Abstract collision systems on G-sets

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Abstract. In this paper, we discuss an abstract collision system (ACS) on a G-set which is an extension of a normal ACS [5, 6]. An ACS is a type of unconventional computing framework that includes collision-based computing, cellular automata (CA), and chemical reaction systems. For a given group G and its subset, we create a set of collisions and a local transition function of an ACS by using the action of G. We first refine definitions of the components of an ACS, and then extend them to the concepts on a G-set. Finally, we define and investigate the operations "union", "division" and "composition" of the ACS on a G-set.

Keywords. collision-based computing, cellular automata.

1. Introduction

Recently, many investigations have been carried out on unconventional computing methods based on the concept of the collision-based computing [1], including cellular automata (CA) and reaction-diffusion systems. It is one of major important subjects to construct an appropriate computational model for investigating the unconventional computing methods.

Conway introduced 'The Game of Life' which is a two-dimensional cellular automaton [2]. In this game, there are some special patterns called "gliders" and he showed that any logical circuit can be simulated by the collision of gliders. Wolfram and Cook [11, 3] found glider patterns in the one dimensional elementary cellular automaton CA110. Cook introduced a cyclic tag system (CTS) as a Turing universal system, and showed that a CTS can be simulated by CA110 by using collisions of gliders in CA110. Recently, Martínez et. al. investigated glider phenomena from the viewpoint of regular language [7]. Morita [8] introduced a reversible one dimensional CA which simulated CTS.

We previously introduced the notion of an abstract collision system (ACS) as a tool for investigating collision phenomena including glider collisions in 'The Game of Life' and 'CA110', and we proved that it is universal for computation [5]. Moreover, we investigated the simulation of ACS by CA, and determined conditions that make this possible [6].

The notion of automata on groups was first treated as a special case of automata on graphs (Cayley graphs) which represent groups [10, 9, 12]. Fujio [4] introduced the composition of CA on groups in order to reduce a complex behaved dynamics into simpler ones. As an example, he showed that rule 90 (3 neighborhood) CA can be factorized into the composition of double XORs, which are rule 6 (2 neighborhood) CA.

In this paper, we introduce an ACS on a G-set, and we investigate the properties of this extended system. Generally, the set of collisions, which is a domain of the local transition function, is very large. However, in the notion of an ACS on a G-set, we use a small set V and a function l named the "base function". We induce the set of collisions $\mathcal{C}$ and the local transition function f_l from V and l.

Next, we consider ACS operations such as "union", "division" and "composition", and introduce a sufficient condition that allows an ACS on a group to be dividable. Furthermore, we proved that the operation "composition" is right-distributive over "union", but the operation "composition" is not left-distributive. We provide a counterexample concerning this left-distributive law. In addition, we reformalize CA on groups by using an ACS on a G-set.

This paper consists of the following sections. In Section 2, we introduce the concept of an ACS. First, we define a set of collisions $\mathcal{C}$ on a non-empty set S. The set $\mathcal{C}$ specifies all combinations of elements in S which cause collisions. Next, we define an ACS using S, $\mathcal{C}$ and f, where $f : \mathcal{C} \to 2^S$ is a local transition function.

In Section 3, we define an ACS on a G-set. Let G be a group which acts on S. When $V \subseteq G$ and a map from 2^V to 2^S are given, we construct a set of collisions $\mathcal{C}$ on S and extend the map to a local transition function $f : \mathcal{C} \to 2^S$ by using the action of G. Moreover, we investigate the behavior of the global transition function from the viewpoint of this extension.

In Section 4, we define the operations "union" and "division" of the ACS. Moreover, we give a sufficient condition allowing an ACS on a group to be dividable.

In Section 5, we discuss the composition of ACSs on G-sets. We define this composition as an operation of base functions of two ACSs and we prove that this definition induces the composition of local (resp. global) transition functions.

In Section 6, we prove that composition is right-distributive over union, but is not left-distributive. We also provide a counter-example.

2. Abstract collision systems

In this section, we define an abstract collision system. Let S be a non-empty set. First, we define a set of collisions on S.

Proposition 1. *Let $\mathcal{C} \subseteq 2^S$. The following two conditions (a) and (b) are equivalent.*

(a) *The set $\mathcal{C}$ satisfies:*

- **(SC1)** *$\{s\} \in \mathcal{C}$ for all $s \in S$.*
- **(SC2)** *For all $\mathcal{X} \subseteq \mathcal{C}$, $(\cup\mathcal{X}) \in \mathcal{C}$ if $(\cap\mathcal{X}) \neq \phi$,*

(b) *The set $\mathcal{C}$ satisfies:*

- **(SC1)** *$\{s\} \in \mathcal{C}$ for all $s \in S$.*
- **(SC'2)** *For all X_1 and $X_2 \in \mathcal{C}$, $X_1 \cup X_2 \in \mathcal{C}$ if $X_1 \cap X_2 \neq \phi$.*
- **(SC'3)** *$[p]^A_{\mathcal{C}} \in \mathcal{C}$ for all $A \in 2^S$ and $p \in A$,*

where

$$\bigcap \mathcal{X} = \bigcap \{X \mid X \in \mathcal{X}\},$$
$$\bigcup \mathcal{X} = \bigcup \{X \mid X \in \mathcal{X}\}, \text{ and}$$
$$[p]^A_{\mathcal{C}} = \bigcup \{X \mid X \in \mathcal{C}, p \in X, X \subseteq A\}. \tag{1}$$

Proof. We prove (SC2) $\Leftrightarrow$ ((SC'2) $\wedge$ (SC'3)).

(SC2) $\Rightarrow$ (SC'2) First, we prove (SC'2) from (SC2). Suppose that $X_1, X_2 \in \mathcal{C}$, $X_1 \cap X_2 \neq \phi$. Let $\mathcal{X} = \{X_1, X_2\} \subseteq \mathcal{C}$. Since $(\cap\mathcal{X}) = X_1 \cap X_2 \neq \phi$, we have $X_1 \cup X_2 = (\cup\mathcal{X}) \in \mathcal{C}$ by (SC2). Therefore we have proved (SC'2).

(SC2) $\Rightarrow$ (SC'3) Next, we show (SC'3) from (SC2). Let

$$\mathcal{X} = \{X \mid X \in \mathcal{C}, p \in X, X \subseteq A\}.$$

Since $p \in (\cap\mathcal{X})$, we have $(\cap\mathcal{X}) \neq \phi$. Therefore $[p]^A_{\mathcal{C}} = (\cup\mathcal{X}) \in \mathcal{C}$ by (SC2).

(SC'2) $\wedge$ (SC'3) $\Rightarrow$ (SC2) Finally, we prove (SC2) from (SC'2) and (SC'3). For all $\mathcal{X} \subseteq \mathcal{C}$, we assume that $(\cap\mathcal{X}) \neq \phi$. Let $x_0 \in (\cap\mathcal{X})$ and

$$A = \bigcup \mathcal{X}. \tag{2}$$

Since $x_0 \in A$, we have

$$[x_0]^A_{\mathcal{C}} \in \mathcal{C} \tag{3}$$

from (SC'3). We see that $[x_0]^A_{\mathcal{C}} \subseteq A$ from the definition of $[x_0]^A_{\mathcal{C}}$. On the other hand, for all $X \in \mathcal{X}$, since $X \in \mathcal{X} \subseteq \mathcal{C}$, we have $X \in \mathcal{C}$. Moreover, since $x_0 \in (\cap\mathcal{X})$ and $A = (\cup\mathcal{X})$, we have $x_0 \in X$ and $X \subseteq A$. Hence we have

$$X \subseteq \bigcup \{X \mid X \in \mathcal{C}, x_0 \in X, X \subseteq A\} = [x_0]^A_{\mathcal{C}}.$$

Therefore we have

$$A = [x_0]^A_{\mathcal{C}}. \tag{4}$$

Hence we have

$$\left(\bigcup \mathcal{X}\right) = A = [x_0]^A_{\mathcal{C}} \in \mathcal{C},$$

by (2), (3) and (4). Therefore we have proved (SC2). □

Definition 1 (Set of collisions). A set $\mathcal{C} \subseteq 2^S$ is called a **set of collisions** on S iff it satisfies conditions of Proposition 1.

Proposition 2. *Let $\mathfrak{C}$ be a family of sets of collisions on S. Then a set*

$$\bigcap \mathfrak{C} = \bigcap_{\mathcal{C} \in \mathfrak{C}} \mathcal{C}$$

is a set of collisions on S.

Proof. We check conditions (SC1) and (SC2).

(SC1) First, we prove (SC1). We have $\{s\} \in \mathcal{C}$ for all $s \in S$ and $\mathcal{C} \in \mathfrak{C}$. Therefore we have $\{s\} \in (\cap\mathfrak{C})$.

(SC2) Next, we prove (SC2). For all $\mathcal{X} \subseteq (\cap\mathfrak{C})$, and $\mathcal{C} \in \mathfrak{C}$, we assume that $(\cap\mathcal{X}) \neq \phi$. The set $\mathcal{C}$ satisfies the assumption of (SC2), i.e.,

$$\mathcal{X} \subseteq (\cap\mathfrak{C}) \subseteq \mathcal{C},$$
$$(\cap\mathcal{X}) \neq \phi.$$

Therefore we have $(\cup\mathcal{X}) \in \mathcal{C}$ from (SC2). Hence we have $(\cup\mathcal{X}) \in (\cap\mathfrak{C})$. □

Definition 2. For a subset $\widetilde{\mathcal{C}}$ of 2^S, we define

$$\mathfrak{C}(\widetilde{\mathcal{C}}) = \bigcap \left\{\mathcal{C} \mid \mathcal{C} \text{ is a set of collisions on } S, \widetilde{\mathcal{C}} \subseteq \mathcal{C}\right\}. \tag{5}$$

By Proposition 2, this set is a set of collisions on S, and it includes the set $\widetilde{\mathcal{C}}$. Moreover, this set is a smallest set in all of sets of collisions on S which includes $\widetilde{\mathcal{C}}$.

Proposition 3. *Let $\mathcal{C}$ be a set of collisions on S. For all $A \in 2^S$ and $p, q \in A$, we have the followings:*

(1) *$[p]^A_{\mathcal{C}} \neq \phi$.*

(2) *If $[p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$, then $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$.*

Proof. First, we prove (1). Since $\{p\} \in \mathcal{C}$, $p \in \{p\}$ and $\{p\} \subseteq A$, we have $\{p\} \subseteq [p]^A_{\mathcal{C}}$. Hence $[p]^A_{\mathcal{C}} \neq \phi$.

Next, we prove (2). Since $[p]^A_{\mathcal{C}} \in \mathcal{C}$, $[q]^A_{\mathcal{C}} \in \mathcal{C}$ and $[p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$, we see that

$$[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} \in \mathcal{C}.$$

Moreover since $p \in [p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}}$ and $[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} \subseteq A$, we have $[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} \subseteq [p]^A_{\mathcal{C}}$. Hence we have

$$[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} = [p]^A_{\mathcal{C}}.$$

Similarly, we have $[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$. Hence $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$. □

Next, we define abstract collision systems.

Definition 3 (An abstract collision system)**.** Let S be a non-empty set, $\mathcal{C}$ a set of collisions on S and f a function $f : \mathcal{C} \to 2^S$. We define an **abstract collision system** M by $M = (S, \mathcal{C}, f)$. We call the function f and the set 2^S a **local transition function** and a **configuration** of M, respectively. We define a **global transition function** $F_M : 2^S \to 2^S$ of M by

$$F_M(A) = \bigcup_{p \in A} (f([p]_{\mathcal{C}}^A))$$

Lemma 1. *Let F_M be the global transition function of an abstract collision system $M = (S, \mathcal{C}, f)$. If $A \in \mathcal{C}$, we have*

$$F_M(A) = f(A).$$

Proof. Since $A \in \mathcal{C}$, we see that $[p]_{\mathcal{C}}^A = A$ for any $p \in A$. Therefore we have

$$F_M(A) = \bigcup_{p \in A} (f([p]_{\mathcal{C}}^A)) = \bigcup_{p \in A} (f(A)) = f(A)$$

□

Example 1. We describe a one dimensional billiard ball system as an example of ACS.

