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Derivation of the distribution of sample regression coefficient by using computer algebra

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The purpose of this article is to give a higher order asymptotic expansion of the distribution of sample regression coefficient for nonnormal populations. The order of expansion is $1/n\sqrt{n}$. In deriving higher order terms, a symbolic algorithm for obtaining moments of symmetric statistics has been used.

1. Introduction

Sample regression coefficient is one the most important statistics, which is also one of the symmetric statistics. Regression coefficient b, based on a random sample $X_1=(x_1, y_1)$, $X_2=(x_2, y_2)$,..., $X_n=(x_n, y_n)$ from a population, whose distribution function is F, is designated by

$$b=\frac{\sum_{i=1}^{n} x_i y_i-\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n} x_i^2-\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$$

Concerning the distribution of b, Cook [2),3)] calculated approximate moments using k-statisitcs. From his results, we have the asymptotic expansion only up to order $1/\sqrt{n}$ To obtain a higher order asymptotic expansion, we require more approximate cumulants or moments, having further high order terms.

The need of huge computation, however, have prevented us from getting them. To overcome this, computer algera which is powerful tool for these kinds of calculation helps us. In fact, simple algorithms appropriate to computer algebra can be seen in Niki[5)]. They have been implemented as LISP functions with interface to REDUCE[4)], a computer algebra system used worldwide. A library of formulae required in obtaining asymptotic expansions has been also prepared.

The purpose of this article is to obtain the asymptotic expansion for the distribution of b up to order $1/n\sqrt{n}$ for nonnormal populations.

2. Higher order cumulants of sample regression coefficient

It is assume that F has finite moments of requisite order. Let κ_{10}, κ_{01}, κ_{20}, κ_{11}, κ_{02}, $\cdots$ denote the cumulants of F.

In order to obtain approximate moments of b, we use auxiliary variables as follows:

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$$w_x = \frac{\sqrt{n}\ (s_x^2 - \kappa_{20})}{\kappa_{20}},\ s_x^2 = \frac{1}{n}\left\{ \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right\},$$

$$w_{xy} = \frac{\sqrt{n}\ (s_{xy} - \kappa_{11})}{\kappa_{11}},\ s_{xy} = \frac{1}{n}\left\{ \sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\right\}.$$

Note that w_x and w_{xy} have the limiting normal distributions in law as $n \to \infty$. Clearly, s_x^2 and s_{xy} are polynomials of power sums, and so w_x are and w_{xy}. We can expand b as power series in terms of $1/\sqrt{n}$ whose coefficients are polynomials of w_x and w_{xy}:

$$\begin{aligned} b &= \frac{\kappa_{11}}{\kappa_{20}}\left(1+\frac{w_{xy}}{\sqrt{n}}\right)\left(1+\frac{w_x}{\sqrt{n}}\right)^{-1} \\ &= \frac{\kappa_{11}}{\kappa_{20}}\left\{1+\frac{1}{\sqrt{n}}(w_{xy}-w_x)+\frac{1}{n}(-w_{xy}w_x+w_x^2)\right. \\ &\qquad \left. +\frac{1}{n\sqrt{n}}(w_{xy}w_x^2-w_x^3)+\frac{1}{n^2}(-w_{xy}w_x^3+w_x^4)\right\} \\ &\qquad +O\left(\frac{1}{n^2\sqrt{n}}\right). \end{aligned}$$

Then the first approximate moment of b is calculated from the apove expression by taking expectation term by term. The r-th ore is similarly given from b^r.

On taking expectation $E[w_x^j w_{xy}^k]$ $(j \geq 0,\ k \geq 0,\ j+k \leq 6)$, we have used the package due to Niki[5] and obtained the first four approximate moments of b up to order $1/n^3$. Our results coincide with Cook's results up to order $1/n^2$.

The approximate cumulants λ_i $(i=1, \ldots, 4)$ of b are given from the relations between moments and cumulants. The lower order parts of them are as follows:

$$\lambda_1 = \frac{1}{\sqrt{n}}\kappa_{20}^{-3}(\kappa_{40}\kappa_{11} - \kappa_{31}\kappa_{20}) + O\left(\frac{1}{n\sqrt{n}}\right)$$

$$\begin{aligned} \lambda_2 = {} & \kappa_{20}^{-4}(\kappa_{40}\kappa_{11}^2 - 2\kappa_{31}\kappa_{20}\kappa_{11} + \kappa_{22}\kappa_{20}^2 + \kappa_{20}^3\kappa_{02} - \kappa_{20}^2\kappa_{11}^2) \\ & + \frac{1}{n}\kappa_{20}^{-6}(- 2\kappa_{60}\kappa_{20}\kappa_{11}^2 + 4\kappa_{51}\kappa_{20}^2\kappa_{11} - 2\kappa_{42}\kappa_{20}^3 + 8\kappa_{40}^2\kappa_{11}^2 - 16\kappa_{40}\kappa_{31}\kappa_{20}\kappa_{11} \\ & \qquad + 3\kappa_{40}\kappa_{22}\kappa_{20}^2 + \kappa_{40}\kappa_{20}^3\kappa_{02} - 5\kappa_{40}\kappa_{20}^2\kappa_{11}^2 + 5\kappa_{31}^2\kappa_{20}^2 + 8\kappa_{31}\kappa_{20}^3\kappa_{11} \\ & \qquad - 8\kappa_{30}^2\kappa_{20}\kappa_{11}^2 + 16\kappa_{30}\kappa_{21}\kappa_{20}^2\kappa_{11} - 4\kappa_{30}\kappa_{20}^3\kappa_{12} - 4\kappa_{22}\kappa_{20}^4 - 4\kappa_{21}^2\kappa_{20}^3 \\ & \qquad + 3\kappa_{20}^5\kappa_{02} - 3\kappa_{20}^4\kappa_{11}^2) \\ & + O\left(\frac{1}{n^2}\right) \end{aligned}$$

$$\lambda_3 = \frac{1}{\sqrt{n}}\kappa_{20}^{-7}(- \kappa_{60}\kappa_{20}\kappa_{11}^3 + 3\kappa_{51}\kappa_{20}^2\kappa_{11}^2 - 3\kappa_{42}\kappa_{20}^3\kappa_{11} + 6\kappa_{40}^2\kappa_{11}^3$$
$$- 18\kappa_{40}\kappa_{31}\kappa_{20}\kappa_{11}^2 + 6\kappa_{40}\kappa_{22}\kappa_{20}^2\kappa_{11} + 3\kappa_{40}\kappa_{20}^3\kappa_{11}\kappa_{02} - 6\kappa_{40}\kappa_{20}^2\kappa_{11}^3$$
$$+ \kappa_{33}\kappa_{20}^4 + 12\kappa_{31}^2\kappa_{20}^2\kappa_{11} - 6\kappa_{31}\kappa_{22}\kappa_{20}^3 - 3\kappa_{31}\kappa_{20}^4\kappa_{02} + 12\kappa_{31}\kappa_{20}^3\kappa_{11}^2$$
$$- 4\kappa_{30}^2\kappa_{20}\kappa_{11}^3 + 12\kappa_{30}\kappa_{21}\kappa_{20}^2\kappa_{11}^2 + \kappa_{30}\kappa_{20}^4\kappa_{03} - 6\kappa_{30}\kappa_{20}^3\kappa_{12}\kappa_{11}$$
$$- 9\kappa_{22}\kappa_{20}^4\kappa_{11} - 6\kappa_{21}^2\kappa_{20}^3\kappa_{11} + 3\kappa_{21}\kappa_{20}^4\kappa_{12} + 3\kappa_{20}^5\kappa_{13}) + O(\frac{1}{n\sqrt{n}})$$

