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ESTIMATING THE LARGEST MEAN OF THREE NORMAL POPULATIONS

Hong-Jie SUN* and Chooichiro ASANO**

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Let π_i be a normal population N (μ_i, σ^2) with an unknown mean μ_i and a common known variance σ^2 for $i=1, 2, 3$. We consider the problem of estimating $\overset{*}{\mu}=\max(\mu_1, \mu_2, \mu_3)$, based on the sample means X, Y, Z drawn from π_i for $i=1, 2, 3$. This paper presents two estimators for $\overset{*}{\mu}$, namely, the Pitman estimator $\hat{\mu}_p$, which has never been derived and suspected by Blumenthal (1984) to be very complex so that Monte Carlo simulation is only a possible way to investigate its performances, and the estimator $\hat{\mu}$, which is derived by considering the class of linear combination of X, Y, and Z. Numerical comparisons of $\hat{\mu}_p$, $\hat{\mu}$ and an ordinal estimator $\hat{\mu}_m=\max(X, Y, Z)$ are made on the criteria of the bias (BIAS) and mean square error (MSE).

1. Introduction

Let π_i be a normal population N (μ_i, σ^2) with an unknown mean μ_i and a common known variance σ^2 for $i=1, 2, 3$, and let

$$\overset{*}{\mu}=\max(\mu_1, \mu_2, \mu_3), \tag{1}$$

which denotes the largest mean. Let X, Y, Z be the sample means drawn from π_i for $i=1$, 2, 3. The problem in the present paper is to estimate $\overset{*}{\mu}$ based on X, Y, Z. For two normal populations, the similar problem has been considered by Blumenthal and Cohen (1968a, b), Dudewicz (1971), Dhariyal, Dudewicz and Blumenthal (1982), Sun and Asano (1987). For more than three populations, this problem becomes very complex. Blumenthal (1984) stated that the Pitman estimator and the maximum likelihood estimator have never been derived for three populations, and suspected that Monte Carlo simulation is only a possible way to investigate their performances for estimating $\overset{*}{\mu}$. In fact, if one follows the lines of Blumenthal and Cohen (1968b) to derive the Pitman estimator, it is almost impossible. This is the reason that max (X_1, X_2) can be simply represented for two independent normal populations as $(X_1+X_2)/2+|X_1-X_2|/2$, and the distributions of $(X_1+X_2)/2$ and $|X_2-X_2|/2$ can be simply obtained. For more than three populations, although the estimator $\max(X_1,\cdots, X_n)$ can be simply represented as a sum of the summation of X_i's and the absolute differences of X_i's and X_j's, but it is very hard to obtain the correspondent distributions. In this paper, we derive the Pitman estimator in a general way and show that the Pitman estimator can be simply obtained in section 2. In section 3, we derive another estimator $\hat{\mu}$ by considering the class of linear combination of X, Y, and Z. In section 4, numerical com-

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parisons among $\hat{\mu}_p$, $\hat{\mu}$ and an ordinal estimator $\hat{\mu}_m = \max(X, Y, Z)$ are made on BIAS and MSE. In section 5, we make some concluding remarks.

2. Derivation of the Pitman estimator $\hat{\mu}_p$

The largest mean $\overset{*}{\mu}$ defined by (1) has symmetric and location translation invariant properties, *i. e.*

$$\overset{*}{\mu}(\mu_1, \mu_2, \mu_3) = \overset{*}{\mu}(\mu_2, \mu_1, \mu_3) = \cdots = \overset{*}{\mu}(\mu_3, \mu_2, \mu_1), \tag{2a}$$
$$\overset{*}{\mu}(\mu_1 + a,\ \mu_2 + a,\ \mu_3 + a) = \overset{*}{\mu}(\mu_1, \mu_2, \mu_3) + a, \qquad a \in \boldsymbol{R}. \tag{2b}$$

It is natural for us to consider estimators which are of the above properties. It is well known that Pitman estimator is suitable to solve this kind of problem. Here we derive the Pitman estimator $\hat{\mu}_p$ for $\overset{*}{\mu}$. The idea is that we first obtain the cumulative distribution function of

$$T = \max(X, Y, Z), \tag{3}$$

and then calculate its expectation, because Pitman estimator is unbiased for the location parameter, so is to $E(T)$. By substituting the sufficient statistics X, Y, Z to μ_1, μ_2, μ_3 in $E(t)$, we can then obtain the Pitman estimator $\hat{\mu}_p$. The following are some calculations.

