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## **Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field**

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# Scaling Limits for the System of Semi-Relativistic Particles Coupled to a Scalar Bose Field

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**Abstract.** In this paper the Hamiltonian for the system of semi-relativistic particles interacting with a scalar bose field is investigated. A scaled total Hamiltonian of the system is defined and its scaling limit is considered. Then the semi-relativistic Schrödinger operator with an effective potential is derived.

Mathematics Subject Classification 2010 : 81Q10, 62M15.

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## 1 Introduction

In this paper we consider the Hamiltonian of the system of  $N$  particles linearly coupled to a scalar bose field. We assume that particles obey the semi-relativistic Schrödinger operator

$$H_p = \sum_{j=1}^N \sqrt{-\Delta_j + M^2},$$

where  $M > 0$  is a rest mass. There has been many results on the spectral properties of  $H_p$ . Refer to e.g. [21, 6, 3, 4, 5, 20]. The free Hamiltonian  $H_b$  of the scalar bose field is defined by the second quantization of the multiplication operator  $\omega$ , which is formally expressed by

$$H_b = \int_{\mathbf{R}^d} \omega(\mathbf{k}) a^*(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}$$

The state space of the interacting system is defined by  $\mathcal{H} = L^2(\mathbf{R}_x^{dN}) \otimes \mathcal{F}_b(L^2(\mathbf{R}_k^d))$  where  $\mathcal{F}_b(L^2(\mathbf{R}_k^d))$  is the boson Fock space on  $L^2(\mathbf{R}_k^d)$ . The total Hamiltonian is given by

$$H = H_p \otimes I + I \otimes H_b + \kappa H_I, \quad \kappa \in \mathbf{R}. \quad (1)$$

Here the interaction  $H_I$  is denoted by formally

$$H_I = \sum_{j=1}^N \frac{1}{\sqrt{2}} \int_{\mathbf{R}^d} \left( \overline{f_{x_j}(\mathbf{k})} \otimes a(\mathbf{k}) + f_{x_j}(\mathbf{k}) \otimes a^*(\mathbf{k}) \right),$$

where  $f_x$  is an multiplication operator on  $L^2(\mathbf{R}_x^d)$ .

We consider the scaled Hamiltonian

$$H(\Lambda) = H_p \otimes I + \Lambda^2 I \otimes H_b + \kappa \Lambda H_I, \quad \Lambda > 0. \quad (2)$$

We investigate the asymptotic behavior of  $H(\Lambda)$  as  $\Lambda \rightarrow \infty$ . The unitary evolution  $e^{-itH(\Lambda)}$  generated by  $H(\Lambda)$  is given by

$$e^{-itH(\Lambda)} = e^{-i\Lambda^2 t \left( \sum_{j=1}^N \sqrt{\left(\frac{\hat{\mathbf{p}}_j}{\Lambda^2}\right)^2 + \left(\frac{M}{\Lambda^2}\right)^2} + H_b + \left(\frac{\kappa}{\Lambda}\right) H_I \right)}.$$

Here  $\Lambda^2 t$  denotes the scaled time,  $\Lambda^{-2} \hat{\mathbf{p}}$  the scaled momentum for  $\hat{\mathbf{p}} = -i\nabla$ ,  $\Lambda^{-2} M$  the scaled mass, and  $\Lambda^{-1} \kappa$  the scaled coupling constant. As far as we know scaling limits of the Hamiltonians of the form (2) is initiated by E. B. Davies [2], where  $H(\Lambda)$  with semi-relativistic Schrödinger operator replaced by a standard Schrödinger operator is considered and a Schrödinger operator with an effective potential is derived as  $\Lambda \rightarrow \infty$ . This model is called the Nelson model, and our result can be regarded as a semi-relativistic version of [2]. In [1], a general theory of scaling limits is established and it is applied to scaling limits of a spin-boson model and non-relativistic QED models. In [10], by removing ultraviolet cutoffs and taking a scaling limit of the Nelson model *simultaneously*, a Schrödinger operator with the Yukawa potential or the Coulomb potential is derived. Refer to see also [8, 17, 18, 16, 19].

In the main theorem, it is shown that for  $z \in \mathbf{Z} \setminus \mathbf{R}$ ,

$$s - \lim_{\Lambda \rightarrow \infty} (H(\Lambda) - z)^{-1} = \left( \sum_{j=1}^N \sqrt{-\Delta_j + M^2} + V_{\text{eff}}(\mathbf{x}_1, \dots, \mathbf{x}_N) - z \right)^{-1} P_{\Omega_b}, \quad (3)$$

where

$$V_{\text{eff}}(\mathbf{x}_1, \dots, \mathbf{x}_N) = -\frac{\kappa^2}{4} \sum_{j,l} \int_{\mathbf{R}^d} \frac{\overline{f_{\mathbf{x}_j}(\mathbf{k})} f_{\mathbf{x}_l}(\mathbf{k}) + \overline{f_{\mathbf{x}_l}(\mathbf{k})} f_{\mathbf{x}_j}(\mathbf{k})}{\omega(\mathbf{k})} d\mathbf{k},$$

and  $P_{\Omega_b}$  is the projection onto the closed subspace spanned by the Fock vacuum  $\Omega_b$  of the bose field.