$$\lambda_4 = \frac{1}{n}\kappa_{20}^{-10}(\kappa_{80}\kappa_{20}^2\kappa_{11}^4 - 4\kappa_{71}\kappa_{20}^3\kappa_{11}^3 + 6\kappa_{62}\kappa_{20}^4\kappa_{11}^2 - 24\kappa_{60}\kappa_{40}\kappa_{20}\kappa_{11}^4$$
$$+ 36\kappa_{60}\kappa_{31}\kappa_{20}^2\kappa_{11}^3 - 12\kappa_{60}\kappa_{22}\kappa_{20}^3\kappa_{11}^2 - 6\kappa_{60}\kappa_{20}^4\kappa_{11}^2\kappa_{02} + 12\kappa_{60}\kappa_{20}^3\kappa_{11}^4$$
$$- 4\kappa_{53}\kappa_{20}^5\kappa_{11} + 60\kappa_{51}\kappa_{40}\kappa_{20}^2\kappa_{11}^3 - 84\kappa_{51}\kappa_{31}\kappa_{20}^3\kappa_{11}^2 + 24\kappa_{51}\kappa_{22}\kappa_{20}^4\kappa_{11}$$
$$+ 12\kappa_{51}\kappa_{20}^5\kappa_{11}\kappa_{02} - 36\kappa_{51}\kappa_{20}^4\kappa_{11}^3 + 32\kappa_{50}\kappa_{30}\kappa_{20}^2\kappa_{11}^4 - 48\kappa_{50}\kappa_{21}\kappa_{20}^3\kappa_{11}^3$$
$$- 4\kappa_{50}\kappa_{20}^5\kappa_{11}\kappa_{03} + 24\kappa_{50}\kappa_{20}^4\kappa_{12}\kappa_{11}^2 + \kappa_{44}\kappa_{20}^6 - 48\kappa_{42}\kappa_{40}\kappa_{20}^3\kappa_{11}^2$$
$$+ 60\kappa_{42}\kappa_{31}\kappa_{20}^4\kappa_{11} - 12\kappa_{42}\kappa_{22}\kappa_{20}^5 - 6\kappa_{42}\kappa_{20}^6\kappa_{02} + 42\kappa_{42}\kappa_{20}^5\kappa_{11}^2$$
$$- 80\kappa_{41}\kappa_{30}\kappa_{20}^3\kappa_{11}^3 + 96\kappa_{41}\kappa_{21}\kappa_{20}^4\kappa_{11}^2 + 4\kappa_{41}\kappa_{20}^6\kappa_{03} - 36\kappa_{41}\kappa_{20}^5\kappa_{12}\kappa_{11}$$
$$+ 72\kappa_{40}^3\kappa_{11}^4 - 288\kappa_{40}^2\kappa_{31}\kappa_{20}\kappa_{11}^3 + 84\kappa_{40}^2\kappa_{22}\kappa_{20}^2\kappa_{11}^2 + 36\kappa_{40}^2\kappa_{20}^3\kappa_{11}^2\kappa_{02}$$
$$- 76\kappa_{40}^2\kappa_{20}^2\kappa_{11}^4 + 12\kappa_{40}\kappa_{33}\kappa_{20}^4\kappa_{11} + 348\kappa_{40}\kappa_{31}^2\kappa_{20}^2\kappa_{11}^2 - 168\kappa_{40}\kappa_{31}\kappa_{22}\kappa_{20}^3\kappa_{11}$$
$$- 72\kappa_{40}\kappa_{31}\kappa_{20}^4\kappa_{11}\kappa_{02} + 232\kappa_{40}\kappa_{31}\kappa_{20}^3\kappa_{11}^3 - 96\kappa_{40}\kappa_{30}^2\kappa_{20}\kappa_{11}^4 + 240\kappa_{40}\kappa_{30}\kappa_{21}\kappa_{20}^2\kappa_{11}^3$$
$$+ 12\kappa_{40}\kappa_{30}\kappa_{20}^4\kappa_{11}\kappa_{03} - 96\kappa_{40}\kappa_{30}\kappa_{20}^3\kappa_{12}\kappa_{11}^2 + 12\kappa_{40}\kappa_{22}^2\kappa_{20}^4 + 12\kappa_{40}\kappa_{22}\kappa_{20}^5\kappa_{02}$$
$$- 108\kappa_{40}\kappa_{22}\kappa_{20}^4\kappa_{11}^2 - 96\kappa_{40}\kappa_{21}^2\kappa_{20}^3\kappa_{11}^2 + 36\kappa_{40}\kappa_{21}\kappa_{20}^4\kappa_{12}\kappa_{11} + \kappa_{40}\kappa_{20}^6\kappa_{04}$$
$$+ 3\kappa_{40}\kappa_{20}^6\kappa_{02}^2 + 16\kappa_{40}\kappa_{20}^5\kappa_{13}\kappa_{11} - 12\kappa_{40}\kappa_{20}^5\kappa_{11}^2\kappa_{02} + 12\kappa_{40}\kappa_{20}^4\kappa_{11}^4$$
$$- 12\kappa_{33}\kappa_{31}\kappa_{20}^5 - 24\kappa_{33}\kappa_{20}^6\kappa_{11} + 72\kappa_{32}\kappa_{30}\kappa_{20}^4\kappa_{11}^2 - 60\kappa_{32}\kappa_{21}\kappa_{20}^5\kappa_{11} + 12\kappa_{32}\kappa_{20}^6\kappa_{12}$$
$$- 120\kappa_{31}^3\kappa_{20}^3\kappa_{11} + 60\kappa_{31}^2\kappa_{22}\kappa_{20}^4 + 24\kappa_{31}^2\kappa_{20}^5\kappa_{02} - 168\kappa_{31}^2\kappa_{20}^4\kappa_{11}^2$$
$$+ 144\kappa_{31}\kappa_{30}^2\kappa_{20}^2\kappa_{11}^3 - 336\kappa_{31}\kappa_{30}\kappa_{21}\kappa_{20}^3\kappa_{11}^2 - 12\kappa_{31}\kappa_{30}\kappa_{20}^5\kappa_{03} + 120\kappa_{31}\kappa_{30}\kappa_{20}^4\kappa_{12}\kappa_{11}$$
$$+ 144\kappa_{31}\kappa_{22}\kappa_{20}^5\kappa_{11} + 120\kappa_{31}\kappa_{21}^2\kappa_{20}^4\kappa_{11} - 36\kappa_{31}\kappa_{21}\kappa_{20}^5\kappa_{12} - 20\kappa_{31}\kappa_{20}^6\kappa_{13}$$
$$+ 12\kappa_{31}\kappa_{20}^6\kappa_{11}\kappa_{02} - 24\kappa_{31}\kappa_{20}^5\kappa_{11}^3 - 48\kappa_{30}^2\kappa_{22}\kappa_{20}^3\kappa_{11}^2 - 24\kappa_{30}^2\kappa_{20}^4\kappa_{11}^2\kappa_{02} + 48\kappa_{30}^2\kappa_{20}^3\kappa_{11}^4$$
$$- 28\kappa_{30}\kappa_{23}\kappa_{20}^5\kappa_{11} + 96\kappa_{30}\kappa_{22}\kappa_{21}\kappa_{20}^4\kappa_{11} - 24\kappa_{30}\kappa_{22}\kappa_{20}^5\kappa_{12} + 48\kappa_{30}\kappa_{21}\kappa_{20}^5\kappa_{11}\kappa_{02}$$
$$- 144\kappa_{30}\kappa_{21}\kappa_{20}^4\kappa_{11}^3 + 4\kappa_{30}\kappa_{20}^6\kappa_{14} - 12\kappa_{30}\kappa_{20}^6\kappa_{12}\kappa_{02} - 12\kappa_{30}\kappa_{20}^6\kappa_{11}\kappa_{03}$$
$$+ 72\kappa_{30}\kappa_{20}^5\kappa_{12}\kappa_{11}^2 + 6\kappa_{24}\kappa_{20}^7 + 12\kappa_{23}\kappa_{21}\kappa_{20}^6 - 21\kappa_{22}^2\kappa_{20}^6$$
$$- 24\kappa_{22}\kappa_{21}^2\kappa_{20}^5 - 6\kappa_{22}\kappa_{20}^7\kappa_{02} + 24\kappa_{22}\kappa_{20}^6\kappa_{11}^2 - 12\kappa_{21}^2\kappa_{20}^6\kappa_{02}$$
$$+ 96\kappa_{21}^2\kappa_{20}^5\kappa_{11}^2 + 12\kappa_{21}\kappa_{20}^7\kappa_{03} - 84\kappa_{21}\kappa_{20}^6\kappa_{12}\kappa_{11} + 3\kappa_{20}^8\kappa_{04} + 6\kappa_{20}^8\kappa_{02}^2$$
$$- 12\kappa_{20}^7\kappa_{13}\kappa_{11} + 12\kappa_{20}^7\kappa_{12}^2 - 12\kappa_{20}^7\kappa_{11}^2\kappa_{02} + 6\kappa_{20}^6\kappa_{11}^4)$$
$$+ O(\frac{1}{n^2})$$