The cumulative distribution function of T, $F(t)$, is given by

$$F(t) \equiv Pr\{T \leqq t\} = Pr\{\max(X, Y, Z) \leqq t\} = Pr\{X \leqq t,\ Y \leqq t,\ Z \leqq t\}$$
$$= Pr\{X \leqq t\}\, Pr\{Y \leqq t\}\, Pr\{Z \leqq t\} = \prod_{i=1}^{3} \Phi\left(\frac{t - \mu_i}{\sigma}\right), \tag{4a}$$

where $\Phi(x) = 1/\sqrt{2\pi}\int_{-\infty}^{x} \exp(-\frac{1}{2}t^2)dt$. The density function of T is

$$f(t) = \frac{dF(t)}{dt} = \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{3} f(t; \mu_i)\,\Phi\left(\frac{t-\mu_j}{\sigma}\right)\Phi\left(\frac{t-\mu_k}{\sigma}\right), \tag{4b}$$

where $f(t; \mu_i)$ is the density function of X, Y, Z for $i = 1, 2, 3$, respectively. The expectation of T is

$$E(T) = \int_{-\infty}^{\infty} tf(t)dt = \int_{-\infty}^{\infty} t \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{3} f(t; \mu_i)\,\Phi\left(\frac{t-\mu_j}{\sigma}\right)\Phi\left(\frac{t-\mu_k}{\sigma}\right)dt$$

$$\mathrm{A}(\mu_1, \mu_2, \mu_3) \equiv \int_{-\infty}^{\infty} tf(t; \mu_1)\,\Phi\left(\frac{t-\mu_2}{\sigma}\right)\Phi\left(\frac{t-\mu_3}{\sigma}\right)dt$$
$$= \int_{-\infty}^{\infty} tf(t; \mu_1) \int_{-\infty}^{0}\int \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x+t-\mu_2)^2 - \frac{1}{2\sigma^2}(x+t-\mu_3)^2} dx\, dy\, dt$$
$$= \frac{1}{\sqrt{(2\pi\sigma^2)^3}} \int_{-\infty}^{\infty}\int_{-\infty}^{0}\int e^{-\frac{1}{2\sigma^2}(t+x-\mu_2)^2 - \frac{1}{2\sigma^2}(t+y-\mu_3)^2 - \frac{1}{2\sigma^2}(t-\mu_1)^2} dx\, dy\, dt$$
$$= \int_{-\infty}^{0}\int \frac{\mu_1}{2\pi\sigma^2\sqrt{3}} e^{-\frac{1}{3\sigma^2}\{(x-\mu_2+\mu_1)^2 - (x-\mu_2+\mu_1)(y-\mu_3+\mu_1) + (y-\mu_3+\mu_1)^2\}} dx\, dy$$

$$-\int_{-\infty}^{\mu_1-\mu_2}\int_{-\infty}^{\mu_1-\mu_3}\frac{x+y}{6\pi\sigma^2\sqrt{3}}e^{-\frac{1}{3\sigma^2}[x^2-xy+y^2]}dy\,dx$$

$$=\mu_1\Phi(\frac{\mu_1-\mu_2}{\sqrt{2}\sigma},\frac{\mu_1-\mu_3}{\sqrt{2}\sigma};\frac{1}{2})+\frac{\sigma}{2\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(\mu_1-\mu_2)^2}\Phi(\frac{\mu_1+\mu_2-2\mu_3}{\sqrt{6}\sigma})$$

$$+\frac{\sigma}{2\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(\mu_1-\mu_3)^2}\Phi(\frac{\mu_1+\mu_3-2\mu_2}{\sqrt{6}\sigma}),$$

where

$$\Phi(h,k;\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{h}\int_{-\infty}^{k}e^{-\frac{1}{2(1-\rho^2)}(u^2-2\rho uv+v^2)}dudv.$$