For the strategy of the proof of the main theorem, we use a unitary transformation, called *the dressing transformation*. Then we apply the general theory investigated in [1] to the unitary transformed Hamiltonian  $U(\Lambda)^{-1} H(\Lambda) U(\Lambda)$ , and the consider the asymptotic behavior of  $U(\Lambda)^{-1} H(\Lambda) U(\Lambda)$  as  $\Lambda \rightarrow \infty$ .

This paper is organized as follows. In Section 2, the theory of boson Fock space is described. Then the total state space and the total Hamiltonian is defined, and the main results are stated. In Section 3, the proof of the main theorem is given.

## 2 Main Results

### 2.1 Boson Fock Spaces

In this subsection we give the mathematically rigorous definition of the bose field. The state space of the bose field is given by the boson Fock space  $\mathcal{F}_b(L^2(\mathbf{R}^d)) = \bigoplus_{n=0}^{\infty} (\bigotimes_s^n L^2(\mathbf{R}^d))$ , where  $\bigotimes_s^n L^2(\mathbf{R}^d)$  denotes the  $n$ -fold symmetric tensor product of  $L^2(\mathbf{R}^d)$  with  $\bigotimes_s^0 L^2(\mathbf{R}^d) := \mathbf{C}$ . The Fock vacuum is defined by  $\Omega_b = \{1, 0, 0, \dots\} \in \mathcal{F}_b(L^2(\mathbf{R}^d))$ . The finite particle subspace  $\mathcal{F}_b^{\text{fin}}(\mathcal{D})$  on the subspace  $\mathcal{D} \subset L^2(\mathbf{R}^d)$  is defined by the set of  $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty}$  satisfying that  $\Psi^{(n)} \in \bigotimes_s^n \mathcal{D}$ ,  $n \geq 0$ , and  $\Psi^{(n')} = 0$  for all  $n' > N$  with some  $N \geq 0$ . Let  $a(\xi)$ ,  $\xi \in L^2(\mathbf{R}^3)$ , and  $a^*(\eta)$ ,  $\eta \in L^2(\mathbf{R}^d)$ , be the annihilation operator and the creation operator on  $\mathcal{F}_b(L^2(\mathbf{R}^d))$ , respectively. Then they satisfy the canonical commutation relations on  $\mathcal{F}_b^{\text{fin}}(L^2(\mathbf{R}^d))$ :

$$[a(\xi), a^*(\eta)] = (\xi, \eta), \quad [a(\xi), a(\eta)] = [a^*(\xi), a^*(\eta)] = 0.$$

Let  $S$  be a self-adjoint operator on  $L^2(\mathbf{R}^d)$ . The second quantization of  $S$  is defined by

$$d\Gamma(S) = \bigoplus_{n=0}^{\infty} \left( \sum_{j=1}^n (I \otimes \dots \otimes I \otimes \underbrace{S}_{j\text{th}} \otimes I \dots \otimes I) \right),$$

For  $\eta \in \mathcal{D}(S^{-1/2})$ , it is seen that  $a(\eta)$  and  $a^*(\eta)$  are relatively bounded with respect to  $d\Gamma(S)^{1/2}$  with the bound

$$\|a(\eta)\Psi\| \leq \|S^{-1/2}\eta\| \|d\Gamma(S)^{1/2}\Psi\|, \quad \Psi \in \mathcal{D}(d\Gamma(S)^{1/2}), \quad (4)$$

$$\|a^*(\eta)\Psi\| \leq \|S^{-1/2}\eta\| \|d\Gamma(S)^{1/2}\Psi\| + \|\eta\| \|\Psi\|, \quad \Psi \in \mathcal{D}(d\Gamma(S)^{1/2}). \quad (5)$$

The field operator and its conjugate operator are defined by

$$\phi(\xi) = \frac{1}{\sqrt{2}} (a(\xi) + a^*(\xi)), \quad \Pi(\eta) = \frac{i}{\sqrt{2}} (-a(\eta) + a^*(\eta)).$$

### 2.2 Main Theorem

In this subsection we define the total Hamiltonian and state the main results. The state space of the system for the  $N$ -particles coupled to bose field is defined by

$$\mathcal{H} = L^2(\mathbf{R}_x^{dN}) \otimes \mathcal{F}_b(L^2(\mathbf{R}_k^d)).$$

The free Hamiltonian of particles and the bose field are defined by

$$H_p = \sum_{j=1}^N \sqrt{-\Delta_j + M^2}, \quad H_b = d\Gamma_b(\omega),$$

where  $M > 0$  is a rest mass and  $\omega$  denotes the multiplication operator by the function  $\omega(\mathbf{k})$ , which describes the energy of the boson with momentum  $\mathbf{k}$ . We assume the following condition :

**(A.1)**  $\omega$  is non-negative.

The interaction  $H_I$  is defined by

$$H_I = \sum_{j=1}^N \phi(f_{\mathbf{x}_j}),$$

where  $f_{\mathbf{x}}$  is the multiplication operator satisfying the following condition :

$$(A.2) \quad \sup_{\mathbf{x} \in \mathbf{R}^d} \int_{\mathbf{R}^d} |f_{\mathbf{x}}(\mathbf{k})|^2 d\mathbf{k} < \infty, \quad \text{and} \quad \sup_{\mathbf{x} \in \mathbf{R}^d} \int_{\mathbf{R}^d} \frac{|f_{\mathbf{x}}(\mathbf{k})|^2}{\omega(\mathbf{k})} d\mathbf{k} < \infty.$$

