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Lee, Hyen Yeal
Department of Computer Science, Pusan University : Department of Information Systems, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University

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On Finite Cellular Automata with Dynamics of Gauss-Seidel Type

Hyen Yeal Lee*

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Abstract

Representative questions of interest include the existence and number of fixed points of transition functions of finite cellular automata. In this paper we shall show some results about the above questions for finite cellular automata *GS* (m) with dynamics of Gauss-Seidel type, that is, (a) The length of maximum cycle of state diagram of a finite cellular automaton *GS* (m) is a linear function of the number m of sites, and (b) The forms of fixed points of transition function vary with boundary conditions of the automata.

1. Introduction

In general cellular automata can be defined as a spatial lattice of sites whose values at each time step are determined as a mapping of the values of the neighboring states at the previous time step. In [1] Huzino has recently explored the behaviour of cellular automata with Gauss-Seidel processing, in which he illustrated that their behaviour are different from those of usual cellular automata (with synchronous processing), if dynamics of cellular sutomata are Gauss-Seidel type. The essence of this system is as follows: a state of the cellular automaton is processed in a definite order. After that, it makes immediate use of the processed result for the next processing. This paper will explore an analysis of finite cellular automata with iterative dynamics of Gauss-Seidel type. Representative questions of interest in this context include the existence of fixed points, the number of fixed points, and length of maximum cycle. The site values are restricted to a finite set of integers, namely the variables at each site may take values 0 *or* 1, and specification of the mapping provides the rule governing dynamical behaviour of the automata. These cellular automata evolve in time according to the following rule: The value of a site at a particular time step is simply the sum *modulo* 2 of the values of its two adjacent sites on the previous time step and its left site on the current time step. The time evolution of the complete cellular automata is obtained by sequential applications of this rule at each site for each time step from left to right.

2. Cellular Automata of Gauss-Seidel Type

First of all we define a finite cellular automaton with dynamics of Gauss-Seidel type, denoted by *GS* (m), as follows:

* Department of Computer Science, Pusan Unobersity, Pusan 605, Korea. (Present Address: Department of Information Systems, Interdisciplinary Graduate School of Engeneering Sciences, Kyushu University 39, Fukuoka 816, Japan.)

Definition 2.1 *Let m be a positive integer and B a set* $\{0, 1\}$. *A finite cellular automaton GS (m) with dynamics of Gauss-Seidel type is an iterative processing system* $<B^m, \tau>$, *where* B^m *is an m-th cartesian product of B and* $\tau: B^m \rightarrow B^m$ *is a mapping from* B^m *into* B^m *defined by*

$$(\tau(b))_i = (\tau(b))_{i-1} + b_i + b_{i+1} \bmod 2$$

for each $b = (b_1, b_2, \cdots, b_m) \in B^m$ *and* $i = 1, 2, \cdots, m$, *where* $(\tau(b))_0 = 0$ *and* $b_{m+1} = 0$. *The addition + means modulo 2 addition. The cellular automaton GS (m) has m sites (cells) and states of the sites vary with 0 or 1. We call the mapping* τ *the global dynamics of GS (m).*

Definition 2.2 *Let* Z_{2m} *be a set* $\{0, 1, \cdots, 2^m-1\}$. *Define a mapping* $\psi: B^m \rightarrow Z_{2m}$ *by*

$$\psi(b) = 2^{m-1}b_1 + 2^{m-2}b_2 + \cdots + 2b_{m-1} + b_m$$

for each $b = (b_1, b_2, \cdots, b_m) \in B^m$. *The mapping* ψ *is bijective from* B^m *onto* Z_{2m}. *Let us define a function f by* $f = \psi \circ \tau \circ \psi^{-1}$. *A system* $D = <Z_{2m}, f>$ *is a dynamical system associated with the original system* $<B^m, \tau>$. *The following diagram commutes:*

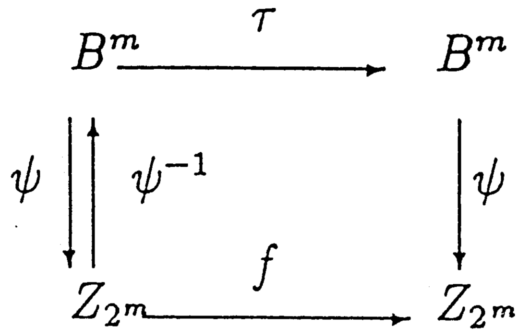

Fig. 2.1.

