The Golden Quandratic Equations

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The Golden Quadratic Equations

Seiichi Iwamoto

Abstract

This paper considers a sequence of quadratic equations (QEs), which are called *Golden quadratic equations*. The first Golden QE is a well known Fibonacci QE. We introduce a general form of Golden QE — the *n-th* Golden quadratic equation — by solving an eigenvalue problem of 2×2 integer symmetric matrices. We translate the eigenvalue problem into an equivalent optimization problem.

1 Introduction

The Fibonacci sequence is defined by a difference equation, which is one of the most famous second-order linear difference equations [3, 4, 12]. The difference equation has a quadratic equation (QE) as its characteristic equation. Sometimes the QE is called *Fibonacci* quadratic equation.

In this paper we propose a *sequence* of Golden QEs, which gives a general form of the *single* Fibonacci QE.

2 The Fibonacci Quadratic Equation

The Fibonacci sequence $\{F_n\}$ is defined by the difference equation

$$F_{n+2} - F_{n+1} - F_n = 0 \qquad (F_0 = 0,\ F_1 = 1). \tag{1}$$

The equation has the characteristic equation

$$x^2 - x - 1 = 0. \tag{2}$$

Sometimes (2) is called *Fibonacci quadratic equation* [3,4,12]. The Fibonacci quadratic equation has two real solutions: ϕ and its *conjugate* $1-\phi$, where

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

is the *Golden number* [3, 4, 8−10, 12]. We note that

$$\phi + (1-\phi) = 1, \qquad \phi(1-\phi) = -1.$$

Lemma 2.1 *It holds that for any real values a, b*

$$(a + b\phi)(a + b - b\phi) = a^2 + ab - b^2.$$

Thus the two numbers $a + b\phi$ and $a + b(1-\phi)$ are called each other's *conjugate* number.

Definition 2.1 *We define for any pair of real values* (a, b)

$$\overline{a+b\phi} := a+b(1-\phi). \tag{3}$$

Thus we have

$$\overline{a+b\phi} = a+b-b\phi, \qquad \overline{\phi} = -\phi^{-1}.$$

The conjugation is the transformation on R^2 :

$$(a, b) \longrightarrow (a+b, -b).$$

Let $x, y, z, \ldots$ be real values represented in form $a+b\phi$ where (a, b) is a pair of real values. Then we have the following relations.

Lemma 2.2

(i) $\overline{\overline{x}} = x$

(ii) $\overline{x+y} = \bar{x} + \bar{y}$

(iii) $\overline{cx} = c\bar{x} \qquad c \in R^1$

(iv) $\overline{xy} = \bar{x}\,\bar{y}$

(v) $\overline{\left(\frac{1}{x}\right)} = \frac{1}{\bar{x}}$

Definition 2.2 *We define for* $x = a + b\phi$, *where* $a, b \in R^1$

$$\|x\| := x\bar{x}. \tag{4}$$

Lemma 2.3 *It holds that for any* $x = a+b\phi$, *where* $a, b \in R^1$

$$\|x\| = a^2 + ab - b^2.$$

We note that $\|x\| = 0$ implies $\frac{b}{a} = \phi$ or $\frac{b}{a} = \overline{\phi}$, provided that $a \neq 0$.

We have for any nontrivial pair of real values (a, b)

$$\frac{1}{a+b\phi} = \frac{a+b}{a^2+ab-b^2} - \frac{b}{a^2+ab-b^2}\phi.$$

For instance we have a list of linear expressions for fractional forms of two linear forms in ϕ as follows :

$$\frac{1}{\phi} = -1+\phi = \frac{-1+\sqrt{5}}{2} \approx 0.61803$$

$$\frac{1}{1+\phi} = 2-\phi = \frac{3-\sqrt{5}}{2} \approx 0.38197$$

$$\frac{1}{2+\phi} = \frac{1}{5}(3-\phi) = \frac{5-\sqrt{5}}{10} \approx 0.27639$$

and

$$\frac{-7+4\phi}{4-3\phi} = \frac{-4+3\phi}{3-\phi} = \frac{-1+2\phi}{2+\phi} = -1+\phi \approx 0.61803$$

$$\frac{3-\phi}{2+\phi}=\frac{2+\phi}{3+4\phi}=2-\phi\approx 0.38197$$

$$\frac{1+3\phi}{2+\phi}=\frac{3+4\phi}{1+3\phi}=\phi\approx 1.61803$$

$$\frac{2+\phi}{3-\phi}=\frac{3+4\phi}{2+\phi}=1+\phi=\frac{3+\sqrt{5}}{2}\approx 2.61803$$

Let us now consider some variants of Fibonacci quadratic equation.

