Z_s^i)ds\Big\rangle \\
&\quad+(n-1)^{-1}\sum_{i=2}^{n}\Big\langle P_n, \int_0^t (\nabla_2 k_q)(Z_s^1-Z_s^2, Z_s^1-Z_s^3)\cdot(\nabla^\perp G)(Z_s^1-Z_s^i)ds\Big\rangle \\
&\quad-(n-1)^{-1}\sum_{\substack{i\neq 2\\ i=1}}^{n}\Big\langle P_n, \int_0^t (\nabla_1 k_q)(Z_s^1-Z_s^2, Z_s^1-Z_s^3)\cdot(\nabla^\perp G)(Z_s^2-Z_s^i)ds\Big\rangle \\
&\quad-(n-1)^{-1}\sum_{\substack{i\neq 3\\ i=1}}^{n}\Big\langle P_n, \int_0^t (\nabla_2 k_q)(Z_s^1-Z_s^2, Z_s^1-Z_s^3)\cdot(\nabla^\perp G)(Z_s^3-Z_s^i)ds\Big\rangle \\
&\quad+4\nu\Big\langle P_n, \int_0^t i_q^*(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\Big\rangle.
\end{aligned}\tag{3.22}$$

Let $I_{p,q}$ $(p=1, 2, \cdots, 6)$ denote the p-th term of (3.22). We first see that $\sup_{|z_2|\geqq 2}|k_q(z_2)|<\infty$ and that $\int_{|z_2|<1}|k_q(z_2)|dz_2<\infty$. Then, by $k_q\leqq 0$ and (0.4), we have

$$\varlimsup_{n\to\infty} I_{1,q}\leqq -\varliminf_{n\to\infty}\langle P_n, k_q(Z_0^1-Z_0^2, Z_0^1-Z_0^3)\rangle<\infty. \tag{3.23}$$

It is clear

$$|\nabla_1 k_1|\leqq(1/2\pi^2)h_{1,1} \quad\text{and}\quad |\nabla_1 k_2|\leqq(1/2\pi^2)h_{3/2,1/2}. \tag{3.24}$$

Then by Lemma 3.5 and symmetry of $\{(Z_t^1, \cdots, Z_t^n)\}$, we have for $p=2, 3, \cdots, 5$

$$\begin{aligned}
|I_{p,1}|\leqq&(2C_1/\pi^2(n-1))\Big\langle P_n, \int_0^t i_2(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\Big\rangle \\
&+(C_1/\pi^2)\Big\langle P_n, \int_0^t i_1(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\Big\rangle
\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}
|I_{p,2}|\leqq&(C_1/\pi^2(n-1))\Big\langle P_n, \int_0^t i_2(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\Big\rangle \\
&+(C_1/\pi^2)\Big\langle P_n, \int_0^t i_1(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\Big\rangle.
\end{aligned}\tag{3.26}$$

Now $\sum_{q=1,2}|i_q-i_q^*|(z_1, z_2)\leqq r(z_1)+r(z_2)+r^{3/2}(z_1)+r^{1/2}(z_2)$. Then by Lemma 3.2, we have

$$\overline{\lim_{n\to\infty}}\sum_{q=1,2}\left\langle P_n, \int_0^t |i_q-i_q^*|(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\right\rangle<\infty. \tag{3.27}$$

Combining (3.25) and (3.26) with (3.22) yields

$$\begin{aligned}\sum_{q=1,2} 4(\nu-C_1/\pi^2)&\left\langle P_n, \int_0^t i_q(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\right\rangle\\ &\leqq-\sum_{q=1,2}\left\langle P_n, k_q(Z_0^1-Z_0^2, Z_0^1-Z_0^3)\right\rangle\\ &\quad+(12C_1/\pi^2(n-1))\left\langle P_n, \int_0^t i_2(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\right\rangle\\ &\quad+4\nu\sum_{q=1,2}\left\langle P_n, \int_0^t |i_q-i_q^*|(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\right\rangle.\end{aligned} \tag{3.28}$$

Now assume $\nu>C_1/\pi^2$. Then (3.28) together with (3.23) and (3.27) yields

$$\overline{\lim_{n\to\infty}}\sum_{q=1,2}\left\langle P_n, \int_0^t i_q(Z_s^1-Z_s^2, Z_s^1-Z_s^3)ds\right\rangle<\infty,$$

which completes the proof of (i). (ii) can be shown similarly and (iii) follows from (ii) immediately.

Let $g_\lambda(z)=(2\pi\lambda)^{-1}\exp(-|z|^2/2\lambda)$ and $g_\lambda(z_1, \cdots, z_p)=\prod_{i=1}^p g_\lambda(z_p)$.