3. Asymptotic expansion

Now we derive the asymptotic expansion for the distribution of the standardized variate B of b, given by

$$B=\frac{\sqrt{n}\ (b-\beta)}{\sigma},$$

where

$$\beta=\frac{\kappa_{11}}{\kappa_{20}},$$

$$\sigma=\frac{\sqrt{\kappa_{40}\kappa_{11}^2-2\kappa_{31}\kappa_{20}\kappa_{11}+\kappa_{22}\kappa_{20}^2+\kappa_{20}^3\kappa_{02}-\kappa_{20}^2\kappa_{11}^2}}{\kappa_{20}^2}.$$

From the choice of σ, approximate cumulants of B calculated from above λ_j's satisfy the Cornish-Fisher assumption. Therefore, we can obtain the asymptotic expansion, namely Edgeworth expansion, for the distribution of B, where so-called Delta method, is used. We note that the validity of Delta method is proved by Bhattacharya and Ghosh[1]. Requisite formulae of the higher order Edgeworth expansion have already derived by Niki and Konishi[6]. Substituting of them to formulae, we get the asymptotic expansion up to order $1/n\sqrt{n}$. A Cornish-Fisher inverse expansion for percentiles has been also obtained up to the same order.

Let $\Phi(x)$ and $\phi(x)$ be the standard normal distribution and its density function, respectively. The Edgeworth expansion is shown in the following:

$$Pr[B<x]=\Phi(x)-\phi(x)\left\{\frac{1}{\sqrt{n}}C_1+\frac{1}{n}C_2\right\}+O\left(\frac{1}{n\sqrt{n}}\right).$$

C_1 and C_2 are polynomials which consist of j-th Hermite polynomial $H_j(x)$ and population cumulants κ_{pq}; for detail, as below:

$$\begin{aligned}C_1 = \frac{H_2(x)}{\sigma^3\kappa_{20}^7}\Big(& -\frac{1}{6}\kappa_{60}\kappa_{20}\kappa_{11}^3+\frac{1}{2}\kappa_{51}\kappa_{20}^2\kappa_{11}^2-\frac{1}{2}\kappa_{42}\kappa_{20}^3\kappa_{11}+\kappa_{40}^2\kappa_{11}^3-3\kappa_{40}\kappa_{31}\kappa_{20}\kappa_{11}^2\\ & +\kappa_{40}\kappa_{22}\kappa_{20}^2\kappa_{11}+\frac{1}{2}\kappa_{40}\kappa_{20}^3\kappa_{11}\kappa_{02}-\kappa_{40}\kappa_{20}^2\kappa_{11}^3+\frac{1}{6}\kappa_{33}\kappa_{20}^4+2\kappa_{31}^2\kappa_{20}^2\kappa_{11}\\ & -\kappa_{31}\kappa_{22}\kappa_{20}^3-\frac{1}{2}\kappa_{31}\kappa_{20}^4\kappa_{02}+2\kappa_{31}\kappa_{20}^3\kappa_{11}^2-\frac{2}{3}\kappa_{30}^2\kappa_{20}\kappa_{11}^3+2\kappa_{30}\kappa_{21}\kappa_{20}^2\kappa_{11}^2\\ & +\frac{1}{6}\kappa_{30}\kappa_{20}^4\kappa_{03}-\kappa_{30}\kappa_{20}^3\kappa_{12}\kappa_{11}-\frac{3}{2}\kappa_{22}\kappa_{20}^4\kappa_{11}-\kappa_{21}^2\kappa_{20}^3\kappa_{11}+\frac{1}{2}\kappa_{21}\kappa_{20}^4\kappa_{12}+\frac{1}{2}\kappa_{20}^5\kappa_{13}\Big)\\ & +\frac{1}{\sigma\kappa_{20}^3}(\kappa_{40}\kappa_{11}-\kappa_{31}\kappa_{20})\end{aligned}$$

$$C_2 = \frac{H_5(x)}{\sigma^6 \kappa_{20}^{14}}\Big(\frac{1}{72}\kappa_{60}^2\kappa_{20}^2\kappa_{11}^6 - \frac{1}{12}\kappa_{60}\kappa_{51}\kappa_{20}^3\kappa_{11}^5 + \frac{1}{12}\kappa_{60}\kappa_{42}\kappa_{20}^4\kappa_{11}^4 - \frac{1}{6}\kappa_{60}\kappa_{40}^2\kappa_{20}\kappa_{11}^6$$

$$+ \frac{1}{2}\kappa_{60}\kappa_{40}\kappa_{31}\kappa_{20}^2\kappa_{11}^5 - \frac{1}{6}\kappa_{60}\kappa_{40}\kappa_{22}\kappa_{20}^3\kappa_{11}^4 - \frac{1}{12}\kappa_{60}\kappa_{40}\kappa_{20}^4\kappa_{11}^4\kappa_{02} + \frac{1}{6}\kappa_{60}\kappa_{40}\kappa_{20}^3\kappa_{11}^6$$

$$- \frac{1}{36}\kappa_{60}\kappa_{33}\kappa_{20}^5\kappa_{11}^3 - \frac{1}{3}\kappa_{60}\kappa_{31}^2\kappa_{20}^3\kappa_{11}^4 + \frac{1}{6}\kappa_{60}\kappa_{31}\kappa_{22}\kappa_{20}^4\kappa_{11}^3 + \frac{1}{12}\kappa_{60}\kappa_{31}\kappa_{20}^5\kappa_{11}^3\kappa_{02}$$

$$- \frac{1}{3}\kappa_{60}\kappa_{31}\kappa_{20}^4\kappa_{11}^5 + \frac{1}{9}\kappa_{60}\kappa_{30}^2\kappa_{20}^2\kappa_{11}^6 - \frac{1}{3}\kappa_{60}\kappa_{30}\kappa_{21}\kappa_{20}^3\kappa_{11}^5 - \frac{1}{36}\kappa_{60}\kappa_{30}\kappa_{20}^5\kappa_{11}^3\kappa_{03}$$

$$+ \frac{1}{6}\kappa_{60}\kappa_{30}\kappa_{20}^4\kappa_{12}\kappa_{11}^4 + \frac{1}{4}\kappa_{60}\kappa_{22}\kappa_{20}^5\kappa_{11}^4 + \frac{1}{6}\kappa_{60}\kappa_{21}^2\kappa_{20}^4\kappa_{11}^4$$

