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Abstract. The system of semi-relativistic particles coupled to a scalar bose field is investigated. A renormalized Hamiltonian is defined by subtracting a divergent term from the total Hamiltonian. We consider taking the scaling limit and removing the ultraviolet cutoffs simultaneously for the renormalized Hamiltonian. By applying an abstract scaling limit theory on self-adjoint operators, we derive the Yukawa potential as an effective potential of semi-relativistic particles.

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1 Introduction

We analyze the system of the N semi-relativistic particles coupled to a scalar bose field. We are interested in the asymptotic behavior of the renormalized Hamiltonian which is defined by subtracting a renormalization term from the total Hamiltonian for the system. The scaling limit of the total Hamiltonian with fixed ultraviolet cutoffs is considered in [22]. In this paper, we take the scaling limit and remove the ultraviolet cutoffs simultaneously for the renormalized Hamiltonian. In the main theorem, we derive the semi-relativistic Schrödinger operator with the Yukawa potential. To prove the main theorem, we apply an extension of the abstract scaling limit theory on self-adjoint operators obtained by Arai [2].

The state space for the system of N semi-relativistic particles coupled to a scalar bose field is defined by $\mathcal{H} = L^2(\mathbf{R}^{3N}_{\mathbf{x}}) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbf{R}^3_{\mathbf{k}}))$ where $\mathcal{F}_{\mathrm{b}}(L^2(\mathbf{R}^3_{\mathbf{k}}))$ is the boson Fock space on $L^2(\mathbf{R}^3_{\mathbf{k}})$. Here we assume $N \geq 2$. The free Hamiltonian of the semi-relativistic particle is defined by $H_{\mathrm{p}} = \sum_{j=1}^{N} \sqrt{-\triangle_j + M^2}$ where $M > 0$ denotes the fixed mass of the particles. Then the total Hamiltonian for the interacting system

$$H = H_{\mathrm{p}} \otimes I + I \otimes H_{\mathrm{b}} + \kappa H_{\mathrm{I}}(\Lambda_0), \qquad \kappa \in \mathbf{R}.$$

where H_{b} is the free Hamiltonian of the bose field, and $H_{\mathrm{I}}(\Lambda_0) = \sum_{j=1}^{N} \phi_{\Lambda_0}(\mathbf{x}_j)$. Here $\phi_{\Lambda_0}(\mathbf{x})$ denotes the field operator with the fixed parameter $\Lambda_0 > 0$ of the ultraviolet cutoff.

Let us introduce the scaled total Hamiltonian $H(\Lambda)$ defined by

$$H(\Lambda) \;=\; H_{\mathrm{p}} \otimes I \;+\; \Lambda^2 I \otimes H_{\mathrm{b}} \;+\; \kappa \Lambda H_{\mathrm{I}}(\Lambda^{\alpha}), \qquad\qquad \alpha > 0. \tag{1}$$

We are interested in taking the scaling limit and removing the ultraviolet cutoff simultaneously for the renormalized Hamiltonian $H(\Lambda) - E_N(\Lambda)$ where $E_N(\Lambda)$ is the divergent term defined by $E_N(\Lambda) = \frac{N}{(2\pi)^3} \left(\int_{\mathbf{R}^3} \frac{\chi_{\Lambda_\alpha}(\mathbf{k})}{\mathbf{k}^2 + m^2} d\mathbf{k} \right)^{1/2}$.

Historically Davies [3] investigates a scaled Hamiltonian of the form (1) with replacing the semi-relativistic Schrödinger operator with the standard Shcrödinger operator. Then he derives the Schrödinger operators with effective potentials by taking the scaling limit as $\Lambda \to \infty$. In [2], Arai investigates an abstract scaling limit theory, and apply it to a spin-boson model and the non-relativistic QED models in the dipole approximation. In [11, 12], the scaling limit of the N-body Schrödinger operator coupled to a scalar bose field with removing ultraviolet cutoff is considered. As is mentioned above, the scaling limit of the Hamiltonian of (1) with the fixed ultraviolet is considered in [22] . For other results on this subject, refer to [9, 10, 13, 14, 18, 19, 17, 21].

Now we state the main theorem. Assume that $0 < \alpha < \frac{1}{2}$ and $0 < |\kappa| < \sqrt{\frac{4\pi}{\sqrt{N-1}}}$ follows. Then it follows that for $z \in \mathbf{C} \backslash \mathbf{R}$,

$$s - \lim_{\Lambda \to \infty} (H(\Lambda) - E_N(\Lambda) - z)^{-1} \;=\; \left(\sum_{j=1}^{N} \sqrt{-\triangle_j + M^2} \;+\; \kappa^2 V_{\mathrm{eff}} \;-\; z \right)^{-1} P_{\Omega_{\mathrm{b}}}, \tag{2}$$

where

$$V_{\mathrm{eff}}(\mathbf{x}_1, \cdots, \mathbf{x}_N) \;=\; -\frac{1}{16\pi} \sum_{j<l} \frac{1}{|\mathbf{x}_j - \mathbf{x}_l|} e^{-m\,|\mathbf{x}_j - \mathbf{x}_l|},$$

and $P_{\Omega_{\mathrm{b}}}$ is the projection onto the closed subspace spanned by the Fock vacuum Ω_{b}.

We prove the main theorem by the following strategy. We consider an extension of the abstract scaling limit theory on self-adjoint operators obtained in [2]. Then we investigate its applition to the Hamiltonian $U(\Lambda)^{-1} \left(H(\Lambda) - E_N(\Lambda) \right) U(\Lambda)$ where U (Λ) is a unitary transformation, called *the dressing transformation*.

This paper is organized as follows. In Section 2, we consider an abstract scaling limit theory on a self-adjoint operators. In section 3, we define the system of N semi-relativistic particles interacting with a scalar bose field and the main results are stated. In Section 3, we prove the main theorem by applying the dressing transformation and the abstract scaling limit theory.

2 Abstract Scaling Limit

In this section we consider a generalized scaling limit on self-adjoint operators. Let A and B are operators on $\mathcal{X}$ and $\mathcal{Y}$, respectively. We analyze the operator $X(\Lambda)$ on $\mathcal{Z} = \mathcal{X} \otimes \mathcal{Y}$, which is defined by

$$X(\Lambda) \;=\; X_0(\Lambda) \,+\, C_{\mathrm{I}}(\Lambda) \,+\, C_{\mathrm{II}}(\Lambda) \otimes I, \qquad\qquad \Lambda > 0,$$

where $X_0(\Lambda) \;=\; A \otimes I \,+\, \Lambda I \otimes B$. It is note that $\mathcal{D}(X_0(\Lambda)) = \mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B)$. We assume the following conditions.

(A.1) A and B are non-negative and self-adjoint. Ker $B \neq \{0\}$.
(A.2) For all $\varepsilon > 0$ there exists a constant $\Lambda(\varepsilon) > 0$ such that for all $\Lambda > \Lambda(\varepsilon)$, $\mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B) \subset \mathcal{D}(C_{\mathrm{I}}(\Lambda))$ follows, and for $\Phi \in \mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B)$,

$$\|C_{\mathrm{I}}(\Lambda)\Phi\| \;\leq\; \varepsilon \|X_0(\Lambda)\Phi\| \,+ b(\varepsilon)\|\Phi\|$$

follows, where $b(\varepsilon) \geq 0$ is a constant independent of $\Lambda > \Lambda(\varepsilon)$. There exists operator C_{I} on $\mathcal{X}$ such that $\mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B) \subset \mathcal{D}(C_{\mathrm{I}})$, and for $\Phi \in \mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B)$,

$$\mathrm{s} - \lim_{\Lambda \to \infty} C_{\mathrm{I}}(\Lambda)\Phi \;=\; C_{\mathrm{I}}\Phi.$$

