CR多様体上のサブラプラシアンに付随する拡散過程

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Diffusion Processes Associated with Sub-Laplacian on CR Manifolds

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Abstract

A diffusion process associated with the real sub-Laplacian Δ_b, the real part of the complex Kohn-Spencer laplacian $\Box_b$, on a strictly pseudoconvex CR manifold is constructed via the method of Eells, Elworthy and Malliavin by taking advantage of the metric connection due to Tanaka and Webster.

Using the diffusion process and the Malliavin calculus, we study the Dirichlet problem for Δ_b in a probabilistic manner and investigate distributions of stochastic line integrals along the diffusion process.

Moreover, we investigate diagonal short time asymptotics of the heat kernel corresponding to the diffusion process by using Watanabe's asymptotic expansion and give a representation for the asymptotic expansion of heat kernels which shows a relationship to the geometric structure.

Notation

- $\mathbb{N} = \{1, 2, 3, \dots\}$: the set of all positive integers.
- $\mathbb{Z}$: the set of all integers.
- $\mathbb{Z}_{\geq 0} = \{a \in \mathbb{Z}; a \geq 0\}$: the set of all non-negative integers.
- $\mathbb{R}$: the set of all real numbers.
- i: the imaginary unit.
- $\mathbb{C} = \{a + b\mathrm{i}; a, b \in \mathbb{R}\}$: the set of all complex numbers.
- $\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$: Kronecker's delta.
- $O(n)$: the group of $n \times n$ orthogonal matrices.
- $U(n)$: the group of $n \times n$ unitary matrices.
- $\mathfrak{o}(n)$: the set of $n \times n$ skew-symmetric matrices.
- $\mathfrak{u}(n)$: the set of $n \times n$ skew Hermitian matrices.
- $T_x M$: the tangent space of a manifold M at $x \in M$.
- $TM = \coprod_{x \in M} T_x M$: the tangent bundle of a manifold M.
- $\Gamma^\infty(V)$: the space of C^∞ cross sections of a vector bundle V.

- $C^\infty(X;E)$: the set of all E-valued C^∞ functions defined on X.

  $C^\infty(X) = C^\infty(X;\mathbb{R})$.

- $C_b^\infty(X;E) = \{f \in C^\infty(X;E); \text{itself and its derivatives of all orders are bounded}\}$.

  $C_b^\infty(X) = C_b^\infty(X;\mathbb{R})$.

- $C_0^\infty(X;E) = \{f \in C^\infty(X;E); \text{support of } f \text{ is compact}\}$.

  $C_0^\infty(X) = C_0^\infty(X;\mathbb{R})$.

- $L^p(X;E)$: the set of all E-valued L^p functions defined on X.

  $L^p(X) = L^p(X;\mathbb{R})$.

Contents

Chapter 1

Introduction

1.1 Background

The theory of probability and stochastic analysis has close relationship to geometry in various aspects. One of ideas connecting probability and geometry is that geometric information of some kind of manifolds can be extracted via analysis of diffusion processes on them. In this direction, diffusion processes called Brownian motions on Riemannian manifolds have been constructed and deeply investigated. The objective of this thesis is to develop analogous analysis as Riemannian manifolds on CR manifolds.

Let M be an oriented strictly pseudoconvex CR manifold of dimension $2n+1$, and Δ_b be the real sub-Laplacian on M, i.e. the real part of the Kohn-Spencer Laplacian $\Box_b$. The aims of this thesis are to construct the diffusion process generated by $-\Delta_b/2$ by extending the method of Eells, Elworthy and Malliavin, and to apply the diffusion processes.

The Brownian motion on a Riemannian manifold is defined as a diffusion process generated by the Laplace-Beltrami operator. The method of Eells, Elworthy and Malliavin is one of constructions of the Brownian motion on a Riemannian manifold, and realizes the Brownian motion as the projection of the solution of the stochastic differential equation (SDE in abbreviation) on the orthonormal frame bundle over the Riemannian manifold (see, for example, [4, 11, 12, 16, 18]). We will carry out this method on a CR manifold, but in the present thesis we use a complex unitary frame bundle instead of a real orthonormal frame bundle. This comes from the fact the CR structure is defined as a complex subbundle $T_{1,0}$ of the complexified tangent bundle $\mathbb{C}TM$.

To be more precise, in the method of Eells, Elworthy and Malliavin on a Riemannian manifold, the vector fields governing the SDE on the orthonormal bundle are constructed with the help of the Riemannian connection. The SDE corresponds to the stochastic parallel translation on the Riemannian manifold. In our construction on a CR manifold, we take advantage of the metric connection on the complex subbundle $T_{1,0}$ due to Tanaka [22] and Webster [25] to have vector fields $L_1, \ldots, L_n$ on the unitary bundle $U(T_{1,0})$ over M. Solving the SDE on $U(T_{1,0})$ governed by $L_1, \ldots, L_n$ and projecting its solution onto M, we arrive at the diffusion process $\mathbb{X} = \{(\{X(t)\}_{t\geq 0}, P_x); x \in M\}$ on M generated by $-\Delta_b/2$.

For applications of this construction of diffusion processes, we will obtain, by using the partial hypoelliplicity argument in the Malliavin calculus, the heat kernel related to this diffusion process. Precisely speaking, we will show the transition probability function of $\mathbb{X}$ has a smooth density function $p(t, x, y)$. Next, we will give a sufficient condition for distributions of stochastic line integrals of 1-forms on M along the diffusion process $\mathbb{X}$ to have smooth density functions.

We will also consider the Dirichlet problem associated with Δ_b. Let G be a relatively compact open set in M with C^3-boundary. We shall show in a probabilistic manner that, for each $f \in C(\partial G)$, there is a $u \in C(\overline{G})$ such that

$$\Delta_b u = 0 \quad \text{on } G \text{ in the weak sense,} \quad \text{and } u = f \text{ on } \partial G.$$

Together with hypoelliplicity of Δ_b, this u is a classical solution to the Dirichlet problem. In the proof, a key role is played by the local representation of the sub-Laplacian Δ_b so that, on every sufficiently small coordinate neighborhood U, there are $a^\alpha \in \mathbb{C}$ and C^∞-vector fields Z_α with $\mathbb{C}$-valued coefficients on U so that

$$\Delta_b = -\sum_{\alpha=1}^{n} (Z_\alpha \overline{Z_\alpha} + \overline{Z_\alpha} Z_\alpha) + \sum_{\alpha=1}^{n} (a^\alpha Z_\alpha + \overline{a^\alpha}\,\overline{Z_\alpha}),$$

and

$$\text{span}_{\mathbb{C}}\{(Z_\alpha)_x, (\overline{Z_\alpha})_x, [Z_\alpha, \overline{Z_\alpha}]_x; \alpha = 1, \ldots, n\} = \mathbb{C}T_x M, \quad x \in U,$$

where $[\cdot, \cdot]$ denotes the Lie product.

Moreover, we investigate diagonal short time asymptotics of the heat kernel corresponding to the diffusion process. Since the diffusion process is obtained as a strong solution of an SDE, we can utilize the asymptotic expansion theory due to Watanabe (see [12]) to investigate asymptotic behavior of the heat kernel associated with the diffusion process. To be more precise, if the solution U_t^ε, $t \in [0, \infty)$ of the SDE which is obtained by putting a parameter $\varepsilon > 0$ into the SDE defining the diffusion process has a suitable expansion in ε, then composing with the delta function and taking the generalized expectation give an asymptotic expansion of $p(\varepsilon^2, x, x)$ in ε.

When proceeding in this approach, the key task to get over is to get the asymptotic expansion of U_t^ε. On this point, Takanobu [21] gives a quite general result taking advantage of the stochastic Taylor expansion. In this thesis, we will show in the case of CR manifolds that the process U_t^ε can be asymptotically expanded in a way that is different from the stochastic Taylor expansion and is more close to the geometric structure of CR manifolds. Our expansion takes advantage of the fact that, in a local coordinate system introduced by Folland-Stein [6], behavior of CR manifolds can be seen as a perturbation of the Heisenberg group. To obtain the desired expansion, we will refine the result of [6], that is, we will determine all higher order terms of the asymptotic behavior in this coordinate.

The result in this part of the thesis is that, in the asymptotic expansion of the heat kernel associated with $-\Delta_b/2$, the leading term depends only on the dimension of M, and coefficients of higher degree are represented in terms of the Takana-Webster connection on M. The result seems of interest in the point that the coefficients of the asymptotic expansion can be written in a concrete way with the expectation of Wiener functionals. This will be described through some examples.

1.2 Structure of the thesis

We first review in Chapter 2 some definitions and facts on the stochastic analysis on manifolds which shall be used later. This chapter also includes a brief review on the construction of Brownian motions on Riemannian manifolds. We also review in Chapter 3 CR geometry briefly. In the same chapter we also study on the asymptotic behavior of the local orthonormal frame written in the Folland-Stein local coordinate, which will be used in Chapter 6.

In Chapter 4, we shall construct vector fields on the unitary bundle $U(T_{1,0})$ over M associated with the metric connection due to Tanaka and Webster, and using these vector fields we will construct a diffusion process $\mathbb{X}$ generated by $-\Delta_b/2$.

In Chapter 5 we shall see some applications of construction of the diffusion process $\mathbb{X}$, which include the heat kernel, distributions of stochastic line integrals and Dirichlet problems associated with Δ_b.

We investigate in Chapter 6 the diagonal short time asymptotics of the heat kernel associated with $-\Delta_b/2$. We shall describe a relationship between the coefficient of the asymptotic expansion of the heat kernel and the Tanaka-Webster connection. We will finally study examples of the asymptotic expansion, in the cases of the Heisenberg group and the CR sphere.

Chapter 2

Preliminaries

2.1 Diffusion processes on manifolds

We begin with a brief review on stochastic differential equations (SDE in abbreviation) and diffusion processes on manifolds. We basically follow [12].

Let M be a connected C^∞ manifold. We denote by $\widehat{M}$ a one-point compactification of M: If M is compact, we have $\widehat{M} = M$. Otherwise we can write $\widehat{M} = M \amalg \{\infty\}$. We write

$$\begin{aligned}\widehat{W}(M) = \{w\colon [0,\infty) \to \widehat{M}; &\text{continuous, } w(0) \in M, \text{ and} \\ &w(t) = \infty \text{ implies } w(t') = \infty \text{ for } t' \geq t\}\end{aligned}$$

and equip $\widehat{W}(M)$ with the σ-field generated by Borel cylinder sets.

Definition 2.1. Let $A_0, \ldots, A_r$ be C^∞ vector fields on M and $(B^1(t), \ldots, B^r(t))$ be a Brownian motion of dimension r.

We say that a $\widehat{W}(M)$-valued random variable X is a *solution* of the SDE

$$dX(t) = \sum_{i=1}^{r} A_i(X(t)) \circ dB^i(t) + A_0(X(t))dt$$

if for any $f \in C_0^\infty(M)$ and $t \in [0, \infty)$

$$f(X(t)) - f(X(0)) = \sum_{i=1}^{r} \int_0^t (A_i f)(X(s)) \circ dB^i(s) + \int_0^t (A_0 f)(X(s))ds$$

holds, where we set $f(\infty) = 0$ and we denote by the symbol $\circ$ the Stranovich integral.

Let $(W^r, \mathcal{B}(W^r), P^r)$ be the r-dimensional Wiener space, where

$$W^r = \{w = (w^1, \ldots, w^r) \colon [0, \infty) \to \mathbb{R}^r; \text{continuous}, \, w(0) = 0\},$$

$\mathcal{B}(W^r)$ is the σ-field generated by Borel cylinder sets, and P^r is the r-dimensional Wiener measure, i.e. the unique probability measure satisfying

$$\begin{aligned} &P^r(w(t_1) \in E_1, \ldots, w(t_n) \in E_n) \\ &\quad = \int_{E_1} dx_1 \cdots \int_{E_n} dx_n \prod_{j=1}^{n} (2\pi(t_j - t_{j-1}))^{-\frac{r}{2}} \exp\left(-\frac{|x_j - x_{j-1}|^2}{2(t_j - t_{j-1})}\right) \end{aligned}$$

for $n \geq 1$, $0 = t_0 < t_1 < \cdots < t_n$ and Borel sets $E_1, \ldots, E_n \subset \mathbb{R}^r$. Note that $w(t) = (w^1(t), \ldots, w^r(t))$ is a Brownian motion of dimension r.

It is known that for any $x \in M$ there exists a unique solution $X = X_x \colon W^r \to \widehat{W}(M)$ of

$$dX(t) = \sum_{i=1}^{r} A_i(X(t)) \circ dw^i(t) + A_0(X(t))dt$$

such that $X_x(0) = x$. Moreover, the family $\{P_x\}_{x \in M}$ of probability laws on $\widehat{W}(M)$ defined by X_x is a *diffusion* genareted by the differential operation

$$A = \frac{1}{2} \sum_{i=1}^{r} A_i^2 + A_0,$$

i.e. $X(w) = w$ $(w \in \widehat{W}(M))$ satisfies

- $P_x(X(0) = x) = 1$,
- the strong Markov property, and
- $f(X(t)) - f(X(0)) - \int_0^t (Af)(X(s))ds$ is a martingale for any $f \in C_0^\infty(M)$.

$\{(\{X(t)\}_{t\geq 0}, P_x); x \in M\}$ satisfying the conditions above is called a *diffusion process* generated by A.

Note that a Brownian motion of dimension r is a diffusion process generated by

$$\frac{1}{2}\Delta_{\mathbb{R}^r} = \frac{1}{2}\sum_{i=1}^{r}\Big(\frac{\partial}{\partial x^i}\Big)^2,$$

where $(x^i)_{i=1,\ldots,r}$ is the standard coordinate of $\mathbb{R}^r$. From this, a Brownian motion on a Riemannian manifold is defined as follows:

Definition 2.2. Let M be a Riemannian manifold and Δ_M be the Laplace-Beltrami operator on M. A diffusion process generated by Δ_M is called a *Brownian motion* on M.

2.2 Brownian motions on Riemannian manifolds

In this section, following [18], we review the method by Eells, Elworthy and Malliavin which constructs the Brownian motion on a Riemannian manifold. We will modify this method to construct diffusion processes on CR manifolds later.

Let M be a Riemannian manifold of dimension n and ∇^M be the connection of Levi-Civita, or the Riemannian connection. The Riemannian metric on M is denoted by g.

2.2.1 Parallel sections along a curve

Let $p\colon [a, b] \to M$, where $a < b$, be a smooth curve. We say that a smooth curve $W\colon [a, b] \to TM$, where TM is the tangent bundle over M, is a *parallel section* along p if $W(t) \in T_{p(t)}M$ and $\nabla^M_{\dot{p}} W = 0$, where $\dot{p}$ means the differentiation of p in t.

For any $v \in T_{p(a)}M$, there exists a unique parallel section W with $W(a) = v$. Parallel sections can be represented locally as follows:

Let $\{E_i\}_{i=1,\dots,n}$ be a local orthonormal frame on an open set U, that is, E_i is a smooth vector field defined on U and $g(E_i, E_j) = \delta_{ij}$. The *Christoffel symbol* Γ_{ij}^k for $i, j, k \in \{1, \dots, n\}$ by

$$\nabla_{E_i}^M E_j = \sum_{k=1}^n \Gamma_{ij}^k E_k.$$

Note that $\Gamma_{ij}^k = -\Gamma_{ik}^j$, $i, j, k \in \{1, \dots, n\}$.

Let $O(n)$ be the group of $n \times n$ orthogonal matrices. We denote the $n \times n$ identity matrix my I_n. Then we have the following:

Lemma 2.3. *Let $\{E_i\}_{i=1,\dots,n}$ be a local orthonormal frame on an open set U and suppose that $p([a,b]) \subset U$. Then there exists a unique smooth curve $\Lambda_p \colon [a,b] \to O(n)$ such that $\Lambda_p(a) = I_n$ and*

$$\dot{\Lambda}_p(t)_j^k + \sum_{i,l=1}^n g(\dot{p}(t), (E_i)_{p(t)}) \Gamma_{il}^k(p(t)) = 0, \qquad j, k = 1, \dots, n$$

holds, where $\dot{\Lambda}_p(t) = (\dot{\Lambda}_p(t)_j^k)_{k,j=1,\dots,n}$.

Moreover, for given $v \in T_{p(a)}M$,

$$W(t) = \sum_{j,k=1}^n \Lambda_p(t)_j^k g(v, (E_j)_{p(a)})(E_k)_{p(t)}$$

is a parallel section along p and satisfies $W(a) = v$.

2.2.2 Horizontal lift to the orthonormal frame bundle

Define bundles $L(M)$, $O(M)$ over M by

$$L(M) = \coprod_{x \in M} \{r \colon \mathbb{R}^n \to T_x M; r \text{ is a nonsingular linear map}\},$$
$$O(M) = \{r \in L(M); r \text{ is isometric}\}.$$

For $r \in L(M)$ with $r\colon \mathbb{R}^n \to T_xM$, let $\pi(r) = x$. We write $r\xi$ for the image of $\xi \in \mathbb{R}^n$ by $r \in L(M)$.

The Lie group $O(n)$ acts on $O(M)$. For each $\Lambda \in O(n)$ we have the map $R_\Lambda\colon O(M) \to O(M)$ defined by

$$(R_\Lambda r)(\xi) = r\Lambda\xi, \qquad r \in O(M), \quad \xi \in \mathbb{R}^n.$$

Moreover, if $\Lambda\colon [a,b] \to O(n)$ is a smooth curve with $\Lambda(a) = I_n$ and $r \in O(M)$, then $\dot{\Lambda}(a)$ is a skew-symmetric matrix and

$$\left.\frac{d}{dt}\right|_{t=a} R_{\Lambda(t)} r = \lambda(\dot{\Lambda}(a))_r,$$

where λ is given by

$$\lambda(o)_r = \left.\frac{d}{ds}\right|_{s=0} R_{\exp(so)} r, \qquad o \in \mathfrak{o}(n).$$

Here we denote the set of $n \times n$ skew-symmetric matrices by $\mathfrak{o}(n)$.

For smooth curves $p\colon [a,b] \to M$ and $\widehat{p}\colon [a,b] \to O(M)$, we say that $\widehat{p}$ is a *horizontal lift* of p to $O(M)$ if $\pi \circ \widehat{p} = p$ and $\widehat{p}(t)e_i$ is a parallel section for $i = 1, \ldots, n$, where $\{e_i\}_{i=1,\ldots,n}$ is the standard coordinate of $\mathbb{R}^n$. For $v \in T_xM$ and $r \in \pi^{-1}(x)$, there exists a unique $\eta_r(v) \in T_rO(M)$ such that there exist a smooth curve p on M and a smooth curve $\widehat{p}$ on $O(M)$ which is a horizontal lift of p, satisfying

$$\widehat{p}(0) = r, \qquad \dot{\widehat{p}}(0) = \eta, \qquad (\pi_*)_r\eta = v.$$

$\eta_r(v)$ is called the *horizontal lift* of v.

The horizontal lift is described locally as follows: Let $\{E_i\}_{i=1,\ldots,n}$ be a local orthonormal frame for $O(M)$ on U and $E\colon U \to O(M)$ be the associated section, i.e. $E(x)e_i = (E_i)_x$. Define $\Phi\colon T_xM \to \mathfrak{o}(n)$ by

$$\Phi(v) = \left(\sum_{i=1}^{n} g(v, (E_i)_x)\Gamma_{ij}^k(x)\right)_{j,k=1,\ldots,n}.$$

Then we have

$$\eta_r(v) = (R_{E(x)^{-1}\circ r})_*(E_*(v) - \lambda(\Phi(v))_{E(x)}) \in T_rO(M), \qquad v \in T_xM, \quad r \in \pi^{-1}(x).$$

2.2.3 Canonical vector fields

For each $r \in O(M)$, the *horizontal subspace* at r is defined by

$$\mathrm{Hor}_r O(M) = \{\eta_r(v); v \in T_{\pi(r)}M\} \subset T_rO(M).$$

If we set the vertical subspace $\mathrm{Ver}_r O(M)$ by

$$\mathrm{Ver}_r O(M) = \mathrm{Ker}(\pi_* \colon T_rO(M) \to T_{\pi(r)}M),$$

then

$$T_rO(M) = \mathrm{Ver}_r O(M) \oplus \mathrm{Hor}_r O(M)$$

holds.