Lemma 2.4 *Eq.(2) has equivalent forms as follows (see Fig.1-8 in Appendix):*

(i) $x=-1+x^2$

(ii) $x=\pm\sqrt{1+x}$

(iii) $x=1+\frac{1}{x}$

(iv) $x=\frac{1}{x-1}$

(v) $x=2-\frac{1}{x+1}$

(vi) $x=-1+\frac{1}{2-x}$

(vii) $x=1+\cfrac{1}{1+\cfrac{1}{x}}\ \left(=\frac{2x+1}{x+1}\right)$

(viii) $x=\cfrac{1}{-1+\cfrac{1}{-1+x}}\ \left(=\frac{-1+x}{2-x}\right)$

(ix) $\frac{1}{x-1}=1+\frac{1}{x}.$

Hence each equation has solutions ϕ and $1-\phi$.

The two solutions ϕ and $1-\phi$ are called *Golden*. We call each of (i)-(ix) *Golden equation*. Because it has the Golden solutions.

3 The Fibonacci Sequence

A Fibonacci sequence $\{F_n\}$ is defined by second-order linear difference equation

$$F_{n+2}-F_{n+1}-F_n=0.$$

Then we have a famous relation.

Lemma 3.1

$$\phi^n=F_n\phi+F_{n-1}$$

$$n=\cdots,-1,0,1,\cdots$$

$$(\phi-1)^n=F_{-n}\phi+F_{-n-1}$$

where $\{F_n\}$ *is the Fibonacci sequence with* $F_0=0,\ \ F_1=1.$

Lemma 3.2 (Three Identities) *The Fibonacci sequence satisfies*

$$\text{(Cassini's)} \qquad F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$
$$\text{(Catalan's)} \qquad F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2$$
$$\text{(d'Ocagne's)} \qquad F_mF_{n+1} - F_nF_{m+1} = (-1)^n F_{m-n}$$

where $n = \cdots, -1, 0, 1, \cdots$.

The Fibonacci sequence is tabulated in Table 1:

n	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
F_n	233	−144	89	−55	34	−21	13	−8	5	−3	2	−1	1	0	1

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597

Table 1 Fibonacci sequence $\{F_n\}$

On the other hand, the Fibonacci sequence has the analytic form

Lemma 3.3

$$F_n = \frac{1}{2\phi - 1}\{\phi^n - (1-\phi)^n\} \quad n = \cdots, -1, 0, 1, \cdots$$

Then we have

$$F_n = (-1)^{n+1}F_{-n} \quad n = 0, 1, 2, \cdots$$

Lemma 3.4

$$\begin{array}{rl} 1 = 1+0\phi = (\phi-1)^0 = 1 = & \phi^0 = 1+0\phi = 1 \\ 0.61803 \approx -1+1\phi = (\phi-1)^1 & \phi^1 = 0+1\phi \approx 1.61803 \\ 0.381966 \approx 2-1\phi = (\phi-1)^2 & \phi^2 = 1+1\phi \approx 2.61803 \\ 0.236068 \approx -3+2\phi = (\phi-1)^3 & \phi^3 = 1+2\phi \approx 4.23606 \\ 0.145898 \approx 5-3\phi = (\phi-1)^4 & \phi^4 = 2+3\phi \approx 6.85410 \\ 0.090170 \approx -8+5\phi = (\phi-1)^5 & \phi^5 = 3+5\phi \approx 11.0902 \\ 0.055728 \approx 13-8\phi = (\phi-1)^6 & \phi^6 = 5+8\phi \approx 17.9443 \\ 0.034442 \approx -21+13\phi = (\phi-1)^7 & \phi^7 = 8+13\phi \approx 29.0344 \\ 0.021286 \approx 34-21\phi = (\phi-1)^8 & \phi^8 = 13+21\phi \approx 46.9787 \\ 0.013156 \approx -55+34\phi = (\phi-1)^9 & \phi^9 = 21+34\phi \approx 76.0132 \\ 0.008131 \approx 89-55\phi = (\phi-1)^{10} & \phi^{10} = 34+55\phi \approx 122.992 \end{array}$$

left-right is invese

Lemma 3.5

(i) $$\frac{\phi}{1} = \frac{1+\phi}{\phi} = \frac{1+2\phi}{1+\phi} = \frac{2+3\phi}{1+2\phi} = \frac{3+5\phi}{2+3\phi} = \cdots = \frac{F_n + F_{n+1}\phi}{F_{n-1}+F_n\phi} \approx 1.61803$$

(ii) $$\frac{F_n + F_{n+1}\phi}{F_{n-1}+F_n\phi} = \cdots = \frac{2\phi-3}{5-3\phi} = \frac{2-\phi}{2\phi-3} = \frac{\phi-1}{2-\phi} = \frac{1}{\phi-1} = \frac{\phi}{1}$$

(iii) $\frac{0.236}{0.146} \approx \frac{0.382}{0.236} \approx \frac{0.618}{0.382} \approx \frac{1}{0.618} \approx \frac{1.618}{1} \approx \frac{2.618}{1.618} \approx \frac{4.236}{2.618} \approx \cdots \approx 1.618$

where $\{F_n\}$ *is the Fibonacci sequence.*

4 The Golden Quadratic Equations

In this section we introduce the sequence of Golden quadratic equations [3-7, 11, 12].