(A.3) For all $\Lambda > 0$, $\mathcal{D}(A) \subset \mathcal{D}(C_{\mathrm{II}}(\Lambda))$ follows. There exist $0 < c < \frac{1}{2}$ and $d \geq 0$, which are independent of $\Lambda > \Lambda_0$ with some Λ_0, such that for $\Phi \in \mathcal{D}(A)$,

$$\|C_{\mathrm{II}}(\Lambda)\Phi\| \;\leq\; c\|A\Phi\| \,+ d\|\Phi\|.$$

There exists operator C_{II} on $\mathcal{X}$ satisfying $\mathcal{D}(A) \subset \mathcal{D}(C_{\mathrm{II}})$, and for $\Phi \in \mathcal{D}(A)$,

$$\|C_{\mathrm{II}}\Phi\| \;\leq\; c\|A\Phi\| \,+ d\|\Phi\|,$$

and

$$\mathrm{s} - \lim_{\Lambda \to \infty} C_{\mathrm{II}}(\Lambda)\Phi \;=\; C_{\mathrm{II}}\Phi.$$

Proposition 2.1 *Assume **(A.1) - (A.3)**. Then **(i)-(iii)** follows.*
(i) *$X(\Lambda)$ is self-adjoint and essentially self-adjoint on any core of $A \otimes I + I \otimes B$.*
(ii) *The operator*

$$X_\infty \;=\; A \otimes I \,+\, (I \otimes P_{kerB})C_I(I \otimes P_{kerB}) \,+\, C_{II} \otimes P_{kerB}$$

is self-adjoint and essentially self-adjoint on any core of $A \otimes I$ where P_{kerB} is the projection onto kerB.
(iii) *For all $z \in \mathbf{C} \backslash \mathbf{R}$,*

$$s - \lim_{\Lambda \to \infty} \Big(X(\Lambda) - z \Big)^{-1} \;=\; (X_\infty \,-\, z)^{-1}(I \otimes P_{kerB}).$$

Remark 2.1
Proposition 2.1 with the condition $C_{II} = 0$ is investigated in ([2], Theorem 2.1).

(Proof)
(i) Let $\Phi \in \mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B)$. By **(A.2),(A.3)** and $\|(A \otimes I)\Phi\| \leq \|X_0(\Lambda)\Phi\|$, it is seen that for sufficiently large $\Lambda > 0$,

$$\left\| \Big(C_{\mathrm{I}}(\Lambda) + C_{\mathrm{II}}(\Lambda) \otimes I \Big) \Phi \right\| \leq (\varepsilon + c) \left\| X_0(\Lambda)\Phi \right\| + (b(\varepsilon) + d)\|\Phi\|. \tag{3}$$

Then for sufficiently small $\varepsilon > 0$, $\varepsilon + c < 1$ follows, and hence **(i)** follows from the Kato-Rellich theorem.

(ii) From **(A.2)** and **(A.3)** we see that

$$\|C_{\mathrm{I}}(\Lambda)(X_0(\Lambda) - z)^{-1}\| \;\leq\; \varepsilon\|X_0(\Lambda)(X_0(\Lambda) - z)^{-1}\| \;+\; b(\varepsilon)\|(X_0(\Lambda) - z)^{-1}\|, \tag{4}$$

$$\|(C_{\mathrm{II}}(\Lambda) \otimes I)(X_0(\Lambda) - z)^{-1}\| \;\leq\; c\|(A \otimes I)(X_0(\Lambda) - z)^{-1}\| \;+\; d\|(X_0(\Lambda) - z)^{-1}\|. \tag{5}$$

Then we have

$$\begin{aligned}&\left\| \Big(C_{\mathrm{I}}(\Lambda) + C_{\mathrm{II}}(\Lambda) \otimes I \Big) \Big(X_0(\Lambda) - z \Big)^{-1} \Psi \right\| \\ &\qquad \leq \Big(\varepsilon|z| + b(\varepsilon) + d \Big) \|(X_0(\Lambda) - z)^{-1}\Psi\| + c\|(A \otimes I)(X_0(\Lambda) - z)^{-1}\| + \varepsilon\|\Psi\|. \end{aligned}\tag{6}$$

We see that

$$\mathrm{s} - \lim_{\Lambda \to \infty} (X_0(\Lambda) - z)^{-1} \;=\; (A - z)^{-1} \otimes P_{\mathrm{ker}B}, \;\; \mathrm{s} - \lim_{\Lambda \to \infty} (A \otimes I)(X_0(\Lambda) - z)^{-1} \;=\; A(A - z)^{-1} \otimes P_{\mathrm{ker}B}.$$

Then we have

$$\left\| \Big(C_{\mathrm{I}} + C_{\mathrm{II}} \otimes I \Big) \Big((A - z)^{-1} \otimes P_{\mathrm{ker}B} \Big) \Psi \right\| \leq \Big((\varepsilon + c)|z| + b(\varepsilon) + d \Big) \|((A - z)^{-1} \otimes P_{\mathrm{ker}B})\Psi\| + (\varepsilon + c)\|\Psi\|. \tag{7}$$

Then we obtain for $\Phi \in \mathcal{D}(A)$,

$$\left\| (I \otimes P_{\mathrm{ker}B}) \Big(C_{\mathrm{I}} + C_{\mathrm{II}} \otimes I \Big) (I \otimes P_{\mathrm{ker}B})\Phi \right\| \leq (\varepsilon + c)\|(A \otimes I)\Phi\| + \Big(2(\varepsilon + c)|z| + b(\varepsilon) + d \Big) \|\Phi\|. \tag{8}$$

Since $\varepsilon + c < 1$ follows for sufficiently small $\varepsilon > 0$, then we obtain **(ii)** from the Kato-Rellich theorem.

(iii) Let $z \in \mathbf{C} \backslash \mathbf{R}$ satisfying $\mathrm{Re}\, z = 0$. We see that

$$C_{\mathrm{I}}(\Lambda) \Big(X_0(\Lambda) - z \Big)^{-1} \;=\; C_{\mathrm{I}}(\Lambda)\, (A - z)^{-1} \otimes P_{\mathrm{ker}B} \;+\; C_{\mathrm{I}}(\Lambda) \Big(X_0(\Lambda) - z \Big)^{-1} (I - I \otimes P_{\mathrm{ker}B}). \tag{9}$$

By (4), we have for $\varepsilon > 0$,

$$\begin{aligned}&\left\| C_{\mathrm{I}}(\Lambda) \Big(X_0(\Lambda) - z \Big)^{-1} (I - I \otimes P_{\mathrm{ker}B})\Psi \right\| \\ &\qquad \leq \Big(\varepsilon|z| + b(\varepsilon) \Big) \|(X_0(\Lambda) - z)^{-1}(I - I \otimes P_{\mathrm{ker}B})\Psi\| \;+ \varepsilon\; \|(I - I \otimes P_{\mathrm{ker}B})\Psi\|. \end{aligned}\tag{10}$$

Then for sufficiently small $\varepsilon > 0$, we obtain

$$\mathrm{s} - \lim_{\Lambda \to \infty} C_{\mathrm{I}}(\Lambda) \Big(X_0(\Lambda) - z \Big)^{-1} (I - I \otimes P_{\mathrm{ker}B}) \;=\; 0. \tag{11}$$

Hence by (9), (11) and **(A.2)** , we have

$$C_{\rm I}(\Lambda)\left(X_0(\Lambda)-z\right)^{-1} = C_{\rm I}\,(A-z)^{-1}\otimes P_{{\rm ker}B}. \tag{12}$$

We also see that

$$\begin{aligned}&\lim_{\Lambda\to\infty}\left\|(C_{\rm II}(\Lambda)\otimes I)\left(X_0(\Lambda)-z\right)^{-1}(I-I\otimes P_{{\rm ker}B})\Psi\right\|\\ &\leq \lim_{\Lambda\to\infty}\left(c\left\|\left(X_0(\Lambda)-z\right)^{-1}(I-I\otimes P_{{\rm ker}B})(A\otimes I)\Psi\right\|+d\left\|\left(X_0(\Lambda)-z\right)^{-1}(I-I\otimes P_{{\rm ker}B})\Psi\right\|\right)=0. \end{aligned}\tag{13}$$

Hence by (13), we have

$$\text{s-}\lim_{\Lambda\to\infty}(C_{\rm II}(\Lambda)\otimes I)\left(X_0(\Lambda)-z\right)^{-1} = C_{\rm II}\,(A-z)^{-1}\otimes P_{{\rm ker}B}. \tag{14}$$