A ball has a velocity, a label and a position. We consider discrete time transitions. A ball moves to left or right according its velocity within an unit time. Let $(2, A, 1)$ be a ball with the velocity 2, the label 'A' and the position 1. At the next step, the ball becomes $(2, A, 3)$ if it does not crash with any other balls (cf. Figure 1).

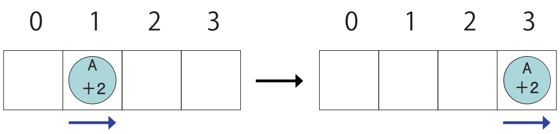

Figure 1: Moving

On the other hand, some balls may crash in some unit time. In this example, we do not describe a crash using positions and velocities. We define a set of balls which cause collisions and assign the result of the collisions.

We describe this example more concretely. Let

$$\begin{aligned} V &= \{-1, 2\}, \\ S &= \{(u, A, x) \mid x \in Z, u \in V\} \\ &\quad \cup \{(v, B, y) \mid y \in Z, v \in V\}. \\ \mathcal{C} &= \{\{(2, A, 1), (-1, B, 2)\}\} \\ &\quad \cup \{\{(u, A, x)\} \mid u \in V, x \in \mathbb{Z}\} \\ &\quad \cup \{\{(v, B, y)\} \mid v \in V, y \in Z\} \end{aligned}$$

We define f by Table. 1. For example,

$$f(\{(2, A, 1), (-1, B, 2)\}) = \{(2, B, 3), (-1, A, 1)\}$$

is shown in Figure 2.

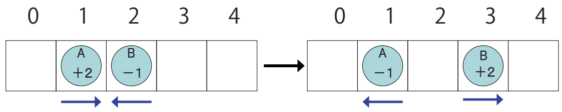

Figure 2: Collision

Table 1: Collision and its result

c	$f(c)$
$\{(2, A, 1), (-1, B, 2)\}$	$\{(2, B, 3), (-1, A, 1)\}$
$\{(u, A, x)\}$	$\{(u, A, x + u)\}$
$\{(v, B, y)\}$	$\{(v, B, y + v)\}$

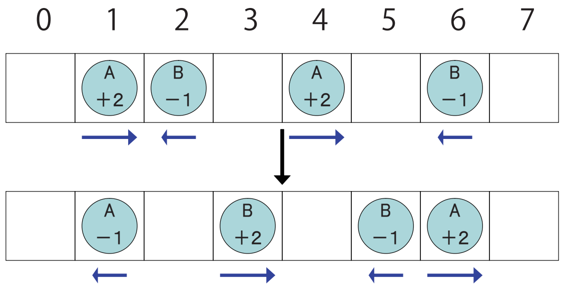

Figure 3: Transition

Let $M = (S, \mathcal{C}, f)$. Then an example of transition is

$$\begin{aligned} &F_M(\{(2, A, 1), (-1, B, 2), (2, A, 4), (-1, B, 6)\}) \\ &= \{(-1, A, 1), (2, B, 3), (-1, B, 5), (2, A, 6)\}, \end{aligned}$$

and it is shown in Figure 3.

We note that the set of two balls $\{(4, A, 2), (-1, B, 6)\}$ does not cause collisions, because it is not an element of $\mathcal{C}$.

Definition 4. Let $M_1 = (S, \mathcal{C}_1, f_1)$ and $M_2 = (S, \mathcal{C}_2, f_2)$ be abstract collision systems. We say that M_1 and M_2 are **equivalent** if they satisfy

$$F_{M_1}(A) = F_{M_2}(A)$$

for all $A \in 2^S$. When M_1 and M_2 are equivalent, we write $M_1 \equiv M_2$.

Lemma 2. *Let $M_1 = (S, \mathcal{C}, f_1)$ and $M_2 = (S, \mathcal{C}, f_2)$ be abstract collision systems. If $f_1 = f_2$ then $M_1 \equiv M_2$.*

Proof. Let F_{M_1} and F_{M_2} be global transition functions of M_1 and M_2, respectively. Suppose that $A \in 2^S$. Then we see that

$$\begin{aligned} &F_{M_1}(A) \\ = &\bigcup_{p \in A} f_1\left([p]_{\mathcal{C}}^A\right) \\ = &\bigcup_{p \in A} f_2\left([p]_{\mathcal{C}}^A\right) \\ = &F_{M_2}(A). \end{aligned}$$

Therefore we have $M_1 \equiv M_2$. □

3. Abstract collision systems on G-set

In this section, we consider an action of a group. Let G be a group, and S a non-empty set.

Definition 5. A map from $G \times S$ to G,

$$G \times S \to S \quad ((g, s) \mapsto gs) \tag{6}$$

is called an **action** of G on S, iff it satisfies:

(1) $(gh)s = g(hs) \quad (g, h \in G, s \in S)$

(2) $es = s \quad$ (e is an identity element of G).

Then we say that the group G **acts on** the set S. Moreover, the set S is called **G-set**.

When a group G acts on a set S, we define an action of G on 2^S by

$$gX = \{gx \mid x \in X\} \quad (g \in G, X \in 2^S). \tag{7}$$

We have the following proposition about this action.

Proposition 4. *For all $g \in G$ and $X, Y \subseteq S$, we have:*

$$\text{If } X \subseteq Y, \text{ then } (gX) \subseteq (gY). \tag{8}$$
$$g(X \cup Y) = (gX) \cup (gY) \tag{9}$$
$$g(X \cap Y) = (gX) \cap (gY) \tag{10}$$

Proof. (8) is clear.

Next prove (9). Since $X, Y \subseteq X \cup Y$, we have $gX \subseteq g(X \cup Y)$ and $gY \subseteq g(X \cup Y)$. Therefore we have

$$(gX) \cup (gY) \subseteq g(X \cup Y).$$

On the other hand, for all $z \in g(X \cup Y)$, there exists $w \in X \cup Y$ such that $z = gw$. Then w satisfies $w \in X$ or $w \in Y$. If $w \in X$ (resp. $w \in Y$), we have $z \in gX$ (resp. $z \in gY$). Therefore $z \in (gX) \cup (gY)$. Hence we have

$$g(X \cup Y) \subseteq (gX) \cup (gY).$$

Finally, we prove (10). Since $X \cap Y \subseteq X, Y$, we have $g(X \cap Y) \subseteq (gX)$ and $g(X \cap Y) \subseteq (gY)$. Therefore we have

$$g(X \cap Y) \subseteq (gX) \cap (gY).$$

On the other hand, for all $z \in (gX) \cap (gY)$, there exists $x \in X$ and $y \in Y$ such that $z = gx$ and $z = gy$. Since $g^{-1}z = x = y$, we have $x = y \in X \cap Y$, which implies

$$z = gx = gy \in g(X \cap Y).$$

Hence we have

$$(gX) \cap (gY) \subseteq g(X \cap Y).$$

All claims of Proposition 4 are proved. □

Definition 6. Let G be a group, S a non-empty G-set, V a non-empty subset of S and l a function $l : 2^V \to 2^S$. Then let

$$\widetilde{\mathcal{C}}_V = \{gX \mid g \in G, X \in 2^V\} \tag{11}$$

and $\mathcal{C}$ be a set of collisions on S which includes $\widetilde{\mathcal{C}}_V$. Then we define a local transition function $f_l : \mathcal{C} \to 2^S$ by

$$f_l(X) = \bigcup_{g \in G} gl((g^{-1}X) \cap V). \tag{12}$$

We call an abstract collision system $M = (S, \mathcal{C}, f_l)$ an **abstract collision system on a G-set** made by V and l. Moreover, we call the function l a **base function** of M. In addition, we call f_l an **induced local transition function** by V and l on G, and denoted by $f_l = \mathrm{Ind}(G, V, l)$.

Definition 7. Let $V' \subseteq V$. We call a set V' **essential domain** of l iff it satisfies

$$l(X) = l(X \cap V')$$

for all $X \in 2^V$.

We investigate the behavior of the global transition function of an abstract collision system on a G-set. We prepare the following lemmas.

Lemma 3. *Let $A \in 2^S$ and $g \in G$. If*

$$g^{-1}[p]^A_{\mathcal{C}} \cap V = \phi$$

for all $p \in A$, then we have

$$g^{-1}A \cap V = \phi.$$

Proof. We assume that $g^{-1}A \cap V \neq \phi$. Then there exists $x \in g^{-1}A \cap V$. Let $p = gx$. Since $p = gx \in A$ and $x \in V$, we have

$$x = g^{-1}p \in g^{-1}[p]^A_{\mathcal{C}} \cap V.$$

Hence we have $g^{-1}[p]^A_{\mathcal{C}} \cap V \neq \phi$, this contradicts the assumption of the lemma. □

Lemma 4. *Let $A \in 2^S$, $p \in A$ and $g \in G$. If*

$$g^{-1}[p]^A_{\mathcal{C}} \cap V \neq \phi,$$

then we have

$$g^{-1}[p]^A_{\mathcal{C}} \cap V = g^{-1}A \cap V.$$

Proof. From the definition of $[p]^A_{\mathcal{C}}$, it is clear that $[p]^A_{\mathcal{C}} \subseteq A$ Hence we have

$$g^{-1}[p]^A_{\mathcal{C}} \cap V \subseteq g^{-1}A \cap V.$$

Let $x \in g^{-1}A \cap V$ and $q \in [p]^A_{\mathcal{C}} \cap gV$. We show

$$x \in g^{-1}[p]^A_{\mathcal{C}} \cap V.$$

Since $q \in [p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$, we have $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$ by Proposition 3. Let

$$\begin{aligned} X_1 &= [q]^A_{\mathcal{C}}, \\ X_2 &= ([q]^A_{\mathcal{C}} \cap gV) \cup \{gx\}. \end{aligned} \tag{13}$$

Then it is clear that $X_1 \in \mathcal{C}$ by (SC'3). Since

$$[q]^A_{\mathcal{C}} \cap gV \subseteq gV, \quad gx \in A \cap gV \subseteq gV,$$

we have $X_2 \subseteq gV$. Hence we have

$$X_1, X_2 \in \widetilde{\mathcal{C}}_V \subseteq \mathcal{C}.$$

Since $q \in [q]^A_{\mathcal{C}}$ and $q \in gV$, we have $q \in X_1$ and $q \in X_2$. Hence $X_1 \cap X_2 \neq \phi$. Therefore we have $X_1 \cup X_2 \in \mathcal{C}$ by (SC2). Moreover, since $x \in g^{-1}A \cap V$, we have $gx \in A$, i.e., $\{gx\} \subseteq A$. Since

$$X_1 \cup X_2 \in \mathcal{C}, \quad q \in X_1 \cup X_2, \quad X_1 \cup X_2 \subseteq A,$$

we have $[q]^A_{\mathcal{C}} \supseteq X_1 \cup X_2$ by (1). Hence $gx \in [q]^A_{\mathcal{C}}$, which implies $x \in g^{-1}[q]^A_{\mathcal{C}}$. Moreover, since $x \in g^{-1}A \cap V$, it is clear that $x \in V$. Therefore we have

$$x \in g^{-1}[q]^A_{\mathcal{C}} \cap V = g^{-1}[p]^A_{\mathcal{C}} \cap V.$$

□

Lemma 5. *For all $g \in G$, $A \in 2^S$, $p, q \in A$, we assume that*

$$(g^{-1}[p]^A_{\mathcal{C}} \cap V) \neq \phi, \quad (g^{-1}[q]^A_{\mathcal{C}} \cap V) \neq \phi.$$

Then we have

$$[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}.$$

Proof. By Lemma 4, we see that

$$(g^{-1}[p]^A_{\mathcal{C}} \cap V) = (g^{-1}[q]^A_{\mathcal{C}} \cap V) = (g^{-1}A \cap V).$$

Hence for all $x \in (g^{-1}[p]^A_{\mathcal{C}} \cap V)$, since $gx \in [p]^A_{\mathcal{C}}, [q]^A_{\mathcal{C}}$, we have $[p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$. Therefore we have $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$ by Proposition 3. □

By these lemmas, we see the following, immediately.

