On the exponential moments of additive processes with the structure of semimartingales

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Abstract. In this paper, the exponential moments of $\mathbb{R}$-valued additive processes with the strucure of semimartingales, which are regarded as the Laplace transforms of the laws of these additive processes, will be explicitly represented by their characteristics. Note that the additive processes investigated here will not necessarily be assumed to be stochastically continuous. To prove the result, a criterion proposed in [5], which is described by the modified Laplace cumulant, will be applied.

Keywords. additive process, semimartingale, exponential moment, Laplace cumulant, modified Laplace cumulant

1. INTRODUCTION

Let $(Y_t)_{t \in [0,T]}$, $T > 0$, be an $\mathbb{R}$-valued additive process that is also a semimartingale and $(C_t^Y, n^Y(dtdy), B_t^Y)$ the characteristics of (Y_t) associated with a truncation function h_1 on $\mathbb{R}$.

The problem which we would like to discuss in this paper is to propose a condition on the measure $n^Y(dtdy)$ which ensures the integrability of the random variable e^{Y_t} and furthermore to express the exponential moment $E[e^{Y_t}]$ via the characteristics $(C_t^Y, n^Y(dtdy), B_t^Y)$ of (Y_t).

This problem is classic because the exponential moment $E[e^{Y_t}]$ can be regarded as the Laplace transform at 1 of the law of Y_t. In fact, when (Y_t) is a Lévy process (stochastically continuous additive process with stationary increments), a complete answer to the problem is stated as Theorem 25.17 in [6] (p.165). Note that if (Y_t) is a Lévy process, then it is in nature a semimartingale that does not have any fixed point of discontinuity. It is one of our objectives to extend the result for Lévy processes to more general processes that do not necessarily have stationary increments or that might have fixed times of discontinuity.

The main result of this paper is Theorem 1, which states that, under the condition

$$\int_{(0,T]} \int_{\{y>1\}} e^y \, n^Y(dudy) < \infty,$$

it holds that

$$E[e^{Y_t}] = e^{K^Y(1)_t}$$

for each $t \in [0,T]$, where

$$
\begin{aligned}
K^Y&(1)_t \\
&= \frac{1}{2} C_t^Y + B_t^Y + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \left(e^y - 1 - h_1(y) \right) n^Y(dudy) \\
&\quad + \sum_{u \in (0,t]} \Big\{ \log \left(1 + \int_{\mathbb{R}\setminus\{0\}} (e^y - 1) \, n^Y(\{u\}, dy) \right) \\
&\qquad\qquad - \int_{\mathbb{R}\setminus\{0\}} (e^y - 1) \, n^Y(\{u\}, dy) \Big\}.
\end{aligned}
$$

$K^Y(1)_t$ is said to be the modified Laplace cumulant of (Y_t) at 1. See [5] and [4] for the concept of the modified Laplace cumulant.

There might be several ways of establishing the result above. In this paper, we will prove it explicitly by applying Theorem 3.2 in [5], which provides us a criterion for the uniform integrability of $(e^{Y_t - K^Y(1)_t})$.

Furthermore, it is another of our objectives to apply the result Corollary 1 (a generalization of Theorem 1) to determine and express the minimal entropy martingale measure for the price process defined by $(S_t := S_0 \, e^{Y_t})$. In this step, it is indispensable to establish the integrability of the random variable defined as the exponential of an additive process transformed from (Y_t). See [2] and [1] for this aspect in the case when (Y_t) is a Lévy process and a stochastically continuous additive process that is also a semimartingale, respectively. We will discuss this application on the stage of mathematical finance in a separate paper.

2. Additive processes with the structure of semimartingales and their exponential moments

Let $(X_t)_{t\in[0,T]}$, $T > 0$, be an $\mathbb{R}^d$-valued additive process that is also a semimartingale, which is supposed to be defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)$ that satisfies the usual condition. See [4] I.1.2 (p.2) for the definition of the usual condition.

To be precise, (X_t) is an $\mathbb{R}^d$-valued adapted càdlàg process with $X_0 = 0$ that has independent increments: for all $s \le t$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s$, and it is also a semimartingale with respect to the filtration $(\mathcal{F}_t)$. According to [4], we will call such a stochastic process as (X_t) a d-dimensional PII-semimartingale.

We would like to emphasize that we do *not necessarily* assume that (X_t) is stochastically continuous. Note that, in our scheme, the stochastic continuity is equivalent to the property of having no fixed time of discontinuity and also to the quasi-left continuity. See [3] Corollary 11.28 (p.308) and [4] Theorem II.4.18 (p.107).

Let $(C_t, n(dtdx), B_t)$ be the characteristics of (X_t) associated with the truncation function $h(x) := xI_{\{|x|\le 1\}}(x)$ on $\mathbb{R}^d$.

Note that, owing to the property that (X_t) has independent increments, all of the components of the characteristics $(C_t, n(dtdx), B_t)$ are deterministic. In particular,

$$\int_{(0,T]} \int_{\mathbb{R}_0^d} (|x|^2 \wedge 1)\, n(dudx) < \infty,$$

where $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$ and $a \wedge b := \min\{a, b\}$.