Therefore, the expectation of T is obtained as

$$E(T)=A(\mu_1,\mu_2,\mu_3)+A(\mu_2,\mu_1,\mu_3)+A(\mu_3,\mu_1,\mu_2)$$

$$=\mu_1\Phi(\frac{\mu_1-\mu_2}{\sqrt{2}\sigma},\frac{\mu_1-\mu_3}{\sqrt{2}\sigma};\frac{1}{2})+\mu_2\Phi(\frac{\mu_2-\mu_1}{\sqrt{2}\sigma},\frac{\mu_2-\mu_3}{\sqrt{2}\sigma};\frac{1}{2})$$

$$+\mu_3\Phi(\frac{\mu_3-\mu_1}{\sqrt{2}\sigma},\frac{\mu_3-\mu_2}{\sqrt{2}\sigma};\frac{1}{2})+\frac{\sigma}{\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(\mu_1-\mu_2)^2}\Phi(\frac{\mu_1+\mu_2-2\mu_3}{\sqrt{6}\sigma})$$

$$+\frac{\sigma}{\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(\mu_1-\mu_3)^3}\Phi(\frac{\mu_1+\mu_3-2\mu_2}{\sqrt{6}\sigma})+\frac{\sigma}{\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(\mu_2-\mu_3)^2}\Phi(\frac{\mu_2+\mu_3-2\mu_1}{\sqrt{6}\sigma}).$$

(5)

By substituting X_i to μ_i in (5), we obtain the Pitman estimator as follows,

$$\hat{\mu}_p=X\Phi(\frac{X-Y}{\sqrt{2}\sigma},\frac{X-Z}{\sqrt{2}\sigma};\frac{1}{2})+Y\Phi(\frac{Y-X}{\sqrt{2}\sigma},\frac{Y-Z}{\sqrt{2}\sigma};\frac{1}{2})+Z\Phi(\frac{Z-X}{\sqrt{2}\sigma},\frac{Z-Y}{\sqrt{2}\sigma};\frac{1}{2})$$

$$+\frac{\sigma}{\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(X-Y)^2}\Phi(\frac{X+Y-2Z}{\sqrt{6}\sigma})+\frac{\sigma}{\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(X-Z)^2}(\frac{X+Z-2Y}{\sqrt{6}\sigma})$$

$$+\frac{\sigma}{\sqrt{\pi}}e^{-\frac{1}{4\sigma^2}(X-Z)^2}(\frac{Y+Z-2X}{\sqrt{6}\sigma}); \qquad (6)$$

Remark 2.1. The above method can be applied to obtain the Pitman estimator for the general $n(\geqq 3)$ normal populations, but the estimator itself is a function of $n-1$, $n-2,\cdots$, 1 variates normal cumulalative functions. Since it is inconvenient to calculate normal probabilities, the Pitman estimator for $n(\geqq 3)$ becomes hard for practical use.