Lemma 6. *For all $g \in G$ and $A \in 2^S$, suppose that $A \notin \mathcal{C}$. Then we have*

$$\bigcup_{p \in A} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) = g\left(l\left(g^{-1}A \cap V\right) \cup l\left(\phi\right)\right) \tag{14}$$

Proof. Suppose that $g^{-1}[p]^A_{\mathcal{C}} \cap V = \phi$ for all $p \in A$. Then we have $g^{-1}A \cap V = \phi$ by Lemma 3. Hence the left hand side of (14) equals to

$$\bigcup_{p \in A} gl(\phi) = gl(\phi) = g(l(\phi) \cup l(\phi)) = g(l(g^{-1}A \cap V) \cup l(\phi)).$$

This equals to the right hand side.

Next, we assume that there exists $p_1 \in A$ such that $g^{-1}[p_1]^A_{\mathcal{C}} \cap V \neq \phi$. Let

$$A' = \left\{p \in A \mid g^{-1}[p]^A_{\mathcal{C}} \cap V \neq \phi.\right\},$$
$$A'' = \left\{p \in A \mid g^{-1}[p]^A_{\mathcal{C}} \cap V = \phi.\right\}.$$

Then we have $p_1 \in A'$, which implies $A' \neq \phi$. Since $A \notin \mathcal{C}$, there exists $q_1 \in A$ such that $[p_1]^A_{\mathcal{C}} \neq [q_1]^A_{\mathcal{C}}$. Hence we have $q_1 \in A''$ by Lemma 5. This implies $A'' \neq \phi$. Therefore the left hand side of (14) equals

$$\begin{aligned}
&\bigcup_{p \in A} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) \\
&= \bigcup_{p \in A'} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) \cup \bigcup_{p \in A''} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) \\
&= \bigcup_{p \in A'} gl\left(g^{-1}A \cap V\right) \cup \bigcup_{p \in A'} gl\left(\phi\right) \quad \text{(by Lemma 4)} \\
&= gl\left(g^{-1}A \cap V\right) \cup gl\left(\phi\right).
\end{aligned}$$

This equals the right hand side of (14). □

This lemma induces the following theorem.

Theorem 1. *Let $M = (S, \mathcal{C}, f_l)$ be an abstract collision system on a G-set. made by V and l. Let F_M be the global transition function of M. If $A \in \mathcal{C}$, then F_M satisfies*

$$F_M(A) = \bigcup_{g \in G} gl\left(g^{-1}A \cap V\right). \tag{15}$$

If $A \notin \mathcal{C}$, then

$$F_M(A) = \bigcup_{g \in G} g\left(l\left(g^{-1}A \cap V\right) \cup l(\phi)\right). \tag{16}$$

Proof. First, suppose that $A \in \mathcal{C}$. (15) is clear by Lemma 1 and (12). Next, suppose that $A \notin \mathcal{C}$. By Lemma 6, we see that

$$\begin{aligned}
F_M(A) &= \bigcup_{p \in A} \left(f_l([p]^A_{\mathcal{C}})\right) \\
&= \bigcup_{p \in A} \bigcup_{g \in G} \left(gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right)\right) \\
&= \bigcup_{g \in G} \bigcup_{p \in A} \left(gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right)\right) \\
&= \bigcup_{g \in G} g\left(l\left(g^{-1}A \cap V\right) \cup l(\phi)\right).
\end{aligned}$$

Hence the theorem follows. □

Corollary 1. *Especially, if $l(\phi) = \phi$, then we have*

$$F_M(A) = \bigcup_{g \in G} gl\left(g^{-1}A \cap V\right)$$

for all $A \in 2^S$.

Corollary 2. *We assume that $l(\phi) = \phi$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be sets of collisions on S. Suppose that $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_1$ and $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_2$. We make abstract collision systems $M_1 = (S, \mathcal{C}_1, f_l)$ and $M_2 = (S, \mathcal{C}_2, f_l)$. Then we have*

$$M_1 \equiv M_2.$$

Proof. Let F_{M_1} and F_{M_2} be global transition functions of M_1 and M_2, respectively. By Corollary 1,

$$F_{M_1}(A) = F_{M_2}(A) = \bigcup_{g \in G} gl\left(g^{-1}A \cap V\right)$$

for all $A \in 2^S$. Hence the corollary follows. □

In the following of this paper, we suppose that $l(\phi) = \phi$. Let $M = (S, \mathcal{C}, f_l)$ be an abstract collision system on a G-set made by V and l. By Theorem 1, the abstract collision system M is determined by only G, S, V and l, i.e., M does not depend on the set of collisions $\mathcal{C}$. Therefore we denote the abstract collision system M by $M = GACS(G, S, V, l)$

Then by Corollary 2, we have the following proposition.

Proposition 5. *We have*

$$GACS(G, S, V_1, l_1) \equiv GACS(G, S, V_2, l_2)$$

if $V_1 = V_2$ *and* $l_1 = l_2$.

By Definition 7 and Theorem 1, we show the following proposition.

Proposition 6. *Let* V' *be an essential domain of* l. *Suppose that* $l(\phi) = \phi$. *Then we have*

$$GACS(G, S, V, l) \equiv GACS(G, S, V', l'),$$

where l' *is restriction of* l *onto* $2^{V'}$.

Proof. Let M and M' be abstract collision systems on a G-set,

$$M = GACS(G, S, V, l), \quad M' = GACS(G, S, V, l'),$$

respectively. Let F_M and $F_{M'}$ be global transition functions of M and M', respectively. By Definition 7 and Theorem 1, we see that

$$\begin{aligned} F_M(A) &= \bigcup_{g\in G} gl\left(g^{-1}A \cap V\right) \\ &= \bigcup_{g\in G} gl\left(\left(g^{-1}A \cap V\right) \cap V'\right) \\ &= \bigcup_{g\in G} gl\left(g^{-1}A \cap V'\right), \\ F_{M'}(A) &= \bigcup_{g\in G} gl'\left(g^{-1}A \cap V'\right) \end{aligned}$$

for all $A \in 2^S$. □

In the followings of this section, we will investigate about cellular automata using the notion of ACS on a G-set.

Definition 8. Let G be a group, V a subset of G and l a function from 2^V to 2^G. We assume that

$$\begin{aligned} &l(X) \subseteq 2^{\{e\}} \quad (\text{for all } X \in 2^V), \\ &l(\phi) = \phi \end{aligned}$$

Then we call $GACS(G, G, V, l)$ a **cellular automaton on the group** G.

We consider the following one-to-one mapping:

$$(x_0, \ldots, x_n) \leftrightarrow \{i \in V \mid x_i = 1\}.$$

We denote a map l by

$$l(\{i \in V \mid x_i = 1\}) = l(x_0, \ldots, x_n),$$

for example $l(\{0, 1, 3\}) = l(1, 1, 0, 1, 0, \ldots, 0)$. Moreover, we denote ϕ by 0 and $\{0\}$ by 1, i.e,

$$\begin{cases} l(x_0, \ldots, x_n) = 0 & l(\{i \in V \mid x_i = 1\}) = \phi \\ l(x_0, \ldots, x_n) = 1 & l(\{i \in V \mid x_i = 1\}) = \{0\} \end{cases} \tag{17}$$

Let n be a positive integer, $G = \mathbb{Z}$, $V = \{0, 1, \ldots, n-1\}$ and l a function $l : 2^V \to 2^G$. Then the **rule number** of l, which is also known as the *Wolfram number*, is defined by the number whose binary expression with the length 2^n

$$l(1, \ldots, 1, 1)\, l(1, \ldots, 1, 0) \cdots l(0, \ldots, 0, 1)\, l(0, \ldots, 0, 0),$$

i.e.,

$$\sum_{x_0, \ldots x_{n-1} \in \{0,1\}^n} l(x_0, \ldots, x_{n-1}) \cdot 2^{\sum_{k=0}^{n-1} x_k 2^{n-1-k}}.$$

Moreover, we denote the function l with rule number r by $l_r^{(n)}$. For example, let $n = 2$ and $V = \{0, 1\}$. Then a function $l_4^{(2)}$, whose rule number is 4, is

$$\begin{aligned} l_4^{(2)}(1, 1) &= 0, \\ l_4^{(2)}(1, 0) &= 1, \\ l_4^{(2)}(0, 1) &= 0, \\ l_4^{(2)}(0, 0) &= 0. \end{aligned}$$

We note that the binary expression of 4 is 0100. Moreover, we call the cellular automata on group $M_{nCA-r} = GACS(\mathbb{Z}, \mathbb{Z}, \{0, \ldots, n-1\}, l_r^{(n)})$ (1 dimensional 2 states) n **neighborhood cellular automata with rule number** r.

Example 2. Let $G = \mathbb{Z}$, $V = \{0, 1\}$. We define l by

$$\begin{aligned} &l(\{0, 1\}) = \phi, \quad l(\{0\}) = \{0\}, \\ &l(\{1\}) = \{0\}, \quad l(\phi) = \phi. \end{aligned}$$

By using notation of (17), we denote this function by

$$\begin{aligned} &l(1, 1) = 0, \quad l(1, 0) = 1, \\ &l(0, 1) = 1, \quad l(0, 0) = 0, \end{aligned}$$

i.e., $l(x_0, x_1) = x_0 \oplus x_1$. We note that the rule number of l is 6 and denote l by $l_6^{(2)}$ Then an abstract collision system $M_{2CA-6} = GACS(\mathbb{Z}, \mathbb{Z}, \{0, 1\}, l_6^{(2)})$ is a 1 dimensional, 2 state, 2 neighborhood cellular automaton with rule number 6.

Example 3. Similarly, we can construct other 1 dimensional, 2 states, n neighborhood cellular automata. Let $Q = \{0, 1\}$ and $f_{n\ CA-r} : Q^n \to Q$. Suppose that

$$f_{n\ CA-r}(0, \ldots, 0) = 0.$$

Let $G = \mathbb{Z}$ and $V = \{0, 1, \ldots, n-1\}$.

We define $l_r^{(n)} : 2^V \to 2^{\mathbb{Z}}$ by

$$l_r^{(n)}(x_0, \ldots, x_{n-1}) = \begin{cases} \phi & f_{n\ CA-r}(x_0, \ldots, x_{n-1}) = 0 \\ \{0\} & f_{n\ CA-r}(x_0, \ldots, x_{n-1}) = 1 \end{cases}$$

for all $(x_0, \ldots, x_{n-1}) \in Q^n$.

By using notation of (17), we denote $l_k^{(n)}$ by

$$l_k^{(n)}(x_0, \ldots, x_{n-1}) = \begin{cases} 0 & f_{n\ CA-r}(x_0, \ldots, x_{n-1}) = 0 \\ 1 & f_{n\ CA-k}(x_0, \ldots, x_{n-1}) = 1 \end{cases}$$
$$= f_{n\ CA-r}(x_0, \ldots, x_{n-1}).$$

Then an abstract collision system a G-set

$$M_{n\ CA-r} = GACS(\mathbb{Z}, \mathbb{Z}, V, l_r^{(n)})$$

is a 1 dimensional 2 states n neighborhood cellular automaton with rule number r.

In the above definition and example, we can construct only 2-state cellular automata. We describe how to make other general cellular automata.

Example 4. Let Q be a non-empty set, $G = \mathbb{Z}$ and $S = \mathbb{Z} \times Q$. We define

$$z_1(z_2, q) = (z_1 + z_2, q)$$

for all $z_1 \in G$ and $(z_2, q) \in S$. We choose a subset $H \subseteq \mathbb{Z}$. and define $V = H \times Q$. Suppose that

$$l(X) \subseteq \{0\} \times Q$$

for all $X \in 2^V$ and $l(\phi) = \phi$. Then an abstract collision system

$$M = GACS(\mathbb{Z}, \mathbb{Z} \times Q, H \times Q, l)$$

is a 1 dimensional, Q state, H neighborhood cellular automaton.

Since $l(\phi) = \phi$, we note that we can construct any cellular automata which has the rule $f_{CA}(0, 0, \ldots, 0) = 1$.

If $l(\phi) \neq \phi$, we can not construct cellular automata on groups. By Theorem 1, the behavior of the global transition function depends on configurations.

First of all, we describe a theorem with respect to the set $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$. From this theorem, we can evaluate the set $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$.