By (3) and $\|(X_0-z)^{-1}\|\leq\frac{1}{|z|}$, we have

$$\left\|\left(C_{\rm I}(\Lambda)+C_{\rm II}(\Lambda)\otimes I\right)\left(X_0(\Lambda)-z\right)^{-1}\right\|\leq 2(\varepsilon+c)+\frac{b(\varepsilon)+d}{|z|}. \tag{15}$$

Note that $0<c<\frac{1}{2}$. Then $\|(C_{\rm I}(\Lambda)+C_{\rm II}(\Lambda)\otimes I)(X_0(\Lambda)-z)^{-1}\|<1$ follows for sufficiently small $\varepsilon>0$ and sufficiently large $|z|$. Hence by (12) and (14), we have

$$\begin{aligned}\text{s-}\lim_{\Lambda\to\infty}(X_0(\Lambda)-z)^{-1}&=\text{s-}\lim_{\Lambda\to\infty}\sum_{n=0}^{\infty}(X_0(\Lambda)-z)^{-1}\left\{\left(C_{\rm I}(\Lambda)+C_{\rm II}(\Lambda)\otimes I\right)\left(X_0(\Lambda)-z\right)^{-1}\right\}^n\\ &=\sum_{n=0}^{\infty}\left((A-z)^{-1}\otimes P_{{\rm ker}B}\right)\left\{\left(C_{\rm I}+C_{\rm II}\otimes I\right)\left((A-z)^{-1}\otimes P_{{\rm ker}B}\right)\right\}^n\\ &=(X_\infty-z)^{-1}(I\otimes P_{{\rm ker}B}).\end{aligned}$$

Thus we obtain **(iii)**. ■

3 Interacting System and Main Result

In this section, we define the Hamiltonian for the N semi-relaivistic particles system coupled to bose field, and state the main results. The total Hilbert space for the system is given by

$$\mathcal{H} = L^2(\mathbf{R}^{3N}_{\mathbf{x}})\otimes\mathcal{F}_{\rm b}(L^2(\mathbf{R}^3_{\mathbf{k}})),$$

where $\mathcal{F}_{\rm b}(L^2(\mathbf{R}^3_{\mathbf{k}})) = \oplus_{n=0}^{\infty}(\otimes_{\rm s}^n L^2(\mathbf{R}^3_{\mathbf{k}}))$ and $\otimes_{\rm s}^n L^2(\mathbf{R}^3_{\mathbf{k}})$ denotes the n-fold symmetric tenser product of $L^2(\mathbf{R}^3_{\mathbf{k}})$ with $\otimes_s^0 L^2(\mathbf{R}^3_{\mathbf{k}}):=\mathbf{C}$. The free particle Hamiltonian is given by

$$H_{\rm p} = \sum_{j=1}^{N}\sqrt{-\triangle_j+M^2},$$

where $M>0$ denotes the fixed mass. The free Hamiltonian of the scalar bose field is given by

$$H_{\rm b} = \bigoplus_{n=0}^{\infty}\left(\sum_{j=1}^{n}(I\otimes\cdots I\otimes\underbrace{\omega}_{jth}\otimes I\cdots\otimes I)\right),$$

where $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$, $m > 0$, is the energy of the boson with momentum $\mathbf{k} \in \mathbf{R}^3$. For $f, g \in L^2(\mathbf{R}^3)$, $a(f)$ and $a^*(g)$ denote the annihilation operator and the creation operator, respectively. The field operator with the cutoff parameter $\Lambda > 0$ is given by

$$\phi_\Lambda(\mathbf{x}) = \frac{1}{\sqrt{2}} \left(a(\frac{\rho_{\Lambda,\mathbf{x}}}{\sqrt{\omega}}) + a^*(\frac{\rho_{\Lambda,\mathbf{x}}}{\sqrt{\omega}}) \right).$$

Here we set $\rho_{\Lambda,\mathbf{x}}(\mathbf{k}) = \rho_\Lambda(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}$ and $\rho_\Lambda(\mathbf{k}) = \chi_\Lambda(|\mathbf{k}|)$ where $\chi_\Lambda(|\mathbf{k}|)$ is the characteristic function on $[-\Lambda, \Lambda] \subset \mathbf{R}$.

The total Hamiltonian is defined by

$$H = H_{\mathrm{p}} \otimes I + I \otimes H_{\mathrm{b}} + \kappa H_{\mathrm{I}}(\Lambda_0),$$

where $\kappa > 0$ is the coupling constant, and $H_{\mathrm{I}}(\Lambda_0) = \sum_{j=1}^{N} \phi_{\Lambda_0}(\mathbf{x}_j)$ with the fixed parameter $\Lambda_0 \geq 0$

Let us consider the self-adjointness of H. For $f \in \mathcal{D}(\omega^{-1/2})$, it is seen that $a(f)$ and $a^*(f)$ are relatively bounded with respect to $H_{\mathrm{b}}^{1/2}$ with

$$\|a(f)\Psi\| \leq \|\frac{f}{\sqrt{\omega}}\| \, \|H_{\mathrm{b}}^{1/2}\Psi\|, \qquad \Psi \in \mathcal{D}(H_{\mathrm{b}}^{1/2}), \tag{16}$$

$$\|a^*(f)\Psi\| \leq \|\frac{f}{\sqrt{\omega}}\| \|H_{\mathrm{b}}^{1/2}\Psi\| + \|f\|\|\Psi\|, \qquad \Psi \in \mathcal{D}(H_{\mathrm{b}}^{1/2}). \tag{17}$$

By (16) and (17), it is seen that $H_{\mathrm{I}}(\Lambda_0)$ is relatively bounded with respect to $I \otimes H_{\mathrm{b}}^{1/2}$. Then $H_{\mathrm{I}}(\Lambda_0)$ is relatively bounded with respect to H_0 with infinitely small bound. From the Kato-Rellich theorem, it is proven that H is self-adjoint and essentially self-adjoint on any core of H_0. Then in particular, H is essentially self-adjoint on $\mathcal{D}_0 = C_0^\infty(\mathbf{R}^{3N}) \hat{\otimes} \mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{D}(\omega))$, where $\hat{\otimes}$ denotes the algebraic tensor product, and $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{D}(\omega))$ denotes the finite particle subspace on $\mathcal{D}(\omega)$. In this paper, the finite particle subspace $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{M})$ on the subspace $\mathcal{M} \subset L^2(\mathbf{R}^3)$ is defined by the set of $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty$ satisfying that $\Psi^{(n)} \in \otimes_{\mathrm{s}}^n \mathcal{M}$, $n \geq 0$, and $\Psi^{(n')} = 0$ for all $n' > N$ with some $N \geq 0$.

Now we introduce the scaled total Hamiltonian defined by

$$H(\Lambda) = H_{\mathrm{p}} \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + \kappa\Lambda \, H_{\mathrm{I}}(\Lambda^\alpha).$$

Let us consider the renormalization term $E_N(\Lambda)$ defined by

$$E_N(\Lambda) = \frac{N}{(2\pi)^3} \left(\int_{\mathbf{R}^3} \frac{\chi_{\Lambda_\alpha}(\mathbf{k})}{\mathbf{k}^2 + m^2} d\mathbf{k} \right)^{1/2}.$$

We consider the asymptotic behavior of the renormalized Hamiltonian $H(\Lambda) - E_N(\Lambda)$ as $\Lambda \to \infty$ with removing the ultraviolet cutoffs. Now we state the main theorem.

Theorem 3.1 *Assume* $0<\alpha<\frac{1}{2}$, *and* $0<|\kappa|<\sqrt{\frac{4\pi}{\sqrt{N-1}}}$. *Then it follows that for* $z\in\mathbf{C}\backslash\mathbf{R}$,

$$s-\lim_{\Lambda\to\infty}\left(H(\Lambda)-\kappa^2E_N(\Lambda)-z\right)^{-1} = \left(\sum_{j=1}^{N}\sqrt{-\triangle_j+M^2} + \kappa^2V_{eff} - z\right)^{-1}\otimes P_{\Omega_b}, \tag{18}$$

where

$$V_{eff}(\mathbf{x}_1,\cdots,\mathbf{x}_N) = -\frac{1}{16\pi}\sum_{j<l}\frac{1}{|\mathbf{x}_j-\mathbf{x}_l|}e^{-m\,|\mathbf{x}_j-\mathbf{x}_l|},$$

and P_{Ω_b} *is the projection onto the closed subspace spanned by the Fock vacuum* Ω_b.