The total Hamiltonian of this system is given by

$$H = H_0 + \kappa H_I,$$

where  $H_0 = H_p \otimes I + I \otimes H_b$ . By (4), (5), and the assumption (A.2), it is seen that the  $H_I$  is relatively bounded with respect to  $I \otimes H_b^{1/2}$ . Hence  $H_I$  is relatively bounded with respect to  $H_0$  with infinitely small bound. Then the Kato-Rellich theorem shows that  $H$  is self-adjoint and essentially self-adjoint any core of  $H_0$ . Then in particular,  $H$  is essentially self-adjoint on  $\mathcal{D}_0 = C_0^\infty(\mathbf{R}^{dN}) \hat{\otimes} \mathcal{F}_b^{\text{fin}}(\mathcal{D}(\omega))$ , where  $\hat{\otimes}$  denotes the algebraic tensor product.

Let us introduce the scaled total Hamiltonian

$$H(\Lambda) = H_0(\Lambda) + \kappa \Lambda H_I,$$

where  $H_0(\Lambda) = H_p \otimes I + \Lambda^2 I \otimes H_b$ . We introduce an additional assumption on the interaction.

$$(A.3) \quad \sup_{\mathbf{x} \in \mathbf{R}^d} \int_{\mathbf{R}^d} \frac{|f_{\mathbf{x}}(\mathbf{k})|^2}{\omega(\mathbf{k})^2} d\mathbf{k} < \infty, \quad \text{and} \quad \sup_{\mathbf{x} \in \mathbf{R}^d} \int_{\mathbf{R}^d} \frac{|f_{\mathbf{x}}(\mathbf{k})|^2}{\omega(\mathbf{k})^3} d\mathbf{k} < \infty.$$

$$(A.4) \quad \sup_{\mathbf{x} \in \mathbf{R}^d} \int_{\mathbf{R}^d} \frac{|\partial_{\mathbf{x}^v} f_{\mathbf{x}}(\mathbf{k})|^2}{\omega(\mathbf{k})^2} d\mathbf{k} < \infty, \quad \sup_{\mathbf{x} \in \mathbf{R}^d} \int_{\mathbf{R}^d} \frac{|\triangle f_{\mathbf{x}}(\mathbf{k})|^2}{\omega(\mathbf{k})^2} d\mathbf{k} < \infty \quad \text{and} \quad (\frac{\partial_{\mathbf{x}^v} f_{\mathbf{x}}}{\omega}, \frac{f_{\mathbf{y}}}{\omega}) \in \mathbf{R}, \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^d.$$

Under the condition  $(\frac{\partial_{\mathbf{x}^v} f_{\mathbf{x}}}{\omega}, \frac{f_{\mathbf{y}}}{\omega}) \in \mathbf{R}$  in (A.4), it follows that  $[\Pi(\frac{\partial_{\mathbf{x}^v} f_{\mathbf{x}}}{\omega}), \Pi(\frac{f_{\mathbf{y}}}{\omega})] = 0, \mathbf{x}, \mathbf{y} \in \mathbf{R}^d$ .

**Remark 2.1** Let us define that  $f_{\mathbf{x}}(\mathbf{k}) = \frac{\chi_R(|\mathbf{k}|)}{\sqrt{\omega(\mathbf{k})}} e^{-i\mathbf{k} \cdot \mathbf{x}}$  with  $\omega(\mathbf{k}) = \omega(-\mathbf{k})$ . Here  $\chi_R$  denotes the characteristic function on  $[0, R)$ . Then the conditions (A.1)-(A.4) are satisfied, and the interaction  $H_I$  is formally expressed by

$$H_I = \sum_{j=1}^N \int_{\mathbf{R}^d} \frac{\chi_R(|\mathbf{k}|)}{\sqrt{2\omega(\mathbf{k})}} \left( a(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}_j} + a^*(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}_j} \right) d\mathbf{k}.$$

The main theorem in this paper is as follows

**Theorem 2.1** Assume (A.1)-(A.4). Then for  $z \in \mathbb{C} \setminus \mathbb{R}$  it follows that

$$s - \lim_{\Lambda \rightarrow \infty} (H(\Lambda) - z)^{-1} = (H_p + V_{\text{eff}}(\mathbf{x}_1, \dots, \mathbf{x}_n) - z) \otimes P_{\Omega_b},$$

where

$$V_{\text{eff}}(\mathbf{x}_1, \dots, \mathbf{x}_n) = -\frac{\kappa^2}{4} \sum_{j,l} \int_{\mathbb{R}^d} \frac{\overline{f_{\mathbf{x}_j}(\mathbf{k})} f_{\mathbf{x}_l}(\mathbf{k}) + \overline{f_{\mathbf{x}_l}(\mathbf{k})} f_{\mathbf{x}_j}(\mathbf{k})}{\omega(\mathbf{k})} d\mathbf{k},$$

and  $P_{\Omega_b}$  is the projection onto the closed subspace spanned by the Fock vacuum  $\Omega_b$ .