3. The estimator based on the linear combination of *X, Y,* and *Z*

We now consider a class of estimators

$$\hat{\mu}=\alpha_1X+\alpha_2Y+\alpha_3Z, \qquad \text{for} \qquad 0\leqq\alpha_i\leqq 1, \qquad \sum_{i=1}^{3}\alpha_i=1. \qquad (7)$$

In order to determine an optimal α_i so as to minimize

$$\mathrm{MSE}(\hat{\mu})=E(\hat{\mu}-\overset{*}{\mu})^2=E\{[\alpha_1X+\alpha_2Y+\alpha_3Z]-\overset{*}{\mu}\}^2, \tag{8}$$

by the use of Lagrange multiplier method, we can get

$$\begin{aligned}
&\alpha_i=\Delta\alpha_i/\Delta, \qquad i=1, 2, 3.\\
&\Delta=3\sigma^2+\{(\mu_1-\mu_2)^2+(\mu_1-\mu_3)^2+(\mu_2-\mu_3)^2\},\\
&\Delta\alpha_1=\sigma^2+\mu_2(\mu_2-\mu_1)+\mu_3(\mu_3-\mu_1)+(2\mu_1-\mu_2-\mu_3)\overset{*}{\mu},\\
&\Delta\alpha_2=\sigma^2+\mu_1(\mu_1-\mu_2)+\mu_3(\mu_3-\mu_2)+(2\mu_2-\mu_1-\mu_3)\overset{*}{\mu},\\
&\Delta\alpha_3=\sigma^2+\mu_1(\mu_1-\mu_3)+\mu_2(\mu_2-\mu_3)+(2\mu_3-\mu_1-\mu_2)\overset{*}{\mu}.
\end{aligned} \tag{9}$$

Substituting the statistics X, Y, Z and $\max(X, Y, Z)$ of μ_1, μ_2, μ_3 and $\overset{*}{\mu}$, respectively, we obtain

$$\hat{\mu}=\frac{\sigma^2(X+Y+Z)+\{(X-Y)^2+(X-Z)^2+(Y-Z)^2\}\max(X, Y, Z)}{3\sigma^2+\{(X-Y)^2+(X-Z)^2+(Y-Z)^2\}} \tag{10}$$

Remark 3.1. For the general $k(\geqq 2)$ normal populations, *i. e.* $N(\mu_i, \sigma^2)$ for $i=1, 2, \cdots, k$, on the basis of the above lines of consideration, the estimator of the largest mean is equal to

$$\hat{\mu}=\frac{\sigma^2\sum_{i<j=1}^{k}X_i+\max(X_1,\cdots,X_k)\sum_{i<j=1}^{k}(X_i-X_j)^2}{k\sigma^2+\sum_{i<j=1}^{k}(X_i-x_j)^2}. \tag{11}$$

4. Comparisons among $\hat{\mu}_p$, $\hat{\mu}$ and $\hat{\mu}_m$

In this section, we compare the performances of $\hat{\mu}_p$, $\hat{\mu}$ with the ordinal estimator

$$\hat{\mu}_m=\max(X, Y, Z), \tag{12}$$

on BIAS and MSE. We note that all the estimators are of symmetric and location translation invariant properties, *i. e.* the BIAS and MSE can be represented as

$$\begin{aligned}
&\mathrm{BIAS}(\hat{\theta})=E(\hat{\theta}-\theta)=\int\hat{\theta}\prod_{i=1}^{3}N(\theta_i-\theta, \sigma^2)dx,\\
&\mathrm{MSE}(\hat{\theta})=E(\hat{\theta}-\theta)^2=\int\hat{\theta}^2\prod_{i=1}^{3}N(\theta_i-\theta, \sigma^2)dx,
\end{aligned} \tag{13}$$

for the estimator $\hat{\theta}$ of θ.

Comparisons are made in both cases of

$$\mu_{[1]}=\mu_{[2]}\leqq\mu_{[3]} \text{ and } \mu_{[1]}\leqq\mu_{[2]}=\mu_{[3]}, \tag{14}$$

where $\mu_{[i]}$ denotes the order statistics of μ_1, μ_2, μ_3, and the two cases are referred to as

Table 1 BIAS and MSE of $\hat{\mu}_p$, $\hat{\mu}$ and $\hat{\mu}_m$ for $\mu_{[1]}=\mu_{[2]}\leqq\mu_{[3]}$