Definition 4.1 *The* n*-th Golden quadratic equation is defined as*

$$x^2 - (F_{n-1}+F_{n+1})x + (-1)^n = 0 \tag{5}$$

where $\{F_n\}$ *is the Fibonacci sequence with* $F_0=0$, $F_1=1$.

For instance we have the following Golden quadratic equations (see Fig's.9, 10 in Appendix):

$$\begin{array}{ll}
 & \iddots \\
(-6)\text{-th} & \underline{x^2 - 18x + 1 = 0} \\
(-5)\text{-th} & x^2 + 11x - 1 = 0 \\
(-4)\text{-th} & \underline{x^2 - 7x + 1 = 0} \\
(-3)\text{-rd} & x^2 + 4x - 1 = 0 \\
(-2)\text{-nd} & \underline{x^2 - 3x + 1 = 0} \\
(-1)\text{-st} & x^2 + x - 1 = 0 \\
0\text{-th} & x^2 - 2x + 1 = 0 \\
\text{first} & x^2 - x - 1 = 0 \\
\text{second} & \underline{x^2 - 3x + 1 = 0} \\
\text{third} & x^2 - 4x - 1 = 0 \\
\text{fourth} & \underline{x^2 - 7x + 1 = 0} \\
\text{fifth} & x^2 - 11x - 1 = 0 \\
\text{sixth} & \underline{x^2 - 18x + 1 = 0} \\
 & \ddots
\end{array} \tag{6}$$

We remark that for even n both n-th and $(-n)$-th Golden quadratic equations are identical, as the above underlines show:

$$x^2 - (F_{n-1}+F_{n+1})x + 1 = 0. \tag{7}$$

For odd n we have the n-th Golden quadratic equation:

$$x^2 - (F_{n-1}+F_{n+1})x - 1 = 0 \tag{8}$$

where

$$\begin{cases} F_{n-1}+F_{n+1} > 0 & \text{for} \quad n > 0 \\ F_{n-1}+F_{n+1} < 0 & \text{for} \quad n < 0 \end{cases}$$

and

$$F_{n-1}+F_{n+1} = F_{-(n-1)}+F_{-(n+1)}.$$

Theorem 4.1 *The n-th Golden quadratic equation (5) has two solutions*

$$\phi^n \text{ and } (1-\phi)^n \tag{9}$$

where ϕ is the Golden number.

We note that

$$\begin{cases} \phi^n = F_{n-1}+F_n\phi \\ (1-\phi)^n = F_{n+1}-F_n\phi. \end{cases}$$

5 Golden Matrices

Let us take four integer symmetric 2×2 matrices as follows:

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \; G = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

We consider eigenvalue problems associated with these matrices [8]. Then we have the following result.

Theorem 5.1 *The four matrices have a common characteristic equation which is the first Golden quadratic equation. Conversely, an integer symmetric 2×2 matrix whose characteristic equation reduces the first Golden quadratic equation is restricted to these matrices.*

5.1 Matrix A

Let us consider the integer symmetric matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}. \tag{10}$$

The matrix A is called *Golden*. Two reasons will be clarified as follows.

First we see that the characteristic equation of A is

$$\lambda^2 - \lambda - 1 = 0, \tag{11}$$

which is the *first Golden* quadratic equation in (6). This gives *one* reason why A is Golden. The Eq.(11) has the eigenvalues

$$\lambda_1 = \phi, \;\; \lambda_2 = 1-\phi \tag{12}$$

with the corresponding eigenvectors

$$x_1 = c\begin{pmatrix} 1 \\ 1-\phi \end{pmatrix}, \;\; x_2 = d\begin{pmatrix} 1 \\ \phi \end{pmatrix} \tag{13}$$

where $-\infty < c,\ d < \infty$. For the normality, in what follows, we take

$$c = \frac{1}{\sqrt{3-\phi}},\quad d = \frac{1}{\sqrt{2+\phi}}.$$

Then the eigenvectors $x_1,\ x_2$ are orthonormal.