Lemma 3.7. *Suppose* ν *satisfies the same assumption in Lemma* 3.6. *Then*

(i) $$\overline{\lim_{\lambda\to 0}}\left\langle \bar{P}_\infty, \left\langle m, \int_0^t g_\lambda(Z_s^1-Z_s^2)ds\right\rangle\right\rangle<\infty,$$

(ii) $$\overline{\lim_{\lambda\to 0}}\left\langle \bar{P}_\infty, \left\langle m, \int_0^t g_\lambda(Z_s^1-Z_s^2, Z_s^1-Z_s^3, Z_s^4-Z_s^5)ds\right\rangle\right\rangle<\infty,$$

for all $t<\infty$.

Proof. Set $f_\lambda=-G*g_\lambda$. Then we have

$$\begin{aligned}\langle P_n, f_\lambda(Z_\cdot^1-Z_\cdot^2)\rangle\Big|_0^t&=\frac{2(n-2)}{(n-1)}\left\langle P_n, \int_0^t (\nabla f_\lambda)(Z_s^1-Z_s^2)\cdot(\nabla^\perp G)(Z_s^1-Z_s^3)ds\right\rangle\\ &\quad+\frac{2}{n-1}\left\langle P_n, \int_0^t (\nabla f_\lambda)(Z_s^1-Z_s^2)\cdot(\nabla^\perp G)(Z_s^1-Z_s^2)ds\right\rangle\\ &\quad+2\nu\left\langle P_n, \int_0^t g_\lambda(Z_s^1-Z_s^2)ds\right\rangle.\end{aligned} \tag{3.29}$$

Letting n to infinity yields

$$\langle \bar{P}_\infty, \langle m, f_\lambda(Z^1_\cdot - Z^2_\cdot)\rangle\rangle\Big|_0^t$$

$$(3.30) \qquad = 2\Big\langle \bar{P}_\infty, \Big\langle m, \int_0^t (\nabla f_\lambda)(Z^1_s - Z^2_s)\cdot(\nabla^\perp G)(Z^1_s - Z^3_s)ds\Big\rangle\Big\rangle$$

$$+ 2\nu\Big\langle \bar{P}_\infty, \Big\langle m, \int_0^t g_\lambda(Z^1_s - Z^2_s)ds\Big\rangle\Big\rangle.$$

Here we used Lemma 3.2 to show the convergence of the first two terms of the right hand side of (3.29).

Now we see

$$(3.31) \qquad -G(z) < f_\lambda(z) < f_{\lambda'}(z) \qquad \text{for } 0<\lambda<\lambda'$$

and

$$(3.32) \qquad |\nabla f_\lambda(z)| \leqq Cr_1(z) \qquad \text{for } 0<\lambda<\infty.$$

Indeed, (3.31) follows from $\nabla_\lambda f_\lambda = g_\lambda > 0$ immediately and the proof of (3.32) is as follows. Let $F_\lambda = |\nabla f_\lambda|^2$. Then

$$\nabla_\lambda F_\lambda = 2(\nabla g_\lambda)\cdot(\nabla f_\lambda) \leqq 2|\nabla g_\lambda| F_\lambda^{1/2}.$$

Hence

$$F_\lambda^{1/2} \leqq F_0^{1/2} + \int_0^\lambda |\nabla g_s| ds \leqq Cr_1,$$

where $C = 1/2\pi + \int_0^\infty |\nabla g_s|(z_0)ds$ with $|z_0|=1$. which shows (3.32).

Let I_p $(p=1, 2, 3)$ denote the p-th term of (3.30). Then by (3.31) we have

$$(3.33) \qquad \overline{\lim_{\lambda\to 0}}\, I_1 \leqq \langle \bar{P}_\infty, \langle m, -G(Z^1_\cdot - Z^2_\cdot)\rangle\rangle\Big|_0^t < \infty.$$

Here we used Lemma 1.3 and (0.4) to show the second term of (3.33) is bounded from above. By (i) of Lemma 3.6 and (3.32), we have $\overline{\lim}_{\lambda\to 0}|I_2| < \infty$. Collecting these results yields (i). The proof of (ii) is similar to that of (i). Hence we omit it.

Lemma 3.8. *Let* $v_\lambda(t, z, m)$ *denote* $(g_\lambda \circ \tilde{m}_t)(z)$, *For* $\bar{P}_\infty$ *a.e.* $m \in \mathcal{M}(C^6)$, $\tilde{m}_t(dz)$ *has a density* $v(t, z, m)$ *for a.e.t such that*

$$\text{(i)} \qquad \int_0^T \int_{R^2} v(t, z, m)^2 dz dt < \infty \qquad \textit{for all } T < \infty.$$

Moreover for $\bar{P}_\infty$ a.e.m

(ii) $$\lim_{\lambda\to 0} v_\lambda(t, z, m) = v(t, z, m) \qquad in\ L^2((0, T)\times \boldsymbol{R}^2).$$