By using the theorem ([19] ; Lemma 2.7), the we obtain the following corollary

Corollary 3.2 *Assume* $0<\alpha<\frac{1}{2}$, *and* $0<|\kappa|<\sqrt{\frac{4\pi}{\sqrt{N-1}}}$. *Then it follows that*

$$s-\lim_{\Lambda\to\infty}e^{-it\left(\,H(\Lambda)-\kappa^2E_N(\Lambda)\right)}\left(I\otimes P_{\Omega_b}\right) = e^{-it\left(\sum_{j=1}^{N}\sqrt{-\triangle_j+M^2}+\kappa^2V_{eff}\right)}\otimes P_{\Omega_b}.$$

4 Proof of Main Theorem

Let

$$\pi_{\Lambda^\alpha}(\mathbf{x}) = \frac{-i}{\sqrt{2}}\left(-a(\frac{\rho_{\Lambda^\alpha,\mathbf{x}}}{\omega^{3/2}}) + a^*(\frac{\rho_{\Lambda^\alpha,\mathbf{x}}}{\omega^{3/2}})\right).$$

The unitary transformation, called the dressing transformation, is given by

$$U(\Lambda) = e^{i\left(\frac{\kappa}{\Lambda}\right)\sum_{j=1}^{N}\pi_{\Lambda^\alpha}(\mathbf{x}_j)}.$$

It is seen that on the finite particle subspace on $\mathcal{F}_{\rm b}^{\rm fin}(L^2(\mathbf{R}^3))$,

$$[\pi_{\Lambda^\alpha}(\mathbf{x}),H_{\rm b}] = -i\,\phi_{\Lambda^\alpha}(\omega\xi),$$
$$[\pi_{\Lambda^\alpha}(\mathbf{x}),\phi_{\Lambda^\alpha}(\mathbf{y})] = \frac{-i}{(2\pi)^3}\int_{\mathbf{R}^3}\frac{\rho_{\Lambda^\alpha}(\mathbf{k})}{\omega(\mathbf{k})^2}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d\mathbf{k}.$$

Then we have

$$U(\Lambda)^{-1}H(\Lambda)U(\Lambda) = H_0(\Lambda) + E_N(\Lambda) + K_{\rm I}(\Lambda)+K_{\rm II}(\Lambda),$$

where

$$K_{\rm I}(\Lambda) = U(\Lambda)^{-1}\left(H_{\rm p}\otimes I\right)U(\Lambda) - H_{\rm p}\otimes I$$
$$K_{\rm II}(\Lambda) = -\frac{\kappa^2}{2}\frac{1}{(2\pi)^3}\sum_{j<l}^{N}\int_{\mathbf{R}^3}\frac{\rho_{\Lambda^\alpha}(\mathbf{k})}{\omega(\mathbf{k})^2}e^{i\mathbf{k}\cdot(\mathbf{x}_j-\mathbf{x}_l)}d\mathbf{k}.$$

Then by the spectral decomposition theorem, we have for $z\in\mathbf{C}\backslash\mathbf{R}$,

$$\left(H(\Lambda)-E_N(\Lambda)-z\right)^{-1} = U(\Lambda)\left(H_0(\Lambda) + K_{\rm I}(\Lambda)+K_{\rm II}(\Lambda) - z\right)^{-1}U(\Lambda)^{-1}. \tag{19}$$

We shall show that $H_0(\Lambda)$, $K_{\rm I}(\Lambda)$ and $K_{\rm II}(\Lambda)$ satisfy the conditions **(A.1)-(A.3)** in Section 2.

Proposition 4.1 *Assume $0 < \alpha < \frac{1}{2}$. Then the following **(1)** and **(2)** hold.*
***(1)** For $\varepsilon > 0$, there exists $\Lambda_0 \geq 0$ such that for all $\Lambda > \Lambda_0$,*

$$\|K_I(\Lambda)\Psi\| \;\leq\; \varepsilon\|H_0(\Lambda)\Psi\| + \nu(\varepsilon)\|\Psi\|, \qquad \Psi \in \mathcal{D}(H_0(\Lambda)) \tag{20}$$

follows, where $\nu(\varepsilon) \geq 0$ is a constant independent of $\Lambda > \Lambda_0$.
***(2)** It follows that*

$$s-\lim_{\Lambda\to\infty} K_I(\Lambda)\Psi \;=\; 0, \qquad \Psi \in \mathcal{D}(H_0(\Lambda)). \tag{21}$$

(Proof)
Let W_j, $j = 1, 2$, be the non-negative and self-adjoint operators with $\mathcal{D}(W_2) \subset \mathcal{D}(W_1)$. Then it is seen that for $\psi \in \mathcal{D}(W_1) \cap \mathcal{D}(W_2)$, $\sqrt{W_j}\psi = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{\lambda}}(W_j+\lambda)^{-1}W_j\,\psi\,d\lambda$, and $(\sqrt{W_1}-\sqrt{W_2})\psi = \frac{1}{\pi}\int_0^\infty \sqrt{\lambda}(W_1+\lambda)^{-1}(W_1-W_2)(W_2+\lambda)^{-1}\psi d\lambda$. Then we have

$$\|(\sqrt{W_1}-\sqrt{W_2})\psi\| \leq \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{\lambda+E_0(W_1)}\left\|\,(W_1-W_2)(W_2+\lambda)^{-1}\psi\right\| d\lambda, \tag{22}$$

where $E_0(X)$ is the infimum of the spectrum of the operator X. Let us use the notation $\hat{\mathbf{p}} = (\hat{p}^1, \cdots, \hat{p}^d) = (-i\frac{\partial}{\partial x^1}, \cdots, -i\frac{\partial}{\partial x^d})$. By the commutation relation $[\Pi(f_{\mathbf{x}}), \hat{p}^\nu] = i\Pi(\partial_{x^\nu} f_{\mathbf{x}})$,
we have for $\Psi \in \mathcal{D}_0$,

$$\Big(U(\Lambda)^{-1}(\mathbf{p}_j \otimes I)U(\Lambda)\Big)^2\Psi \;=\; \left(\sum_{\nu=1}^{d}\left(\hat{p}_j^\nu \otimes I + \left(\frac{\kappa}{\Lambda}\right)\Pi\Big(\frac{\partial_{x_j^\nu} f_{\mathbf{x}_j}}{\omega}\Big)\right)^2\right)\Psi. \tag{23}$$

Then by applying $U(\Lambda)^{-1}(\hat{\mathbf{p}}_j^2 \otimes I)U(\Lambda) + M^2$ to W_1, and $\hat{\mathbf{p}}_j^2 + M^2$ to W_2 in (22), we have

$$\begin{aligned}\|K_{\rm I}(\Lambda)\Psi\| \leq & \frac{1}{\pi}\left(\frac{\kappa}{\Lambda}\right)\sum_{j=1}^{N}\int_0^\infty \frac{\sqrt{\lambda}}{\lambda+M^2}\left\|\,S_j(\Lambda)(\hat{\mathbf{p}}_j^2\otimes I \,+\, M^2+\lambda)^{-1}\Psi\right\|\,d\lambda \\ & + \frac{1}{\pi}\left(\frac{\kappa}{\Lambda}\right)^2\sum_{j=1}^{N}\int_0^\infty \frac{\sqrt{\lambda}}{\lambda+M^2}\left\|T_j(\Lambda)(\hat{\mathbf{p}}_j^2\otimes I \,+\, M^2+\lambda)^{-1}\Psi\right\|\,d\lambda,\end{aligned} \tag{24}$$

where

$$S_j(\Lambda) \;=\; -2\sum_{\nu=1}^{3}\pi\left(\frac{ik^\nu\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)(\hat{\mathbf{p}}_j{}^\nu\otimes I) \;-\; \pi\left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right), \tag{25}$$

$$T_j(\Lambda) \;=\; \sum_{\nu=1}^{3}\pi\left(\frac{ik^\nu\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)^2. \tag{26}$$

