On a Normal Form of Non-singular Quartic Surfaces in the Three Dimensional Projective Space

Takahashi, Tadashi
Department of Mathmatics, Gunma College of Technology

Nakagawa Shigekazu
Department of Information Systems, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University

Niki, Naoto
Department of Information Systems, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University

https://doi.org/10.15017/17211

出版情報：九州大学大学院総合理工学報告. 12 (4), pp.381-384, 1991-03-01. 九州大学大学院総合理工学研究科
バージョン：
権利関係：

九州大学大学院総合理工学研究科報告
第12巻 第4号 381-384頁 平成3年3月

Engineering Sciences Reports, Kyushu University
(KYUSHU DAIGAKU SOGORIKOGAKU KENKYUKA HOKOKU)
Vol. 12, No. 4, pp. 381 - 384, March 1991

On a Normal Form of Non-singular Quartic Surfaces in the Three Dimensional Projective Space

Tadashi TAKAHASHI*, Shigekazu NAKAGAWA** and Naoto NIKI***

(Received November 30, 1990)

A normal form of homogeneous polynomials which provides definitions for non-singular quartic surfaces in the three dimensional complex projective space is given. The parameter-space of the polynomials is $\mathbf{C}^{19}$ and which corresponds the ***K3*** surfaces as discussed in Kodaira (1963). Computer algebra has shown its utility in this research field and promises a breakthrough.

1. Introduction and Preliminaries

Let $\boldsymbol{P}^{n-1}$ be a $(n-1)$-dimensional complex projective space with the coordinate $[x_1, x_2, \cdots, x_n]$ and let ${}^m\boldsymbol{H}_n$ be the set of homogeneous polynomials of degree m in $\boldsymbol{P}^{n-1}$. For any $f \in {}^m\boldsymbol{H}_n$, we define the analytic set

$$V_n = \{(x_1, x_2, \cdots, x_n) \mid f=0\}$$

which is called a complex projective surface of degree n. In this article, we call V_4 a quartic surface in $\boldsymbol{P}^3$ (see Hartshore (1977)).

The following proposition is most fundamental:

Proposition 1.1. *Let $g(x_1, x_2, \cdots, x_n)$ be a polynomial in $\boldsymbol{C}^n$ and let U be an analytic set such that*

$$U = \{(x_1, x_2, \cdots, x_n) \mid g(x_1, x_2, \cdots, x_n) = 0\}.$$

Then for any singular point $(x_1, x_2, \cdots, x_n)$ of U, there exists a linear coordinate transformation

$$\psi : [x_1, x_2, \cdots, x_n] \to [y_1, y_2, \cdots, y_n]$$

such that

$$(\psi g)(y_1, y_2, \cdots, y_n) = 0$$

and

$$\frac{\partial (\psi g)(y_1, y_2, \cdots, y_n)}{\partial y_i} = 0, \qquad i=1, \cdots, n-1.$$

This work was completed while the first author was visiting at Research Institute of Fundamental Information Science, Kyushu University.

*Department of Mathematics, Gunma College of Technology
**Department of Information Systems, Graduate Student
***Department of Information Systems

Hence, we can assume, without loss of generality, that for each singular point $(x_1, x_2, \cdots, x_n)$ of U satisfies

$$g(x_1, x_2, \cdots, x_n)=0$$

and

$$\frac{\partial g(x_1, x_2, \cdots, x_n)}{\partial x_i}=0, \qquad i=1, \cdots, n-1.$$

This assumption will be present throughout the article. Clearly, if these equations hold simultaneously then U has a singularity at that point.

The following propositions are also fundamental and are required in obtaining the main results as well as in designing an algorithm to get the normal forms of polynomials in ${}^m\boldsymbol{H}_n$.

Proposition 1.2. *Let U be the same set defined in Proposition 1.1. If U has an isolated singular point at the origin $O=(0, \cdots, 0)$ in $\boldsymbol{C}^n$ then, for any $i\,(i=1, 2, \cdots, n)$,*

(i) *the defining polynomial g contains a monomial of the form $c_i x_i^{a_i}$ where c_i is a constant coefficient and $a_i \geq 2$;*

or

(ii) *g contains a monomial of the form $c_{ij}\, x_i^{a_i} x_j$ where c_{ij} is a constant, $i \neq j$ and $a_i \geq 1$.*

Proposition 1.3. *If the defining polynomial g is homogeneous with degree n, then the analytic set $\boldsymbol{U}$ defined in Proposition 1.1 has a singularity at O.*

Remark 1.1. When V_n has an isolated singular point (or points), these propositions enable us to assume, without loss of generality, that it (or one of them) is the point $(1, 0, \cdots, 0)$. If not so, a suitable linear transformation is the only need. In that case, we can write $f \in {}^m\boldsymbol{H}_n$ *as*

$$f=f_2(x_2, \cdots, x_n)\, x_1^{n-2}+f_3(x_2, \cdots, x_n)\, x_1^{n-3}+\cdots+f_n(x_2, \cdots, x_n)$$

where f_i denotes a homogeneous polynomial of degree $i\,(i=2, \cdots, n)$.

A ***K3*** surface is characterized with $K=0$ and $q=0$ and satisfies $p_g=p_a=1$ (see, *e. g.*, Hartshone (1977)), where K, p_g, p_a, q are the canonical divisor, the geometric genus, the arithmetic genus and the irregularity for an algebraic surface, respectively. Each ***K3*** surface is simply connected and can be obtained by deformation of some non-singular quartic surface in $\boldsymbol{P}^3$. ***K3*** surfaces compose families of 19 dimensions (see Kodaira (1963)).