**Remark 2.2** When  $f_{\mathbf{x}}(\mathbf{k}) = \frac{\chi_R(|\mathbf{k}|)}{\sqrt{\omega(\mathbf{k})}} e^{-i\mathbf{k} \cdot \mathbf{x}}$  with  $\omega(\mathbf{k}) = \omega(-\mathbf{k})$ , the effective potential is given by

$$V_{\text{eff}}(\mathbf{x}_1, \dots, \mathbf{x}_n) = -\frac{\kappa^2}{2} \sum_{j,l} \int_{\mathbb{R}^d} \frac{|\chi_R(\mathbf{k})|^2}{\omega(\mathbf{k})^2} e^{-i\mathbf{k} \cdot (\mathbf{x}_j - \mathbf{x}_l)} d\mathbf{k},$$

By using the norm convergence theorem considered in ([18] ; Lemma 2.7), the the next corollary follows.

**Corollary 2.2** Assume (A.1)-(A.4). Then for  $z \in \mathbb{C} \setminus \mathbb{R}$  it follows that

$$s - \lim_{\Lambda \rightarrow \infty} e^{-itH(\Lambda)} (I \otimes P_{\Omega_b}) = e^{-it(H_p + V_{\text{eff}}(\mathbf{x}_1, \dots, \mathbf{x}_n))} \otimes P_{\Omega_b}.$$

### 3 Proof of Main Theorem

The outline of the proof of Theorem 2.1 is as follows. A unitary transformation  $U(\Lambda)$ , called the dressing transformation, is defined and we consider the unitarily transformed Hamiltonian  $U(\Lambda)^{-1}H(\Lambda)U(\Lambda)$ . Then we apply the general theory on scaling limits in [1] to  $U(\Lambda)^{-1}H(\Lambda)U(\Lambda)$ .

Under the condition (A.3), the following unitary operator can be defined :

$$U(\Lambda) = e^{i\left(\frac{\kappa}{\Lambda}\right) \sum_{j=1}^N \Pi\left(\frac{f_{\mathbf{x}_j}}{\omega}\right)}.$$

It is seen that on the finite particle subspace

$$[\Pi(\xi), H_b] = -i\phi(\omega\xi), \quad \xi \in \mathcal{D}(\omega), \quad (6)$$

$$[\Pi(\xi), \phi(\eta)] = \frac{-i}{2} \left( (\xi, \eta) + (\eta, \xi) \right), \quad \xi, \eta \in L^2(\mathbb{R}^3). \quad (7)$$

By (6) and (7), we have

$$U(\Lambda)^{-1}H(\Lambda)U(\Lambda) = H_0(\Lambda) + K(\Lambda) \quad (8)$$

where

$$K(\Lambda) = U(\Lambda)^{-1} (H_p \otimes I) U(\Lambda) - H_p \otimes I + V_{\text{eff}}(\mathbf{x}_1, \dots, \mathbf{x}_N). \quad (9)$$

Now we apply the general theory on scaling limits investigated in [1]. Let us set the total Hilbert space by  $\mathcal{Z} = \mathcal{X} \otimes \mathcal{Y}$ . Let  $A$  and  $B$  be non-negative self-adjoint operators on  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Here we assume that  $\ker B \neq \{0\}$ . We consider a family of symmetric operators  $\{C(\Lambda)\}_{\Lambda>0}$  satisfying the conditions :

**(S.1)** For all  $\varepsilon > 0$  there exists a constant  $\Lambda(\varepsilon) > 0$  such that for all  $\Lambda > \Lambda(\varepsilon)$ ,  $\mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B) \subset \mathcal{D}(C(\Lambda))$ , and there exists  $b(\varepsilon) \geq 0$  such that

$$\|C(\Lambda)\Phi\| \leq \varepsilon\|(A \otimes I + \Lambda I \otimes B)\Phi\| + b(\varepsilon)\|\Phi\|.$$

**(S.2)** There exists a symmetric operator  $C$  on  $\mathcal{Z}$  such that  $\mathcal{D} \otimes \ker B \subset \mathcal{D}(C)$  and for all  $z \in \mathbf{C} \setminus \mathbf{R}$ ,

$$s - \lim_{\Lambda \rightarrow \infty} C(\Lambda)(A \otimes I + \Lambda I \otimes B - z) = C(A - z)^{-1} \otimes P_B,$$

where  $P_B$  is the orthogonal projection from  $\mathcal{Y}$  onto  $\ker B$ .

**Proposition A** ([1] ; Theorem 2.1) Assume **(S.1)** and **(S.2)**. Then (i)-(iii) follows.

(i) There exists  $\Lambda_0 \geq 0$  such that for all  $\Lambda > \Lambda_0$ ,

$$X(\Lambda) = A \otimes I + \Lambda I \otimes B + C(\Lambda)$$

is self-adjoint on  $\mathcal{D}(A \otimes I) \cap \mathcal{D}(I \otimes B)$  and uniformly bounded from below for  $\Lambda$ , furthermore  $X(\Lambda)$  is essentially self-adjoint on any core of  $A \otimes I + I \otimes B$ .

(ii) Let  $X = A \otimes I + (I \otimes P_B)C(I \otimes P_B)$ . Then  $X$  is self-adjoint on  $\mathcal{D}(A \otimes I)$  and bounded from below, and essentially self-adjoint on any core of  $A \otimes I$ .