Therefore we have the two-dimensional variational characterization of $\lambda_1,\ \lambda_2$ through the Rayleigh quotient $(x,\ Ax)/(x,\ x)$ ([1, p.72], [2]):

Lemma 5.1 *It holds that*

$$\begin{aligned}\phi &= \max_x \frac{(x,\ Ax)}{(x,\ x)}\\ \bar{\phi} &= \min_x \frac{(x,\ Ax)}{(x,\ x)}.\end{aligned} \tag{14}$$

In the case, $x^* = \begin{pmatrix} \phi \\ -1 \end{pmatrix}$ is a maximizer and $\hat{x} = \begin{pmatrix} 1 \\ \phi \end{pmatrix}$ is a minimizer.

Thus we have represented two pairs of eigenvalue and eigenvector in terms of a single term, the Golden ratio, ϕ. This is *the other* reason.

Furhermore, we consider the maximization and minimization of the quadratic form $(x,\ Ax) = x^2 - 2xy$ under the unit constraint $x^2 + y^2 = 1$:

$$\mathrm{OA}_1 \qquad \begin{array}{ll} \text{Maximize and minimize} & x^2 - 2xy \\ \text{subject to} & \text{(i)}\ \ x^2 + y^2 = 1 \\ & \text{(ii)}\ \ -\infty < x,\ y < \infty. \end{array}$$

Then OA_1 has the maximum value $M = \phi$ at the points $(x^*,\ y^*) = \pm\frac{1}{\sqrt{2+\phi}}(\phi,\ -1)$ and the minimum value $m = \bar{\phi}$ at the points $(\hat{x},\ \hat{y}) = \pm\frac{1}{\sqrt{2+\phi}}(1,\ \phi)$. Both the maximum points and minimum points constitute the Golden ratio, whereupon the maximum value and the minimum value are the Golden number and its conjugate, respectively. In this sense both solutions - pairs of optimum value and optimum point - are also Golden.

6 The Golden Identities

Let $\{F_n\}$ be the Fibonacci sequence with $F_0 = 0,\ \ F_1 = 1$:

$$F_{n+2} - F_{n+1} - F_n = 0 \qquad n = \cdots,\ -1,\ 0,\ 1,\ \cdots.$$

First we consider the integer symmetric matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & -F_1 \\ -F_1 & F_0 \end{pmatrix}$$

Then we have

$$A^{-1}=\begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix},$$

$$A^{2}=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix},\ A^{3}=\begin{pmatrix}3 & -2\\ -2 & 1\end{pmatrix},\ A^{4}=\begin{pmatrix}5 & -3\\ -3 & 2\end{pmatrix},\ \cdots$$

and

$$A^{-2}=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix},\ A^{-3}=\begin{pmatrix}-1 & -2\\ -2 & -3\end{pmatrix},\ A^{-4}=\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix},\ \cdots$$

where $A^0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Lemma 6.1

$$A^{n}=\begin{pmatrix}F_{n+1} & -F_n\\ -F_n & F_{n-1}\end{pmatrix},\qquad A^{-n}=(-1)^n\begin{pmatrix}F_{n-1} & F_n\\ F_n & F_{n+1}\end{pmatrix}$$

where $n=0,1,\cdots$

We can easily find two pairs of eigenvalue and eigenvector for matrix A^n. The two pairs yield both the maximum solution and minimum solution for the corresponding optimization problem:

$$\text{OA}_\text{n}\qquad \begin{array}{ll}\text{Maximize and minimize} & F_{n+1}x^2-2F_nxy+F_{n-1}y^2\\ \text{subject to} & \text{(i)}\ \ x^2+y^2=1\\ & \text{(ii)}\ \ -\infty<x,\ y<\infty.\end{array}$$

Then OA_n has the maximum value $M=\phi^n$ at the points $(x^*,y^*)=\pm\frac{1}{\sqrt{2+\phi}}(-1,\phi)$ and the minimum value $m=\bar{\phi}^n$ at the points $(\hat{x},\hat{y})=\pm\frac{1}{\sqrt{2+\phi}}(1,\phi)$. Thus we have the sequence of identities as follows:

Theorem 6.1 (The Sequence of Golden Identities A) *It holds that*

$$\text{(i)}\quad (F_{n+1}-F_n\phi)(x^2+y^2)+F_n(\phi-1)(y-\phi x)^2=F_{n+1}x^2-2F_nxy+F_{n-1}y^2$$

and

$$\text{(ii)}\quad F_{n+1}x^2-2F_nxy+F_{n-1}y^2+F_n(\phi-1)(x+\phi y)^2=(F_{n-1}+F_n\phi)(x^2+y^2)$$

for $n=\cdots,-1,0,1,\cdots$.

