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Fukumoto, Yasuhide
Faculty of Mathematics and Mathematical Research Center for Industrial Technology, Kyushu University

Hirota, Makoto
Japan Atomic Energy Agency

Mie, Youichi
Faculty of Mathematics, Kyushu University

出版情報：MI Preprint Series. 2010-20, 2010-04-28. Faculty of Mathematics, Kyushu University
バージョン：
権利関係：

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

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Yasuhide Fukumoto,
Makoto Hirota &
Youichi Mie

MI 2010-20

(Received April 28, 2010)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

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Y. Fukumoto[1], **M. Hirota**[2] **and Y. Mie**[1]

[1]Faculty of Mathematics and Mathematical Research Center for Industrial Technology, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan

[2]Japan Atomic Energy Agency, Naka, Ibaraki 311-0193, Japan

E-mail: `yasuhide@math.kyushu-u.ac.jp`

Abstract. Rotating flows with elliptically strained streamlines suffer from a parametric resonance instability between a pair of Kelvin waves whose azimuthal wavenumbers are separated by two. We address the weakly nonlinear evolution of amplitude of three-dimensional Kelvin waves, in resonance, on a flow confined in a cylinder of elliptical cross-section. In a traditional Eulerian approach, derivation of the mean-flow induced by nonlinear interaction of Kelvin waves stands as an obstacle. We show how topological idea, or the Lagrangian approach, facilitates calculation of the wave-induced mean flow. A steady incompressible Euler flow is characterized as a state of the maximum of the total kinetic energy with respect to perturbations constrained to an isovortical sheet, and the isovortical perturbation is handled only in terms of the Lagrangian variables. The criticality in energy of a steady flow allows us to work out the wave-induced mean flow only from the linear Lagrangian displacement. With the mean flow at hand, the Lagrangian approach provides us with a bypass to enter into weakly nonlinear regime of amplitude evolution of three-dimensional disturbances. Unlike the Eulerian approach, the amplitude equations are available directly in the Hamiltonian normal form.

1. Introduction

It is well known that a vortex tube embedded in a strain flow suffers from three-dimensional (3D) instability, being referred to as the Moore-Saffman-Tsai-Widnall (MSTW) instability [17, 20, 3, 5]. The MSTW instability is typically a parametric resonance, driven by the imposed shear, between left- and right-handed helical waves. The waves on a circular cylindrical vortex tube are called the Kelvin waves or the inertial waves. In general, a vortex tube with elliptic core goes through a parametric resonance when two Kelvin waves, with difference in azimuthal wave numbers m being 2, are simultaneously excited. Fukumoto [5] showed on the ground of the Hamiltonian spectral theory that all the intersection points of dispersion curves of the Kelvin waves with m and $m+2$ result in instability. The $(m, m+2) = (1, 3)$ and $(0, 2)$ resonances were detected in a confined geometry [4, 12]. Malkus [14] created a rotating flow with strained streamlines in a water-filled flexible cylinder pressed by two stationary rollers (see also [4]). His experiment showed that the MSTW modes grow, followed by excitation of a number of waves and then by eventual disruption. A knowledge of nonlinear growth of linearly unstable modes is indispensable for describing a route to the collapse.

Nonlinear effect comes into play at a matured stage of exponential growth of disturbance amplitude and modifies evolution of the MSTW instability. Waleffe [22] and Sipp [19] showed that the weakly nonlinear effect acts to saturate the amplitude of the Kelvin waves. Mason and Kerswell [15] proceeded to the secondary instability of the MSTW instability. We shall show that their procedure is incomplete in the sense that they did not determine, to the full detail, the mean flow induced by nonlinear interactions of the Kelvin waves. Rodrigues and Luca [18] dealt with the case where mean flow is absent, and found chaotic orbits.

The Lagrangian displacement field is instrumental in handling interaction of waves [2, 9, 10]. Fukumoto and Hirota [7] developed the Lagrangian approach to derive the wave-induced mean flow. The Lagrangian approach allows us to give the mean flow solely in terms of the Lagrangian displacement of first order in amplitude. We rest on this approach to deduce weakly nonlinear amplitude equations. The purpose of this paper is to amend the previous Eulerian treatment and thereby to manipulate the amplitude equations for weakly nonlinear evolution of the MSTW instability. We limit ourselves to the stationary resonance of left- and right-handed helical waves.

In §2 and 3, we develop the Lagrangian approach to calculate the energy of and the mean flow induced by waves on a steady flow. We recollect the Kelvin waves in §4 and inquire into the mean flow induced by nonlinear interactions of Kelvin waves in §5. We recollect the MSTW instability in §6 and 7, and enter into the weakly nonlinear regime in §8. We close with conclusions (§9).