Proof. The key point of the proof is to notice that (i) of Lemma 3.7 follows from

$$\varlimsup_{\lambda\to 0} \left\langle \bar{P}_\infty, \int_0^T \int_{\boldsymbol{R}^2} v_\lambda(t, z, m)^2 dzdt \right\rangle < \infty \tag{3.34}$$

for all $T<\infty$. Indeed, by (3.34), we can choose a convergent subsequence $\{v_{\lambda'}\}$ endowed with weak L^2-topology with a limit $v(t, z, m) \in L^2((0. T)\times \boldsymbol{R}^2\times \mathcal{M}(C^6))$;

$$\lim_{\lambda'\to 0} \int v_{\lambda'}\phi\, dtdzd\bar{P}_\infty = \int v\phi\, dtdzd\bar{P}_\infty$$

for all $\phi(t, z, m) \in L^2((0, T)\times \boldsymbol{R}^2\times \mathcal{M}(C^6))$. Here the integration is taken over $(0, T)\times \boldsymbol{R}^2\times \mathcal{M}(C^6)$. Then it is clear that for $\bar{P}_\infty$ a.e. m,

$$v(t, z, m)dz = \tilde{m}_t(dz) \qquad \text{for a.e. t.}$$

Thus (i) of Lemma 3.7 follows from (3.34).

Now, by the semigroup property of g_λ and Lemma 3.1, we have

$$\begin{aligned}\left\langle m, \int_0^t g_{2\lambda}(Z_s^1 - Z_s^2)ds \right\rangle &= \left\langle m, \int_0^t \int_{\boldsymbol{R}^2} g_\lambda(Z_s^1 - z)g_\lambda(z - Z_s^2)dzds \right\rangle \\ &= \int_0^t \int_{\boldsymbol{R}^2} v_\lambda(s, z, m)^2 dzds.\end{aligned}$$

Then by (i) of Lemma 3.6, we obtain (3.34), which completes the proof of (i). It is clear that (ii) follows from (i).

Proof of Proposition 3.2. We observe first that $g_{\lambda/2}(z-\xi, z-\eta) = g_{\lambda/4}(z-(\xi+\eta)/2)g_\lambda(\xi-\eta)$. Hence

$$\begin{aligned}&\left\langle \bar{P}_\infty, \left\langle m, \int_0^t g_\lambda(Z_s^1 - Z_s^2, Z_s^1 - Z_s^3)ds \right\rangle\right\rangle \\ &= \left\langle \bar{P}_\infty, \int_0^t \int_{\boldsymbol{R}^4} g_\lambda(\xi-\eta)v_{\lambda/4}(s, (\xi+\eta)/2, m)v_{\lambda/2}(s, \xi, m)v_{\lambda/2}(s, \eta, m)d\xi d\eta ds \right\rangle.\end{aligned}$$

It follows from Lemma 3.8 that for $\bar{P}_\infty$ a.e.m

$$\lim_{\lambda\to 0} \int_{\boldsymbol{R}^2} g_\lambda(\xi-\eta)v_{\lambda/4}(s, (\xi+\eta)/2, m)v_{\lambda/2}(s, \eta, m)d\eta = v(s, \xi, m)^2 \text{ a.e. } (s, \xi)$$

and

$$\lim_{\lambda\to 0} v_{\lambda/2}(s, \xi, m) = v(s, \xi, m) \qquad \text{in } L^2((0, T)\times \boldsymbol{R}^2).$$

Then by Fatou's lemma we obtain

$$\varliminf_{\lambda\to 0} \left\langle \bar{P}_\infty, \left\langle m, \int_0^t g_\lambda(Z_s^1 - Z_s^2, Z_s^1 - Z_s^3) ds \right\rangle \right\rangle \geqq \left\langle \bar{P}_\infty, \int_0^t \int_{\boldsymbol{R}^2} v(s, z, m)^3 dz ds \right\rangle.$$

Combining this with (ii) of Lemma 3.7 yields

$$\int_0^t \int_{\boldsymbol{R}^2} v(s, z, m)^3 dz ds < \infty \qquad \text{for } \bar{P}_\infty \text{ a.e.m.}$$

Proposition 3.2 follows from this immediately.

Proof of Theorem. By Proposition 2.1, 3.1 and 3.2, $\bar{P}_n$ converges to $\delta_{P\otimes\cdots\otimes P}$, where P is the distribution of McKean process associated with (0.1) which satisfies the uniqueness condition (W.3), which completes the proof.

