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# Weakly nonlinear saturation of stationary resonance of a rotating flow in an elliptic cylinder

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**Abstract.** We address weakly nonlinear stability of a uniformly rotating flow confined in a cylinder of elliptic cross-section to three-dimensional disturbances. A Lagrangian approach is developed to derive unambiguously the drift current induced by nonlinear interaction of isovortical disturbances. This approach rescues the insufficiency inherent in the Eulerian approach and provides a direct path to reach the amplitude equations in the Hamiltonian normal form. The nonlinear effect saturates the stationary instability mode, and asymptotic form of its saturation amplitude is gained, in a tidy form, in the short-wavelength regime.

*Keywords.* elliptical instability, weakly nonlinear stability, Lagrangian approach, mean flow

## 1. INTRODUCTION

It is well known that flows with elliptic streamlines suffer from three-dimensional (3D) instability called the elliptical instability [2, 24, 11]. This is the short-wave limit of the Moore-Saffman-Tsai-Widnall (MSTW) instability [18, 21, 4, 5]. It is often instability rather than stability of vortices and rotating flows that is useful in engineering designs as listed by trailing vortices of an aircraft wing and mixing in gasoline engines. Imposing straining field holds a key to the flow control by breaking up otherwise robust vortices.

The MSTW instability is typically a parametric resonance, driven by the imposed shear, between left- and right-handed helical waves. The waves on a circular cylindrical vortex tube are called the Kelvin waves or the inertial waves. In general, a vortex tube with elliptic core goes through a parametric resonance when two Kelvin waves with difference in azimuthal wave numbers  $m$  being 2, having common axial wavenumber and frequency  $(k, \omega)$ , are simultaneously excited. The  $(m, m + 2) = (1, 3)$  and  $(0, 2)$  resonances were detected in a confined geometry [3, 11]. The stability of a uniformly rotating flow is solvable in the sense that the eigen-values and functions are written out in full in terms of the Bessel functions [5, 6]. The short-wave stability analysis spotlights local character, and hence its formulation and result carry over to bounded flows. Malkus [14] created a rotating flow with strained streamlines in a water-filled flexible cylinder pressed by two stationary rollers (see also [3]). His experiment showed that the MSTW modes grow, followed by excitation of a number of waves possibly via secondary and tertiary instabilities and then by eventual disruption. A knowledge of nonlinear growth of linearly unstable modes is indispensable for de-

scribing a possible route to the collapse of a rotating flow.

An account for the linear instability was given for a general columnar vortex embedded in a strain field by Moore and Saffman [18], for two stationary Kelvin waves of the same axial wavenumber  $k$  and azimuthal wavenumber  $m = 1$  and  $-1$ , and a detailed analysis was made for the Rankine vortex [21, 4, 5]. Fukumoto [5] showed on the ground of the Hamiltonian spectral theory that all the intersection points of dispersion curves of the Kelvin waves with  $m$  and  $n + 2$  result in instability.

Nonlinear effect comes into play at a matured stage of exponential growth of disturbance amplitude and modifies evolution of the MSTW instability. Waleffe [23] and Sipp [20] showed that the weakly nonlinear effect acts to saturate the amplitude of the Kelvin waves. Mason and Kerswell [15] calculated the secondary instability of the MSTW instability, but they also disregarded the mean flow. We shall show that their procedure is incomplete in the sense that they did not determine, to the full detail, the mean flow induced by nonlinear interactions of the Kelvin waves. Rodrigues and Luca [19] dealt with the case where mean flow is absent, and found chaotic orbits.

The Euler equations admit arbitrary azimuthal velocity profile of a circular vortex, and in conjunction this, are not capable of uniquely determining the wave-induced mean flow by themselves. The Lagrangian displacement field is instrumental in handling interaction of waves [1, 12, 8, 9]. Fukumoto and Hirota [7] developed the Lagrangian approach to derive the mean flow.

The purpose of this paper is to amend the previous Eulerian treatment and thereby to manipulate the amplitude equations for weakly nonlinear evolution of the MSTW instability. We limit ourselves to the stationary resonance of

left- and right-handed helical waves.

In §2, we give the basic flow and the formulation of the problems. We recollect the Kelvin waves in §3, and the MSTW instability, or the  $(m, m+2)$  parametric resonance, in §4. In §5, we inquire into the mean flow induced by nonlinear interactions of Kelvin waves. The Lagrangian approach [8, 9, 7] allows us to give the mean flow solely in terms of the Lagrangian displacement of first order in amplitude. We rest on this approach to deduce weakly nonlinear amplitude equations in §6. In Appendix C, a comparison is made between the Eulerian and the Lagrangian approaches.

## 2. STABILITY PROBLEM OF ROTATING FLOW IN ELLIPTIC CYLINDER

Malkus [14] found collapse of rotating flow in an elliptic cylinder. We write elliptic cross-section as

$$(1) \quad \frac{x^2}{1+\epsilon} + \frac{y^2}{1-\epsilon} = 1,$$

where the parameter  $\epsilon$  means the elliptic distortion. Let us introduce cylindrical coordinates  $(r, \theta, z)$  with the  $z$ -axis along the centerline. Let the  $r$  and  $\theta$  components of 2D basic velocity field be  $U$  and  $V$ , and the pressure  $P$ . Assuming that the fluid is inviscid and incompressible, a planar perturbation solution of the Euler equations are written as

$$(2) \quad \begin{aligned} U &= \epsilon U_1(r, \theta), & V &= V_0(r) + \epsilon V_1(r, \theta), \\ P &= P_0(r) + \epsilon P_1(r, \theta). \end{aligned}$$

The subscript designates order in elliptic parameter  $\epsilon$ . The leading-order term of the basic rotating flow  $\mathbf{U}_0$  is a rigidly rotating flow.

$$(3) \quad V_0 = r, \quad P_0 = \frac{r^2}{2} - 1.$$

The first order perturbation is a quadrupole field, as given by

$$(4) \quad U_1 = -r \sin 2\theta, \quad V_1 = -r \cos 2\theta, \quad P_1 = 0,$$

representing the elliptic strain. This velocity field is corresponding to the field consisting of strain field whose stretching direction lies along  $\theta = -\pi/4$  and whose direction of contraction is along  $\theta = \pi/4$ . We add disturbed flow  $\tilde{\mathbf{u}}$  to 2D basic flow.

The disturbance field  $\tilde{\mathbf{u}}$  satisfies the Euler equations,

$$(5) \quad \frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\mathbf{U} \cdot \nabla) \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{U} + \nabla \tilde{p} = \mathbf{0},$$

where  $\tilde{p}$  is the disturbance pressure field and the disturbance filling the elliptic cylinder is incompressible,

$$(6) \quad \nabla \cdot \tilde{\mathbf{u}} = 0.$$

Following Sipp [20], we consider asymptotic expansions of the velocity field in two small parameters  $\epsilon$  and  $\alpha$  as

$$(7) \quad \begin{aligned} \mathbf{u} &= \mathbf{U} + \tilde{\mathbf{u}} \\ &= \mathbf{U}_0 + \epsilon \mathbf{U}_1 + \alpha \mathbf{u}_{01} + \alpha^2 \mathbf{u}_{02} + \alpha^3 \mathbf{u}_{03} \\ &\quad + \epsilon \alpha \mathbf{u}_{11} + \epsilon \alpha^2 \mathbf{u}_{12} + \dots, \end{aligned}$$

where  $\alpha$  signifies the measure of the amplitude of the linearized disturbance. This expansion as well as that for the pressure field  $p$  are inserted into the incompressible Euler equations (5). Here, the velocity field  $\mathbf{u}_{mn}$  occurs at  $O(\epsilon^m \alpha^n)$ .