2. Lagrangian approach

The signature of the energy of the waves is a key ingredient for the Hamiltonian bifurcation theory. In the presence of a basic flow, the calculation of energy of waves is unattainable in the framework of the traditional approach of using the Eulerian variables. A way out is to use the derivative of the dispersion relation with respect to the frequency [5], but without justification. A steady state of the Euler flows is characterized as an extremal of the kinetic energy with respect to isovortical disturbance [2]. The use of criticality facilitates the calculation of the excess energy by which the kinetic energy increases in the presence of an induced wave. The isovortical disturbance, for which the disturbance vorticity is frozen into the flow, is expressible faithfully in terms of the Lagrangian variables. Mathematical construction of the energy, along with the verification of its relation with the derivative of the dispersion relation, was carried through in our previous papers [9, 10]. The disturbance velocity of second-order in amplitude suggests the existence of a mean flow [7]. This section presents a brief sketch of a geometric approach to this [8].

Recent studies in the field of weakly nonlinear analyses, sometimes, associated with the wave-mean flow interaction, revealed that the Lagrangian approach of incorporating the topological nature of a fluid is more effective than the traditional Eulerian approach [2, 11]. Our method [9, 10, 7], given below, has the same spirits as those studies, though with some novelty in mathematical formulation.

The motion of an inviscid incompressible fluid is regarded as an orbit on $S\mathrm{Diff}(\mathcal{D})$, the group of the volume-preserving diffeomorphisms on the domain $\mathcal{D} \subset \mathbb{R}^3$. Its Lie algebra $\mathfrak{g}$ is the velocity field of the fluid. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$ with respect to natural pair $< u, v >$ between $u \in \mathfrak{g}$ and $v \in \mathfrak{g}^*$. In the context of fluid flows, an element of $\mathfrak{g}^*$ is a vector field, and $< , >$ is simply the scalar product between vector fields. The Lie bracket $[\ ,\ ]$ of $\mathfrak{g}$ is the adjoint representation, which operates on the vector field as

$$\mathrm{ad}(u_1)u_2 = [u_1, u_2] = (\boldsymbol{u}_2 \cdot \nabla)\boldsymbol{u}_1 - (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_2 \quad \text{for } u_1, u_2 \in \mathfrak{g}, \tag{1}$$

where the bold symbol is used for clarification of it being a vector field.

Define the Lie-Poisson bracket for functions F_1 and F_2 on $\mathfrak{g}^*$ by

$$\{F_1, F_2\} = \left\langle \left[\frac{\delta F_1}{\delta v}, \frac{\delta F_2}{\delta v} \right], v \right\rangle. \tag{2}$$

The Euler equations for an inviscid incompressible is written in the form of the Poisson equation $\partial F/\partial t = \{F, H\}$. By an introduction of the dual operator ad^* of ad by $\langle u, \mathrm{ad}(\xi)^* v\rangle = \langle \mathrm{ad}(\xi)u, v\rangle$ with $\xi \in \mathfrak{g}$, the Poisson equation provides us with an evolution equation for $v \in \mathfrak{g}^*$ [1]:

$$\frac{\partial v}{\partial t} = -\mathrm{ad}^* \left(\frac{\delta H}{\delta v} \right) v. \tag{3}$$

If $\delta H/\delta v$ is replaced by an arbitrary $u(t) \in \mathfrak{g}$, (3) is called the Euler-Poincaré equation

[11]. The operation of ad* reads in component wise, for the Euler equations,

$$[\mathrm{ad}^*(\boldsymbol{\xi})\boldsymbol{v}]_i = [-\boldsymbol{\xi} \times (\nabla \times \boldsymbol{v}) + \nabla f]_i \,, \tag{4}$$

where f is a function on $\mathcal{D}$ and $\boldsymbol{\xi}$ is a vector field on $\mathcal{D}$. We may identify $\boldsymbol{v} \in \mathfrak{g}^*$ as a solenoidal vector field on $\mathcal{D}$ which is tangent to the boundary $\mathcal{D}$ by a suitable adjustment of f. The solution of (3) is a coadjoint orbit $v(t) = \mathrm{Ad}^*\left(\varphi_t^{-1}\right) v(0)$. Here φ_t is a one-parameter subgroup of $S\mathrm{Diff}(\mathcal{D})$ generated by $\delta H/\delta v$. The collection of the coadjoint orbit $\{\mathrm{Ad}^*(\varphi)v(0) \in \mathfrak{g}^* | \varphi \in S\mathrm{Diff}(\mathcal{D})\}$ is alternatively called the isovortical sheet. The velocity field is written by

$$u(t_0) = \left.\frac{\partial}{\partial t}\right|_{t_0} \left(\varphi_t \circ \varphi_{t_0}^{-1}\right) = \frac{\delta H}{\delta v}. \tag{5}$$

