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Yasutake, Kazunori
Faculty of Mathematics, Kyushu University

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Kazunori Yasutake

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ON THE CLASSIFICATION OF RANK 2 ALMOST FANO BUNDLES
ON PROJECTIVE SPACE

KAZUNORI YASUTAKE

ABSTRACT. An almost Fano bundle is a vector bundle on a smooth projective variety
that its projectivization is an almost Fano variety. In this paper, we prove that almost
Fano bundles exist only on almost Fano manifolds and study rank 2 almost Fano bundles
over projective spaces.

INTRODUCTION

An almost Fano variety is a smooth projective variety whose anti-canonical line bundle
is nef and big. This is a natural generalization of Fano varieties and often appears in the
study of deformation of a Fano variety ([12], [15]). Almost Fano surfaces were completely
classified by Demazure [3]. Recently Jahnke, Peternel and Radloff classified almost Fano
threefolds with picard number 2 whose pluri-anti-canonical morphism is divisorial in [8].
In [19], Takeuchi studied almost Fano threefolds with del pezzo fibration structure whose
pluri-anti-canonical morphism is small. In higher dimensional case, Jahnke and Peternell
classified almost del Pezzo varieties, which are almost Fano n-folds with index n − 1 i.e.
its anti-canonical line bundle is divisible by n − 1 in the Picard group.

The aim of this paper is to study ruled almost Fano varieties $M$ of dimension $n \geq 3$
over nonsingular variety $S$ i.e. there is a vector bundle $E$ on $S$ such that $M$ is isomorphic
to its projectivization $\mathbb{P}_S(E)$.

Szurek and Wisniewski introduced the notion of Fano bundle in [16]. As an almost
Fano version, we introduce the notion of almost Fano bundle as below.

DEFINITION 0.1. Let $E$ be a vector bundle on a smooth complex projective variety $M$.
We say that $E$ is almost Fano if its projectivization $\mathbb{P}_M(E)$ is an almost Fano variety.

Such bundles always exist on an almost Fano variety $M$. In fact, we notice that the
trivial rank $r$ vector bundle is almost Fano since $\mathbb{P}_M(\mathcal{O}_M^r) \cong M \times \mathbb{P}^{r-1}$ is also an almost
Fano variety. In [16, Theorem 1.6], it is shown that Fano bundles are only on Fano
manifolds. We consider the almost Fano case and obtain the following theorem.

THEOREM A. If $E$ is an almost Fano bundle over a smooth complex projective variety
$M$, then $M$ is an almost Fano variety.

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On projective spaces, rank 2 Fano bundles are completely classified in [1], [16] and [17]. Using their methods, we study the classification of rank 2 almost Fano bundles on projective spaces and have the list mentioned below.

**Theorem B.** Let $\mathcal{E}$ be a rank 2 normalized (i.e. $c_1(\mathcal{E}) = 0$ or $-1$) almost Fano bundle on $\mathbb{P}^n$. Assume that $\mathcal{E}$ is not Fano. Then, $\mathcal{E}$ is isomorphic to one of the following:

1. $\mathcal{O}_{\mathbb{P}^n}(\lfloor \frac{n}{2} \rfloor) \oplus \mathcal{O}_{\mathbb{P}^n}(\lceil -\frac{n}{2} \rceil)$, where $\lfloor \frac{n}{2} \rfloor$ is the largest integer $\leq \frac{n}{2}$.
2. a stable bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 2$.
3. a stable bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$.
4. a vector bundle on $\mathbb{P}^2$ determined by the exact sequence: $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \mathcal{I}_p(-1) \to 0$, where $\mathcal{I}_p$ is the ideal sheaf of a point $p$.
5. a stable bundle on $\mathbb{P}^2$ with $c_1 = -1$, $2 \leq c_2 \leq 5$.
6. a stable bundle on $\mathbb{P}^2$ with $c_1 = 0$, $4 \leq c_2 \leq 6$.