By the following Lemma 4.2, there exists constants $\beta_j(\delta) \geq 0$ for $0 < \delta < \frac{1}{10}$, $\gamma_j \geq 0$, and $\nu_j \geq 0$, such that

$$\left\|\,S_j(\Lambda)(\hat{\mathbf{p}}_j^2\otimes I \,+\, M^2+\lambda)^{-1}\Psi\right\| \;\leq\; \left(\beta_j\frac{\Lambda^\alpha}{\lambda^{\frac{1}{2}+\delta}} + \gamma_j\frac{\Lambda^{2\alpha}}{\lambda+M^2}\right)\left(\left\|\,H_0(\Lambda)\Psi\right\| + \|\Psi\|\right) \tag{27}$$

$$\left\|\,T_j(\Lambda)(\hat{\mathbf{p}}_j^2\otimes I \,+\, M^2+\lambda)^{-1}\Psi\right\| \;\leq\; \mu_j\frac{\Lambda^{2\alpha}}{\lambda+M^2}\left(\left\|\,H_0(\Lambda)\Psi\right\| + \|\Psi\|\right) \tag{28}$$

Then by (24), (27) and (28), we have

$$\|K_{\mathrm{I}}(\Lambda)\Psi\| \ \leq \left(\frac{C_1}{\Lambda^{1-\alpha}} + \frac{C_2}{\Lambda^{1-2\alpha}} + \frac{C_2}{\Lambda^{2(1-\alpha)}} \right) \left(\left\| H_0(\Lambda)\Psi \right\| + \|\Psi\| \right), \qquad \Psi \in \mathcal{D}_0, \tag{29}$$

where $C_1 = \sum\limits_{j=1}^{N} \frac{\kappa\beta_j(\delta)}{\pi} \int_0^\infty \frac{1}{(\lambda+M^2)\lambda^\delta} d\lambda$, $C_2 = \sum\limits_{j=1}^{N} \frac{\kappa\gamma_j}{\pi} \int_0^\infty \frac{1}{(\lambda+M^2)^2} d\lambda$, and $C_3 = \sum\limits_{j=1}^{N} \frac{\kappa^2\mu_j}{\pi} \int_0^\infty \frac{1}{(\lambda+M^2)^2} d\lambda$. Since $\mathcal{D}_0$ is a core of $H_0(\Lambda)$, (29) follows for all $\mathcal{D}(H_0)$. Then **(1)** follows for $0<\alpha<\frac{1}{2}$. We see that **(2)** follows from (29). ■

Lemma 4.2 *There exist constants $\beta_j(\delta) \geq 0$ for $0<\delta<\frac{1}{10}$, $\gamma_j \geq 0$, and $\nu_j \geq 0$ such that*

(i)
$$\left\| \pi \left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (\hat{p}_j^\nu \otimes I)(\hat{\mathbf{p}}_j^2 \otimes I + M^2+\lambda)^{-1}(H_0(\Lambda)+1)^{-1} \right\| \leq \beta_j(\delta) \frac{\Lambda^\alpha}{\lambda^{\frac{1}{2}+\delta}}, \tag{30}$$

(ii)
$$\left\| \pi \left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (\hat{\mathbf{p}}_j^2 \otimes I + M^2+\lambda)^{-1}(H_0(\Lambda)+1)^{-1} \right\| \leq \gamma_j \frac{\Lambda^{2\alpha}}{\lambda+M^2}, \tag{31}$$

(iii)
$$\left\| \pi \left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right)^2 (\hat{\mathbf{p}}_j^2 \otimes I + M^2+\lambda)^{-1}(H_0(\Lambda)+1)^{-1} \right\| \leq \mu_j \frac{\Lambda^{2\alpha}}{\lambda+M^2}. \tag{32}$$

(Proof)
(i) By Proposition B in Appendix, it is seen that

$$\begin{aligned}
&\left\| \pi \left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (\hat{\mathbf{p}}_j{}^\nu \otimes I)(\hat{\mathbf{p}}_j^2 \otimes I + M^2+\lambda)^{-1}\Psi \right\| \\
&\leq \left\| \pi \left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (I \otimes H_{\mathrm{b}}+1)^{-1/2} \right\| \left\| (\hat{\mathbf{p}}_j{}^\nu \otimes I)(\hat{\mathbf{p}}_j^2 \otimes I + M^2+\lambda)^{-1}(H_{\mathrm{p}} \otimes I+1)^{-1/2} \right\| \\
&\qquad \times \left\| (I \otimes H_{\mathrm{b}}+1)^{-1/2}(H_{\mathrm{p}} \otimes I+1)^{1/2}\Psi \right\| \\
&\leq \frac{c_j(\delta)}{\lambda^{\frac{1}{2}+\delta}} \left\| \pi \left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (I \otimes H_{\mathrm{b}}+1)^{-1/2} \right\| \left\| (I \otimes H_{\mathrm{b}}+1)^{-1/2}(H_{\mathrm{p}} \otimes I+1)^{1/2}\Psi \right\|
\end{aligned} \tag{33}$$

It is seen that $\left\| \frac{|\mathbf{k}|\rho_{\Lambda^\alpha}}{\omega^{3/2}} \right\| \in O(\Lambda^\alpha)$ and $\left\| \frac{|\mathbf{k}|\rho_{\Lambda^\alpha}}{\omega^2} \right\| \in O(\Lambda^{\frac{\alpha}{2}})$. Then by (16) and (17), we see that

$$\left\| \pi \left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (I \otimes H_{\mathrm{b}}+1)^{-1/2} \right\| \in O(\Lambda^\alpha). \tag{34}$$

By using $\left\| (I \otimes H_{\mathrm{b}}+1)^{-1/2}(H_{\mathrm{p}} \otimes I+1)^{1/2}\Psi \right\| \leq \left\| H_0(\Lambda)\Psi \right\| + \|\Psi\|$ and (34), we have

$$\left\| \pi \left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (\hat{\mathbf{p}}_j{}^\nu \otimes I)(\hat{\mathbf{p}}_j^2 \otimes I + M^2+\lambda)^{-1}\Psi \right\| \leq \left(\beta_j(\delta) \frac{\Lambda^\alpha}{\lambda^{\frac{1}{2}+\delta}} \right) \left(\left\| H_0(\Lambda)\Psi \right\| + \|\Psi\| \right). \tag{35}$$

Thus we obtain **(i)**.
(ii) We see that

$$\left\| \pi \left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (\hat{\mathbf{p}}_j^2 \otimes I + M^2+\lambda)^{-1} \right\| \leq \frac{1}{M^2+\lambda} \left\| \pi \left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}} \right) (I \otimes H_{\mathrm{b}}+1)^{-1/2} \right\| \|(I \otimes H_{\mathrm{b}}+1)^{1/2}\Psi\| \tag{36}$$

Note that $\left\|\frac{\mathbf{k}^2\rho_{\Lambda^\alpha}}{\omega^{3/2}}\right\| \in O(\Lambda^{2\alpha})$ and $\left\|\frac{\mathbf{k}^2\rho_{\Lambda^\alpha}}{\omega^2}\right\| \in O(\Lambda^{\frac{3}{2}\alpha})$. Hence by (16) and (17), we have

$$\left\| \pi\left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)(I\otimes H_{\mathrm{b}}+1)^{-1/2}\right\| \in O(\Lambda^{2\alpha}). \tag{37}$$

Bu using $\|(I\otimes H_{\mathrm{b}}+1)^{1/2}\Psi\| \leq \|(H_0(\Lambda)+1)\Psi\|$ and (37), we obtain

$$\left\| \pi\left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)(\hat{\mathbf{p}}_j^2\otimes I + M^2+\lambda)^{-1}\Psi\right\| \leq \gamma_j \frac{\Lambda^{2\alpha}}{\lambda+M^2}\left(\left\| H_0(\Lambda)\Psi\right\| + \|\Psi\|\right),$$

and thus we have **(ii)**.
(iii) It is seen that

$$\left\| \pi\left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)^2(\hat{\mathbf{p}}_j^2\otimes I + M^2+\lambda)^{-1}\right\| \leq \frac{1}{M^2+\lambda}\left\| \pi\left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)(I\otimes H_{\mathrm{b}}+1)^{-1}\right\| \|(I\otimes H_{\mathrm{b}}+1)^1\Psi\| \tag{38}$$