Given φ_t, an orbit on $S\mathrm{Diff}(\mathcal{D})$, of a basic flow and $v(t)$ as the basic solution in $\mathfrak{g}^*$, suppose that $v(0)$ is disturbed to $v_\alpha(0) = \mathrm{Ad}^*\left(\varphi_{\alpha,0}^{-1}\right) v(0)$ with $\varphi_{\alpha,0} \in S\mathrm{Diff}(\mathcal{D})$. Here $\alpha \in \mathbb{R}$ is a small parameter measuring the amplitude of the disturbance field. The disturbed initial condition $v_\alpha(0)$ is maintained on the same isovortical sheet as $v(t)$, and so is the subsequent orbit $v_\alpha(t)$. It follows that, at each instant t, there exists a diffeomorphism $\varphi_{\alpha,t} \in S\mathrm{Diff}(\mathcal{D})$ such that $v(t)$ is disturbed to

$$v_\alpha(t) = \mathrm{Ad}^*\left(\varphi_{\alpha,t}^{-1}\right) v(t) = \mathrm{Ad}^*\left((\varphi_{\alpha,t} \circ \varphi_t)^{-1}\right) v(0) \tag{6}$$

For small values of α, $\varphi_{\alpha,t}$ is a near-identity map, and there exists a generator $\xi_\alpha(t) \in \mathfrak{g}$ for it, namely, $\varphi_{\alpha,t} = \exp \xi_\alpha(t)$. We expand ξ_α in a power series in α, to $O(\alpha^2)$, as $\xi_\alpha = \alpha\xi_1 + \alpha^2\xi_2/2 + \cdots$.

Using $\mathrm{Ad}^*\left(\varphi_{\alpha,t}^{-1}\right) = \sum_{n=0}^{\infty} \left[-\mathrm{ad}^*(\xi_\alpha)\right]^n /n!$, (6) is expanded as $v_\alpha = v + \alpha v_1 + \alpha^2 v_2/2 + \cdots$;

$$v_1 = -\mathrm{ad}^*(\xi_1)v, \quad v_2 = -\mathrm{ad}^*(\xi_2)v + \mathrm{ad}^*(\xi_1)\mathrm{ad}^*(\xi_1)v. \tag{7}$$

In the language of vector calculus, this is translated into

$$\boldsymbol{v}_1 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times \boldsymbol{\omega}\right], \quad \boldsymbol{v}_2 = \mathcal{P}\left[\boldsymbol{\xi}_1 \times (\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega})) + \boldsymbol{\xi}_2 \times \boldsymbol{\omega}\right], \tag{8}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$, with identification of $\boldsymbol{v} = v \in \mathfrak{g}^*$, and $\mathcal{P}$ is an operator projecting to solenoidal vector field. Likewise, the velocity field of the disturbed orbit $\varphi_{\alpha,t} \circ \varphi_t$ is expanded as

$$\begin{aligned} u_\alpha(t_0) &= \left.\frac{\partial}{\partial t}\right|_{t_0} \left(\varphi_{\alpha,t} \circ \varphi_t \circ \varphi_{t_0}^{-1} \circ \varphi_{\alpha,t_0}^{-1}\right) \\ &= u(t_0) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left[\mathrm{ad}(\xi_\alpha)\right]^n \left(\frac{\partial \xi_\alpha}{\partial t} - \mathrm{ad}(v)\xi_\alpha\right). \end{aligned} \tag{9}$$

The first few terms of $u_\alpha = u + \alpha u_1 + \alpha^2 u_2/2 + \cdots$ are

$$\begin{aligned} u_1 &= \frac{\partial \xi_1}{\partial t} - \mathrm{ad}(u)\xi_1, \\ u_2 &= \frac{\partial \xi_2}{\partial t} - \mathrm{ad}(u)\xi_2 + \mathrm{ad}(\xi_1)\left(\frac{\partial \xi_1}{\partial t} - \mathrm{ad}(u)\xi_1\right). \end{aligned} \tag{10}$$

A vector field $u_\alpha \in \mathfrak{g}$ and its dual $v_\alpha \in \mathfrak{g}^*$ is made identifiable as the vector field generated by the Hamiltonian H,

$$u_\alpha(t) = \left.\frac{\delta H}{\delta v}\right|_\alpha (t) = v_\alpha(t), \tag{11}$$

in view of $H = \int_\mathcal{D} v_\alpha^2/2\mathrm{d}V$ with the density of fluid taken as unity.

With this identification, (8) and (10) are combined to yield

$$\frac{\partial \boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{U} = \boldsymbol{v}_1, \tag{12}$$

$$\frac{\partial \boldsymbol{\xi}_2}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{\xi}_2 - (\boldsymbol{\xi}_2\cdot\nabla)\boldsymbol{U} + (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{u}_1 = \boldsymbol{v}_2, \tag{13}$$

with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ provided by (8). The symbol $\boldsymbol{U} = v$ designates the velocity field of the basic flow, and $\boldsymbol{\omega} = \nabla\times\boldsymbol{U}$ is its vorticity. The first-order equation (12) has been well known, but the second-order equation (13) was first derived in our previous paper [7]. As compared with ref [7], this symbolic derivation furnishes a short-cut to reach (13).

