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https://doi.org/10.15017/17205

出版情報：九州大学大学院総合理工学報告. 12 (3), pp.319-328, 1990-12-01. 九州大学大学院総合理工学研究科

九州大学総合理工学研究科報告
第12巻 第3号 319-328頁 平成2年12月

Engineering Sciences Reports, Kyushu University
(KYUSHU DAIGAKU SOGORIKOGAKU KENKYUKA HOKOKU)
Vol. 12, No. 3, pp. 319 - 328, December 1990

Analytical Study on Post-Shock Expansion in Transonic Flow

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(Received August 31, 1990)

A simple analytical method for the interaction between weak normal shock waves and turbulent boundary layers on flat surfaces is presented. The flow is assumed to be adiabatic, two-dimensional and to follow the power-law form. An empirical relation for a normal shock wave in transonic flow is taken into account, and momentum and energy integral equations for the turbulent boundary layer are applied. This study is aimed to analyze the post-shock expansion caused by the shock/boundary layer interaction. The results show that the post-shock expasion is largely positioned outside the edge of boundary layer and the extent of it scales with the flow Mach number, and also show the qualitative agreement with experimental results.

1. Introdution

In transonic flows such as those in aerofoils, turbine blades, compressors, diffusers and nozzles, neary-normal shock waves occur and interact with the boundary layer developing along the wall surface. When the free stream Mach number just upstream of the shock wave excceeds about 1.1 and 1.33[1], respectively, in the cases laminar and turbulent boundary layers, it has been well known that the flow separation occur, i.e. shock-induced separation. There are still many discussions even up to now on whether the shock waves at such Mach numbers can really provoke the flow separation. However, from a number of experimental and numerical knowledgements so far, it is accepted at least as the common sense that the pressure rise experienced at the wall surface underneath the shock wave is always appreciably less than what may be termed the full theorerical normal shock wave. The reason is that the turbulent boundary layer thickens under the shock wave, caused by the shock/turbulent boundary layer interaction, so that just outside the boundary layer the streamtube areas are reduced downstream, with a consequent reduction in pressure recovery.

In these connections, experimental and analytical studies have been worked by many researchers. For example, in 1947 Ackeret et al.[2] and Weise[3] found that downstream the shock wave, there exists the static pressure decrease in streamwise direction, where their experiments were used a curved surface flow as an aerofoil model and a two-dimensional duct, respectively. On the other hand, Bohing & Zierep[4] made a combination of the streamwise inhomogeneous, vibrating equation and the transverse eigen-value problem in the inner and outer layers developing over a curved surface, and solved analytically the post-

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shock expansion as a boundary value problem. Also, Inger[5] presented an approximate approach on a weak normal shock/turbulent boundary layer interaction in plane flow using the triple deck theory based upon Lightill's perturbation equation[6]. Unfortunately, these solutions were however confined only to the boundary layer flows and thus the post-shock expansion in actual flows such as those in internal (diffuser, nozzle, pipe, etc.) flows was not investigated enough to give a satisfiable solution. Form this point of view, kooi's experiment[7] in a uniform flow with a Mach number 1.4 was available to the authors and then gave an impetus to the present study.

In this study, Gadd's method[8] is reviewed in more details and compared with the previous data by the authors in supersonic nozzle flow. As a modle of the simple flow, a plane flow with the flow Mach numbers of 1.1−1.3 and a Reynolds numble of 2×10^6 was analyzed.

2. Theoretical Analysis

2.1 Overall flow pattern

The overall flow pattern is described in **Fig. 1**. This pattern shows the case that the extent of interaction between a shock wave and a turbulent boundary layer is comparatively weak and thus, shock-induced separation is excluded here. The boundary layer thickens under the action of the adverse pressure gradients, and this displaces the external flow away from the wall surface. The streamwise pressure distribution in the external flow is in turn governed by this displacement, and hence the theoretical problem, as with all cases of interactions between shock waves and boundary layers is to determine the conditions