For each $\xi \in \mathbb{R}^n$, define a vector field $L(\xi)$ by $L(\xi)_r = \eta_r(r\xi)$ for $r \in O(M)$. We set

$$L_i = L(e_i), \qquad i = 1, \ldots, n$$

and call $\{L_i\}_{i=1,\ldots,n}$ the *canonical vector fields.*

Let $\{E_i\}_{i=1,\ldots,n}$ be a local orthonormal frame for TM on U. Define $\{e_i^j(r)\} \in \mathbb{R}^n \otimes \mathbb{R}^n$ for $r \in L(M)$ with $\pi(r) \in U$ by

$$r(e_i) = \sum_{j=1}^n e_i^j(r)(E_j)_{\pi(r)}.$$

We can then introduce a local coordinate system $\{(x^k, e_i^j)\}$ of $L(M)$, where $(x^k)_{k=1,\ldots,n}$ is a local coordinate system of M. With this coordinate we can represent tha canonical vector field L_i, $i = 1, \ldots, n$ as

$$(L_i)_r = \sum_{j=1}^n e_i^j E_j - \sum_{j,k,l,m=1}^n \Gamma_{jl}^k e_m^l e_i^j \frac{\partial}{\partial e_m^k}.$$

2.2.4 Construction of a Brownian motion

Take a Brownian motion $\{B(t) = (B^1(t), \dots, B^n(t))\}_{t\geq 0}$ of dimension n. Let $\{r(t) = r(t, r, B)\}_{t\geq 0}$ be the unique strong solution to the SDE on $O(M)$

$$dr(t) = \sum_{i=1}^{n} L_i(r(t)) \circ dB^i(t), \qquad r(0) = r \in O(M). \tag{2.4}$$

Let $\{E_i\}_{i=1,\dots,n}$ be a local orthonormal frame for TM and (x^k, e_i^j) be an associated local coordinate of $L(M)$ as above. Then (2.4) can be rewritten locally as

$$\begin{cases} dx(t) = \displaystyle\sum_{i,j=1}^{n} (e_i^j(t) E_j(x(t)) \circ dB^i(t), \\ de_m^k(t) = -\displaystyle\sum_{i,j,l=1}^{n} \Gamma_{jl}^k(x(t)) e_m^l(t) e_i^j(t) \circ dB^i(t). \end{cases}$$

Hence it follows from the uniqueness of $\{r(t, r, B)\}_{t\geq 0}$ that

$$r(t, r\Lambda, \Lambda^t B) = r(t, r, B)$$

for every $\Lambda \in O(n)$. We have that the induced measures Q_r of $\pi(r(\cdot, r, B))$ on $\widehat{W}(M)$ coincide for all $r \in \pi^{-1}(x)$.

Put

$$P_x = Q_r \circ \pi^{-1}, \qquad r \in \pi^{-1}(x).$$

Since

$$\Delta_M = \sum_{i=1}^{n} (L_i)^2|_M,$$

it is easily seen that

$$f(X(t)) - \frac{1}{2}\int_0^t \Delta_M f(X(s))ds$$

is a martingale under P_x for every $x \in M$ and $f \in C_0^\infty(M)$, where $X(t)$ denotes the position of $X \in \widehat{W}(M)$ at $t \in [0, \infty)$.

To summarize, we have:

Proposition 2.5. *There exists a Brownian motion on M which is obtained via the SDE* (2.4)*.*

2.3 Malliavin calculus

In this section, basically following [12], we briefly review results from the Malliavin calculus, especially Watanabe's asymptotic theory which we shall use to obtain an asymptotic expansion of a heat kernel in Section 6.1.

2.3.1 Sobolev spaces of Wiener functionals

Let $T \in [0, \infty)$, $d \in \mathbb{N}$ and

$$W_T = W_T^d = \{w = (w^1, \dots, w^d)\colon [0, T] \to \mathbb{R}^d; \text{continuous}, \ w(0) = 0\}$$

be the d-dimensional Wiener space. Let

$$H_T = H_T^d = \{h = (h^1, \dots, h^d) \in W_T; \text{absolutely continuous and } \dot{h} \in L^2(\mathbb{R}^d)\}$$

be the *Cameron-Martin subspace* of W_T. H_T is a real separable Hilbert space with the inner product defined by

$$\langle h_1, h_2 \rangle_{H_T} = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle_{\mathbb{R}^d} dt, \qquad h_1, h_2 \in H_T,$$

where $\langle , \rangle_{\mathbb{R}^d}$ is the standard inner product of $\mathbb{R}^d$. If the dual space of a linear topological space E is denoted by E^*, there exist natural inclusion maps

$$W_T^* \subset H_T^* \simeq H_T \subset W_T,$$

where $H_T^* \simeq H_T$ is the identification via the inner product. Moreover, the inclusions $W_T^* \subset H_T^*$ and $H_T \subset W_T$ are continuous and dense.

Define $\mathcal{I}\colon H_T \to L^2(W_T)$ by

$$\mathcal{I}(h)(w) = \sum_{i=1}^{d} \int_0^T \dot{h}^i(t) dw^i(t), \qquad h \in H_T, \quad w \in W_T.$$

$\mathcal{I}$ is an isometry and $\mathcal{I}(h)$ is called the *Wiener integral* of h.

Let $\mathcal{P}$ be the set of all functions $\phi\colon W_T \to \mathbb{R}$ such that there exist $l_1, \ldots, l_m \in W_T^*$ and a polynomial $p\colon \mathbb{R}^m \to \mathbb{R}$ satisfying

$$\phi(w) = p(l_1(w), \ldots, l_m(w)), \qquad w \in W_T.$$

For a real separable Hilbert space E, let $\mathcal{P}(E)$ be the set of all functions $F\colon W_T \to E$ such that there exist $\phi_1, \ldots, \phi_m \in \mathcal{P}$ and $e_1, \ldots, e_m \in E$ satisfying

$$F(w) = \sum_{j=1}^{m} \phi_j(w) e_j, \qquad w \in W_T.$$

Let $\{h_j\}_{j=1}^\infty$ be an orthonormal basis of H_T. For a sequence of nonnegative integer $\boldsymbol{\alpha} = (\alpha_j)_{j=1}^\infty$ such that $\alpha_j \in \mathbb{Z}_{\geq 0}$ and $\displaystyle\sum_{j=1}^{\infty} \alpha_j < \infty$, we define $H_{\boldsymbol{\alpha}}\colon W_T \to \mathbb{R}$ by

$$H_{\boldsymbol{\alpha}}(w) = \prod_{j=1}^{\infty} H_{\alpha_j}(\mathcal{I}(h_j)(w)), \qquad w \in W_T,$$

where H_m denotes the Hermite polynomial:

$$H_m(x) = \frac{(-1)^m}{m!} \exp\left(\frac{1}{2}x^2\right) \frac{d^m}{dx^m}\left(\exp\left(-\frac{1}{2}x^2\right)\right), \qquad x \in \mathbb{R}.$$

Define $\mathcal{H}_n \subset L^2(W_T)$, $n \in \mathbb{Z}_{\geq 0}$ as the closed subspace generated by $\{H_{\boldsymbol{\alpha}}; |\boldsymbol{\alpha}| = n\}$, where $|\boldsymbol{\alpha}| = \displaystyle\sum_{j=1}^{\infty} \alpha_j$ for $\boldsymbol{\alpha} = \{\alpha_j\}_{j=1}^\infty$.

Then $\mathcal{H}_n$ is independet of $\{h_j\}_{j=1}^\infty$ and the following orthogonal decomposition holds:

$$L^2(W_T) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

This is called the *Wiener chaos decomposition*.

Let $J_n\colon L^2(W_T) \to \mathcal{H}_n$ be the orthogonal projection. If $\phi \in \mathcal{P}$, then $J_n \in \mathcal{P}$ for any n and $\phi = \sum_{n=0}^{\infty} J_n\phi$ holds, where the right hand side is a finite sum. We can define $J_n\colon \mathcal{P}(E) \to \mathcal{P}(E)$ by

$$(J_nF)(w) = \sum_{j=1}^{m}(J_n\phi_j)(w)e_j, \qquad w \in W_T,$$

where $F(w) = \sum_{j=1}^{m}\phi_j(w)e_j$, $\phi_j \in \mathcal{P}$, $w \in W_T$.

For $p \in (1,\infty)$, $s \in \mathbb{R}$, a norm $\|\cdot\|_{p,s}$ on $\mathcal{P}(E)$ is defined by

$$\|F\|_{p,s} = \left\|\sum_{n=0}^{\infty}(1+n)^{\frac{s}{2}}J_nF\right\|_p, \qquad F \in \mathcal{P}(E),$$

where $\|\cdot\|_p$ denotes the L^p norm. The completion of $\mathcal{P}(E)$ by $\|\cdot\|_{p,s}$ is denoted by $\mathbb{D}^{p,s}(E)$.

We list some fundamental properties of $\mathbb{D}^{p,s}(E)$:

Proposition 2.6. *Let $p, p', q \in (1,\infty)$ and $s, s' \in \mathbb{R}$.*

(i) $\mathbb{D}^{p,0}(E) = L^p(E)$.

(ii) For $s \geq s'$, $p \geq p'$, there exists a natural continuous injection $\mathbb{D}^{p',s'}(E) \subset \mathbb{D}^{p,s}(E)$.

(iii) If $\dfrac{1}{p} + \dfrac{1}{q} = 1$, there exists a natural isomorphism $(\mathbb{D}^{p,s}(E))^* \simeq \mathbb{D}^{q,-s}(E)$.

Define

$$\mathbb{D}^{\infty}(E) = \bigcap_{p\in(1,\infty),s\in\mathbb{R}} \mathbb{D}^{p,s}(E), \qquad \mathbb{D}^{-\infty}(E) = \bigcup_{p\in(1,\infty),s\in\mathbb{R}} \mathbb{D}^{p,s}(E),$$

We write $\mathbb{D}^{\infty} = \mathbb{D}^{\infty}(\mathbb{R})$, $\mathbb{D}^{-\infty} = \mathbb{D}^{-\infty}(\mathbb{R})$. $\mathbb{D}^{\infty}(E)$ is a Fréchet space and we have a natural isomorphism $\mathbb{D}^{-\infty}(E) \simeq (\mathbb{D}^{\infty}(E))^*$.

Definition 2.7. (i) An element of $\mathbb{D}^{-\infty}(E)$ is called an E-valued *generalized Wiener functional*.

(ii) For $\Phi \in \mathbb{D}^{-\infty}(E)$, $F \in \mathbb{D}^{\infty}(E)$, denote by $\mathbb{E}[\langle \Phi, F\rangle_E]$ the image of (Φ, F) by the composition $\mathbb{D}^{-\infty}(E) \times \mathbb{D}^{\infty}(E) \simeq (\mathbb{D}^{\infty}(E))^* \times \mathbb{D}^{\infty}(E) \to \mathbb{R}$, where the right map is the evaluation map. In particular, for $\Phi \in \mathbb{D}^{-\infty}$, we write $\mathbb{E}[\Phi]$ for $\mathbb{E}[\langle \Phi, 1\rangle_{\mathbb{R}}]$, where $1 \in \mathbb{D}^{\infty}$ is the constant map. $\mathbb{E}[\Phi]$ is called the *generalized expectation* of Φ.

We note that if $\Phi \in \mathbb{D}^{p,0}(E) = L^p(W_T; E)$ for some $p \in (1, \infty)$, $\mathbb{E}[\langle \Phi, F\rangle_E]$ coincides with the ordinary expectation with respect to the Wiener probability measure.

2.3.2 Derivative operator

For real separable Hilbert spaces E_1 and E_2, we write $E_1 \otimes E_2$ for the Hilbert space consisting of all Hilbert-Schmidt operators $E_1 \to E_2$.

Let $\nabla \colon \mathcal{P}(E) \to \mathcal{P}(H_T \otimes E)$ be the unique linear map such that

$$\nabla F(w) = \sum_{j=1}^{m} (\partial_j p)(\mathcal{I}(h_1)(w), \ldots, \mathcal{I}(h_m)(w)) h_j \otimes e, \qquad w \in W_T,$$

if $F(w) = p(\mathcal{I}(h_1)(w), \ldots, \mathcal{I}(h_m)(w))e$ with a polynomial p, orthogonal elements $h_1, \ldots, h_m \in H_T$ and $e \in E$. Here ∂_j denotes the partial differentiation with respect to the j-th variable.

∇ is uniquely extended to an operator

$$\nabla \colon \mathbb{D}^{-\infty}(E) \to \mathbb{D}^{-\infty}(H_T \otimes E)$$

which by restriction yields a continuous map $\mathbb{D}^{p,s+1}(E) \to \mathbb{D}^{p,s}(H_T \otimes E)$ for $p \in (1, \infty)$, $s \in \mathbb{R}$.

We can define the adjoint operator

$$\nabla^* \colon \mathbb{D}^{-\infty}(H_T \otimes E) \to \mathbb{D}^{-\infty}(E),$$

which by restriction yields a continuous map $\mathbb{D}^{p,s+1}(H_T \otimes E) \to \mathbb{D}^{p,s}(E)$ for $p \in (1,\infty)$, $s \in \mathbb{R}$.

Definition 2.8. Suppose $n \in \mathbb{N}$ and $F = (F^i)_{i=1,\dots,n} \in \mathbb{D}^\infty(\mathbb{R}^n)$.

(i) $\sigma_F = (\sigma_F^{ij})_{i,j=1,\dots,n}(\mathbb{R}^n \otimes \mathbb{R}^n)$ defined by

$$\sigma_F^{ij}(w) = \langle \nabla F^i(w), \nabla F^j(w) \rangle_{H_T}, \qquad i = 1, \dots, n$$

is called the *Malliavin covariance* of F.

(ii) F is *non-degenerate (in the sense of Malliavin)* if

$$(\det \sigma_F(w))^{-1} \in \bigcap_{p \in (1,\infty)} L^p(W_T).$$

2.3.3 Pull-back of Schwartz distributions

For a non-degenerate Wiener functional, the pull-back of Schwartz distribution can be defined. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing C^∞ functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of slowly increasing distributions on $\mathbb{R}^n$. For $F = (F^i)_{i=1,\dots,n} \in \mathbb{D}^\infty(\mathbb{R}^n)$, the pull-back

$$\mathcal{S}(\mathbb{R}^n) \to \mathbb{D}^\infty; \phi \mapsto \phi \circ F$$

is extended to a continuous linear map

$$\mathcal{S}'(\mathbb{R}^n) \to \mathbb{D}^{-\infty}.$$

The image of $T \in \mathcal{S}'(\mathbb{R}^n)$ by this map is denoted by $T(F)$.

The probability density function can be represented using the pull-back of the Dirac distribution. We write $\delta_x \in \mathcal{S}'(\mathbb{R}^n)$ for the Dirac distribution concentrated at $x \in \mathbb{R}^n$.

Proposition 2.9. *Suppose $F \in \mathbb{D}^\infty(\mathbb{R}^n)$ is non-degenerate.*

(i) $p_F \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$p_F(x) = \mathbb{E}[\delta_x(F)]$$

is the C^∞ density function with respect to the Lebesgue measure of F.

(ii) If $G \in \mathbb{D}^\infty$,

$$\mathbb{E}[\delta_x(F)G] = p_F(x)\mathbb{E}[G|F = x]$$

holds for almost every $x \in \mathbb{R}^n$ satisfying $p_F(x) > 0$.

The follwing chain rule for ∇ holds:

Proposition 2.10. *For $T \in \mathcal{S}'(\mathbb{R}^n)$ and $F = (F^1, \ldots, F^n) \in \mathbb{D}^\infty(\mathbb{R}^n)$, it holds that*

$$\nabla T(F) = \sum_{i=1}^{n} (\partial_i T)(F)\nabla F_i.$$

2.3.4 Asymptotic expansion

The following result about the asymptotic expansion of Wiener functionals will be used to investigate asymptotic behavior of heat kernels.

Definition 2.11. Let $\{F^\varepsilon\}_{\varepsilon\in(0,1]}$ be a family of $\mathbb{R}^n$-valued generalized Wiener functionals.

(i) We say that the family $\{F^\varepsilon\}_{\varepsilon\in(0,1]}$ is *uniformly non-degenerate* if $F^\varepsilon \in \mathbb{D}^\infty(\mathbb{R}^n)$ for $\varepsilon \in (0,1]$ and

$$\sup_{\varepsilon\in(0,1]} \mathbb{E}[(\det\sigma^\varepsilon)^{-p}] < \infty$$

holds for any $p \in (1,\infty)$, where $\sigma^\varepsilon = \sigma_{F^\varepsilon}$ is the Malliavin covariance of F^ε.

(ii) Let $\phi_a \in \mathbb{D}^\infty(\mathbb{R}^n)$ (*resp.* $\mathbb{D}^{-\infty}(\mathbb{R}^n)$) for $a \in \mathbb{Z}_{\geq 0}$ and $F^\varepsilon \in \mathbb{D}^\infty(\mathbb{R}^n)$ (*resp.* $\mathbb{D}^{-\infty}(\mathbb{R}^n)$). We write

$$F^\varepsilon \sim \sum_{a=0}^\infty \varepsilon^a\phi_a \quad \text{in } \mathbb{D}^\infty(\mathbb{R}^n) \quad (\textit{resp.} \text{ in } \mathbb{D}^{-\infty}(\mathbb{R}^n))$$

if, for any $a \in \mathbb{Z}_{\geq 0}$, $p \in (1,\infty)$, $s \in \mathbb{R}$ (*resp.* for any $a \in \mathbb{Z}_{\geq 0}$ and for some $p \in (1,\infty)$, $s \in \mathbb{R}$),

$$\sup_{\varepsilon\in(0,1]} \varepsilon^{-a-1}\left\| F^\varepsilon - \sum_{b=0}^a \varepsilon^b\phi_b \right\|_{p,s} < \infty$$

holds.

Proposition 2.12 ([12, Theorem V.9.4])**.** *Let $\{F^\varepsilon\}_{\varepsilon\in(0,1]}$, $F^\varepsilon \in \mathbb{D}^\infty(\mathbb{R}^n)$ be a uniformly non-degenerate family of Wiener functionals and suppose that*

$$F^\varepsilon \sim \sum_{a=0}^\infty \varepsilon^a\phi_a \quad \textit{in } \mathbb{D}^\infty(\mathbb{R}^n)$$

holds for $\phi_a = (\phi_a^1,\ldots,\phi_a^n) \in \mathbb{D}^\infty(\mathbb{R}^n)$, $a \in \mathbb{Z}_{\geq 0}$.