The side wall (1) of the cylinder is  $r = 1 + \epsilon \cos 2\theta/2 + O(\epsilon^2)$  when the elliptic strain  $\epsilon$  is small. The boundary condition is to be imposed at the rigid side wall is

$$(8) \quad \mathbf{u} \cdot \mathbf{n} = 0, \quad \text{at } r = 1 + \epsilon \cos 2\theta/2,$$

where  $\mathbf{n}$  is the outward normal vector to the side wall boundary.

At every order,  $O(\alpha), O(\epsilon\alpha), O(\alpha^2), \dots$ , by integrating inhomogeneous linear differential equations whose forcing terms consist of the quantities of lower orders and the boundary condition (8), we obtain  $\mathbf{u}_{ij}$  and  $p_{ij}$ .

## 3. KELVIN WAVE

We briefly recall the Kelvin waves, linearized disturbances of  $O(\epsilon^0 \alpha)$  on a circular core of a vortex. A Kelvin wave with azimuthal wavenumber  $m$  and axial wavenumber  $k$  is a normal mode of the form

$$(9) \quad \mathbf{u}_{01}^{(m)} = A_m(t) \mathbf{u}_{01}^{(m)}(r) e^{im\theta} e^{ikz}. \quad A_m(t) \propto e^{-i\omega_0 t},$$

where  $A_m$  is a complex function of time  $t$ ,  $\mathbf{u}_{01}^{(m)}$  is a radial structure function and  $\omega_0$  is frequency. The radial function obeys

$$(10) \quad \mathcal{L}_{m,k} \mathbf{u}_{01}^{(m)} + \nabla p_{01}^{(m)} = \mathbf{0}, \quad \nabla \cdot \mathbf{u}_{01}^{(m)} = 0,$$

where

$$(11) \quad \mathcal{L}_{m,k} = \begin{pmatrix} -i(\omega_0 - m) & -2 & 0 \\ 2 & -i(\omega_0 - m) & 0 \\ 0 & 0 & -i(\omega_0 - m) \end{pmatrix}.$$

The boundary condition reads

$$(12) \quad \mathbf{u}_{01} \cdot \mathbf{n} = 0 \quad \text{on } r = 1.$$

Introducing (9) into (10), we obtain a solution of the linearized equation as

$$(13) \quad \begin{aligned} p_{01}^{(m)} &= J_m(\eta_m r), \\ \mathbf{u}_{01}^{(m)} &= \frac{i}{\omega_0 - m + 2} \left\{ -\frac{m}{r} J_m(\eta_m r) \right. \\ &\quad \left. + \frac{\omega_0 - m}{\omega_0 - m - 2} \eta_m J_{m+1}(\eta_m r) \right\}, \\ v_{01}^{(m)} &= \frac{1}{\omega_0 - m + 2} \left\{ \frac{m}{r} J_m(\eta_m r) \right. \\ &\quad \left. + \frac{2\eta_m}{\omega_0 - m - 2} J_{m+1}(\eta_m r) \right\}, \\ w_{01}^{(m)} &= \frac{k}{\omega_0 - m} J_m(\eta_m r), \end{aligned}$$

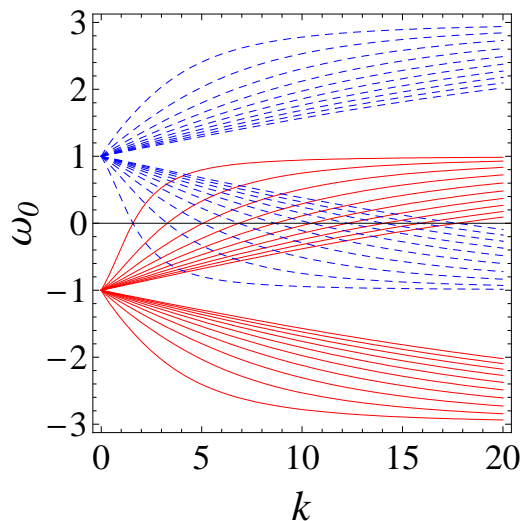


Figure 1: dispersion relation of Kelvin waves, of  $m = -1$  (red solid lines) and of  $m = +1$  (blue dashed lines), in an elliptic cylinder.

where  $\eta_m$  is the radial wavenumber

$$(14) \quad \eta_m^2 = \left[ \frac{4}{(\omega_0 - m)^2} - 1 \right] k^2,$$

and  $J_m$  is the  $m$ -th Bessel function of the first kind. The boundary condition (12) requires the dispersion relation

$$(15) \quad J_{m+1}(\eta_m) = \frac{(\omega_0 - m - 2)m}{(\omega_0 - m)\eta_m} J_m(\eta_m).$$

Figure 1 displays the dispersion relation of bending waves  $m = \pm 1$ . Curves for  $m = -1$  are drawn with solid red lines, while those for  $m = +1$  are drawn with dashed blue lines. Its eigenfunction has a simple radial structure. Infinitely many branches emanate from  $(k, \omega_0) = (0, 1)$  for  $m = 1$  and from  $(k, \omega_0) = (0, -1)$  for  $m = -1$ , among which twenty branches for each, ten upward and ten downward, are displayed. Note that the dispersion relation we consider Figure 1 has no isolated mode, unlike the case of the Rankine vortex [5, 21].

#### 4. MOORE-SAFFMAN-TSAI-WIDNALL INSTABILITY

We explore the effect of elliptic strain (4) at  $O(\epsilon)$  [17, 21, 5]. If the given disturbance flow, in the absence of elliptic strain, has Kelvin waves with  $e^{im\theta}$  and  $e^{i(m+2)\theta}$ , at  $O(\epsilon\alpha)$ , interaction of these waves with the strain (4), through the convective terms of the Euler equations (5) excite again Kelvin waves with  $e^{im\theta}$  and  $e^{i(m+2)\theta}$  [21]. This coincidence indicates occurrence of parametric instability. Fukumoto[5] made a thorough analysis of the 3D instability of the Rankine vortex embedded in a plane shear field and showed that the parameter resonance at  $O(\alpha)$  instability occurs at all

intersection points  $(k, \omega)$  of dispersion curves of a combination of Kelvin waves  $m$  and  $m + 2$ . In the following, we show that this is the case with the rotating flow confined in an elliptic cylinder.

##### 4.1. PARAMETRIC RESONANCE IN ELLIPTIC CYLINDER

Envisaging the parametric resonance of a Hamiltonian system, we start, to  $O(\epsilon^0\alpha)$ , with a superposition of the  $m$  and the  $m + 2$  waves:

$$(16) \quad \mathbf{u}_{01} = A_m(t)\mathbf{u}_{01}^{(m)}(r)e^{im\theta}e^{ikz} + A_{m+2}(t)\mathbf{u}_{01}^{(m+2)}(r)e^{i(m+2)\theta}e^{ikz} + c.c.,$$

in which  $c.c.$  designates the complex conjugate.

We then consider the behavior at  $O(\epsilon\alpha)$ , the influence of the elliptic strain  $\epsilon$ . Excited at  $O(\epsilon\alpha)$  by interaction of the straining field of the basic flow is

$$(17) \quad \mathbf{u}_{11} = B_m\mathbf{u}_{11}^{(m-2)}e^{i(m-2)\theta}e^{ikz} + B_{m+2}\mathbf{u}_{11}^{(m)}e^{im\theta}e^{ikz} + B_m\mathbf{u}_{11}^{(m+2)}e^{i(m+2)\theta}e^{ikz} + B_{m+2}\mathbf{u}_{11}^{(m+4)}e^{i(m+4)\theta}e^{ikz} + c.c.,$$

and similarly for  $p_{11}$ , the pressure at  $O(\epsilon\alpha)$ . The radial functions  $\mathbf{u}_{11}^{(m)}(r)$  and  $\mathbf{u}_{11}^{(m+2)}(r)$  are determined by inhomogeneous linear ordinary differential equations, derived from the Euler equations supplemented by the continuity equations,

$$(18) \quad \begin{aligned} & \mathcal{L}_{m,k}\mathbf{u}_{11}^{(m)} + \nabla p_{11}^{(m)} \\ &= \frac{1}{2} \left( \begin{array}{l} ir \frac{\partial \mathbf{u}_{01}^{(m+2)}}{\partial r} + i(m+3)\mathbf{u}_{01}^{(m+2)} \\ 2\mathbf{u}_{01}^{(m+2)} + ir \frac{\partial \mathbf{v}_{01}^{(m+2)}}{\partial r} + i(m+1)\mathbf{v}_{01}^{(m+2)} \\ ir \frac{\partial \mathbf{w}_{01}^{(m+2)}}{\partial r} + i(m+2)\mathbf{w}_{01}^{(m+2)} \end{array} \right) \\ & - \frac{1}{A_{m+2}} \frac{\partial A_m}{\partial t_{10}} \mathbf{u}_{01}^{(m)}, \\ & \nabla \cdot \mathbf{u}_{01}^{(m)} = 0, \end{aligned}$$

and similarly for the  $m + 2$  wave, where  $t_{10} = \epsilon t$ .