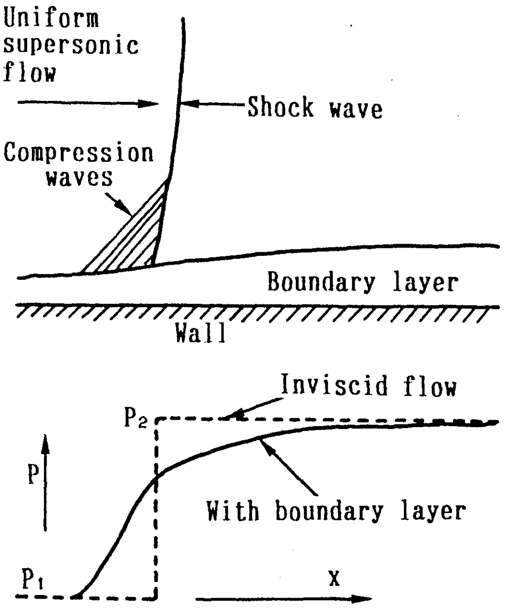

Fig. 1 Flow pattern and typical pressure distribution

under which the pressure distribution can be matched to the boundary layer thickening. It is assumed that the compression wave region shown in **Fig. 1** is of the simple wave type, with the Mach waves emanating from the edge of the boundary layer intersecting the shock wave and being terminated by it. This terminating shock wave is vanishingly weak at the edge of the boundary later, so that the layer is not called upon to support any discontinuous jump in pressure. Away from the wall surface, the shock wave becomes stronger. It is assumed that the distance of the shock from the line $x=0$ perpendicular to the wall suface and passing through the end of the shock at the edge of the boundary layer is everywhere fairly small. Then to a rough approximation the boundary conditions imposed by the shock on the downstream flow may be taken as applying along line $x=0$. These boundary conditions are, in the outer part of the flow, the relations between the downstream pressure and the flow angle appropriate to a shock with a Mach number M_1 just upstream of it. Nearer the edge of the boundary layer the relations are in theory more complicated, since the flow deflection is then made up of a part achieved continuously in the simple wave flow, and of a part occurring abruptly through the shock.

2.2 The relations between pressure and flow angle

In the simple compression wave region, let a parameter K, related to the pressure p, be defined by

$$\frac{p}{p_1}=1+\frac{2\gamma}{\gamma+1}(M_1^2-1)(1-K) \tag{1}$$

where suffix 1 denotes conditions in the external flow upstream of the shock, γ is the specific heat ratio, and M is the flow Mach number. Thus $K=1$ upstream, and $K\to 0$ downstream, where the pressure must approach the full normal shock value. In other words,

$$1-K=(p-p_1)/(p_2-p_1) \tag{2}$$

where p_2 is the pressure far downstream. In the external flow the relationship between the Mach number M and K may be derived in the energy balance. The external flow is everywhere approximately isentropic if M_1 is not too large ($M_1\leqq 1.3$) and hence, from Eqn (1)

$$\frac{I_1}{I}=\left(\frac{p}{p_1}\right)^{-\frac{\gamma-1}{\gamma}}\fallingdotseq 1-\frac{2(\gamma-1)}{\gamma+1}(1-K)(M_1^2-1)+\frac{(\gamma-1)(2\gamma-1)}{2\gamma^2}\times\left\{\frac{2\gamma}{\gamma+1}(1-K)(M_1^2-1)\right\}^2 \tag{3}$$

where I is the enthalpy.

Here $M_1^2-1\ll 1$ and $1-K\ll 1$, thus the last term in the right hand is negligibly small. Hence approximately

$$M^2-1=(2K-1)(M_1^2-1) \tag{4}$$

For M_1 as large as 1.3, this relation is fairly accurate upstream of the shock, where $1/2\leqq K\leqq 1$, but it is inaccurate downstream where, however, it is not required.