Then for $T \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$T(F^\varepsilon) \sim \sum_{a=0}^\infty \varepsilon^a\psi_a \quad \textit{in } \mathbb{D}^{-\infty}(\mathbb{R}^n),$$

where each ψ_a is obtained by the formal expansion in ε

$$T\left(\sum_{a=0}^\infty \varepsilon^a\phi_a\right) = \sum_{a=0}^\infty \frac{1}{a!} \sum_{i_1,\ldots,i_a=1}^n (\partial_{i_1}\cdots\partial_{i_a}T)(\phi_0)(\varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots)$$

$$= \psi_0 + \varepsilon\psi_1 + \cdots .$$

In particular, we have

$$\mathbb{E}[T(F^\varepsilon)] = \sum_{a=0}^{\infty} \varepsilon^a \mathbb{E}[\psi_a]$$

for sufficiently small ε*.*

Remark 2.13. For example, we have

$$\begin{aligned}
\psi_0 &= T(\phi_0), \\
\psi_1 &= \sum_{i=1}^{n} \phi_1^i (\partial_i T)(\phi_0), \\
\psi_2 &= \sum_{i=1}^{n} \phi_2^i (\partial_i T)(\phi_0) + \frac{1}{2} \sum_{i,j=1}^{n} \phi_1^i \phi_1^j (\partial_i \partial_j T)(\phi_0).
\end{aligned}$$

Chapter 3

CR geometry

3.1 CR manifolds

In this section, we list the results on CR manifolds which we shall use later, following Dragomir-Tomassini [3] and Lee [15].

A CR manifold M is a real differentiable manifold together with a complex subbundle $T_{1,0}$ of the complexified tangent bundle $\mathbb{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$T_{1,0} \cap T_{0,1} = \{0\}, \quad \text{where } T_{0,1} = \overline{T_{1,0}},$$
$$[T_{1,0}, T_{1,0}] \subset T_{1,0}.$$

We consider the case that M is orientable and of real dimension $2n+1$ with $n \in \mathbb{N}$ and $T_{1,0}$ is of complex dimension n, i.e. the CR codimension is 1.

Set

$$H = \mathrm{Re}(T_{1,0} \oplus T_{0,1}),$$

which is called the *Levi distribution* of $(M, T_{1,0})$. There exists a *pseudo-Hermitian structure*, that is, a real non-vanishing 1-form θ on M which annihilates H. For such θ, the *Levi form* L_θ of θ is defined by

$$L_\theta(Z, W) = -\mathrm{i}d\theta(Z, W), \qquad Z, W \in \Gamma^\infty(T_{1,0} \oplus T_{0,1}),$$

where $\Gamma^\infty(V)$ stands for the space of C^∞ cross sections of a vector bundle V. Hereinafter, we assume that M is *strictly pseudoconvex*, that is, the Levi form L_θ is positive definite. Then $T_{1,0}$ is an Hermitian fiber bundle with Hermitian fiber metric L_θ.

Let T be the *characteristic direction*, that is, the unique real vector field on M transverse to H, defined by

$$T \rfloor d\theta = 0, \quad T \rfloor \theta = 1,$$

where $T \rfloor \omega$ is the interior product: $T \rfloor \omega(X_1, \ldots, X_{p-1}) = \omega(T, X_1, \ldots, X_{p-1})$ for a p-form ω.

As M is strictly pseudoconvex, the $(2n+1)$-form $\psi = \theta \wedge (d\theta)^n$ on M determines a volume form, where we have chosen the orientation of M so that ψ is a positive form. Then it induces the L^2-inner product on functions:

$$\langle u, v \rangle_\theta = \int_M u\overline{v}\psi, \qquad u, v \in C_0^\infty(M; \mathbb{C}).$$

The Levi form induces a metric on H (denoted by L_θ again), and the dual metric L_θ^* on H^*. Then the L^2-inner product on sections of H^* is given by

$$\langle \omega, \eta \rangle_\theta = \int_M L_\theta^*(\omega, \eta)\psi, \qquad \omega, \eta \in \Gamma^\infty(H^*).$$

Denoting by $r \colon T^*M \to H^*$ the natural restriction mapping, we define a section $d_b u$ for $u \in C^\infty(M)$ by $d_b u = r \circ du$. The *real sub-Laplacian* Δ_b on functions is given by

$$\langle \Delta_b u, v \rangle_\theta = \langle d_b u, d_b v \rangle_\theta, \qquad v \in C_0^\infty(M).$$

Similarly, denoting by $\overline{\partial}_b u$ the projection of du onto $T_{0,1}^*$ for $u \in C^\infty(M; \mathbb{C})$, we introduce the *Kohn-Spencer Laplacian* $\Box_b$ defined by

$$\langle \Box_b u, v \rangle_\theta = \langle \overline{\partial}_b u, \overline{\partial}_b v \rangle_\theta, \qquad v \in C_0^\infty(M; \mathbb{C}).$$

These two operators are related to each other by

$$\Box_b = \Delta_b + \mathrm{i}nT \qquad \text{on } C^\infty(M;\mathbb{C}).$$

3.2 Tanaka-Webster connection

We now review the connection due to Tanaka [22] and Webster [25].

Let $J\colon H \to H$ be the complex structure related to $(M, T_{1,0})$; that is, the $\mathbb{C}$-linear extension of J is the multiplication by i on $T_{1,0}$ and $-\mathrm{i}$ on $T_{0,1}$, where we have used the fact that $H \otimes_{\mathbb{R}} \mathbb{C} = T_{1,0} \oplus T_{0,1}$. Moreover, we extend J linearly to TM by $J(T) = 0$.

Since $TM = H \oplus \mathbb{R}T = \{X + aT \mid X \in H, a \in \mathbb{R}\}$, there exists the unique Riemannian metric g_θ on M satisfying that

$$g_\theta(X,Y) = d\theta(X, JY), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1$$

for $X, Y \in H$. g_θ is called the *Webster metric*. We extend g_θ to $\mathbb{C}TM$ $\mathbb{C}$-bilinearly.

Definition 3.1. There exists the unique linear connection ∇ on M satisfying that

$$\nabla_X Y \in \Gamma^\infty(H), \qquad X \in \Gamma^\infty(TM), Y \in \Gamma^\infty(H), \tag{3.2}$$

$$\nabla J = 0, \qquad \nabla g_\theta = 0, \tag{3.3}$$

$$T_\nabla(Z,W) = 0, \qquad Z, W \in \Gamma^\infty(T_{1,0}), \tag{3.4}$$

$$T_\nabla(Z,W) = 2\mathrm{i}L_\theta(Z,W)T, \qquad Z \in \Gamma^\infty(T_{1,0}), W \in \Gamma^\infty(T_{0,1}), \tag{3.5}$$

$$T_\nabla(T, J(X)) + J(T_\nabla(T,X)) = 0, \qquad X \in \Gamma^\infty(TM),$$

where ∇_X is the covariant derivative in the direction of X and T_∇ is the torsion tensor field of ∇: $T_\nabla(Z,W) = \nabla_Z W - \nabla_W Z - [Z,W]$.

∇ is called the *Tanaka-Webster connection*.

Let $\{Z_\alpha\}_{\alpha=1,\dots,n}$ be a local orthonormal frame for $T_{1,0}$ on an open set U, that is, Z_α is a $T_{1,0}$-valued section defined on U and $g_\theta(Z_\alpha, Z_{\overline{\beta}}) = \delta_{\alpha\beta}$, where $Z_{\overline{\beta}} = \overline{Z_\beta}$. If we set $\langle\!\langle n\rangle\!\rangle = \{0, 1, \dots, n, \overline{1}, \dots, \overline{n}\}$ and $Z_0 = T$, then $\{Z_A\}_{A\in\langle\!\langle n\rangle\!\rangle}$ is a local frame for $\mathbb{C}TM$. We define *Christoffel symbols* Γ^C_{AB} for $A, B, C \in \langle\!\langle n\rangle\!\rangle$ by

$$\nabla_{Z_A} Z_B = \sum_{C\in\langle\!\langle n\rangle\!\rangle} \Gamma^C_{AB} Z_C.$$

Note that $\Gamma^C_{AB} = 0$ unless $(B, C) \in \{(\beta, \gamma), (\overline{\beta}, \overline{\gamma}); \beta, \gamma = 1, \dots, n\}$, because $\nabla_X(\Gamma^\infty(T_{1,0})) \subset \Gamma^\infty(T_{1,0})$ and $\nabla T = 0$ by the conditions (3.2) and (3.3). We also have that

$$\Gamma^\gamma_{A\beta} + \Gamma^{\overline{\beta}}_{A\overline{\gamma}} = 0, \qquad \beta, \gamma = 1, \dots, n, \quad A \in \langle\!\langle n\rangle\!\rangle \tag{3.6}$$

by the condition (3.3).

3.3 Folland-Stein normal coordinate

In this section, we study the asymptotic behavior of the local orthonormal frame written in a local coordinate, introduced by Folland-Stein [6].

3.3.1 Definition

Let $\{Z_\alpha\}_{\alpha=1,\dots,n}$ be a local orthonormal frame around a fixed point $x \in M$. We set

$$\widehat{X}_\alpha = \sqrt{2}\mathrm{Re} Z_\alpha = \frac{1}{\sqrt{2}}(Z_\alpha + Z_{\overline{\alpha}}),$$
$$\widehat{X}_{n+\alpha} = \sqrt{2}\mathrm{Im} Z_\alpha = -\frac{\mathrm{i}}{\sqrt{2}}(Z_\alpha - Z_{\overline{\alpha}})$$

for $\alpha = 1, \ldots, n$ and $\widehat{X}_{2n+1} = T$. Given $u = (u^1, \ldots, u^{2n+1}) \in \mathbb{R}^{2n+1}$, let $C_u \colon [0,1] \to M$ be the integral curve of the tangent vector field

$$\widehat{X}_u = \sum_{j=1}^{2n+1} u^j \widehat{X}_j$$

starting at x, i.e. C_u is defined by

$$\frac{dC_u}{dt}(t) = \widehat{X}_u(C_u(t)), \qquad C_u(0) = x.$$

We set $E_x(u) = C_u(1)$. By a standard theory of ordinary differential equations, E_x defines a diffeomorphism of a neighborhood U_x of $0 \in \mathbb{R}^{2n+1}$ onto a neighborhood V_x of $x \in M$. The inverse map $E_x^{-1} \colon V_x \to U_x$ defines a local chart of M and the resulting local coordinates $u^j \colon V_x \to \mathbb{R}$, $j = 1, \ldots, 2n+1$, are referred to as the *Folland-Stein normal coordinates*. We hereinafter regard a function on V_x as a function of $u \in U_x$ via E_x.

3.3.2 Asymptotic behavior

To describe an asymptotic behavior of vector fields with respect to these coordinates, the following asymptotic notation is introduced:

Definition 3.7. For $a \geq 1$, a function f on V_x is said to be O^a if

$$f(u) = O\left(\left(\sum_{j=1}^{2n} |u^j| + |u^{2n+1}|^2\right)^a\right)$$

as $u = (u^1, \ldots, u^{2n+1}) \to 0$. We may simply write $f = O^a$.

Define the structure functions $C^i_{jk} \colon U_x \to \mathbb{R}$, $i, j, k = 1, \ldots, 2n+1$ by

$$[\widehat{X}_j, \widehat{X}_k] = \sum_{i=1}^{2n+1} C^i_{jk} \widehat{X}_i.$$

Definition 3.8. For a multiple index $J = (j_1, \dots, j_a)$, $j_1, \dots, j_a \in \{1, \dots, 2n+1\}$, define

$$|J| = a,$$
$$\|J\| = a + \#\{b; j_b = 2n+1\},$$
$$u^J = u^{j_1} \cdots u^{j_a}.$$

Theorem 3.9. *$\widehat{X}_j$, $j = 1, \dots, 2n$ can be written with respect to the Folland-Stein normal coordinates $(u^j)_{j=1,\dots,2n+1}$ as follows:*

$$\begin{aligned}\widehat{X}_\alpha = \frac{\partial}{\partial u^\alpha} &- u^{n+\alpha}\frac{\partial}{\partial u^{2n+1}} \\ &+ \sum_{i=1}^{2n}\Bigg(\sum_{1\le\|J\|\le a} c_{\alpha,i,J}u^J + O^{a+1}\Bigg)\frac{\partial}{\partial u^i} \\ &+ \Bigg(\sum_{2\le\|J\|\le a} c_{\alpha,2n+1,J}u^J + O^{a+1}\Bigg)\frac{\partial}{\partial u^{2n+1}},\end{aligned}$$

$$\begin{aligned}\widehat{X}_{n+\alpha} = \frac{\partial}{\partial u^{n+\alpha}} &+ u^{\alpha}\frac{\partial}{\partial u^{2n+1}} \\ &+ \sum_{i=1}^{2n}\Bigg(\sum_{1\le\|J\|\le a} c_{n+\alpha,i,J}u^J + O^{a+1}\Bigg)\frac{\partial}{\partial u^i} \\ &+ \Bigg(\sum_{2\le\|J\|\le a} c_{n+\alpha,2n+1,J}u^J + O^{a+1}\Bigg)\frac{\partial}{\partial u^{2n+1}}\end{aligned}$$

for $\alpha = 1, \dots, n$. Moreover, each $c_{j,i,J}$, $i = 1, \dots, 2n+1$, $j = 1, \dots, 2n$, $\|J\| = a$, is an at most a-th degree polynomial of $\left.\frac{d^b}{ds^b}C^\alpha_{\beta\gamma}(su)\right|_{s=0}$ for $b \le a$.

To prove the theorem, write

$$\widehat{X}_k = \sum_{j=1}^{2n+1} G^j_k \frac{\partial}{\partial u^j}, \qquad k = 1, \dots, 2n$$

using the Folland-Stein normal coordinates, where G^j_k is a smooth function on U_x. Note that $G^j_k(0) = \delta_{jk}$ by definition of E_x. Since $G(u) = (G^j_k(u))_{j,k=1,\dots,2n+1}$ is an invertible matrix for every $u \in U_x$, we can define $F(u) = (F^j_k(u))_{j,k=1,\dots,2n+1}$, $F^j_k \colon U_x \to \mathbb{R}$ by $F = G^{-1}$.

Let us consider the matrices

$$\mathcal{A} = (\mathcal{A}^j_k)_{j,k=1,\dots,2n+1}, \quad \Xi = (\Xi^j_k)_{j,k=1,\dots,2n+1}, \qquad \mathcal{A}^j_k, \Xi^j_k \colon [-1,1] \times U_x \to \mathbb{R}$$

defined by

$$\mathcal{A}^j_k(s,u) = sF^j_k(su), \qquad \Xi^j_k(s,u) = \sum_{l=1}^{2n+1} C^j_{lk}(su)u^l.$$

We use the following lemma [3, Lemma 3.3]:

Lemma 3.10. *We have*

$$\frac{\partial}{\partial s}\mathcal{A} = I_{2n+1} - \Xi\mathcal{A}.$$

By Taylor's theorem, we can write for $A = 0, 1, \dots,$

$$G^j_k(su) = \sum_{a=0}^{A} \frac{s^a}{a!} \sum_{|J|=a} \frac{\partial^a G^j_k}{\partial u^J}(0)u^J + \frac{s^{A+1}}{(A+1)!} \sum_{|J|=A+1} \frac{\partial^{A+1} G^j_k}{\partial u^J}(csu)u^J$$

for some $c \in (0,1)$. Since

$$\frac{\partial^{a+1} G^j_k}{\partial u^J}(csu)u^J = O^{a+1}$$

for each J with $|J| = a+1$, we have

$$G^j_k(u) = \sum_{a=0}^{A} G^{(a)}(u)^j_k + O^{A+1},$$

where

$$G^{(a)}(u)^j_k = \frac{1}{a!} \sum_{|J|=a} \frac{\partial^a G^j_k}{\partial u^J}(0)u^J.$$

To compute $G^{(a)}(u) = (G^{(a)}(u)^j_k)_{j,k=1,\dots,2n+1}$, we note that the Taylor expansion yields

$$G(su) \sim \sum_{a=0}^{\infty} s^a G^{(a)}(u), \qquad F(su) \sim \sum_{a=0}^{\infty} s^a F^{(a)}(u)$$

with $G^{(0)}(u) = F^{(0)}(u) = I_{2n+1}$ and

$$\sum_{b=0}^{a} G^{(b)}(u)F^{(a-b)}(u) = 0, \qquad a \geq 1, \tag{3.11}$$

because $GF = I_{2n+1}$. Hereinafter "$\sim \sum_{a=0}^{\infty}$" denotes an asymptotic expansion of any order, and does not mean an infinite sum. We have by definition of $\mathcal{A}$ an expansion

$$\mathcal{A}(s,u) \sim \sum_{a=0}^{\infty} s^{a+1}F^{(a)}(u).$$

By Lemma 3.10, we have

$$(a+1)F^{(a)}(u) = -\sum_{b=0}^{a-1} \Xi^{(b)}(u)F^{(a-b-1)}(u), \quad a \geq 1. \tag{3.12}$$

We have from (3.11) and (3.12) that

$$G^{(a)}(u) = \sum_{b_1+\cdots+b_i=a} c_{b_1,\ldots,b_i}\Xi^{(b_1-1)}(u)\cdots\Xi^{(b_i-1)}(u)$$

for some constants $c_{b_1,\ldots,b_i}$. Therefore we have the following expression:

Proposition 3.13. *Each component $G^{(a)}(u)^j_k$ can be written as a linear combination of terms*

$$\prod_{q=1}^{r} \frac{d^{b_q}}{ds^{b_q}} C^{\alpha_q}_{\beta_q\gamma_q}(su)\Big|_{s=0} u^{m_1}\cdots u^{m_i}$$

with $b_1 + \cdots + b_r + m_1 + \cdots + m_i = a$. For small a we have

$$G^{(1)}(u)^j_k = \frac{1}{2}\sum_{l=1}^{2n+1} C^j_{lk}(0)u^l,$$

$$G^{(2)}(u)^j_k = \frac{1}{12}\sum_{l,m,p=1}^{2n+1} C^j_{lp}(0)C^p_{mk}(0)u^l u^m + \frac{1}{3}\sum_{l=1}^{2n+1} \frac{d}{ds}C^j_{lk}(su)\Big|_{s=0} u^l,$$

$$G^{(3)}(u)^j_k = \frac{1}{8}\sum_{l,m,p=1}^{2n+1} C^j_{lp}(0)\frac{d}{ds}C^p_{mk}(su)\Big|_{s=0} u^l u^m + \frac{1}{8}\sum_{l=1}^{2n+1} \frac{d^2}{ds^2}C^j_{lk}(su)\Big|_{s=0} u^l.$$

To establish Theorem 3.9, it remains to show that

$$G^{(1)}(u)^{\alpha}_{2n+1} = -u^{n+\alpha} + O^2, \qquad G^{(1)}(u)^{n+\alpha}_{2n+1} = u^{\alpha} + O^2$$

for $\alpha = 1, \ldots, n$. To this end, it is enough to show the following:

Proposition 3.14 ([3])**.** *For $\alpha, \beta = 1, \ldots, n$ we have*

$$C^{2n+1}_{\alpha\beta} = C^{2n+1}_{n+\alpha,n+\beta} = 0, \qquad C^{2n+1}_{\alpha,n+\beta} = -C^{2n+1}_{n+\alpha,\beta} = 2\delta_{\alpha\beta}$$

on U_x.