Note that $\left\|\frac{|\mathbf{k}|\rho_{\Lambda^\alpha}}{\omega^{3/2}}\right\|^2 \in O(\Lambda^{2\alpha})$, $\left\|\frac{|\mathbf{k}|\rho_{\Lambda^\alpha}}{\omega^{3/2}}\right\|\left\|\frac{|\mathbf{k}|\rho_{\Lambda^\alpha}}{\omega^2}\right\| \in O(\Lambda^{\frac{3}{2}\alpha})$ and $\left\|\frac{\mathbf{k}^2\rho_{\Lambda^\alpha}}{\omega^2}\right\|^2 \in O(\Lambda^{\alpha})$. Hence from Lemma A in Appendix, we have

$$\left\| \pi\left(\frac{\mathbf{k}^2\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)(I\otimes H_{\mathrm{b}}+1)^{-1}\right\| \in O(\Lambda^{2\alpha}). \tag{39}$$

Bu using $\|(I\otimes H_{\mathrm{b}}+1)^{1/2}\Psi\| \leq \|(H_0(\Lambda)+1)\Psi\|$ and (39), it follows that

$$\left\| \pi\left(\frac{|\mathbf{k}|\rho_{\Lambda^\alpha,\mathbf{x}_j}}{\omega^{3/2}}\right)^2(\hat{\mathbf{p}}_j^2\otimes I + M^2+\lambda)^{-1}\Psi\right\| \leq \mu_j \frac{\Lambda^{2\alpha}}{\lambda+M^2}\left(\left\| H_0(\Lambda)\Psi\right\| + \|\Psi\|\right).$$

Thus we obtain **(iii)**. ■

(Proof of Theorem 3.1)
Let us show that $H(\Lambda)$ satisfies the conditions (**A.1**)-(**A.3**). It is easy to see that $H(\Lambda)$ satisfies (**A.1**) and (**A.2**) from Proposition 4.1. Then, let us consider the condition (**A.3**). By the spherical symmetry of ρ_{Λ^α}, we see that $\int_{\mathbf{R}^3}\frac{\rho_{\Lambda^\alpha}(\mathbf{k})}{\omega(\mathbf{k})^2}e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k} = \frac{2\pi}{|\mathbf{x}|}\int_0^\infty \frac{\chi_{\Lambda^\alpha}(r)}{r^2+m^2}r\sin(|x|r)dr = \frac{\pi}{|\mathbf{x}|}\int_{-\Lambda^\alpha}^{\Lambda^\alpha}\frac{1}{r^2+m^2}r\sin(|x|r)dr$. Then we have

$$K_{\mathrm{II}}(\Lambda) = -\frac{\kappa^2}{16\pi^2}\sum_{j<l}^{N}\frac{1}{|\mathbf{x}_j-\mathbf{x}_l|}G_{\Lambda^\alpha}(\mathbf{x}_j-\mathbf{x}_l), \tag{40}$$

where $G_{\Lambda^\alpha}(\mathbf{x}) = \mathrm{Im}\int_{-\Lambda^\alpha}^{\Lambda^\alpha}\frac{1}{r^2+m^2}e^{i|\mathbf{x}|z}dz$. Let $\Gamma_{\Lambda^\alpha} = \{z=t \mid -\Lambda^\alpha \leq t \leq \Lambda^\alpha\}\cup C_{\Lambda^\alpha}$ with $C_{\Lambda^\alpha} = \{z = \Lambda^\alpha e^{i\theta} \mid 0\leq\theta<\pi\}$. We see that $\int_{\Gamma_{\Lambda^\alpha}}\frac{z}{z^2+m^2}e^{i|\mathbf{x}|z}dz = i\pi e^{-m\mathbf{x}}$ and $\left|\int_{C_{\Lambda^\alpha}}\frac{z}{z^2+m^2}e^{i|\mathbf{x}|z}dz\right| \leq \frac{\Lambda^{2\alpha}}{\Lambda^{2\alpha}-m^2}\int_0^\pi e^{-|\mathbf{x}|\Lambda^\alpha\sin\theta}d\theta$. Then we have

$$\left| G_{\Lambda^\alpha}(\mathbf{x}_j-\mathbf{x}_l) - \pi e^{-m|\mathbf{x}_j-\mathbf{x}_l|}\right| \leq 2\frac{\Lambda^{2\alpha}}{\Lambda^{2\alpha}-m^2}\delta_{\Lambda^\alpha}(\mathbf{x}_j-\mathbf{x}_l), \tag{41}$$

where $\delta_{\Lambda^\alpha}(\mathbf{x}) = \int_0^{\frac{\pi}{2}} e^{-|\mathbf{x}|\Lambda^\alpha \sin\theta} d\theta$. By Lemma C in Appendix, we have for $\psi \in C_0^\infty(\mathbf{R}^{3N})$,

$$\int_{\mathbf{R}^{3N}} \left| \sum_{j<l} \frac{1}{|\mathbf{x}_j - \mathbf{x}_l|} \, \psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) \right|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N \;\leq\; 4(N-1) \|H_{\mathrm{p}} \psi\|^2. \tag{42}$$

Hence by the uniform boundness $\delta_{\Lambda^\alpha}(\mathbf{x}) \leq \frac{\pi}{2}$, (41) and (42), it follows that for sufficiently large $\Lambda > 0$,

$$\|K_{\mathrm{II}}(\Lambda)\psi\| \;\leq\; \kappa^2 \frac{\sqrt{(N-1)}}{8\pi} \|H_{\mathrm{p}} \psi\| \,+\, \frac{\kappa^2}{16\pi} \|\psi\| \tag{43}$$

follows for $\psi \in C_0^\infty(\mathbf{R}^{3N})$. Since $C_0^\infty(\mathbf{R}^{3N})$ is a core of H_{p}, (43) follows for all $\psi \in \mathcal{D}(H_{\mathrm{p}})$. Here it is noted that $\kappa^2 \frac{\sqrt{(N-1)}}{8\pi} < \frac{1}{2}$ follows for $|\kappa| < \sqrt{\frac{4\pi}{\sqrt{N-1}}}$. We also see that $\delta_{\Lambda^\alpha}(\mathbf{x}) \leq \int_0^{\frac{\pi}{2}} e^{-|\mathbf{x}|\Lambda^\alpha \frac{2}{\pi}\theta} d\theta \leq \frac{\pi}{2|\mathbf{x}|\Lambda^\alpha}$. Then $\lim_{\Lambda \to \infty} \delta_{\Lambda^\alpha}(\mathbf{x}) \;=\; 0$ follows for $\mathbf{x} \neq 0$. Hence we have

$$\lim_{\Lambda \to \infty} \left| \, G_{\Lambda^\alpha}(\mathbf{x}_j - \mathbf{x}_l) \;-\; \pi e^{-m|\mathbf{x}_j - \mathbf{x}_l|} \, \right| \;=\; 0, \qquad \text{a.e. } (\mathbf{x}_1, \cdots, \mathbf{x}_N) \in \mathbf{R}^{3N}. \tag{44}$$

Note that $\| \frac{1}{|\mathbf{x}_i - \mathbf{x}_l|} \psi \|_{L^2} < \infty$ follows for $\psi \in C_0^\infty(\mathbf{R}^{3N})$. Then by the Lebesgue dominated convergence theorem and (44), we see that for $\psi \in C_0^\infty(\mathbf{R}^{3N})$,

$$\lim_{\Lambda \to \infty} \int_{\mathbf{R}^{3N}} \left| \, e^{-m|\mathbf{x}_i - \mathbf{x}_l|} - G_{\Lambda^\alpha}(\mathbf{x}_i - \mathbf{x}_l) \right|^2 \left| \frac{1}{|\mathbf{x}_i - \mathbf{x}_l|} \right|^2 \left| \, \psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) \right|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N \;=\; 0. \tag{45}$$

Hence by (45), we obtain for $\psi \in C_0^\infty(\mathbf{R}^{3N})$

$$\mathrm{s} - \lim_{\Lambda \to \infty} K_{\mathrm{II}}(\Lambda)\psi \;=\; \kappa^2 V_{\mathrm{eff}} \psi. \tag{46}$$