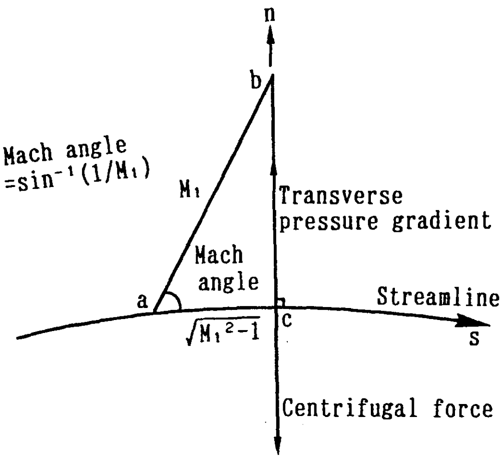

Fig. 2 Streamline in simple compression region

Figure 2 shows a streamline in the simple wave region. The line *ac* is a short length of a streamline, the line *ab* is a Mach line so that K is constant along the line *ab*, and *bc* is normal to *ac*. If n represents distance along the normal and S distance along the streamline,

$$\frac{\partial K}{\partial n}=\frac{K_b-K_c}{bc}=\frac{K_a-K_c}{bc}=\frac{K_a-K_c}{ac}\sqrt{(2K-1)^2(M_1^2-1)}=-\sqrt{(2K-1)(M_1^2-1)}\frac{\partial K}{\partial s} \tag{5}$$

The pressure gradient normal to the streamline balances the centrifugal force associated with the streamline curvature. Thus if the angle of the streamline to the wall is α

$$\frac{\partial p}{\partial n}=-\rho q^2\frac{\partial \alpha}{\partial s}=-\gamma pM^2\frac{\partial \alpha}{\partial s} \tag{6}$$

where ρ is the density and q the speed along the streamline. The product pM^2 does not enormously vary throughout the interaction region. Thus approximately

$$\frac{\partial \alpha}{\partial s}=\frac{2(M_1^2-1)}{(\gamma+1)M_1^2}\frac{\partial K}{\partial n}=-\frac{2(M_1^2-1)^{3/2}}{(\gamma+1)M_1^2}(2K-1)^{1/2}\frac{\partial K}{\partial s} \tag{7}$$

Hence

$$\alpha=\frac{2}{3}\frac{(M_1^2-1)^{3/2}}{(\gamma+1)M_1^2}\left[1-(2K-1)^{3/2}\right] \tag{8}$$

since $\alpha=0$ when $K=1$ in the undisturbed free stream. Equation (8) is of course only valid for $1/2\leqq K\leqq 1$, the range of K for which the flow is supersonic.

On the other hand, in general the shock wave is inclined to the free stream direction. The deflection which streamlines undergo on passing through the shock will depend on upstream properties. Four equations, namely those of energy, momentum parallel to the shock wave, momentum perpendicular to the shock and continuity can be available and the oblique shock relation[9] can also be used. The resulting relation becomes approximately

$$\alpha_d=\frac{2(M_1^2-1)^{3/2}}{(\gamma+1)M_1^2}K_d^{1/2}(1-K_d) \tag{9}$$

where suffix d denotes downstream of the shock. This relation for α_d is assumed to apply along the line $x=0$, through the foot of the shock perpendicular to the wall, and this forms one boundary condition for the flow downstream of the shock. On the other hand, the conditions behind the normal shock in inviscid flow and the balances of the pressure gradient normal to the streamlines with the centrifugal force associated with their curvature lead to following Eqn. (10)

$$\frac{\partial\alpha}{\partial x}=\frac{2(1-M_2^2)}{(\gamma+1)M_2^2}\frac{\partial K}{\partial y} \tag{10}$$

where suffix 2 denotes the condition behind the normal shock. Another relation between α and k can be obtained from the continuity equation

$$\frac{\partial}{\partial x}(\rho q)+\frac{\partial}{\partial y}(\rho q\alpha)=0 \tag{11}$$

since the external flow is everywhere almost isentropic, as long as $M\leqq 1.3$

$$\frac{\rho}{\rho_2}=\left(\frac{p}{p_2}\right)^{\frac{1}{\gamma}},\quad \frac{I}{I_2}=\left(\frac{p}{p_2}\right)^{\frac{\gamma-1}{\gamma}} \tag{12}$$

Also, from the energy equation and Eqn. (11), the authors obtain approximately

$$\frac{\rho q}{\rho_2 q_2}\fallingdotseq 1+\frac{2(1-M_2^2)^2}{(\gamma+1)M_2^2}K(1-K)+O(1-M_2^2)^3 \tag{13}$$