$$- \frac{1}{12}\kappa_{60}\kappa_{21}\kappa_{20}^5\kappa_{12}\kappa_{11}^3 - \frac{1}{12}\kappa_{60}\kappa_{20}^6\kappa_{13}\kappa_{11}^3 + \frac{1}{8}\kappa_{51}^2\kappa_{20}^4\kappa_{11}^4 - \frac{1}{4}\kappa_{51}\kappa_{42}\kappa_{20}^5\kappa_{11}^3$$

$$+ \frac{1}{2}\kappa_{51}\kappa_{40}^2\kappa_{20}^2\kappa_{11}^5 - \frac{3}{2}\kappa_{51}\kappa_{40}\kappa_{31}\kappa_{20}^3\kappa_{11}^4 + \frac{1}{2}\kappa_{51}\kappa_{40}\kappa_{22}\kappa_{20}^4\kappa_{11}^3 + \frac{1}{4}\kappa_{51}\kappa_{40}\kappa_{20}^5\kappa_{11}^3\kappa_{02}$$

$$- \frac{1}{2}\kappa_{51}\kappa_{40}\kappa_{20}^4\kappa_{11}^5 + \frac{1}{12}\kappa_{51}\kappa_{33}\kappa_{20}^6\kappa_{11}^2 + \kappa_{51}\kappa_{31}^2\kappa_{20}^4\kappa_{11}^3 - \frac{1}{2}\kappa_{51}\kappa_{31}\kappa_{22}\kappa_{20}^5\kappa_{11}^2$$

$$- \frac{1}{4}\kappa_{51}\kappa_{31}\kappa_{20}^6\kappa_{11}^2\kappa_{02} + \kappa_{51}\kappa_{31}\kappa_{20}^5\kappa_{11}^4 - \frac{1}{3}\kappa_{51}\kappa_{30}^2\kappa_{20}^3\kappa_{11}^5 + \kappa_{51}\kappa_{30}\kappa_{21}\kappa_{20}^4\kappa_{11}^4$$

$$+ \frac{1}{12}\kappa_{51}\kappa_{30}\kappa_{20}^6\kappa_{11}^2\kappa_{03} - \frac{1}{2}\kappa_{51}\kappa_{30}\kappa_{20}^5\kappa_{12}\kappa_{11}^3 - \frac{3}{4}\kappa_{51}\kappa_{22}\kappa_{20}^6\kappa_{11}^3 - \frac{1}{2}\kappa_{51}\kappa_{21}^2\kappa_{20}^5\kappa_{11}^3$$

$$+ \frac{1}{4}\kappa_{51}\kappa_{21}\kappa_{20}^6\kappa_{12}\kappa_{11}^2 + \frac{1}{4}\kappa_{51}\kappa_{20}^7\kappa_{13}\kappa_{11}^2 + \frac{1}{8}\kappa_{42}^2\kappa_{20}^6\kappa_{11}^2 - \frac{1}{2}\kappa_{42}\kappa_{40}^2\kappa_{20}^3\kappa_{11}^4$$

$$+ \frac{3}{2}\kappa_{42}\kappa_{40}\kappa_{31}\kappa_{20}^4\kappa_{11}^3 - \frac{1}{2}\kappa_{42}\kappa_{40}\kappa_{22}\kappa_{20}^5\kappa_{11}^2 - \frac{1}{4}\kappa_{42}\kappa_{40}\kappa_{20}^6\kappa_{11}^2\kappa_{02} + \frac{1}{2}\kappa_{42}\kappa_{40}\kappa_{20}^5\kappa_{11}^4$$

$$- \frac{1}{12}\kappa_{42}\kappa_{33}\kappa_{20}^7\kappa_{11} - \kappa_{42}\kappa_{31}^2\kappa_{20}^5\kappa_{11}^2 + \frac{1}{2}\kappa_{42}\kappa_{31}\kappa_{22}\kappa_{20}^6\kappa_{11} + \frac{1}{4}\kappa_{42}\kappa_{31}\kappa_{20}^7\kappa_{11}\kappa_{02}$$

$$- \kappa_{42}\kappa_{31}\kappa_{20}^6\kappa_{11}^3 + \frac{1}{3}\kappa_{42}\kappa_{30}^2\kappa_{20}^4\kappa_{11}^4 - \kappa_{42}\kappa_{30}\kappa_{21}\kappa_{20}^5\kappa_{11}^3 - \frac{1}{12}\kappa_{42}\kappa_{30}\kappa_{20}^7\kappa_{11}\kappa_{03}$$

$$+ \frac{1}{2}\kappa_{42}\kappa_{30}\kappa_{20}^6\kappa_{12}\kappa_{11}^2 + \frac{3}{4}\kappa_{42}\kappa_{22}\kappa_{20}^7\kappa_{11}^2 + \frac{1}{2}\kappa_{42}\kappa_{21}^2\kappa_{20}^6\kappa_{11}^2 - \frac{1}{4}\kappa_{42}\kappa_{21}\kappa_{20}^7\kappa_{12}\kappa_{11}$$

$$- \frac{1}{4}\kappa_{42}\kappa_{20}^8\kappa_{13}\kappa_{11} + \frac{1}{2}\kappa_{40}^4\kappa_{11}^6 - 3\kappa_{40}^3\kappa_{31}\kappa_{20}\kappa_{11}^5 + \kappa_{40}^3\kappa_{22}\kappa_{20}^2\kappa_{11}^4 + \frac{1}{2}\kappa_{40}^3\kappa_{20}^3\kappa_{11}^4\kappa_{02}$$

$$- \kappa_{40}^3\kappa_{20}^2\kappa_{11}^6 + \frac{1}{6}\kappa_{40}^2\kappa_{33}\kappa_{20}^4\kappa_{11}^3 + \frac{13}{2}\kappa_{40}^2\kappa_{31}^2\kappa_{20}^2\kappa_{11}^4 - 4\kappa_{40}^2\kappa_{31}\kappa_{22}\kappa_{20}^3\kappa_{11}^3$$

$$- 2\kappa_{40}^2\kappa_{31}\kappa_{20}^4\kappa_{11}^3\kappa_{02} + 5\kappa_{40}^2\kappa_{31}\kappa_{20}^3\kappa_{11}^5 - \frac{2}{3}\kappa_{40}^2\kappa_{30}^2\kappa_{20}\kappa_{11}^6 + 2\kappa_{40}^2\kappa_{30}\kappa_{21}\kappa_{20}^2\kappa_{11}^5$$

$$+ \frac{1}{6}\kappa_{40}^2\kappa_{30}\kappa_{20}^4\kappa_{11}^3\kappa_{03} - \kappa_{40}^2\kappa_{30}\kappa_{20}^3\kappa_{12}\kappa_{11}^4 + \frac{1}{2}\kappa_{40}^2\kappa_{22}^2\kappa_{20}^4\kappa_{11}^2 + \frac{1}{2}\kappa_{40}^2\kappa_{22}\kappa_{20}^5\kappa_{11}^2\kappa_{02}$$

$$- \frac{5}{2}\kappa_{40}^2\kappa_{22}\kappa_{20}^4\kappa_{11}^4 - \kappa_{40}^2\kappa_{21}^2\kappa_{20}^3\kappa_{11}^4 + \frac{1}{2}\kappa_{40}^2\kappa_{21}\kappa_{20}^4\kappa_{12}\kappa_{11}^3 + \frac{1}{8}\kappa_{40}^2\kappa_{20}^6\kappa_{11}^2\kappa_{02}^2$$

$$+ \frac{1}{2}\kappa_{40}^2\kappa_{20}^5\kappa_{13}\kappa_{11}^3 - \frac{1}{2}\kappa_{40}^2\kappa_{20}^5\kappa_{11}^4\kappa_{02} + \frac{1}{2}\kappa_{40}^2\kappa_{20}^4\kappa_{11}^6 - \frac{1}{2}\kappa_{40}\kappa_{33}\kappa_{31}\kappa_{20}^5\kappa_{11}^2$$