It is important to recognize that the Lévy-Khinchin formula (see [4] Theorem II.4.15 (p.106)) states that the law of (X_t) is described by the characteristics: for all $\xi \in \mathbb{R}^d$,

$$E^P[e^{\sqrt{-1}\,\xi\cdot(X_t - X_s)}]$$
$$= \exp\Big[-\frac{1}{2}\xi(C_t - C_s)\xi + \sqrt{-1}\,\xi\cdot(B_t - B_s)$$
$$+ \int_{(s,t]}\int_{\mathbb{R}_0^d}(e^{\sqrt{-1}\xi\cdot x} - 1 - \sqrt{-1}\,\xi\cdot h(x))I_{J^c}(u)\,n(dudx)\Big]$$
$$\times \prod_{u\in(s,t]}\Big\{e^{-\sqrt{-1}\,\xi\cdot\Delta B_u}\Big[1 + \int_{\mathbb{R}_0^d}(e^{\sqrt{-1}\xi\cdot x} - 1)n(\{u\}, dx)\Big]\Big\},$$

where $a \cdot b$ denotes the inner product of $a, b \in \mathbb{R}^d$. Also, $J := \{t > 0;\ n(\{t\}, \mathbb{R}_0^d) > 0\}$ denotes the set of all fixed times of discontinuity of (X_t). Note that J is not empty in general.

Let $(Y_t)_{t\in[0,T]}$ be a 1-dimensional PII-semimartingale with $Y_0 = 0$ and $(C_t^Y, n^Y(dtdy), B_t^Y)$ the characteristics of (Y_t) associated with the truncation function $h_1(y) := yI_{\{|y|\le 1\}}(y)$ on $\mathbb{R}$.

The main purpose of this paper is to give an explicit proof of the following result:

Theorem 1. *Suppose that*

$$(1) \qquad \int_{(0,T]}\int_{\{y>1\}} e^y\, n^Y(dudy) < \infty.$$

Then, $(e^{Y_t - K^Y(1)_t})_{t\in[0,T]}$ is a uniformly integrable martingale with mean 1, where $(K^Y(1)_t)$ is the modified Laplace cumulant of (Y_t) at 1:

$$(2)\quad K^Y(1)_t$$
$$= \frac{1}{2}C_t^Y + B_t^Y$$
$$+ \int_{(0,t]}\int_{\mathbb{R}_0}(e^y - 1 - h_1(y))\,n^Y(dudy)$$
$$+ \sum_{u\in(0,t]}\Big\{\log\Big(1 + \int_{\mathbb{R}_0}(e^y - 1)\,n^Y(\{u\}, dy)\Big)$$
$$- \int_{\mathbb{R}_0}(e^y - 1)\,n^Y(\{u\}, dy)\Big\},$$

where $\mathbb{R}_0 := \mathbb{R}\setminus\{0\}$. In particular,

$$(3) \qquad E[e^{Y_t}] = e^{K^Y(1)_t}.$$

Remark 1. See [5] and [4] for the definition and properties of the modified Laplace cumulant in the framework of the theory of semimartingales.

Remark 2. We denote by $J^Y := \{t > 0;\ n^Y(\{t\}, \mathbb{R}_0) > 0\}$ the set of all fixed times of discontinuity of (Y_t). Note that

$$\int_{(0,t]}\int_{\mathbb{R}_0}(e^y - 1 - h_1(y))I_{J^Y}(u)\,n^Y(dudy)$$
$$= -\sum_{u\in(0,t]}\Delta B_u^Y + \sum_{u\in(0,t]}\int_{\mathbb{R}_0}(e^y - 1)\,n^Y(\{u\}, dy),$$

since

$$\Delta B_u^Y = \int_{\mathbb{R}_0} h_1(y)\,n^Y(\{u\}, dy).$$

Therefore, the equation (3) can be rewritten as:

$$E[e^{Y_t}]$$
$$= \exp\Big[\frac{1}{2}C_t^Y + B_t^Y$$
$$+ \int_{(0,t]}\int_{\mathbb{R}_0}(e^y - 1 - h_1(y))I_{(J^Y)^c}(u)\,n^Y(dudy)\Big]$$
$$\times \prod_{u\in(0,t]}\Big\{e^{-\Delta B_u^Y}\Big[1 + \int_{\mathbb{R}_0}(e^y - 1)\,n^Y(\{u\}, dy)\Big]\Big\}.$$

This expression is nothing but the one formally obtained by replacing ξ by $(-\sqrt{-1})$ and taking $s = 0$ in the Lévy-Khinchin formula for (Y_t):

$$E[e^{\sqrt{-1}\xi(Y_t - Y_s)}]$$
$$= \exp\Big[-\frac{1}{2}\xi^2(C_t^Y - C_s^Y) + \sqrt{-1}\,\xi(B_t^Y - B_s^Y)$$
$$+ \int_{(s,t]}\int_{\mathbb{R}_0}(e^{\sqrt{-1}\xi y} - 1 - \sqrt{-1}\,\xi h_1(y))$$
$$\times I_{(J^Y)^c}(u)\,n^Y(dudy)\Big]$$
$$\times \prod_{u\in(s,t]}\Big\{e^{-\sqrt{-1}\xi\Delta B_u^Y}$$
$$\times \Big[1 + \int_{\mathbb{R}_0}(e^{\sqrt{-1}\xi y} - 1)\,n^Y(\{u\}, dy)\Big]\Big\}.$$

Let $(X_t)_{t \in [0,T]}$ be a d-dimensional PII-semimartingale and $(C_t, n(dtdx), B_t)$ the characteristics of (X_t) associated with the truncation function $h(x)$.