For all subsets $X, Y \subseteq G$, we define

$$X \otimes Y^{-1} = \left\{ xy^{-1} \mid x \in X, y \in Y \right\}.$$

We define a set $\mathcal{C}_V$ by

$$\mathcal{C}_V = \left\{ X \;\middle|\; \begin{array}{c} \text{for all } Y_1 \text{ and } Y_2 \subseteq X, \\ (Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) \neq \phi \\ \text{if } Y_1 \neq \phi, Y_2 \neq \phi \text{ and } Y_1 \cup Y_2 = X. \end{array} \right\}. \tag{18}$$

Then, we can show the following two lemmas.

Lemma 7. *$\mathcal{C}_V$ is a set of collisions on G.*

Proof. We check the condition (SC1). For all $s \in S$, let $X = \{s\}$. For all $Y_1, Y_2 \subseteq X$, we assume that $Y_1 \neq \phi$, $Y_2 \neq \phi$, and $Y_1 \cup Y_2 = X$. Then, since $Y_1 = Y_2 = \{s\} = X$, we have

$$(Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) = X \otimes V^{-1} \neq \phi.$$

Hence $\{s\} \in \mathcal{C}_V$.

We check the condition (SC2). Let $\mathcal{X} \subseteq \mathcal{C}_V$ and $(\cap\mathcal{X}) \neq \phi$. We assume that $(\cup\mathcal{X}) \notin \mathcal{C}_V$, that is, there exist Y_1 and $Y_2 \subseteq (\cup\mathcal{X})$ such that $Y_1 \neq \phi$, $Y_2 \neq \phi$, $Y_1 \cup Y_2 = (\cup\mathcal{X})$,

$$(Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) = \phi.$$

Then, since $(\cap\mathcal{X}) \neq \phi$, there exists $s_0 \in (\cap\mathcal{X})$. We can assume that $s_0 \in Y_1$ without loss of generality.

Since $(\cup\mathcal{X}) = Y_1 \cup Y_2$ and $Y_2 \neq \phi$, we have

$$\left(\bigcup \mathcal{X}\right) \cap Y_2 = (Y_1 \cup Y_2) \cap Y_2 = Y_2 \neq \phi.$$

Hence there exists $X \in \mathcal{X}$ such that $Y_2 \cap X \neq \phi$.

Since $X \in \mathcal{X} \subseteq \mathcal{C}_V$, we have

$$X \in \mathcal{C}_V. \tag{19}$$

Since $s_0 \in (\cap\mathcal{X}) \subseteq X$, $s_0 \in Y_1$, we have

$$s_0 \in Y_1 \cap X \neq \phi. \tag{20}$$

Let $Y_1' = Y_1 \cap X$, $Y_2' = Y_2 \cap X$. Then we have $Y_1 \cap X \neq \phi$, $Y_2 \cap X \neq \phi$. Moreover, we see that

$$\begin{aligned} &Y_1' \cup Y_2' \\ =&(Y_1 \cup Y_2) \cap X \\ =&(\cup\mathcal{X}) \cap X = X, \\ &(Y_1' \otimes V^{-1}) \cap (Y_2' \otimes V) \\ \subseteq&(Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) \\ =&\phi. \end{aligned}$$

Hence $X \notin \mathcal{C}_V$, this contradicts (19). Therefore we have $(\cup\mathcal{X}) \in \mathcal{C}_V$, □

Lemma 8. *The set $\mathcal{C}_V$ includes the set $\widetilde{\mathcal{C}}_V$, i.e.,*

$$\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_V$$

Proof. For all $g \in G$ and $X \in 2^V$, we show that $gX \in \mathcal{C}_V$. If $X = \phi$ or $\#X = 1$, then we can see easily. We assume that $\#X \geq 2$. Let Y_1 and Y_2 be subsets of gX. Suppose that

$$Y_1 \neq \phi, \quad Y_2 \neq \phi, \quad Y_1 \cup Y_2 = gX.$$

For all $y_1 \in Y_1$, we have

$$y_1 \in Y_1 \subseteq gX \subseteq gV.$$

Therefore there exists $h_1 \in V$ such that $y_1 = gh_1$. Hence we have

$$g = y_1 {h_1}^{-1} \in Y_1 \otimes V^{-1}.$$

Similarly, we have $g \in Y_2 \otimes V^{-1}$. Therefore we have

$$(Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) \neq \phi.$$

Hence $gX \in \mathcal{C}_V$. Therefore we have $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_V$. □

We can prove the following proposition easily, from these two lemmas,

Proposition 7. *We have*

$$\mathfrak{C}(\widetilde{\mathcal{C}}_V) \subseteq \mathcal{C}_V.$$

Let $V = \{0, 1, 2\}$, $l(\phi) \neq \phi$ and M be an abstract collision system $M = (S, \mathfrak{C}(\widetilde{\mathcal{C}}_V), f_l)$ made by V and l. Let F_M be the global transition function of M. For example, a configuration $\mathbf{c}_1 = \{0\}$ is an element of $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$. Therefore by Theorem 1, we see that

$$F_M(\mathbf{c}_1) = f_l(\mathbf{c}_1).$$

However, we consider another configuration $\mathbf{c}_2 = \{0, 3\}$. By Proposition 7, we can see $\mathbf{c}_2 \notin \mathfrak{C}(\widetilde{\mathcal{C}}_V)$. Therefore by Theorem 1, we see that

$$F_M(\mathbf{c}_2) = \bigcup_{g \in \mathbb{Z}} g\{0\} = \mathbb{Z}.$$

4. Union and division of abstract collision systems

In this section, we discuss about union and division of abstract collision systems.

Definition 9 (Union)**.** Let M_1 and M_2 be abstract collision systems $M_1 = (S_1, \mathcal{C}_1, f_1)$ and $M_2 = (S_2, \mathcal{C}_2, f_2)$. We define $f_1 \cup f_2$ by

$$(f_1 \cup f_2)(X) = F_{M_1}\left(X \cap 2^{S_1}\right) \cup F_{M_2}\left(X \cap 2^{S_2}\right),$$

where F_{M_1} and F_{M_2} are global transition functions of M_1 and M_2, respectively. We define **union** of M_1 and M_2, which is denoted by $M_1 \cup M_2$, by

$$M_1 \cup M_2 = (S_1 \cup S_2, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_1 \cup f_2). \tag{21}$$

Definition 10 (Division)**.** Let M be an abstract collision system $M = (S, \mathcal{C}, f)$. We say that M is **dividable** iff there exists two abstract collision systems $M_1 \neq M$ and $M_2 \neq M$ such that $M \equiv M_1 \cup M_2$.

Proposition 8.

$$\begin{aligned}&GACS(G, S, V, l_1) \cup GACS(G, S, V, l_2)\\ \equiv& GACS(G, S, V, l_1 \cup l_2),\end{aligned}$$

where

$$(l_1 \cup l_2)(X) = l_1(X) \cup l_2(X)$$

for all $X \in 2^V$.

Proof. We choose arbitrary set of collisions $\mathcal{C}_1$ and $\mathcal{C}_2$ includes $\widetilde{\mathcal{C}}_V$. Let

$$\begin{aligned}f_{l_1} &= \mathrm{Ind}(G, V, l_1),\\ f_{l_2} &= \mathrm{Ind}(G, V, l_2),\\ f_{l_1 \cup l_2} &= \mathrm{Ind}(G, V, l_1 \cup l_2).\end{aligned}$$

Let $M_1 = (S, \mathcal{C}_1, f_{l_1})$ and $M_2 = (S, \mathcal{C}_2, f_{l_2})$. For all $X \in 2^V$, we see that

$$\begin{aligned}&f_{l_1 \cup l_2}(X)\\ =& \bigcup_{g \in G} g\left(l_1 \cup l_2\right)\left(g^{-1}X \cap V\right)\\ =& \bigcup_{g \in G} g\left\{l_1\left(g^{-1}X \cap V\right) \cup l_2\left(g^{-1}X \cap V\right)\right\}\\ =& \bigcup_{g \in G} gl_1\left(g^{-1}X \cap V\right) \cup \bigcup_{g \in G} gl_2\left(g^{-1}X \cap V\right)\\ =& F_{l_1}(X \cap S) \cup F_{l_2}(X \cap S)\\ =& (f_{l_1} \cup f_{l_2})(X).\end{aligned}$$

Therefore we have

$$\begin{aligned}&M_1 \cup M_2\\ =&(S, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_{l_1} \cup f_{l_2})\\ \equiv&(S, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_{l_1 \cup l_2}).\end{aligned}$$

Moreover, since $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_1, \mathcal{C}_2$, we have $\widetilde{\mathcal{C}}_V \subseteq \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2)$. Hence we have

$$\begin{aligned}&(S, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_{l_1 \cup l_2})\\ \equiv& GACS(G, S, V, l_1 \cup l_2)\end{aligned}$$

Therefore we have

$$M_1 \cup M_2 \equiv GACS(G, S, V, l_1 \cup l_2).$$

□

Corollary 3. *Let*

$$M_1 \equiv GACS(G, S, V, l_1), \quad M_2 \equiv GACS(G, S, V, l_2).$$

Let F_{M_1}, F_{M_2} and $F_{M_1 \cup M_2}$ be global transition functions of M_1, M_2 and $M_1 \cup M_2$, respectively. Then we have

$$F_{M_1 \cup M_2}(A) = F_{M_1}(A) \cup F_{M_2}(A)$$

for all $A \in 2^S$

Proof. Let $A \in 2^S$. From Proposition 8, we have

$$M_1 \cup M_2 \equiv GACS(G, S, V, l_1 \cup l_2).$$

Therefore we see that

$$\begin{aligned}&F_{M_1 \cup M_2}(A)\\ =& \bigcup_{g \in G} g\left(l_1 \cup l_2\right)\left(g^{-1}A \cap V\right)\\ =& \bigcup_{g \in G} gl_1\left(g^{-1}A \cap V\right) \cup \bigcup_{g \in G} gl_2\left(g^{-1}A \cap V\right)\\ =& F_{M_1}(A) \cup F_{M_2}(A)\end{aligned}$$

by Theorem 1. □

Corollary 4. *Let*

$$M_1 = GACS(G, S, V, l_1),$$
$$M_2 = GACS(G, S, V, l_2),$$
$$M_3 = GACS(G, S, V, l_3).$$

We have

$$M_1 \cup M_3 \equiv M_2 \cup M_3$$

if $M_1 \equiv M_2$.

Proof. Let F_{M_1}, F_{M_2}, F_{M_3} $F_{M_1 \cup M_3}$ and $F_{M_1 \cup M_3}$ be global transition functions of M_1, M_2, M_3, $F_{M_1 \cup M_3}$ and $F_{M_2 \cup M_3}$ respectively. Then we have $F_{M_1} = F_{M_2}$. For all $A \in 2^S$, we see that

$$\begin{aligned} & F_{M_1 \cup M_3}(A) \\ =& F_{M_1}(A) \cup F_{M_3}(A) \\ =& F_{M_2}(A) \cup F_{M_3}(A) \\ =& F_{M_2 \cup M_3}(A) \end{aligned}$$

from Corollary 3. Hence we have $M_1 \cup M_3 = M_2 \cup M_3$. □

Next, we consider to divide the set $\mathcal{C}$ into some partitions.

Proposition 9. *Let* $\mathcal{C}$ *be a set of collisions on* S. *The following three conditions are equivalent.*

(a) *There exists two sets of collisions* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *on* S *which satisfies:*

(22) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$,

(23) *for all* $X_1 \in \mathcal{C}_1, X_2 \in \mathcal{C}_2$, *if* $\#X_1 \geq 2$ *and* $\#X_2 \geq 2$, *then* $X_1 \cap X_2 = \phi$,

where $\#X_i$ *is the number of elements in* X_i.

(b) *There exists subsets* $\widetilde{\mathcal{C}}_1$ *and* $\widetilde{\mathcal{C}}_2$ *of* 2^S *which satisfy*

(24) $\mathcal{C} = \widetilde{\mathcal{C}}_1 \cup \widetilde{\mathcal{C}}_2$,

(25) $X_1 \in \widetilde{\mathcal{C}}_1, X_2 \in \widetilde{\mathcal{C}}_2 \Rightarrow X_1 \cap X_2 = \phi$.