	$\hat{\mu}_p$		$\hat{\mu}$		$\hat{\mu}_m$	
μ^*	BIAS	MSE	BIAS	MSE	BIAS	MSE
0.00	1.19683	1.86184	0.55009	0.85546	0.83186	1.26633
0.20	1.06754	1.57115	0.42716	0.73721	0.71772	1.07880
0.40	0.94655	1.33390	0.31273	0.66378	0.60357	0.93694
0.60	0.83409	1.14474	0.20600	0.62981	0.48942	0.84073
0.80	0.73030	0.99816	0.11872	0.62717	0.40716	0.78082
1.00	0.63522	0.88869	0.04225	0.64803	0.33197	0.75068
1.20	0.54876	0.81096	−0.02713	0.68864	0.25678	0.75062
1.40	0.47077	0.75982	−0.08143	0.73712	0.20270	0.75917
1.60	0.40098	0.73044	−0.12404	0.78917	0.16080	0.77518
1.80	0.33903	0.71839	−0.16117	0.84685	0.11891	0.80795
2.00	0.28451	0.71966	−0.18922	0.90134	0.08698	0.84025
2.20	0.23694	0.73076	−0.20699	0.94596	0.06757	0.86192
2.40	0.19580	0.74868	−0.22125	0.98916	0.04816	0.89136
2.60	0.16052	0.77090	−0.23138	1.02784	0.03167	0.92145
2.80	0.13054	0.79539	−0.23437	1.05253	0.02428	0.93482
3.00	0.10530	0.82056	−0.23556	1.07413	0.01689	0.95114
3.20	0.08424	0.84522	−0.23539	1.09326	0.00971	0.96978
3.40	0.06684	0.86851	−0.23148	1.10144	0.00736	0.97558
3.60	0.05258	0.88988	−0.22688	1.10730	0.00507	0.98214
3.80	0.04101	0.90902	−0.22191	1.11167	0.00278	0.98962
4.00	0.03172	0.92580	−0.21589	1.11160	0.00185	0.99268
4.20	0.02432	0.94023	−0.20960	1.10968	0.00128	0.99468
4.40	0.01848	0.95243	−0.20338	1.10717	0.00071	0.99691
4.60	0.01392	0.96258	−0.19714	1.10356	0.00038	0.99824
4.80	0.01040	0.97089	−0.19094	1.09911	0.00027	0.99872
5.00	0.00770	0.97761	−0.18498	1.09461	0.00015	0.99923
5.20	0.00565	0.98297	−0.17925	1.09011	0.00006	0.99966
5.40	0.00410	0.98718	−0.17372	1.08548	0.00005	0.99975
5.60	0.00296	0.99046	−0.16845	1.08104	0.00003	0.99984
5.80	0.00211	0.99297	−0.16343	1.07683	0.00001	0.99994
6.00	0.00149	0.99487	−0.15865	1.07278	0.00001	0.99996
6.20	0.00105	0.99630	−0.15410	1.06897	0.00000	0.99997
6.40	0.00073	0.99735	−1.14977	1.06539	0.00000	0.99999
6.60	0.00050	0.99813	−0.14566	1.06204	0.00000	1.00000
6.80	0.00034	0.99869	−0.14174	1.05890	0.00000	1.00000
7.00	0.00023	0.99909	−0.13802	1.05597	0.00000	1.00000
7.20	0.00015	0.99938	−0.13447	1.05323	0.00000	1.00000
7.40	0.00010	0.99958	−0.13109	1.05067	0.00000	1.00000
7.60	0.00007	0.99971	−0.12786	1.04827	0.00000	1.00000
7.80	0.00004	0.99981	−0.12478	1.04603	0.00000	1.00000
8.00	0.00003	0.99987	−0.12184	1.04394	0.00000	1.00000
8.20	0.00002	0.99992	−0.11903	1.04197	0.00000	1.00000
8.40	0.00001	0.99995	−0.11634	1.04013	0.00000	1.00000
8.60	0.00001	0.99997	−0.11376	1.03841	0.00000	1.00000
8.80	0.00000	0.99998	−0.11130	1.03678	0.00000	1.00000
9.00	0.00000	0.99999	−0.10893	1.03526	0.00000	1.00000
9.20	0.00000	0.99999	−0.10663	1.03382	0.00000	1.00000
9.40	0.00000	0.99999	−0.10448	1.03247	0.00000	1.00000
9.60	0.00000	1.00000	−0.10238	1.03120	0.00000	1.00000
9.80	0.00000	1.00000	−0.10037	1.02999	0.00000	1.00000
10.00	0.00000	1.00000	−0.09843	1.02886	0.00000	1.00000