Let the canonical representation of (X_t) associated with $h(x)$ be given as follows:

$$X_t = X_t^c + B_t + \int_{(0,t]} \int_{\mathbb{R}_0^d} h(x) \, \widetilde{N}(dudx)$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \check{h}(x) \, N(dudx).$$

Here, (X_t^c) is a continuous local martingale with $X_0^c = 0$ and $\langle X^{c,i}, X^{c,j} \rangle_t = C_t^{ij}$. $N(dudx)$ denotes the counting measure of the jumps of (X_t):

$$N((0,t], A) := \sharp\{u \in (0,t]; \ \Delta X_u := X_u - X_{u-} \in A\}$$

for $A \in \mathcal{B}(\mathbb{R}_0^d)$, where $X_{u-} := \lim_{v \uparrow u} X_v$ and $\mathcal{B}(\mathbb{R}_0^d)$ is the Borel σ-field on $\mathbb{R}_0^d$. The measure $n(dudx)$ is the compensator of $N(dudx)$. We denote by $\widetilde{N}(dudx) := N(dudx) - n(dudx)$ the compensated measure of $N(dudx)$. Also, $\check{h}(x) := x - h(x)$. See [4] Theorem II.2.34 (p.84) for the canonical representation.

Let $(\theta_u = (\theta_u^1, \dots, \theta_u^d))$ be an $\mathbb{R}^d$-valued Borel measurable function. Note that it is deterministic. We say that (θ_u) is integrable with respect to (X_t) if the following conditions (i)$\sim$(iii) are satisfied:

(i) $\displaystyle \int_{(0,T]} \theta_u \, dC_u \theta_u := \sum_{i,j=1}^d \int_{(0,T]} \theta_u^i \, dC_u^{ij} \theta_u^j < \infty,$

(ii) $\displaystyle \sum_{i=1}^d \int_{(0,T]} |\theta_u^i| \, d(\mathrm{Var}(B^i))_u < \infty$, where $\mathrm{Var}(A)_t$ denotes the total variation of the function (A_u) on the interval $[0,t]$,

(iii) $\displaystyle \int_{(0,T]} \int_{\mathbb{R}_0^d} |\theta_u \cdot h(x)|^2 \, n(dudx) < \infty.$

We denote by $L(X)$ the set of all integrable functions with respect to (X_t). Note that an arbitrary bounded measurable function belongs to $L(X)$.

Let $(\theta_u) \in L(X)$. Then, we can define an integral $(\int_{(0,t]} \theta_u \cdot dX_u)$ of (θ_u) based on (X_t) by

$$(4) \quad \int_{(0,t]} \theta_u \cdot dX_u := \int_{(0,t]} \theta_u \cdot dX_u^c + \int_{(0,t]} \theta_u \cdot dB_u$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot h(x) \, \widetilde{N}(dudx)$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot \check{h}(x) \, N(dudx).$$

Proposition 1. Let $(\theta_u) \in L(X)$ and $(Y_t := \int_{(0,t]} \theta_u \cdot dX_u)$. Then, (Y_t) is a 1-dimensional PII-semimartingale

with the characteristics (C^Y, n^Y, B^Y) (associated with h_1 on $\mathbb{R}$)) given by

$$(5) \qquad C_t^Y = \int_{(0,t]} \theta_u \, dC_u \theta_u;$$

$$(6) \quad n^Y((0,t], A) = \int_{(0,t]} \int_{\mathbb{R}_0^d} I_A(\theta_u \cdot x) \, n(dudx),$$
$$A \in \mathcal{B}(\mathbb{R}_0);$$

$$(7) \quad B_t^Y = \int_{(0,t]} \theta_u \cdot dB_u$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \big(h_1(\theta_u \cdot x) - \theta_u \cdot h(x)\big) \, n(dudx).$$

Proof. It is clear that the stochastic process (Y_t) is a semimartingale, since

$$\Big(\int_{(0,t]} \theta_u \cdot dX_u^c + \int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot h(x) \, \widetilde{N}(dudx)\Big)$$

is a square integrable martingale and

$$\Big(\int_{(0,t]} \theta_u \cdot dB_u + \int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot \check{h}(x) \, N(dudx)\Big)$$

is a process with finite variation on $[0,T]$. It is also clear that the stochastic process (Y_t) has independent increments, since (θ_u) is deterministic.

Since $\Delta Y_u = \theta_u \cdot \Delta X_u$, we have

$$N^Y((0,t], A) := \sharp\{u \in (0,t]; \ \Delta Y_u \in A\}$$
$$= \int_{(0,t]} \int_{\mathbb{R}_0^d} I_A(\theta_u \cdot x) N(dudx)$$

for $A \in \mathcal{B}(\mathbb{R}_0^d)$. Hence, it is immediate that (6) holds.

Next, we will show that (5) and (7) hold. Since (Y_t) is a semimartingale as we have seen above, it holds that

$$\int_{(0,T]} \int_{\mathbb{R}_0^d} (|\theta_u \cdot x|^2 \wedge 1) \, n(dudx)$$
$$= \int_{(0,T]} \int_{\mathbb{R}_0} (|y|^2 \wedge 1) \, n^Y(dudy) < \infty.$$

Then, it follows from this property and the third one (iii) in the definition of $L(X)$ that

$$(8) \quad \int_{(0,t]} \int_{\mathbb{R}_0^d} |h_1(\theta_u \cdot x) - \theta_u \cdot h(x)| \, n(dudx) < \infty.$$

Therefore, we see that

$$\int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot h(x) \, \widetilde{N}(dudx)$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \theta_u \cdot \check{h}(x) \, N(dudx)$$
$$= \int_{(0,t]} \int_{\mathbb{R}_0^d} h_1(\theta_u \cdot x) \, \widetilde{N}(dudx)$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \check{h}_1(\theta_u \cdot x) \, N(dudx)$$
$$- \int_{(0,t]} \int_{\mathbb{R}_0^d} \{\theta_u \cdot h(x) - h_1(\theta_u \cdot x)\} \, n(dudx).$$