We get the following results.

Proposition 2.3 *The global dynamics* $\tau: B^m \rightarrow B^m$ *of GS (m) is bijective.*

PROOF. It is sufficient to show that dynamics τ is injective, since the set B^m is a finite set. Assume that $\tau(a) = \tau(b)$ for $a = (a_1, a_2, \cdots, a_m)$ and $b = (b_1, b_2, \cdots, b_m) \in B^m$, that is, $(\tau(a))_i = (\tau(b))_i (i = 1, 2, \cdots, m)$. For $i = m$, we have by definition

$$(\tau(a))_m = (\tau(a))_{m-1} + a_m \bmod 2$$

and

$$(\tau(b))_m = (\tau(b))_{m-1} + b_m \bmod 2.$$

Hence, by the assumption, we get $a_m = b_m$. And similarly discussing, we have for $i = m-1$,

$$(\tau(a))_{m-1} = (\tau(a))_{m-2} + a_{m-1} + a_m \bmod 2$$

and

$$(\tau(b))_{m-1} = (\tau(b))_{m-2} + b_{m-1} + b_m \bmod 2.$$

Thus we get $a_{m-1} = b_{m-1}$. By sucessively doing those, we have $a_i = b_i$ for all $i = 1, 2, \cdots, m$, that is, $a = b$. Hence the mapping τ is injective, and surjective. Therefore, this completes the proof. □

Proposition 2.4 *The dynamics* $\tau : B^m \to B^m$ *of* $GS(m)$ *is additive.*

PROOF. It will be proved by mathematical induction on $i = 1, 2, \cdots, m$ as follows:

$$(\tau(a+b))_i = (\tau(a))_i + (\tau(b))_i \bmod 2.$$

Basic step $(i = 1)$:

$$\begin{aligned}(\tau(a+b))_1 &= (a+b)_1 + (a+b)_2 \bmod 2\\ &= a_1 + b_1 + a_2 + b_2 \bmod 2\\ &= \tau(a)_1 + \tau(b)_1 \bmod 2\end{aligned}$$

Induction step: Assuming that it holds at $i-1$, it is enough to show that it holds at i. By definition

$$\begin{aligned}(\tau(a+b))_i &= (\tau(a+b))_{i-1} + (a+b))_i + (a+b))_{i+1} \bmod 2\\ &= (\tau(a))_{i-1} + (\tau(b))_{i-1} + a_i + b_i + a_{i+1} + b_{i+1} \bmod 2\\ &= (\tau(a))_i + (\tau(b))_i \bmod 2.\end{aligned}$$

Hence it holds for all $i = 1, 2, \cdots, m$, and the proof is completed. □

Thus the function f of the associated system $D = \langle Z_{2m}, f \rangle$ is bijective from Z_{2m} onto Z_{2m}. So, by observing the function f, we can analyze the behaviour of the automaton $GS(m)$.

Theorem 2.5 *The function f of the system D is as follows:*

$$f(p) = \begin{cases} 2p & (p = 0, 1, 2, \cdots, 2^{m-1}-1)\\ 2^{m+1} - (2p+1) & (p = 2^{m-1}, 2^{m-1}+1, \cdots, 2^m - 1).\end{cases}$$

PROOF. For each $p \in Z_{2m}$ there exists uniquely an element $b = (b_1, b_2, \cdots, b_m)$ satisfying $\psi(b) = p$. Then we get

$$\begin{aligned}f(p) &= \psi(\tau(\psi^{-1}(p)))\\ &= \psi(\tau(b_1, b_2, \cdots, b_m))\\ &= \psi((b_1+b_2 \bmod 2, b_1+b_3 \bmod 2, \cdots, b_1+b_m \bmod 2, b_1))\\ &= 2^{m-1}(b_1+b_2 \bmod 2) + 2^{m-2}(b_1+b_3 \bmod 2) + \cdots + 2(b_1+b_m \bmod 2) + b_1\end{aligned}$$

For $p = 0, 1, \cdots, 2^{m-1}-1$ we get $\psi^{-1}(p) = b = (0, b_2, b_3, \cdots, b_m)$. Hence

$$\begin{aligned} f(p) &= \psi(\tau(b)) \\ &= 2^{m-1}b_2 + 2^{m-2}b_3 + \cdots 2b_m \\ &= 2(2^{m-1}\cdot 0 + 2^{m-2}b_2 + \cdots + b_m) \\ &= 2p \end{aligned}$$