Moreover, we show that all cases stated above really exist, except the case when $c_2 = 6$ in (6). Note that these varieties are of index 1 or 2. On three dimensional projective space, the most difficult part is a construction of almost Fano bundles satisfying the condition in (3). To obtain this, we use Maruyama’s theory of elementary transformation of vector bundles. On projective plane, the case $c_1 = -1$ was classified in [7]. Therefore we treat the case $c_1 = 0$ i.e. $\mathbb{P}(\mathcal{E})$ is of index 1. In particular, we study almost Fano threefolds of index 1 whose pluri-anti-canonical morphism is small, having $\mathbb{P}^1$-bundle structure over $\mathbb{P}^2$.

Ruled varieties play an important role in the classification theory of projective varieties. So we may expect that our results also have applications.

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**Notation**

Throughout this paper $\mathcal{E}$ is a vector bundle on a smooth complex projective variety $M$ and $\xi_{\mathcal{E}}$ is the tautological line bundle on $X = \mathbb{P}_M(\mathcal{E})$. By $\pi$ we denote the projection $\pi : \mathbb{P}_M(\mathcal{E}) \to M$ and by $H$ the pull-back of hyperplane if $M = \mathbb{P}^n$ (i.e. $\mathcal{O}_{\mathbb{P}_M(\mathcal{E})}(H) \cong \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$). For a curve $C$ in $M$, we denote by $[C]$ the numerical equivalence class of $C$ in $M$.

1. **Proof of Theorem A**

In this section we prove Theorem A. Before starting the proof, we prepare some facts.
Definition 1.1. Let $X$ be a normal projective variety and $\Delta$ an effective $\mathbb{Q}$-divisor on $X$. Let $\varphi : Y \to X$ be a log resolution of $(X, \Delta)$. We set

$$K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,$$

where $E_i$ is a prime divisor. The pair $(X, \Delta)$ is called kawamata log terminal (klt, for short) if $a_i > -1$ for all $i$.

Definition 1.2. Let $X$ be a normal projective variety and $\Delta$ an effective $\mathbb{Q}$-divisor on $X$. We say that the pair $(X, \Delta)$ is a log Fano variety if $(X, \Delta)$ is klt and $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-divisor.

Lemma 1.3. If $X$ is an almost Fano manifold, there is an effective $\mathbb{Q}$-divisor $\Delta$ such that $(X, \Delta)$ is a log Fano variety.

Proof. For any ample divisor $A$, there are an integer $m$ and an effective divisor $E$ such that $-mK_X = A + E$ by [9, Lemma 2.60]. Put $\Delta = \frac{1}{l}E$ for $l \gg 0$, then $(X, \Delta)$ is klt from [9], corollary 2.35 and

$$-lm(K_X + \Delta) = m(l - m)(-K_X) + mA$$

is ample. \hfill \Box

Using this lemma, we get the following results by [9] and [21].

Theorem 1.4. Let $X$ be an almost Fano manifold. Then,

(1) (Basepoint-free Theorem)
Any nef divisor $D$ on $X$ is semiample (i.e. $bD$ is basepoint free for $b \gg 0$).

(2) (Cone Theorem)
There are finitely many rational curves $C_j$ on $X$ such that

$$NE(X) = \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j].$$

(3) $X$ is rationally connected i.e. for any two points in $X$ there exists a rational curve which passes through them.

The next lemma is also needed.

Lemma 1.5. (c.f. [20, Lemma 3.3]). Let $\pi : X = \mathbb{P}_M(\mathcal{E}) \to M$ be the projectivization of a rank $r$ almost Fano bundle $\mathcal{E}$ and $C$ an extremal rational curve on $X$ not contracted by $\pi$. Then, we have $0 \leq -K_X.C \leq -K_M.\pi(C)$.