$$+ \frac{1}{6}\kappa_{40}\kappa_{33}\kappa_{22}\kappa_{20}^6\kappa_{11} + \frac{1}{12}\kappa_{40}\kappa_{33}\kappa_{20}^7\kappa_{11}\kappa_{02} - \frac{1}{6}\kappa_{40}\kappa_{33}\kappa_{20}^6\kappa_{11}^3 - 6\kappa_{40}\kappa_{31}^3\kappa_{20}^3\kappa_{11}^3$$

$$+ 5\kappa_{40}\kappa_{31}^2\kappa_{22}\kappa_{20}^4\kappa_{11}^2 + \frac{5}{2}\kappa_{40}\kappa_{31}^2\kappa_{20}^5\kappa_{11}^2\kappa_{02} - 8\kappa_{40}\kappa_{31}^2\kappa_{20}^4\kappa_{11}^4 + 2\kappa_{40}\kappa_{31}\kappa_{30}^2\kappa_{20}^2\kappa_{11}^5$$

$$- 6\kappa_{40}\kappa_{31}\kappa_{30}\kappa_{21}\kappa_{20}^3\kappa_{11}^4 - \frac{1}{2}\kappa_{40}\kappa_{31}\kappa_{30}\kappa_{20}^5\kappa_{11}^2\kappa_{03} + 3\kappa_{40}\kappa_{31}\kappa_{30}\kappa_{20}^4\kappa_{12}\kappa_{11}^3 - \kappa_{40}\kappa_{31}\kappa_{22}^2\kappa_{20}^5\kappa_{11}$$

$$- \kappa_{40}\kappa_{31}\kappa_{22}\kappa_{20}^{6}\kappa_{11}\kappa_{02} + \frac{15}{2}\kappa_{40}\kappa_{31}\kappa_{22}\kappa_{20}^{5}\kappa_{11}^{3} + 3\kappa_{40}\kappa_{31}\kappa_{21}^{2}\kappa_{20}^{4}\kappa_{11}^{3} - \frac{3}{2}\kappa_{40}\kappa_{31}\kappa_{21}\kappa_{20}^{5}\kappa_{12}\kappa_{11}^{2}$$

$$- \frac{1}{4}\kappa_{40}\kappa_{31}\kappa_{20}^{7}\kappa_{11}\kappa_{02}^{2} - \frac{3}{2}\kappa_{40}\kappa_{31}\kappa_{20}^{6}\kappa_{13}\kappa_{11}^{2} + \frac{3}{2}\kappa_{40}\kappa_{31}\kappa_{20}^{6}\kappa_{11}^{3}\kappa_{02} - 2\kappa_{40}\kappa_{31}\kappa_{20}^{5}\kappa_{11}^{5}$$

$$- \frac{2}{3}\kappa_{40}\kappa_{30}^{2}\kappa_{22}\kappa_{20}^{3}\kappa_{11}^{4} - \frac{1}{3}\kappa_{40}\kappa_{30}^{2}\kappa_{20}^{4}\kappa_{11}^{4}\kappa_{02} + \frac{2}{3}\kappa_{40}\kappa_{30}^{2}\kappa_{20}^{3}\kappa_{11}^{6} + 2\kappa_{40}\kappa_{30}\kappa_{22}\kappa_{21}\kappa_{20}^{4}\kappa_{11}^{3}$$

$$+ \frac{1}{6}\kappa_{40}\kappa_{30}\kappa_{22}\kappa_{20}^{6}\kappa_{11}\kappa_{03} - \kappa_{40}\kappa_{30}\kappa_{22}\kappa_{20}^{5}\kappa_{12}\kappa_{11}^{2} + \kappa_{40}\kappa_{30}\kappa_{21}\kappa_{20}^{5}\kappa_{11}^{3}\kappa_{02} - 2\kappa_{40}\kappa_{30}\kappa_{21}\kappa_{20}^{4}\kappa_{11}^{5}$$

$$+ \frac{1}{12}\kappa_{40}\kappa_{30}\kappa_{20}^{7}\kappa_{11}\kappa_{03}\kappa_{02} - \frac{1}{2}\kappa_{40}\kappa_{30}\kappa_{20}^{6}\kappa_{12}\kappa_{11}^{2}\kappa_{02} - \frac{1}{6}\kappa_{40}\kappa_{30}\kappa_{20}^{6}\kappa_{11}^{3}\kappa_{03} + \kappa_{40}\kappa_{30}\kappa_{20}^{5}\kappa_{12}\kappa_{11}^{4}$$

$$- \frac{3}{2}\kappa_{40}\kappa_{22}^{2}\kappa_{20}^{6}\kappa_{11}^{2} - \kappa_{40}\kappa_{22}\kappa_{21}^{2}\kappa_{20}^{5}\kappa_{11}^{2} + \frac{1}{2}\kappa_{40}\kappa_{22}\kappa_{21}\kappa_{20}^{6}\kappa_{12}\kappa_{11} + \frac{1}{2}\kappa_{40}\kappa_{22}\kappa_{20}^{7}\kappa_{13}\kappa_{11}$$

$$- \frac{3}{4}\kappa_{40}\kappa_{22}\kappa_{20}^{7}\kappa_{11}^{2}\kappa_{02} + \frac{3}{2}\kappa_{40}\kappa_{22}\kappa_{20}^{6}\kappa_{11}^{4} - \frac{1}{2}\kappa_{40}\kappa_{21}^{2}\kappa_{20}^{6}\kappa_{11}^{2}\kappa_{02} + \kappa_{40}\kappa_{21}^{2}\kappa_{20}^{5}\kappa_{11}^{4}$$

$$+ \frac{1}{4}\kappa_{40}\kappa_{21}\kappa_{20}^{7}\kappa_{12}\kappa_{11}\kappa_{02} - \frac{1}{2}\kappa_{40}\kappa_{21}\kappa_{20}^{6}\kappa_{12}\kappa_{11}^{3} + \frac{1}{4}\kappa_{40}\kappa_{20}^{8}\kappa_{13}\kappa_{11}\kappa_{02} - \frac{1}{2}\kappa_{40}\kappa_{20}^{7}\kappa_{13}\kappa_{11}^{3}$$

$$+ \frac{1}{72}\kappa_{33}^{2}\kappa_{20}^{8} + \frac{1}{3}\kappa_{33}\kappa_{31}^{2}\kappa_{20}^{6}\kappa_{11} - \frac{1}{6}\kappa_{33}\kappa_{31}\kappa_{22}\kappa_{20}^{7} - \frac{1}{12}\kappa_{33}\kappa_{31}\kappa_{20}^{8}\kappa_{02}$$

$$+ \frac{1}{3}\kappa_{33}\kappa_{31}\kappa_{20}^{7}\kappa_{11}^{2} - \frac{1}{9}\kappa_{33}\kappa_{30}^{2}\kappa_{20}^{5}\kappa_{11}^{3} + \frac{1}{3}\kappa_{33}\kappa_{30}\kappa_{21}\kappa_{20}^{6}\kappa_{11}^{2} + \frac{1}{36}\kappa_{33}\kappa_{30}\kappa_{20}^{8}\kappa_{03}$$

$$- \frac{1}{6}\kappa_{33}\kappa_{30}\kappa_{20}^{7}\kappa_{12}\kappa_{11} - \frac{1}{4}\kappa_{33}\kappa_{22}\kappa_{20}^{8}\kappa_{11} - \frac{1}{6}\kappa_{33}\kappa_{21}^{2}\kappa_{20}^{7}\kappa_{11} + \frac{1}{12}\kappa_{33}\kappa_{21}\kappa_{20}^{8}\kappa_{12}$$