For $p = 2^{m-1}, 2^{m-1}+1, \cdots, 2^m-1$ we get $b_1 = 1$. Hence

$$\begin{aligned} f(p) &= 2^{m-1}\overline{b_2} + 2^{m-2}\overline{b_3} + \cdots + 2\overline{b_m} + 1 \\ &= 2^{m-1}(1-b_2) + 2^{m-2}(1-b_3) + \cdots + 2(1-b_m) + 1 \\ &= 2^m + 2^{m-1} + \cdots + 2 + 1 - (2^m + 2^{m-1}b_2 + \cdots + 2b_m) \\ &= 2^{m+1} - (2p+1) \end{aligned}$$

Hence the proof is completed. □

The following *table* 2.1. shows the values of the function f.

p	0	1	2	$\cdots$	$2^{m-1}-1$	2^{m-1}	$2^{m-1}+1$	$\cdots$	2^m-2	2^m-1
f(p)	0	2	4	$\cdots$	2^m-2	2^m-1	2^m-3	$\cdots$	3	1

Table 2.1

From the above table , we get

$$1 \xrightarrow{f} 2 \xrightarrow{f} 2^2 \xrightarrow{f} 2^3 \xrightarrow{f} \cdots \xrightarrow{f} 2^{m-1} \xrightarrow{f} 2^m - 1 \xrightarrow{f} 1,$$

that is, $f^{m+1}(1) = 1$. This shows that the state diagram of the automaton *GS* (m) contains transition cycles of length $m+1$. See Figure 2.2:

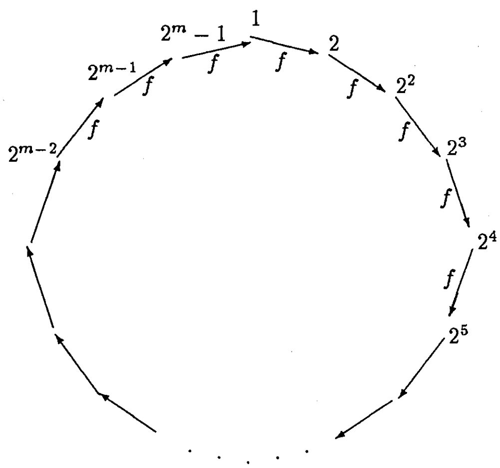

Figure 2.2

Example 2.6 *Let us consider the case GS* (2) $= < B^2, \tau >$ *when* $m = 2$. *Then the associated function f from* Z_{2^2} *onto* Z_{2^2} *takes the following values:*

p	0 1 2 3
$f(p)$	0 2 3 1

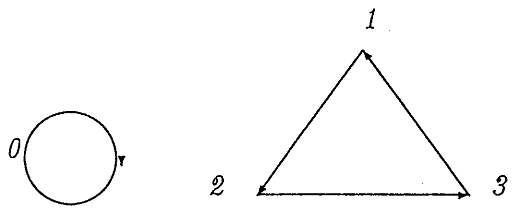

Table 2.2

So, we can observe that the state diagram of GS (2) *has a cycle of lenght* 3 *and a fixed point.*

Example 2.7 *Let us consider the case GS* (3) $= < B^3, \tau >$ *when* $m = 3$.
Then the function f takes the following values:

p	0 1 2 3 4 5 6 7
$f(p)$	0 2 4 6 7 5 3 1

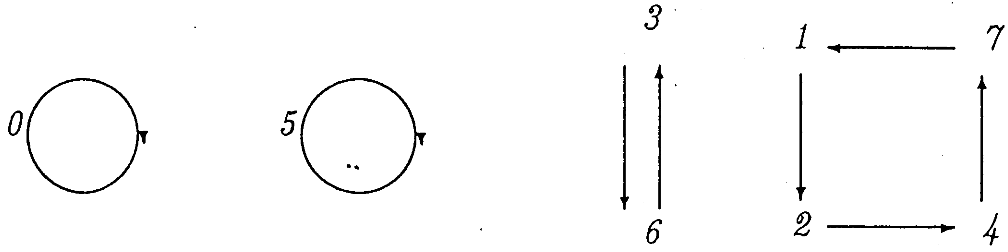

Table 2.3

The state diagram of GS (3) *has a cycle of length* 4, *a cycle of length* 2 *and two fixed points.*