Proof. Let $C$ be an extremal rational curve on $X$ not contracted by $\pi$ and $\varphi_C$ the corresponding elementary contraction map. Then $\varphi$ satisfies the assumption in [20, Lemma 3.3]. Hence we obtain the inequality in the lemma. \hfill \Box
Proof of Theorem A. Put $X = \mathbb{P}_M(E)$. From Theorem 1.4, we can find finitely many extremal rational curves $C_0, C_1, \ldots, C_\rho$ in $X$ which generate the Kleiman-Mori cone $\overline{NE}(X)$. Let $C_0$ be contained in a fiber of the projection $\pi$. Then we see that $\overline{NE}(M) = \sum_{i=1}^{\rho} \mathbb{R}_{\geq 0}[\pi(C_i)]$. From Lemma 1.5, it follows that

$$-K_M, \pi(C_i) \geq -K_X.C_i \geq 0$$

for $1 \leq i \leq \rho$. Therefore $-K_M$ is nef. Next we show the bigness of $-K_M$. Applying Theorem 1.4 (1) to $D := \pi^*(-K_M)$, we know $D$ is semiample. Because $\pi$ is projective space bundle, $-K_M$ is also semiample. Let $\varphi = \varphi_{[-lK_M]} : M \to W$ be a morphism induced by $-lK_M$ for $l \gg 0$. Suppose that $\dim M > \dim W$. Take the Stein factorization, we may assume a fiber of $\varphi$ is connected. We denote its general fiber by $F$. Then $F$ is smooth and we see that $-K_M|_F = -K_F$ holds. From this,

$$-K_X|_{\pi^{-1}(F)} = (r\xi_E - \pi^* (K_M + c_1(E)))|_{\pi^{-1}(F)}$$

$$= r\xi_{E|_F} - \pi^* (K_F + c_1(E|_F)) = -K_{E|_F}.$$

Therefore we may only consider $\varphi(M)$ is a point. In this case, Kodaira dimension $\kappa(M)$ of $M$ is equal to 0. On the other hand, $X$ is rationally connected due to Theorem 1.4 (3). Since $\pi$ is surjective, $M$ is also rationally connected. Hence we have $\kappa(M) = -\infty$. This is a contradiction. \hfill \Box

Remark 1.6. (1) This theorem is proved in [2] if $\dim X = 2$ and $\text{rank} E = 2$.

(2) Recently Fujino and Gongyo prove if $X$ is almost Fano and $f : X \to Y$ is a smooth morphism, then $Y$ should be almost Fano [5].

2. Proof of Theorem B

In this section, we study the structure of almost Fano bundles on projective space.

First we consider almost Fano bundles which are decomposed into a direct sum of line bundles. In this case, we can characterize almost Fano bundles for any rank.

Proposition 2.1. Let $E \cong \mathcal{O} \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \mathcal{O}(a_r)$ be a vector bundle on $\mathbb{P}^n$, where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_r$. Then, $E$ is almost Fano if and only if $0 \leq c_1(E) = \sum_{i=1}^{r} a_i \leq n + 1$. Moreover $E$ is not Fano if and only if $c_1(E) = n + 1$.

Proof. Put $X = \mathbb{P}_{\mathbb{P}^n}(E)$. Then, we have $-K_X = r\xi_E - (n + 1 - c_1(E))H$. From the choice of $E$, we can check naturally that $E$ is Fano if and only if $0 \leq c_1(E) \leq n$. Next we will establish the latter part. It is easy to see that $-K_M$ is nef but not ample if and only if $c_1(E) = n + 1$. Therefore it is sufficient to show that $-K_M$ is big if $c_1(E) = n + 1$. In this case, $H^0(\xi_E - H) \cong H^0(E(-1)) \neq 0$. By Kodaira’s lemma, $-K_M = r\xi_E = ((r-1)\xi_E + H) + (\xi_E - H)$ is big. \hfill \Box
From now on, we give a proof of Theorem B. The proof is divided into three parts, (I) \( n \geq 4 \), (II) \( n = 3 \) and (III) \( n = 2 \).

(I) \( n \geq 4 \).

At first, we consider the case where \( n \geq 4 \). The claim is as follows.

**Proposition 2.2.** Let \( \mathcal{E} \) be an almost Fano 2-bundle on \( \mathbb{P}^n \), \( n \geq 4 \). Then \( \mathcal{E} \) is a direct sum of two line bundles.

These bundles are classified in Proposition 2.1. To show this, we need the next two lemmata.