Substituing the approximate relation for ρq into Egn. (11)

$$\frac{\partial\alpha}{\partial y}=-\frac{2(1-M_2^2)^2}{(\gamma+1)M_2^2}\frac{\partial}{\partial x}(K-K^2) \tag{14}$$

Hence, Eqn. (11) and the above Eqn. (14) are simplified to

$$\frac{\partial^2K}{\partial x^2}+\frac{\partial^2K}{\partial\bar{y}^2}=\frac{\partial^2K^2}{\partial x^2} \tag{15}$$

where

$$\bar{y}=\sqrt{1-M_2^2}\cdot y \tag{16}$$

In terms of this variable, Equs. (10) and (13) beome

$$\frac{\partial\alpha}{\partial x}=\frac{2(1-M_2^2)^{3/2}}{(\gamma+1)M_2^2}\frac{\partial K}{\partial\bar{y}} \tag{17}$$

$$\frac{\partial \alpha}{\partial \bar{y}} = -\frac{2(1-M_2^2)^{3/2}}{(\gamma+1)M_2^2}\frac{\partial}{\partial x}(K-K^2) \tag{18}$$

To faciliate matching these relations to the Eqn. (8) and (9) for α in the simple wave flow region and behind the shock, the authors note that $(1-M_2^2)^{3/2}/M_2^2$ is not very different from $(M_1^2-1)^{3/2}/M_1^2$. Hence, it was also available in the external flow downstream of the shock.

2.3 Turbulent boundary layer in interaction region

In the present analysis, the velocity profiles for a constant pressure boundary layer are assumed to be fitted by a power law form with $n=1/7$. Also, it is assumed that the rate of entrainment of fluid into the boundary layer from the external flow is the same as just upstream of the interaction region. This is probably an underestimate, since Seddon's interesting paper[10] concerning the interaction with a normal shock with $M_1=1.47$ shows that in that case, the rate of mass entrainment becomes much larger downstream than upstream. However it is difficult to formulate a more accurate relation for entrainment, and since the effect is only important in the downstream part of the interaction region, where the solution is necessarily crude, the simple assumption of a constant rate is probably good enough. Thus

$$\rho_e u_e \delta = (\rho_e u_e)_a \left[\delta_a + (x-x_a)\left(\frac{d\delta}{dx}\right)_a\right] \tag{19}$$

where suffix 'e' denotes the boundary layer edge. To a good approximation, the product term inside the square brackets here may be neglected upstream of the shock, and downstream it may be replaced by $x(d\delta/dx)_a$, x being measured from the foot of the shock. An additional relation is needed to determine the two unknowns, n and δ, in terms of M_e. This is provided by the boundary layer integral equation.

On the other hand, the procedure the authors adopt to solve Eqn. (15) is to choose a suitable mathematical form for the distribution of K in the x, y plane downstream of the shock. This distribution has a number of disposal constants which are chosen by making the distribution fit various integral conditions derived from the equations and the boundary conditions. The distribution used for K is

$$K=\frac{e^{-x/A}+Ze^{-x/B}}{2(1+Z)}\cdot\left\{\left(1+\frac{\bar{y}}{C}\right)\left(1-\frac{x}{D}e^{-x/E}\right)e^{-\bar{y}/C}+\frac{x}{D}e^{-x/E}\cdot e^{-\bar{y}/F}\right\} \tag{20}$$

where Z, A, $B\cdots F$ are constants. This makes $k=1/2$ at $x=\bar{y}=O$, satisfting the condition that the pressure is sonic at the foot of the shock, $K\to 0$ as $x, \bar{y}\to\infty$, satisfying the boundary conditions at infinity, and $\partial K/\partial\bar{y}=0$ at $x=\bar{y}=0$.