§ 4. Moment bound

The purpose of this section is to prove estimates for fundamental solutions of G.D.F. which are independent of dimension n and to prove Lemma 1.3. In case of divergence form Lemma 1.3 has been already shown in [5]. Our argument here is based on that of in [5]. However, we have to consider moments for $p(x_i)$ which is a smooth approximation of $|x_i|$ because the space of test functions for G.D.F. is $H^2(\boldsymbol{R}^n)$ (not $H^1(\boldsymbol{R}^n)$) and $|x_i| \in H^1_{\mathrm{loc}} \cap (H^2_{\mathrm{loc}})^c$.

We give a reduction of Lemma 1.3:

Proposition 4.1. *Let $A \in G_0(n; \alpha, \beta, \gamma)$ and p be the fundamental solution of $\partial_t - A$. Then*

$$\sum_{i=1}^n \int_{\boldsymbol{R}^n} p(0, 0, 1, x) |x_i|^k dx \leqq Cn, \tag{4.1}$$

where C is a positive constant depending only on α, β, γ and k. Especially C is independent of dimension n and smoothness of coefficients.

Indeed, let $A = m^{-1}\{\sum_{i,j} \nabla_i a_{ij} \nabla_j + (\nabla_i c_{ij})\nabla_j\} \in G(n; \alpha, \beta, \gamma)$. Define $A^{\varepsilon,h} = m(x/\varepsilon + h)^{-1}\{\sum_{i,j} \nabla_i a(x/\varepsilon + h)\nabla_j + (\nabla_i c_{ij}(x/\varepsilon + h))\nabla_j\}$. Then

$$A^{\varepsilon,h} \in G(n; \alpha, \beta, \gamma) \qquad \text{for all } \varepsilon \text{ and } h. \tag{4.2}$$

Hence Lemma 1.3 follows from Proposition 4.1.

Since the coefficients of A are smooth, it follows that p is unique and $\partial_t p$ exists. Moreover, for $0<t\leqq T$

$$C_1 t^{-n/2}\exp[-C_2|x|^2/t]\leqq p(0,0,t,x)\leqq C_3 t^{-n/2}\exp[-C_4|x|^2/t], \tag{4.3}$$

where $C_1, \cdots, C_4$ are positive constants depending on the smoothness of the coefficients, dimension n and T.

Lemma 4.1. *For $t>0$*

$$p(0,0,t,x)\leqq(\kappa t)^{-n/2}, \tag{4.4}$$

where κ is a positive constant depending only on α and γ.

See Lemma 3.1 in [8] for proof.

We now prepare notations. Let $\rho(t)$ be a smooth function on $\boldsymbol{R}^1$ satisfying the following conditions;

(i) $0\leqq\rho\leqq|t|$ for $-\infty<t<\infty$, $\rho(t)=\rho(-t)$,

(ii) $\rho'(0)=\rho(0)=0$,

(iii) $\rho(t)=t$ for $t\geqq 1/2$.

Define $\rho_i=\rho(x_i)$, $Y_p=\sum_{i=1}^n\rho_i^p$ and $Y^q=(Y_1)^q$. Similarly we set $X_p=\sum_{i=1}^n|x_i|^p$ and $X^p=(X_1)^p$. It should be noted that

$$Y_1\leqq X_1\leqq Y_1+n. \tag{4.5}$$

In the following, an integration with respect to space variable is always taken over $\boldsymbol{R}^n$ and P denotes $p(0,0,t,x)$. For integers p and q, we define

$$M(p,q)=\int X_pX^qPmdx, \qquad \tilde{M}(p,q)=\int Y_pY^qPmdx,$$

$$E(p,q)=-\int X_pX^qP(\log P)mdx, \qquad \tilde{E}(p,q)=-\int Y_pY^qP(\log P)mdx,$$

$$G(p,q)=\int Y_pY^q\sum_{i=1}^n|\nabla_i(\log P)|^2Pdx,$$

$$\tilde{H}(p,q)=\int Y_pY^q(\log P)^2Pdx, \qquad H(p,q)=\int X_pX^q(\log P)^2Pdx.$$

Moreover

$$J(p,q)=\int Y_pY^q\sum_{i=1}^n|(\nabla_iP)(\log P)|Pdx,$$

$$K(p,q)=\int Y_pY^q\sum_{i=1}^n|\nabla_iP|Pdx,$$

$$L(p, q)=\int Y_p Y^q P|\log P|dx.$$

The last three quantities are managed by $M(p, q)$, $G(p, q)$ and $H(p, q)$. In fact, by the Schwarz inequality we have

Lemma 4.2.

$$J(p, q)\leqq(nG(p, q)\tilde{H}(p, q))^{1/2},$$
$$K(p, q)\leqq(n\gamma G(p, q)\tilde{M}(p, q))^{1/2},$$
$$L(p, q)\leqq(n\gamma\tilde{M}(p, q)\tilde{H}(p, q))^{1/2}.$$