Example 2.8 *Let us consider the case GS* (4) $= < B^4, \tau >$ *when* $m = 4$. *Then the associated function f from* Z_{2^4}, *onto* Z_{2^4} *takes the following values:*

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f(p)	0	2	4	6	8	10	12	14	15	13	11	9	7	5	3	1

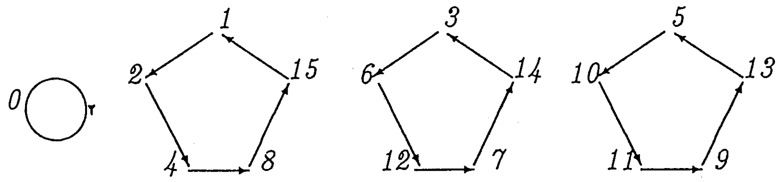

Table 2.4

Hence the state diagram of GS (4) has three cycles of length 5 and a fixed point. Here we can convert the boundary conditions $(\tau(b))_0 = b_{m+1} = 0$ *into the boundary conditions* $(\tau(b))_0 = 0$ *and* $b_{m+1} = 1$. *Then, we can observe the another configuration of the behaviour of automaton GS (m). Under the converted boundary conditions, we can orgnize a new system which is associated with the original system* $C = \langle B^m, \tau \rangle$. *In some cases, it does'nt have any fixed point, and in other cases, it has one fixed point.*

Let us define a function g by $g = \psi \tau \psi^{-1}$. A system $E = \langle Z_{2^m}, g \rangle$ is a dynarnical system associated with the original system $C = \langle B^m, \tau \rangle$. The following diagram commutes:

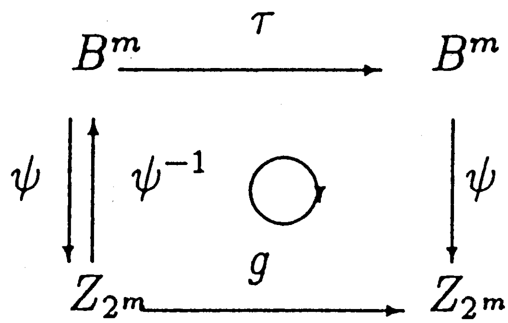

Figure 2.3

We get the following result.

Theorem 2.9 *The function g of the system* $E = \langle Z_{2^m}, g \rangle$ *is as follows:*

$$g(p) = \begin{cases} 2p+1 & (p = 0, 1, 2, \cdots, 2^{m-1}-1) \\ 2^{m+1}-2(p+1) & (p = 2^{m-1}, 2^{m-1}+1, \cdots, 2^{m-1}-1). \end{cases}$$

PROOF. For each $p \in Z_{2m}$, there exists uniquely an element $b = (b_1, b_2, \cdots, b_m) \in B^m$ satisfying $\psi(b) = p$. Tthen we get

$$\begin{aligned} g(p) &= \psi(\tau(\psi^{-1}(p))) \\ &= \psi(\tau(b_1, b_2, \cdots, b_m)) \\ &= \psi((b_1+b_2 \bmod 2, b_1+b_3 \bmod 2, \cdots, b_1+b_m \bmod 2, b_1+1 \bmod 2)) \end{aligned}$$

For $p = 0, 1, 2, \cdots, 2^{m-1}-1$, $\psi^{-1}(p) = b = (0, b_2, b_3, \cdots)$. Hence

$$\begin{aligned} g(p) &= 2^{m-1}b_2+2^{m-2}b_3+ \cdots 2b_m+1 \\ &= 2(2^{m-1} \cdot 0+2^{m-2}b_2+ \cdots +b_m)+1 \\ &= 2p+1 \end{aligned}$$

For $p = 2^{m-1}, 2^{m-1}+1, \cdots, 2^m-1$ we get $b_1 = 1$. Hence

$$\begin{aligned} g(p) &= 2^{m-1}\overline{b_2} + 2^{m-2}\overline{b_3} + \cdots + 2\overline{b_m} + (1+1 \bmod 2) \\ &= 2^{m-1}(1-b_2)+2^{m-2}(1-b_3)+ \cdots +2(1-b_m) \\ &= 2^m+2^{m-1}+ \cdots +2+1-(2^m+2^{m-1}b_2+ \cdots +2b_m+1) \\ &= 2^{m+1}-(2p+1) \end{aligned}$$