$$+ \frac{1}{12}\kappa_{33}\kappa_{20}^{9}\kappa_{13} + 2\kappa_{31}^{4}\kappa_{20}^{4}\kappa_{11}^{2} - 2\kappa_{31}^{3}\kappa_{22}\kappa_{20}^{5}\kappa_{11} - \kappa_{31}^{3}\kappa_{20}^{6}\kappa_{11}\kappa_{02}$$

$$+ 4\kappa_{31}^{3}\kappa_{20}^{5}\kappa_{11}^{3} - \frac{4}{3}\kappa_{31}^{2}\kappa_{30}^{2}\kappa_{20}^{3}\kappa_{11}^{4} + 4\kappa_{31}^{2}\kappa_{30}\kappa_{21}\kappa_{20}^{4}\kappa_{11}^{3} + \frac{1}{3}\kappa_{31}^{2}\kappa_{30}\kappa_{20}^{6}\kappa_{11}\kappa_{03}$$

$$- 2\kappa_{31}^{2}\kappa_{30}\kappa_{20}^{5}\kappa_{12}\kappa_{11}^{2} + \frac{1}{2}\kappa_{31}^{2}\kappa_{22}^{2}\kappa_{20}^{6} + \frac{1}{2}\kappa_{31}^{2}\kappa_{22}\kappa_{20}^{7}\kappa_{02} - 5\kappa_{31}^{2}\kappa_{22}\kappa_{20}^{6}\kappa_{11}^{2}$$

$$- 2\kappa_{31}^{2}\kappa_{21}^{2}\kappa_{20}^{5}\kappa_{11}^{2} + \kappa_{31}^{2}\kappa_{21}\kappa_{20}^{6}\kappa_{12}\kappa_{11} + \frac{1}{8}\kappa_{31}^{2}\kappa_{20}^{8}\kappa_{02}^{2} + \kappa_{31}^{2}\kappa_{20}^{7}\kappa_{13}\kappa_{11}$$

$$- \kappa_{31}^{2}\kappa_{20}^{7}\kappa_{11}^{2}\kappa_{02} + 2\kappa_{31}^{2}\kappa_{20}^{6}\kappa_{11}^{4} + \frac{2}{3}\kappa_{31}\kappa_{30}^{2}\kappa_{22}\kappa_{20}^{4}\kappa_{11}^{3} + \frac{1}{3}\kappa_{31}\kappa_{30}^{2}\kappa_{20}^{5}\kappa_{11}^{3}\kappa_{02}$$

$$- \frac{4}{3}\kappa_{31}\kappa_{30}^{2}\kappa_{20}^{4}\kappa_{11}^{5} - 2\kappa_{31}\kappa_{30}\kappa_{22}\kappa_{21}\kappa_{20}^{5}\kappa_{11}^{2} - \frac{1}{6}\kappa_{31}\kappa_{30}\kappa_{22}\kappa_{20}^{7}\kappa_{03} + \kappa_{31}\kappa_{30}\kappa_{22}\kappa_{20}^{6}\kappa_{12}\kappa_{11}$$

$$- \kappa_{31}\kappa_{30}\kappa_{21}\kappa_{20}^{6}\kappa_{11}^{2}\kappa_{02} + 4\kappa_{31}\kappa_{30}\kappa_{21}\kappa_{20}^{5}\kappa_{11}^{4} - \frac{1}{12}\kappa_{31}\kappa_{30}\kappa_{20}^{8}\kappa_{03}\kappa_{02} + \frac{1}{2}\kappa_{31}\kappa_{30}\kappa_{20}^{7}\kappa_{12}\kappa_{11}\kappa_{02}$$

$$+ \frac{1}{3}\kappa_{31}\kappa_{30}\kappa_{20}^{7}\kappa_{11}^{2}\kappa_{03} - 2\kappa_{31}\kappa_{30}\kappa_{20}^{6}\kappa_{12}\kappa_{11}^{3} + \frac{3}{2}\kappa_{31}\kappa_{22}^{2}\kappa_{20}^{7}\kappa_{11} + \kappa_{31}\kappa_{22}\kappa_{21}^{2}\kappa_{20}^{6}\kappa_{11}$$

$$- \frac{1}{2}\kappa_{31}\kappa_{22}\kappa_{21}\kappa_{20}^{7}\kappa_{12} - \frac{1}{2}\kappa_{31}\kappa_{22}\kappa_{20}^{8}\kappa_{13} + \frac{3}{4}\kappa_{31}\kappa_{22}\kappa_{20}^{8}\kappa_{11}\kappa_{02} - 3\kappa_{31}\kappa_{22}\kappa_{20}^{7}\kappa_{11}^{3}$$

$$+ \frac{1}{2}\kappa_{31}\kappa_{21}^{2}\kappa_{20}^{7}\kappa_{11}\kappa_{02} - 2\kappa_{31}\kappa_{21}^{2}\kappa_{20}^{6}\kappa_{11}^{3} - \frac{1}{4}\kappa_{31}\kappa_{21}\kappa_{20}^{8}\kappa_{12}\kappa_{02}$$

$$+ \kappa_{31}\kappa_{21}\kappa_{20}^{7}\kappa_{12}\kappa_{11}^{2} - \frac{1}{4}\kappa_{31}\kappa_{20}^{9}\kappa_{13}\kappa_{02} + \kappa_{31}\kappa_{20}^{8}\kappa_{13}\kappa_{11}^{2} + \frac{2}{9}\kappa_{30}^{4}\kappa_{20}^{2}\kappa_{11}^{6}$$

$$- \frac{4}{3}\kappa_{30}^{3}\kappa_{21}\kappa_{20}^{3}\kappa_{11}^{5} - \frac{1}{9}\kappa_{30}^{3}\kappa_{20}^{5}\kappa_{11}^{3}\kappa_{03} + \frac{2}{3}\kappa_{30}^{3}\kappa_{20}^{4}\kappa_{12}\kappa_{11}^{4} + \kappa_{30}^{2}\kappa_{22}\kappa_{20}^{5}\kappa_{11}^{4}$$

$$+ \frac{8}{3}\kappa_{30}^{2}\kappa_{21}^{2}\kappa_{20}^{4}\kappa_{11}^{4} + \frac{1}{3}\kappa_{30}^{2}\kappa_{21}\kappa_{20}^{6}\kappa_{11}^{2}\kappa_{03} - \frac{7}{3}\kappa_{30}^{2}\kappa_{21}\kappa_{20}^{5}\kappa_{12}\kappa_{11}^{3} + \frac{1}{72}\kappa_{30}^{2}\kappa_{20}^{8}\kappa_{03}^{2}$$

$$- \frac{1}{6}\kappa_{30}^{2}\kappa_{20}^{7}\kappa_{12}\kappa_{11}\kappa_{03} - \frac{1}{3}\kappa_{30}^{2}\kappa_{20}^{6}\kappa_{13}\kappa_{11}^{3} + \frac{1}{2}\kappa_{30}^{2}\kappa_{20}^{6}\kappa_{12}^{2}\kappa_{11}^{2} - 3\kappa_{30}\kappa_{22}\kappa_{21}\kappa_{20}^{6}\kappa_{11}^{3}$$

$$- \frac{1}{4}\kappa_{30}\kappa_{22}\kappa_{20}^{8}\kappa_{11}\kappa_{03} + \frac{3}{2}\kappa_{30}\kappa_{22}\kappa_{20}^{7}\kappa_{12}\kappa_{11}^{2} - 2\kappa_{30}\kappa_{21}^{3}\kappa_{20}^{5}\kappa_{11}^{3} - \frac{1}{6}\kappa_{30}\kappa_{21}^{2}\kappa_{20}^{7}\kappa_{11}\kappa_{03}$$