Thus, we obtain the canonical representation of (Y_t) associated with h_1:

$$Y_t = \int_{(0,t]} \theta_u \cdot dX_u^c + \Big\{ \int_{(0,t]} \theta_u \cdot dB_u$$
$$- \int_{(0,t]} \int_{\mathbb{R}_0^d} \{\theta_u \cdot h(x) - h_1(\theta_u \cdot x)\}\, n(dudx) \Big\}$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} h_1(\theta_u \cdot x)\, \widetilde{N}(dudx)$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \check{h}_1(\theta_u \cdot x)\, N(dudx),$$

which implies that (5) and (7) hold. $\qquad\square$

Combining Theorem 1 with Proposition 1, we obtain the following result:

Corollary 1. *Let $(\theta_u) \in L(X)$ and suppose that*

$$(9) \qquad \int_{(0,T]} \int_{\{\theta_u \cdot x > 1\}} \mathrm{e}^{\theta_u \cdot x}\, n(dudx) < \infty.$$

Then, $(\mathrm{e}^{\int_{(0,t]} \theta_u \cdot dX_u - K^X(\theta)_t})_{t \in [0,T]}$ is a uniformly integrable martingale with mean 1, where $(K^X(\theta)_t)$ is the modified Laplace cumulant of (X_t) at (θ_u):

$$(10)$$
$$K^X(\theta)_t = \frac{1}{2} \int_{(0,t]} \theta_u\, dC_u\, \theta_u + \int_{(0,t]} \theta_u \cdot dB_u$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \{\mathrm{e}^{\theta_u \cdot x} - 1 - \theta_u \cdot h(x)\}\, n(dudx)$$
$$+ \sum_{u \in (0,t]} \Big\{ \log\Big(1 + \int_{\mathbb{R}_0^d} \{\mathrm{e}^{\theta_u \cdot x} - 1\}\, n(\{u\}, dx)\Big)$$
$$- \int_{\mathbb{R}_0^d} \{\mathrm{e}^{\theta_u \cdot x} - 1\}\, n(\{u\}, dx) \Big\}.$$

In particular,

$$(11) \qquad E[\mathrm{e}^{\int_{(0,t]} \theta_u \cdot dX_u}] = \mathrm{e}^{K^X(\theta)_t}.$$

Remark 3. The result of this corollary is an extension of those of a part of Theorem 25.17 in [6](p.165) and of Theorem 2.2 in [1].

Remark 4. If (θ_u) is a bounded measurable function, the hypothesis of this colollary can be replaced by the following one:

$$\int_{(0,T]} \int_{\{|x|>1\}} \mathrm{e}^{\theta_u \cdot x}\, n(dudx) < \infty.$$

Proof. Let $(Y_t := \int_{(0,t]} \theta_u \cdot dX_u)$. It is easy to see that the integrability condition (1) is satisfied, since it follows from Proposition 1 and the hypothesis (9) that

$$\int_{(0,T]} \int_{\{y>1\}} \mathrm{e}^{y}\, n^Y(dudy)$$
$$= \int_{(0,T]} \int_{\{\theta_u \cdot x > 1\}} \mathrm{e}^{\theta_u \cdot x}\, n(dudx) < \infty.$$

Therefore, applying Theorem 1 to (Y_t), we see that e^{Y_t} ($t \in [0,T]$) is integrable and that $E[\mathrm{e}^{Y_t}] = \mathrm{e}^{K^Y(1)_t}$. Furthermore, if we note that $K^Y(1)_t = K^X(\theta)_t$ (see Lemma 2.17 in [5] (p.404)), then it is immediate to get the conclusion (11).

However, we can deduce the conclusion directly as follows. By Proposition 1, we see that

$$K^Y(1)_t = \frac{1}{2} \int_{(0,t]} \theta_u\, dC_u\, \theta_u + \Big\{ \int_{(0,t]} \theta_u \cdot dB_u$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \big(h_1(\theta_u \cdot x) - \theta_u \cdot h(x)\big)\, n(dudx) \Big\}$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \big(\mathrm{e}^{\theta_u \cdot x} - 1 - h_1(\theta_u \cdot x)\big)\, n(dudx)$$
$$+ \sum_{u \in (0,t]} \Big\{ \log\Big(1 + \int_{\mathbb{R}_0^d} (\mathrm{e}^{\theta_u \cdot x} - 1)\, n(\{u\}, dx)\Big)$$
$$- \int_{\mathbb{R}_0^d} (\mathrm{e}^{\theta_u \cdot x} - 1)\, n(\{u\}, dx) \Big\}$$
$$= \frac{1}{2} \int_{(0,t]} \theta_u\, dC_u\, \theta_u + \int_{(0,t]} \theta_u \cdot dB_u$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0^d} \big(\mathrm{e}^{\theta_u \cdot x} - 1 - \theta_u \cdot h(x)\big)\, n(dudx)$$
$$+ \sum_{u \in (0,t]} \Big\{ \log\Big(1 + \int_{\mathbb{R}_0^d} (\mathrm{e}^{\theta_u \cdot x} - 1)\, n(\{u\}, dx)\Big)$$
$$- \int_{\mathbb{R}_0^d} (\mathrm{e}^{\theta_u \cdot x} - 1)\, n(\{u\}, dx) \Big\}$$
$$= K^X(\theta)_t. \qquad\square$$

3. Proof of Theorem 1

In this section, we will give a proof of Theorem 1. There might be several ways of establishing the result. In this paper, we will prove it by applying Theorem 3.2 in [5], which provides us a criterion for the uniform integrability of $(\mathrm{e}^{Y_t - K^Y(1)_t})$.