(iii) Let  $z \in \bigcap_{\Lambda \geq \Lambda_0} \rho(X(\Lambda)) \cap \rho(X)$ , where  $\rho(\mathcal{O})$  denotes the resolvent set of an operator  $\mathcal{O}$ . Then

$$s - \lim_{\Lambda \rightarrow \infty} (X(\Lambda) - z)^{-1} = (X - z)^{-1}(I \otimes P_B).$$

Now we consider  $H(\Lambda)$  again. What we have to prove is that  $H(\Lambda)$  satisfies the condition **(S.1)** and **(S.2)** by applying  $H_0(\Lambda)$  to  $A \otimes I + \Lambda I \otimes B$  and  $K(\Lambda)$  to  $C(\Lambda)$ . First let us consider the term  $U(\Lambda)^{-1}(H_p \otimes I)U(\Lambda)$  in (9). Let us set  $\hat{\mathbf{p}} = (\hat{p}^1, \dots, \hat{p}^d) = (-i \frac{\partial}{\partial x^1}, \dots, -i \frac{\partial}{\partial x^d})$ . Then by the spectral decomposition theorem,

$$U(\Lambda)^{-1}(H_p \otimes I)U(\Lambda) = \sum_{j=1}^N \sqrt{\left( U(\Lambda)^{-1}(\mathbf{p}_j \otimes I)U(\Lambda) \right)^2 + M^2}, \quad (10)$$

follows. We see that

$$[\Pi(f_{\mathbf{x}}), \hat{p}^v] = i\Pi(\partial_{x^v} f_{\mathbf{x}}). \quad (11)$$

Then by **(A.4)**, it follows that for  $\Psi \in \mathcal{D}_0$ ,

$$\left( U(\Lambda)^{-1}(\mathbf{p}_j \otimes I)U(\Lambda) \right)^2 \Psi = \left( \sum_{v=1}^d \left( \hat{p}_j^v \otimes I + \left( \frac{\kappa}{\Lambda} \right) \Pi \left( \frac{\partial_{x_j^v} f_{\mathbf{x}_j}}{\omega} \right) \right)^2 + M^2 \right) \Psi.$$

Then we have

$$\left( U(\Lambda)^{-1}(\mathbf{p}_j \otimes I)U(\Lambda) \right)^2 = \overline{\left( \sum_{v=1}^d \left( \hat{p}_j^v \otimes I + \left( \frac{\kappa}{\Lambda} \right) \Pi \left( \frac{\partial_{x_j^v} f_{\mathbf{x}_j}}{\omega} \right) \right)^2 + M^2 \right)}_{\upharpoonright \mathcal{D}_0}, \quad (12)$$



where  $\bar{Y}$  denotes the closure of the operator  $Y$ . Here we abbreviate as

$$\begin{aligned}\Pi\left(\frac{\nabla f_{\mathbf{x}}}{\omega}\right) \cdot (\hat{\mathbf{p}} \otimes I) &= \sum_{v=1}^d \Pi\left(\frac{\partial_{x^v} f_{\mathbf{x}}}{\omega}\right) (\hat{p}^v \otimes I), \\ \Pi\left(\frac{\nabla f_{\mathbf{x}}}{\omega}\right) \cdot \Pi\left(\frac{\nabla f_{\mathbf{x}}}{\omega}\right) &= \sum_{v=1}^d \Pi\left(\frac{\partial_{x^v} f_{\mathbf{x}}}{\omega}\right) \phi\left(\frac{\partial_{x^v} f_{\mathbf{x}}}{\omega}\right).\end{aligned}$$

Then we see that

$$U(\Lambda)^{-1} (H_p \otimes I) U(\Lambda) = \sum_{j=1}^N \left( \overline{\left( -\Delta_j \otimes I + Q_j(\Lambda) + M^2 \right)}_{\upharpoonright \mathcal{D}_0} \right)^{1/2}, \quad (13)$$

where

$$Q_j(\Lambda) = \left( \frac{\kappa}{\Lambda} \right) \left( 2\Pi\left(\frac{\nabla f_{\mathbf{x}_j}}{\omega}\right) \cdot (\hat{\mathbf{p}}_j \otimes I) - i\Pi\left(\frac{\Delta f_{\mathbf{x}}}{\omega}\right) \right) + \left( \frac{\kappa}{\Lambda} \right)^2 \Pi\left(\frac{\nabla f_{\mathbf{x}_j}}{\omega}\right) \cdot \Pi\left(\frac{\nabla f_{\mathbf{x}_j}}{\omega}\right).$$

**Proposition 3.1** *Assume (A.1)-(A.4). Then for  $\varepsilon > 0$ , there exists  $\Lambda(\varepsilon) \geq 0$  such that for all  $\Lambda > \Lambda(\varepsilon)$ ,*

$$\|U(\Lambda)^{-1} (H_p \otimes I) U(\Lambda) \Psi - (H_p \otimes I) \Psi\| \leq \varepsilon \|H_0(\Lambda) \Psi\| + b(\varepsilon) \|\Psi\| \quad (14)$$

where  $b(\varepsilon)$  is a constant independent of  $\Lambda \geq \Lambda(\varepsilon)$ .

Before proving Proposition 3.1, we show the following lemma.