Hence the proof is completed. □

The following table 2.2. shows the values of the function g.

p	0	1	2	$\cdots$	2^{m-1}	$2^{m-1}+1$	$\cdots$	2^m-2	2^m-1
g(p)	1	3	5	$\cdots$	2^m-2	2^m-4	$\cdots$	2	0

Table 2.5

From the above table , we get

$$0 \xrightarrow{g} 1 \xrightarrow{g} 3 \xrightarrow{g} 7 \xrightarrow{g} \cdots \xrightarrow{g} 2^{m-1} \xrightarrow{g} 2^m-1 \xrightarrow{g} 0,$$

that is, $g^{m+1}(0) = 0$

Example 2.10 *Let us consider the case* $GS(2) = \langle B^2, \tau \rangle$ *when* $m = 2$. *Then the associated function g from* Z_{2^2} *onto* Z_{2^2} *takes the following values:*

p	*0*	*1*	*2*	*3*
g(p)	*1*	*3*	*2*	*0*

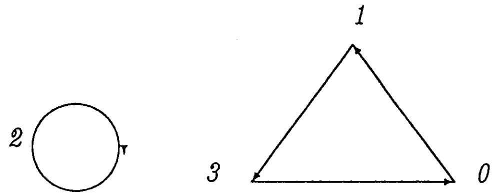

Table 2.6

So, we can observe that the state diagram of GS (2) *under the boundary conditions* $(\tau\ (b))_0 = 0$ *and* $b_{m+1} = 1$ *has a cycle of lenght* 3 *and a fixed point.*

Example 2.11 *Let us consider the case GS* $(3) = < B^3, \tau >$ *when* $m = 3$. *Then the function g takes the following values:*

p	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
f(p)	*1*	*3*	*5*	*7*	*6*	*4*	*2*	*0*

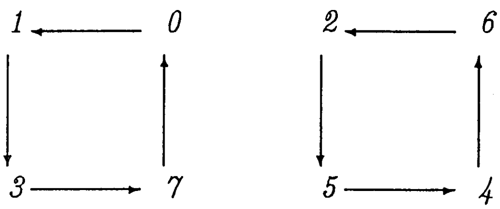

Table 2.7

The state diagram of GS (3) *under the boundary conditions of* $(\tau\ (b))_0 = 0$ *and* $b_{m+1} = 1$ *has only two cycles of length* 4, *which is compared with the state diagram of GS* (3) *under the boundary conditions of* $(\tau\ (b))_0 = 0$ *and* $b_{m+1} = 0$, *that is, it does'nt have any fixed point.*

Example 2.12 *Let us consider the case GS* $(4) = < B^4, \tau >$ *when* $m = 4$. *Then the function g takes the following values:*

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(p)$	1	3	5	7	9	11	13	15	14	12	10	8	6	4	2	0

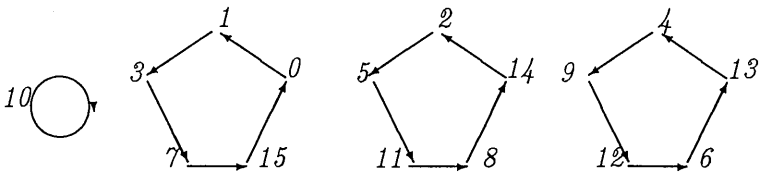

Table 2.8

The state diagram of GS (4) has three cycles of length 5 and a fixed point.

3. Main Theorems

Let $e^{(i)}$ be a configuration $(0,\cdots, 0, 1, 0,\cdots, 0)$ with only one state 1 at its i-th component Note that, by the definition of τ,

(i) $\tau(e^{(i)}) = e^{i-1} (i = 2, 3,\cdots, m)$

(ii) $\tau(e^{(1)}) = \sum_{i=1}^{m} e^{(i)}$,

(iii) $\tau(\sum_{i=1}^{m} e^{(i)}) = e^{(m)}$.