(c) *There exists* S_1 *and* S_2 *which satisfy*

(26) $S_1 \cup S_2 = S$,

(27) $S_1 \cap S_2 = \phi$,

(28) $(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) = \mathcal{C}$.

Proof. We prove (c) ⇔ (a) and (c) ⇔ (b).

(c) ⇒ (b) Let

$$\widetilde{\mathcal{C}}_i = \mathcal{C} \cap 2^{S_i}.$$

Then we have

$$\widetilde{\mathcal{C}}_1 \cup \widetilde{\mathcal{C}}_2 = (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C}_2 \cap 2^{S_2}) = \mathcal{C},$$

by (28). Hence we have (24).

Moreover, for all $X_1 \in \widetilde{\mathcal{C}}_1, X_2 \in \widetilde{\mathcal{C}}_2$, since

$$X_1 \in 2^{S_1}, X_2 \in 2^{S_2}$$

and (27) of (c), we have

$$X_1 \cap X_2 \subseteq S_1 \cap S_2 = \phi.$$

Hence we have (25).

(b) ⇒ (c) Let

$$S_i = \bigcup \widetilde{\mathcal{C}}_i.$$

We prove (26), i.e., $S_1 \cup S_2 = S$. It is clear that $S_1 \cap S_2 \subseteq S$. On the other hand, for all $s \in S$, we have $\{s\} \in \mathcal{C}$. Therefore we have

$$s \in (\cup\widetilde{\mathcal{C}}_1) \cup (\cup\widetilde{\mathcal{C}}_2) = S_1 \cup S_2,$$

by (24). This implies $S \subseteq S_1 \cup S_2$. Therefore we have (26).

Next, we prove (27), i.e., $S_1 \cap S_2 = \phi$. We suppose that $S_1 \cap S_2 \neq \phi$. Then, there exists $s \in S_1 \cap S_2$. Therefore, there exists $X_1 \in \widetilde{\mathcal{C}}_1$ and $X_2 \in \widetilde{\mathcal{C}}_2$ such that $s \in X_1, s \in X_2$. This implies $s \in X_1 \cap X_2 \neq \phi$. This contradicts (25). Hence we have (27). Finally, we prove (28). It is clear that

$$(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \subseteq \mathcal{C}.$$

On the other hand, since $S_1 = (\cup\widetilde{\mathcal{C}}_1)$, we have $X \subseteq (\cup\widetilde{\mathcal{C}}_1) = S_1$ for all $X \in \widetilde{\mathcal{C}}_1$. This implies $X \in 2^{S_1}$. Therefore $\widetilde{\mathcal{C}}_1 \subseteq 2^{S_1}$. Hence we have

$$\widetilde{\mathcal{C}}_1 \subseteq \mathcal{C} \cap 2^{S_1}.$$

Similarly, we have $\widetilde{\mathcal{C}}_2 \subseteq \mathcal{C} \cap 2^{S_2}$. Therefore we have

$$\mathcal{C} = \widetilde{\mathcal{C}}_1 \cup \widetilde{\mathcal{C}}_2 \subseteq (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

Hence we have (28).

(c) ⇒ (a) Let

(29) $\mathcal{C}_i = (\mathcal{C} \cap 2^{S_i}) \cup \{\{s\} \mid s \in S_{3-i}\}, \quad (i = 1, 2).$

For all $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$, suppose that $\#X_1 \geq 2$ and $\#X_2 \geq 2$. Since

$$X_i \notin \{\{s\} \mid s \in S_{3-i}\},$$

we have $X_i \in (\mathcal{C} \cap 2^{S_i})$. By (27), we have (23) as following:

$$X_1 \cap X_2 \subseteq S_1 \cap S_2 = \phi.$$

Moreover, we have (22) as following:

$$\begin{aligned} \mathcal{C}_1 \cup \mathcal{C}_2 &= (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \cup \{\{s\} \mid s \in S\} \\ &= \mathcal{C} \cup \{\{s\} \mid s \in S\} \\ &= \mathcal{C}. \end{aligned}$$

Finally, we show that $\mathcal{C}_1$ and $\mathcal{C}_2$ are sets of collisions on S. By (29), it is easy to show that $\mathcal{C}_i$ satisfies the condition (SC1). We check the condition (SC2). We assume $\mathcal{X} \subseteq \mathcal{C}_1$ without loss of generality. Suppose that $(\cap\mathcal{X}) \neq \phi$.

We suppose that $\#\mathcal{X} = 1$. Then there exists $X \in \mathcal{C}_1$ such that $\mathcal{X} = \{X\}$. Therefore $(\cup\mathcal{X}) = X \in \mathcal{C}_1$.

We suppose that $\#\mathcal{X} \geq 2$. We assume that there exists $X \in \mathcal{X}$ such that $X \notin \mathcal{C} \cap 2^{S_1}$. Then there exists $s_2 \in S_2$ such that $X = \{s_2\}$. Therefore $(\cap\mathcal{X}) \supseteq \{s_2\}$. However, by (29), we have

$$X \in \mathcal{C}_1, X \neq \{s_2\} \Rightarrow s_2 \notin X.$$

Therefore, $(\cap\mathcal{X}) = \phi$. This contradicts $(\cap\mathcal{X}) \neq \phi$. Hence we have $X \in \mathcal{C} \cap 2^{S_1}$ for all $X \in \mathcal{X}$. This implies $\mathcal{X} \subseteq \mathcal{C} \cap 2^{S_1}$. Since $\mathcal{C}$ is a set of collisions on S, we have $(\cup\mathcal{X}) \in \mathcal{C}$ from $\mathcal{X} \subseteq \mathcal{C}$ and $(\cap\mathcal{X}) \neq \phi$. Moreover, we have $(\cup\mathcal{X}) \in 2^{S_1}$ from $\mathcal{X} \subseteq 2^{S_1}$. Therefore, $(\cup\mathcal{X}) \in \mathcal{C} \cap 2^{S_1}$. Hence the set $\mathcal{C}_1$ satisfies the condition (SC2).

(a) $\Rightarrow$ (c) We assume there exists $s \in S$ such that

$$X \in \mathcal{C}, s \in X \Rightarrow X = \{s\}.$$

Then we can easily prove (c), by putting $S_1 = \{s\}$ and $S_2 = S \setminus S_1$. In fact, it is clear that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \phi$. Let $X \in \mathcal{C}$. If $s \in X$ then $X = \{s\} \subseteq S_1$. If $s \notin X$ then $X \subseteq S_2$. This implies that $X \in 2^{S_1} \cup 2^{S_2}$. Therefore we have

$$X \in \mathcal{C} \cap (2^{S_1} \cup 2^{S_2}) = (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

Hence we have $(\mathcal{C} \cap 2^{S_1}) \cap (\mathcal{C} \cap 2^{S_2}) = \mathcal{C}$.

In the following, suppose that for all $s \in S$, there exists $X \in \mathcal{C}$ such that

$$(30) \qquad s \in X, \quad \#X \geq 2.$$

Let

$$(31) \qquad S_i = \bigcup \{X \mid X \in \mathcal{C}_i, \#X \geq 2\}, \quad (i = 1, 2).$$

First, it is clear that $S_i \neq \phi$ and $S_1 \cup S_2 \subseteq S$. We show that $S_1 \cup S_2 \supseteq S$. For all $s \in S$, there exists $X \in \mathcal{C}$ such that $s \in X$, $\#X \geq 2$. Since $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, we have $X \in \mathcal{C}_1$ or $X \in \mathcal{C}_2$. If $X \in \mathcal{C}_1$ (resp. $\mathcal{C}_2$), we have $X \subseteq S_1$ (resp. $X \subseteq S_2$) by $\#X \geq 2$. Therefore we can conclude that $s \in X \subseteq S_1 \cup S_2$. Next, we prove (27). We assume that $S_1 \cap S_2 \neq \phi$. There exist $X_1 \in \mathcal{C}_1(\#X_1 \geq 2)$ and $X_2 \in \mathcal{C}_2(\#X_2 \geq 2)$ such that $s \in X_1$, $s \in X_2$. This implies $X_1 \cap X_2 \neq \phi$. This contradicts (23). Finally, we prove (28). Let $X \in \mathcal{C}$. Suppose that $\#X = 1$. Since $S = S_1 \cup S_2$, we have $X \subseteq S_1$ or $X \subseteq S_2$. Therefore $X \in 2^{S_1} \cup 2^{S_2}$. We suppose that $\#X \geq 2$. Then we have $X \subseteq S_1$ or $X \subseteq S_2$ by (31). This implies $X \in 2^{S_1} \cup 2^{S_2}$. Therefore we have

$$X \in \mathcal{C} \cap (2^{S_1} \cup 2^{S_2}) = (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

Hence we see that

$$\mathcal{C} \subseteq (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

On the other hand, it is clear that

$$(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \subseteq \mathcal{C}.$$

Hence we have (28). $\square$

Definition 11. Let $\mathcal{C}$ be a set of collisions on S. We call the set $\mathcal{C}$ is **dividable** iff it satisfies conditions of Proposition 9.

Proposition 10. *Let $M = (S, \mathcal{C}, f)$ be an abstract collision system. If the set $\mathcal{C}$ is dividable, then M is dividable.*

Proof. Since the set $\mathcal{C}$ is dividable, it satisfies the condition (c). Therefore, there exists S_1 and S_2 such that

$$(32) \quad S_1 \cup S_2 = S, \quad S_1 \cap S_2 = \phi, \quad (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) = \mathcal{C}.$$

Let

$$(33) \qquad M_1 = (S_1, \mathcal{C} \cap 2^{S_1}, f_1), \quad M_2 = (S_2, \mathcal{C} \cap 2^{S_2}, f_2),$$

where f_1 and f_2 is restriction of f onto $\mathcal{C} \cap 2^{S_1}$ and $\mathcal{C} \cap 2^{S_2}$, respectively. Since $\mathcal{C}$ is a set of collision on S, we have

$$\mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2) = \mathfrak{C}(\mathcal{C}) = \mathcal{C}.$$

Therefore for all $X \in \mathcal{C}$, we have $X \in (\mathcal{C} \cap 2^{S_1})$ or $X \in (\mathcal{C} \cap 2^{S_2})$. We suppose that $X \in (\mathcal{C} \cap 2^{S_1})$. Since $X \subseteq S_1$, we have $X \cap S_2 = \phi$. Hence we have

$$(34) \qquad (f_1 \cup f_2)(X) = f_1(X) \cup \phi = f_1(X) = f(X).$$

We can also prove (34) in the same way for $X \in (\mathcal{C} \cap 2^{S_2})$.

Hence we have

$$M_1 \cup M_2 = (S_1 \cup S_2, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_1 \cup f_2) = (S, \mathcal{C}, f_1 \cup f_2).$$

Therefore we have $M_1 \cup M_2 \equiv M$ by Lemma 2 and (34). $\square$

The converse of Proposition 10 does not hold. We show that there exists an abstract collision system $M = (S, \mathcal{C}, f)$ such that $\mathcal{C}$ is not dividable but M is dividable.

Proposition 11. *Let G be a cyclic group and its generator be an element a, i.e.,*

$$G = \langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}.$$

We assume that $V \supseteq \{a^0, a^1\}$. Then any set of collisions $\mathcal{C}$ which includes $\widetilde{\mathcal{C}}_V$ is not dividable.

Proof. First we prove that

$$X_n = \{a^0, a^1, \ldots, a^n\}$$

is an element of $\mathcal{C}$ for all $n \in \mathbb{N}$. We prove this by using mathematical induction. When $n = 1$, since

$$X_1 = \{a^0, a^1\} \in 2^V,$$

we have $X_1 \in \widetilde{\mathcal{C}}_V \subseteq \mathcal{C}$. Let $k \geq 1$ and we assume that $X_k \in \mathcal{C}$. Since $\{a^0, a^1\} \in 2^V$ and $a^k \in G$, we have

$$X'_{k+1} = \{a^k, a^{k+1}\} = a^k \{a^0, a^1\} \in \widetilde{\mathcal{C}}_V \subseteq \mathcal{C}.$$

Therefore we have

$$X_k \in \mathcal{C}, X'_{k+1} \in \mathcal{C}, X_k \cap X'_{k+1} = \{a^k\} \neq \phi.$$

Hence we have

$$X_k \cup X'_{k+1} = X_{k+1} \in \mathcal{C},$$

by (SC2). Similarly, we have

$$\{a^{-m}, \ldots, a^0\} \in \mathcal{C}$$

for all $m \in \mathbb{N}$. Therefore we have

$$\{a^{-m}, \ldots, a^0, \ldots, a^n\} \in \mathcal{C}$$

for all $m, n \in \mathbb{N}$.