The solution of these inhomogeneous linear equations is represented explicitly. The boundary condition (8) at  $O(\epsilon\alpha)$ ,

$$(19) \quad u_{11} - u_{01} \cos 2\theta/2 + v_{01} \sin 2\theta = 0$$

provides equations governing  $B_m$  and  $B_{m+2}$ . Imposition of the solvability condition on these equations gives rise to the growth rate of the  $(m, m + 2)$  resonance.

##### 4.2. PARAMETRIC RESONANCE OF HELICAL WAVES

We focus our attention on the case  $(m, m + 2) = (-1, +1)$ . There are intersection points on the  $k$ -axis ( $\omega_0 = 0$ ) at certain values of  $k$ . For left and right-handed helical wave resonance, the stationary mode  $\omega_0 = 0$  have far greater growth rate than non-stationary modes ( $\omega_0 \neq 0$ ) [21, 4, 5]. We confine ourselves to the stationary mode.

Under the restriction of  $\omega_0 = 0$ , the radial wavenumber (14) reads  $\eta = \sqrt{3}k$ . Then, we represent the disturbance velocity  $\mathbf{u}_{01}$  as

$$(20) \quad \mathbf{u}_{01} = A_- \mathbf{u}_- e^{-i\theta} e^{ikz} + A_+ \mathbf{u}_+ e^{i\theta} e^{ikz} + c.c.$$

Here we use the notation  $A_{\pm}$  in place of  $A_{\pm 1}$ . The boundary condition (19) provide algebraic equations for  $B_{-1}$  and  $B_{+1}$  and the solvability condition gives rise to, with the help of the dispersion relation (15),

$$(21) \quad \frac{1}{A_+} \frac{\partial A_-}{\partial t_{10}} = \frac{-1}{A_-} \frac{\partial A_+}{\partial t_{10}} = i \frac{3(3k^2 + 1)}{8(2k^2 + 1)},$$

where  $k$  is the solution of dispersion relation  $J_1(\eta) = -\eta J_0(\eta)$ . Note that  $k = 0$  is excluded as is read off from Figure 1. In the case of an elliptic vortex in an unbounded flow field, Kelvin waves of  $m = +1$  and  $-1$  have an intersection point at  $(k_0, \omega_0) = (0, 0)$ , and its growth rate is 0.5 [5]. By contrast, in the case of a rotating flow confined in an elliptic cylinder, Kelvin waves of  $m = +1$  and  $-1$  do not have the intersection point at  $(k, \omega) = (0, 0)$ .

The degenerate modes with  $\omega_0 = 0$  necessarily result in parametric resonance whose growth rate is given by

$$(22) \quad \frac{3(3k^2 + 1)}{8(2k^2 + 1)}.$$

Vladimirov *et al.* [22], where the ratio of the amplitude is

$$(23) \quad A_- / A_+ = i.$$

Numerical values of the growth rate for a first few intersection points with  $\omega_0 = 0$  are listed as follows:  $(k, \sigma) = (1.578, 0.5311), (3.286, 0.5542), (5.061, 0.5589), \dots$

When wavenumber  $k$  is very large, the growth rate  $\sigma$  of stationary mode is close to 9/16 [5]. The reason for the predominance of the stationary resonance is that the disturbance vorticity is liable to be original with the stretching direction of the external straining field.

At  $O(\alpha^3)$ , the modes  $e^{im\theta} e^{ikz}$  and  $e^{i(m+2)\theta} e^{ikz}$  again arise, which invites the compatibility conditions. The function  $\mathbf{u}_{03}^{(m)}$  and  $\mathbf{u}_{03}^{(m+2)}$  are governed by

$$(24) \quad \mathcal{L}_{m,k} \begin{pmatrix} \mathbf{u}_{03}^{(m)} \\ p_{03}^{(m)} \end{pmatrix} = \mathcal{N} - \frac{\partial}{\partial t_{02}} \begin{pmatrix} \mathbf{u}_{01}^{(m)} \\ 0 \end{pmatrix},$$

where  $t_{02} = \alpha^2 t$  and  $\mathcal{N}$  is the inhomogeneous. Since the matrix  $\mathcal{L}_{m,k}$  is singular,  $(\partial/\partial t_{02})\mathbf{u}_{01}$  is adjusted for the forcing terms to satisfy the solvability condition. The occurrence of parametric resonance implies the existence of negative-energy waves. This is indeed the case [5].

The energy of Kelvin waves has been efficiently calculated with use of the Lagrangian displacement [8, 9]. As a by-product, the mean flow of  $O(\alpha^2)$  induced by self-interaction of a Kelvin wave is manipulated in a systematic manner [7]. Our procedure does not need elliptic strain for solving the mean flow. On the contrary, the Eulerian treatment necessitates the elliptic strain, and its coefficient is not completely determined [20]. In the following section, we demonstrate the advantage of the Lagrangian approach.

## 5. DRIFT CURRENT

### 5.1. MEAN FLOW OF $O(\alpha^2)$ BY EULERIAN APPROACH

For the disturbance velocity  $\mathbf{u}_{02}$  of  $O(\alpha^2)$ , we may take a separable solution of the form

$$(25) \quad \begin{aligned} \mathbf{u}_{02} = & A_m^2 \mathbf{u}_{02}^{(2m, 2k)} e^{i2m\theta} e^{i2kz} + A_{m+2}^2 \mathbf{u}_{02}^{(2m+4, 2k)} e^{i(2m+4)\theta} e^{i2kz} \\ & + A_m A_{m+2} \mathbf{u}_{02}^{(2m+2, 2k)} e^{i(2m+2)\theta} e^{i2kz} + A_{m+2} \overline{A_m} \mathbf{u}_{02}^{(2, 0)} e^{i2\theta} \\ & + c.c. + \mathbf{u}_{02}^{(0, 0)}, \end{aligned}$$

where  $\mathbf{u}_{02}^{(0, 0)}$  is the mean flow, being independent of  $\theta$  and  $z$ . In order to obtain  $\mathbf{u}_{02}^{(0, 0)}$ , we have to solve the following inhomogeneous linear equation,

$$(26) \quad \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} u_{02}^{(0, 0)} \\ v_{02}^{(0, 0)} \end{pmatrix} + \begin{pmatrix} \partial_r p_{02}^{(0, 0)} \\ 0 \end{pmatrix} = \begin{pmatrix} f(r) \\ 0 \end{pmatrix},$$

$$(27) \quad \partial_r u_{02}^{(0, 0)} + u_{02}^{(0, 0)} / r = 0.$$

The inhomogeneous term introduced in (26) is defined in appendix. At the first glance, the radial component of mean flow is founded to be zero:  $u_{02}^{(0, 0)} = 0$ . But the azimuthal and the axial components,  $v_{02}^{(0, 0)}$  and  $w_{02}^{(0, 0)}$  remain undetermined because  $p_{02}^{(0, 0)}$  may taken to be arbitrary. Entering into a higher order in elliptic strain  $\epsilon$ , i.e. at  $O(\epsilon\alpha^2)$ , Sipp [20] deduced

$$(28) \quad \begin{aligned} \mathcal{L}_{0,0} \mathbf{u}_{12}^{(0, 0)} + \nabla p_{12}^{(0, 0)} = \\ \begin{pmatrix} (A_m \overline{A_{m+2}} + \overline{A_m} A_{m+2}) f_r(r) \\ i (A_m \overline{A_{m+2}} - \overline{A_m} A_{m+2}) f_\theta(r) \\ i (A_m \overline{A_{m+2}} - \overline{A_m} A_{m+2}) f_z(r) \end{pmatrix} - \frac{\partial}{\partial t_{10}} \mathbf{u}_{02}^{(0, 0)}. \end{aligned}$$

Enforcement of the compatibility condition on (28) yields time derivative of the mean flow  $\mathbf{u}_{02}^{(0, 0)}$ , where  $t_{10} = \epsilon t$  is to be remembered.