**Lemma 2.3.** Let \( \mathcal{E} \) be a normalized rank 2 almost Fano bundle on \( \mathbb{P}^n \). If \( n > 4 \), then \( \mathcal{E}(n) \) is generated by its global sections.

**Proof.** The proof is in the similar fashion as in [1, Proposition 2.6]. We give an outline of the proof in the case where \( n \) is even and \( c_1 = -1 \). Put \( n = 2k \) and \( X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{E}) \), then we have

\[
-K_X = 2\xi + (2k + 2)H = 2(\xi + (k + 1)H)
\]

is nef and big. Therefore \( \mathcal{E}(k+2) \) is ample vector bundle. By Le Potier vanishing theorem,

\[
H^i(\mathcal{E}(k+2+j) \otimes K_{\mathbb{P}^n}) = H^i(\mathcal{E}(j-k+1)) = 0
\]

for any \( i \geq 2 \) and \( j \geq 0 \). Especially letting \( j = 3k - i - 1 \), we have \( H^i(\mathcal{E}(n-i)) = 0 \) for \( i \geq 2 \). Moreover

\[
H^1(\mathbb{P}_{\mathbb{P}^n}(\mathcal{E}), 3(\xi + (k + 1)H) + (k - 2)H + K_{\mathbb{P}^n}(\mathcal{E})) = H^1(\mathbb{P}^n, \mathcal{E}(n-1)) = 0
\]

from Kawamata-Vieweg vanishing theorem. Combining above, then we see that \( H^i(\mathcal{E}(n-i)) = 0 \) for \( i \geq 1 \) namely \( \mathcal{E} \) is \( n \)-regular. By means of Castelnuovo-Mumford lemma, \( \mathcal{E}(n) \) is generated by its global sections. Other cases are proved in the same way. \( \square \)

**Lemma 2.4.** [1] Let \( \mathcal{E} \) be a globally generated 2-bundle on \( \mathbb{P}^n \). Then we have

1. If \( \mathcal{E} \) is not stable and \( c_2(\mathcal{E}) < (n - 1)(c_1(\mathcal{E}) - n + 2) \), then \( \mathcal{E} \) is split into a direct sum of two line bundles.
2. If \( n \geq 6 \) and \( c_1(\mathcal{E})^2 < 4c_2(\mathcal{E}) \), then we have \( c_1(\mathcal{E}) \geq 2n + 3 \).

**Proof of Proposition 2.2.** Applying Lemma 2.4 to \( \mathcal{E}(n) \), we can show immediately that \( \mathcal{E} \) is split except for \( n = 4 \) and 5 essentially in the same as in the proof of Proposition 3.1 and Proposition 5.1 in [1]. If \( n = 4 \) (resp. \( n = 5 \)), then \( \mathcal{E}(3) \) (resp. \( \mathcal{E}(4) \)) is nef. From [1, Proposition 9.2] (resp. [1, Proposition 9.4]), \( \mathcal{E} \) is split. \( \square \)

(II) \( n = 3 \).

Next, we consider the case where \( n = 3 \). To start with, we demonstrate rank 2 almost Fano bundle on \( \mathbb{P}^3 \) is one of vector bundles below.
Proposition 2.5. Let $E$ be a normalized almost Fano 2-bundle on $\mathbb{P}^3$. Then $E$ is isomorphic to a direct sum of two line bundles or one of the following:

1. stable vector bundle with $c_1 = 0$, $c_2 = 1$.
2. stable vector bundle with $c_1 = 0$, $c_2 = 2$.
3. stable vector bundle with $c_1 = 0$, $c_2 = 3$.

Proof. We shall discuss the two cases $c_1 = 0$ and $c_1 = -1$ separately.

First we treat $c_1 = -1$. Since $-K_X = 2\xi + 5H$ is nef and big, we have that $E(3)$ is ample. We can apply the argument in [16, Theorem 2.2], to this case and we can show that $E$ is decomposed into a direct sum of two line bundles.