Let $(K^Y(1-\delta)_t)$, $\delta \in (0,1)$, be the modified Laplace cumulant of Y at $(1-\delta)$:

$$(12)$$
$$K^Y(1-\delta)_t$$
$$:= \frac{1}{2}(1-\delta)^2 C_t^Y + (1-\delta)B_t^Y$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0} \big(\mathrm{e}^{(1-\delta)y} - 1 - (1-\delta)h_1(y)\big)\, n^Y(dudy)$$
$$+ \sum_{u \in (0,t]} \Big\{ \log\Big(1 + \int_{\mathbb{R}_0} (\mathrm{e}^{(1-\delta)y} - 1)\, n^Y(\{u\}, dy)\Big)$$
$$- \int_{\mathbb{R}_0} (\mathrm{e}^{(1-\delta)y} - 1)\, n^Y(\{u\}, dy) \Big\}.$$

Note that both of $(K^Y(1)_t)$ and $(K^Y(1-\delta)_t)$ are deterministic. Hence, if we localize our discussion on the finite

interval $[0, T]$, the condition $I(0, 1-)$ in [5] is reduced to

$$I(0, 1-)_T : \quad \lim_{\delta \downarrow 0} \sup_{t \in [0,T]} \left\{ (1-\delta) K^Y(1)_t - K^Y(1-\delta)_t \right\} = 0.$$

According to Theorem 3.2 in [5] (p.411), we have the following result:

Proposition 2. *Let $T > 0$ be fixed. Suppose that the condition $I(0, 1-)_T$ holds. Then*

$$(\mathrm{e}^{Y_t - K^Y(1)_t})_{t \in [0,T]}$$

is a uniformly integrable martingale with mean 1.

Therefore, in order to complete our proof of Theorem 1, it is sufficient to show that if we assume that the condition of (1) holds, then so does the condition $I(0, 1-)_T$.

Remark 5. In [5], another condition $I(0, 1)$ for the uniform integrability of $(\mathrm{e}^{Y_t - K^Y(1)_t})$ is proposed. It is actually more tractable than the condition $I(0, 1-)$ is. However, in order to make the condition $I(0, 1)$ hold in our setting, we need to assume that

$$\int_{(0,T]} \int_{\{y > 1\}} y\, \mathrm{e}^y\, n^Y(dudy) < \infty,$$

which is clearly stronger than (1). Also it looks superfluous in view of the case when (Y_t) is a Lévy process or more generally a stochastically continuous PII-semimartingale. See, for example, Theorem 2.1 in [1].

We will prove Theorem 1 by deviding into three parts: Lemmas 1 ∼ 3.

Lemma 1. *The function $\left((1-\delta) K^Y(1)_t - K^Y(1-\delta)_t \right)$ is nonnegative-valued for each $\delta \in (0, 1)$.*

Proof. By Proposition 3.13-(2) in [5] (p.416), for any predictable process (V_t) with finite variation,

$$(1-\delta) K^Y(1)_t - K^Y(1-\delta)_t$$
$$= (1-\delta) K^{Y+V}(1)_t - K^{Y+V}(1-\delta)_t.$$

Moreover, by Proposition 3.13-(1) in [5] (p.416),

$$K^{Y+V}(1)_t = K^Y(1)_t + V_t.$$

Hence, taking $V_t := -K^Y(1)_t$, it follows from that

$$(1-\delta) K^Y(1)_t - K^Y(1-\delta)_t$$
$$= (1-\delta) K^{Y - K^Y(1)}(1)_t - K^{Y - K^Y(1)}(1-\delta)_t$$
$$= (1-\delta)\{K^Y(1)_t - K^Y(1)_t\} - K^{Y - K^Y(1)}(1-\delta)_t$$
$$= -K^{Y - K^Y(1)}(1-\delta)_t.$$

Set $Z_t := Y_t - K^Y(1)_t$. Then it is a 1-dimensional semimartingale. Let (C^Z, n^Z, B^Z) be the characteristics of (Z_t) associated with h_1.

Let $(\widetilde{K}^Z(1)_t))$ be the Laplace cumulant of (Z_t) at 1:

$$\widetilde{K}^Z(1)_t$$
$$:= \frac{1}{2} C_t^Z + B_t^Z + \int_{(0,t]} \int_{\mathbb{R}_0} \left(\mathrm{e}^z - 1 - h_1(z) \right) n^Z(dudz).$$

Since $K^Z(1)_t \equiv 0$,

$$0 = \Delta K^Z(1)_u = \log\left(1 + \Delta \widetilde{K}^Z(1)_u \right)$$

for any u, which implies that $\Delta \widetilde{K}^Z(1)_u = 0$. Hence, we see that

$$\widetilde{K}^Z(1)_t = \widetilde{K}^Z(1)_t + \sum_{u \in (0,t]} \left\{ \log(1 + \Delta \widetilde{K}^Z(1)_u) - \Delta \widetilde{K}^Z(1)_u \right\}$$
$$= K^Z(1)_t = 0.$$

On the other hand,

$$(1-\delta) \widetilde{K}^Z(1)_t - \widetilde{K}^Z(1-\delta)_t$$
$$= (1-\delta)\left\{ \frac{1}{2} C_t^Z + B_t^Z \right.$$
$$\left. + \int_{(0,t]} \int_{\mathbb{R}_0} \left(\mathrm{e}^z - 1 - h_1(z) \right) n^Z(dudz) \right\}$$
$$- \left\{ \frac{1}{2}(1-\delta)^2 C_t^Z + (1-\delta) B_t^Z \right.$$
$$\left. + \int_{(0,t]} \int_{\mathbb{R}_0} \left(\mathrm{e}^{(1-\delta)z} - 1 - (1-\delta) h_1(z) \right) n^Z(dudz) \right\}$$
$$= \frac{1}{2}(1-\delta)\delta C_t^Z$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0} \left\{ (1-\delta)\mathrm{e}^z - \mathrm{e}^{(1-\delta)z} + \delta \right\} n^Z(dudz).$$

Here, for fixed $\delta \in (0, 1)$, $(1-\delta)\mathrm{e}^z - \mathrm{e}^{(1-\delta)z} + \delta \geq 0$ for all $z \in \mathbb{R}$. Thererfore, we see that

$$\left(-\widetilde{K}^Z(1-\delta)_t = (1-\delta) \widetilde{K}^Z(1)_t - \widetilde{K}^Z(1-\delta)_t \right)$$

is nondecreasing.