Table 2 BIAS and MSE of $\hat{\mu}_p$, $\hat{\mu}$ and $\hat{\mu}_m$ for $\mu_{[1]} \leqq \mu_{[2]} = \mu_{[3]}$

	$\hat{\mu}_p$		$\hat{\mu}$		$\hat{\mu}_m$	
$\mathring{\mu}$	BIAS	MSE	BIAS	MSE	BIAS	MSE
0.00	1.19683	1.86184	0.55009	0.85546	0.83186	1.26633
0.20	1.13408	1.71749	0.49280	0.79744	0.78191	1.17688
0.40	1.07891	1.60069	0.44261	0.75986	0.73195	1.10742
0.60	1.03084	1.50733	0.39994	0.73932	0.68199	1.05794
0.80	0.98933	1.43365	0.37029	0.73271	0.65326	0.02760
1.00	0.95385	1.37628	0.34769	0.73505	0.62927	1.00796
1.20	0.92381	1.33224	0.33066	0.74359	0.60527	0.99791
1.40	0.89863	1.29893	0.32088	0.75540	0.59052	0.99263
1.60	0.87774	1.27414	0.31627	0.76891	0.58119	0.98988
1.80	0.86058	1.25601	0.31502	0.78301	0.57189	0.99087
2.00	0.84663	1.24300	0.31716	0.79692	0.56533	0.99270
2.20	0.83542	1.23386	0.32209	0.80944	0.56245	0.99326
2.40	0.82650	1.22759	0.32853	0.82099	0.55956	0.99498
2.60	0.81948	1.22341	0.33615	0.83127	0.55717	0.99704
2.80	0.81401	1.22072	0.34475	0.84007	0.55647	0.99750
3.00	0.80981	1.21906	0.35365	0.84785	0.55577	0.99823
3.20	0.80661	1.21810	0.36264	0.85469	0.55509	0.99921
3.40	0.80420	1.21759	0.37159	0.86060	0.55494	0.99935
3.60	0.80241	1.21737	0.38028	0.86587	0.55481	0.99955
3.80	0.80109	1.21732	0.38860	0.87062	0.55468	0.99980
4.00	0.80013	1.21737	0.39652	0.87492	0.55463	0.99988
4.20	0.79944	1.21745	0.40398	0.87889	0.55461	0.99991
4.40	0.79895	1.21755	0.41098	0.88259	0.55459	0.99996
4.60	0.79861	1.21764	0.41752	0.88608	0.55458	0.99998
4.80	0.79837	1.21773	0.42362	0.88939	0.55458	0.99999
5.00	0.79820	1.21780	0.42930	0.89225	0.55458	0.99999
5.20	0.79809	1.21785	0.43458	0.89557	0.55458	1.00000
5.40	0.79802	1.21789	0.43949	0.89847	0.55457	1.00000
5.60	0.79797	1.21792	0.44407	0.90126	0.55457	1.00000
5.80	0.79794	1.21795	0.44839	0.90394	0.55457	1.00000
6.00	0.79792	1.21796	0.45230	0.90652	0.55457	1.00000
6.20	0.79791	1.21797	0.45602	0.90900	0.55457	1.00000
6.40	0.79790	1.21798	0.45949	0.91138	0.55457	1.00000
6.60	0.79789	1.21799	0.46274	0.91367	0.55457	1.00000
6.80	0.79789	1.21799	0.46579	0.91587	0.55457	1.00000
7.00	0.79789	1.21799	0.46865	0.91798	0.55457	1.00000
7.20	0.79789	1.21799	0.47135	0.92000	0.55457	1.00000
7.40	0.79788	1.21799	0.47389	0.92195	0.55457	1.00000
7.60	0.79788	1.21799	0.47628	0.92382	0.55457	1.00000
7.80	0.79788	1.21799	0.47854	0.92561	0.55457	1.00000
8.00	0.79788	1.21799	0.48069	0.92733	0.55457	1.00000
8.20	0.79788	1.21799	0.48271	0.92899	0.55457	1.00000
8.40	0.79788	1.21799	0.48464	0.93058	0.55457	1.00000
8.60	0.79788	1.21799	0.48647	0.93211	0.55457	1.00000
8.80	0.79788	1.21799	0.48820	0.93358	0.55457	1.00000
9.00	0.79788	1.21799	0.48986	0.93499	0.55457	1.00000
9.20	0.79788	1.21799	0.49143	0.93635	0.55457	1.00000
9.40	0.79788	1.21799	0.49293	0.93766	0.55457	1.00000
9.60	0.79788	1.21799	0.49437	0.93893	0.55457	1.00000
9.80	0.79788	1.21799	0.49574	0.94014	0.55457	1.00000
10.00	0.79788	1.21799	0.49705	0.94132	0.55457	1.00000