3. Energy and drift current of disturbance field

The restriction to isovortical disturbances also facilitates the calculation of increased energy, originating from the superposition of disturbance, $H(v_\alpha) = H(v) + \alpha H_1 + \alpha^2 H_2/2 + \cdots$, in a power series in α. For a steady flow $\partial v/\partial t = 0$, the first-order term is shown, by use of (7), to vanish

$$H_1 = \left\langle \frac{\delta H}{\delta v}, v_1 \right\rangle = \left\langle \frac{\delta H}{\delta v}, -\mathrm{ad}^*(\xi_1)v \right\rangle = -\left\langle \xi_1, \frac{\partial v}{\partial t}\right\rangle = 0, \tag{14}$$

consistently with the fact that a steady Euler flow is an extremal of the kinetic energy with respect to disturbances constrained on an isovortical sheet [1, 2]. Then the excess energy is dominated by the second-order term

$$H_2 = -\left\langle \xi_1, \frac{\partial v_1}{\partial t}\right\rangle = \int \boldsymbol{\omega}\cdot\left(\frac{\partial \boldsymbol{\xi}_1}{\partial t}\times\boldsymbol{\xi}_1\right)\mathrm{d}V. \tag{15}$$

The advantage of the reduced form (15) of the wave energy cannot be overemphasized; (15) dispenses with the solution $\boldsymbol{\xi}_2(\boldsymbol{x}, t)$ of (13). The same form as (15) was derived for the wave energy [2], but the second field $\boldsymbol{\xi}_2$ had previously gone unnoticed.

Along the same line with the derivation of the energy, we are able to deduce the drift current driven at $O(\alpha^2)$ by nonlinear interaction of waves [8]. For a given $\eta \in \mathfrak{g}$, consider the momentum $J =< \eta, v >$ in this direction. By the Hamiltonian Noether theorem, if the Hamiltonian is invariant with respect to the transformation $\exp\eta$, J is constant. We define $J_\alpha =< \eta, v_\alpha >$ for the disturbed field v_α, and calculate the increment of J_α relative to J by expanding in powers of α as $J_\alpha =< \eta, v > +\alpha J_1 + \alpha^2 J_2/2 + \cdots$;

$$J_1 = \langle \eta, v_1\rangle = \langle \eta, -\mathrm{ad}^*(\xi_1)v\rangle = \langle \xi_1, \mathrm{ad}^*(\eta)v\rangle\,,$$
$$J_2 = \langle \eta, v_2\rangle = \langle \xi_2, \mathrm{ad}^*(\eta)v\rangle + \langle \xi_1, \mathrm{ad}^*(\eta)v_1\rangle\,, \tag{16}$$

where use has been made of (7).

If the basic flow $v(t)$ has a symmetry such that $\mathrm{ad}^*(\eta)v = 0$, then $J_1 = 0$ and J_2 becomes

$$J_2 = \langle \mathrm{ad}(\eta)\xi_1, -\mathrm{ad}^*(\xi_1)v\rangle = \int \boldsymbol{\omega}\cdot(\boldsymbol{\xi}_1 \times \mathcal{L}_{\boldsymbol{\eta}}\boldsymbol{\xi}_1)\,\mathrm{d}V. \tag{17}$$

Here $\mathcal{L}_{\boldsymbol{\eta}}\boldsymbol{\xi}_1 = -\mathrm{ad}(\eta)\xi_1$ is the Lie derivative of $\boldsymbol{\xi}_1$ with respect to $\boldsymbol{\eta}$. Notably, the excess momentum J_2 is expressible solely in terms of the first-order quantity $\boldsymbol{\xi}_1$.

4. Kelvin wave

We briefly recall the Kelvin waves, linearized disturbances of $O(\alpha)$ on the Rankine vortex in a confined geometry. At the outset, we take, as the basic flow, the rigid-body rotation of an inviscid incompressible fluid confined in a cylinder of circular cross-section of unit radius, though later allowance is made for elliptic deformation.

Let us introduce cylindrical coordinates (r, θ, z) with the z-axis along the centerline. Let the r and θ components of 2D basic velocity field $\boldsymbol{U}_0$ be U_0 and V_0, and the pressure P_0. The suffix 0 signifies that these quantities pertain to the case of circular cross-section. The basic flow is confined to $r \le 1$, with the velocity field given by

$$U_0 = 0, \quad V_0 = r, \quad P_0 = r^2/2 - 1. \tag{18}$$

Take, as the disturbance field, $\tilde{\boldsymbol{u}} = \alpha\boldsymbol{u}_{01}$. We focus our attention on a normal mode

$$\boldsymbol{u}_{01}^{(m)} = A_m(t)\boldsymbol{u}_{01}^{(m)}(r)\mathrm{e}^{im\theta}\mathrm{e}^{ikz}, \quad A_m(t) \propto \mathrm{e}^{-i\omega_0 t}, \tag{19}$$

where A_m is a complex function of time t and ω_0 is frequency. This velocity field represents a Kelvin wave with azimuthal wavenumber m and axial wavenumber k. The linearized Euler equations supplies equations for the radial function $\boldsymbol{u}_{01}^{(m)}$ as

$$\mathcal{L}_{m,k}\boldsymbol{u}_{01}^{(m)} + \nabla p_{01}^{(m)} = \boldsymbol{0}, \quad \nabla\cdot\boldsymbol{u}_{01}^{(m)} = 0, \tag{20}$$

where

$$\mathcal{L}_{m,k} = \begin{pmatrix} -i(\omega_0 - m) & -2 & 0 \\ 2 & -i(\omega_0 - m) & 0 \\ 0 & 0 & -i(\omega_0 - m) \end{pmatrix}. \tag{21}$$