$$+ 2\kappa_{30}\kappa_{21}^{2}\kappa_{20}^{6}\kappa_{12}\kappa_{11}^{2} + \frac{1}{12}\kappa_{30}\kappa_{21}\kappa_{20}^{8}\kappa_{12}\kappa_{03} + \kappa_{30}\kappa_{21}\kappa_{20}^{7}\kappa_{13}\kappa_{11}^{2} - \frac{1}{2}\kappa_{30}\kappa_{21}\kappa_{20}^{7}\kappa_{12}^{2}\kappa_{11}$$

$$+ \frac{1}{12}\kappa_{30}\kappa_{20}^{9}\kappa_{13}\kappa_{03} - \frac{1}{2}\kappa_{30}\kappa_{20}^{8}\kappa_{13}\kappa_{12}\kappa_{11} + \frac{9}{8}\kappa_{22}^{2}\kappa_{20}^{8}\kappa_{11}^{2} + \frac{3}{2}\kappa_{22}\kappa_{21}^{2}\kappa_{20}^{7}\kappa_{11}^{2}$$

$$- \frac{3}{4}\kappa_{22}\kappa_{21}\kappa_{20}^{8}\kappa_{12}\kappa_{11} - \frac{3}{4}\kappa_{22}\kappa_{20}^{9}\kappa_{13}\kappa_{11} + \frac{1}{2}\kappa_{21}^{4}\kappa_{20}^{6}\kappa_{11}^{2} - \frac{1}{2}\kappa_{21}^{3}\kappa_{20}^{7}\kappa_{12}\kappa_{11}$$

$$- \frac{1}{2}\kappa_{21}^{2}\kappa_{20}^{8}\kappa_{13}\kappa_{11} + \frac{1}{8}\kappa_{21}^{2}\kappa_{20}^{8}\kappa_{12}^{2} + \frac{1}{4}\kappa_{21}\kappa_{20}^{9}\kappa_{13}\kappa_{12} + \frac{1}{8}\kappa_{20}^{10}\kappa_{13}^{2})$$

$$+ \frac{H_3(x)}{\sigma^4\kappa_{20}^{10}}(\frac{1}{24}\kappa_{80}\kappa_{20}^{2}\kappa_{11}^{4} - \frac{1}{6}\kappa_{71}\kappa_{20}^{3}\kappa_{11}^{3} + \frac{1}{4}\kappa_{62}\kappa_{20}^{4}\kappa_{11}^{2} - \frac{7}{6}\kappa_{60}\kappa_{40}\kappa_{20}\kappa_{11}^{4}$$

$$+ \frac{5}{3}\kappa_{60}\kappa_{31}\kappa_{20}^{2}\kappa_{11}^{3} - \frac{1}{2}\kappa_{60}\kappa_{22}\kappa_{20}^{3}\kappa_{11}^{2} - \frac{1}{4}\kappa_{60}\kappa_{20}^{4}\kappa_{11}^{2}\kappa_{02} + \frac{1}{2}\kappa_{60}\kappa_{20}^{3}\kappa_{11}^{4}$$

$$- \frac{1}{6}\kappa_{53}\kappa_{20}^{5}\kappa_{11} + 3\kappa_{51}\kappa_{40}\kappa_{20}^{2}\kappa_{11}^{3} - 4\kappa_{51}\kappa_{31}\kappa_{20}^{3}\kappa_{11}^{2} + \kappa_{51}\kappa_{22}\kappa_{20}^{4}\kappa_{11}$$

$$+ \frac{1}{2}\kappa_{51}\kappa_{20}^{5}\kappa_{11}\kappa_{02} - \frac{3}{2}\kappa_{51}\kappa_{20}^{4}\kappa_{11}^{3} + \frac{4}{3}\kappa_{50}\kappa_{30}\kappa_{20}^{2}\kappa_{11}^{4} - 2\kappa_{50}\kappa_{21}\kappa_{20}^{3}\kappa_{11}^{3}$$

$$- \frac{1}{6}\kappa_{50}\kappa_{20}^{5}\kappa_{11}\kappa_{03} + \kappa_{50}\kappa_{20}^{4}\kappa_{12}\kappa_{11}^{2} + \frac{1}{24}\kappa_{44}\kappa_{20}^{6} - \frac{5}{2}\kappa_{42}\kappa_{40}\kappa_{20}^{3}\kappa_{11}^{2}$$

$$+ 3\kappa_{42}\kappa_{31}\kappa_{20}^{4}\kappa_{11} - \frac{1}{2}\kappa_{42}\kappa_{22}\kappa_{20}^{5} - \frac{1}{4}\kappa_{42}\kappa_{20}^{6}\kappa_{02} + \frac{7}{4}\kappa_{42}\kappa_{20}^{5}\kappa_{11}^{2}$$

$$- \frac{10}{3}\kappa_{41}\kappa_{30}\kappa_{20}^{3}\kappa_{11}^{3} + 4\kappa_{41}\kappa_{21}\kappa_{20}^{4}\kappa_{11}^{2} + \frac{1}{6}\kappa_{41}\kappa_{20}^{6}\kappa_{03} - \frac{3}{2}\kappa_{41}\kappa_{20}^{5}\kappa_{12}\kappa_{11}$$

$$+ 4\kappa_{40}^{3}\kappa_{11}^{4} - 16\kappa_{40}^{2}\kappa_{31}\kappa_{20}\kappa_{11}^{3} + \frac{9}{2}\kappa_{40}^{2}\kappa_{22}\kappa_{20}^{2}\kappa_{11}^{2} + 2\kappa_{40}^{2}\kappa_{20}^{3}\kappa_{11}^{2}\kappa_{02}$$

$$- \frac{25}{6}\kappa_{40}^{2}\kappa_{20}^{2}\kappa_{11}^{4} + \frac{2}{3}\kappa_{40}\kappa_{33}\kappa_{20}^{4}\kappa_{11} + \frac{39}{2}\kappa_{40}\kappa_{31}^{2}\kappa_{20}^{2}\kappa_{11}^{2} - 9\kappa_{40}\kappa_{31}\kappa_{22}\kappa_{20}^{3}\kappa_{11}$$

$$- 4\kappa_{40}\kappa_{31}\kappa_{20}^{4}\kappa_{11}\kappa_{02} + \frac{38}{3}\kappa_{40}\kappa_{31}\kappa_{20}^{3}\kappa_{11}^{3} - \frac{14}{3}\kappa_{40}\kappa_{30}^{2}\kappa_{20}\kappa_{11}^{4} + 12\kappa_{40}\kappa_{30}\kappa_{21}\kappa_{20}^{2}\kappa_{11}^{3}$$

$$+ \frac{2}{3}\kappa_{40}\kappa_{30}\kappa_{20}^{4}\kappa_{11}\kappa_{03} - 5\kappa_{40}\kappa_{30}\kappa_{20}^{3}\kappa_{12}\kappa_{11}^{2} + \frac{1}{2}\kappa_{40}\kappa_{22}^{2}\kappa_{20}^{4} + \frac{1}{2}\kappa_{40}\kappa_{22}\kappa_{20}^{5}\kappa_{02}$$

$$- 6\kappa_{40}\kappa_{22}\kappa_{20}^{4}\kappa_{11}^{2} - 5\kappa_{40}\kappa_{21}^{2}\kappa_{20}^{3}\kappa_{11}^{2} + 2\kappa_{40}\kappa_{21}\kappa_{20}^{4}\kappa_{12}\kappa_{11} + \frac{1}{24}\kappa_{40}\kappa_{20}^{6}\kappa_{04}$$