The compatibility conditions of this equation read that  $e_\theta$  and  $e_z$  components of forcing terms should be zero, i.e.

$$(29) \quad i (A_m \overline{A_{m+2}} - \overline{A_m} A_{m+2}) \begin{pmatrix} f_\theta(r) \\ f_z(r) \end{pmatrix} = \frac{\partial}{\partial t_{10}} \begin{pmatrix} v_{02}^{(0, 0)} \\ w_{02}^{(0, 0)} \end{pmatrix}.$$

The mean flow is suggested to be  $\mathbf{u}_{02}^{(0, 0)}(t, r) = C(t) \mathbf{u}_C(r)$ , with  $\mathbf{u}_C(r) = (0, f_\theta(r), f_z(r))^T$  and obtained time derivative of its amplitude as

$$(30) \quad \frac{dC}{dt} = \epsilon i (A_m \overline{A_{m+2}} - \overline{A_m} A_{m+2}),$$

[20]. If the amplitude  $C(t)$  of the mean flow can be integrated, subject to the initial condition  $C(0)$ , the mean flow is determined. But the initial value of mean flow  $C(0)$  cannot be given freely, because the mean flow  $\mathbf{u}_{02}^{(0, 0)}(t, r)$  is induced by nonlinear interaction of Kelvin waves,  $\mathbf{u}_{01}$ . Since the linear operator of Euler equation (26) is degenerate, information of dependence on  $\mathbf{u}_{01}$  is vanished.

The Lagrangian approach enables us to obtain the mean flow of  $O(\alpha^2)$ , without having to proceed to higher order,  $O(\epsilon\alpha^2)$  [7]. Subsequently, We give a sketch of this Lagrangian approach.

## 5.2. LAGRANGIAN APPROACH

Arnold [1] showed that a steady state of the Euler flow is characterized as an extremal of the energy functional with respect to isovortical disturbances. By virtue of this structure, the second variation of energy and the mean flow induced by nonlinear interaction of Kelvin waves is expressible solely in terms of the first order Lagrangian displacement field [8, 7].

First, we give a concise description of the Lagrangian representation of the first and second-order the velocity disturbance. Next, we calculate the energy and the mean flow to second order in disturbance amplitude.

## 5.3. DISTURBANCE VELOCITY FIELD

Let  $S\text{Diff}(\mathcal{D})$  be the group of the volume-preserving diffeomorphisms of the fluid contained in a domain  $\mathcal{D}$ . Motion of fluid particles inside  $\mathcal{D}$  is expressed by a one-parameter family of the fluid flow map, elements of  $S\text{Diff}(\mathcal{D})$  [1]. We assume that disturbance deforms basic the particle position by following map.

$$(31) \quad \psi(\mathbf{x}, \alpha) = \mathbf{x} + \alpha \boldsymbol{\xi} + \frac{\alpha^2}{2} [(\boldsymbol{\xi} \cdot \nabla) \boldsymbol{\xi} + \boldsymbol{\eta}] + O(\alpha^3).$$

Here  $\boldsymbol{\xi}$  and  $\boldsymbol{\eta}$  are the Lagrangian displacement of  $O(\alpha)$  and  $O(\alpha^2)$ , respectively. Writing down the disturbance velocity to second order explicitly,

$$(32) \quad \begin{aligned} \mathbf{u}_{01} &= \mathcal{P} [\boldsymbol{\xi} \times \boldsymbol{\omega}_0], \\ \mathbf{u}_{02} &= \mathcal{P} [\boldsymbol{\xi} \times (\nabla \times (\boldsymbol{\xi} \times \boldsymbol{\omega}_0)) + \boldsymbol{\eta} \times \boldsymbol{\omega}_0] / 2, \end{aligned}$$

where  $\mathcal{P}$  is the projection operator keeping the velocity field divergence-free [7]. The previous treatment, for example [12], forgot the second-order disturbance  $\boldsymbol{\eta}$ , and is incomplete.

The advantage of the Lagrangian approach is manifested in evaluation of wave energy. The kinetic energy  $H$  is

$$(33) \quad H = \frac{1}{2} \int \|\mathbf{u}\|^2 d^3x,$$

We can expand the kinetic energy (33) as follows,

$$(34) \quad H = H_0 + \alpha \delta H + \frac{\alpha^2}{2} \delta^2 H + O(\alpha^3).$$

$H_0$  is constant because of steady flow  $\mathbf{u}_0$ . The first variation of the kinetic energy vanishes identically:  $\delta H = 0$  [1]. The second variation of kinetic energy is reducible to

$$(35) \quad \begin{aligned} \delta^2 H &= \int (\|\mathbf{u}_{01}\|^2 + 2\mathbf{u}_0 \cdot \mathbf{u}_{02}) d^3x \\ &= \int \mathcal{P} [\boldsymbol{\xi} \times \boldsymbol{\omega}_0] \cdot \partial_t \boldsymbol{\xi} d^3x. \end{aligned}$$

Fortunately, the second-order Lagrangian displacement  $\boldsymbol{\eta}$  is excluded from (35). The advantage of the Lagrangian approach cannot be overemphasized. This disperses with the second-order Lagrangian displacement, implying that we can obtain the second variation of kinetic energy if we solve linear equation to  $O(\alpha)$ .

The second of (32) reveals that the mean flow  $\overline{\delta^2 \mathbf{u}}$  satisfies the following equation

$$(36) \quad \overline{\delta^2 \mathbf{u}} = \overline{\mathcal{P} [\boldsymbol{\xi} \times \delta \boldsymbol{\omega}]} = \overline{\mathcal{P} [\boldsymbol{\xi} \times (\nabla \times (\boldsymbol{\xi} \times \boldsymbol{\omega}_0))]}.$$

In next subsection, we calculate the Lagrangian displacement  $\boldsymbol{\xi}$ , and thereby deduce the mean flow.

## 5.4. LAGRANGIAN DISPLACEMENT

It is well known that equation describing the evolution of the Lagrangian displacement  $\boldsymbol{\xi}$  is

$$(37) \quad \mathbf{u}_{01} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\mathbf{u}_0 \cdot \nabla) \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{u}_0.$$

For a circular cylindrical vortex, we seek the normal mode solution of the form,

$$(38) \quad \boldsymbol{\xi} = \hat{\boldsymbol{\xi}} e^{i(-\omega_0 t + m\theta + kz)}, \quad \hat{\boldsymbol{\xi}} = (\hat{\xi}_r, \hat{\xi}_\theta, \hat{\xi}_z).$$

When the basic flow  $\mathbf{u}_0$  is rigidly rotating flow,  $\mathbf{u}_0 = r e_\theta$ , the RHS of (37) is reduced to  $-i(\omega_0 - m) \hat{\boldsymbol{\xi}}$ .