We set

$$M^*(p, q)=M(p, q) \text{ if } p\geqq 0 \text{ and } q\geqq 0, =0 \text{ if } p<0 \text{ or } q<0,$$

and similarly for $E^*(p, q)$, $G^*(p, q)$, $\cdots$, $L^*(p, q)$. Throughout this section C always denotes the unified symbol which represents a positive constant depending only on p, q, α, β, γ and the function ρ. Moreover, $f\lesssim g$ will denote $f\leqq Cg$ for some positive constant C as above. Let $\nabla_i=\frac{\partial}{\partial x_i}$ and $\nabla=(\nabla_1, \cdots, \nabla_n)$. We define

$$|\nabla f|=\sum_{i=1}^n|\nabla_i f| \quad \text{and} \quad |\nabla\nabla f|=\sum_{i,j=1}^n|\nabla_i\nabla_j f|.$$

We start with some simple calculations.

Lemma 4.3. *Let* $(Y_pY^q)^*=Y_pY^q$ *if* $p, q\geqq 0$ *and* $(Y_pY^q)^*=0$ *if* $p<0$ *or* $q<0$. *Then for* $p, q=0, 1, 2, \cdots$

(i) $|\nabla Y_pY^q|\lesssim(Y_{p-1}Y^q)^*+(nY_pY^{q-1})^*$.

(ii) $|\nabla\nabla Y_pY^q|\lesssim(Y_{p-2}Y^q)^*+(Y_{p-1}Y^q)^*+n(Y_{p-1}Y^{q-1})^*$
$+n(Y_pY^{q-1})^*+n^2(Y_pY^{q-2})^*$.

(iii) *Let* f *be a smooth function defined on* $(0, \infty)$ *with*

$$\int_0^\infty|f|(1+|t|^k)^{-1}dt<\infty$$

for some $k>0$. *Then*

$$\left|\int\sum_{i,j}^n\nabla_i(Y_pY^q)a_{ij}^1\nabla_jf(P)dx\right|$$
$$\lesssim\int(Y_{p-2}Y^q)^*|f(P)|dx+\int(Y_{p-1}Y^q)^*|f(P)|dx$$
$$+\int(Y_{p-1}Y^{q-1})^*|f(P)|dx+n\int(Y_pY^{q-1})^*|f(P)|dx$$
$$+n^2\int(Y_pY^{q-2})^*|f(P)|dx.$$

Proof. The proof of (i) and (ii) is elementary. Hence we omit it.

$$\left|\int \sum_{i,j}^{n} \nabla_i(Y_pY^q)a^1_{ij}\nabla_j f(P)dx\right| = \left|\int \sum_{i,j}^{n} \nabla_j\nabla_i(Y_pY^q)a^1_{ij}f(P)dx\right|$$
$$\leqq \left|\int \sum_{i=1}^{n} l_i a^1_{ii} Y^q f(P)dx\right| + 2\left|\int \sum_{i,j}^{n} (\nabla_i Y_p)(\nabla_j Y^q)a^1_{ij}f(P)dx\right|$$
$$+ \left|\int \sum_{i,j}^{n} Y_p(\nabla_i\nabla_j Y^q)a^1_{ij}f(P)dx\right|.$$

Here $l_i=\rho''(x_i)p\rho(x_i)^{p-1}+\rho'(x_i)^2p(p-1)\rho(x_i)^{p-2}$. This together with

$$\sup_j \sum_{i=1}^{n} |a^1_{ij}| \leqq \beta$$

yields (iii).

In the following succesive four lemmas, we shall estimate $\tilde{M}(p, q)$ from above.

Lemma 4.4. *Let p and q be non-negative integers. Then*

$$|\partial_t \tilde{M}(p, q)| \lesssim U_1 + \{nG^*(p, q-1)\tilde{M}^*(p, q-1)\}^{1/2},$$

where

$$U_1 = \tilde{M}^*(p-2, q) + \tilde{M}^*(p-1, q) + \tilde{M}^*(p-1, q-1) + n\tilde{M}^*(p, q-1)$$
$$+ n^2\tilde{M}^*(p, q-2) + n^{-1}K^*(p-1, q).$$

Proof.

$$\begin{aligned}\partial_t\tilde{M}(p, q) = &-\int \sum_{i,j} \nabla_i(Y_pY^q)a^1_{ij}\nabla_j P dx - \int \sum_{i,j} \nabla_i(Y_pY^q)a^2_{ij}\nabla_j P dx \\ &+ \int \sum_{i,j} \nabla_j(Y_pY^q)c_{ij}\nabla_i P dx + \int \sum_{i,j} \nabla_i\nabla_j(Y_pY^q)c_{ij} P dx.\end{aligned} \tag{4.6}$$