Proof. By (3.4), we have

$$0 = T_{\nabla}(Z_\alpha, Z_\beta) = \nabla_{Z_\alpha} Z_\beta - \nabla_{Z_\beta} Z_\alpha - [Z_\alpha, Z_\beta].$$

This is rewritten using Christoffel symbols as

$$[Z_\alpha, Z_\beta] = \sum_{\gamma=1}^{n} (\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}) Z_\gamma.$$

Taking conjugate of this equality yields

$$[Z_{\overline{\alpha}}, Z_{\overline{\beta}}] = \sum_{\gamma=1}^{n} (\Gamma^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} - \Gamma^{\overline{\gamma}}_{\overline{\beta}\overline{\alpha}}) Z_{\overline{\gamma}}.$$

Next we have from (3.5) that

$$2\mathrm{i} L_\theta(Z_\alpha, Z_{\overline{\beta}}) T = T_{\nabla}(Z_\alpha, Z_{\overline{\beta}}) = \nabla_{Z_\alpha} Z_{\overline{\beta}} - \nabla_{Z_{\overline{\beta}}} Z_\alpha - [Z_\alpha, Z_{\overline{\beta}}].$$

Together with $L_\theta(Z_\alpha, Z_{\overline{\beta}}) = \delta_{\alpha\beta}$, we have

$$[Z_\alpha, Z_{\overline{\beta}}] = -\sum_{\gamma=1}^{n} \Gamma^{\gamma}_{\overline{\beta}\alpha} Z_\gamma + \sum_{\gamma=1}^{n} \Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}} Z_{\overline{\gamma}} - 2\mathrm{i}\delta_{\alpha\beta} T, \tag{3.15}$$

$$[Z_{\overline{\alpha}}, Z_\beta] = \sum_{\gamma=1}^{n} \Gamma^{\gamma}_{\overline{\alpha}\beta} Z_\gamma - \sum_{\gamma=1}^{n} \Gamma^{\overline{\gamma}}_{\beta\overline{\alpha}} Z_{\overline{\gamma}} + 2\mathrm{i}\delta_{\alpha\beta} T.$$

The proposition follows by rewriting these equalities using definition of $\widehat{X}_j$. □

Theorem 3.9 has been proved from Propositions 3.13 and 3.14.

Remark 3.16. We note that an idea using exponential maps associated with vector fields is also seen in Ben Arous [2].

One of difference between [2] and the present paper is the way in which exponential maps are used. In [2], exponential maps are used to transform the solution of an SDE associated with vector fields. On the other hand, we have represented the vector fields in the Folland-Stein coordinate which is obtained by exponential maps, then in the next section we will consider an SDE in this coordinate and apply a perturbation argument taking advantage of Theorem 3.9.

Chapter 4

Construction of diffusion processes on CR manifolds

4.1 Canonical vector fields

To construct diffusion processes on CR manifolds in the next section, we introduce suitable vector bundles and principal bundles on them. The method employed there is a modification of the method of Eells, Elworthy and Malliavin, which we have reviewed in Section 2.2.

4.1.1 Parallel sections along a curve

Let $p\colon [a,b] \to M$, where $a < b$, be a smooth curve. We say that a smooth curve $W\colon [a,b] \to T_{1,0}$ is a parallel section along p if $W(t) \in (T_{1,0})_{p(t)}$ and $\nabla_{\dot{p}} W = 0$.

For any $v \in (T_{1,0})_{p(a)}$, there exists a unique parallel section W with $W(a) = v$. This can be seen by the localization argument as follows: Let $\{Z_\alpha\}_{\alpha=1,\dots,n}$ be a local orthonormal frame for $T_{1,0}$ on U and suppose that $p([a,b]) \subset U$. Then for a smooth curve $W\colon [a,b] \to T_{1,0}$ satisfying $W(t) \in (T_{1,0})_{p(t)}$, it holds that

$$W(t) = \sum_{\alpha=1}^{n} c^\alpha(t)(Z_\alpha)_{p(t)},$$

where $c^\alpha(t) = g_\theta(W(t), (Z_{\overline{\alpha}})_{p(t)})$ for $\alpha = 1, \dots, n$. By the very definition of the covariant derivative,

$$\begin{aligned}\nabla_{\dot p}W(t) &= \sum_{\alpha=1}^{n}(\dot c^{\alpha}(t)(Z_\alpha)_{p(t)} + c^{\alpha}(t)\nabla_{\dot p}Z_\alpha(t))\\ &= \sum_{\alpha=1}^{n}\dot c^{\alpha}(t)(Z_\alpha)_{p(t)} + \sum_{A\in\langle\langle n\rangle\rangle}\sum_{\alpha,\beta=1}^{n}c^{\alpha}(t)g_\theta(\dot p(t),(Z_{\overline{A}})_{p(t)})\Gamma^{\beta}_{A\alpha}(p(t))(Z_\beta)_{p(t)},\end{aligned}$$

where we have used the convention that $\overline{0}=0$. Therefore $\nabla_{\dot p}W=0$ if and only if

$$\dot c^{\beta}(t) + \sum_{A\in\langle\langle n\rangle\rangle}\sum_{\alpha=1}^{n}c^{\alpha}(t)g_\theta(\dot p(t),(Z_{\overline{A}})_{p(t)})\Gamma^{\beta}_{A\alpha}(p(t)) = 0$$

for each $\beta = 1,\ldots,n$. Now, as an elementary application of the theory of ordinary differential equations, given $v\in(T_{1,0})_{p(a)}$ there exists a unique parallel section W along p such that $W(a)=v$.

Let $U(n)$ be the group of $n\times n$ unitary matrices. Parallel sections can be represented locally as follows:

Lemma 4.1. *Let $\{Z_\alpha\}_{\alpha=1,\ldots,n}$ be a local orthonormal frame for $T_{1,0}$ on U and suppose that $p([a,b])\subset U$. Then there exists a unique $\Lambda_p\colon [a,b]\to U(n)$ such that $\Lambda_p(a)=I_n$ and*

$$\dot\Lambda_p(t)^{\gamma}_{\beta} + \sum_{A\in\langle\langle n\rangle\rangle}\sum_{\delta=1}^{n}\Lambda_p(t)^{\delta}_{\beta}g_\theta(\dot p(t),(Z_{\overline{A}})_{p(t)})\Gamma^{\gamma}_{A\delta}(p(t)) = 0,\qquad \beta,\gamma=1,\ldots,n \quad (4.2)$$

holds, where $\dot\Lambda_p(t) = (\dot\Lambda_p(t)^{\gamma}_{\beta})_{\gamma,\beta=1,\ldots,n}$.

Moreover, for given $v\in(T_{1,0})_{p(a)}$,

$$W(t) = \sum_{\beta,\gamma=1}^{n}\Lambda_p(t)^{\gamma}_{\beta}g_\theta(v,(Z_{\overline{\beta}})_{p(a)})(Z_\gamma)_{p(t)}$$

is a parallel section along p and satisfies $W(a)=v$.

Proof. It is clear that the condition for Λ_p defines a unique curve on $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the group of $n\times n$ complex matrices. By (3.6) and (4.2) it is easy to check that

$$\frac{d}{dt}\left(\sum_{\gamma=1}^{n}\Lambda_p(t)^\gamma_\alpha\overline{\Lambda_p(t)^\gamma_\beta}\right)=0$$

holds for $\alpha,\beta=1,\dots,n$, which in conjunction with $\Lambda_p(a)=I_n$ implies that $\Lambda_p(t)\in U(n)$.

We next show the second assertion. Recall that

$$\nabla_{\dot p}W(t)=\sum_{\beta,\gamma=1}^{n}g_\theta(v,(Z_{\overline{\beta}})_{p(a)})(\dot\Lambda_p(t)^\gamma_\beta(Z_\gamma)_{p(t)}+\Lambda_p(t)^\gamma_\beta\nabla_{\dot p}Z_\gamma(t)).$$

Plugging (4.2) and the identity

$$\nabla_{\dot p}Z_\gamma(t)=\sum_{A\in\langle\langle n\rangle\rangle}g_\theta(\dot p(t),(Z_{\overline{A}})_{p(t)})(\nabla_{Z_A}Z_\gamma)_{p(t)}$$

into this, we obtain the desired equality $\nabla_{\dot p}W=0$. □

4.1.2 Horizontal lift to the unitary frame bundle

Now we introduce the bundles over M given by

$$\begin{aligned}L(T_{1,0})&=\coprod_{x\in M}\{r\colon\mathbb{C}^n\to(T_{1,0})_x;r\text{ is a non-singular complex-linear map}\},\\ U(T_{1,0})&=\{r\in L(T_{1,0});r\text{ is isometric}\}.\end{aligned}$$

For $r\in L(T_{1,0})$ with $r\colon\mathbb{C}^n\to(T_{1,0})_x$, let $\pi(r)=x$. We write $r\xi$ for the image of $\xi\in\mathbb{C}^n$ by $r\in L(T_{1,0})$.

The Lie group $U(n)$ acts on $U(T_{1,0})$. For each $\Lambda\in U(n)$ we have the map $R_\Lambda\colon U(T_{1,0})\to U(T_{1,0})$ defined by

$$(R_\Lambda r)(\xi)=r\Lambda\xi,\qquad r\in U(T_{1,0}),\quad\xi\in\mathbb{C}^n.$$

Moreover, if $\Lambda\colon[a,b]\to U(n)$ is a smooth curve with $\Lambda(a)=I_n$ and $r\in U(T_{1,0})$, then $\dot\Lambda(a)$ is a skew Hermitian matrix and

$$\left.\frac{d}{dt}\right|_{t=a}R_{\Lambda(t)}r=\lambda(\dot\Lambda(a))_r,$$

where λ is given by

$$\lambda(u)_r = \frac{d}{ds}\bigg|_{s=0} R_{\exp(su)} r, \qquad u \in \mathfrak{u}(n).$$

Here we denote the set of $n \times n$ skew Hermitian matrices by $\mathfrak{u}(n)$. It should be remarked that, while the fiber of $U(T_{1,0})$ is a complex vector space and complex group $U(n)$ acts on it, $U(T_{1,0})$ is a real manifold since so is the base manifold M.

For smooth curves $p\colon [a,b] \to M$ and $\widehat{p}\colon [a,b] \to U(T_{1,0})$, we say that $\widehat{p}$ is a *horizontal lift* of p to $U(T_{1,0})$ if $\pi \circ \widehat{p} = p$ and $\widehat{p}(t)e_\alpha$ is a parallel section for any $\alpha = 1, \ldots, n$, where $\{e_\alpha\}_{\alpha=1,\ldots,n}$ is the standard coordinate of $\mathbb{C}^n$. For $v \in T_xM$, $r \in \pi^{-1}(x)$ and $\eta \in T_rU(T_{1,0})$, we say that η is a *horizontal lift* of v if there exist a smooth curve p on M and a smooth curve $\widehat{p}$ on $U(T_{1,0})$ which is a horizontal lift of p, satisfying

$$\widehat{p}(0) = r, \qquad \dot{\widehat{p}}(0) = \eta, \qquad (\pi_*)_r\eta = v.$$

For each $v \in T_xM$, there exists a unique horizontal lift $\eta_r(v) \in T_rU(T_{1,0})$. Indeed, let $\{Z_\alpha\}_{\alpha=1,\ldots,n}$ be a local orthonormal frame for $T_{1,0}$ on U and suppose the curve p is contained in U. Let $Z\colon U \to U(T_{1,0})$ be the section determined by $\{Z_\alpha\}_{\alpha=1,\ldots,n}$, i.e. $Z(x)e_\alpha = (Z_\alpha)_x$. Let $p\colon [a,b] \to M$ be a smooth curve. By virtue of Lemma 4.1, $\widehat{p}\colon [a,b] \to U(T_{1,0})$ is a horizontal lift of p if and only if $\pi(\widehat{p}(a)) = p(a)$ and

$$\widehat{p}(t) = R_{Z(p(a))^{-1}\circ\widehat{p}(a)} \circ R_{\Lambda_p(t)} Z(p(t)), \tag{4.3}$$

where $Z(p(a))^{-1} \circ \widehat{p}(a)\colon \mathbb{C}^n \to \mathbb{C}^n$ is regarded as an element of $U(n)$ and Λ_p is the curve defined in Lemma 4.1.

Under the identification of $Z(x) \in U(T_{1,0})$ with $((Z_1)_x, \ldots, (Z_n)_x) \in (T_{1,0})_x^n$, we have $R_\Lambda Z(x) = Z(x)\Lambda$ for $\Lambda \in U(n)$. Then we can calculate as

$$\frac{d}{dt}(R_{\Lambda_p(t)} Z(p(t))) = \frac{d}{dt}(Z(p(t))\Lambda_p(t)) = Z_*(\dot{p}(t))\Lambda_p(t) + Z(p(t))\dot{\Lambda}_p(t).$$

By differentiating (4.3) and substituting the above identity, we arrive at the unique horizontal lift of v:

$$\eta_r(v) = (R_{Z(x)^{-1}\circ r})_*(Z_*(v) - \lambda(\Phi(v))_{Z(x)}) \in T_r U(T_{1,0}), \tag{4.4}$$

where $\Phi\colon T_x M \to \mathfrak{u}(n)$ is defined by

$$\Phi(v) = \left(\sum_{A \in \langle\langle n \rangle\rangle} g_\theta(v, (Z_{\overline{A}})_x) \Gamma^\gamma_{A\beta}(x) \right)_{\beta,\gamma=1,\dots,n}.$$

4.1.3 Canonical vector fields

For each $r \in U(T_{1,0})$, the *horizontal subspace* at r is defined by

$$\mathrm{Hor}_r U(T_{1,0}) = \{\eta_r(v); v \in T_{\pi(r)} M\} \subset T_r U(T_{1,0}).$$

If we set the vertical subspace $\mathrm{Ver}_r U(T_{1,0})$ by

$$\mathrm{Ver}_r U(T_{1,0}) = \mathrm{Ker}(\pi_*\colon T_r U(T_{1,0}) \to T_{\pi(r)} M),$$

then

$$T_r U(T_{1,0}) = \mathrm{Ver}_r U(T_{1,0}) \oplus \mathrm{Hor}_r U(T_{1,0})$$

holds.

η_r extends naturally to a $\mathbb{C}$-linear map from $T_x M \otimes_{\mathbb{R}} \mathbb{C}$ to $\mathrm{Hor}_r U(T_{1,0}) \otimes_{\mathbb{R}} \mathbb{C}$. Then for each $\xi \in \mathbb{C}^n$, we can define the canonical vector field $L(\xi)$ by $L(\xi)_r = \eta_r(r\xi)$ for $r \in U(T_{1,0})$. We set

$$L_\alpha = L(e_\alpha), \qquad \alpha = 1, \dots, n$$

and call $\{L_\alpha\}_{\alpha=1,\dots,n}$ the *canonical vector fields*.

Let $\{Z_\alpha\}_{\alpha=1,\dots,n}$ be a local orthonormal frame for $T_{1,0}$ on U. Define $\{e^\beta_\alpha(r)\} \in \mathbb{C}^n \otimes \mathbb{C}^n$ for $r \in L(T_{1,0})$ with $\pi(r) \in U$ by

$$r(e_\alpha) = \sum_{\beta=1}^n e^\beta_\alpha(r)(Z_\beta)_{\pi(r)}.$$

We can then introduce a local coordinate system $\{(x^k, e^\beta_\alpha)\}$ of $L(T_{1,0})$, where $(x^k)_{k=1,\dots,2n+1}$ is a local coordinate system of M. With respect to this coordinate we represent the canonical vector field L_α, $\alpha = 1, \dots, n$ as follows.

Recall that $U(T_{1,0})$ can be identified with $M \times U(n)$ locally, and under this identification $R_\Lambda((x, e)) = (x, e\Lambda)$ for $(x, e) \in M \times U(n)$. Therefore it holds that

$$(R_{Z(x)^{-1}\circ r})_* Z_*(v) = v, \qquad v \in T_x M \otimes_{\mathbb{R}} \mathbb{C}, \quad r \in U(T_{1,0}),$$

where $Z\colon U \to U(T_{1,0})$ is the section determined by $\{Z_\alpha\}_{\alpha=1,\dots,n}$ as before. Since

$$\begin{aligned}
\Phi\Bigl(\operatorname{Re}\sum_{\beta=1}^n e^\beta_\alpha Z_\beta\Bigr) &= \Bigl(\frac{1}{2}\sum_{\beta=1}^n (e^\beta_\alpha \Gamma^\gamma_{\beta\delta} + e^{\overline{\beta}}_{\overline{\alpha}} \Gamma^\gamma_{\overline{\beta}\delta})\Bigr)_{\gamma,\delta=1,\dots,n}, \\
\Phi\Bigl(\operatorname{Im}\sum_{\beta=1}^n e^\beta_\alpha Z_\beta\Bigr) &= \Bigl(\frac{1}{2\mathrm{i}}\sum_{\beta=1}^n (e^\beta_\alpha \Gamma^\gamma_{\beta\delta} - e^{\overline{\beta}}_{\overline{\alpha}} \Gamma^\gamma_{\overline{\beta}\delta})\Bigr)_{\gamma,\delta=1,\dots,n},
\end{aligned}$$

we have

$$\begin{aligned}
\lambda\Bigl(\Phi\Bigl(\operatorname{Re}\sum_{\beta=1}^n e^\beta_\alpha Z_\beta\Bigr)\Bigr) &= \frac{1}{2}\sum_{\beta,\gamma,\delta=1}^n \Bigl((e^\beta_\alpha \Gamma^\gamma_{\beta\delta} + e^{\overline{\beta}}_{\overline{\alpha}} \Gamma^\gamma_{\overline{\beta}\delta})\frac{\partial}{\partial e^\gamma_\delta} + (e^{\overline{\beta}}_{\overline{\alpha}} \Gamma^{\overline{\gamma}}_{\overline{\beta}\overline{\delta}} + e^\beta_\alpha \Gamma^{\overline{\gamma}}_{\beta\overline{\delta}})\frac{\partial}{\partial e^{\overline{\gamma}}_{\overline{\delta}}}\Bigr), \\
\lambda\Bigl(\Phi\Bigl(\operatorname{Im}\sum_{\beta=1}^n e^\beta_\alpha Z_\beta\Bigr)\Bigr) &= \frac{1}{2\mathrm{i}}\sum_{\beta,\gamma,\delta=1}^n \Bigl((e^\beta_\alpha \Gamma^\gamma_{\beta\delta} - e^{\overline{\beta}}_{\overline{\alpha}} \Gamma^\gamma_{\overline{\beta}\delta})\frac{\partial}{\partial e^\gamma_\delta} - (e^{\overline{\beta}}_{\overline{\alpha}} \Gamma^{\overline{\gamma}}_{\overline{\beta}\overline{\delta}} - e^\beta_\alpha \Gamma^{\overline{\gamma}}_{\beta\overline{\delta}})\frac{\partial}{\partial e^{\overline{\gamma}}_{\overline{\delta}}}\Bigr).
\end{aligned}$$

Therefore we have from (4.4) that

$$(L_\alpha)_r = \sum_{\beta=1}^n e^\beta_\alpha Z_\beta - \sum_{\beta,\gamma,\delta,\varepsilon=1}^n \Gamma^\gamma_{\beta\delta} e^\delta_\varepsilon e^\beta_\alpha \frac{\partial}{\partial e^\gamma_\varepsilon} - \sum_{\beta,\gamma,\delta,\varepsilon=1}^n \Gamma^{\overline{\gamma}}_{\beta\overline{\delta}} e^{\overline{\delta}}_{\overline{\varepsilon}} e^\beta_\alpha \frac{\partial}{\partial e^{\overline{\gamma}}_{\overline{\varepsilon}}}. \tag{4.5}$$

4.2 Construction of a diffusion process

In this section, we construct a diffusion process $\mathbb{X} = \{(\{X(t)\}_{t\geq 0}, P_x); x \in M\}$ generated by $-\Delta_b/2$.