**Lemma 3.2** *For  $\lambda > 0$  and  $\delta \in (0, \frac{1}{10})$ , there exists  $M_v(\delta)$ ,  $v = 1, \dots, d$ , such that*

$$\|\hat{p}^v (-\Delta + M^2 + \lambda)^{-1} (\sqrt{-\Delta + M^2 + 1})^{-1/2}\| \leq \frac{1}{\lambda^{\frac{1}{2}+\delta}} M_v(\delta). \quad (15)$$

**(Proof)** For  $\mathbf{p} = (p^1, \dots, p^d) \in \mathbf{R}^d$ ,  $v = 1, \dots, d$ , we see that

$$|p^v (\mathbf{p}^2 + M^2 + \lambda)^{-1} (\sqrt{\mathbf{p}^2 + M^2 + 1})^{-1/2}| = \frac{1}{\lambda^{\frac{1}{2}+\delta}} |\lambda^{\frac{1}{2}+\delta} p^v| (\mathbf{p}^2 + M^2 + \lambda)^{-1} (\sqrt{\mathbf{p}^2 + M^2 + 1})^{-1/2}.$$

We shall show that

$$\sup_{\lambda > 0, \mathbf{p} \in \mathbf{R}^d} |\lambda^{\frac{1}{2}+\delta} p^v| (\mathbf{p}^2 + M^2 + \lambda)^{-1} (\sqrt{\mathbf{p}^2 + M^2 + 1})^{-1/2} < \infty, \quad (16)$$

and hence (15) follows from the spectral decomposition theorem. The Young's inequality shows that for  $q > 1$  and  $\tilde{q} > 1$  satisfying  $\frac{1}{q} + \frac{1}{\tilde{q}} = 1$ ,

$$\lambda^{\frac{1}{2}+\delta} |p^v| \leq \frac{1}{q} \lambda^{(\frac{1}{2}+\delta)q} + \frac{1}{\tilde{q}} |p^v|^{\tilde{q}} \quad (17)$$

follows. Let us take  $q = (\frac{1}{2} + \delta)^{-1}$  for  $\delta \in (0, \frac{1}{10})$ , and hence  $\tilde{q} = (\frac{1}{2} - \delta)^{-1}$ . Then we have

$$\lambda^{\frac{1}{2}+\delta} |p^v| \leq \left(\frac{1}{2} + \delta\right) \lambda + \left(\frac{1}{2} - \delta\right) |p^v|^{(\frac{1}{2}-\delta)^{-1}}. \quad (18)$$

Note that

$$\sup_{\lambda > 0, \mathbf{p} \in \mathbf{R}^d} \lambda (\mathbf{p}^2 + M^2 + \lambda)^{-1} (\sqrt{\mathbf{p}^2 + M^2} + 1)^{-1/2} < \infty. \quad (19)$$

Since  $0 < \delta < \frac{1}{10}$ , we see that  $(\frac{1}{2} - \delta)^{-1} < \frac{5}{2}$ , and hence

$$\sup_{\mathbf{p} \in \mathbf{R}^d} |p^\nu|^{(\frac{1}{2} - \delta)^{-1}} (\mathbf{p}^2 + M^2)^{-1} (\sqrt{\mathbf{p}^2 + M^2} + 1)^{-1/2} < \infty. \quad (20)$$

Then we have

$$\begin{aligned} \sup_{\lambda > 0, \mathbf{p} \in \mathbf{R}^d} |p^\nu|^{(\frac{1}{2} - \delta)^{-1}} (\mathbf{p}^2 + M^2 + \lambda)^{-1} (\sqrt{\mathbf{p}^2 + M^2} + 1)^{-1/2} \\ \leq \sup_{\mathbf{p} \in \mathbf{R}^d} |p^\nu|^{(\frac{1}{2} - \delta)^{-1}} (\mathbf{p}^2 + M^2)^{-1} (\sqrt{\mathbf{p}^2 + M^2} + 1)^{-1/2} < \infty. \end{aligned} \quad (21)$$

By (18), (19) and (21), we obtain (16). ■

### (Proof of Proposition 3.1)

It follows that for a nonnegative self-adjoint operator  $S$ ,

$$\sqrt{S}\Phi = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\lambda}} (S + \lambda)^{-1} S \Phi d\lambda, \quad \Phi \in \mathcal{D}(S). \quad (22)$$

Let

$$\begin{aligned} A_j(\Lambda) &= \overline{\left( -\Delta_j \otimes I + Q_j(\Lambda) + M^2 \right)}_{\upharpoonright \mathcal{D}_0}, \\ B_j &= -\Delta_j \otimes I + M^2. \end{aligned}$$

Then we have for  $\Psi \in \mathcal{D}_0$ ,

$$\begin{aligned} \left( U(\Lambda)^{-1} (H_p \otimes I) U(\Lambda) - H_p \otimes I \right) \Psi &= \sum_{j=1}^N \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\lambda}} \{ (A_j(\Lambda) + \lambda)^{-1} A_j(\Lambda) - (B_j + \lambda)^{-1} B_j \} \Psi d\lambda \\ &= \sum_{j=1}^N \frac{1}{\pi} \int_0^\infty \sqrt{\lambda} (A_j(\Lambda) + \lambda)^{-1} (A_j(\Lambda) - B_j) (B_j + \lambda)^{-1} \Psi d\lambda \\ &= \sum_{j=1}^N \frac{1}{\pi} \int_0^\infty \sqrt{\lambda} (A_j(\Lambda) + \lambda)^{-1} Q_j(\Lambda) (B_j + \lambda)^{-1} \Psi d\lambda. \end{aligned} \quad (23)$$