Let I_k denote the k-th term of the right-hand side of (4.6). Then by (iii) of Lemma 4.3, we have

$$\begin{aligned}|I_1| \lesssim\, &\tilde{M}^*(p-2, q) + \tilde{M}^*(p-1, q) + \tilde{M}^*(p-1, q-1) \\ &+ n\tilde{M}^*(p, q-1) + n^2\tilde{M}^*(p, q-2).\end{aligned} \tag{4.7}$$

By (i) of Lemma 4.3 we have

$$|I_2| \lesssim (1/n)\int |\nabla(Y_pY^q)||\nabla P|dx \lesssim (1/n)K^*(p-1, q) + K^*(p, q-1). \tag{4.8}$$

Similarly we have

$$|I_3| \lesssim (1/n)K^*(p-1, q) + K^*(p, q-1),$$

$$\begin{aligned} |I_4| &\lesssim (1/n) \int |\nabla\nabla(Y_p Y^q)| P dx \\ &\lesssim (1/n)\tilde{M}^*(p-2, q) + (1/n)\tilde{M}^*(p-1, q) + \tilde{M}^*(p-1, q-1) \\ &\quad + \tilde{M}^*(p, q-1) + n\tilde{M}^*(p, q-2). \end{aligned}$$

Combining these results with Lemma 4.2 concludes Lemma 4.4.

Lemma 4.5. *Let $p \geqq 0$ and $q \geqq 1$. Then for $0 < t \leqq 1$ we have*

$$G(p, q-1) \lesssim \partial_t \tilde{E}(p, q-1) + U_2 + U_3 + U_4 + \{n\tilde{H}^*(p, q-2) G^*(p, q-2)\}^{1/2},$$

where

$$\begin{aligned} U_2 &= n(|\log t| + 1)\{\tilde{M}^*(p-2, q-1) + \tilde{M}^*(p-1, q-1) \\ &\quad + n\tilde{M}^*(p, q-2) + n\tilde{M}^*(p, q-3)\}, \\ U_3 &= n^{-1} J^*(p-1, q-1) + n^{-1} K^*(p-1, q-1) + K^*(p, q-2), \\ U_4 &= n^{-1} J^*(p-2, q-1) + L^*(p-1, q-2) + nL^*(p, q-3). \end{aligned}$$

Proof. We obtain

$$\begin{aligned} \partial_t \tilde{E}(p, q-1) &= -\int Y_p Y^{q-1} \partial_t (P \log P) m dx \\ &= \int Y_p Y^{q-1} P \sum_{i,j} (\nabla_i \log P)(\nabla_j \log P) a_{ij} dx \\ &\quad + \int \sum_{i,j} \nabla_i (Y_p Y^{q-1}) a_{ij}^1 \nabla_j (P \log P) dx \\ &\quad + \int \sum_{i,j} \nabla_i (Y_p Y^{q-1}) a_{ij}^2 \nabla_j (P \log P) dx \\ &\quad - \int \sum_{i,j} \nabla_j (Y_p Y^{q-1}) c_{ij} \nabla_i (P \log P) dx \\ &\quad - \int \sum_{i,j} \nabla_i \nabla_j (Y_p Y^{q-1}) c_{ij} (P \log P) dx. \end{aligned} \tag{4.9}$$

Let I_k denote the k-th term of the right-hand side of (4.9). Then by the uniform ellipticity of $\{a_{ij}\}$ we have

$$I_1 \geqq \alpha^{-1} G(p, q-1). \tag{4.10}$$

Moreover by (iii) of Lemma 4.3 and $|\log P| \lesssim n|\log t| + n$, which follows from Lemma 4.1, we have

$$(4.11)\qquad |I_2|\lesssim U_2.$$

On account of $|a_{ij}^2|$, $|c_{ij}|\leqq\beta/n$ and (i) of Lemma 4.3, we have

$$(4.12)\qquad \begin{aligned} |I_3|,\ |I_4|&\lesssim\beta n^{-1}\int|\nabla(Y_pY^{q-1})||\nabla(P\log P)|dx\\ &\lesssim n^{-1}J^*(p-1,q-1)+n^{-1}K^*(p-1,q-1)\\ &\quad+K^*(p,q-2)+J^*(p,q-2)\\ &\lesssim U_3+\{n\tilde{H}^*(p,q-2)G^*(p,q-2)\}^{1/2}.\end{aligned}$$

Here we used Lemma 4.2. By (ii) of Lemma 4.3 and $|c_{ij}|\leqq\beta/n$, we have

$$(4.13)\qquad |I_5|\leqq\beta n^{-1}\int|\nabla\nabla Y_pY^{q-1}||P\log P|dx=U_4.$$

Combining (4.10), $\cdots$, (4.13) with (4.9) completes the proof.