Second we consider the integer symmetric matrix

$$G=\begin{pmatrix}0 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}F_0 & -F_1\\ -F_1 & F_2\end{pmatrix}.$$

Through the analysis of eigenvalue problem for matrix G, we have

Corollary 6.1 (The Sequence of Golden Identities G) *It holds that*

$$\text{(i)}\quad (F_{n+1}-F_n\phi)(x^2+y^2)+F_n(\phi-1)(x-\phi y)^2=F_{n-1}x^2-2F_nxy+F_{n+1}y^2$$

and

$$\text{(ii)}\quad F_{n-1}x^2-2F_nxy+F_{n+1}y^2+F_n(\phi-1)(y+\phi x)^2=(F_{n-1}+F_n\phi)(x^2+y^2)$$

for $n = \cdots, -1, 0, 1, \cdots$.

This result is obtained through the transformation $(x, y) \longrightarrow (y, x)$ on Theorem 6.1.

Third we consider the integer symmetric matrix

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix}.$$

Then we have the following sequence of Golden identities.

Corollary 6.2 (The Sequence of Golden Identities B) *It holds that*

$$\text{(i)} \quad (F_{n+1} - F_n\phi)(x^2 + y^2) + F_n(\phi - 1)(y + \phi x)^2 = F_{n+1}x^2 + 2F_n xy + F_{n-1}y^2$$

and

$$\text{(ii)} \quad F_{n+1}x^2 + 2F_n xy + F_{n-1}y^2 + F_n(\phi - 1)(\phi y - x)^2 = (F_{n-1} + F_n\phi)(x^2 + y^2)$$

for $n = \cdots, -1, 0, 1, \cdots$.

This result is obtained through the transformation $(x, y) \longrightarrow (-x, y)$ on Theorem 6.1.

Finally we consider the integer symmetric matrix

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}.$$

Corollary 6.3 (The Sequence of Golden Identities C) *It holds that*

$$\text{(i)} \quad (F_{n+1} - F_n\phi)(x^2 + y^2) + F_n(\phi - 1)(x + \phi y)^2 = F_{n-1}x^2 + 2F_n xy + F_{n+1}y^2$$

and

$$\text{(ii)} \quad F_{n-1}x^2 + 2F_n xy + F_{n+1}y^2 + F_n(\phi - 1)(\phi x - y)^2 = (F_{n-1} + F_n\phi)(x^2 + y^2)$$

for $n = \cdots, -1, 0, 1, \cdots$.

This is obtained through the transformation $(x, y) \longrightarrow (x, -y)$ on Corollary 6.1.

Finally we remark as follows.

Lemma 6.2 *The four matrices A, B, C and G have a common characteristic equation*

$$\lambda^2 - \lambda - 1 = 0. \tag{15}$$

This is nothing but the *Golden* quadratic equation. Thus the four matrices are also called *Golden*. These Golden matrices have two common eigenvalues

$$\lambda_1 = \phi, \qquad \lambda_2 = 1 - \phi. \tag{16}$$

First the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ has the corresponding eigenvectors

$$x_1 = c\begin{pmatrix} 1 \\ 1 - \phi \end{pmatrix}, \qquad x_2 = d\begin{pmatrix} 1 \\ \phi \end{pmatrix}$$

where $c \neq 0$, $d \neq 0$. For the normality, in what follows, we take

$$c = \frac{1}{\sqrt{3 - \phi}}, \qquad d = \frac{1}{\sqrt{2 + \phi}}.$$

Then the eigenvectors x_1, x_2 are orthonormal. Sometimes, however, we take a simple form

$$x_1 = \begin{pmatrix} -\phi \\ 1 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 1 \\ \phi \end{pmatrix}. \tag{17}$$

This is called ϕ-*expression*.

Second the matrix $G = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ has the corresponding eigenvectors

$$x_1 = c\begin{pmatrix} 1-\phi \\ 1 \end{pmatrix}, \qquad x_2 = d\begin{pmatrix} \phi \\ 1 \end{pmatrix}$$

where $c \neq 0$, $d \neq 0$. A simple form is ϕ-expression:

$$x_1 = \begin{pmatrix} 1 \\ -\phi \end{pmatrix}, \qquad x_2 = \begin{pmatrix} \phi \\ 1 \end{pmatrix}. \tag{18}$$

Third the matrix $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has the eigenvectors in ϕ-expression:

$$x_1 = \begin{pmatrix} \phi \\ 1 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 1 \\ -\phi \end{pmatrix}. \tag{19}$$

Finally the matrix $C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ has the following ϕ-expression:

$$x_1 = \begin{pmatrix} 1 \\ \phi \end{pmatrix}, \qquad x_2 = \begin{pmatrix} -\phi \\ 1 \end{pmatrix}. \tag{20}$$

Concluding Remarks

Each of four corresponding pairs of eigenvectors (17)-(20) and each of the eight vectors are also called *Golden*. Thus we call the characteristic equation (15) *Golden*.