Note that $C_0^\infty(\mathbf{R}^{3N})$ is a core of H_{p}. Then by using (43), we see that (46) follows for all $\psi \in \mathcal{D}(H_{\mathrm{p}})$. By (43) and (46), we also see that

$$\|V_{\mathrm{eff}} \psi\| \;\leq\; \kappa^2 \frac{\sqrt{(N-1)}}{8\pi} \|H_{\mathrm{p}} \psi\| \,+\, \frac{\kappa^2}{16\pi} \|\psi\| \tag{47}$$

Then by (43), (46) and (47) , it is proven that $H(\Lambda)$ satisfies the condition (**A.3**). ■

Appendix

Lemma A Let $f, \, g \, \in \mathcal{D}(\omega^{-1/2})$. Then it follows that for $\Psi \in \mathcal{D}(d\Gamma_{\mathrm{b}}(\omega))$,

(**1**) $\|a(f)a(g)\Psi\| \leq \left\| \frac{f}{\sqrt{\omega}} \right\| \left\| \frac{g}{\sqrt{\omega}} \right\| \|H_{\mathrm{b}} \Psi\|$,

(**2**) $\|a^*(f)a(g)\Psi\| \leq \left\| \frac{f}{\sqrt{\omega}} \right\| \left\| \frac{g}{\sqrt{\omega}} \right\| \|H_{\mathrm{b}} \Psi\| + \|f\| \left\| \frac{g}{\sqrt{\omega}} \right\| \|H_{\mathrm{b}}^{1/2} \Psi\|$,

(**3**) $\|a(f)a^*(g)\Psi\| \leq \left\| \frac{f}{\sqrt{\omega}} \right\| \left\| \frac{g}{\sqrt{\omega}} \right\| \|H_{\mathrm{b}} \Psi\| + \left\| \frac{f}{\sqrt{\omega}} \right\| \|g\| \, \|H_{\mathrm{b}}^{1/2} \Psi\| + \|f\| \, \|g\| \, \|\Psi\|$,

(**4**) $$\|a^*(f)a^*(g)\Psi\| \leq \left\| \frac{f}{\sqrt{\omega}} \right\| \left\| \frac{g}{\sqrt{\omega}} \right\| \|H_{\mathrm{b}} \Psi\| + \left(\|f\| \left\| \frac{g}{\sqrt{\omega}} \right\| + \left\| \frac{f}{\sqrt{\omega}} \right\| \|g\| \right) \|H_{\mathrm{b}}^{1/2} \Psi\| + 2\|f\| \|g\| \|\Psi\|.$$

(Proof) Since $\mathcal{F}_{\rm b}^{\rm fin}(\mathcal{D}(\omega))$ is a core of $H_{\rm b}$, it is enough to show that **(1) - (4)** follows for $\Psi \in \mathcal{F}_{\rm b}^{\rm fin}(\mathcal{D}(\omega))$.
(1) For $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm b}^{\rm fin}(\mathcal{D}(d\Gamma_{\rm b}(\omega)))$, we see that

$$(a(f)a(g)\Psi)^{(n)}(d\mathbf{k}_1, \cdots, d\mathbf{k}_n) = \sqrt{n+1}\sqrt{n+2}\int \overline{f(\mathbf{p})}\ \overline{f(\mathbf{q})}\Psi^{(n+2)}(\mathbf{p}, \mathbf{q}, \mathbf{k}_1, \cdots, \mathbf{k}_n)\, d\mathbf{p}\, d\mathbf{p}.$$

Then by the Schwartz inequality, we see that

$$\begin{aligned}&\left| \, (a(f)a(g)\Psi)^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n)\right| \\ &\qquad \leq \sqrt{n+1}\sqrt{n+2}\left\| \frac{f}{\sqrt{\omega}}\right\| \left\| \frac{g}{\sqrt{\omega}}\right\| \left\{ \int \omega(\mathbf{p})\omega(\mathbf{q}) \left| \Psi^{(n+2)}(\mathbf{p}, \mathbf{q}, \mathbf{k}_1, \cdots, \mathbf{k}_n s)\right|^2 d\mathbf{p}\, d\mathbf{p}\right\}^{1/2},\end{aligned}$$

and thus

$$\begin{aligned}\| a(f)a(g)\Psi^{(n)}\|^2_{\otimes^n L^2(\mathbf{R}^3)} &= \int \left| \, (a(f)a(g)\Psi)^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n)\right|^2 d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &\leq (n+1)\ (n+2)\left\| \frac{f}{\sqrt{\omega}}\right\|^2 \left\| \frac{g}{\sqrt{\omega}}\right\|^2 \int \omega(\mathbf{p})\omega(\mathbf{q}) \left| \Psi^{(n+2)}(\mathbf{p}, \mathbf{q}, \mathbf{k}_1, \cdots, \mathbf{k}_n)\right|^2 d\mathbf{p}\, d\mathbf{p}\, d\mathbf{k}_1 \cdots d\mathbf{k}_n.\end{aligned}$$

It is noted that

$$\begin{aligned}&(n+1)\ (n+2)\int \omega(\mathbf{p})\omega(\mathbf{q}) \left| \Psi^{(n+2)}(\mathbf{p}, \mathbf{q}, \mathbf{k}_1, \cdots, \mathbf{k}_n)\right|^2 d\mathbf{p}\, d\mathbf{p}\, d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &= (n+2)\int \omega(\mathbf{p}) \left(\omega(\mathbf{q}) + \sum_{j=1}^{n} \omega(\mathbf{k}_j)\right) \left| \Psi^{(n+2)}\left(\mathbf{p}, \mathbf{q}, \mathbf{k}_1, \cdots, \mathbf{k}_n\right)\right|^2 d\mathbf{p}\, d\mathbf{p}\, d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &\leq (n+2)\int \omega(\mathbf{p}) \left(\omega(\mathbf{p}) + \omega(\mathbf{q}) + \sum_{j=1}^{n} \omega(\mathbf{k}_j)\right) \left| \Psi^{(n+2)}(\mathbf{p}, \mathbf{q}, \mathbf{k}_1, \cdots, \mathbf{k}_n)\right|^2 d\mathbf{p}\, d\mathbf{p}\, d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &= \int \left(\omega(\mathbf{p}) + \omega(\mathbf{q}) + \sum_{j=1}^{n} \omega(\mathbf{k}_j)\right)^2 \left| \Psi^{(n+2)}(\mathbf{p}, \mathbf{q}, \mathbf{k}_1, \cdots, \mathbf{k}_n)\right|^2 d\mathbf{p}\, d\mathbf{p}\, d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &= \left\| \, (H_{\rm b}\Psi)^{(n+2)}\right\|^2_{\otimes^{n+2} L^2(\mathbf{R}^3)}\ .\end{aligned}$$

Hence, we see that $\sum_{n=0}^{\infty} \| a(f)a(g)\Psi^{(n)}\|^2_{\otimes^n L^2(\mathbf{R}^3)} \leq \sum_{n=0}^{\infty} \left\| \, (H_{\rm b}\Psi)^{(n+2)}\right\|^2_{\otimes^{n+2} L^2(\mathbf{R}^3)} \leq \left\| \, H_{\rm b}\Psi\right\|^2$, and thus **(1)** is obtained.

(2) By the canonical commutation relations, we have for $\Psi \in \mathcal{F}_{\rm b}^{\rm fin}(\mathcal{D}(\omega))$,

$$\| a^*(f)a(g)\Psi\|^2 = (a(f)a^*(f)a(g)\Psi\ , a(g)\Psi) = \| f\|^2 \| a(g)\Psi\|^2 + \| a(f)a(g)\Psi\|^2.$$

Then we have $\| a^*(f)a(g)\Psi\| \leq \| f\| \| a(g)\Psi\| + \| a(f)a(g)\Psi\|$. By using **(1)**, we have **(2)**.

(3) Since $a(f)a^*(g)\Psi = a^*(g)a(f)\Psi + (f, g)\Psi$ follows for $\Psi \in \mathcal{F}_{\rm b}^{\rm fin}(\mathcal{D}(\omega))$, we have $\| a(f)a^*(g)\Psi\| \leq \| a^*(g)a(f)\Psi\| + \| f\| \, \| g\| \| \Psi\|$. Then by applying **(2)**, we have **(3)**.