Lemma 4.6. *Let $p\geqq 0$ and $q\geqq 2$.* Then for $0<t\leqq 1$ we have

(i) $\quad H(p,q-2)\lesssim M(p,q)+n^2(1+|\log t|^2)M(p,q-2)+n^2.$

(ii) $\quad \tilde{H}(p,q-2)\lesssim\tilde{M}(p,q)+U_5,$

where $U_5=n^2(1+|\log t|^2)M(p,q-2)+n^{q+1}(1+|\log t|^2)$.

Proof. See Lemma 2.3 in [5] for the proof of (i). On account of $\tilde{H}(p,q-2)\leqq H(p,q-2)$ and $M(p,q)\leqq\tilde{M}(p,q)+n^{q+1}$, (ii) follows from (i).

By means of the previous three lemmas, we conclude the following:

Lemma 4.7. *It holds that*

$$\begin{aligned}|\partial_t\tilde{M}(p,q)|\lesssim U_1+[n\tilde{M}^*(p,q-1)\{\partial_t\tilde{E}^*(p,q-1)+U_2+U_3+U_4\\ +(n\tilde{G}^*(p,q-2)\{\tilde{M}(p,q)+U_5\})^{1/2}\}]^{1/2}.\end{aligned}$$

Now we estimate $M(p,q)$ from below by $E(p,q-1)$ and $M(p,q-1)$.

Lemma 4.8. *Let $p, q\geqq 0$. Then we have for* $0<t\leqq 1$

(i) $$Cn^{q+1}\exp\left[\frac{E(p,q-1)-2n^{q+1}}{CM(p,q-1)}\right]\leqq M(p,q).$$

(ii) *Let $F=E(0,0)-(n/2)\log\kappa t\geqq 0$. Then for* $0<t\leqq 1$

$$n^2t^{1/2}\exp[F/n^2]\lesssim M(0,1).$$

See Lemma 2.5 in [5] for proof.

Proposition 4.2. *Let p and q be non-negative integers. Then*

(i) $$M(p, q) \lesssim n^{q+1} t^{(p+q)/2} \quad \text{for } 0<t\leqq 1,$$

(ii) $$\int_0^1 t^{(p+q-1)/2} |\tilde{E}^*(p, q-1)| dt \lesssim n^{q+1},$$

(iii) $$\int_0^1 [G^*(p, q-1)]^{1/2} dt \lesssim n^{(q+1)/2},$$

(iv) $$\int_0^1 \tilde{H}^*(p, q-2) dt \lesssim n^{q+1}.$$

Proof. We remark first that (i) and the following are equivalent:

(i)′ $$M(p, q)(1) \lesssim n^{q+1}.$$

(i)″ $$\tilde{M}(p, q)(1) \lesssim n^{q+1}.$$

This follows from (4.2) and (4.5).

We introduce the following order $\rightarrow$ on the set $\{(p, q); p, q=0, 1, \cdots\}$.

$$(p, q) \rightarrow (p', q')$$

if and only if

$$p+q < p'+q'$$

or

$$p+q = p'+q' \quad \text{and} \quad p < p'.$$

We shall prove Proposition 4.2 by induction on (p, q) with respect to this order.

If $(p, q)=0$, every estimate is trivial. In case $(p, q)=(0, 1)$, (iv) is clear and the proof of the other statements is as follows: By Lemma 4.7, we see

$$|\partial_t \tilde{M}(0, 1)| \lesssim n^2 + n[\partial_t \tilde{E}(0, 0)]^{1/2} = n^2 + n[(\partial_t F) + n^2/2t]^{1/2}. \tag{4.14}$$

Here we used $F = E(0, 0) - (n/2) \log \kappa t$ and $E(0, 0) = \tilde{E}(0, 0)$. Since $\lim_{t\to 0} \tilde{M}(0, 1) = 0$, we have

$$\tilde{M}(0, 1)(1) \lesssim n^2 + n \int_0^1 [(2t)^{1/2}(2n)^{-1}\partial_t F + n(2t)^{-1/2}]^{1/2} dt \lesssim n^2 + F(1). \tag{4.15}$$

Here we used the fact that $F \geqq 0$ and that $(a+b)^{1/2} \leqq a/(2b^{1/2}) + b^{1/2}$ for $b>0$ and $a+b>0$. By (ii) of Lemma 4.8, we have

$$n^2 \exp[F(1)/n^2] \leqq M(0, 1)(1). \tag{4.16}$$

Since $M \lesssim \tilde{M}$, we obtain from (4.15) and (4.16) that

$$n^2 \exp [F(1)/n^2] \lesssim n^2 + F(1),$$

which implies

$$F(1) \lesssim n^2. \tag{4.17}$$

This together with (4.15) yields (i)$''$. (ii) follows from $|\tilde{E}(0, 0)| \lesssim n^2 |\log \kappa t|$, and (iii) follows from $G(0, 0) \lesssim \partial_t \tilde{E}(0, 0)$, (4.14), (4.15) and (4.17).