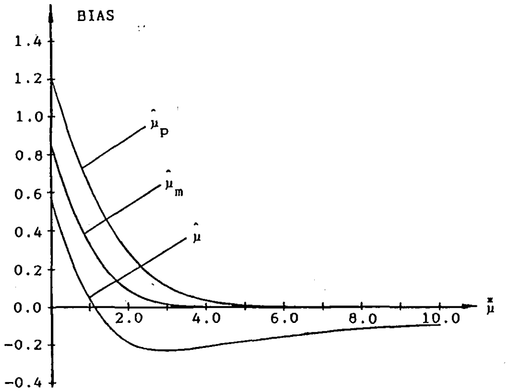

Fig. 1. BIAS of $\hat{\mu}_p$, $\hat{\mu}_m$ for $\mu_{[1]}=\mu_{[2]}\leqq\mu_{[3]}$.

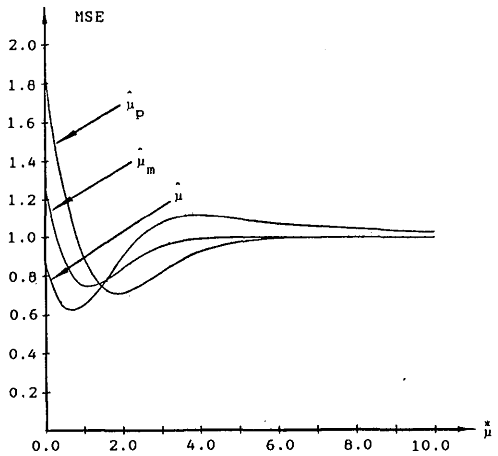

Fig. 2. MSE of $\hat{\mu}_p$, $\hat{\mu}$ and $\hat{\mu}_m$ for $\mu_{[1]}=\mu_{[2]}\leqq\mu_{[3]}$.

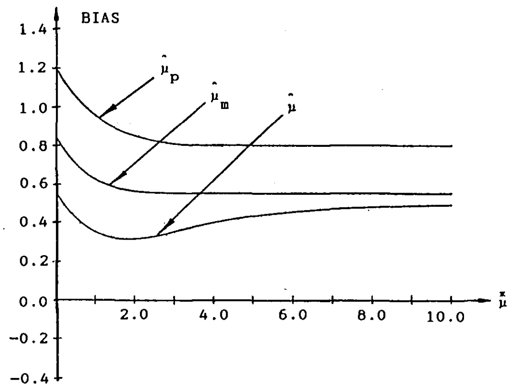

Fig. 3. BIAS of $\hat{\mu}_p$, $\hat{\mu}$ and $\hat{\mu}_m$ for $\mu_{[1]}\leqq\mu_{[2]}=\mu_{[3]}$.