When we chose disturbance flow  $\delta \mathbf{u}$  as linear coupling of Kelvin waves,  $\delta \mathbf{u} = \sum A_m \mathbf{u}_{01}^{(m)} e^{i(m\theta + kz)}$ , the Lagrangian displacement  $\boldsymbol{\xi}$  is given by

$$(39) \quad \boldsymbol{\xi} = \sum \frac{iA_m(t)}{\omega_0 - m} \mathbf{u}_{01}^{(m)}(r) e^{im\theta} e^{ikz},$$

with the incompressibility constraint

$$(40) \quad \nabla \cdot \boldsymbol{\xi} = 0.$$

## 5.5. MEAN FLOW OF $O(\alpha^2)$

In the case  $\mathbf{u}_0 = r e_\theta$ , the mean flow is specialized to

$$(41) \quad \begin{aligned} \overline{\delta^2 \mathbf{u}} &= \overline{\mathcal{P} [\boldsymbol{\xi} \times (\nabla \times (\boldsymbol{\xi} \times (\nabla \times \mathbf{u}_0))]} / 2 \\ &= \overline{\boldsymbol{\xi} \times \frac{\partial \boldsymbol{\xi}}{\partial z}} \\ &= \sum \frac{2ik}{(\omega_0 - m)^2} |A_m|^2 (\mathbf{u}_{01}^{(m)})^* \times \mathbf{u}_{01}^{(m)}. \end{aligned}$$

Substitution from the Lagrangian displacement  $\boldsymbol{\xi}$  yields

$$(42) \quad (\mathbf{u}_{01}^{(m)})^* \times \mathbf{u}_{01}^{(m)} = 2 \begin{pmatrix} 0 \\ u_{01}^{(m)} w_{01}^{(m)} \\ -u_{01}^{(m)} v_{01}^{(m)} \end{pmatrix},$$

where  $*$  denotes complex conjugate. In the same manner as for the second-order energy variation  $\delta^2 H$  the mean-flow  $\overline{\delta^2 \mathbf{u}}$  of  $O(\alpha^2)$  is determined by the first-order Lagrangian displacement  $\boldsymbol{\xi}$  only. The direct construction of the wave

induced mean flow provides us with a by-path to reach the Hamiltonian normal form of amplitude equations. In next section, we derive weakly nonlinear amplitude equations.

It is worthwhile to compare the Lagrangian approach with the Eulerian one. Whereas, we have to introduce a new parameter  $C$  in the Eulerian approach, with only  $dC/dt$  being available, we can represent the mean flow directly through  $A_+$  and  $A_-$  in the Lagrangian approach. Unlike the Eulerian approach, the Lagrangian approach is capable of calculating the mean flow at any points satisfied by the dispersion relation  $(k, \omega_0)$ .

This exhibits a marked difference in the amplitude equations. In the Eulerian approach, they do not take the final form, or the Hamiltonian normal form. They suffer from redundancy.

### 6. AMPLITUDE EQUATION

The boundary condition (8) at  $O(\epsilon\alpha)$  and at  $O(\alpha^3)$  presents the terms in the amplitude equations stemming from ellipticity and weakly nonlinearity. The Lagrangian approach does not deal with elliptic deformation of streamlines to reach the Hamiltonian normal form of amplitude equations [10].

#### 6.1. AMPLITUDE EQUATIONS OF STATIONARY RESONANCE

The mean flow induced by nonlinear interactions of stationary helical Kelvin waves is

$$(43) \quad 4ik \begin{pmatrix} 0 \\ (|A_-|^2 + |A_+|^2) u_{01}^{(+)} w_{01}^{(+)} \\ (|A_-|^2 - |A_+|^2) u_{01}^{(+)} v_{01}^{(+)} \end{pmatrix}.$$

For general  $(m, m+2)$  parametric resonance, only the radial component of mean flow is zero because of (42). But, in the case of stationary helical-wave parametric resonance, the axial components of mean flow is zero, since  $|A_-| = |A_+|$  is necessary condition of parametric resonance occurring elliptical instability, [20]. The boundary condition (8) gives Hamiltonian normal form of amplitude equations

$$(44) \quad \frac{dA_{\pm}}{dt} = \mp i [\epsilon a A_{\mp} + \alpha^2 A_{\pm} (b|A_{\pm}|^2 + c|A_{\mp}|^2)],$$

where

$$(45) \quad \begin{aligned} a &= \frac{3(3k^2+1)}{8(2k^2+1)}, \\ b &= \frac{-2k^4}{3(2k^2+1)} \left[ \frac{4}{J_0(\eta)^2} \int_0^1 r J_0(\eta r)^2 J_1(\eta r)^2 dr \right. \\ &\quad \left. - (11k^4 + 13k^2 + 5) J_0(\eta)^2 \right], \\ c &= \frac{k^2}{12(2k^2+1)} \left[ \frac{64k^2}{J_0(\eta)^2} \int_0^1 r J_0(\eta r)^2 J_1(\eta r)^2 dr \right. \\ &\quad \left. + (20k^6 + 97k^4 + 14k^2 - 27) J_0(\eta)^2 \right]. \end{aligned}$$

As nonlinear terms  $|A_{\pm}|^2 A_{\pm}, |A_{\mp}|^2 A_{\pm}$  comprehend the effect of mean flow (41), the Lagrangian approach leads us directly to the Hamiltonian normal form. We need not any more introduce a new constant associated with the mean

flow. By virtue of compact form available, the coefficients of these equations are calculated with ease and are listed in Table 1 for  $k_0 = 1.5788, 3.2859, \dots$  which satisfy the dispersion relation with  $\omega_0 = 0$ .

It follows from (35) that the energy of the stationary mode is zero to  $O(\alpha^2)$ ;  $E_{02} = 0$ . Comparison of the coefficients of (44) with that of (C.3), with the value in Table 1 substituted, shows that

$$(46) \quad b \neq b' - d'/(2a), \quad c \neq c' - d'/(2a).$$

It appears that (C.3) is not consistent with (44). But this is not the case because parametric resonance condition,  $|A_-|^2 = |A_+|^2$  for the stationary resonance, where

$$(47) \quad \begin{aligned} b + c &= \left( b' - \frac{d'}{2a} \right) + \left( c' - \frac{d'}{2a} \right) \\ &= \frac{-k^2}{4(2k^2+1)} [(4k^2 + 3)(9k^4 + 10k^2 - 3) J_0(\eta)^2 \\ &\quad + \frac{32k^2}{3J_0(\eta)^2} \int_0^1 r J_0(\eta r)^2 J_1(\eta r)^2 dr]. \end{aligned}$$

#### 6.2. FOUR-DIMENSIONAL DYNAMICAL SYSTEM

As  $A_-$  and  $A_+$  are complex functions of  $t$ , the amplitude equations (44) constitute a four-dimensional dynamical system. Following Knobloch *et al.* [10], we proceed to analysis of the amplitude equations (44). Define

$$(48) \quad \begin{aligned} N &= |A_-|^2 + |A_+|^2, \quad w = |A_-|^2 - |A_+|^2, \\ u &= 2\text{Re} [A_- A_+], \quad v = 2\text{Im} [A_- A_+]. \end{aligned}$$

By inspection, we find that  $w$  and

$$(49) \quad T = -2au - \frac{b+c}{2} N^2$$

are real constant quantities. Then timewise development of  $N$  and  $v$  obey

$$(50) \quad \frac{dN}{dt} = -2\epsilon av,$$

$$(51) \quad \frac{dv}{dt} = \epsilon N \left[ \frac{b+c}{2a} \left( T - \frac{4a^2}{b+c} \right) + \frac{(b+c)^2}{4a} N^2 \right].$$

Since  $T \leq 2a^2/(b+c)$  and  $N \geq |v|$ , the equilibrium point of  $(v, N) = (0, 0)$  is unstable, while the stable equilibrium point of  $(v, N)$  is

$$(52) \quad \left( 0, \left[ \frac{2}{b+c} \left( \frac{4a^2}{b+c} - T \right) \right]^{1/2} \right).$$

As is observed from Figure 2,  $N$  has the upper bound prescribed by the initial value of the complex amplitudes,  $A_-$  and  $A_+$ . This guarantees the systems (44) to be non-divergent.