As a result we have introduced the adjective *Golden* on the basisi of the Golden number ϕ for the following terms: (1) matrix, (2) quadratic equation, (3) characteristic equation, (4) eigenvalue, (5) eigenvector, (6) inequality and (7) identity.

References

[1] E.F. Beckenbach and R. Bellman, *Inequalities*, 3rd revised printing, Springer, New York, 1971.

[2] R.E. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, NY, 1970 (Second Edition is a SIAM edition 1997).

[3] A. Beutelspacher and B. Petri, *Der Goldene Schnitt 2., überarbeitete und erweiterte Auflange*, Elsevier GmbH, Spectrum Akademischer Verlag, Heidelberg, 1996.

[4] R.A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific Publishing Co.Pte. Ltd., 1977.

2nd ed., Cambridge Univ. Press, London and New York, 1952.

[5] S. Iwamoto, Cross dual on the Golden optimum solutions, *Proceedings of the Workshop in Mathematical Economics, RIMS Kokyu Roku* No.1443, pp.27-43, Kyoto University, 2005.

[6] S. Iwamoto, The Golden optimum solution in quadratic programming, *Proc. of the Fourth Intl Conference on Nonlinear Analysis and Convex Analysis (NACA 2005)*, Yokohama Publishers, Yokohama, 2007, pp.109-205.

[7] S. Iwamoto, The Golden trinity — optimality, inequality, identity —, *Proceedings of the Workshop in Mathematical Economics, RIMS Kokyu Roku* No.1488, pp.1-14, Kyoto University, 2006.

[8] S. Iwamoto, Golden quadruplet: optimization-inequality-identity-operator, *Modeling Decisions for Artificial Intelligence*, Springer-Verlag Lecture Notes in Artificial Intelligence, Vol. 4617, 2007, pp.14-23.

[9] A. Kira and S. Iwamoto, Golden complementary dual in quadratic optimization, *Modeling Decisions for Artificial Intelligence*, Springer-Verlag Lecture Notes in Artificial Intelligence, Vol.5285, 2008, pp.191-202.

[10] S. Iwamoto and M. Yasuda, Golden optimal path in discrete-time dynamic optimization processes, *Advanced Studies in Pure Mathematics* 53, 2009, *Advances in Discrete Dynamic Systems*, pp.77-86.

[11] S. Iwamoto and M. Yasuda, “Dynamic programming creates the Golden Ratio, too,” *Proc. of the Sixth Intl Conference on Optimization: Techniques and Applications (ICOTA 2004)*, Ballarat, Australia, December 2004.

[12] H. Walser, *Der Goldene Schnitt*, B.G. Teubner, Leibzig, 1996.

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Appendix: Graphic representation