By (13) and the spectral decomposition theorem,  $\|(A_j(\Lambda) + \lambda)^{-1}\| \leq \frac{1}{\lambda + M^2}$ ,  $\lambda > 0$  follows, and then we have

$$\left\| \left( U(\Lambda)^{-1} (H_p \otimes I) U(\Lambda) - H_p \otimes I \right) \Psi \right\| \leq \sum_{j=1}^N \frac{1}{\pi} \int_0^\infty \frac{\sqrt{\lambda}}{\lambda + M^2} \|Q_j(\Lambda) (B_j + \lambda)^{-1} \Psi\| d\lambda. \quad (24)$$

We see that

$$\begin{aligned} \|Q_j(\Lambda) (B_j + \lambda)^{-1} \Psi\| &\leq \left( \frac{\kappa}{\Lambda} \right) \left( \left\| \Pi \left( \frac{\nabla f_{\mathbf{x}_j}}{\omega} \right) \cdot (\mathbf{p}_j \otimes I) (B_j + \lambda)^{-1} \Psi \right\| + \left\| \Pi \left( \frac{\Delta f_{\mathbf{x}_j}}{\omega} \right) (B_j + \lambda)^{-1} \Psi \right\| \right) \\ &\quad + \left( \frac{\kappa}{\Lambda} \right)^2 \left\| \Pi \left( \frac{\nabla f_{\mathbf{x}_j}}{\omega} \right) \cdot \Pi \left( \frac{\nabla f_{\mathbf{x}_j}}{\omega} \right) (B_j + \lambda)^{-1} \Psi \right\|. \end{aligned} \quad (25)$$

Note that

$$\begin{aligned}
& \|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot (\hat{\mathbf{p}}_j \otimes I)(B_j + \lambda)^{-1} \Psi\| \\
& \leq \sum_v \|\Pi(\frac{\partial_{x_j^v} f_{\mathbf{x}_j}}{\omega})(I \otimes H_b + 1)^{-1/2}\| \|(\hat{\mathbf{p}}_j^v \otimes I)(B_j + \lambda)^{-1}(H_p \otimes I + 1)^{-1/2}\| \|(H_p \otimes I + 1)^{1/2}(I \otimes H_b + 1)^{1/2} \Psi\|
\end{aligned} \tag{26}$$

Here we used the boundness (4) and (5). Applying the Lemma 3.2 to  $\|(\hat{\mathbf{p}}_j^v \otimes I)(B_j + \lambda)^{-1}(H_p \otimes I + 1)^{-1/2}\|$  in (26), it is seen that for  $\delta \in (0, \frac{1}{10})$ , there exist  $\alpha_j(\delta) \geq 0$  such that

$$\|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot (\hat{\mathbf{p}}_j \otimes I)(B_j + \lambda)^{-1} \Psi\| \leq \frac{\alpha_j(\delta)}{\lambda^{\frac{1}{2}+\delta}} \|(H_p \otimes I + 1)^{1/2}(I \otimes H_b + 1)^{1/2} \Psi\|,$$

and hence we have

$$\|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot (\hat{\mathbf{p}}_j \otimes I)(B_j + \lambda)^{-1} \Psi\| \leq \frac{\alpha_j(\delta)}{\lambda^{\frac{1}{2}+\delta}} \left( \|H_p \Psi\| + \|H_b \Psi\| + \|\Psi\| \right). \tag{27}$$

Since  $(\|H_p \otimes I \Psi\| + \|I \otimes H_b \Psi\|)^2 \leq 2\|H_0(\Lambda)\|^2$ , we have

$$\int_0^\infty \frac{\sqrt{\lambda}}{\lambda + M^2} \|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot (\hat{\mathbf{p}}_j \otimes I)(B_j + \lambda)^{-1} \Psi\| d\lambda \leq \alpha_j(\delta) \left( \int_0^\infty \frac{1}{(\lambda + M^2)\lambda^\delta} d\lambda \right) \left( \sqrt{2}\|H_0(\Lambda)\Psi\| + \|\Psi\| \right). \tag{28}$$

By  $\|\Pi(\frac{\Delta f_{\mathbf{x}_j}}{\omega})(I \otimes H_b + 1)^{-1/2}\| < \infty$  and  $\|(B_j + \lambda)^{-1}\| \leq \frac{1}{\lambda + M^2}$ , we have

$$\begin{aligned}
\|\Pi(\frac{\Delta f_{\mathbf{x}_j}}{\omega})(B_j + \lambda)^{-1} \Psi\| & \leq \|\Pi(\frac{\Delta f_{\mathbf{x}_j}}{\omega})(I \otimes H_b + 1)^{-1/2}\| \|(B_j + \lambda)^{-1}\| \|(I \otimes H_b + 1)^{1/2} \Psi\| \\
& \leq \frac{1}{\lambda + M^2} \|\Pi(\frac{\Delta f_{\mathbf{x}_j}}{\omega})(I \otimes H_b + 1)^{-1/2}\| \|(I \otimes H_b + 1)^{1/2} \Psi\|.
\end{aligned} \tag{29}$$

Then by  $\|(I \otimes H_b + 1)^{1/2} \Psi\| \leq \|H_0(\Lambda)\Psi\| + \|\Psi\|$ , we have

$$\begin{aligned}
& \int_0^\infty \frac{\sqrt{\lambda}}{\lambda + M^2} \|\Pi(\frac{\Delta f_{\mathbf{x}_j}}{\omega})(B_j + \lambda)^{-1} \Psi\| d\lambda \\
& \leq \|\Pi(\frac{\Delta f_{\mathbf{x}_j}}{\omega})(I \otimes H_b + 1)^{-1/2}\| \left( \int_0^\infty \frac{\sqrt{\lambda}}{(\lambda + M^2)^2} d\lambda \right) \left( \|H_0(\Lambda)\Psi\| + \|\Psi\| \right).
\end{aligned} \tag{30}$$