The stationary helical-mode resonance condition requires  $|A_-| = |A_+|$ . As (44) preserves  $|A_-|^2 - |A_+|^2 = \text{constant}$ , if we set initially absolute value of complex amplitudes  $A_-$  and  $A_+$  to be the same value  $|A_-(t)|$  and  $|A_+(t)|$  have the same value,  $|A(t)|$  say, at all time. Introducing the

Table 1: The coefficients of amplitude equation (44)

$k$	1.5788	3.2859	5.0614	6.8562	8.6587	10.465	12.274	14.084	15.894
$a$	0.53117	0.55420	0.55891	0.56053	0.56126	0.56165	0.56188	0.56203	0.56213
$-b$	0.39757	8.2860	40.448	118.36	266.09	509.51	875.92	1393.8	2092.5
$c$	5.2217	53.388	212.75	562.10	1185.4	2170.3	3607.1	5588.6	8209.5

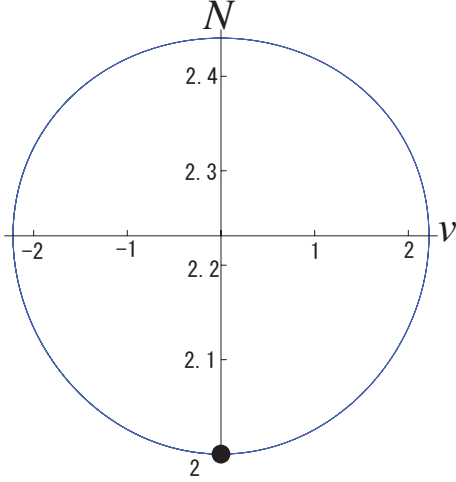


Figure 2: Trajectory in the  $(v, N)$  space in the case  $k = 1.578$  and  $\epsilon = 0.1$ . The thick dot is the initial amplitude of  $(A_-, A_+) = (1, 1)$ . The intersection point of  $v$ -axes and  $N$ -axes is an equilibrium point (52)

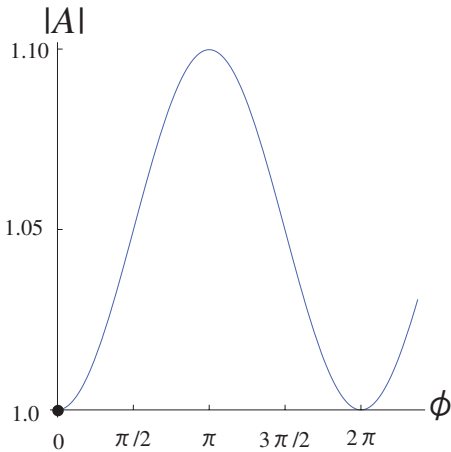


Figure 3: The modulus  $|A|$  as a function of  $\phi = \phi_- - \phi_+$  in the case  $k = 1.578$  and  $\epsilon = 0.1$ , obtained by integrating equations (53) and then by eliminating  $t$ . The thick dot is the initial amplitude of  $(A_-, A_+) = (1, 1)$ .

modulus  $|A|$  and the phase  $\phi$  by  $A_- = |A| \exp(i\phi_-)$  and  $A_+ = |A| \exp(i\phi_+)$ , (44) is reduced to

$$(53) \quad \frac{d|A|}{dt} = \epsilon a |A| \sin \phi, \quad \frac{d\phi}{dt} = 2\epsilon a \cos \phi + 2\alpha^2 (b + c) |A|^2,$$

where  $\phi = \phi_- - \phi_+$ . The solution of equations (53) is drawn in Figure 3. The nonlinear effect promotes decoherence of phase by supplying unidirectional rotation of  $\phi$ . This, in turn, acts to weaken the linear stability.

### 6.3. RESTRICTION TO TWO DIMENSIONS

The amplitude equations (44) admits restriction of the phase space to a two-dimensional subspace with  $A = A_- = -\overline{A_+}$ , whereby a detailed analysis becomes feasible. The complex amplitude equations (44) collapses, by a choice of  $\alpha^2 = \epsilon$ , to

$$(54) \quad \frac{dA}{dt} = i\epsilon (-a\overline{A} + \beta |A|^2 A),$$

where  $\beta = b + c$  is the sum of the coefficient of nonlinear terms. In this equation, the effect of the mean flow is incorporated into the nonlinear terms. If  $A = |A|e^{i\phi}$ , the leading-order term of the three-dimensional disturbance field is  $\sqrt{\epsilon} \mathbf{u}_{01}$ , which is written down in component wise as

$$(55) \quad \mathbf{u}_{01} = 4|A| \begin{pmatrix} u \sin(\theta - \phi) \cos kz \\ v \cos(\theta - \phi) \cos kz \\ w \sin(\theta - \phi) \sin kz \end{pmatrix},$$

where  $u, v$  and  $w$  are each components of stationary disturbed velocity field with azimuthal wavenumber  $m = -1$ ,

$$(56) \quad \begin{aligned} u &= \frac{-1}{3} (\eta J_0(\eta r) + \frac{1}{r} J_1(\eta r)), \\ v &= \frac{-1}{3} (2\eta J_0(\eta r) - \frac{1}{r} J_1(\eta r)), \\ w &= -kJ_1(\eta r). \end{aligned}$$

The corresponding components of the vorticity are

$$(57) \quad \boldsymbol{\omega}_{01} = 4|A| \begin{pmatrix} (kv + w/r) \cos(\theta - \phi) \sin kz \\ -(ku + \partial_r w) \sin(\theta - \phi) \sin kz \\ (-u/r + v/r + \partial_r v) \cos(\theta - \phi) \cos kz \end{pmatrix}.$$

The three-dimensional disturbance creates horizontal components of the vorticity. The direction of the horizontal vorticity is tied with  $\phi$ . In the case of  $\phi = -\pi/4$ , the above expression is the same as that of Leweke and Williamson [13]. Figure 4 illustrates trajectory of fluid particles pro-



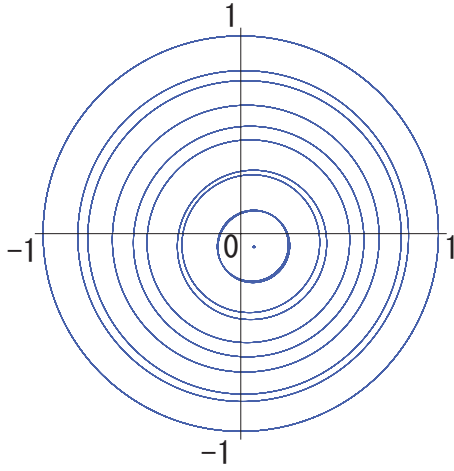


Figure 4: Trajectories of fluid particles in total flow,  $\mathbf{U}_0 + \alpha \mathbf{u}_{01}$