Next, we show that $\mathcal{C}$ is not dividable. We assume that $\mathcal{C}$ is dividable. Then there exist two set S_1 and S_2 such that they satisfy 3 conditions of (c) in Proposition 9.

We assume $a^0 \in S_1$ without loss of generality. Since $S_2 \neq \phi$, we can take an element $a^n \in S_2$. Then the set

$$Y_n = \left\{a^{-|n|}, \ldots, a^0, \ldots, a^{|n|}\right\}$$

is an element of $\mathcal{C}$. Since $a^0 \in Y_n$, $a^0 \in S_1$, $a^n \in Y_n$, $a^n \in S_2$ and $S_1 \cap S_2 = \phi$, we have

$$Y_n \notin 2^{S_1}, \quad Y_n \notin 2^{S_2}.$$

This implies

$$(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \neq \mathcal{C}.$$

This contradicts (28). Hence $\mathcal{C}$ is not dividable. $\square$

Example 5. We consider a 1 dimensional, 2 states, 2 neighborhood cellular automaton rule number 6:

$$f_{CA6}(x_0, x_1) = x_0 \oplus x_1.$$

We note that

$$x_0 \oplus x_1 = (x_0 \wedge \neg x_1) \vee (\neg x_0 \wedge x_1).$$

Let $G = \mathbb{Z}$. We define $l_6^{(2)}$ by

$$l_6^{(2)}(\{0,1\}) = \phi, \quad l_6^{(2)}(\{0\}) = \{0\},$$
$$l_6^{(2)}(\{1\}) = \{0\}, \quad l_6^{(2)}(\phi) = \phi$$

By using the notation of (17), we denote $l_6^{(2)}$ by

$$l_6^{(2)}(1,1) = 0, \quad l_6^{(2)}(1,0) = 1,$$
$$l_6^{(2)}(0,1) = 1, \quad l_6^{(2)}(0,0) = 0,$$

i.e., $l_6^{(2)}(x_0, x_1) = x_0 \oplus x_1$. Let $V = \{0, 1\}$. Then we see that the set of collisions $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$ is not dividable from Proposition 11.

Moreover, we define two functions $l_2^{(2)}$ and $l_4^{(2)}$ by

$$l_2^{(2)}(1,1) = 0, \quad l_2^{(2)}(1,0) = 0,$$
$$l_2^{(2)}(0,1) = 1, \quad l_2^{(2)}(0,0) = 0,$$
$$l_4^{(2)}(1,1) = 0, \quad l_4^{(2)}(1,0) = 1,$$
$$l_4^{(2)}(0,1) = 0, \quad l_4^{(2)}(0,0) = 0,$$

i.e.,

$$l_2^{(2)}(x_0, x_1) = \neg x_0 \vee x_1,$$
$$l_4^{(2)}(x_0, x_1) = x_0 \vee \neg x_1.$$

Let

$$\begin{aligned} M_{2CA-6} &= GACS(\mathbb{Z}, \mathbb{Z}, V, l_6^{(2)}), \\ M_{2CA-2} &= GACS(\mathbb{Z}, \mathbb{Z}, V, l_2^{(2)}), \\ M_{2CA-4} &= GACS(\mathbb{Z}, \mathbb{Z}, V, l_4^{(2)}). \end{aligned}$$

Then have $M_{2CA-6} \equiv M_{2CA-2} \cup M_{2CA-4}$.

The results of 1 dimensional 2 states 2 neighborhood cellular automata are listed in Table 2. From this table, we see that the rule numbers of cellular automata which is dividable are 6, 10, 12 and 14.

Example 6. We consider 1 dimensional 2 state 3 neighborhood cellular automaton CA 222, i.e.,

$$V = \{0, 1, 2\},$$
$$l_{222}^{(3)}(x_0, x_1, x_2) = (x_0 \oplus x_2) \vee x_1,$$
$$M_{3CA-222} = GACS(\mathbb{Z}, \mathbb{Z}, V, l_{222}^{(3)}).$$

Then we see that $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$ $\mathcal{C}$ is not dividable.

On the other hand, we define two functions

$$l_{90}^{(3)}(x_0, x_1, x_2) = x_0 \oplus x_2,$$
$$l_{204}^{(3)}(x_0, x_1, x_2) = x_1,$$

and make abstract collision systems

$$M_{3CA-90} = GACS(\mathbb{Z}, \mathbb{Z}, \{0, 2\}, l_{90}^{(3)}),$$
$$M_{3CA-204} = GACS(\mathbb{Z}, \mathbb{Z}, \{1\}, l_{204}^{(3)}).$$

Then we can easily prove that

$$M_{3CA-222} \equiv M_{3CA-90} \cup M_{3CA-204}.$$

Table 2: union of two 2 neighborhood CA

$l_2 \backslash l_1$	0	2	4	6	8	10	12	14
0	0	2	4	6	8	10	12	14
2	2	2	6	6	10	10	14	14
4	4	6	4	6	12	14	12	14
6	6	6	6	6	14	14	14	14
8	8	10	12	14	8	10	12	14
10	10	10	14	14	10	10	14	14
12	12	14	12	14	12	14	12	14
14	14	14	14	14	14	14	14	14

Finally, we show a sufficient condition with which ACS is dividable.

Theorem 2. *Let G be a group. We consider an abstract collision system on a G-set, $GACS(G, G, V, l)$. We assume that there exists a normal subgroup H of G and $d \in G$ such that $H \neq G$ and $dV \subseteq H$. Then the set $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$ is dividable.*

Proof. Without loss of generality, we can assume that $d = e$ (e is the identity element of G), and the index $\#(G/H)$ is 2. In other cases, we can prove similarly. We prove (c) of Proposition 9. Let $h \in G \setminus H$, and

$$S_1 = H, \quad S_2 = hH.$$

It is clear that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$.

Next, we prove that $\mathfrak{C}(\widetilde{\mathcal{C}}) \subseteq 2^H \cup 2^{hH}$. To prove this, we show that $2^H \cup 2^{hH}$ is a set of collisions on S and

$$\widetilde{\mathcal{C}}_V = \left\{ gX \mid g \in G, X \in 2^V \right\} \subseteq (2^H \cup 2^{hH}). \tag{35}$$

It is clear that $2^H \cup 2^{hH}$ is a set of collisions on S. We prove (35). Let $Y \in \widetilde{\mathcal{C}}_V$. There exists $g \in G$, $X \in 2^V$ such that $Y = gX$. Since $V \subseteq H$, we have $X \in 2^H$, i.e., $X \subseteq H$.

Hence $Y = gX \subseteq gH$. Since gH equals to H or hH, 2^{gH} equals to 2^H or 2^{hH}. Therefore we have

$$Y \in 2^{gH} \subseteq 2^H \cup 2^{hH}.$$

Hence we have $Y \in (2^H \cup 2^{hH})$ for all $Y \in \widetilde{\mathcal{C}}_V$. This implies (35). Finally, let $C_1 = \mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^H$ and $C_2 = \mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^{hH}$. Then we see that

$$\begin{aligned} C_1 \cup C_2 &= (\mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^H) \cup (\mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^{hH}) \\ &= \mathfrak{C}(\widetilde{\mathcal{C}}) \cap (2^H \cup 2^{hH}) \\ &= \mathfrak{C}(\widetilde{\mathcal{C}}). \end{aligned}$$

Hence we have (28). □

Example 7. We consider a 1 dimensional 2 state 3 neighborhood cellular automata CA 90. Let $l_{90}^{(3)}$ be

$$l_{90}^{(3)}(x_0, x_1, x_2) = x_0 \oplus x_2.$$

First,it seems to be able to divide cells into cells which position are even and odd. We see intuitively that this division is able if the number of cells is infinite or even. We describe this facts by using Theorem 2.

First, we suppose that the number of cells is infinite, i.e., $G = \mathbb{Z}$. Let $V = \{0, 2\}$. Therefore we choose $H = 2\mathbb{Z}$, we see that $\{0, 2\} \subseteq H$. Hence the abstract collision system $GACS(\mathbb{Z}, \mathbb{Z}, \{0, 2\})$ is dividable.

Next, we suppose that the number of cells is finite and even i.e., $G = \mathbb{Z}/(2n)\mathbb{Z}$. Similarly, we choose $H = \{2n \mid n \in G\}$. Then H is a subgroup of $\mathbb{Z}$ and we have $H \neq G$ and $\{0, 2\} \subseteq H$.

5. Composition of abstract collision systems on G-sets

In this section, we discuss about compositions of abstract collision systems.

Definition 12. Let $l : 2^V \to 2^S$. The **range** of l, which is denoted by Range l, is defined by

$$\text{Range } l = \bigcup \left\{ l(X) \mid X \in 2^V \right\} \subseteq S.$$

Let $S = G$.

Definition 13 (Composition)**.** Let $V_1 \subseteq G$, $V_2 \subseteq G$, $l_1 : 2^{V_1} \to 2^S$ and $l_2 : 2^{V_2} \to 2^S$. We define a set $V_2(l_1)$ by

$$V_2(l_1) = \begin{cases} \{v_2 \in G \mid (v_2(\text{Range } l_1)) \cap V_2 \neq \phi\}, & \\ & \text{if Range } l_1 \neq \phi \\ V_2, & \text{if Range } l_1 = \phi. \end{cases} \tag{36}$$

Moreover, we define a set $V_2(l_1) \otimes V_1$ and a function $l_2 \Diamond l_1 : 2^{V_2(l_1) \otimes V_1} \to 2^S$ by

$$V_2(l_1) \otimes V_1 = \{v_2 v_1 \mid v_2 \in V_2(l_1), v_1 \in V_1\}, \tag{37}$$

$$l_2 \Diamond l_1 (X) = l_2 \left(\bigcup_{v \in V_2(l_1)} v l_1((v^{-1}X) \cap V_1) \cap V_2 \right). \tag{38}$$

Lemma 9. *The two sets in the Definition 13 satisfy*

$$V_2(l_1) \neq \phi, \quad V_2(l_1) \otimes V_1 \neq \phi.$$

Proof. We prove $V_2(l_1) \neq \phi$. If Range $l_1 = \phi$, we have $V(l_1) = V_2 \neq \phi$ from (36). Suppose that Range $l_1 \neq \phi$. For all $x \in$ Range l_1 and $y \in V_2$, let $v_2 = yx^{-1}$. Then $y = v_2 x$. Since $v_2 x \in v_2$(Range l_1) and $y \in V_2$, we have

$$y \in (v_2(\text{Range } l_1)) \cap V_2.$$

This implies

$$(v_2(\text{Range } l_1)) \cap V_2 \neq \phi.$$

Hence $v_2 \in V_2(l_1)$. Therefore we have $V_2(l_1) \neq \phi$. □

Lemma 10. *For all $v_2 \in V_2(l_1)$, we have*

$$V_1 \subset {v_2}^{-1}(V_2(l_1) \otimes V_1).$$

Especially, we have

$$({v_2}^{-1}(V_2(l_1) \otimes V_1)) \cap V_1 = V_1. \tag{39}$$

Proof. Let $v_1 \in V_1$. We have $v_1 = {v_2}^{-1}(v_2 v_1)$. Since $v_2 \in V_2(l_1)$, we have

$$(v_2 v_1) \in V_2(l_1) \otimes V_1.$$

Therefore we have

$$v_1 = (v_2)^{-1}(v_2 v_1) \in {v_2}^{-1}(V_2(l_1) \otimes V_1).$$

Hence we have

$$V_1 \subset {v_2}^{-1}(V_2(l_1) \otimes V_1)$$

□

Lemma 11. *Let $h \in G$. For all $g \in G \setminus h(V_2(l_1))$ and $X \subset V_1$, we have*

$$(h^{-1} g l_1(X)) \cap V_2 = \phi. \tag{40}$$

Proof. Suppose that Range $l_1 = \phi$. Since $l_1(X) = \phi$ for all $X \subseteq V_1$, our claim is clear. Suppose that Range $l_1 \neq \phi$. We assume that

$$(h^{-1}gl_1(X)) \cap V_2 \neq \phi. \tag{41}$$

Since $l_1(X) \subseteq$ Range l_1, we have

$$(h^{-1}gl_1(X)) \cap V_2 \subset (h^{-1}g(\text{Range } l_1)) \cap V_2.$$

Therefore we have

$$(h^{-1}g(\text{Range } l_1)) \cap V_2 \neq \phi$$

from (41). Hence we can conclude that $h^{-1}g \in V_2(l_1)$. This implies $g \in hV_2(l_1)$. This contradicts $g \in G \setminus (hV_2(l_1))$. □