See figure **3.1**.:

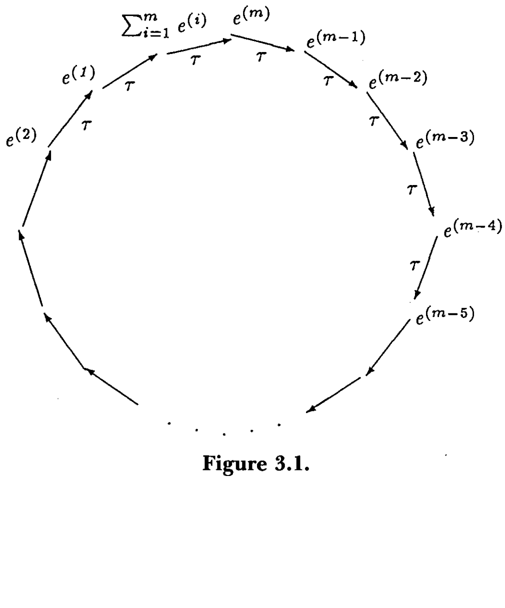

Figure 3.1.

From (i), (i) and (iii), we ger $\tau^{(m+1)}(e^{(i)}) = e^{(i)}$. The following theorem describes that the maximum length of cycles of transition diagrams of the automaton GS (m) is a linear funcion of m.

Theorem 3.1 *The maximum length of cycles of transition diagrams of the automaton GS* (m) *is equal to* $m+1$, *i. e.*

$$\tau^{m+1}(b) = b$$

for each $b \in B^m$. (*This shows that each length of cycles of transition diagram of GS* (m) *is a factor of* $m+1$.)

PROOF. For each $b = (b_1, b_2, \cdots, b_m) \in B^m$, we have

$$b = \sum_{i=1}^{m} e^{(i)} b_i \bmod 2.$$

Then, by the linearity of τ,

$$\tau(b) = \sum_{i=1}^{m} \tau(e^{(i)}) b_i$$

and

$$\tau^{m+1}(b) = \sum_{i=1}^{m} \tau^{m+1}(e^{(i)}) b_i$$

$$= \sum_{i=1}^{m} e^{(i)} b_i$$

$$= b.$$

This completes the proof. □

We are able to find the fixed points of the dynamics τ.

Theorem 3.2 *The dynamics* τ *of the automaton GS* (m) *has fixed points, and the number of fixed points are one for even m and two for odd m.*

PROOF. Consider the associated dynamical system $D = \langle Z_{2m}, f \rangle$ of the automaton GS (m). The number of fixed points of the function f is equal to those of the dynamics τ, since ψ is bijective. Fixed points of the function f are

(i) 0 for even m

and

(ii) 0 and $(2^{m+1}-1)/3$ for odd m.

Clearly, by the form of f, 0 is fixed point of f: $f(0)=0$. Another fixed point, if it exists, has to satisfy the relation $f(p)=p$. Hence the number p satisfying $f(p)=2^{m+1}-(2p+1)=p$ is a candidate of fixed point of f. When m is an old number, according to an elementary analysis of the number theory, $2^{m+1}-1$ is divisible by 3. Since $(2^{m+1}-1)/3>2^{m-1}$,

$$f((2^{m+1}-1)/3)=2^{m+1}-[2(2^{m+1}-1)/3+1]$$
$$=(2^{m+1}-1)/3.$$

Therefore, the point $(2^{m+1}-1)/3$ is a fixed point of the function f. Hence we have the conclusion.

Remark.

$$\psi^{-1}(0)=(0,\cdots,0)$$

and

$$\psi^{-1}((2^{m+1}-1)/3)=(1,0,1,0,\cdots 1,0,1)\ (\text{for odd } m).$$

These are fixed points of τ.

Theorm 3.3 *Under the boundary conditions* $(\tau(b))_0=0$ *and* $b_{m+1}=1$, *the dynamics* τ *of the automaton GS* (m) *has fixed point, and the number of fixed point is one for even m.*

PROOF. Consider the associated dynamical system $E=<Z_{2m}, g>$ of the automaton $GS(m)$. The number of fixed point of the function g is equal to those of the dynamics τ. Since ψ is bijective, the fixed point of the function g is $2(2^m-1)/3$. It has to satisfy $g(p)=p$. Hence the number p satisfying $g(p)=2^{m+1}-2(p+l)=p$ is a fixed point of g. When m is an even number, according to an elementary analysis of number theory, $2^{m+1}-2$ is divisible by 3. Since $(2^{m+1}-2)/3>2^{m-1}$,

$$g((2^{m+1}-2)/3)=2^{m+1}-2[(2^{m+1}-2)/3+1)$$
$$=(2^{m+1}-2)/3.$$

Therefore the point $(2^{m+1}-2)/3$ is a fixed point of the function g. Hence the proof is completed. □

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