2. Main Results

Theorem 2.1. *There exist non-singular quartic surfaces in $\boldsymbol{P}^3$ having the normal form*

$$x_1^3x_4+(x_2^2x_4+x_3^3+a_1x_3^2x_4+(a_2x_2+a_3x_3)\,x_4^2+a_4x_4^3)\,x_1+f_4(x_2, x_3, x_4)$$

where $f_4(x_2, x_3, x_4)$ *is a homogeneous polynomial of variables* x_2, x_3, x_4 *of degree* 4.

The proof is straightforward and is omitted.

Theorem 2.2. *If* $f \in {}^4\boldsymbol{H}_4$ *provides a non-singular quartic surface, then f is linearly equivalent to the normal form in Theorem 2.1.*

The proof is lengthy and is given in Takahashi (1991).

Remark 2.1. The parameter-space of the defining equation given in Theorem 2.1 is C^{19}, since the parameter-space of the equation $f_4(x_2, x_3, x_4) = 0$ is C^{15}. All ***K3*** surfaces compose families of 20 dimensions but if expressing them as algebraic ones the number of dimensions is reduced to 19 (see Kodaira (1963)). Theorem 2.1 gives the normal form of defining polynomials of ***K3*** surfaces as an explicit expression.

Remark 2.2. Conversely, from Remark 1.1, the polynomial given in Theorem 2.1 does not have any isolated singular point other than O in C^4.

Remark 2.3. In the most general expression, the parameter-space of the defining equation in Theorem 2.2 is nominally C^{35}. Theorem 2.2 implies that this parameter-space can be reduced to C^{19} by applying series of linear transformations.

3. Use of Computer Algebra

During the research a huge amount of algebraic calculation is needed, including linear transformation, modular arithmetic modulo a polynomial and resultant calculus. Each comprises various kinds of polynomial arithmetics covering differentiation, simplification, solution of algebraic equations (or systems of equations) and manipulation of structures.

For example, the following program written in Mathematica language (Wolfram (1988)) is to confirm that the parameter-space of the defining equation given in Theorem 2.1 is C^{19} as stated in Remark 2.1.

```
(*the defining equation*)
  f=X4 X1^3+(X3 X2^2+a1 X4^2 X2+X3^3+a2 X3^2 X4+a3 X3 X4^2+a4 X4^3) X1+
      a5 X2^4+
     (a6 X3+a7 X4) X2^3+
     (a8 X3^2+a9 X3 X4+a10 X4^2) X2^2+
     (a11 X3^3+a12 X3^2 X4+a13 X3 X4^2+a14 X4^3) X2+
      a15 X3^4+a16 X16 X3^3 X4+a17 X3^2 X4^2+a18 X3 X4^3+a19 X4^4;

(*linear transformation for finding singularity if exists*)
  f1=Expand [f/.{X1->u1 x1+u2 x2+u3 x3+x4,
                 X2->u4 x1+u5 x2+x3,
                 X3->u6 x1+x2, X4->x1}];
```

```
(*variables and works*)
  x= {x1, x2, x3, x4}; u= {u1, u2, u3, u4, u5, u6};
  Array [c, 4]; Array [g, 4];
  Array [ch, 10]; Array [cg1, 10]; Array [cg2, 10];

(*extract and print the coefficients of the transformed polynomial*)
  Do [c[i] =Coefficient [f1, x1^3 x[[i]]], {i, 4}];
  Do [Print [c[i]], {i, 4}];

(*transform f1 again forcing the leading coefficient to be unity*)
  h=c[4] -1;
  Do [g[i] =PolynomialRemainder [c[i], h, u1], {i, 4}];
  Do [Print [g[i], {i, 4}]];

(*reparametrize*)
  u3=- (q[3] -u3);
  Do [Expand [g[i]], {i, 4}];
  Do [Print [g[i]], {i, 4}];

(*extract the coefficients of the reparametrized polynomials*)
  ch=CoefficientList [h, u1];
  cg1=CoefficientList [g[1], u1];
  ca2=CoefficientList [g[2], u1];

(*resultants modulo u3 and h*)
  g[1] =Expand [(ch[[2]] cg1[[2]]-2 ch[[3]] cg1[[1]])^2
    - (ch[[2]]^2-4 ch[[3]] ch[[1]]) cg1[[2]]^2];
  g[2] =Expand [(ch[[2]] cg2[[2]]-2 ch[[3]] cg2[[1]])^2
    - (ch[[2]]^2-4 ch[[3]] ch[[1]]) cg2[[2]]^2];

(*print the results*)
  Do [Print [CoefficientList [g[i], u[[j]]]], {i, 2}, {j, 2}];
  Do [Print [CoefficientList [g[i], u[[j]]]], {i, 2}, {j, 4, 6}];
```

After execution of the above program we have obtained the fact that Coefficient [g[2], u2^2] =36 and this term does not contain any parameter. Therefore, the number of parameters remains 19 as the former.

References

1) Hartshone, R. (1977): *Algebraic Geometry*, GTM 52, Springer-Verlag.
2) Kodaira, K. (1963): On Compact Complex Analytic Surface Ⅲ, *Ann. of Math.*, **78**, pp. 1—40.
3) Takahashi, T. (1991): Normal Form of Non-singular Quartic Surface and it's Moduli, *To appear.*
4) Wolfram, S. (1988): *Mathematica™, A System for Doing Mathematics by Computer,* Addison-Wesley, California.