Theorem 3. *Let f_{l_1}, f_{l_2} and $f_{l_2 \Diamond l_1}$ be induced local transition functions by V_1 and l_1, V_2 and l_2, $V_2(l_1) \otimes V_1$ and $l_2 \Diamond l_1$, respectively, i.e.,*

$$\begin{aligned} f_{l_1} &= \mathrm{Ind}(G, V_1, l_1), \\ f_{l_2} &= \mathrm{Ind}(G, V_2, l_2), \\ f_{l_2 \Diamond l_1} &= \mathrm{Ind}(G, V_2(l_1) \otimes V_1, l_2 \Diamond l_1). \end{aligned}$$

Then we have

$$f_{l_2 \Diamond l_1} = f_{l_2} \circ f_{l_1}. \tag{42}$$

Proof. First, we assume that Range $l_1 \neq \phi$. For all $X \in 2^{V_2(l_1) \otimes V_1}$, we compute $f_{l_2 \Diamond l_1}$ and $f_{l_2} \circ f_{l_1}$.

$$\begin{aligned} & f_{l_2 \Diamond l_1}(X) \\ &= \bigcup_{g_2 \in G} g_2 \, (l_2 \Diamond l_1) \left(g_2{}^{-1} X \cap (V_2(l_1) \otimes V_1)\right) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(\bigcup_{v \in V_2(l_1)} v \\ &\qquad l_1 \left(v^{-1}(g_2{}^{-1} X \cap (V_2(l_1) \otimes V_1)) \cap V_1\right) \cap V_2 \Bigg) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(\bigcup_{v \in V_2(l_1)} v \\ &\qquad l_1 \left((g_2 v)^{-1} X \cap v^{-1}(V_2(l_1) \otimes V_1) \cap V_1\right) \cap V_2 \Bigg) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(\bigcup_{v \in V_2(l_1)} g_2{}^{-1}(g_2 v) \\ &\qquad l_1 \left((g_2 v)^{-1} X \cap V_1\right) \cap V_2 \Bigg), \end{aligned} \tag{43}$$

and

$$\begin{aligned} & f_{l_2} \circ f_{l_1}(X) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \left(g_2{}^{-1} f_{l_1}(X) \cap V_2\right) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(g_2{}^{-1} \\ &\qquad \Big(\bigcup_{g_1 \in G} g_1 l_1 \left(g_1{}^{-1} X \cap V_1\right) \Big) \cap V_2 \Bigg) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(\bigcup_{g_1 \in G} \left(g_2{}^{-1} g_1 l_1 \left(g_1{}^{-1} X \cap V_1\right)\right) \cap V_2 \Bigg) \end{aligned} \tag{44}$$

To show that they are equal, we prove that

$$\begin{aligned} & \bigcup_{v \in V_2(l_1)} g_2{}^{-1}(g_2 v) l_1 \left((g_2 v)^{-1} X \cap V_1\right) \cap V_2 \\ &= \bigcup_{g_1 \in G} g_2{}^{-1} g_1 l_1 \left(g_1{}^{-1} X \cap V_1\right) \\ &= \bigcup_{g_1 \in g_2 V_2(l_1)} \left(g_2{}^{-1} g_1 l_1 \left(g_1{}^{-1} X \cap V_1\right)\right) \cap V_2 \\ &\quad \cup \bigcup_{g_1 \in G \setminus g_2 V_2(l_1)} \left(g_2{}^{-1} g_1 l_1 \left(g_1{}^{-1} X \cap V_1\right)\right) \cap V_2 \end{aligned} \tag{45}$$

for all $g_2 \in G$. Since we have

$$\bigcup_{g_1 \in G \setminus g_2 V_2(l_1)} \left(g_2{}^{-1} g_1 l_1 \left(g_1{}^{-1} X \cap V_1\right)\right) \cap V_2 = \phi$$

by Lemma 11, we show

$$\begin{aligned} & \bigcup_{v \in V_2(l_1)} g_2{}^{-1}(g_2 v) l_1 \left((g_2 v)^{-1} X \cap V_1\right) \cap V_2 \\ &= \bigcup_{g_1 \in g_2 V_2(l_1)} \left(g_2{}^{-1} g_1 l_1 \left(g_1{}^{-1} X \cap V_1\right)\right) \cap V_2 \end{aligned} \tag{46}$$

instead of (45).

However, $v \in V_2(l_1)$ and $g_1 \in g_2(V_2(l_2))$ is one-to-one with $g_1 = g_2 v$. Hence we have (46).

Next, we assume that Range $l_1 = \phi$. For all $X \in 2^{V_2 \otimes V_1}$, (38) becomes

$$l_2 \Diamond l_1(X) = l_2(\phi). \tag{47}$$

On the other hand, f_{l_1} satisfies $f_{l_1}(Y) = \phi$ for all $Y \in 2^S$. Therefore we have

$$f_{l_2} \circ f_{l_1}(Y) = f_{l_2}(\phi).$$

Hence for all $Y \in 2^{V_2 \otimes V_1}$, we have

$$\begin{aligned} f_{l_2 \Diamond l_1}(Y) &= \bigcup_{g \in G} g l_2 \Diamond l_1((g^{-1} Y) \cap (V_2 \otimes V_1)) \\ &= \bigcup_{g \in G} g l_2(\phi) \\ &= \bigcup_{g \in G} g l_2((g^{-1}\phi) \cap V_2) \\ &= f_{l_2}(\phi) \\ &= f_{l_2} \circ f_{l_1}(Y). \end{aligned}$$

That's our claim. □

Definition 14. Let

$$M_1 = GACS(G,G,V_1,l_1),$$
$$M_2 = GACS(G,G,V_2,l_2).$$

We define an abstract collision system $M_2 \lozenge M_1$ by

$$M_2 \lozenge M_1 = GACS(G,G,V_2(l_1)\otimes V_1, l_2 \lozenge l_1).$$

Theorem 4. *Let*

$$M_1 = GACS(G,G,V_1,l_1),$$
$$M_2 = GACS(G,G,V_2,l_2).$$

Let F_{M_1}, F_{M_2} and $F_{M_2\lozenge M_1}$ be global transition functions of M_1, M_2 and $M_2\lozenge M_1$, respectively. Then we have

$$F_{M_2\lozenge M_1}(A) = F_{M_2}\circ F_{M_1}(A)$$

for all $A \in 2^G$.

Proof. We see that

$$\begin{aligned}&F_{M_2\lozenge M_1}(A)\\ =&\bigcup_{g_2\in G} g_2\left(l_2\lozenge l_1\right)\left({g_2}^{-1}A\cap\left(V_2(l_1)\otimes V_1\right)\right),\\ &F_{M_2}\circ F_{M_1}(A)\\ =&\bigcup_{g_2\in G} g_2 l_2\Bigg({g_2}^{-1}\\ &\left(\bigcup_{g_1\in G} g_1 l_1\left({g_1}^{-1}X\cap V_1\right)\right)\cap V_2\Bigg)\end{aligned}$$

from Theorem 1. The right hand sides of these formulae are appeared in (43) and (44) in the proof of Theorem 3, and we proved they are equal. Hence we have

$$F_{M_2\lozenge M_1}(A) = F_{M_2}\circ F_{M_1}(A).$$

□

Corollary 5. *Let*

$$M_1 = GACS(G,G,V,l_1),$$
$$M_2 = GACS(G,G,V,l_2),$$
$$M_3 = GACS(G,G,V',l_3).$$

Then we have

$$M_3\lozenge M_1 \equiv M_3\lozenge M_2,$$
$$M_1\lozenge M_3 \equiv M_2\lozenge M_3$$

if $M_1 \equiv M_2$.

Proof. Let F_{M_1}, F_{M_2}, F_{M_3} $F_{M_3\lozenge M_1}$ and $F_{M_3\lozenge M_2}$ be global transition functions of M_1, M_2, M_3, $M_3\lozenge M_1$ and $M_3\lozenge M_2$, respectively. Then we have $F_{M_1} = F_{M_2}$. Therefore for all $A\in 2^S$, we have

$$\begin{aligned}&F_{M_3\lozenge M_1}(A)\\ =&F_{M_3}\circ F_{M_1}(A)\\ =&F_{M_3}\circ F_{M_2}(A)\\ =&F_{M_3\lozenge M_2}(A)\end{aligned}$$

from Theorem 4. Hence we have $M_3\lozenge M_1 \equiv M_3\lozenge M_2$. Similarly, we have $M_1\lozenge M_3 \equiv M_2\lozenge M_3$. □

Example 8. Let M_{2CA-i}, M_{2CA-j} and M_{3CA-k} be cellular automata on groups

$$M_{2CA-i} = GACS(\mathbb{Z},\mathbb{Z},\{0,1\},l_i^{(2)}),$$
$$M_{2CA-j} = GACS(\mathbb{Z},\mathbb{Z},\{0,1\},l_j^{(2)}),$$
$$M_{3CA-k} = GACS(\mathbb{Z},\mathbb{Z},\{0,1,2\},l_k^{(3)}).$$

Then we have

$$V(l_j^{(2)})\otimes V = \{0,1,2\},$$
$$l_i^{(2)}\lozenge l_j^{(2)}(x_0,x_1,x_2) = l_i^{(2)}\left(l_j^{(2)}(x_0,x_1), l_j^{(2)}(x_1,x_2)\right),$$

by Theorem 3. This means that we can construct a 3 neighborhood cellular automaton by composing two 2 neighborhood cellular automata.

The result of compositions of 2 neighborhood cellular automata are listed in Table 3. For example,

$$M_{2CA-6}\lozenge M_{2CA-6} = M_{3CA-90},$$
$$M_{2CA-8}\lozenge M_{2CA-4} = M_{3CA-0}.$$

Since

$$l_6^{(2)}(x_0,x_1) = x_0\oplus x_1,$$
$$l_{90}^{(3)}(x_0,x_1,x_2) = x_0\oplus x_2,$$

the first example shows

$$\begin{aligned}&(x_0\oplus x_1)\oplus(x_1\oplus x_2)\\ =&l_6^{(2)}\left(l_6^{(2)}(x_0,x_1), l_6^{(2)}(x_1,x_2)\right)\\ =&l_{90}^{(3)}(x_0,x_1,x_2)\\ =&x_0\oplus x_2.\end{aligned}$$

Similarly, the second example shows

$$\begin{aligned}&(x_0\wedge\neg x_1)\wedge(x_1\wedge\neg x_2)\\ =&l_8^{(2)}\left(l_4^{(2)}(x_0,x_1), l_4^{(2)}(x_1,x_2)\right)\\ =&l_0^{(3)}(x_0,x_1,x_2)\\ =&0.\end{aligned}$$

Table 3: The composition of 2 neighborhood CA, $l_i\Diamond l_j$.

$l_i \backslash l_j$	0	2	4	6	8	10	12	14
0	0	0	0	0	0	0	0	0
2	0	34	68	66	8	34	12	2
4	0	12	48	24	64	68	48	16
6	0	46	116	90	72	102	60	18
8	0	0	0	36	128	136	192	236
10	0	34	68	102	136	170	204	238
12	0	12	48	60	192	204	240	252
14	0	46	116	126	200	238	252	254

We assumed that $S=G$ in order to simplify the discussion. The following of this section, we extend the definition of composition in the case of $S\neq G$.