Now, by a fundamental relation between the Laplace cumulant and the modified one (see Definition 2.16 in [5] (p.403)),

$$\mathrm{e}^{K^Z(1-\delta)_t} = \mathcal{E}(\widetilde{K}^Z(1-\delta))_t$$
$$= 1 + \int_{(0,t]} \mathcal{E}(\widetilde{K}^Z(1-\delta))_{u-}\, d\left(\widetilde{K}^Z(1-\delta) \right)_u$$
$$= 1 + \int_{(0,t]} \mathrm{e}^{K^Z(1-\delta)_{u-}}\, d\left(\widetilde{K}^Z(1-\delta) \right)_u,$$

where $(\mathcal{E}(X)_t)$ denotes the Doléans-Dade exponential of the semimartingale (X_t). Since $\mathrm{e}^{K^Z(1-\delta)_{u-}} > 0$ and $\left(\widetilde{K}^Z(1-\delta)_t \right)$ is nonincreasing, $\left(\int_{(0,t]} \mathrm{e}^{K^Z(1-\delta)_{u-}}\, d\left(\widetilde{K}^Z(1-\delta) \right)_u \right)$ is nonincreasing, and hence $\left(\mathrm{e}^{K^Z(1-\delta)_t} \right)$ is also nonincreasing.

Thus we see that $\left(-K^Z(1-\delta)_t \right)$ is nondecreasing, and hence $\left(-K^Z(1-\delta)_t \right)$ is nonnegative.

By the argument above, we have shown that

$$\left((1-\delta) K^Y(1)_t - K^Y(1-\delta)_t = -K^Z(1-\delta)_t \right)$$

is nonnegative. □

In order to simplify the notation in the following argument, we set

$$W_u^\delta := \int_{\mathbb{R}_0} (e^{(1-\delta)y} - 1)\, n^Y(\{u\}, dy)$$

for $u \in (0, T]$ and $\delta \in [0, 1)$, and $W_u := W_u^0$.

Next, we will prepare the following lemma, which assures that W_u^δ converges to W_u uniformly in u as $\delta \downarrow 0$.

Lemma 2. *Suppose that the condition of (1) holds. Then*

$$(13) \qquad \lim_{\delta \downarrow 0} \sup_{t \in [0,T]} \sum_{u \in (0,t]} |(1-\delta)W_u - W_u^\delta| = 0.$$

Proof. We devide the term $\big((1-\delta)W_u - W_u^\delta\big)$ into three parts as follows:

$$(1-\delta)W_u - W_u^\delta$$
$$= \int_{\mathbb{R}_0} \big\{ (1-\delta)e^y - e^{(1-\delta)y} + \delta \big\}\, n^Y(\{u\}, dy)$$
$$= \sum_{k=1}^3 \int_{\mathbb{R}_0} f_k^\delta(y)\, n^Y(\{u\}, dy),$$

where

$$f_1^\delta(y) := \big\{ (1-\delta)e^y - e^{(1-\delta)y} + \delta \big\} I_{\{|y|\le 1\}}(y);$$
$$f_2^\delta(y) := \big\{ (1-\delta)e^y - e^{(1-\delta)y} + \delta \big\} I_{\{y>1\}}(y);$$
$$f_3^\delta(y) := \big\{ (1-\delta)e^y - e^{(1-\delta)y} + \delta \big\} I_{\{y<-1\}}(y).$$

Concerning the first term, since

$$f_1^\delta(y) = \Big\{ (1-\delta)(1 + y + \int_0^1 (1-t)e^{yt}\, dt \times y^2)$$
$$- \big(1 + (1-\delta)y + \int_0^1 (1-t)e^{(1-\delta)yt}\, dt$$
$$\times ((1-\delta)y)^2 \big) + \delta \Big\} I_{\{|y|\le 1\}}(y)$$
$$= (1-\delta)\Big\{ \int_0^1 (1-t)e^{yt}\, dt$$
$$- (1-\delta)\int_0^1 (1-t)e^{(1-\delta)yt}\, dt \Big\} \times y^2 I_{\{|y|\le 1\}}(y),$$

we obtain

$$\sup_{\delta \in (0,1)} |f_1^\delta(y)| \le 2e \times |y|^2 I_{\{|y|\le 1\}}(y).$$

Moreover, since $|y|^2 I_{\{|y|\le 1\}}(y)$ is integrable with respect to the measure $I_{(0,T]\cap J}(u)\, n^Y(dudy)$ on $(0,T] \times \mathbb{R}_0$, it follows from the dominated convergence theorem that

$$(14) \qquad \lim_{\delta \downarrow 0} \sum_{u \in (0,T]} \int_{\mathbb{R}_0} |f_k^\delta(y)|\, n^Y(\{u\}, dy) = 0$$

for $k = 1$.