The solution is found with ease and the radial component is

$$u_{01}^{(m)} = \frac{i}{\omega_0 - m + 2}\left\{-\frac{m}{r}J_m(\eta_m r) + \frac{\omega_0 - m}{\omega_0 - m - 2}\eta_m J_{m+1}(\eta_m r)\right\}, \tag{22}$$

where η_m is the radial wavenumber, $\eta_m^2 = [4/(\omega_0 - m)^2 - 1]\,k^2$, and J_m is the m-th Bessel function of the first kind. The boundary condition $u_{01}^{(m)} = 0$ at $r = 1$ gives rise to the dispersion relation [21, 16]

$$J_{m+1}(\eta_m) = \frac{(\omega_0 - m - 2)m}{(\omega_0 - m)\eta_m}J_m(\eta_m). \tag{23}$$

Figure 1 displays the dispersion relation of helical waves $m = \pm 1$. Curves for $m = -1$ are drawn with solid lines, while those for $m = +1$ are drawn with dashed lines. Infinitely many branches emanate from $(k, \omega_0) = (0, 1)$ for $m = 1$ and from $(k, \omega_0) = (0, -1)$ for $m = -1$. Notice the absence of isolated modes, as opposed to the unbounded case [5, 20].

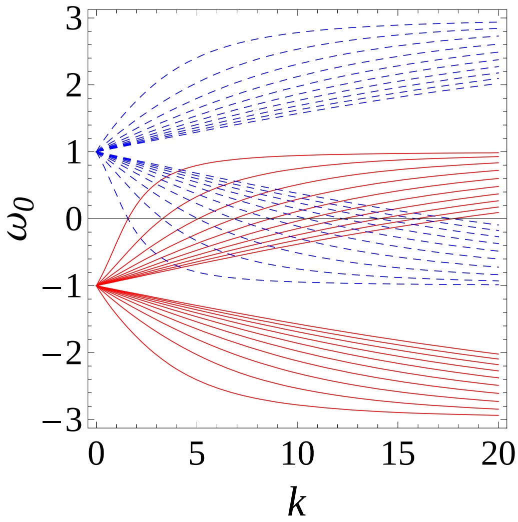

Figure 1. dispersion relation of Kelvin waves, of $m = -1$(solid lines) and of $m = +1$(dashed lines), in an elliptic cylinder.

5. Drift current

The second-order term $\alpha^2 \boldsymbol{u}_{02}$ of the disturbance field $\tilde{\boldsymbol{u}}$ includes a mean flow induced by nonlinear interaction of Kelvin waves $\alpha \boldsymbol{u}_{01}$. In keeping with the fact that the basic flow, with circular cylindrical symmetry, admits arbitrary radial profiles of azimuthal and axial velocity. By introducing the elliptical strain of strength ϵ, the functional form of its azimuthal component is somehow manipulated by the solvability condition at $O(\epsilon\alpha^2)$, though limited to intersection points of the dispersion curves [19]. Moreover its coefficient includes an arbitrary parameter, of integration-constant origin, yet to be determined [16].

The Lagrangian approach rescues this difficulty, by producing the mean flow of $O(\alpha^2)$ without having to proceed to a higher order $O(\epsilon\alpha^2)$ [7]. Subsequently, we give a sketch of this Lagrangian approach.

In the context of Kelvin waves, (12) reads

$$\boldsymbol{u}_{01} = \frac{\partial \boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{U}_0 \cdot \nabla)\, \boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1 \cdot \nabla)\, \boldsymbol{U}_0. \tag{24}$$

For our flow $\boldsymbol{U}_0 = r\boldsymbol{e}_\theta$, the RHS of (24) is reduced to $-i(\omega_0 - m)\boldsymbol{\xi}_1$, and upon substitution from a linear combination of Kelvin waves (19), (24) yields

$$\boldsymbol{\xi}_1 = \mathrm{Re}\left[\sum \frac{iA_m(t)}{\omega_0 - m} \boldsymbol{u}_{01}^{(m)}(r) \mathrm{e}^{im\theta} \mathrm{e}^{ikz}\right], \tag{25}$$

supplemented by the incompressibility constraint $\nabla \cdot \boldsymbol{\xi}_1 = 0$. Taking the real part should be born in mind.

For the rigid-body rotation $\boldsymbol{U}_0 = r\boldsymbol{e}_\theta$, it is advantageous to calculate the mean flow directly from the spatial average of (8) rather than from (17), since simply $\nabla \times \boldsymbol{U}_0 = 2\boldsymbol{e}_z$.