In Definition 13, we would like to reset $V_1, V_2 \subseteq G$ by $V_1, V_2 \subseteq S$. However, since the set S has no operation, (37) is not well-defined. We would like to define (37) by using the action of G on S. First, let $V\subseteq S$ and $H\subseteq G$, we define

$$HV = \{hv \mid h\in H, v\in V\}\,. \tag{48}$$

Next, we take $H_1, H_2 \subseteq G$. We replace V_1 and V_2 in (37) by H_1V and H_2V, respectively.

Definition 15. Let $V\subseteq S$, $H_1\subseteq G$, $H_2\subseteq G$, $l_1 : 2^{H_1V}\to 2^S$ and $l_2 : 2^{H_2V}\to 2^S$. We define a set $H_2(l_1)$ by

$$H_2(l_1) = \begin{cases} \{h\in G \mid (h(\text{Range } l_1))\cap H_2V \neq \phi\}\,, & \\ \qquad\qquad \text{if Range } l_1 \neq \phi \\ H_2, \qquad\quad \text{if Range } l_1 = \phi \end{cases} \tag{49}$$

Moreover, we define a sets $H_2(l_1)\otimes H_1$ and a function $l_2\Diamond l_1 : 2^{H_2(l_1)\otimes H_1V}\to 2^S$ by

$$\begin{aligned} &H_2(l_1)\otimes H_1\\ &=\{h_2h_1 \mid h_2\in H_2(l_1), h_1\in H_1\}\,, \end{aligned} \tag{50}$$

$$\begin{aligned} &l_2\Diamond l_1(X)\\ &=l_2\left(\bigcup_{h\in H_2(l_1)} hl_1((h^{-1}X)\cap H_1V)\cap H_2V\right) \end{aligned} \tag{51}$$

Theorem 5. *Let f_{l_1}, f_{l_2} and $f_{l_2\Diamond l_1}$ be induced local transition function by H_1V and l_1 H_2V and l_2, $(H_2(l_1)\otimes H_1)V$ and $l_2\Diamond l_1$, respectively, i.e.,*

$$\begin{aligned} f_{l_1} &= \mathrm{Ind}(G, H_1V, l_1),\\ f_{l_2} &= \mathrm{Ind}(G, H_2V, l_2),\\ f_{l_2\Diamond l_1} &= \mathrm{Ind}(G, (H_2(l_1)\otimes H_1)V, l_2\Diamond l_1). \end{aligned}$$

Then we have

$$f_{l_2\Diamond l_1} = f_{l_2}\circ f_{l_1}. \tag{52}$$

Proof. First of all, by the similar way of the proof of Lemma 10, we can easily get

$$\left(h_2{}^{-1}(H_2(l_1)\otimes H_1)V\right)\cap H_1V = H_1V \tag{53}$$

for all $h_2\in H_2(l_1)$. Moreover, we can also get

$$\left(h^{-1}gl_1(X)\right)\cap H_2V = \phi \tag{54}$$

for all $h\in G$, $g\in G\setminus h(H_2(l_1))$ and $X\subset H_1V$, instead of Lemma 11.

Suppose that Range $l_1\neq\phi$. Let $X\in 2^{H_2(l_1)\otimes H_1}$. By the similar way of the proof of Theorem 3, we can get

$$\begin{aligned} &f_{l_2\Diamond l_1}\\ &=\bigcup_{g_2\in G} g_2l_2\Bigg(\bigcup_{h\in H_2(l_1)} g_2{}^{-1}(g_2h)\\ &\qquad\qquad l_1\left((g_2h)^{-1}X\cap H_1V\right)\cap H_2V\Bigg),\\ &f_{l_2}\circ f_{l_1}(X)\\ &=\bigcup_{g_2\in G} g_2l_2\Bigg(\bigcup_{g_1\in G}\left(g_2{}^{-1}g_1l_1\left(g_1{}^{-1}X\cap H_1V\right)\right)\cap H_2V\Bigg). \end{aligned}$$

To show they are equal, we prove that

$$\begin{aligned} &\bigcup_{h\in H_2(l_1)} g_2{}^{-1}(g_2h)l_1\left((g_2h)^{-1}X\cap H_1V\right)\cap H_2V\\ &=\bigcup_{g_1\in g_2H_2(l_1)}\left(g_2{}^{-1}g_1l_1\left(g_1{}^{-1}X\cap H_1V\right)\right)\cap H_2V.\\ &\cup\bigcup_{g_1\in G\setminus g_2H_2(l_1)}\left(g_2{}^{-1}g_1l_1\left(g_1{}^{-1}X\cap H_1V\right)\right)\cap H_2V \end{aligned} \tag{55}$$

Since we have (54), we show

$$\begin{aligned} &\bigcup_{h\in H_2(l_1)} g_2{}^{-1}(g_2h)l_1\left((g_2h)^{-1}X\cap H_1V\right)\cap H_2V\\ &=\bigcup_{g_1\in g_2H_2(l_1)}\left(g_2{}^{-1}g_1l_1\left(g_1{}^{-1}X\cap H_1V\right)\right)\cap H_2V \end{aligned} \tag{56}$$

instead of (55). However, it is very easy to show (56).

In the case of Range $l_1=\phi$, we can easily prove with the similar way of the proof of Theorem 3. ☐

Example 9. Let $H=\{0,1\}$ Let M_1 and M_2 be 1 dimensional Q state, H neighborhood cellular automata, defined by Example 4. Then we have

$$l_2\Diamond l_1(x_0,x_1,x_2) = l_2\left(l_1(x_0,x_1), l_1(x_1,x_2)\right) \tag{57}$$

by composing M_1 and M_2.

6. Distributive Law

In this section, we consider that two operations, union and composition of ACSs on G-sets, and check the distributive law. We consider the most easy case, cellular automata on groups.

Example 10. Let M_{2CA-i} and M_{3CA-j} be cellular automata on groups

$$M_{2CA-i} = GACS(\mathbb{Z}, \mathbb{Z}, \{0,1\}, l_i^{(2)}),$$
$$M_{3CA-j} = GACS(\mathbb{Z}, \mathbb{Z}, \{0,1,2\}, l_j^{(3)}),$$

respectively. From Table 2 and Table 3, we have $M_{2CA-2} \cup M_{2CA-4} = M_{2CA-6}$ and $M_{2CA-6} \Diamond M_{2CA-6} = M_{3CA-90}$. Moreover, we have

$$\begin{aligned} M_{2CA-6} \Diamond M_{2CA-2} &= M_{3CA-46}, \\ M_{2CA-6} \Diamond M_{2CA-4} &= M_{3CA-116}, \\ M_{2CA-2} \Diamond M_{2CA-6} &= M_{3CA-66}, \\ M_{2CA-4} \Diamond M_{2CA-6} &= M_{3CA-24} \end{aligned}$$

from Table 3. Furthermore, we can compute easily

$$\begin{aligned} M_{3CA-46} \cup M_{3CA-116} &= M_{3CA-126}, \\ M_{3CA-66} \cup M_{3CA-24} &= M_{3CA-90}. \end{aligned}$$

Therefore we see that

$$\begin{aligned} & M_{2CA-6} \Diamond (M_{2CA-2} \cup M_{2CA-4}) \\ =& (M_{2CA-2} \cup M_{2CA-4}) \Diamond M_{2CA-6} \\ =& M_{2CA-6} \Diamond M_{2CA-6} \\ =& M_{3CA-90}, \\ & (M_{2CA-6} \Diamond M_{2CA-2}) \cup (M_{2CA-6} \Diamond M_{2CA-4}) \\ =& M_{3CA-46} \cup M_{3CA-116} \\ =& M_{3CA-126}, \\ & (M_{2CA-2} \Diamond M_{2CA-6}) \cup (M_{2CA-4} \Diamond M_{2CA-6}) \\ =& M_{3CA-66} \cup M_{3CA-24} \\ =& M_{3CA-90}. \end{aligned}$$

Hence we have

$$\begin{aligned} & M_{2CA-6} \Diamond (M_{2CA-2} \cup M_{2CA-4}) \\ \neq & (M_{2CA-6} \Diamond M_{2CA-2}) \cup (M_{2CA-6} \Diamond M_{2CA-4}), \\ & (M_{2CA-2} \cup M_{2CA-4}) \Diamond M_{2CA-6} \\ = & (M_{2CA-2} \Diamond M_{2CA-6}) \cup (M_{2CA-4} \Diamond M_{2CA-6}). \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} & (M_{2CA-j} \cup M_{2CA-k}) \Diamond M_{2CA-i} \\ = & (M_{2CA-j} \Diamond M_{2CA-i}) \cup (M_{2CA-k} \Diamond M_{2CA-i}) \end{aligned}$$

for all rule number i, j and k. However the equation

$$\begin{aligned} & M_{2CA-k} \Diamond (M_{2CA-i} \cup M_{2CA-j}) \\ = & (M_{2CA-k} \Diamond M_{2CA-i}) \cup (M_{2CA-k} \Diamond M_{2CA-j}) \end{aligned}$$

is not always hold for rule number i, j and k.

Theorem 6. *Let*

$$\begin{aligned} M_1 &= GACS(G, G, V, l_1), \\ M_2 &= GACS(G, G, V, l_2), \\ M_3 &= GACS(G, G, V', l_3). \end{aligned}$$

Then we have

$$(M_1 \cup M_2) \Diamond M_3 = (M_1 \Diamond M_3) \cup (M_2 \Diamond M_3). \tag{58}$$

Proof. We see that

$$\begin{aligned} & (M_1 \cup M_2) \Diamond M_3 \\ \equiv\ & GACS(G, G, V, l_1 \cup l_2) \Diamond GACS(G, G, V_3, l_3) \\ & \qquad \text{(by Cor. 5)} \\ \equiv\ & GACS(G, G, V(l_3) \otimes V_3, (l_1 \cup l_2) \Diamond l_3) \quad \text{(by Def. 14),} \\ & (M_1 \Diamond M_3) \cup (M_2 \Diamond M_3) \\ \equiv\ & GACS(G, G, V_3(l_3) \otimes V, l_1 \Diamond l_3) \\ \cup\ & GACS(G, G, V_3(l_3) \otimes V, l_2 \Diamond l_3) \qquad \text{(by Cor. 4)} \\ \equiv\ & GACS(G, G, V_3(l_3) \otimes V, (l_1 \Diamond l_3) \cup (l_2 \Diamond l_3)) \\ & \qquad \text{(by Prop. 8).} \end{aligned}$$

Moreover, we see that

$$\begin{aligned} & (l_1 \cup l_2) \Diamond l_3 (X) \\ = & (l_1 \cup l_2) \left(\bigcup_{v \in V(l_3)} v l_3 \left(v^{-1} X \cap V_3 \right) \cap V \right) \\ = & l_1 \left(\bigcup_{v \in V(l_3)} v l_3 \left(v^{-1} X \cap V_3 \right) \cap V \right) \\ \cup & l_2 \left(\bigcup_{v \in V(l_3)} v l_3 \left(v^{-1} X \cap V_3 \right) \cap V \right) \\ = & l_1 \Diamond l_3 (X) \cup l_2 \Diamond l_3 (X) \end{aligned}$$

for all $X \in 2^{V(l_3) \otimes V_3}$. Hence we have (58). $\square$

This theorem says that the operation $\diamond$ is **right-distributive** over $\cup$, but $\diamond$ is **not left-distributive** over $\cup$.

7. Conclusion

We introduced abstract collision systems on G-sets, and investigated their properties. First, we proved that if $l(\phi) = \phi$, the global function does not depend on the set of collisions $\mathcal{C}$.

Next, we defined operations "union" and "division" of ACS. We determined a sufficient condition that an ACS on a G-set is dividable. Finally, by using actions of groups, we introduced the new concept "composition" of ACS on a G-set. We proved the global transition function of the composed ACS is the usual composition of global transition functions of two ACSs. We proved that "composition" is right-distributive over "union", but is not left-distributive.

The union of cellular automata on groups is corresponding the cellular automaton with a local transition rule defined by the "logical or" of given local transition rules. The composition of cellular automata with ACS is an extension of the composition of local transition rules of cellular automata in [4]. We enumerated all 3 neighborhood CAs defined by the composition of two 2 neighborhood CAs.

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