Let $\{L_\alpha\}_{\alpha=1,\dots,n}$ be the canonical vector fields on $U(T_{1,0})$ constructed in the previous section. Take a $2n$-dimensional Brownian motion

$$\{B(t) = (B^1(t), \dots, B^{2n}(t))\}_{t\geq 0},$$

and let

$$\xi^\alpha(t) = \frac{1}{\sqrt{2}}(B^\alpha(t) + \mathrm{i}B^{n+\alpha}(t)), \quad \xi^{\overline{\alpha}}(t) = \frac{1}{\sqrt{2}}(B^\alpha(t) - \mathrm{i}B^{n+\alpha}(t)), \qquad \alpha = 1, \dots, n.$$

Let $\{r(t) = r(t, r, \xi)\}_{t\geq 0}$ be the unique solution to the SDE on $U(T_{1,0})$:

$$dr(t) = \sum_{\alpha=1}^{n}(L_\alpha(r(t)) \circ d\xi^\alpha(t) + L_{\overline{\alpha}}(r(t)) \circ d\xi^{\overline{\alpha}}(t)), \quad r(0) = r \in U(T_{1,0}), \tag{4.6}$$

or equivalently

$$\begin{aligned} dr(t) &= \sum_{\alpha=1}^{n}(\sqrt{2}\,\mathrm{Re}L_\alpha(r(t)) \circ dB^\alpha(t) + \sqrt{2}\,\mathrm{Im}L_\alpha(r(t)) \circ dB^{n+\alpha}(t)), \\ r(0) &= r \in U(T_{1,0}), \end{aligned} \tag{4.7}$$

where

$$\mathrm{Re}L_\alpha = \frac{L_\alpha + L_{\overline{\alpha}}}{2}, \qquad \mathrm{Im}L_\alpha = \frac{L_\alpha - L_{\overline{\alpha}}}{2\mathrm{i}}.$$

The process $r(t)$ may explode. The manipulation of taking the real part on the right hand side of the SDE (4.6) is due to that $U(T_{1,0})$ is a real manifold. Let $\{Z_\alpha\}_{\alpha=1,\dots,n}$ be a local orthonormal frame for $T_{1,0}$ and (x^k, e_α^β) be the associated local coordinate of $L(T_{1,0})$ as in the previous section. Then (4.6) can be rewritten locally as

$$\begin{cases} dx(t) = \displaystyle\sum_{\alpha,\beta=1}^{n} (e_{\alpha}^{\beta}(t) Z_{\beta}(x(t)) \circ d\xi^{\alpha}(t) + e_{\overline{\alpha}}^{\overline{\beta}}(t) Z_{\overline{\beta}}(x(t)) \circ d\xi^{\overline{\alpha}}(t)), \\ de_{\varepsilon}^{\gamma}(t) = -\displaystyle\sum_{\alpha,\beta,\delta=1}^{n} (\Gamma_{\beta\delta}^{\gamma}(x(t)) e_{\varepsilon}^{\delta}(t) e_{\alpha}^{\beta}(t) \circ d\xi^{\alpha}(t) \\ \qquad\qquad\qquad + \Gamma_{\overline{\beta}\delta}^{\gamma}(x(t)) e_{\varepsilon}^{\delta}(t) e_{\overline{\alpha}}^{\overline{\beta}}(t) \circ d\xi^{\overline{\alpha}}(t)). \end{cases} \tag{4.8}$$

Hence it follows from the uniqueness of $\{r(t,r,\xi)\}_{t\geq 0}$ that

$$r(t, r\Lambda, \overline{\Lambda}^{t}\xi) = r(t, r, \xi)$$

for every unitary matrix Λ. We have that the induced measures Q_r of $\pi(r(\cdot, r, \xi))$ on $\widehat{W}(M)$ coincide for all $r \in \pi^{-1}(x)$. Put

$$P_x = Q_r \circ \pi^{-1}, \qquad r \in \pi^{-1}(x).$$

Set

$$\mathcal{L} = \frac{1}{2} \sum_{\alpha=1}^{n} (L_{\alpha} L_{\overline{\alpha}} + L_{\overline{\alpha}} L_{\alpha})|_{M}.$$

It is easily seen that

$$f(X(t)) - \int_0^t \mathcal{L} f(X(s)) ds$$

is a martingale under P_x for every $x \in M$ and $f \in C_0^{\infty}(M)$, where $X(t)$ denotes the position of $X \in \widehat{W}(M)$ at time t. By a straightforward computation, we have a local representation of $\mathcal{L}$ as follows:

$$\mathcal{L} = \frac{1}{2}\Bigg(\sum_{\alpha=1}^{n} (Z_{\alpha} Z_{\overline{\alpha}} + Z_{\overline{\alpha}} Z_{\alpha}) - \sum_{\alpha,\beta=1}^{n} (\Gamma_{\overline{\beta}\beta}^{\alpha} Z_{\alpha} + \Gamma_{\beta\overline{\beta}}^{\overline{\alpha}} Z_{\overline{\alpha}}) \Bigg). \tag{4.9}$$

Let $\{\theta, \theta^{\alpha}, \theta^{\overline{\alpha}}\}_{\alpha=1,\ldots,n}$ be the dual basis to $\{T, Z_{\alpha}, Z_{\overline{\alpha}}\}_{\alpha=1,\ldots,n}$. Recall, moreover, an identity that

$$d_b f = \sum_{\alpha=1}^{n} (Z_{\alpha} f \theta^{\alpha} + Z_{\overline{\alpha}} f \theta^{\overline{\alpha}})$$

and Greenleaf's result [9] that

$$\langle Z_\alpha f, \overline{g}\rangle_\theta = \left\langle f, \overline{\left(-Z_{\overline{\alpha}} + \sum_{\beta=1}^{n} \Gamma^{\alpha}_{\overline{\beta}\beta}\right) g} \right\rangle_\theta .$$

Plugging these into (4.9), we see that

$$\mathcal{L} = -\frac{1}{2}\Delta_b.$$

Thus we have shown that

Theorem 4.10. *There exists a diffusion process* $\mathbb{X} = \{(\{X(t)\}_{t\geq 0}, P_x); x \in M\}$ *generated by* $-\Delta_b/2$ *and which is obtained via the SDE* (4.6).

Example 4.11. Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Heisenberg group with a coordinate system (z, t), $z = (z^1, \ldots, z^n) \in \mathbb{C}^n$, $t \in \mathbb{R}$. Define

$$\theta = \frac{1}{2}\left(dt - \mathrm{i}\sum_{\alpha=1}^{n}(\overline{z^\alpha}dz^\alpha - z^\alpha d\overline{z^\alpha})\right),$$

$$T_{1,0} = \bigoplus_{\alpha=1}^{n} \mathbb{C}Z_\alpha,$$

where

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + \mathrm{i}\overline{z^\alpha}\frac{\partial}{\partial t}.$$

Then $\mathbb{H}_n$ is a strictly pseudoconvex CR manifold, see [3]. Since $\{Z_\alpha\}_{\alpha=1,\ldots,n}$ is a global orthonormal frame for $T_{1,0}$ and

$$d\theta = \mathrm{i}\sum_{\alpha=1}^{n} dz^\alpha \wedge d\overline{z^\alpha}, \quad d(dz^\alpha) = 0,$$

the associated covariant derivation is a null mapping. In particular,

$$\Delta_b = -\sum_{\alpha=1}^{n}\left(Z_\alpha Z_{\overline{\alpha}} + Z_{\overline{\alpha}} Z_\alpha\right).$$

The diffusion process described in Theorem 4.10 is exactly the same one as that studied by Gaveau in [7].

Remark 4.12. Diffusion processes on sub-Riemannian manifolds, which include CR manifolds, are studied from the point of view of sub-Riemannian geometry. For example, in Gordina-Laetsch [8] diffusion processes on sub-Riemannian manifolds are constructed as the limit of random walks constructed piecewisely via the Hamiltonian-flow associated with a sub-Riemannian structure. The method of Eells, Elworthy and Malliavin for the sub-Laplacian is used in Grong-Thalmaier [10].

Chapter 5

Applications of diffusion processes

5.1 Heat kernel and stochastic line integral

In this section, we apply the result [23] on partial hypoellipticity to the diffusion process constructed in the previous section and stochastic line integrals along the diffusion process.

5.1.1 Heat kernel

We first consider the heat equation

$$\frac{\partial}{\partial t}u = -\frac{1}{2}\Delta_b u, \qquad u(0,x) = f(x), \quad f \in C_b^\infty(M), \tag{5.1}$$

via the diffusion process $\mathbb{X} = \{(\{X(t)\}_{t\geq 0}, P_x); x \in M\}$ constructed in Theorem 4.10.

By Whitney's embedding theorem, we may think of $U(T_{1,0})$ as a closed submanifold of $\mathbb{R}^k$ for some k. We further assume that

(H) there exist C^∞ vector fields L'_α, $\alpha = 1, \ldots, n$, on $\mathbb{R}^k$ with $\mathbb{C}$-valued coefficients such that

(i) $L_\alpha = L'_\alpha$ on $U(T_{1,0})$, and

(ii) the coefficients of L'_α and their derivatives of all orders are bounded.

The hypothesis (H) implies that $r(t)$ does not explode. For example, this hypothesis is fulfilled if M is compact. We shall establish

Theorem 5.2. *Assume that (H) holds. Then there is a* $p \in C^\infty((0,\infty)\times M \times M)$ *such that*

$$P_x(X(t) \in dy) = p(t,x,y)\psi(dy).$$

Proof. Recall the expression (4.7). By virtue of [23, Theorem 3.1] and [24, Lemma 3.1], it suffices to show that

$$\mathrm{span}_{\mathbb{R}}\{(\pi_*)_r \mathrm{Re}L_\alpha, (\pi_*)_r \mathrm{Im}L_\alpha, (\pi_*)_r[\mathrm{Re}L_\alpha, \mathrm{Im}L_\alpha]; \alpha = 1,\dots,n\} = T_{\pi(r)}M \quad (5.3)$$

for every $r \in U(T_{1,0})$, where $\mathrm{span}_{\mathbb{R}}$ stands for taking all real linear combinations.

To see this, let $\{Z_\alpha\}_{\alpha=1,\dots,n}$ be a local orthonormal frame for $T_{1,0}$, and $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ be the dual basis of $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$. By (4.5), it holds that

$$(\pi_*)L_\alpha = \sum_{\beta=1}^n e_\alpha^\beta Z_\beta. \quad (5.4)$$

We next observe that

$$(\pi_*)[L_\alpha, L_{\overline{\alpha}}] = -2\mathrm{i}T \quad \mod \{Z_\alpha, Z_{\overline{\alpha}}\}_\alpha, \quad (5.5)$$

where we have meant by "$A = B \mod \{Z_\alpha, Z_{\overline{\alpha}}\}_\alpha$" that

$$A = B + \sum_{\alpha=1}^n a^\alpha Z_\alpha + \sum_{\alpha=1}^n b^{\overline{\alpha}} Z_{\overline{\alpha}}$$

for some $a^\alpha, b^{\overline{\alpha}} \in \mathbb{C}$. For this purpose, recall that

$$d\theta(Z,W) = \frac{1}{2}(Z(\theta(W)) - W(\theta(Z)) - \theta([Z,W])), \qquad Z, W \in \mathbb{C}TM.$$

Since $\theta(T_{1,0} \oplus T_{0,1}) = 0$, it holds that

$$\theta([Z_\alpha, Z_{\overline{\beta}}]) = -2d\theta(Z_\alpha, Z_{\overline{\beta}}) = -2\mathrm{i}L_\theta(Z_\alpha, Z_{\overline{\beta}}) = -2\mathrm{i}\delta_{\alpha\beta}. \tag{5.6}$$

Hence

$$[Z_\alpha, Z_{\overline{\beta}}] = -2\mathrm{i}\delta_{\alpha\beta}T \mod \{Z_\alpha, Z_{\overline{\alpha}}\}_\alpha, \tag{5.7}$$

which yields that (5.5) holds.

(5.3) follows from (5.4) and (5.5). □

Remark 5.8. By Theorem 5.2, a bounded solution to the heat equation (5.1) can be written as

$$u(t,x) = E_x[f(X(t))] = \int_M f(y)p(t,x,y)\psi(dy).$$

5.1.2 Stochastic line integral

We next investigate stochastic line integrals. Let Ξ be a 1-form on M, which, under the imbedding made in the assumption (H), can be extended to a 1-form on $\mathbb{R}^k$ such that its derivatives of all orders are bounded.

Denote by $\int_{X[0,t]} \Xi$ the stochastic line integral of Ξ along $\{X(t)\}_{t\geq 0}$ from time 0 to t. For definition, see [12]. It is easily checked that

$$\int_{X[0,t]} \Xi = \sum_{A\in\langle\langle n\rangle\rangle\setminus\{0\}} \int_0^t (\pi^*\Xi)_{r(s)}(L_A) \circ d\xi^A(s),$$

where $\pi^*\Xi$ is the pull-back of Ξ through $\pi\colon U(T_{1,0}) \to M$ and $(\pi^*\Xi)_r(L_A)$ is the pairing of cotangent vector $(\pi^*\Xi)_r$ and tangent vector $(L_A)_r$ at $r \in U(T_{1,0})$. Thus, $\left\{\widetilde{r}(t) = \left(r(t), \int_{X[0,t]} \Xi\right)\right\}_{t\geq 0}$ obeys the SDE

$$d\widetilde{r}(t) = \sum_{A\in\langle\langle n\rangle\rangle\setminus\{0\}} \widetilde{L}_A(\widetilde{r}(t)) \circ d\xi^A(t),$$

where $\widetilde{L}_A$'s are vector fields on $U(T_{1,0}) \times \mathbb{R}$ defined by

$$\widetilde{L}_A = L_A + (\pi^*\Xi)(L_A)\frac{\partial}{\partial \xi},$$

and ξ is the coordinate on $\mathbb{R}$.

For $x \in M$, take a local orthonormal frame $\{Z_\alpha\}_{\alpha=1,\dots,n}$ for $T_{1,0}$ on U, and set $\Xi_A = \Xi(Z_A)$ for $A \in \langle\langle n \rangle\rangle \setminus \{0\}$. For $A_1, \dots, A_m \in \langle\langle n \rangle\rangle \setminus \{0\}$, define

$$\Phi_{A_1,\dots,A_m}(\Xi) \colon U \to \mathbb{C}$$

successively by

$$\Phi_{A_1}(\Xi) = \Xi_{A_1},$$
$$\Phi_{A_1,\dots,A_m}(\Xi) = Z_{A_1}\Phi_{A_2,\dots,A_m}(\Xi) - [Z_{A_2}, [\dots, [Z_{A_{m-1}}, Z_{A_m}]\dots]]\Xi_{A_1}.$$

Theorem 5.9. *Suppose that (H) holds and for each $x \in M$ there exist $A_1, \dots, A_m \in \langle\langle n \rangle\rangle \setminus \{0\}$ such that $\Phi_{A_1,\dots,A_m}(\Xi)(x) \neq 0$. Then the distribution of $\int_{X[0,t]} \Xi$ under P_x admits a smooth density function with respect to the Lebesgue measure on $\mathbb{R}$ for every $x \in M$.*

Proof. Under the same notation as used in (4.5), set $(f_\alpha^\beta)_{\alpha,\beta=1,\dots,n} = (e_\alpha^\beta)^{-1}_{\alpha,\beta=1,\dots,n}$ and define locally

$$\widehat{L}_\alpha = \sum_{\beta=1}^n f_\alpha^\beta \widetilde{L}_\beta.$$

Then it is easily seen that

$$\begin{aligned}
&\operatorname{span}_{\mathbb{C}}\big\{(\widetilde{L}_A)_r, ([\widetilde{L}_{A_1}, [\dots, [\widetilde{L}_{A_{m-1}}, \widetilde{L}_{A_m}]\dots]])_r; \\
&\qquad\qquad A, A_1, \dots, A_m \in \langle\langle n \rangle\rangle \setminus \{0\}, m = 2, 3, \dots\big\} \\
&= \operatorname{span}_{\mathbb{C}}\big\{(\widehat{L}_A)_r, ([\widehat{L}_{A_1}, [\dots, [\widehat{L}_{A_{m-1}}, \widehat{L}_{A_m}]\dots]])_r; \\
&\qquad\qquad A, A_1, \dots, A_m \in \langle\langle n \rangle\rangle \setminus \{0\}, m = 2, 3, \dots\big\}.
\end{aligned}$$

We have

$$(\widetilde{\pi}_*)_r(\widehat{L}_\alpha)_r = \sum_{\beta,\gamma=1}^{n} f_\alpha^\beta e_\beta^\gamma \Xi_\gamma(\pi(r))\frac{\partial}{\partial\xi} = \Phi_\alpha(\Xi)(\pi(r))\frac{\partial}{\partial\xi},$$

where $\widetilde{\pi}\colon U(T_{1,0})\times\mathbb{R}\to\mathbb{R}$ is the natural projection. By induction on m we have

$$(\widetilde{\pi}_*)_r([\widehat{L}_{A_1},[\ldots,[\widehat{L}_{A_{m-1}},\widehat{L}_{A_m}]\ldots]]_r) = \Phi_{A_1,\ldots,A_m}(\Xi)(\pi(r))\frac{\partial}{\partial\xi}.$$

Hence, applying [23, Theorem 3.1], we obtain the desired result. □

Remark 5.10. Although Z_A's in the definition of $\Phi_{A_1,\ldots,A_m}(\Xi)$ are all in $T_{1,0}\oplus T_{0,1}$, the direction T appears in $\Phi_{A_1,\ldots,A_m}(\Xi)$'s because the expression $[Z_\alpha, Z_{\overline{\alpha}}]$ contains T-part by (5.7). Hence, for example, even if $\Xi_A(x)=0$ for each $A\in\langle\langle n\rangle\rangle\setminus\{0\}$, the assumption $\Phi_{A_1,\ldots,A_m}(\Xi)(x)\neq 0$ may be satisfied.

5.2 Dirichlet problem

In this section, we study Dirichlet problems related to Δ_b. For $f\in C(\partial G)$, what to be found is a $u_f\in C^2(G)\cap C(\overline{G})$ such that

$$\Delta_b u_f = 0 \quad\text{and}\quad u_f|_{\partial G} = f.$$

We first establish a weak solution in a probabilistic manner following Stroock and Varadhan [20]. As will be seen in Remark 5.18, we indeed obtain a classical solution stated above.

Let $\mathbb{X}=\{(\{X(t)\}_{t\geq 0},P_x);x\in M\}$ be the diffusion process obtained in Theorem 4.10. Let G be a relatively compact connected open set in M with C^3 boundary. Define

$$\tau' = \inf\{t\geq 0; X(t)\notin\overline{G}\}.$$

We shall show that

Theorem 5.11. *For $f \in C(\partial G)$, define $u_f(x) = E_x[f(X(\tau'))]$. Then $u_f \in C(\overline{G})$ and satisfies that*

$$\langle u_f, \Delta_b v \rangle_\theta = 0 \quad \text{for any } v \in C_0^\infty(G), \quad \text{and} \quad u_f = f \quad \text{on } \partial G.$$

Due to the result by Stroock and Varadhan [20], the theorem is verified once we have established the following two lemmas.