3. Discussions of Results

In the present study, the calculated distributions of $(p-p_1)/(p_2-p_1)$ which is proportional to $1-K$ are plotted in **Figs. 3** to **5** as a function of $x=x/\delta_a$. It can be seen that the pressure distributions at δ_a distance from the wall has a simple increase through the shock. Although not indicated in figure, the pressure distributions at the wall also showed such a tendency. As already pointed out, the pressure gradients caused by the shock don't become infinite in reality. Thus it is not unreasonable that the theory should predict that the position of maximun pressure gradient occurs upstream of the sonic point at the higher Mach numbers. Also these figures show that the pressure rise by the shock decrease as M_1 is increased. Just behind the shock, the pressure at a sufficient distance from the wall

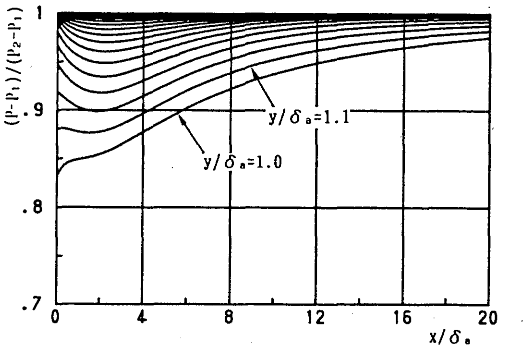

Fig. 3 Streamline pressure distributions in case of $M_1=1.10$ ($y/\delta_a=0.1$ upward intervals from $y/\delta_a=1.0$)

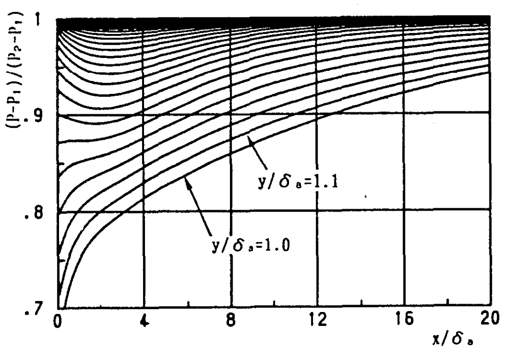

Fig. 4 Streamwise pressure distributions in case of $M_1=1.20$ ($y/\delta_a=0.1$ upward intervals from $y/\delta_a=1.0$)

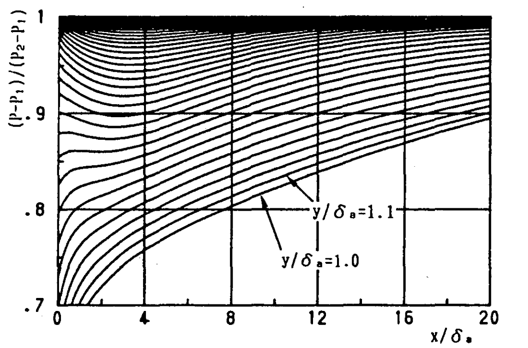

Fig. 5 Streamline pressure distributions in case of $M_1=1.30$ ($y/\delta_a=0.1$ upward intervals from $y/\delta_a=1.0$)

becomes close to the full downstream pressure behind a normal shock but the pressure gradient is initially falling. This is because the stream tubes must contract to be consistent with the streamlines being roughly parallel to the wall a long way from it, and inclined with a positive slope at the boundary layer edge. This tendency for there to exist regions where the pressure falls in the stream-wise direction only operates downstream of the shock. Upstream, the contraction of the stream tubes outside the boundary layer due to the latter's thickening is consistent with favorable pressure gradients everywhere, because in supersonic flows the stream tubes contract on encountering a pressure rise. Furthermore, as shown in **Fig. 6** based on Figs. 3 to 5, the post-shock expansion (P_2/P_3) [11] in transverse direction behind the shock wave is replotted.

As the flow Mach number increase, the post-shock expansion becomes wider and is positioned at distances further away from the wall, but the maximun strength of the local post-shock expansion is decreased. Unfortunately the authors did not find any information giving the available evidence for this tendency. This is the reason that with the increase of M_1, the interaction of the shock with boundary layer becomes stronger and thus turbulent boundary layer under the shock thicker. Therefore, the streamlines seem to be curved at distances further away from the wall surface.