Next, note that

$$\sup_{\delta \in (0,1)} |f_2^\delta(y)| \le 3e^y I_{\{y>1\}}(y).$$

and that, by the hypothesis (1), $e^y I_{\{y>1\}}(y)$ is integrable with respect to the measure $I_{(0,T]\cap J}(u)\, n^Y(dudy)$. Hence, it follows from the dominated convergence theorem again that (14) holds for $k = 2$.

Similarly, since

$$\sup_{\delta \in (0,1)} |f_3^\delta(y)| \le 3 I_{\{y<-1\}}(y)$$

and $I_{\{y<-1\}}(y)$ is integrable with respect to the measure $I_{(0,T]\cap J}(u)\, n^Y(dudy)$, we see that (14) holds for $k = 3$. Thus, we have shown that

$$\lim_{\delta \downarrow 0} \sum_{u \in (0,T]} |(1-\delta)W_u - W_u^\delta| = 0,$$

which immediately implies the conclusion (13). $\qquad \square$

We can complete our proof of Theorem 1 if we combine Lemma 1 and the following lemma:

Lemma 3. *Suppose that the condition of (1) holds. Then*

$$(15) \qquad \lim_{\delta \downarrow 0} \sup_{t \in [0,T]} |(1-\delta)K^Y(1)_t - K^Y(1-\delta)_t| = 0.$$

Proof. Note that

$$(16)\ (1-\delta)K^Y(1)_t - K^Y(1-\delta)_t$$
$$= \frac{1}{2}(1-\delta)\delta\, C_t^Y$$
$$+ \int_{(0,t]} \int_{\mathbb{R}_0} \big\{ (1-\delta)e^y - e^{(1-\delta)y} + \delta \big\}\, n^Y(dudy)$$
$$+ \sum_{u \in (0,t]} \Big\{ (1-\delta)\big\{ \log(1+W_u) - W_u \big\}$$
$$- \big\{ \log(1+W_u^\delta) - W_u^\delta \big\} \Big\}.$$

For the first term in the right hand side of (16), it is easy to see that

$$(17) \qquad \lim_{\delta \downarrow 0} \sup_{t \in [0,T]} \frac{1}{2}(1-\delta)\delta\, |C_t^Y| = 0,$$

since (C_t^Y) is a continuous function on the interval $[0,T]$.

Concerning the second term in the right hand side of (16), we can show that
$$(18)$$
$$\lim_{\delta \downarrow 0} \sup_{t \in [0,T]} \Big| \int_{(0,t]} \int_{\mathbb{R}_0} \big\{ (1-\delta)e^y - e^{(1-\delta)y} + \delta \big\}\, n^Y(dudy) \Big| = 0$$

by a similar way to the proof of Lemma 2.

Finally, we will investigate the third term in the right hand side of (16). We devide it into two parts as follows:

$$\sup_{t \in [0,T]} \Big| \sum_{u \in (0,t]} \Big\{ (1-\delta)\big\{ \log(1+W_u) - W_u \big\}$$
$$- \big\{ \log(1+W_u^\delta) - W_u^\delta \big\} \Big\} \Big|$$
$$\le \sum_{u \in (0,T]} |(1-\delta)\big\{ \log(1+W_u) - W_u \big\}$$

$$- \big\{ \log(1 + W_u^\delta) - W_u^\delta \big\} I_{\{|W_u| \le 1/2\}}(u)$$

$$+ \sum_{u \in (0,T]} \big| (1-\delta)\big\{ \log(1 + W_u) - W_u \big\}$$

$$- \big\{ \log(1 + W_u^\delta) - W_u^\delta \big\} \big| I_{\{|W_u| > 1/2\}}(u).$$

Since

$$\log(1 + y) = y - \int_0^1 \frac{1-t}{(1+yt)^2} \, dt \times y^2$$

for any $y(> -1)$, we see that

$$(19) \quad |(1-\delta)\big\{ \log(1 + W_u) - W_u \big\} | I_{\{|W_u| \le 1/2\}}(u)$$

$$\le \int_0^1 \frac{1-t}{|1 + W_u t|^2} \, dt \times |W_u|^2 I_{\{|W_u| \le 1/2\}}(u)$$

$$\le 2 |W_u|^2 I_{\{|W_u| \le 1/2\}}(u).$$

On the other hand, it follows from Lemma 2 that there exists $\delta_0 > 0$ (which is independent of u) such that, for any $\delta \in (0, \delta_0)$ and for any $u \in \{u \in (0, T]; |W_u| \le 1/2\}$, $|W_u^\delta| \le 3/4$.

Therefore, for any $\delta \in (0, \delta_0)$ and $u \in (0, T]$,

$$(20) \quad | \log(1 + W_u^\delta) - W_u^\delta | I_{\{|W_u| \le 1/2\}}(u)$$

$$\le \int_0^1 \frac{1-t}{|1 + W_u^\delta t|^2} \, dt \times |W_u^\delta|^2 I_{\{|W_u| \le 1/2\}}(u)$$

$$\le 8 |W_u^\delta|^2 I_{\{|W_u| \le 1/2\}}(u).$$

Now, note that, for any $\delta \in [0, 1)$,

$$(21) \quad |W_u^\delta|$$

$$\le \big| \int_{\{|y| \le 1\}} (e^{(1-\delta)y} - 1) \, n^Y(\{u\}, dy) \big|$$

$$+ \big| \int_{\{y > 1\}} (e^{(1-\delta)y} - 1) \, n^Y(\{u\}, dy) \big|$$

$$+ \big| \int_{\{y < -1\}} (e^{(1-\delta)y} - 1) \, n^Y(\{u\}, dy) \big|$$

$$\le \int_{\{|y| \le 1\}} (1-\delta)|y| \int_0^1 e^{(1-\delta)yt} \, dt \, n^Y(\{u\}, dy)$$