Next we consider the case $q=0$ and $p>0$. In this case (ii), (iii) and (iv) are trivial. By Lemma 4.4, we have

$$|\partial_t \tilde{M}(p, 0)| \lesssim U_1 = \tilde{M}^*(p-2, 0) + \tilde{M}^*(p-1, 0) + (1/n)\tilde{K}^*(p-1, 0). \tag{4.18}$$

Applying the hypothesis of the induction to (4.18) yields

$$\int_0^1 U_1(t) dt \lesssim n,$$

which together with $\lim_{t \to 0} \tilde{M}(p, 0) = 0$ shows (i)$''$.

We finally consider the case $q \geqq 1$ and $(p, q) \neq (0, 1)$. By applying the hypothesis of the induction to Lemma 4.7 and $\lim_{t \to 0} \tilde{M}(p, q) = 0$, we have

$$\begin{aligned} \sup_{0<t\leqq 1} \tilde{M}(p, q) &\leqq \int_0^1 |\partial_t \tilde{M}(p, q)| dt \\ &\lesssim n^{q+1} + \Big[n^{q+1} \sup_{0<t\leqq 1} \Big\{ \int_0^t s^{(p+q-1)/2} \partial_s \tilde{E}(p, q-1) ds \Big\} \\ &\quad + C n^{2(q+1)} + C n^{3(q+1)/2} (\sup_{0<t\leqq 1} \tilde{M}(p, q))^{1/2} \Big]^{1/2}. \end{aligned} \tag{4.19}$$

Now, we see

$$\begin{aligned} &\int_0^t s^{(p+q-1)/2} \partial_s \tilde{E}(p, q-1) ds \\ &\quad = [s^{(p+q-1)/2} \tilde{E}(p, q-1)]_0^t - \frac{1}{2}(p+q-1) \int_0^t s^{(p+q-3)/2} \tilde{E}(p, q-1) ds. \end{aligned}$$

By (4.3), we have $\lim_{t \to 0} t^{(p+q-1)/2} \tilde{E}(p, q-1) = 0$. Moreover we have by Lemma 4.1 and the hypothesis of the induction

$$\tilde{E}(p, q-1) \geqq (n/2) \tilde{M}(p, q-1) \log (\kappa t) \gtrsim -n^{q+1} \quad \text{for } 0<t\leqq 1. \tag{4.20}$$

Hence

$$\sup_{0<t\leqq 1}\left\{\int_0^t s^{(p+q-1)/2}\partial_s\tilde{E}(p, q-1)ds\right\} \leqq \sup_{0<t\leqq 1}\{\tilde{E}(p, q-1)-2n^{q+1}\}+Cn^{q+1}. \tag{4.21}$$

Now we prove (i)″ and (ii). For this, we divide the situation into two cases. First we suppose that $\sup_{0<t\leqq 1}\{\tilde{E}(p, q-1)-2n^{q+1}\}\leqq 0$. Then (4.19) and (4.21) gives

$$\sup_{0<t\leqq 1}\tilde{M}(p, q)\lesssim n^{q+1}+[n^{2(q+1)}+n^{3(q+1)/2}(\sup_{0<t\leqq 1}\tilde{M}(p, q))^{1/2}]^{1/2}.$$

(i)″ follows from this immediately. Combining (4.20) and

$$\sup_{0<t\leqq 1}\tilde{E}(p, q-1)\leqq 2n^{q+1}$$

yields (ii). Next we suppose that

$$\sup_{0<t\leqq 1}\{\tilde{E}(p, q-1)-2n^{q+1}\}>0.$$

Let

$$\lambda=\sup_{0<t\leqq 1}\{\tilde{E}(p, q-1)-2n^{q+1}\} \quad \text{and} \quad \eta=\sup_{0<t\leqq 1}\tilde{M}(p, q)/n^{q+1}.$$

Then we have by Lemma 4.8 and $\lambda>0$ that

$$\exp[C\lambda/n^{q+1}]\lesssim \sup_{0<t\leqq 1}M(p, q)/n^{q+1}\lesssim \eta+1. \tag{4.22}$$

On the other hand, by (4.19) and (4.21) we have

$$\eta\lesssim 1+[\lambda/n^{q+1}+1+\eta^{1/2}]^{1/2}. \tag{4.23}$$

It follows from (4.22) and (4.23) that

$$\lambda/n^{q+1}\lesssim 1 \quad \text{and} \quad \eta\lesssim 1,$$

which implies (i)″. This together with Lemma 4.1 yields (ii).

As for the proof of (iii) and (iv) we proceed as follows. (iv) is an immediate consequence of (i) and Lemma 4.7. (iii) follows from (iv) and Lemma 4.5. The proof of Proposition 4.2 is thus completed.

Proof of Proposition 4.1. Proposition 4.1 follows from Proposition 4.2 immediately.

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