Lemma 5.12. *It holds that*

$$\sup_{x \in \overline{G}} E_x[\tau'] < \infty.$$

Lemma 5.13. *Every boundary point is τ'-regular, that is,*

$$P_x(\tau' = 0) = 1, \quad x \in \partial G.$$

Proof of Lemma 5.12. On account of [20, Remark 5.2], it suffices to show that

$$P_x(\tau' < T) > 0, \quad x \in \overline{G} \text{ and } T > 0. \tag{5.14}$$

To do this, take a family $\{U_j\}_{j=1}^N$ of coordinate neighborhoods of M such that $\overline{G} \subset \bigcup_{j=1}^N U_j$. Let $\Lambda = \{j; U_j \cap \partial G \neq \emptyset\}$. Take $j \in \Lambda$ and a local orthonormal frame $\{Z_\alpha\}_{\alpha=1,\dots,n}$ for $T_{1,0}$ on U_j. Then, by virtue of (4.9), we may assume that the part of $\{X(t)\}_{t\geq 0}$ on U_j is governed by an SDE

$$\begin{aligned} dX(t) = \sum_{\alpha=1}^n (\sqrt{2}\,\mathrm{Re}Z_\alpha(X(t)) \circ dB^\alpha(t) + \sqrt{2}\,\mathrm{Im}Z_\alpha(X(t)) \circ dB^{n+\alpha}(t)) \\ + b(X(t))dt, \end{aligned} \tag{5.15}$$

where $(B^1(t), \dots, B^{2n}(t))$ is an $\mathbb{R}^{2n}$-valued Brownian motion and

$$b = -\sum_{\alpha,\beta=1}^n (\Gamma^\alpha_{\overline{\beta}\beta} Z_\alpha + \Gamma^{\overline{\alpha}}_{\beta\overline{\beta}} Z_{\overline{\alpha}}).$$

Due to (5.7), applying the support theorem (cf. [13, Theorem 3.2]), we obtain that

$$P_x(\tau' < T) > 0, \qquad x \in U_j, \quad j \in \Lambda, \quad T > 0. \tag{5.16}$$

For U_k such that $k \notin \Lambda$ and $U_k \cap U_j \neq \emptyset$ for some $j \in \Lambda$, by the same reasoning as above, applying the support theorem again, we have that

$$P_x(X(t) \text{ hits } U_j \text{ before } T) > 0, \qquad x \in U_k, \quad T > 0.$$

Combined with (5.16) and the strong Markov property, this yields that

$$P_x(\tau' < T) > 0, \qquad x \in U_k, \quad T > 0.$$

Repeating this argument successively, we can conclude (5.14). □

Proof of Lemma 5.13. Let $x \in \partial G$ and U be a coordinate neighborhood of x. For a local orthonormal frame $\{Z_\alpha\}_{\alpha=1,\ldots,n}$ for $T_{1,0}$ defined on U, we may and will assume that the part of $\{X(t)\}_{t\geq 0}$ on U obeys the SDE (5.15).

Let φ be a locally defining function of G around x, which means that there is an open set V containing x such that $\varphi \in C^3(V)$, $V \cap G = \{y \in V; \varphi(y) < 0\}$, and $d\varphi(y) \neq 0$ for $y \in \partial G \cap V$. If either $(\mathrm{Re}Z_\alpha)\varphi(x) \neq 0$ or $(\mathrm{Im}Z_\alpha)\varphi(x) \neq 0$, then by [19, Corollary 4], x is τ'-regular. Now we suppose that

$$(\mathrm{Re}Z_\alpha)\varphi(x) = (\mathrm{Im}Z_\alpha)\varphi(x) = 0, \qquad \alpha = 1, \ldots, n. \tag{5.17}$$

Since $\{\mathrm{Re}Z_\alpha, \mathrm{Im}Z_\alpha, T\}_{\alpha=1,\ldots,n}$ forms a local basis of TM on U, this implies that $T\varphi(x) \neq 0$. Moreover, in conjunction with (5.7) and (5.17) it also implies that

$$[\mathrm{Re}Z_\alpha, \mathrm{Im}Z_\alpha]\varphi(x) = T\varphi(x) \neq 0.$$

Hence it follows that, for each α, either

$$(\mathrm{Re}Z_\alpha)(\mathrm{Im}Z_\alpha)\varphi(x) \neq 0 \qquad \text{or} \qquad (\mathrm{Im}Z_\alpha)(\mathrm{Re}Z_\alpha)\varphi(x) \neq 0$$

and that a matrix

$$\begin{pmatrix} (\mathrm{Re}Z_\alpha)(\mathrm{Re}Z_\beta)\varphi(x) & (\mathrm{Re}Z_\alpha)(\mathrm{Im}Z_\beta)\varphi(x) \\ (\mathrm{Im}Z_\alpha)(\mathrm{Re}Z_\beta)\varphi(x) & (\mathrm{Im}Z_\beta)(\mathrm{Im}Z_\beta)\varphi(x) \end{pmatrix}_{\alpha,\beta=1,\ldots,n}$$

is not symmetric. Applying [19, Corollary 7], we see that x is τ'-regular. □

Remark 5.18. Since Δ_b is hypoelliptic ([3, Theorem 2.1]), that is, if $\Delta_b v = g$ and $g \in C^\infty(U)$ then $v \in C^\infty(U)$, u_f is a classical solution to the Dirichlet problem, namely it holds that $u_f \in C^\infty(G) \cap C(\overline{G})$, $\Delta_b u_f = 0$ and $u_f|_{\partial G} = f$.

Chapter 6

Diagonal short time asymptotics of heat kernels

6.1 Asymptotic expansion of the heat kernel

In this section, we consider the heat equation

$$\frac{\partial}{\partial t}v = -\frac{1}{2}\Delta_b u, \quad v(0,x) = f(x), \quad f \in C_b^\infty(M).$$

We assume that M is compact. As we have shown in Theorem 5.2, there is a function $p \in C^\infty((0,\infty) \times M \times M)$ such that

$$P_x(X(t) \in dy) = p(t,x,y)\psi(dy), \quad x, y \in M.$$

$p(t,x,y)$ is the heat kernel associated to $-\frac{1}{2}\Delta_b$.

Now we state the main theorem of this chapter:

Theorem 6.1. *For each $x \in M$, there exists constants $c_a^M(x)$ $(a = 0, 1, \ldots)$ such that*

$$p(t,x,x) \sim t^{-n-1}\sum_{a=0}^{\infty} c_a^M(x)t^a \quad \text{as } t \downarrow 0.$$

Furthermore,

(i) $c_0^M(x) = c_0 = \dfrac{1}{(2\pi)^{n+1}}\displaystyle\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^n d\tau$, *which depends only on n.*

(ii) *For* $a \geq 1$, $c_a^M(x)$ *can be written as an at most* $2a$*-th degree polynomial of at most* $(2a+2)$*-th derivatives of Christoffel symbols of* M.

To prove the theorem, we first note that it is sufficient to consider the problem locally. To be precise, let $\{Z_\alpha\}_{\alpha=1,\dots,n}$ be a local orthonormal frame for $T_{1,0}$ on a relatively compact open neighborhood V of x. Let $\widehat{X}_j$, $j = 1, \dots, 2n$ be as in Section 3.3 and

$$\widehat{X}_0 = -\sum_{\alpha,\beta=1}^{n} (\Gamma^{\alpha}_{\overline{\beta}\beta} Z_\alpha + \Gamma^{\overline{\alpha}}_{\beta\overline{\beta}} Z_{\overline{\alpha}}) = -\sqrt{2}\sum_{\alpha,\beta=1}^{n} (\mathrm{Re}\Gamma^{\alpha}_{\overline{\beta}\beta}\widehat{X}_\alpha - \mathrm{Im}\Gamma^{\alpha}_{\overline{\beta}\beta}\widehat{X}_{n+\alpha}).$$

Let $\{((U_t)_{t\geq 0}, \widetilde{P}_y); y \in V\}$ be the minimal diffusion process on V generated by

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{2n} \widehat{X}_j^2 + \widehat{X}_0 \tag{6.2}$$

and $\widetilde{p}$ be the density function of $\widetilde{P}_y$ with respect to ψ. A repetition of the argument in [12, Section V.10] yields the following:

Lemma 6.3. *There exist constants* $c_1, c_2 > 0$ *such that*

$$|p(t,x,x) - \widetilde{p}(t,x,x)| \leq c_1 \exp\left(-\frac{c_2}{t}\right), \qquad t > 0.$$

Remark 6.4. We comment on some preceding results related to Theorem 6.1.

By Lemma 6.3, the analysis of $p(t,x,x)$ can be reduced to an analysis on local coordinates with a generator (6.2). This type of analysis was considered in [21] using the stochastic Taylor expansion as mentioned in Introduction. We employ a perturbation argument using the Folland-Stein normal coordinate, so our method is more (CR-)geometric than the general theory in [21].

Diagonal short time asymptotics of heat kernels without drift term (i.e. $\widehat{X}_0 = 0$) are also obtained in [2], which cannot be directly used to our case because a non-trivial drift term appears in the local expression of the generator.

We also note that asymptotics regarding heat equations on CR manifolds are deeply investigated in the geometric context. The asymptotic expansion of heat kernels for the complex Kohn-Spencer Laplacian $\Box_b$ acting on (p, q) forms are established by Beals, Greiner and Stanton [1], using pseudodifferential calculus. Using this result, Stanton [17] obtained the asymptotic expansion for the real sub-Laplacian Δ_b, which we study in this thesis (note that Δ_b is the real part of $\Box_b$ acting on $(0, 0)$ forms, i.e. on functions).

By Lemma 6.3, for a proof of Theorem 6.1 it suffices to show the theorem replacing p by the heat kernel associated with $\mathcal{L}$. Moreover, by Theorem 3.9 we can assume that $\widehat{X}_j$'s are vector fields on $\mathbb{R}^{2n+1}$ which satisfy the condition of Theorem 3.9 and $x = 0 \in \mathbb{R}^{2n+1}$. For simplicity we write p by the heat kernel associated with $\mathcal{L}$ in the sequel.

We write

$$\widehat{X}_j = X_j + A_j, \quad j = 1, \dots, 2n,$$

where

$$X_\alpha = \frac{\partial}{\partial u^\alpha} - u^{n+\alpha} \frac{\partial}{\partial u^{2n+1}}, \qquad X_{n+\alpha} = \frac{\partial}{\partial u^{n+\alpha}} + u^\alpha \frac{\partial}{\partial u^{2n+1}}, \quad \alpha = 1, \dots, n$$

and

$$A_j = \sum_{i=1}^{2n+1} a_j^i \frac{\partial}{\partial u^i}, \quad j = 1, \dots, 2n.$$

We also write

$$\widehat{X}_0 = \sum_{i=1}^{2n+1} b^i \frac{\partial}{\partial u^i}.$$

For later use, for $\varepsilon \in (0, 1]$ we define

$$\widehat{X}_j^\varepsilon = X_j + A_j^\varepsilon, \qquad A_j^\varepsilon = \sum_{i=1}^{2n+1} a_j^{i,\varepsilon} \frac{\partial}{\partial u^i}, \quad i = 1, \dots, 2n, \tag{6.5}$$

$$\widehat{X}_0^\varepsilon = \sum_{i=1}^{2n+1} b^{i,\varepsilon} \frac{\partial}{\partial u^i} \tag{6.6}$$

with

$$a_j^{i,\varepsilon}(u) = a_j^i(\lambda_\varepsilon(u)), \qquad i = 1, \ldots, 2n, \quad j = 1, \ldots, 2n,$$
$$a_j^{2n+1,\varepsilon}(u) = \varepsilon^{-1} a_j^{2n+1}(\lambda_\varepsilon(u)), \qquad j = 1, \ldots, 2n,$$
$$b^{i,\varepsilon}(u) = \varepsilon b^i(\lambda_\varepsilon(u)), \qquad b^{2n+1,\varepsilon}(u) = b^{2n+1}(\lambda_\varepsilon(u)), \quad i = 1, \ldots, 2n,$$

where $\lambda_\varepsilon(u) = (\varepsilon u^1, \ldots, \varepsilon u^{2n}, \varepsilon^2 u^{2n+1})$ for $u = (u^1, \ldots, u^{2n+1})$. Note that

$$\widehat{X}_j^\varepsilon = \widehat{X}_j, \quad j = 1, \ldots, 2n+1 \quad \text{if } \varepsilon = 1.$$

Then we have from Theorem 3.9 that

$$a_j^{i,\varepsilon}(0) = 0, \quad i = 1, \ldots, 2n+1, \quad j = 1, \ldots, 2n, \quad \varepsilon \in (0,1], \tag{6.7}$$

$$\frac{\partial a_j^{2n+1,\varepsilon}}{\partial u^k}(0) = 0, \quad j = 1, \ldots, 2n, \quad k = 1, \ldots, 2n+1, \quad \varepsilon \in (0,1]. \tag{6.8}$$

To state the next lemma, let

$$S^{2n} = \{\xi \in \mathbb{R}^{2n+1}; |\xi| = 1\}$$

be the $2n$-dimensional sphere and denote the standard inner product in $\mathbb{R}^{2n+1}$ by $\langle , \rangle_{\mathbb{R}^{2n+1}}$. We identify $T_0\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n+1}$ by $\sum_{j=1}^{2n+1} x^j \left(\frac{\partial}{\partial u^j}\right)_0 \mapsto (x^j)_{j=1,\ldots,2n+1}$.

Lemma 6.9. *For $\alpha = 1, \ldots, n$, we have*

$$\inf_{\varepsilon \in (0,1]} \inf_{\xi \in S^{2n}} \left(\sum_{j=1}^{2n} \langle (\widehat{X}_j^\varepsilon)_0, \xi \rangle_{\mathbb{R}^{2n+1}}^2 + \langle [\widehat{X}_\alpha^\varepsilon, \widehat{X}_{n+\alpha}^\varepsilon]_0, \xi \rangle_{\mathbb{R}^{2n+1}}^2 \right) > 0,$$

where the tangent vector at 0 of a tangent vector field V is denoted by V_0. In particular, the set $\{(\widehat{X}_j)_0; j = 1, \ldots, 2n\} \cup \{[\widehat{X}_\alpha, \widehat{X}_{n+\alpha}]_0\}$ spans $T_0\mathbb{R}^{2n+1}$ over $\mathbb{R}$.

Proof. We have $(\widehat{X}_j)_0 = \left(\dfrac{\partial}{\partial u^j}\right)_0$ for $j = 1, \dots, 2n$ by (6.7). If we write

$$\eta^i_\alpha = \frac{\partial a^i_{n+\alpha}}{\partial u^\alpha}(0) - \frac{\partial a^i_\alpha}{\partial u^{n+\alpha}}(0), \quad i = 1, \dots, 2n,$$

we have

$$\begin{aligned}[\widehat{X}_\alpha, \widehat{X}_{n+\alpha}]_0 &= 2\left(\frac{\partial}{\partial u^{2n+1}}\right)_0 + \varepsilon \sum_{i=1}^{2n} \eta^i_\alpha \left(\frac{\partial}{\partial u^i}\right)_0 \\ &\equiv 2\left(\frac{\partial}{\partial u^{2n+1}}\right)_0 \qquad \text{mod span}_{\mathbb{R}}\{(\widehat{X}_j)_0; j = 1, \dots, 2n\}\end{aligned}$$

by (6.8). Then

$$\begin{aligned}&\sum_{j=1}^{2n} \langle (\widehat{X}^\varepsilon_j)_0, \xi\rangle^2_{\mathbb{R}^{2n+1}} + \langle [\widehat{X}^\varepsilon_\alpha, \widehat{X}^\varepsilon_{n+\alpha}]_0, \xi\rangle^2_{\mathbb{R}^{2n+1}} \\ &\qquad = \sum_{j=1}^{2n} (\xi^j)^2 + \left(2\xi^{2n+1} + \varepsilon \sum_{j=1}^{2n} \eta^j_\alpha \xi^j\right)^2\end{aligned}$$

can be seen as a continuous function on the compact set $S^{2n} \times [0,1] \ni (\xi, \varepsilon)$ which never attains zero. The lemma follows. □

For $\varepsilon > 0$, let $\{U^\varepsilon_t\}_{t \geq 0}$ be the solution of SDE

$$dU^\varepsilon_t = \varepsilon \sum_{j=1}^{2n} \widehat{X}_j(U^\varepsilon_t) \circ dB^j_t + \varepsilon^2 \widehat{X}_0(U^\varepsilon_t) dt, \quad U^\varepsilon_0 = 0,$$

where $(B^1_t, \dots, B^{2n}_t)_{t \geq 0}$ is a Brownian motion of $2n$ dimension. Then $\{U^1_t\}_{t \geq 0}$ is a diffusion process with generator $\mathcal{L}$.

Since $\widehat{X}_0, \widehat{X}_1, \dots, \widehat{X}_{2n}$ satisfies the Hörmander condition by Lemma 6.9, we have $U^\varepsilon_t \in \mathbb{D}^\infty(\mathbb{R}^{2n+1})$ is non-degenerate in the sense of the Malliavin calculus for $t > 0$. Moreover, U^ε_t and $U^1_{\varepsilon^2 t}$ have the same law. Then we can define $\delta_0(U^\varepsilon_t)$ as a generalized Wiener functionals and we have

$$p(\varepsilon^2, 0, 0) = \mathbb{E}[\delta_0(U^\varepsilon_t)] = \mathbb{E}[\delta_0(U^1_{\varepsilon^2 t})],$$

where the expectation is denoted by $\mathbb{E}$ and δ_0 is the Dirac delta function at $0 \in \mathbb{R}^{2n+1}$.

To investigate the asymptotic behavior of $\mathbb{E}[\delta_0(U^1_{\varepsilon^2 t})]$, we first introduce the multiple Wiener integral:

Definition 6.10. For a multiple index $J = (j_1, \ldots, j_b)$, $j_1, \ldots, j_b \in \{0, 1, \ldots, 2n\}$, define B^J_t by

$$B^{(j_1)}_t = B^{j_1}_t, \qquad B^{(j_1,\ldots,j_b)}_t = \int_0^t B^{(j_1,\ldots,j_{b-1})}_s \circ dB^{j_b}_s \quad \text{for } b \geq 2,$$

where we set $B^0_t = t$.

By applying the Itô formula repetitively, we have the following expansion formula (cf. [12]):

Proposition 6.11. *For $i = 1, \ldots, 2n+1$, $A = 1, 2, \ldots$ and $\varepsilon > 0$, we have*

$$U^{\varepsilon,i}_t = \sum_{a=1}^{A} \varepsilon^a \sum_{\|J\|=a} (\widehat{X}_J u^i)(0) B^J_t + \varepsilon^{A+1} \Phi^{A,\varepsilon}_t,$$

where we write $U^\varepsilon_t = (U^{\varepsilon,1}_t, \ldots, U^{\varepsilon,2n+1}_t)$ and

$$\|J\| = b + \#\{\nu; j_\nu = 0\}, \qquad \widehat{X}_J = \widehat{X}_{j_1} \cdots \widehat{X}_{j_b}$$

for $J = (j_1, \ldots, j_b)$, $j_1, \ldots, j_b \in \{0, 1, \ldots, 2n\}$.

Moreover, we have $\Phi^{A,\varepsilon}_t \in \mathbb{D}^\infty(\mathbb{R}^{2n+1})$ and

$$\sup_{\varepsilon \in (0,1)} \mathbb{E}\left[\sup_{t \in [0,1]} |\Phi^{A,\varepsilon}_t|^p\right] < \infty, \qquad p \in (1, \infty).$$

Let us write $\Psi^{A,\varepsilon}_t = (\Psi^{A,\varepsilon,1}_t, \ldots, \Psi^{A,\varepsilon,2n+1}_t)$ with

$$\Psi^{A,\varepsilon,i}_t = \begin{cases} \Phi^{A,\varepsilon,i}_t, & i = 1, \ldots, 2n, \\ \Phi^{A+1,\varepsilon,2n+1}_t, & i = 2n+1. \end{cases}$$

Using Theorem 3.9, we have

$$\sum_{\|J\|=1} (\widehat{X}_J u^i)(0)B_t^J = \sum_{j=1}^{2n} (\widehat{X}_j u^i)(0)B_t^j = \begin{cases} B_t^i, & i = 1, \dots, 2n, \\ 0, & i = 2n+1 \end{cases}$$

and

$$\begin{aligned}\sum_{\|J\|=2} (\widehat{X}_J u^{2n+1})(0)B_t^J &= \sum_{j,k=1}^{2n} (\widehat{X}_j \widehat{X}_k u^{2n+1})(0)B_t^{(j,k)} + (\widehat{X}_0 u^{2n+1})(0)t \\ &= \sum_{\alpha=1}^{n} S_t^\alpha,\end{aligned}$$

where

$$S_t^\alpha = B_t^{(\alpha,n+\alpha)} - B_t^{(n+\alpha,\alpha)} = \int_0^t (B_s^\alpha \circ dB_s^{n+\alpha} - B_s^{n+\alpha} \circ dB_s^\alpha)$$

is *Lévy's stochastic area.*

From this we can write

$$\begin{aligned} U_t^{\varepsilon,i} &= \varepsilon B_t^i + \sum_{a=2}^{A} \varepsilon^a \varphi_t^{a,i} + \varepsilon^{A+1} \Psi_t^{A,\varepsilon,i}, \quad i = 1, \dots, 2n, \\ U_t^{\varepsilon,2n+1} &= \varepsilon^2 \sum_{\alpha=1}^{n} S_t^\alpha + \sum_{a=2}^{A} \varepsilon^{a+1} \varphi_t^{a,2n+1} + \varepsilon^{A+2} \Psi_t^{A,\varepsilon,2n+1}, \end{aligned}$$

where

$$\varphi_t^{a,i} = \begin{cases} \sum_{\|J\|=a} (\widehat{X}_J u^i)(0)B_t^J, & i = 1, \dots, 2n, \\ \sum_{\|J\|=a} (\widehat{X}_J u^{2n+1})(0)B_t^J, & i = 2n+1. \end{cases} \tag{6.12}$$

Let us write $\mathbb{X}_t = \left(B_t^1, \dots, B_t^{2n}, \sum_{\alpha=1}^{n} S_t^\alpha \right)$. By the change of variables

$$(\varepsilon u^1, \dots, \varepsilon u^{2n}, \varepsilon^2 u^{2n+1}) \mapsto (v^1, \dots, v^{2n}, v^{2n+1}),$$

we have

$$\mathbb{E}[\delta_0(U_1^\varepsilon)] = \varepsilon^{-2n-2} \mathbb{E}[\delta_0(\mathbb{X}_1 + \varepsilon \Psi_1^{1,\varepsilon})].$$

Let $F_t^\varepsilon = \mathbb{X}_t + \varepsilon \Psi_t^{1,\varepsilon}$ and let σ^ε be the Malliavin covariance of F_1^ε.