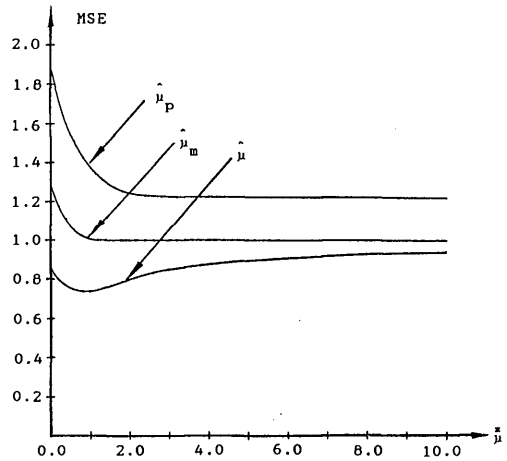

Fig. 4. MSE of $\hat{\mu}_p$, $\hat{\mu}$ and $\hat{\mu}_m$ for $\mu_{[1]}\leqq\mu_{[2]}=\mu_{[3]}$.

least favorable constructions (LFC). **Tables 1** and **2** present the computing results of BIAS and MSE for (6), (10) and (12) in the cases of (14), and **figures 1, 2, 3** and **4** the correspondent curves. Facing to these computations, we applied Gauss-Hermite 24-point formula (see, Stroud and Secret (1966)), algorithm AS 66 (Hill (1973)), algorithm 462 (Donnelly (1973)). From these tables, we make the following comments.

(i). With regard to MSE in the case of $\mu_{[1]}=\mu_{[2]}\leqq\mu_{[3]}$, $\hat{\mu}_p$ has a larger risk than $\hat{\mu}_m$ and $\hat{\mu}$ when $\mu_{[3]}$ is near to $\mu_{[1]}$ and $\mu_{[2]}$, but has a relative smaller risk than $\hat{\mu}_m$ and $\hat{\mu}$ when $\mu_{[3]}$ is far away from $\mu_{[1]}$ and $\mu_{[2]}$. While in the case of $\mu_{[1]}\leqq\mu_{[2]}=\mu_{[3]}$, $\hat{\mu}$ is better than $\hat{\mu}_m$ and jointly $\hat{\mu}_m$ is better than $\hat{\mu}_p$.

(ii). With regard to BIAS in the case of $\mu_{[1]}=\mu_{[2]}\leqq\mu_{[3]}$, $\hat{\mu}_p$ has a larger BIAS than $\hat{\mu}_m$ and $\hat{\mu}$, where the BIAS of $\hat{\mu}_p$ and $\hat{\mu}_m$ is definitely positive. While in the case of $\mu_{[1]}\leqq\mu_{[2]}=\mu_{[3]}$, $\hat{\mu}$ has a smaller BIAS than $\hat{\mu}_m$ and $\hat{\mu}_p$, and jointly $\hat{\mu}_m$ is smaller than $\hat{\mu}_p$.

5. Concluding remarks

In this paper, we derived the Pitman estimator $\hat{\mu}_p$ for estimating the largest mean for three normal populations. This method can applied to derive the Pitman estimator for more than three normal populations as noted in remark 2. 1. Besides this, we proposed an estimator $\hat{\mu}$ which is derived by considering the class of linear combination of sample means. Comparisons are made under the LFC cases among $\hat{\mu}_p$, $\hat{\mu}$ and $\hat{\mu}_m$ which is equal to the largest sample mean. As Blumenthal (1984) pointed out, the maximum likelihood estimator is hard to derive even by using the method in section 2, since there involve transcendental equations which are hard to solve.

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