Next we treat $c_1 = 0$. In this case, $E(2)$ is nef. If $H^0(E(-2)) \neq 0$, then we can take a non-zero section $s \in H^0(E(-2))$. If $Z := \{s = 0\} = \emptyset$, then $E$ is decomposed into a direct sum of line bundles. If $Z \neq \emptyset$, then for a line $L$ meeting $Z$ in a finite number of points we would have

$$E(-2)|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-4 - d), (d \geq 1)$$

which contradicts to the nefness of $E(2)$. If $H^0(E(-2)) = 0$ and $H^0(E(-1)) \neq 0$, then we can take a non-zero section $s \in H^0(E(-1))$. If $Z := \{s = 0\} = \emptyset$, then $E$ is decomposed into a direct sum of line bundles. If $Z \neq \emptyset$, then $Z$ is a curve. Suppose that $\deg Z \geq 2$, we can take a line $L$ intersecting with $Z$ at least two points. Then

$$E(-1)|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-2 - d), (d \geq 2)$$

and contradict to the nefness of $E(2)$. If $\deg Z = 1$, then $Z$ is a line. But,

$$\deg K_Z = \deg(K_{\mathbb{P}^3} + c_1(E(-1)))|_Z = -6.$$  

This is a contradiction. If $H^0(E(-1)) = 0$ and $H^0(E) \neq 0$, then we can take a non-zero section $s \in H^0(E)$. If $Z := \{s = 0\} = \emptyset$, then $E$ is decomposed into a direct sum of two line bundles. If $Z \neq \emptyset$, then $Z$ is a curve and $\deg Z = c_2 \geq 1$. On the other hand, $\xi_E(-K_X)^3 = 8 - 6c_2 \geq 0$. Therefore $\deg Z = 1$ i.e. $Z$ is a line. But,

$$\deg K_Z = \deg(K_{\mathbb{P}^3} + c_1(E))|_Z = -4.$$  

This is a contradiction. Finally we assume $H^0(E) = 0$ i.e. $E$ is stable. In this case, $c_1^2 < 4c_2$ and $(-K_X)^4 = 128(4 - c_2) > 0$ hold. Hence $1 \leq c_2 \leq 3$. $box{}$

It is shown [16] that all stable bundles satisfying $c_1 = 0$, $c_2 = 1$ are Fano. If $c_2 = 2$, then $E$ is 2-regular by [6]. Therefore $-K_X = 2(\xi_E + 2H)$ is nef and big i.e. $E$ is almost Fano. Such $E$ is not Fano bundle [16]. The case $c_2 = 3$ is more complicated. First we show such an almost Fano bundle really exists.

Proposition 2.6. There is an almost Fano stable bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 3$.

To show this, we use the following result.
Theorem 2.7. [13, Proposition 6] There is a nonsingular elliptic curve $C$ on a smooth quartic surface $S \subset \mathbb{P}^3$ and a very ample divisor $A$ on $S$ such that

1. $\text{Pic}(S) \cong \mathbb{Z}[A] \oplus \mathbb{Z}[C]$.
2. $A^2 = 4$, $A.C = 7$, $C^2 = 0$.
3. $C$ is base point free.
4. $S$ does not contain any rational curve.

Proof of Proposition 2.6. Let $(S, C)$ be a pair in Theorem 2.7. Using the theory of elementary transformation [10] and [11], we can construct a rank 2 regular vector bundle $F$ on $\mathbb{P}^3$ where $c_1(F) = S$, $c_2(F) = C$ modulo numerical equivalence. We will prove that $E := F(-2)$ is the bundle we want. Since $F$ has a global sections, we have a following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to F \to I_C(4) \to 0.$$ Twist by $\mathcal{O}_{\mathbb{P}^3}(-2)$, we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to F(-2) \to I_C(2) \to 0.$$ Because $C$ is not contained in any quadric surface, we see that $H^0(I_C(2)) = 0$. Therefore $F$ is stable since $H^0(F(-2)) = 0$ and $c_1(F(-2)) = 0$. Next we show $F$ is nef. Note that $F$ has 2 global sections which induce the generically surjective morphism $\varphi : \mathcal{O}^{\oplus 2} \to F$ where $\varphi$ is isomorphic outside $S$ by the construction. Consequently $F$ is nef over curves not contained in $S$. Over $S$, we get an exact sequence