In addition we also see that

$$\begin{aligned}
\|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot \Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega})(B_j + \lambda)^{-1} \Psi\| & \leq \|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot \Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega})(I \otimes H_b + 1)^{-1}\| \|(B_j + \lambda)^{-1}\| \|(I \otimes H_b + 1) \Psi\| \\
& \leq \frac{1}{\lambda + M^2} \|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot \Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega})(I \otimes H_b + 1)^{-1}\| \|(I \otimes H_b + 1) \Psi\|.
\end{aligned} \tag{31}$$

Since  $\|\Pi(\xi)\Pi(\eta)(H_b + 1)^{-1}\| < \infty$  for  $\xi, \eta \in \mathcal{D}(\omega)$ , we obtain

$$\begin{aligned} & \int_0^\infty \frac{\sqrt{\lambda}}{\lambda + M^2} \|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot \Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega})(B_j + \lambda)^{-1} \Psi\| d\lambda \\ & \leq \|\Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega}) \cdot \Pi(\frac{\nabla f_{\mathbf{x}_j}}{\omega})(H_b + 1)^{-1}\| \left( \int_0^\infty \frac{\sqrt{\lambda}}{(\lambda + M^2)^2} d\lambda \right) (\|H_0(\Lambda)\Psi\| + \|\Psi\|). \end{aligned} \quad (32)$$

Then from (28), (30), (32) and (25), the proposition follows. ■

**Proposition 3.3** Assume (A.1) - (A.4).

(1) For  $\varepsilon > 0$ , there exists  $\Lambda(\varepsilon) \geq 0$  such that for all  $\Lambda > \Lambda(\varepsilon)$ ,

$$\|K(\Lambda)\Psi\| \leq \varepsilon \|H_0(\Lambda)\Psi\| + v(\varepsilon) \|\Psi\|, \quad \Psi \in \mathcal{D}_0, \quad (33)$$

holds, where  $v(\varepsilon)$  is a constant independent of  $\Lambda \geq \Lambda(\varepsilon)$ .

(2) Then for all  $z \in \mathbf{C} \setminus \mathbf{R}$ , it follows that

$$s - \lim_{\Lambda \rightarrow \infty} K(\Lambda) (H_0(\Lambda) - z)^{-1} = V_{\text{eff}} (H_p - z)^{-1} \otimes P_{\Omega_b}, \quad (34)$$

**(Proof)**

(1) By the condition (A.4),  $V_{\text{eff}}$  is bounded. Then (1) follows from Proposition 3.1.

(2) It is seen that

$$K(\Lambda) (H_0(\Lambda) - z)^{-1} = K(\Lambda) (H_p - z)^{-1} \otimes P_{\Omega_b} + K(\Lambda) (H_0(\Lambda) - z)^{-1} (I \otimes (1 - P_{\Omega_b})).$$

By Proposition 3.1, we have

$$s - \lim_{\Lambda \rightarrow \infty} K(\Lambda) \left( (H_p - z)^{-1} \otimes P_{\Omega_b} \right) \Psi = V_{\text{eff}} \left( (H_p - z)^{-1} \otimes P_{\Omega_b} \right) \Psi. \quad (35)$$

By (33), we see that for  $\varepsilon > 0$  there exists  $\Lambda(\varepsilon) \geq 0$  such that for all  $\Lambda > \Lambda(\varepsilon)$

$$\|K(\Lambda) (H_0(\Lambda) - z)^{-1} \Phi\| \leq \varepsilon \|\Phi\| + (\varepsilon|z| + v(\varepsilon)) \| (H_0(\Lambda) - z)^{-1} \Phi \|, \quad \Phi \in \mathcal{H}.$$

Note that  $\lim_{\Lambda \rightarrow \infty} \| (H_0(\Lambda) - z)^{-1} (I \otimes (1 - P_{\Omega_b})) \Psi \| = 0$ , and hence we obtain

$$\lim_{\Lambda \rightarrow \infty} \left\| K(\Lambda) (H_0(\Lambda) - z)^{-1} (I \otimes (1 - P_{\Omega_b})) \Psi \right\| = 0. \quad (36)$$

By (35) and (36), we obtain (34). ■

**(Proof of Theorem 2.1)**

By Proposition 3.3, it is shown that  $H(\Lambda)$  satisfies the condition (S.1) and (S.2) by applying  $H_0(\Lambda)$  to  $A \otimes I + \Lambda I \otimes B$  and  $K(\Lambda)$  to  $C(\Lambda)$ . Hence by the Proposition A, we have for  $z \in \mathbf{C} \setminus \mathbf{R}$ ,

$$s - \lim_{\Lambda \rightarrow \infty} (H(\Lambda) - z)^{-1} = \lim_{\Lambda \rightarrow \infty} U(\Lambda) (H_0(\Lambda) - z)^{-1} U(\Lambda)^{-1} = (H_p + V_{\text{eff}} - z)^{-1} \otimes P_{\Omega_b}.$$

Thus the proof is completed. ■

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