Lemma 6.13. *For any* $p \in [1, \infty)$, *we have* $\sup_{\varepsilon \in (0,1]} \mathbb{E}[(\det \sigma^\varepsilon)^{-p}] < \infty$, *i.e.* $\{F_1^\varepsilon\}_{\varepsilon \in (0,1]}$ *is uniformly non-degenerate.*

Proof. We write $F_t^\varepsilon = (F_t^{\varepsilon,i})_{i=1,\dots,2n+1}$. Since

$$F_t^{\varepsilon,i} = \varepsilon^{-1} U_t^{\varepsilon,i}, \qquad F_t^{\varepsilon,2n+1} = \varepsilon^{-2} U_t^{\varepsilon,2n+1}, \quad i = 1, \dots, 2n,$$

we can see that $\{F_t^\varepsilon\}_{t \geq 0}$ obeys the SDE

$$dF_t^\varepsilon = \sum_{j=1}^{2n} \widehat{X}_j^\varepsilon(F_t^\varepsilon) \circ dB_t^j + \widehat{X}_0^\varepsilon(F_t^\varepsilon) dt,$$

where $\widehat{X}_j^\varepsilon$ is given in (6.5), (6.6).

Note that for any $a \geq 1$, $j_1, \dots, j_a \in \{0, \dots, 2n\}$, the vector field

$$[\widehat{X}_{j_a}^\varepsilon, [\dots, [\widehat{X}_{j_2}^\varepsilon, \widehat{X}_{j_1}^\varepsilon] \dots]]$$

has bounded coefficients uniformly in $\varepsilon \in (0,1]$ on a neighborhood of $0 \in \mathbb{R}^{2n+1}$. To see this, it is enough to show that for each $i \in \{1, \dots, 2n+1\}$, $j \in \{1, \dots, 2n\}$, $a \geq 1$, $j_1, \dots, j_a \in \{0, \dots, 2n\}$ and $\varepsilon' > 0$,

$$\sup_{\varepsilon \in (0,1]} \sup_{|u| \leq \varepsilon'} |a_j^{i,\varepsilon}(u)| < \infty, \qquad \sup_{\varepsilon \in (0,1]} \sup_{|u| \leq \varepsilon'} \left| \frac{\partial}{\partial u^J} a_j^{i,\varepsilon}(u) \right| < \infty, \tag{6.14}$$

$$\sup_{\varepsilon \in (0,1]} \sup_{|u| \leq \varepsilon'} |b^{i,\varepsilon}(u)| < \infty, \qquad \sup_{\varepsilon \in (0,1]} \sup_{|u| \leq \varepsilon'} \left| \frac{\partial}{\partial u^J} b^{i,\varepsilon}(u) \right| < \infty \tag{6.15}$$

hold, where $J = (j_1, \dots, j_a)$ and $\dfrac{\partial}{\partial u^J} = \dfrac{\partial}{\partial u^{j_1}} \cdots \dfrac{\partial}{\partial u^{j_b}}$.

We show (6.14) for $i = 2n+1$ and proofs of other cases are similar or more clear. We have by definition of $a^{2n+1,\varepsilon}$ and (6.7) that

$$a_j^{2n+1,\varepsilon}(u) = \varepsilon^{-1} a_j^{2n+1}(\lambda_\varepsilon(u)) = \sum_{k=1}^{2n} u^k \frac{\partial a_j^{2n+1}}{\partial u^k}(\lambda_{c\varepsilon}(u)) + 2c\varepsilon u^{2n+1} \frac{\partial a_j^{2n+1}}{\partial u^{2n+1}}(\lambda_{c\varepsilon}(u))$$

for some $c \in (0,1)$ depending on ε and u. This shows the first formula in (6.14). We also have

$$\frac{a_j^{2n+1,\varepsilon}}{\partial u^J}(u) = \varepsilon^{\|J\|-1}\frac{\partial a_j^{2n+1}}{\partial u^J}(\lambda_\varepsilon(u)),$$

which shows the second formula in (6.14).

Therefore we apply [14, Theorem 2.17] to $\{\widehat{X}_j^\varepsilon\}_{j=0,\ldots,2n}$, together with Lemma 6.9, we have

$$\begin{aligned}\mathbb{P}(\det\sigma^\varepsilon \le K^{-\frac{1}{3(2n+1)}}) &\le \mathbb{P}(\inf_{\xi\in S^{2n}}\langle\xi,\sigma^\varepsilon\xi\rangle_{\mathbb{R}^{2n+1}} \le K^{-\frac{1}{3}}) \\ &\le \mu_1\exp(-\mu_2 K^{\mu_3}), \quad K\in[1,\infty)\end{aligned}$$

with positive constants μ_1,μ_2,μ_3 which are independent of ε. This implies the desired result. □

By Lemma 6.13 and the Taylor expansion

$$\delta_0(x+y) \sim \delta_0(x) + \sum_{a=1}^{\infty}\frac{1}{a!}\sum_{|J|=a}\mathbb{E}[\partial^J\delta_0(x)y^J],$$

where $y^J = y^{j_1}\cdots y^{j_a}$ and $\partial^J = \dfrac{\partial^a}{\partial u^{j_1}\cdots\partial u^{j_a}}$ for $J=(j_1,\ldots,j_a)$, we apply Proposition 2.12 to have

$$\varepsilon^{2n+2}\mathbb{E}[\delta_0(U_1^\varepsilon)] \sim \mathbb{E}[\delta_0(\mathbb{X}_1)] + \sum_{a=1}^{\infty}\frac{\varepsilon^a}{a!}\sum_{|J|=a}\mathbb{E}[\partial^J\delta_0(\mathbb{X}_1)\Psi_1^{1,\varepsilon,J}]. \tag{6.16}$$

Since we have by definition

$$\Psi_1^{1,\varepsilon} = \sum_{a=2}^{A+1}\varepsilon^{a-2}\varphi_1^a + \varepsilon^A\Psi_1^{A+1,\varepsilon}, \tag{6.17}$$

where $\varphi_t^A = (\varphi_t^{A,1},\ldots,\varphi_t^{A,2n+1})$, we arrive at the expression as follows:

Proposition 6.18. *We have* $\varepsilon^{2n+1}\mathbb{E}[\delta_0(U_1^\varepsilon)] = d_0 + d_1\varepsilon + d_2\varepsilon^2 + \cdots$ *with*

$d_0 = \mathbb{E}[\delta_0(\mathbb{X}_1)]$,

$$d_A = \sum_{a=1}^{A} \frac{1}{a!} \sum_{(*)} (\widehat{X}_{J_1} u^{i_1})(0) \cdots (\widehat{X}_{J_a} u^{i_a})(0) \mathbb{E}[\partial^I \delta_0(\mathbb{X}_1) B^{J_1} \cdots B^{J_a}], \quad A \geq 1, \tag{6.19}$$

where the summation $(*)$ *is taken over the set*

$$\{(I, J_1, \ldots, J_a); I = (i_b)_{b=1,\ldots,a}, 1 \leq i_b \leq 2n+1, \|J_b\| \geq 2 + \delta_{i_b,2n+1},$$
$$\|J_1\| + \cdots + \|J_a\| = A + \|I\|\}.$$

Proof. From (6.16) and (6.17) we have the expansion

$$\varepsilon^{2n+1} \mathbb{E}[\delta_0(U_1^\varepsilon)] \sim d_0 + d_1 \varepsilon + d_2 \varepsilon^2 + \cdots$$

with $d_0 = \mathbb{E}[\delta_0(\mathbb{X}_1)]$ and

$$d_A = \sum_{a=1}^{A} \frac{1}{a!} \sum_{i_1,\ldots,i_a=1}^{2n+1} \sum_{\substack{k_1,\ldots,k_a \geq 0 \\ k_1+\cdots+k_a = A-a}} \mathbb{E}[\partial^{(i_1,\ldots,i_a)} \delta_0(\mathbb{X}_1) \varphi_1^{2+k_1,i_1} \cdots \varphi_1^{2+k_a,i_a}]$$

for $A \geq 1$. (6.12) yields

$$\begin{aligned} &\mathbb{E}[\partial^{(i_1,\ldots,i_a)} \delta_0(\mathbb{X}_1) \varphi_1^{2+k_1,i_1} \cdots \varphi_1^{2+k_a,i_a}] \\ &\quad = \sum_{\|J_1\|=l_1} \cdots \sum_{\|J_a\|=l_a} (\widehat{X}_{J_1} u^{i_1})(0) \cdots (\widehat{X}_{J_a} u^{i_a})(0) \mathbb{E}[\partial^{(i_1,\ldots,i_a)}(\mathbb{X}_1) B_1^{J_1} \cdots B_1^{J_a}] \end{aligned}$$

with $l_b = 2 + k_b + \delta_{i_b,2n+1}$. For fixed $I = (i_1, \ldots, i_a)$ we have

$$\begin{aligned} &k_b \geq 0, k_1 + \cdots + k_a = A - a, \|J_b\| = 2 + k_b + \delta_{i_b,2n+1} \\ &\quad \Longleftrightarrow \|J_b\| \geq 2 + \delta_{i_b,2n+1}, \|J_1\| + \cdots + \|J_a\| = A + \|I\|, \\ &\qquad \|J_b\| = 2 + k_b + \delta_{i_b,2n+1}, \end{aligned}$$

the proposition follows. □

If we set

$$\mathbb{X}'_t = \left(-B_t^1, \ldots, -B_t^{2n}, \sum_{\alpha=1}^{n} S_t^\alpha\right),$$

It is easy to observe that, for each term appealing in (6.19),

$$\begin{aligned}
&\mathbb{E}[\partial^{i_1\cdots i_a}(\mathbb{X}_1')(-B_1^{J_1})\cdots(-B_1^{J_a})] \\
&\quad = (-1)^{\|I\|}(-1)^{\|J_1\|+\cdots\|J_a\|}\mathbb{E}[\partial^{i_1\cdots i_a}(\mathbb{X}_1)B_1^{J_1}\cdots B_1^{J_a}] \\
&\quad = (-1)^{A}\mathbb{E}[\partial^{i_1\cdots i_a}(\mathbb{X}_1)B_1^{J_1}\cdots B_1^{J_a}].
\end{aligned}$$

By considering the Brownian motion $(-B_t^1,\ldots,-B_t^{2n})$ instead of $(B_t^1,\ldots,B_t^{2n})$, it follows that

$$d_A = 0 \quad \text{if } A \text{ is odd.}$$

Using this, if we set $\varepsilon^2 = t$ and $c_A = d_{2A}$, we have the desired expansion in Theorem 6.1. In (6.19), we observe that $(\widehat{X}_{J_b}u^{i_b})(0)$ is not affected by $O^{\|J_b\|+1}$ terms, and if we note that

$$\|J_b\| \le A + \|I\| - 2(a-1) \le A+2,$$

Theorem 3.9 shows that d_A can be written as an at most A-th degree polynomial of at most $(A+2)$-th derivatives of Christoffel symbols.

It remains to calculate the leading term $c_0 = d_0 = \mathbb{E}[\delta_0(\mathbb{X}_1)]$. $\mathbb{E}[\delta_0(\mathbb{X}_1)]$ is equal to the value at 0 of the density function of $\mathbb{X}_1 = \left(B_1^1,\ldots,B_1^{2n},\sum_{\alpha=1}^{n} S_1^\alpha\right)$. By [7, Théorèm 1], we see that

$$c_0 = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^n d\tau$$

and the proof of Theorem 6.1 is complete.

Remark 6.20. By the integration-by-parts formula for generalized Wiener functionals (see for example [12]), we can write

$$\mathbb{E}[\partial^I\delta_0(\mathbb{X}_1)B^{J_1}\cdots B^{J_a}] = \mathbb{E}[\delta_0(\mathbb{X}_1)\Psi] = \mathbb{E}[\Psi|\mathbb{X}_1=0]p_{\mathbb{X}_1}(0)$$

for a function Ψ of $\{B_t\}_{t\in[0,1]}$, where $p_{\mathbb{X}_1}$ is the density function of $\mathbb{X}_1$. Ψ is generally too complicated to write down but some examples of calculation will be shown in Section 6.2.

6.2 Examples

We show two examples of calculating the coefficients in the asymptotic expansion in Theorem 6.1. One is the Heisenberg group, which is a trivial example and all higher terms of the expansion vanish. The other is the CR sphere, in which some non-trivial higher terms appear. We refer to [3], [5] and references therein for some fundamental facts on these two examples.

6.2.1 The Heisenberg group

Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Heisenberg group with a coordinate system (z,t), $z = (z^1,\dots,z^n) \in \mathbb{C}^n$, $t \in \mathbb{R}$. $\mathbb{H}^n$ is a strictly pseudoconvex CR manifold whose CR structure is

$$T_{1,0} = \bigoplus_{\alpha=1}^{n} \mathbb{C}Z_\alpha, \quad Z_\alpha = \frac{\partial}{\partial z^\alpha} + \mathrm{i}\overline{z^\alpha}\frac{\partial}{\partial t}.$$

Note that $\{Z_\alpha\}_{\alpha=1,\dots,n}$ is a global orthonormal frame for $T_{1,0}$. It is easy to see that the associated covariant derivation is a null mapping, i.e.

$$\nabla_{AB}^{C} = 0 \quad \text{for any } A, B, C \in \{0, 1, \dots, n, \overline{1}, \dots, \overline{n}\}.$$

The structure functions introduced in Section 3.3 are

$$C_{\alpha,n+\alpha}^{2n+1} = 2, \qquad C_{n+\alpha,\alpha}^{2n+1} = -2, \quad \alpha = 1,\dots,n$$

and other C_{jk}^{i}'s are zero. Then we have

$$\widehat{X}_\alpha = \frac{\partial}{\partial u^\alpha} - u^{n+\alpha}\frac{\partial}{\partial u^{2n+1}}, \qquad \widehat{X}_{n+\alpha} = \frac{\partial}{\partial u^{n+\alpha}} + u^\alpha\frac{\partial}{\partial u^{2n+1}}, \quad \alpha = 1,\dots,n,$$

therefore we see that $\varphi_t^a = 0$ for any $a \geq 2$. It follows that $C_a^{\mathbb{H}_n} = 0$ for $a \geq 1$.

Remark 6.21. This result also follows immediately from [7, Théorèm 1], which says $p(t,x,x) = t^{-n-1}\dfrac{1}{(2\pi)^{n+1}}\displaystyle\int_{\mathbb{R}}\left(\frac{2\tau}{\sinh 2\tau}\right)^n d\tau$.

6.2.2 The CR sphere

Let

$$S^{2n+1} = \left\{(z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1}; \sum_{\alpha=1}^{n+1} |z^\alpha|^2 = 1\right\} \subset \mathbb{C}^{n+1}$$

be the $(2n+1)$-dimensional unit sphere which we regard as a submanifold of $\mathbb{C}^{n+1}$. S^{2n+1} has a CR structure

$$\begin{aligned}(T_{1,0})_z &= (T^{1,0}\mathbb{C}^{n+1})_z \cap \mathbb{C}T_z S^{2n+1} \\ &= \left\{\sum_{\alpha=1}^{n+1} v^\alpha \frac{\partial}{\partial z^\alpha}; \sum_{\alpha=1}^{n} v^\alpha \overline{z^\alpha} = 0\right\}, \quad z = (z^1, \dots, z^{n+1}) \in S^{2n+1}.\end{aligned}$$

We can take

$$\theta = \mathrm{i}\sum_{\alpha=1}^{n+1}(-\overline{z^\alpha} dz^\alpha + z^\alpha d\overline{z^\alpha}),$$

$$T = \frac{\mathrm{i}}{2}\sum_{\alpha=1}^{n+1}\left(z^\alpha \frac{\partial}{\partial z^\alpha} - \overline{z^\alpha}\frac{\partial}{\partial \overline{z^\alpha}}\right)$$

and then the Levi form is

$$L_\theta = 2\sum_{\alpha=1}^{n+1} dz^\alpha \wedge d\overline{z^\alpha},$$

which is positive definite, thus S^{2n+1} is strictly pseudoconvex.