$$0 \to \mathcal{O}_S(C) \to F|_S \to \mathcal{O}_S(4A-C) \to 0.$$ From the choice of $C$, $\mathcal{O}_S(C)$ is nef. We have only to check the nefness of $\mathcal{O}_S(4A-C)$. Since $(aA + bC)(4A - C) = 9a + 28b$, we must prove that $9a + 28b \geq 0$ if $aA + bC$ is effective. But this is true since $(28A - 9C)^2 = -17 < 0$ and in view of Kleiman-Mori cone. Therefore $-K_X = 2\xi_F$ is nef and big. Namely $F$ is almost Fano. Hence $E = F(-2)$ is a stable 2-bundle with $c_1 = 0$, $c_2 = 3$ which is almost Fano.

Let $\mathcal{M}(0, 3)$ be the moduli space of stable rank 2 vector bundles on $\mathbb{P}^2$ with $c_1 = 0$ and $c_2 = 3$. From Theorem in [4], we see that $\mathcal{M}(0, 3)$ has two irreducible components $\mathcal{M}_0(0, 3)$ and $\mathcal{M}_1(0, 3)$ where $\mathcal{M}_a(0, 3)$ is the moduli space of vector bundles $E$ satisfying the condition $\dim H^1(E(-2)) \equiv \alpha (\text{mod } 2)$. The dimension of each components are 21. Almost Fano bundles constructed in Proposition 2.6 are contained in $\mathcal{M}_0(0, 3)$. The author does not know whether $\mathcal{M}_1(0, 3)$ contains almost Fano bundles or not.

Next we show that each components contain the member which is not almost Fano.

Example 2.8. From Proposition in [14], we see that vector bundles in $\mathcal{M}_0(0, 3)$ which have a maximal order jumping line is of dimension 20. Such a bundle $E$ is decomposed into $\mathcal{O}_L(3) \oplus \mathcal{O}_L(-3)$ over some line $L$. These bundles cannot be almost Fano since $E(2)$ is not nef.

Example 2.9. Let $Y$ be a disjoint union of a plane cubic and a nonsingular space elliptic curve in $\mathbb{P}^3$. By Serre construction, we can construct a rank 2 bundle $F$ on $\mathbb{P}^3$.
with \( c_1 = 4, \ c_2 = 7 \). Then, we can check \( H^0(\mathcal{F}(-2)) = 0 \) due to the exact sequence \( 0 \to \mathcal{O}_{P^3} \to \mathcal{F} \to \mathcal{I}_Y(4) \to 0 \). Hence \( \mathcal{F} \) is stable. Since every nonsingular space elliptic curve is a complete intersection of two quadrics, we have \( H^0(\mathcal{I}_Y(3)) = H^0(\mathcal{F}(-1)) \neq 0 \).

From easy computation, \((\xi_{\mathcal{F}} - H)(-K_{P^2(\mathcal{F})})^3 = -1\). Thus \( \mathcal{E} := \mathcal{F}(-2) \) is a stable vector bundle with \( c_1 = 0, \ c_2 = 3 \) which is not almost Fano. We can check

\[
\dim H^1(\mathcal{E}(-2)) = \dim H^1(\mathcal{I}_Y) = \dim H^0(\mathcal{O}_Y) - 1 = 1
\]

using the exact sequence \( 0 \to \mathcal{I}_Y \to \mathcal{O}_{P^3} \to \mathcal{O}_Y \to 0 \). Hence \( \mathcal{E} \) is contained in \( \mathcal{M}_1(0, 3) \).

\[
(III) \ n = 2.
\]

Finally, we consider the case where \( n = 2 \). If \( c_1 = -1 \), then \( X = P_{P^2}(\mathcal{E}) \) is an almost del Pezzo 3-fold and completely classified in [7]. So we may only study bundles with \( c_1 = 0 \).

The statement is as follows.