$$+ \int_{\{y > 1\}} |e^{(1-\delta)y} - 1| \, n^Y(\{u\}, dy)$$

$$+ \int_{\{y < -1\}} |e^{(1-\delta)y} - 1| \, n^Y(\{u\}, dy)$$

$$\le e \times \int_{\{|y| \le 1\}} |y| \, n^Y(\{u\}, dy)$$

$$+ \int_{\{y > 1\}} (e^y + 1) \, n^Y(\{u\}, dy)$$

$$+ \int_{\{y < -1\}} 1 \, n^Y(\{u\}, dy)$$

$$\le e \times \Big\{ \int_{\{|y| \le 1\}} |y|^2 \, n^Y(\{u\}, dy) \Big\}^{1/2}$$

$$+ \int_{\{y > 1\}} e^y \, n^Y(\{u\}, dy) + \int_{\{|y| > 1\}} n^Y(\{u\}, dy).$$

Note that, in order to obtain the last inequality, we have used the property: $n^Y(\{u\}, \mathbb{R}_0) \le 1$. See II.2.13 in [4] (p.77).

Moreover, note that the second term in the right hand side of (21) is uniformly bounded with respect to $u \in (0, T]$, since it follows from the hypothesis (1) that

$$\int_{\{y > 1\}} e^y \, n^Y(\{u\}, dy)$$

$$\le \int_{(0,T]} \int_{\{y > 1\}} e^y \, n^Y(du \, dy) =: C_1 < \infty.$$

Hence,

$$|W_u^\delta|^2$$

$$\le 4 \times \Big\{ e^2 \int_{\{|y| \le 1\}} |y|^2 \, n^Y(\{u\}, dy)$$

$$+ \big| \int_{\{y > 1\}} e^y \, n^Y(\{u\}, dy) \big|^2 + \big| \int_{\{|y| > 1\}} n^Y(\{u\}, dy) \big|^2 \Big\}$$

$$\le 4 \times \Big\{ e^2 \int_{\{|y| \le 1\}} |y|^2 \, n^Y(\{u\}, dy)$$

$$+ C_1 \int_{\{y > 1\}} e^y \, n^Y(\{u\}, dy) + \int_{\{|y| > 1\}} n^Y(\{u\}, dy) \Big\}.$$

Thus, we see from (19) and (20) that

$$(22) \quad | \log(1 + W_u^\delta) - W_u^\delta | I_{\{|W_u| \le 1/2\}}(u)$$

$$\le C \Big\{ \int_{\{|y| \le 1\}} |y|^2 \, n^Y(\{u\}, dy)$$

$$+ \int_{\{y > 1\}} e^y \, n^Y(\{u\}, dy) + n^Y(\{u\}, \{|y| > 1\}) \Big\},$$

where C is a constant that does not depend either on δ and on u.

Using this estimate (22) for $\delta = 0$ and $\delta \in (0, \delta_0)$, we see that

$$|(1-\delta)\big\{ \log(1 + W_u) - W_u \big\}$$

$$- \big\{ \log(1 + W_u^\delta) - W_u^\delta \big\} | I_{\{|W_u| \le 1/2\}}(u)$$

$$\le 2C \Big\{ \int_{\{|y| \le 1\}} |y|^2 \, n^Y(\{u\}, dy)$$

$$+ \int_{\{y > 1\}} e^y \, n^Y(\{u\}, dy) + n^Y(\{u\}, \{|y| > 1\}) \Big\}.$$

Therefore, as in the proof of Lemma 2, it follows from the dominated convergence theorem that

$$(23)$$

$$\lim_{\delta \downarrow 0} \sum_{u \in (0,T]} |(1-\delta)\big\{ \log(1 + W_u) - W_u \big\}$$

$$- \big\{ \log(1 + W_u^\delta) - W_u^\delta \big\} | I_{\{|W_u| \le 1/2\}}(u) = 0.$$

Let $(\widetilde{K}^Y(1)_t))$ be the Laplace cumulant of (Y_t) at 1:

$$\widetilde{K}^Y(1)_t$$

$$:= \frac{1}{2} C_t^Y + B_t^Y + \int_{(0,t]} \int_{\mathbb{R}_0} \big(e^y - 1 - h_1(y) \big) \, n^Y(du \, dy).$$

It is a càdlàg function and

$$\Delta \widetilde{K}^Y(1)_t = \int_{\mathbb{R}_0} (e^y - 1)\, n^Y(\{u\}, dy) = W_u.$$

Hence, $\{u \in (0, T]; |W_u| > 1/2\}$ is a finite set. Moreover, since

$$\lim_{\delta \downarrow 0} |(1-\delta)\{\log(1+W_u) - W_u\} - \{\log(1+W_u^\delta) - W_u^\delta\}| = 0$$

for each $u \in (0, T]$, we see that

$$(24)$$
$$\lim_{\delta \downarrow 0} \sum_{u \in (0,T]} |(1-\delta)\{\log(1+W_u) - W_u\}$$
$$- \{\log(1+W_u^\delta) - W_u^\delta\}|I_{\{|W_u|>1/2\}}(u) = 0.$$

Thus, it follows from (23) and (24) that

$$(25)\ \lim_{\delta \downarrow 0} \sup_{t \in [0,T]} \sum_{u \in (0,t]} |(1-\delta)\{\log(1+W_u) - W_u\}$$
$$- \{\log(1+W_u^\delta) - W_u^\delta\}| = 0.$$

Finally, by (17), (18) and (25), we obtain the conclusion (15). □

Acknowledgments

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