It turns out that the second-order field $\boldsymbol{\xi}_2$ has no contribution when spatially averaged, leaving

$$\begin{aligned}\overline{\boldsymbol{u}_{02}} &= \overline{\mathcal{P}\left[\boldsymbol{\xi}_1 \times (\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{e}_z))\right]} = \overline{\boldsymbol{\xi}_1 \times \partial \boldsymbol{\xi}_1/\partial z} \\ &= \sum \frac{4ik}{(\omega_0 - m)^2}|A_m|^2(0, u_{01}^{(m)}w_{01}^{(m)}, -u_{01}^{(m)}v_{01}^{(m)}). \qquad (26)\end{aligned}$$

The Lagrangian approach is capable of calculating the mean flow at any points satisfied by the dispersion relation (k, ω_0).

6. Stability of rotating flow in an elliptic cylinder: setting of problem

Rotating flows in an elliptic cylinder go through excitation of a number of growing waves, resulting in disruption [14, 4]. We express elliptic cross-section as

$$\frac{x^2}{1+\epsilon} + \frac{y^2}{1-\epsilon} = 1. \qquad (27)$$

The parameter ϵ designates the elliptic distortion. We assume that $|\epsilon|$ is small. In conjunction with this distortion the basic flow is perturbed as

$$\begin{aligned}&\boldsymbol{U} = \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1 + \cdots, \quad P = P_0 + \epsilon P_1 + \cdots; \\ &U_1 = -r\sin 2\theta, \quad V_1 = -r\cos 2\theta, \qquad P_1 = 0. \qquad (28)\end{aligned}$$

The subscript designates order in elliptic parameter ϵ. The augmented term of $O(\epsilon)$ represents a steady quadrupole field. In other words, $\boldsymbol{U}_1$ corresponds to the field consisting of strain field whose stretching direction lies along $\theta = -\pi/4$ and whose direction of contraction is along $\theta = \pi/4$.

We add three-dimensional disturbance field $\tilde{\boldsymbol{u}}$ to this two-dimensional basic flow. We consider asymptotic expansions of the velocity field in two small parameters ϵ and α as

$$\boldsymbol{u} = \boldsymbol{U} + \tilde{\boldsymbol{u}} = \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1 + \alpha \boldsymbol{u}_{01} + \epsilon\alpha \boldsymbol{u}_{11} + \alpha^2 \boldsymbol{u}_{02} + \alpha^3 \boldsymbol{u}_{03} + \cdots, \qquad (29)$$

specifically to $O(\alpha^3)$ in amplitude. Here, the velocity field $\boldsymbol{u}_{mn}$ occurs at $O(\epsilon^m \alpha^n)$. The side wall (27) of the cylinder is $r = 1 + \epsilon\cos 2\theta/2 + O(\epsilon^2)$ when the elliptic strain ϵ is small. The boundary condition to be imposed at the rigid side wall is

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{at} \quad r = 1 + \epsilon\cos 2\theta/2, \qquad (30)$$

where $\boldsymbol{n}$ is the unit outward normal vector to the cylinder boundary.

7. Moore-Saffman-Tsai-Widnall instability

The Kelvin waves described in §4, neutrally stable oscillations, are made unstable, by breaking the circular symmetry of the cylinder cross-section. We explore the effect of elliptic strain $\epsilon \boldsymbol{U}_1$ upon the Kelvin waves. If the given disturbance flow, in the absence of elliptic strain, has Kelvin waves with $\mathrm{e}^{im\theta}$ and $\mathrm{e}^{i(m+2)\theta}$, interaction of these waves via the strain (28), at $O(\epsilon\alpha)$, through the convective terms of the Euler equations excite again Kelvin waves with $e^{im\theta}$ and $e^{i(m+2)\theta}$. This coincidence indicates occurrence of parametric instability [17, 20, 5]. Fukumoto [5] made a thorough analysis of the 3D instability of the Rankine vortex embedded in a plane shear field, in an unbounded space, and showed that the parameter resonance instability occurs, at $O(\epsilon\alpha)$, at all intersection points (k, ω) of dispersion curves of the m and $m+2$ waves. The same is true for the rotating flow confined in an elliptic cylinder [21].

There are intersection points on the k-axis ($\omega_0 = 0$) at certain values of k as is observed in Figure 1. For left and right-handed helical wave resonance, the stationary mode $\omega_0 = 0$ has far greater growth rate than non-stationary modes ($\omega_0 \neq 0$)[20, 3, 5]. We limit ourselves to the stationary parametric resonance between left and right-handed helical waves $(m, m+2) = (-1, +1)$ occurring at $\omega_0 = 0$.