**Proposition 2.10.** Let \( \mathcal{E} \) be a rank 2 almost Fano bundle on \( P^2 \) with \( c_1 = 0 \). Then, \( \mathcal{E} \) is isomorphic to one of the following

\[
\begin{align*}
(1) \ & \mathcal{O}_{P^2}(1) \bigoplus \mathcal{O}_{P^2}(-1), \\
(2) \ & \mathcal{O}_{P^2} \bigoplus \mathcal{O}_{P^2}, \\
(3) \ & \mathcal{E} \text{ is determined by } 0 \to \mathcal{O}_{P^2} \to \mathcal{E} \to \mathcal{I}_p \to 0, \text{ where } \mathcal{I}_p \text{ is the ideal sheaf of a point}, \\
(4) \ & \text{stable vector bundle with } 2 \leq c_2 \leq 6.
\end{align*}
\]

**Proof.** In this case, \( \mathcal{E}(2) \) is ample. If \( H^0(\mathcal{E}(-1)) \neq 0 \), we take a non-zero section \( s \in H^0(\mathcal{E}(-1)) \). If \( Z := \{ s = 0 \} = \emptyset \), then \( \mathcal{E} \) is decomposed into a direct sum of line bundles. If \( Z \neq \emptyset \), then for a line \( L \) meeting \( Z \) in a finite number of points we would have

\[
\mathcal{E}(-1)|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-2 - d), \ d \geq 1.
\]

This contradict to the ampleness of \( \mathcal{E}(2) \).

If \( H^0(\mathcal{E}(-1)) = 0 \) and \( H^0(\mathcal{E}) \neq 0 \), take a non-zero section \( s \in H^0(\mathcal{E}) \). If \( Z := \{ s = 0 \} = \emptyset \), then \( \mathcal{E} \) is decomposed into a direct sum of line bundles. If \( Z \neq \emptyset \) and \( \deg Z \geq 2 \), then for a line \( L \) intersecting with \( Z \) at least two points we would have

\[
\mathcal{E}|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-d), \ d \geq 2.
\]

This is a contradiction.

If \( \deg Z = 1 \), \( \mathcal{E} \) has an exact sequence \( 0 \to \mathcal{O}_{P^2} \to \mathcal{E} \to \mathcal{I}_p \to 0 \), where \( \mathcal{I}_p \) is the ideal sheaf of a point \( p \). In this case \( \mathcal{E} \) is Fano bundle by [17, Proposition 2.3]. Finally we consider the case \( H^0(\mathcal{E}) = 0 \) i.e. \( \mathcal{E} \) is stable. Then \( 2 \leq c_2 \leq 6 \) since \( (-K_X)^3 = 54 - 8c_2 > 0 \).

We have some comments of Fano bundles with \( c_1 = 0 \). If \( c_2 = 2 \), all stable bundles are Fano bundle from [17]. In the situation \( c_2 = 3 \), there is a stable Fano bundles by [17]. Moreover, we have the following result.
PROPOSITION 2.11. If $E$ is a stable almost Fano bundle on a projective plane with $c_1 = 0$, $c_2 = 3$. Then $E$ is Fano bundle.

PROOF. Let $E$ be a stable almost Fano bundle on a projective plane with $c_1 = 0$, $c_2 = 3$. Using Riemann-Roch theorem, we have $\dim H^0(E(1)) > 0$. Therefore we get an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to E(1) \to I_Z(2) \to 0$ where $Z$ is 4 points in $\mathbb{P}^2$ and $I_Z$ is the ideal sheaf of $Z$. If $E$ is not Fano, then the linear system $|\xi + H|$ has one dimensional base locus $B$ by [17], Claim 2.7. By virtue of Claim 2.10 and 2.11 in [17], we have $H.B \leq 2$ and $(\xi + H).B \leq -1$. Since $0 \leq -K_{\mathbb{P}^2}(E) \cdot B = 2(\xi + H).B + H.B \leq 0$, we obtain $H.B = 2$ and $(\xi + H).B = -1$. If $\pi(B)$ is a line $L$, we have $E(1)|_L \cong \mathcal{O}(d) \oplus \mathcal{O}(2-d)$, $d \geq 3$. This contradicts the ampleness of $E(2)$. Finally we consider the case where $\pi(B)$ is nonsingular conic $C$. Since $(\xi + H).B = -1$, we obtain the splitting $E(1)|_C \cong \mathcal{O}_C(d) \oplus \mathcal{O}_C(4-d), d \geq 5$. This is impossible because $Z$ is only 4 points.