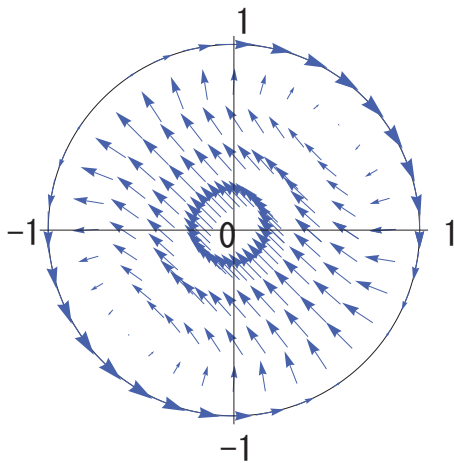


Figure 5: Disturbance vorticity in horizontal space

jected in horizontal space. This show the central point, or velocity null, is displaced to the direction of phase  $\theta = \phi$ . Figure 5 illustrates the direction of vorticity disturbance (57) as the same as that of phase  $\phi$ . We know that the central point is displaced to the direction of vorticity. The stagnation point  $(x_c, y_c)$  is

$$(58) \quad (x_c, y_c) \approx \begin{cases} 2\sqrt{3}\alpha k|A| \cos kz (\cos \phi, \sin \phi) & \text{for } \cos kz > 0 \\ -2\sqrt{3}\alpha k|A| \cos kz (\cos \phi, \sin \phi) & \text{for } \cos kz < 0 \end{cases},$$

because  $v = -\sqrt{3}k/2 + O(r^2)$ . There are two pressure minima in  $(x, y)$ -space within one wavelength. This is consistent with Eloy *et al.*'s experiment (FIG.4(b))[3]. The modulus  $|A|$  and the phase  $\phi$  satisfy the following equations,

$$(59) \quad \frac{d|A|}{dt} = -\epsilon a|A| \sin 2\phi, \quad \frac{d\phi}{dt} = -\epsilon a \cos 2\phi + \epsilon \beta |A|^2.$$

The linear effect is dominant compared with the nonlinear effect for small disturbance amplitude  $|A| (\ll 1)$ . In case the equilibrium point  $A = 0$  is unstable, the direction of disturbance vorticity  $\phi$  is liable to be parallel to the unstable direction,  $\theta = -\pi/4$ . The elliptic strain makes horizontal vortex lines continuously stretched, if they are oriented, on average, in the direction of  $\theta = -\pi/4$ . This is the mechanism for the MSTW instability at the linear stage. When the disturbance grows substantially,  $|A| \approx 1$  say, the nonlinear effect is called into play. In view of (59), the nonlinear effect is exclusively rotation of the phase  $\phi$ . As a consequence, alignment of horizontal vorticity to the direction  $\phi = -\pi/4$  is hindered, which renders the disturbance amplitude saturate.

#### 6.4. SHORT-WAVELENGTH LIMIT

Expanding for very large values of  $k$ , the coefficients of amplitude equations (44) tend to

$$(60) \quad a \rightarrow \frac{9}{16}, \quad b \rightarrow -\frac{k^3 \log k}{\sqrt{3}\pi}, \quad c \rightarrow \frac{2k^3 \log k}{\sqrt{3}\pi},$$

with the aid of

$$(61) \quad J_0(\eta)^2 \rightarrow \frac{2}{\pi\eta^3}, \quad \int_0^1 r J_0(\eta r)^2 J_1(\eta r)^2 dr \rightarrow \frac{1}{2\pi^2} \frac{\log \eta}{\eta^2}.$$

the sign of the coefficients  $a$ ,  $b$ , and  $c$ , for large wavenumber  $k$ , as

$$(62) \quad a > 0, \quad b < 0, \quad c > 0.$$

As verified by Table 1, the sign of the coefficients is maintained to small values of  $k$ .

In the subspace dynamics,  $A_- = -\overline{A_+}$ , amplitude of the saturated state  $|A|_{\text{eq}}$  is found to be

$$(63) \quad |A|_{\text{eq}} = \sqrt{a/\beta} \rightarrow \frac{3}{4} \left( \frac{\sqrt{3}\pi}{k^3 \log k} \right)^{1/2}.$$

## 7. CONCLUSION

In this article, we have made a weakly nonlinear analysis of the short-wave instabilities of rotating flow in a cylinder of elliptic cross-section. We put emphasis on the advantage of the Lagrangian approach, over the Eulerian one, in the derivation of the mean flow induced by nonlinear interactions of the Kelvin waves, which is effected for isovortical disturbances in §5. This approach provides a short-cut to deduce amplitude equations (§6) for the weakly nonlinear evolution of the MSTW instability.

The phase of the complex amplitude  $A_- = |A_-|e^{i\phi_-}$  and  $A_+ = |A_+|e^{i\phi_+}$  of the leading-order term of the 3D disturbance represents the angle, from the  $x$ -axis, of oscillating vorticity disturbances in the horizontal plane. We have analyzed the dynamics of the amplitude equations under the constraint  $A = A_- = -A_+^*$ . For small amplitude, the features of the linear short-wave instability is retrieved that the particular disturbance of  $\phi = \phi_- - \phi_+ = -\pi/4$  is selectively amplified, because the disturbance vorticity is continuously stretched by the ambient strain [21, 5]. The non-linear effect suppresses this monotonic growth by turning the disturbance vorticity out of the stretching direction ( $\phi = -\pi/4$ ) as soon as the amplitude of  $A$  becomes sufficiently large [20]. Owing to this effect of phase shift, the trajectories are constrained to a bounded domain with no exception, and thus the linearly growing modes eventually saturate their amplitude.

However this behavior does not coincide with the vigorous amplification of a number of waves and the ultimate disruption of a strained flow observed in experiments [14, 13, 3]. This indicates that the nonlinear interaction of a single MSTW mode is far from sufficient in describing practical flows. The secondary and the tertiary instability, which may be invited before reaching the stage of nonlinear saturation, will drastically alter the subsequent evolution [15, 6]. These and other nonlinear interactions call for an independent investigation. The Lagrangian approach would be vital for dealing with these higher-order bifurcations, because the Eulerian treatment becomes too involved to carry through.

## ACKNOWLEDGMENTS

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## A. DEFINITION OF FUNCTIONS

This appendix accommodates detailed form of functions appearing in the text. The right-handed side (RHS) of

(26) for the drift current of  $O(\alpha^2)$  is

$$(A.1) \quad (f(r), 0, 0)^T \\ = |A_m|^2 \left( \left\{ \mathbf{u}_{01}^{(m,k)}, \overline{\mathbf{u}_{01}^{(m,k)}} \right\} + \left\{ \overline{\mathbf{u}_{01}^{(m,k)}}, \mathbf{u}_{01}^{(m,k)} \right\} \right) \\ + |A_{m+2}|^2 \left( \left\{ \mathbf{u}_{01}^{(m+2,k)}, \overline{\mathbf{u}_{01}^{(m+2,k)}} \right\} + \left\{ \overline{\mathbf{u}_{01}^{(m+2,k)}}, \mathbf{u}_{01}^{(m+2,k)} \right\} \right),$$

where

$$(A.2) \quad \{\mathbf{u}_1, \mathbf{u}_2\} = -(\mathbf{u}_1 \cdot \nabla) \mathbf{u}_2,$$

$$(A.3) \quad = - \begin{pmatrix} u_1 \partial_r u_2 + v_1/r (im u_2 - v_2) + ik w_1 u_2 \\ u_1 \partial_r v_2 + v_1/r (im v_2 + u_2) + ik w_1 v_2 \\ u_1 \partial_r w_2 + v_1/r (im w_2) + ik w_1 w_2 \end{pmatrix}.$$

RHS of (28) for the drift current of  $O(\epsilon \alpha^2)$  comes from

$$(A.4) \quad A_m \overline{A_{m+2}} \left[ \left\{ \mathbf{u}_{01}^{(m)}, \overline{\mathbf{u}_{11}^{(m)}} \right\} + \left\{ \overline{\mathbf{u}_{01}^{(m+2)}}, \mathbf{u}_{11}^{(m+2)} \right\} \right. \\ \left. + \left\{ \overline{\mathbf{u}_{11}^{(m)}}, \mathbf{u}_{01}^{(m)} \right\} + \left\{ \mathbf{u}_{11}^{(m+2)}, \overline{\mathbf{u}_{01}^{(m+2)}} \right\} \right. \\ \left. + \mathcal{N}^{(-2,0)} \mathbf{u}_{02}^{(-2,0)}/2 \right] + c.c. + \frac{\partial}{\partial t_{10}} \mathbf{u}_{02}^{(0,0)}.$$