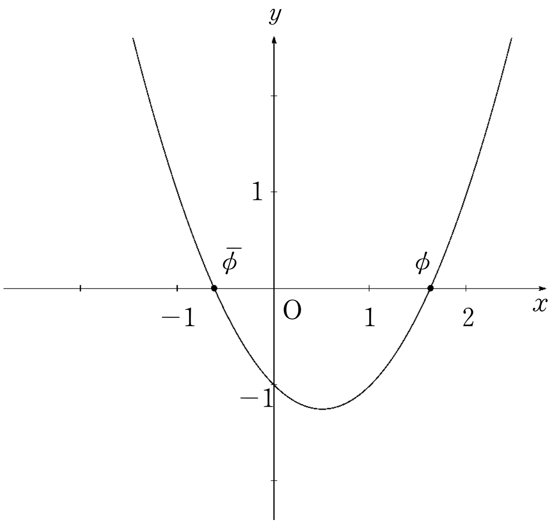

Fig. 1　$y=0$,　$y=x^2-x-1$

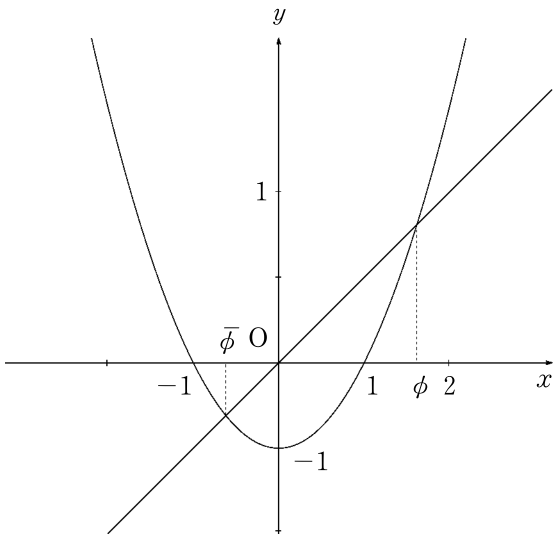

Fig. 2　$y=x$,　$y=-1+x^2$

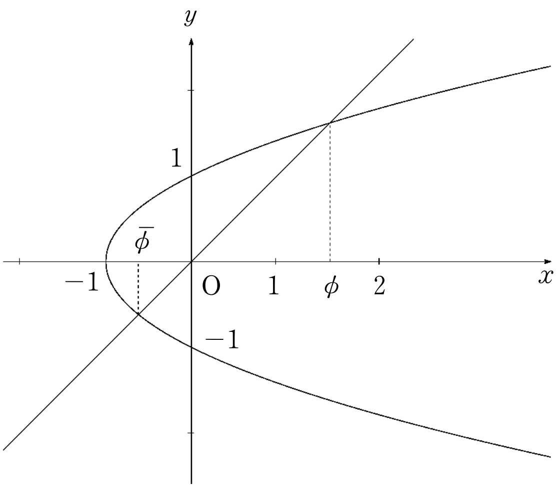

Fig. 3　$y=x$,　$y=\pm\sqrt{1+x}$

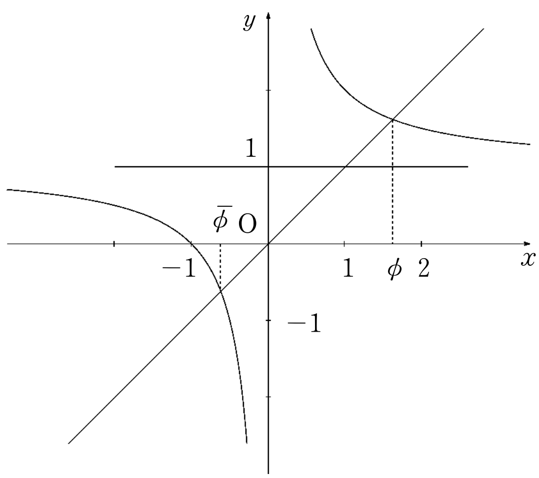

Fig. 4　$y=x$,　$y=1+\dfrac{1}{x}$

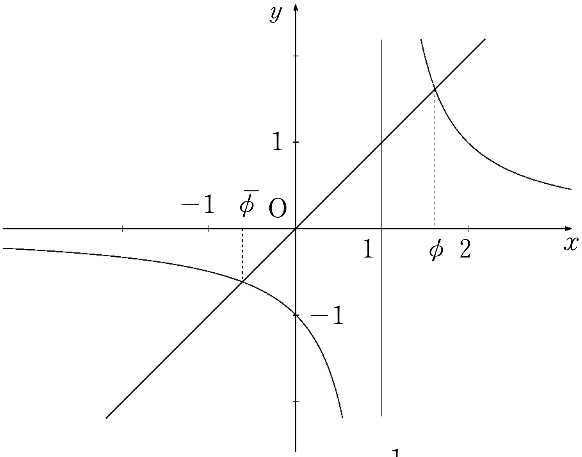

Fig. 5 $y=x$, $y=\dfrac{1}{x-1}$

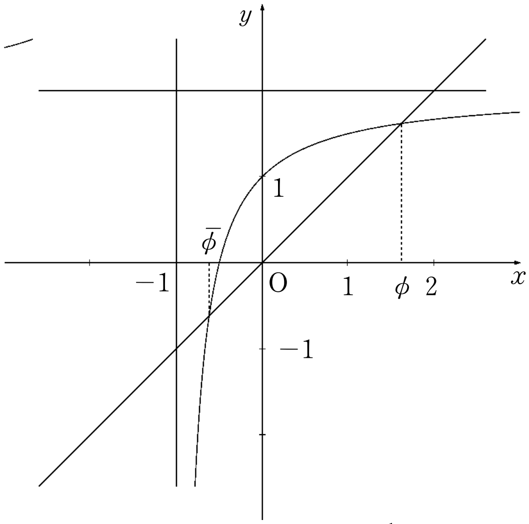