COROLLARY 2.12. Let $E$ be a stable vector bundle on a projective plane with $c_1 = 0$, $c_2 = 3$. If $S^2(E)(3)$ is nef, then $E(1)$ is generated by global sections.

PROOF. If $S^2(E)(3)$ is nef, then $E$ is almost Fano. From Proposition 2.11, $E$ is Fano bundle. By means of Proposition 2.6 in [17], $E(1)$ is generated by global sections.

When $c_2 = 4$, no stable 2-bundle is Fano [17]. We can construct almost Fano 2 bundle with $c_1 = 0, c_2 = 4$ as follows.

EXAMPLE 2.13. Let $Y$ be 5 points in general position and $C$ is a smooth conic containing $Y$. Then the pair $(C, Y)$ yields us a rank 2 regular vector bundle $F$ with $c_1 = C$, $c_2 = Y$ by virtue of an elementary transform by [10] and [11]. We have a following exact sequence

$0 \to \mathcal{O}_{\mathbb{P}^2} \to F \to I_Y(2) \to 0$

where $I_Y$ is the ideal sheaf of $Y$. Twist by $\mathcal{O}_{\mathbb{P}^2}(-1)$, we get

$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to F(-1) \to I_Y(1) \to 0$.

Because there is no line containing $Y$, we have $H^0(I_Y(1)) = 0$. Therefore $F$ is stable since $H^0(F(-1)) = 0$ and $c_1(F(-1)) = 0$. We check $-K_{\mathbb{P}^2}(F) = 2\xi_F + H$ is nef. First we remark that $F$ has 2 global sections which induce the generically surjective morphism $\varphi: \mathcal{O}\otimes^2 \to F$ where $\varphi$ is isomorphic outside $C$ by the construction. Hence we notice that $2\xi_F + H$ is nef outside $\pi^{-1}(C)$. On $C$, we have that $F|_C \cong \mathcal{O}_C(5) \oplus \mathcal{O}_C(-1)$ from the theory of elementary transformation. From this fact, we can check that $(2\xi_F + H).D \geq 0$ for every curves $D$ contained in Hirzebruch surface $\mathbb{P}_C(F|_C)$. The equality holds only for the minimal section associated with the quotient line bundle $F|_C \to \mathcal{O}_C(-1) \to 0$. 


Therefore $-K_{P^{2}}(F)$ is nef and big. Hence $E := F(-1)$ is a stable almost Fano bundle with $c_2 = 4$.

There exists an almost Fano stable bundle with $c_1 = 0, c_2 = 5$ from Theorem 0.19(C) in [18]. Finally we construct stable vector bundles with $c_1 = 0, 3 \leq c_2 \leq 6$ which are not almost Fano.

**Example 2.14.** Let $Y_k = \{p_0, p_1, \cdots , p_k\}$ be the $k + 1$ points $(4 \leq k \leq 7)$ in $P^2$. We assume that $p_0, p_1, p_2$ are lying in a line $L$ and other points are not on $L$. By Serre construction, we have rank 2 vector bundles $E_k$ on $P^2$ with $c_1 = 2, c_2 = k + 1$. $E_k$ has an exact sequence $0 \rightarrow O_{P^2} \rightarrow E_k \rightarrow I_{Y_k}(2) \rightarrow 0$. Because there is no line containing $Y_k$, we see $\dim H^0(E_k(-1)) = \dim H^0(I_{Y_k}) = 0$. Combining with $c_1(E_k(-1)) = 0$, $E_k$ is stable bundle. Restricting each bundles into $L$, we get $E_k|_L \cong O_L(3) \oplus O_L(-1)$. Since $E_k(1)$ is not ample, $E_k$ is not almost Fano.

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Graduate School of Mathematical Sciences
Kyushu University
Fukuoka 819-0395
Japan
E-mail address: k-yasutake@math.kyushu-u.ac.jp
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