Set

$$T_\alpha = \frac{\partial}{\partial z^\alpha} - \overline{z^\alpha}\sum_{\beta=1}^{n+1} z^\beta \frac{\partial}{\partial z^\beta} \in T_{1,0}$$

and $T_{\overline{\alpha}} = \overline{T_\alpha}$, then we have

$$L_\theta(T_\alpha, T_{\overline{\beta}}) = \delta_{\alpha\beta} - \overline{z^\alpha} z^\beta.$$

In the open subset $U = \{z; z^{n+1} \neq 0\}$, if we set

$$Z_\alpha = T_\alpha - \frac{\overline{z^\alpha} z^{n+1}}{|z^{n+1}|(1+|z^{n+1}|)} T_{n+1}, \quad \alpha = 1, \dots, n,$$

then $\{Z_\alpha\}_{\alpha=1,\dots,n}$ is a local orthonormal frame for $T_{1,0}$ on U. A direct calculation shows that for $\alpha, \beta = 1, \dots, n$,

$$\begin{aligned}
[Z_\alpha, Z_\beta] &= \frac{1}{1+|z^{n+1}|}(-\overline{z^\beta} Z_\alpha + \overline{z^\alpha} Z_\beta), \\
[Z_\alpha, Z_{\overline{\beta}}] &= \left(\frac{\delta_{\alpha\beta}}{1+|z^{n+1}|} + \frac{\overline{z^\alpha} z^\beta}{2|z^{n+1}|(1+|z^{n+1}|)^2} \right) \sum_{\gamma=1}^{n} (z^\gamma Z_\gamma + \overline{z^\gamma} Z_{\overline{\gamma}}) \\
&\qquad - 2\mathrm{i}\delta_{\alpha\beta} T, \qquad (6.22) \\
[T, Z_\alpha] &= -\frac{\mathrm{i}}{2} Z_\alpha.
\end{aligned}$$

By rewriting these equalities in terms of $\widehat{X}_i$, we have the structure functions:

$$\begin{aligned}
C_{\alpha\beta}^{\gamma} &= \frac{1}{\sqrt{2}} \left(\frac{-\delta_{\alpha\gamma} x^\beta + \delta_{\beta\gamma} x^\alpha}{1+|z^{n+1}|} + \frac{(x^\alpha y^\beta - x^\beta y^\alpha) y^\gamma}{|z^{n+1}|(1+|z^{n+1}|)^2} \right), \\
C_{\alpha\beta}^{n+\gamma} &= \frac{1}{\sqrt{2}} \left(\frac{-\delta_{\alpha\gamma} y^\beta + \delta_{\beta\gamma} y^\alpha}{1+|z^{n+1}|} + \frac{(x^\alpha y^\beta - x^\beta y^\alpha) x^\gamma}{|z^{n+1}|(1+|z^{n+1}|)^2} \right), \\
C_{n+\alpha,n+\beta}^{\gamma} &= \frac{1}{\sqrt{2}} \left(\frac{\delta_{\alpha\gamma} x^\beta - \delta_{\beta\gamma} x^\alpha}{1+|z^{n+1}|} + \frac{(x^\alpha y^\beta - x^\beta y^\alpha) y^\gamma}{|z^{n+1}|(1+|z^{n+1}|)^2} \right), \\
C_{n+\alpha,n+\beta}^{n+\gamma} &= \frac{1}{\sqrt{2}} \left(\frac{\delta_{\alpha\gamma} y^\beta - \delta_{\beta\gamma} y^\alpha}{1+|z^{n+1}|} + \frac{(x^\alpha y^\beta - x^\beta y^\alpha) x^\gamma}{|z^{n+1}|(1+|z^{n+1}|)^2} \right), \\
C_{\alpha,n+\beta}^{\gamma} &= \frac{1}{\sqrt{2}} \left(\frac{\delta_{\alpha\gamma} y^\beta - \delta_{\beta\gamma} y^\alpha + 2\delta_{\alpha\beta} y^\gamma}{1+|z^{n+1}|} + \frac{(x^\alpha x^\beta + y^\alpha y^\beta) y^\gamma}{|z^{n+1}|(1+|z^{n+1}|)^2} \right), \\
C_{\alpha,n+\beta}^{n+\gamma} &= \frac{1}{\sqrt{2}} \left(\frac{-\delta_{\alpha\gamma} x^\beta + \delta_{\beta\gamma} x^\alpha + 2\delta_{\alpha\beta} x^\gamma}{1+|z^{n+1}|} + \frac{(x^\alpha x^\beta + y^\alpha y^\beta) x^\gamma}{|z^{n+1}|(1+|z^{n+1}|)^2} \right),
\end{aligned}$$

$$C_{\alpha\beta}^{2n+1} = 0, \qquad C_{n+\alpha,n+\beta}^{2n+1} = 0, \qquad C_{\alpha,n+\beta}^{2n+1} = 2\delta_{\alpha\beta},$$

$$C_{2n+1,\alpha}^{\beta} = 0, \qquad C_{2n+1,\alpha}^{n+\beta} = \frac{\delta_{\alpha\beta}}{2}, \qquad C_{2n+1,n+\alpha}^{\beta} = -\frac{\delta_{\alpha\beta}}{2}, \qquad C_{2n+1,n+\alpha}^{n+\beta} = 0,$$

where $\alpha, \beta, \gamma \in \{1, \dots, n\}$ and $x^\alpha = \mathrm{Re} z^\alpha$, $y^\alpha = \mathrm{Im} z^\alpha$.

We can also calculate Christoffel symbols, that is, by comparing (6.22) and (3.15), we have

$$\Gamma^{\gamma}_{\overline{\beta}\alpha} = \left(\frac{\delta_{\alpha\beta}}{1+|z^{n+1}|} + \frac{\overline{z^{\alpha}} z^{\beta}}{2|z^{n+1}|(1+|z^{n+1}|)^2} \right) z^{\gamma},$$
$$\Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}} = \left(\frac{\delta_{\alpha\beta}}{1+|z^{n+1}|} + \frac{\overline{z^{\alpha}} z^{\beta}}{2|z^{n+1}|(1+|z^{n+1}|)^2} \right) \overline{z^{\gamma}}$$

for $\alpha, \beta, \gamma \in \{1, \ldots, n\}$. In particular, we have

$$\Gamma^{\alpha}_{\overline{\beta}\beta} = \left(\frac{1}{1+|z^{n+1}|} + \frac{|z^{\beta}|^2}{2|z^{n+1}|(1+|z^{n+1}|)^2} \right) z^{\alpha}, \quad \alpha, \beta = 1, \ldots, n.$$

Now let $x = (0, \ldots, 0, 1) \in S^{2n+1}$ and we consider the Folland-Stein normal coordinate $\{u^j\}_{j=1,\ldots,2n+1}$ around x. Theorem 3.9 and the calculation of C^i_{jk}'s above show that

$$\begin{aligned}
\widehat{X}_{\alpha} = {} & \frac{\partial}{\partial u^{\alpha}} - u^{n+\alpha} \frac{\partial}{\partial u^{2n+1}} + \frac{1}{12} \sum_{j=1}^{2n} (u^j)^2 \frac{\partial}{\partial u^{\alpha}} + \frac{1}{4} u^{2n+1} \frac{\partial}{\partial u^{n+\alpha}} \\
& - \sum_{j=1}^{2n} \left(\frac{1}{12} u^{\alpha} u^j + O^3 \right) \frac{\partial}{\partial u^j} \\
& + \left(\frac{1}{12} u^{\alpha} u^{2n+1} - \frac{1}{8} u^{n+\alpha} \sum_{j=1}^{2n} (u^j)^2 + O^4 \right) \frac{\partial}{\partial u^{2n+1}}, \\
\widehat{X}_{n+\alpha} = {} & \frac{\partial}{\partial u^{n+\alpha}} - u^{\alpha} \frac{\partial}{\partial u^{2n+1}} - \frac{1}{4} u^{2n+1} \frac{\partial}{\partial u^{\alpha}} + \frac{1}{12} \sum_{j=1}^{2n} (u^j)^2 \frac{\partial}{\partial u^{n+\alpha}} \\
& - \sum_{j=1}^{2n} \left(\frac{1}{12} u^{n+\alpha} u^j + O^3 \right) \frac{\partial}{\partial u^j} \\
& + \left(\frac{1}{12} u^{n+\alpha} u^{2n+1} + \frac{1}{8} u^{\alpha} \sum_{j=1}^{2n} (u^j)^2 + O^4 \right) \frac{\partial}{\partial u^{2n+1}}.
\end{aligned}$$

We also note that $\Gamma^{\alpha}_{\overline{\beta}\beta}(0) = \Gamma^{\overline{\alpha}}_{\beta\overline{\beta}}(0) = 0$. Then we have

$$\varphi^{2,i}_t = 0, \qquad i = 1, \ldots, 2n+1.$$

From this, Theorem 6.1 and Proposition 6.18 show that

$$c_1^{S^{2n+1}}(x) = \sum_{i=1}^{2n+1} \mathbb{E}[\partial^i \delta_0(\mathbb{X}_1)\varphi_1^{3,i}].$$

We also have

$$\begin{aligned}
\varphi_t^{3,\alpha} &= \sum_{j=1}^{2n}\left(\frac{1}{6}B_t^{(j,j,\alpha)} - \frac{1}{12}B_t^{(j,\alpha,j)} - \frac{1}{12}B_t^{(\alpha,j,j)}\right) \\
&\qquad + \sum_{\beta=1}^{n}\left(\frac{1}{4}B_t^{(n+\beta,\beta,n+\alpha)} - \frac{1}{4}B_t^{(\beta,n+\beta,n+\alpha)}\right) - \frac{n}{2}B_t^{(\alpha,0)}, \\
\varphi_t^{3,n+\alpha} &= \sum_{j=1}^{2n}\left(\frac{1}{6}B_t^{(j,j,\alpha)} - \frac{1}{12}B_t^{(j,\alpha,j)} - \frac{1}{12}B_t^{(\alpha,j,j)}\right) \\
&\qquad + \sum_{\beta=1}^{n}\left(-\frac{1}{4}B_t^{(n+\beta,\beta,\alpha)} + \frac{1}{4}B_t^{(\beta,n+\beta,\alpha)}\right) - \frac{n}{2}B_t^{(\alpha,0)}, \\
\varphi_t^{3,2n+1} &= \sum_{\alpha=1}^{n}\sum_{j=1}^{2n}\left(\frac{5}{12}B_t^{(j,j,\alpha,n+\alpha)} - \frac{5}{12}B_t^{(j,j,n+\alpha,\alpha)}\right. \\
&\qquad + \frac{1}{6}B_t^{(j,\alpha,j,n+\alpha)} - \frac{1}{6}B_t^{(j,n+\alpha,j,\alpha)} \\
&\qquad - \frac{1}{12}B_t^{(n+\alpha,j,\alpha,j)} + \frac{1}{12}B_t^{(\alpha,j,n+\alpha,j)} \\
&\qquad \left. - \frac{1}{12}B_t^{(j,n+\alpha,\alpha,j)} + \frac{1}{12}B_t^{(j,\alpha,n+\alpha,j)}\right) \\
&\quad + \sum_{\alpha=1}^{n}\left(-\frac{n}{2}B_t^{(\alpha,0,n+\alpha)} + \frac{n}{2}B_t^{(n+\alpha,0,\alpha)}\right).
\end{aligned}$$

Let us describe $c_1^{S^{2n+1}}(x)$ more specifically using modified derivative operator.

Let $W^{2n} = \{w = (w^1, \dots, w^{2n}) \colon [0,1] \to \mathbb{R}^{2n}; \text{continuous}, w(0) = 0\}$ be the $2n$-dimensional Wiener space and our Brownian motion is regarded as $B^i(w) = w^i$, $w \in W^{2n}$. Let H be the Cameron-Martin subspace of W^{2n} and $\nabla \colon \mathbb{D}^{-\infty}(\mathbb{R}^{2n}) \to \mathbb{D}^{-\infty}(H \otimes \mathbb{R}^{2n})$, $\nabla^* \colon \mathbb{D}^{-\infty}(H \otimes \mathbb{R}^{2n}) \to \mathbb{D}^{-\infty}(\mathbb{R}^{2n})$ be the derivation of generalized Wiener functionals and its dual respectively. Hereinafter, the inner product in H is denoted by $\langle , \rangle$.

Set $S_t = \sum_{\alpha=1}^{n} S_t^\alpha$, then $\mathbb{X}_1 = (B_1^1, \dots, B_1^{2n}, S_1)$ yields $\nabla \mathbb{X}_1 = (\ell_1^1, \dots, \ell_1^{2n}, \nabla S_1)$, where $\ell_1^i \colon W^{2n} \to \mathbb{R}; w \mapsto w^i$, $i = 1, \dots, 2n$ is regarded as a H-valued constant function via $(W^{2n})^* \subset H^* \simeq H$.

Then for $i = 1, \dots, 2n$,

$$\langle \nabla \delta_0(\mathbb{X}_1), \ell_1^i \rangle = \partial^{(i)} \delta_0(\mathbb{X}_1) + \partial^{(2n+1)} \delta_0(\mathbb{X}_1) \langle \nabla S_1, \ell_1^i \rangle.$$

Therefore we have

$$\mathbb{E}[\partial^{(i)} \delta_0(\mathbb{X}_1)\Psi] = \mathbb{E}[\delta_0(\mathbb{X}_1)\nabla^*(\Psi \ell_1^i) - \partial^{(2n+1)} \delta_0(\mathbb{X}_1)\langle S_1, \ell_1^i \rangle \Psi].$$

Define $\widehat{\nabla} \colon \mathbb{D}^{-\infty}(\mathbb{R}^{2n}) \to \mathbb{D}^{-\infty}(H \otimes \mathbb{R}^{2n})$ by $\widehat{\nabla} F = \nabla F - \sum_{i=1}^{2n} \langle \nabla F, \ell_1^i \rangle \ell_1^i$. Then we have $\widehat{\nabla}(\delta_0(\mathbb{X}_1)) = \partial^{(2n+1)} \delta_0(\mathbb{X}_1) \widehat{\nabla} S_1$ and

$$\begin{aligned} \mathbb{E}[\partial^{(2n+1)} \delta_0(\mathbb{X}_1)\Psi] &= \mathbb{E}[\langle \widehat{\nabla}(\delta_0(\mathbb{X}_1)), \widehat{\nabla} S_1 \rangle \|\widehat{\nabla} S_1\|^{-2} \Psi] \\ &= \mathbb{E}[\delta_0(\mathbb{X}_1)\nabla^*(\|\widehat{\nabla} S_1\|^{-2} \widehat{\nabla} S_1 \Psi)]. \end{aligned}$$

A direct calculation shows

$$\begin{aligned} \widehat{\nabla} S_1^\alpha(t) &= -2\int_0^t B_s^{n+\alpha} ds + 2t \int_0^1 B_s^{n+\alpha} ds, \\ \widehat{\nabla} S_1^{n+\alpha}(t) &= 2\int_0^t B_s^\alpha ds - 2t \int_0^1 B_s^\alpha ds \end{aligned}$$

for $\alpha = 1, \dots, n$. Combining these and performing a direct but lengthy calculation, we can write the coefficient $c_1^{S^{2n+1}}(x)$ as follows:

Proposition 6.23. *For $x = (0, \dots, 0, 1) \in S^{2n+1}$, we have $c_1^{S^{2n+1}}(x) = \mathbb{E}[\delta_0(\mathbb{X}_1)\Phi]$, where*

$$\Phi = \frac{1}{6}\sum_{i,j=1}^{2n} \Phi_{jji}^i - \frac{1}{12}\sum_{i,j=1}^{2n} \Phi_{jij}^i - \frac{1}{12}\sum_{i,j=1}^{2n} \Phi_{ijj}^i$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{\alpha,\beta=1}^{n}\Phi^{\alpha}_{n+\beta,\beta,n+\alpha}-\frac{1}{4}\sum_{\alpha,\beta=1}^{n}\Phi^{\alpha}_{\beta,n+\beta,n+\alpha}\\
&-\frac{1}{4}\sum_{\alpha,\beta=1}^{n}\Phi^{n+\alpha}_{n+\beta,\beta,\alpha}+\frac{1}{4}\sum_{\alpha,\beta=1}^{n}\Phi^{n+\alpha}_{\beta,n+\beta,\alpha}\\
&-\frac{n}{2}\sum_{i=1}^{2n}\Phi^{i}_{i,2n+1}-\frac{n}{2}\sum_{\alpha=1}^{n}(\Phi^{2n+1}_{\alpha,2n+1,n+\alpha}-\Phi^{2n+1}_{n+\alpha,2n+1,\alpha})\\
&+\frac{5}{12}\sum_{\alpha=1}^{n}\sum_{j=1}^{2n}(\Phi^{2n+1}_{j,j,\alpha,n+\alpha}-\Phi^{2n+1}_{j,j,n+\alpha,\alpha})+\frac{1}{6}\sum_{\alpha=1}^{n}\sum_{j=1}^{2n}(\Phi^{2n+1}_{j,\alpha,j,n+\alpha}-\Phi^{2n+1}_{j,n+\alpha,j,\alpha})\\
&+\frac{1}{6}\sum_{\alpha=1}^{n}\sum_{j=1}^{2n}(\Phi^{2n+1}_{\alpha,j,j,n+\alpha}-\Phi^{2n+1}_{n+\alpha,j,j,\alpha})+\frac{1}{12}\sum_{\alpha=1}^{n}\sum_{j=1}^{2n}(\Phi^{2n+1}_{\alpha,j,n+\alpha,j}-\Phi^{2n+1}_{n+\alpha,j,\alpha,j})\\
&+\frac{1}{12}\sum_{\alpha=1}^{n}\sum_{j=1}^{2n}(\Phi^{2n+1}_{j,\alpha,n+\alpha,j}-\Phi^{2n+1}_{j,n+\alpha,\alpha,j})-\frac{1}{6}\sum_{\alpha=1}^{n}\sum_{j=1}^{2n}(\Phi^{2n+1}_{\alpha,n+\alpha,j,j}-\Phi^{2n+1}_{n+\alpha,\alpha,j,j}),
\end{aligned}
$$

$$
\mathbb{E}[\partial^{(i)}\delta_0(\mathbb{X}_1)B_1^J]=\mathbb{E}[\delta_0(\mathbb{X}_1)\Phi^i_J],\quad i=1,\dots,2n{+}1, J=(j_1,\dots,j_a), j_b=0,\dots,2n.
$$

Moreover, for $i,j,k,l=1,\dots,2n$ we have

$$
\begin{aligned}
\Phi^i_{jkl}&=-\delta_{ij}B_1^{(0,k,l)}-\delta_{ik}B_1^{(j,0,l)}-\delta_{il}B_1^{(j,k,0)}+(\kappa_1^i\kappa_4+\kappa_2^i\kappa_3)B_1^{(j,k,l)}\\
&\quad-2\kappa_1^i\kappa_3\bigg(\sigma(j)B_1^{(\sigma(j)n+j,0,k,l)}+\sigma(k)\int_0^1\int_0^s B_u^jB_u^{\sigma(k)n+k}du\circ dB_s^l\\
&\qquad+\sigma(l)\int_0^1 B_s^{(j,k)}B_s^{\sigma(l)n+l}ds\bigg),\\
\Phi^i_{i,0}&=(\kappa_1^i\kappa_4+\kappa_2^i\kappa_3)B_1^{(i,0)}-2\sigma(i)\kappa_1^i\kappa_3B_1^{(\sigma(i)n+i,0)},\\
\Phi^{2n+1}_{ijkl}&=-\kappa_4B_1^{(i,j,k,l)}\\
&\quad+2\kappa_3\bigg(\sigma(i)B_1^{(\sigma(i)n+i,0,j,k,l)}\\
&\qquad+\sigma(j)\int_0^1\int_0^s\int_0^u B_v^iB_v^{\sigma(j)n+j}dv\circ dB_v^k\circ dB_s^l\\
&\qquad+\sigma(k)\int_0^1\int_0^s B_u^{(i,j)}B_u^{\sigma(k)n+k}du\circ dB_s^l\\
&\qquad+\sigma(l)\int_0^1 B_s^{(i,j,k)}B_s^{\sigma(l)n+l}ds\bigg),
\end{aligned}
$$

$$\Phi^{2n+1}_{i,0,j} = -\kappa_4 B_1^{(i,0,j)} + 2\kappa_3\left(\sigma(i) B_1^{(\sigma(i)n+i,0,0,j)} + \sigma(j)\int_0^1 B_s^{(i,0)} B_s^{\sigma(j)n+j} ds\right),$$

where

$$\sigma(i) = \begin{cases} 1, & i = 1, \dots, n, \\ -1, & i = n+1, \dots, 2n, \end{cases}$$

$$\begin{aligned}
\kappa_1^i &= -2\sigma(i) B_1^{(\sigma(i)n+i,0)}, \\
\kappa_2^i &= 2B_1^{(i,0)} - 4B_1^{(i,0,0)}, \\
\kappa_3 &= \left(4\sum_{i=1}^{2n}\left(\int_0^1 (B_t^i)^2 dt - (B_1^{(i,0)})^2\right)\right)^{-1}, \\
\kappa_4 &= -8(\kappa_3)^2 \sum_{\alpha=1}^{n}\left(-2\int_0^1 B_t^{(n+\alpha,0)} B_t^{\alpha} dt + 2\int_0^1 B_t^{(\alpha,0)} B_t^{n+\alpha} dt\right) \\
&\quad + 2\sum_{\alpha=1}^{n}(B_1^{(\alpha,0)} B_1^{(n+\alpha,0,0)} - B_1^{(n+\alpha,0)} B_1^{(\alpha,0,0)}).
\end{aligned}$$

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