## B. NONLINEAR TERMS OF AMPLITUDE EQUATIONS OF $O(\alpha^3)$

We restrict our attention to the stationary mode (20). At  $O(\epsilon^0 \alpha^3)$ , the equations (5) and (6) we consider is the following equation

$$(B.1) \quad \mathcal{L} \mathbf{u}_{03} + \nabla p_{03} = \{\mathbf{u}_{01}, \mathbf{u}_{02}\} + \{\mathbf{u}_{02}, \mathbf{u}_{01}\} - \frac{\partial}{\partial t_{02}} \mathbf{u}_{01},$$

$$(B.2) \quad \nabla \cdot \mathbf{u}_{03} = 0,$$

whence we may pose

$$(B.3) \quad \mathbf{u}_{03} = A_- e^{-i\theta} e^{ikz} \left( |A_-|^2 \mathbf{u}_{03}^{(--)} + |A_+|^2 \mathbf{u}_{03}^{(-+)} \right) \\ + A_+ e^{i\theta} e^{ikz} \left( |A_-|^2 \mathbf{u}_{03}^{(+-)} + |A_+|^2 \mathbf{u}_{03}^{(++)} \right) \\ + [\text{non-resonant terms}] + c.c.$$

The same linear operator as that for  $p_{01}$  is shared by the pressure  $p_{03}$  of  $O(\alpha^3)$ .

The solution of equation for  $p_{03}$  is expressible in terms of the Bessel functions and their integrals, from which we

obtain

$$\begin{aligned}
 u_{03}^{(\pm)} &= \mp C_{\pm} \left( \frac{k}{\sqrt{3}} J_0(\eta r) + \frac{1}{3r} J_1(\eta r) \right) \\
 &- \frac{\partial A_{\pm}}{\partial t_{02}} \left\{ \frac{7k}{3\sqrt{3}} J_0(\eta r) + \left( \frac{1}{9r} - \frac{4}{3} k^2 r \right) J_1(\eta r) \right\} \\
 &\pm |A_{\pm}|^2 A_{\pm} \left[ -\frac{50}{9\sqrt{3}} k^5 J_0(\eta r)^3 \right. \\
 &- \left( \frac{2k^4}{9r} - \frac{44}{27} k^6 r \right) J_0(\eta r)^2 J_1(\eta r) \\
 &- \left( \frac{8k^3}{9\sqrt{3}r^2} + \frac{110k^5}{27\sqrt{3}} \right) J_0(\eta r) J_1(\eta r)^2 \\
 &- \left( \frac{8k^2}{81r^3} - \frac{2k^4}{81r} - \frac{44}{27} k^6 r \right) J_1(\eta r)^3 \\
 &- \frac{8\pi k^6}{3r} \int_0^r s J_0(\eta s)^2 J_1(\eta s) \times \\
 &\left. \left\{ J_1(\eta s) (\eta Y_0(\eta r) + Y_1(\eta r)/r) \right. \right. \\
 (B.4) \quad &- \left. \left. Y_1(\eta s) \left( \eta J_0(\eta r) + \frac{1}{r} J_1(\eta r) \right) \right\} ds \right] \\
 &\pm |A_{\mp}|^2 A_{\pm} \left[ 8k^6 r J_0(\eta)^2 J_1(\eta r) \right. \\
 &+ \left( \frac{5k^3}{6\sqrt{3}r^2} - \frac{35k^5}{9\sqrt{3}} \right) J_0(\eta r)^3 \\
 &- \left( \frac{k^2}{r^3} + \frac{k^4}{6r} - \frac{10}{27} k^6 r \right) J_0(\eta r)^2 J_1(\eta r) \\
 &+ \left( \frac{2k}{3\sqrt{3}r^4} - \frac{25k^3}{9\sqrt{3}r^2} - \frac{25k^5}{27\sqrt{3}} \right) J_0(\eta r) J_1(\eta r)^2 \\
 &+ \left( \frac{4}{27r^5} - \frac{29k^2}{162r^3} - \frac{211k^4}{162r} + \frac{10}{27} k^6 r \right) J_1(\eta r)^3 \\
 &+ \frac{16\pi k^6}{3r} \int_0^r s J_0(\eta s)^2 J_1(\eta s) \times \\
 &\left. \left\{ J_1(\eta s) (\eta Y_0(\eta r) + \frac{1}{r} Y_1(\eta r)) \right. \right. \\
 &- \left. \left. Y_1(\eta s) (\eta J_0(\eta r) + J_1(\eta r)/r) \right\} ds \right],
 \end{aligned}$$

where  $\eta = \sqrt{3}k$ .

The radial velocity  $u_{03}$  has to satisfy the boundary condition (8). The solvability condition arising from this procedure brings in the cubic nonlinear terms in the amplitude equations (44).

### C. COMPARISON WITH EULERIAN APPROACH

As emphasized in the text, the Eulerian treatment is unable to fully determine the mean flow induced by nonlinear interactions. Setting the mean flow to be  $C(t)\mathbf{u}_C(r)$ , the boundary condition (8) gives amplitude equations of three variables  $A_-$ ,  $A_+$ , and  $C$  as

$$\begin{aligned}
 (C.1) \quad \frac{dA_{\pm}}{dt} &= \pm i \left[ -\epsilon a' A_{\mp} + \alpha^2 A_{\pm} (b' |A_{\pm}|^2 + c' |A_{\mp}|^2 + d' C) \right], \\
 \frac{dC}{dt} &= i\epsilon (A_- \bar{A}_+ - \bar{A}_- A_+),
 \end{aligned}$$

$$\begin{aligned}
 (C.2) \quad a' &= \frac{3(3k^2+1)}{8(2k^2+1)}, \quad b' \equiv 0, \\
 c' &= \frac{k^2}{4(1+2k^2)} \left[ \frac{32k^2}{3J_0(\eta)^2} \int_0^1 r J_0(\eta r)^2 J_1(\eta r)^2 dr \right. \\
 &- \left. (44k^6 + 75k^4 + 26k^2 - 9) J_0(\eta)^2 \right], \\
 d' &= \frac{-k^4(3k^2+1)}{4(2k^2+1)^2} \left[ \frac{8}{J_0(\eta)^2} \int_0^1 r J_0(\eta r)^2 J_1(\eta r)^2 dr \right. \\
 &- \left. 3(k^2 - k + 1)(k^2 + k + 1) J_0(\eta)^2 \right].
 \end{aligned}$$

Coupled equations (C.1) describe an orbit in a five-dimensional space because  $A_-$  and  $A_+$  are complex functions, but  $C$  is a real function. Equations (C.1) cannot determine an orbit unless the initial conditions  $A_-(0)$ ,  $A_+(0)$  and  $C(0)$  are all specified. However  $C(0)$  may be taken arbitrary.

In order to restore the Hamiltonian normal form for a medium having the rotational and translational symmetries

about a common axis obtained by Knobloch *et al.* [10], we appeal to the energy conservation law [20] and relate the amplitude  $C$  of the mean flow to the energy constant  $E_{02}$  of  $O(\alpha^2)$  via  $|A_+|^2 + |A_-|^2 + 2a'C = E_{02}$ , leaving

$$(C.3) \quad \frac{dA_{\pm}}{dt} = \pm i \left[ \alpha^2 d A_{\pm} - \epsilon a A_{\mp} + \alpha^2 A_{\pm} (b |A_{\pm}|^2 + c |A_{\mp}|^2) \right],$$

where  $a = a'$ ,  $b = b' - d'/2a'$ ,  $c = c' - d'/2a'$  and  $d = E_{02}d'/2a'$ . In keeping with the mean-flow amplitude  $C$ ,  $E_{02}$  cannot be determined within the Eulerian framework.

As far as the isovortical disturbances are concerned, (35) shows that the disturbance energy of  $O(\alpha^2)$   $E_{02}$  is exactly zero for waves with  $\omega_0 = 0$ . By substitution of  $E_{02} = 0$ , (C.3) restores (44).

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