$$+ \frac{1}{8}\kappa_{40}\kappa_{20}^{6}\kappa_{02}^{2} + \frac{7}{6}\kappa_{40}\kappa_{20}^{5}\kappa_{13}\kappa_{11} - \frac{1}{2}\kappa_{40}\kappa_{20}^{5}\kappa_{11}^{2}\kappa_{02} + \frac{1}{2}\kappa_{40}\kappa_{20}^{4}\kappa_{11}^{4}$$

$$- \frac{2}{3}\kappa_{33}\kappa_{31}\kappa_{20}^{5} - \kappa_{33}\kappa_{20}^{6}\kappa_{11} + 3\kappa_{32}\kappa_{30}\kappa_{20}^{4}\kappa_{11}^{2} - \frac{5}{2}\kappa_{32}\kappa_{21}\kappa_{20}^{5}\kappa_{11}$$

$$+ \frac{1}{2}\kappa_{32}\kappa_{20}^{6}\kappa_{12} - 7\kappa_{31}^{3}\kappa_{20}^{3}\kappa_{11} + \frac{7}{2}\kappa_{31}^{2}\kappa_{22}\kappa_{20}^{4} + \frac{3}{2}\kappa_{31}^{2}\kappa_{20}^{5}\kappa_{02}$$

$$- 9\kappa_{31}^{2}\kappa_{20}^{4}\kappa_{11}^{2} + \frac{20}{3}\kappa_{31}\kappa_{30}^{2}\kappa_{20}^{2}\kappa_{11}^{3} - 16\kappa_{31}\kappa_{30}\kappa_{21}\kappa_{20}^{3}\kappa_{11}^{2} - \frac{2}{3}\kappa_{31}\kappa_{30}\kappa_{20}^{5}\kappa_{03}$$

$$+ 6\kappa_{31}\kappa_{30}\kappa_{20}^{4}\kappa_{12}\kappa_{11} + \frac{15}{2}\kappa_{31}\kappa_{22}\kappa_{20}^{5}\kappa_{11} + 6\kappa_{31}\kappa_{21}^{2}\kappa_{20}^{4}\kappa_{11} - 2\kappa_{31}\kappa_{21}\kappa_{20}^{5}\kappa_{12}$$

$$- \frac{4}{3}\kappa_{31}\kappa_{20}^{6}\kappa_{13} + \frac{1}{2}\kappa_{31}\kappa_{20}^{6}\kappa_{11}\kappa_{02} - \kappa_{31}\kappa_{20}^{5}\kappa_{11}^{3} - 2\kappa_{30}^{2}\kappa_{22}\kappa_{20}^{3}\kappa_{11}^{2}$$

$$- \kappa_{30}^{2}\kappa_{20}^{4}\kappa_{11}^{2}\kappa_{02} + 2\kappa_{30}^{2}\kappa_{20}^{3}\kappa_{11}^{4}$$

$$- \frac{7}{6}\kappa_{30}\kappa_{23}\kappa_{20}^{5}\kappa_{11} + 4\kappa_{30}\kappa_{22}\kappa_{21}\kappa_{20}^{4}\kappa_{11} - \kappa_{30}\kappa_{22}\kappa_{20}^{5}\kappa_{12} + 2\kappa_{30}\kappa_{21}\kappa_{20}^{5}\kappa_{11}\kappa_{02}$$

$$- 6\kappa_{30}\kappa_{21}\kappa_{20}^{4}\kappa_{11}^{3} + \frac{1}{6}\kappa_{30}\kappa_{20}^{6}\kappa_{14} - \frac{1}{2}\kappa_{30}\kappa_{20}^{6}\kappa_{12}\kappa_{02} - \frac{1}{2}\kappa_{30}\kappa_{20}^{6}\kappa_{11}\kappa_{03}$$

$$+ 3\kappa_{30}\kappa_{20}^{5}\kappa_{12}\kappa_{11}^{2} + \frac{1}{4}\kappa_{24}\kappa_{20}^{7} + \frac{1}{2}\kappa_{23}\kappa_{21}\kappa_{20}^{6} - \frac{7}{8}\kappa_{22}^{2}\kappa_{20}^{6}$$

$$- \kappa_{22}\kappa_{21}^{2}\kappa_{20}^{5} - \frac{1}{4}\kappa_{22}\kappa_{20}^{7}\kappa_{02} + \kappa_{22}\kappa_{20}^{6}\kappa_{11}^{2} - \frac{1}{2}\kappa_{21}^{2}\kappa_{20}^{6}\kappa_{02}$$

$$+ 4\kappa_{21}^{2}\kappa_{20}^{5}\kappa_{11}^{2} + \frac{1}{2}\kappa_{21}\kappa_{20}^{7}\kappa_{03} - \frac{7}{2}\kappa_{21}\kappa_{20}^{6}\kappa_{12}\kappa_{11} + \frac{1}{8}\kappa_{20}^{8}\kappa_{04}$$

$$+ \frac{1}{4}\kappa_{20}^{8}\kappa_{02}^{2} - \frac{1}{2}\kappa_{20}^{7}\kappa_{13}\kappa_{11} + \frac{1}{2}\kappa_{20}^{7}\kappa_{12}^{2} - \frac{1}{2}\kappa_{20}^{7}\kappa_{11}^{2}\kappa_{02} + \frac{1}{4}\kappa_{20}^{6}\kappa_{11}^{4})$$

$$+ \frac{H_1(x)}{\sigma^2\kappa_{20}^{6}}(- \kappa_{60}\kappa_{20}\kappa_{11}^{2} + 2\kappa_{51}\kappa_{20}^{2}\kappa_{11} - \kappa_{42}\kappa_{20}^{3} + \frac{9}{2}\kappa_{40}^{2}\kappa_{11}^{2} - 9\kappa_{40}\kappa_{31}\kappa_{20}\kappa_{11}$$

$$+ \frac{3}{2}\kappa_{40}\kappa_{22}\kappa_{20}^{2} + \frac{1}{2}\kappa_{40}\kappa_{20}^{3}\kappa_{02} - \frac{5}{2}\kappa_{40}\kappa_{20}^{2}\kappa_{11}^{2} + 3\kappa_{31}^{2}\kappa_{20}^{2} + 4\kappa_{31}\kappa_{20}^{3}\kappa_{11}$$

$$- 4\kappa_{30}^{2}\kappa_{20}\kappa_{11}^{2} + 8\kappa_{30}\kappa_{21}\kappa_{20}^{2}\kappa_{11} - 2\kappa_{30}\kappa_{20}^{3}\kappa_{12} - 2\kappa_{22}\kappa_{20}^{4}$$

$$- 2\kappa_{21}^{2}\kappa_{20}^{3} + \frac{3}{2}\kappa_{20}^{5}\kappa_{02} - \frac{3}{2}\kappa_{20}^{4}\kappa_{11}^{2})$$

4. Normal case

Now we apply above results to the case that F is the normal distribution, that is,

$$\phi\ (x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right\}.$$

Since population cumulants are

$$\kappa_{10} = \kappa_{01} = 0, \kappa_{20} = \kappa_{02} = 1, \kappa_{11} = \rho, \kappa_{pq} = 0 \,(\text{for } p+q>2),$$

the standardized variate is

$$B = \frac{\sqrt{n}\ (b-\rho)}{\sqrt{1-\rho^2}}.$$

Therefore cumulants ν_j of B is reduced as follows;

$$\nu_1 = 0,$$

$$\nu_2 = 1 + \frac{3}{n} + O(\frac{1}{n^2}),$$

$$\nu_3 = 0,$$

$$\nu_4 = \frac{6}{n} + O(\frac{1}{n^2}).$$

The Edgeworth expansion is also much simpler than in the general case;

$$Pr[B<x] = \Phi(x) - \frac{\phi(x)}{n}\left\{\frac{1}{4}H_3(x) + \frac{3}{2}H_1(x)\right\} + O(\frac{1}{n\sqrt{n}}).$$

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