Under the restriction of $\omega_0 = 0$, the radial wavenumber becomes $\eta = \sqrt{3}k$. Then, we send the following disturbance velocity $\boldsymbol{u}_{01}$, a superposition of $m = \pm 1$ waves,

$$\boldsymbol{u}_{01} = A_- \boldsymbol{u}_{01}^{(-)} e^{-i\theta} e^{ikz} + A_+ \boldsymbol{u}_{01}^{(+)} e^{i\theta} e^{ikz} + c.c. \tag{31}$$

We use the notation $A_\pm$ in place of $A_{\pm 1}$. Excited at $O(\epsilon\alpha)$ fueled by $\boldsymbol{U}_1$ is

$$\boldsymbol{u}_{11} = \left\{ B_- \boldsymbol{u}_{11}^{(-)} e^{-i\theta} + B_+ \boldsymbol{u}_{11}^{(+)} e^{i\theta} + B_{-3} \boldsymbol{u}_{11}^{(-3)} e^{-3i\theta} + B_3 \boldsymbol{u}_{11}^{(3)} e^{3i\theta} \right\} e^{ikz} + c.c. \tag{32}$$

The radial functions $\boldsymbol{u}_{11}^{(m)}(r)$ and $\boldsymbol{u}_{11}^{(m+2)}(r)$ are determined by solving inhomogeneous linear ordinary differential equations, derived from the Euler equations and by the continuity equations, subject to the boundary condition (30) at $O(\epsilon\alpha)$,

$$u_{11} - u_{01} \cos 2\theta/2 + v_{01} \sin 2\theta = 0. \tag{33}$$

The boundary condition (33) provides algebraic equations for $B_\pm$ and the solvability condition gives rise to, with the help of the dispersion relation (23),

$$\frac{1}{A_+} \frac{\partial A_-}{\partial t_{10}} = \frac{-1}{A_-} \frac{\partial A_+}{\partial t_{10}} = i \frac{3(3k^2+1)}{8(2k^2+1)} = ia, \tag{34}$$

where $t_{10} = \epsilon t$, the slow time scale, and k is the solution of dispersion relation $J_1(\eta) = -\eta J_0(\eta)$.

The degenerate modes with $\omega_0 = 0$ necessarily result in parametric resonance with growth rate $a = 3(3k^2+1)/[8(2k^2+1)]$ [21] and with amplitude ratio of the eigen-function given by $A_-/A_+ = i$. Numerical values of the growth rate are, for a first few intersection points with $\omega_0 = 0$, $(k, \sigma) \approx (1.578, 0.5311), (3.286, 0.5542), \cdots$. The occurrence of parametric resonance implies the existence of negative-energy waves. This is indeed the case [5]. The energy of Kelvin waves is efficiently calculated from the formula (15) built in the Lagrangian framework [7].

At $O(\alpha^3)$, the modes $e^{\pm i\theta} e^{ikz}$ again arise, which invites the compatibility conditions. The function $\boldsymbol{u}_{03}^{(m)}$ with $m = \pm 1$ is governed by $\mathcal{L}_{m,k} \boldsymbol{u}_{03}^{(m)} = \mathcal{N} - \partial \boldsymbol{u}_{01}^{(m)}/\partial t_{02}$, with $t_{02} = \alpha^2 t$. Since the matrix $\mathcal{L}_{m,k}$ is singular, $(\partial/\partial t_{02})\boldsymbol{u}_{01}^{(m)}$ is adjusted for the forcing terms to satisfy the solvability condition. The calculation of $\mathcal{N}$ requires the precise form of the mean flow of $O(\alpha^2)$.

8. Amplitude equation

We are now ready to derive the weakly nonlinear amplitude equations to $O(\alpha^3)$. The procedure is described at length in ref [16].

The mean flow induced by nonlinear interactions of helical waves is

$$4ik\left(0, \left(|A_-|^2 + |A_+|^2\right) u_{01}^{(+)} w_{01}^{(+)}, \left(|A_-|^2 - |A_+|^2\right) u_{01}^{(+)} v_{01}^{(+)}\right). \tag{35}$$

For general $(m, m+2)$ parametric resonance, only the radial component of mean flow is zero as is seen from (26). But, in the case of stationary helical-wave parametric resonance, the axial components of mean flow is zero, because $|A_-| = |A_+|$ [5, 19]. Taking account of this, the boundary condition at $O(\alpha^3)$ leads us immediately to the amplitude equations

$$\frac{\mathrm{d}A_\pm}{\mathrm{d}t} = \mp i\left[\epsilon a A_\mp + \alpha^2 A_\pm\left(b|A_\pm|^2 + c|A_\mp|^2\right)\right], \tag{36}$$

where a is defined by (34) and

$$b = \frac{-2k^4}{3(2k^2+1)}\left[\frac{4}{J_0(\eta)^2}\int_0^1 rJ_0(\eta r)^2 J_1(\eta r)^2 \mathrm{d}r - (11k^4 + 13k^2 + 5)J_0(\eta)^2\right],$$
$$c = \frac{k^2}{12(2k^2+1)}\left[\frac{64k^2}{J_0(\eta)^2}\int_0^1 rJ_0(\eta r)^2 J_1(\eta r)^2 \mathrm{d}r + (20k^6 + 97k^4 + 14k^2 - 27)J_0(\eta)^2\right]. \tag{37}$$

By virtue of availability of compact form, these coefficients are readily calculated at all the intersection points on the k-axis ($\omega_0 = 0$). For the longest two wavelengths, we have $(k; a, b, c) \approx (1.579; 0.5312, -0.3976, 5.222), (3.286; 0.5542, -8.286, 53.39)$.