Figure 7 shows the streamwise position of the post-shock expansion and the strength of it against the flow Mach numbers. The previous experiment results by the authors is also indicated by ■. With the increase of M_1, the post-shock expansion is positioned at further downstream in streamwise direction and the strengh of it at $y/\delta_a=3.2$ distance from the wall is increased. This is appreciably different from the experiment results in its level, but the tendency to the flow Mach numbers seems to be acceptable, especially in considering that the experiment was conducted in the supersonic nozzle. Moreover, the constraint effect of the bounded flows can play an important role on the strength of the post-shock expansion. Thus the difference in its strength and the pressure recovery must be always recognized as the fact. From these points of view, **Fig. 8** seems to be qualitatively consis-

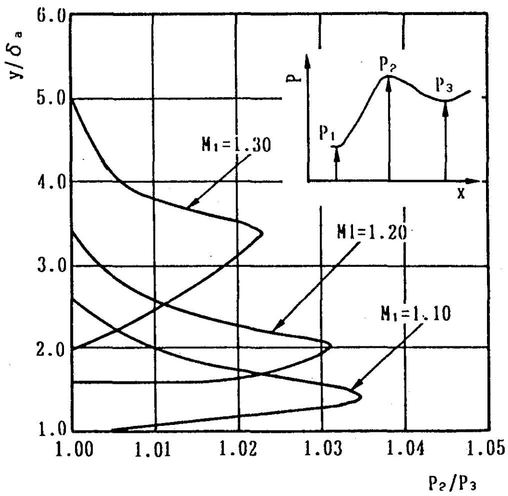

Fig. 6　Post-shock expansion in transverse direction

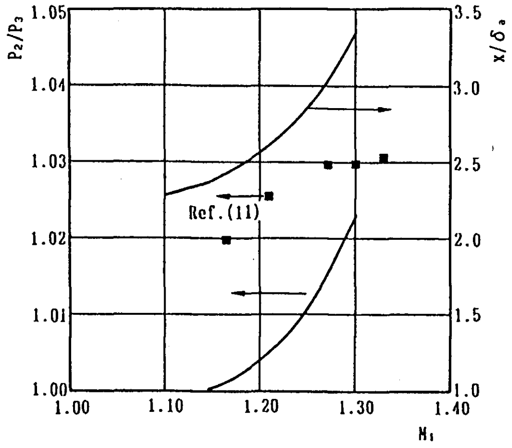

Fig. 7 Post-shock expansion against M1 and streamwise extent of it

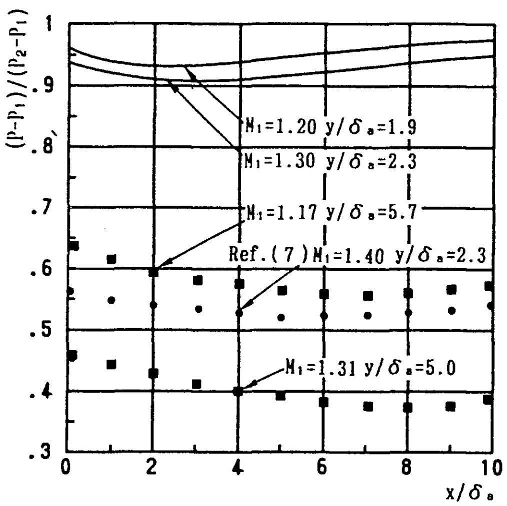

Fig. 8 Pressure distributions behind shock wave

tent with experiment results. Moreover the results by Ref. 7, where experiment was done in straight duct with the flow Mach number 1.40, agree with our results well. Conclusively the study on the post-shock expansion in the present flat surface has a good base for further work in future.

4. Conclusions

A simple theory was presented for the interaction between an weak normal shock wave and a turbulent boundary layer on a flat surface. The post-shock expansion as a whole was well explained by the present theory. The results showed that the generation of the post-shock expansion could be explained by the curved streamline, and that with the increase of the flow Mach number, the post-shock expansion was increased too, its streamwise position located at further downstream and the transverse position of it loacted at distance further away from the wall. Also, the present results were qualitatively consistent with experimental ones.

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