(4) By the canonical commutation relation, it is seen that for $\Psi \in \mathcal{F}_{\rm b}^{\rm fin}(\mathcal{D}(\omega))$,

$$\|a^*(f)a^*(g)\Psi\|^2 = (a(f)a^*(f)a^*(g)\Psi \,, a^*(g)\Psi) = \|f\|^2\|a^*(g)\Psi\|^2 + \|a(f)a^*(g)\Psi\|^2.$$

Thus we have $\|a^*(f)a^*(g)\Psi\| \leq \|f\|\|a^*(g)\Psi\| + \|a(f)a^*(g)\Psi\|$. By **(3)**, we obtain **(4)**. ■

Proposition B ([22], Lemma 3.2)
For $0 < \delta < \frac{1}{10}$, there exists $c_j(\delta) \geq 0,\ j = 1, \cdots, N$, such that for $\lambda > 0$,

$$\|\hat{p}_j^\nu(\hat{\mathbf{p}}_j + M^2 + \lambda)^{-1}(\sum_{l=1}^{N}\sqrt{\hat{\mathbf{p}}_l{}^2 + M^2} + 1)^{-1/2}\| \leq \frac{1}{\lambda^{\frac{1}{2}+\delta}} c_j(\delta). \tag{48}$$

(**Proof**) Let $\mathbf{p}_j = (p_j^\nu)_{\nu=1}^3 \in \mathbf{R}^3$. From the spectral decomposition theorem, it is enough to show that

$$\sup_{\lambda>0,\,\mathbf{p}_l\in\mathbf{R}^3, l=1,\cdots,N} |\lambda^{\frac{1}{2}+\delta} p_j^\nu| \ (\mathbf{p}_j^2 + M^2 + \lambda)^{-1}(\sum_{l=1}^{N}\sqrt{\mathbf{p}_l^2 + M^2} + 1)^{-1/2} \ < \ \infty. \tag{49}$$

From the The Young's inequality, it is seen that for $q > 1$ and $\tilde{q} > 1$ satisfying $\frac{1}{q} + \frac{1}{\tilde{q}} = 1$,

$$\lambda^{\frac{1}{2}+\delta}|p_j^\nu| \ \leq \ \frac{1}{q}\lambda^{(\frac{1}{2}+\delta)q} \ + \ \frac{1}{\tilde{q}}|p_j^\nu|^{\tilde{q}}.$$

Let us set $q = (\frac{1}{2}+\delta)^{-1}$ for $0 < \delta < \frac{1}{10}$. Then $\tilde{q} = (\frac{1}{2}-\delta)^{-1}$ holds, and we see that

$$\lambda^{\frac{1}{2}+\delta}|p_j^\nu| \ \leq \ (\frac{1}{2}+\delta)\lambda \ + \ (\frac{1}{2}-\delta)|p_j^\nu|^{(\frac{1}{2}-\delta)^{-1}}. \tag{50}$$

It is seen that

$$\sup_{\lambda>0,\,\mathbf{p}_l\in\mathbf{R}^3, l=1,\cdots,N} \lambda \ (\mathbf{p}_j^2 + M^2 + \lambda)^{-1}(\sum_{l=1}^{N}\sqrt{\mathbf{p}_l^2 + M^2} + 1)^{-1/2} \ < \ \infty. \tag{51}$$

Note that $(\frac{1}{2}-\delta)^{-1} \ < \frac{5}{2}$ follows for $0 < \delta < \frac{1}{10}$. Then it follows that

$$\sup_{\mathbf{p}_l\in\mathbf{R}^3, l=1,\cdots,N} |p_j^\nu|^{(\frac{1}{2}-\delta)^{-1}}(\mathbf{p}_j^2 + M^2)^{-1}(\sum_{l=1}^{N}\sqrt{\mathbf{p}_l^2 + M^2} + 1)^{-1/2} < \infty. \tag{52}$$

Then by (52), we have

$$\begin{aligned}&\sup_{\lambda>0,\,\mathbf{p}_l\in\mathbf{R}^3, l=1,\cdots,N} |p_j^\nu|^{(\frac{1}{2}-\delta)^{-1}}(\mathbf{p}_j^2 + M^2 + \lambda)^{-1}(\sum_{l=1}^{N}\sqrt{\mathbf{p}_l^2 + M^2} + 1)^{-1/2} \\ &\qquad \leq \sup_{\mathbf{p}_l\in\mathbf{R}^3, l=1,\cdots,N} |p_j^\nu|^{(\frac{1}{2}-\delta)^{-1}}(\mathbf{p}_j^2 + M^2)^{-1}(\sum_{l=1}^{N}\sqrt{\mathbf{p}_l^2 + M^2} + 1)^{-1/2} < \infty. \end{aligned} \tag{53}$$

By (50), (51) and (53), we obtain (49). ■

Lemma C It follows that

$$\int_{\mathbf{R}^{3N}} \frac{1}{|\mathbf{x}_i - \mathbf{x}_l|^2} \Big| \psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) \Big|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N \ \le\ 4 \int_{\mathbf{R}^{3N}} \Big| \nabla_{\mathbf{x}_i} \psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) \Big|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N, \qquad i \neq j.$$

(**Proof**) For $\varepsilon > 0$, let us set $\mathbf{a}_{i,j,\varepsilon}(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \left(a^{\nu}_{i,j,\varepsilon}(\mathbf{x}_1, \cdots, \mathbf{x}_N) \right)^3_{\nu=1}$ with $a^{\nu}_{i,j,\varepsilon}(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \frac{x^{\nu}_i - x^{\nu}_j}{|\mathbf{x}_i - \mathbf{x}_l|^2 + \varepsilon}$. It is seen that $\left\| \left(\nabla_{\mathbf{x}_i} + t\mathbf{a}_{i,j,\varepsilon}\right) \psi \right\|^2 = \|\nabla_{\mathbf{x}_i} \psi\| - t\left(\psi, [\nabla_{\mathbf{x}_i}, \mathbf{a}_{i,j,\varepsilon}]\psi\right) + t^2 \|\mathbf{a}_{i,j,\varepsilon}\psi\|$, for $t > 0$. Note that $[\nabla_{\mathbf{x}_i}, \mathbf{a}_{i,j,\varepsilon}]\psi = \sum_{\nu=1}^{3} [\partial_{x^{\nu}_i}, a^{\nu}_{i,j,\varepsilon}]\psi = \sum_{\nu=1}^{3} \left(\frac{3}{|\mathbf{x}_i - \mathbf{x}_l|^2 + \varepsilon} - \frac{2|\mathbf{x}_i - \mathbf{x}_l|^2}{(|\mathbf{x}_i - \mathbf{x}_l|^2 + \varepsilon)^2} \right) \psi$. Hence we have

$$\|\nabla_{\mathbf{x}_i} \psi\|^2 \ge \int_{\mathbf{R}^{3N}} \left\{ t \left(\frac{3}{|\mathbf{x}_i - \mathbf{x}_l|^2 + \varepsilon} - \frac{2|\mathbf{x}_i - \mathbf{x}_l|^2}{(|\mathbf{x}_i - \mathbf{x}_l|^2 + \varepsilon)^2} \right) - t^2 \frac{|\mathbf{x}_i - \mathbf{x}_l|^2}{(|\mathbf{x}_i - \mathbf{x}_l|^2 + \varepsilon)^2} \right\} \Big| \psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) \Big|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N. \tag{54}$$

Note that $\sup_{\varepsilon > 0} \frac{1}{|\mathbf{x}_i - \mathbf{x}_l|^2 + \varepsilon} \le \frac{3}{|\mathbf{x}_i - \mathbf{x}_l|^2}$. Note that $\int_{\mathbf{R}^{3N}} \frac{1}{|\mathbf{x}_i - \mathbf{x}_l|^2} \Big| \psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) \Big|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N < \infty$ follows for $\psi \in C^{\infty}_0(\mathbf{R}^{3N})$. Then by applying the Lebesgue dominated convergence theorem to the right side of (c) as $\varepsilon \to 0$, we have $\|\nabla_{\mathbf{x}_i} \psi\|^2 \ge (-t^2 + t) \int_{\mathbf{R}^{3N}} \frac{1}{|\mathbf{x}_i - \mathbf{x}_l|^2} \Big| \psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) \Big|^2 d\mathbf{x}_1 \cdots d\mathbf{x}_N$. Since $-t^2 + t \le \frac{1}{4}$, the proof is completed. ■

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