It is remarkable that the resulting equations (36) are exactly Hamiltonian normal form [13]. The normal form shows up at once with nonlinear terms $|A_\pm|^2 A_\pm, |A_\mp|^2 A_\pm$ fully incorporating the effect of mean flow (26). On the other hand, in the Eulerian treatment, the amplitude of the mean flow had to be introduced as an intervening dependent parameter, and its equation produces an undetermined constant [19].

As A_- and A_+ are complex functions of t, the amplitude equations (36) constitute a four-dimensional dynamical system. The sign of coefficients (a, b, c) is unchanged, regardless of the choice of the intersection points: $a > 0$, $b < 0$ and $c > 0$.

The amplitude equations (36) admits restriction of the phase space to a two-dimensional subspace with $A = A_- = -A_+^*$, where $*$ stands for the complex conjugate. The complex amplitude equations (36) collapses, by a choice of $\alpha^2 = \epsilon$, to

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i\epsilon\left(-aA^* + \beta|A|^2 A\right), \tag{38}$$

where $\beta = b + c$. Figure 2 illustrates the trajectory in the phase space $(\mathrm{Re}[A], \mathrm{Im}[A])$. The rigidly rotating state (the origin) is unstable, but the amplitude of the orbit necessarily saturates within a basin of the unstable equilibrium.

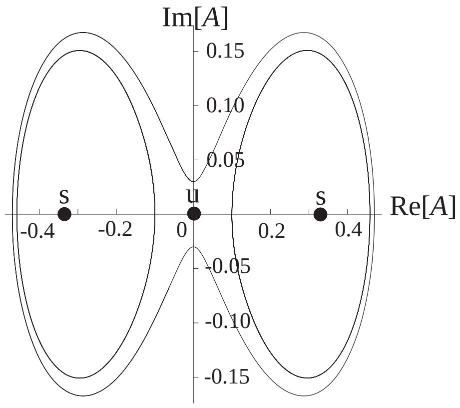

Figure 2. Trajectories in the phase space $(\mathrm{Re}[A], \mathrm{Im}[A])$ for $k = 1.579$. The dots designate equilibria (s: stable, u: unstable).

Let us set $A = |A|\mathrm{e}^{i\phi}$. The modulus $|A|$ and the phase ϕ satisfy the following equations,

$$\frac{\mathrm{d}|A|}{\mathrm{d}t} = -\epsilon a|A|\sin 2\phi, \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = -\epsilon a\cos 2\phi + \epsilon\beta|A|^2. \tag{39}$$

The linear effect predominates over the nonlinear effect for small disturbance amplitude $|A|(\ll 1)$. In case the equilibrium point $A = 0$ is unstable, the direction of disturbance vorticity ϕ is liable to be parallel to the unstable direction $\phi = -\pi/4$. The elliptic strain makes horizontal vortex lines continuously stretched, if they are oriented, on average, in the direction of $\phi = -\pi/4$. This is the mechanism for the MSTW instability at the linear stage. When the disturbance grows substantially, $|A| \approx 1$ say, the nonlinear effect is called into play. In view of (39), the nonlinear effect is exclusively rotating the phase angle ϕ. As a consequence, alignment of horizontal vorticity to the direction $\phi = -\pi/4$ is hindered, which renders the disturbance amplitude saturate.

The compact form of the coefficients (37) makes it feasible to manipulate the short-wavelength asymptotics of equilibrium amplitude [16]. The short-wavelength asymptotics of (36) is found without difficulty, and amplitude of the saturated state is found to be $|A|_{\mathrm{eq}} = \sqrt{a/\beta} \to 3/4\left(\sqrt{3}\pi/(k^3\log k)\right)^{1/2}$.

9. Conclusion

We have made a weakly nonlinear analysis of the short-wave instabilities of rotating flow in a cylinder of elliptic cross-section. We put emphasis on the advantage of the Lagrangian approach, over the Eulerian one, in the derivation of the mean flow induced by nonlinear interactions of the Kelvin waves (§5).

The phase of the complex amplitude of $O(\alpha)$ of the 3D disturbance represents the angle, from the x-axis, of oscillating vorticity disturbances in the horizontal plane. For small amplitude, the features of the linear short-wave instability is retrieved that selectively amplification of the particular is invited, for which the disturbance vorticity is continuously stretched by the ambient strain [20, 5]. The non-linear effect suppresses this monotonic growth by turning the disturbance vorticity out of the stretching direction as soon as the amplitude becomes sufficiently large [19].

However this behavior does not coincide with the vigorous amplification of a number of waves and the ultimate disruption of a strained flow observed in experiments [14, 4]. This indicates that the nonlinear interaction of a single MSTW mode is far from sufficient in describing practical flows. The secondary and the tertiary instability, which may be invited before reaching the stage of nonlinear saturation, will drastically alter the subsequent evolution [15, 6]. The Lagrangian approach would be vital for dealing with these higher-order bifurcations.

Acknowledgments

We are grateful to Yuji Hattori for invaluable suggestions. Y. F. was supported in part by a Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science (Grant No. 21540390).

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