Fig. 6 $y=x$, $y=2-\dfrac{1}{x+1}$

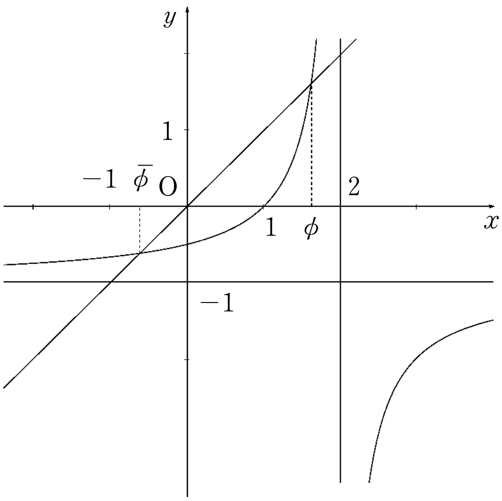

Fig. 7 $y=x$, $y=-1+\dfrac{1}{2-x}$

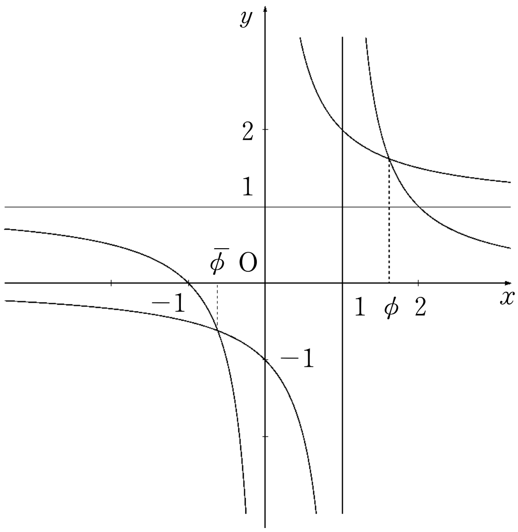

Fig. 8 $y=\dfrac{1}{x-1}$, $y=1+\dfrac{1}{x}$

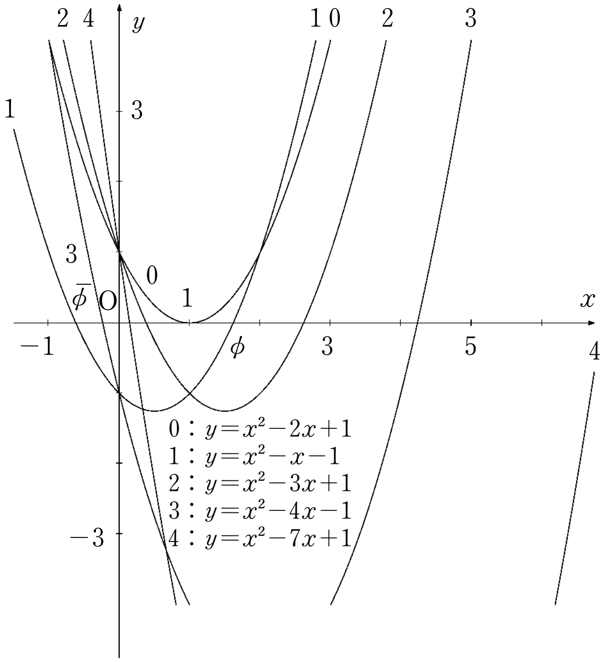

Fig. 9　Graph of the n-th Golden quadratic equation　$n=0, 1, 2, 3, 4$

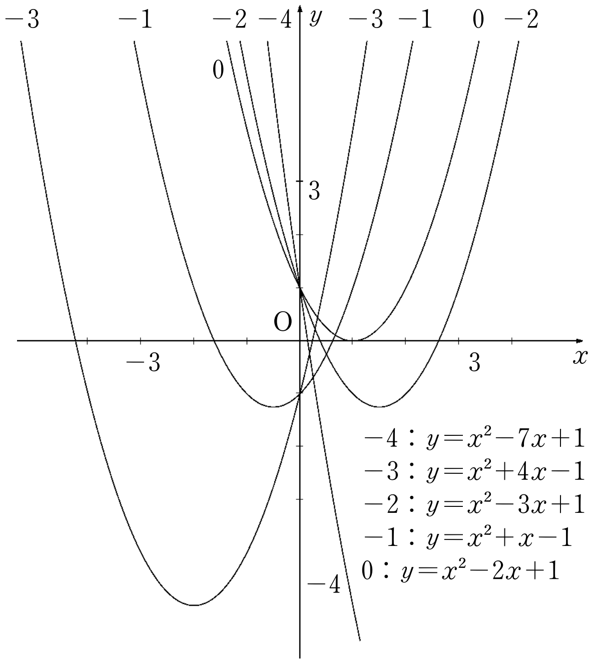

Fig. 10　Graph of the n-th Golden quadratic equation　$n=0, -1, -2, -3, -4$