

On Simulation of Tempered Stable Random Variates

Kawai, Reiichiro
Department of Mathematics, University of Leicester

Masuda, Hiroki
Graduate School of Mathematics, Kyushu University

<https://hdl.handle.net/2324/16782>

出版情報 : MI Preprint Series. 2010-13, 2010-03-09. 九州大学大学院数理学研究院
バージョン :
権利関係 :



MI Preprint Series

**Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub**

On simulation of tempered stable random variates

**Reiichiro Kawai
& Hiroki Masuda**

MI 2010-13

(Received March 9, 2010)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

On Simulation of Tempered Stable Random Variates

REIICHIRO KAWAI* AND HIROKI MASUDA†

Abstract

Various, existing and new, simulation methods for tempered stable random variates are investigated with a view towards practical implementation, in particular simulation of increments over a very short stepsize. The methods under consideration are based on acceptance-rejection sampling, a Gaussian approximation of a small jump component, and infinite shot noise series representations. Numerical results are presented to discuss advantages, limitations and trade-off issues between approximation error and computing effort. With a given computing budget, an approximative acceptance-rejection sampling technique [2] is both most efficient and handiest, and any desired level of accuracy may be attained with a small amount of additional computing effort.

Keywords: acceptance-rejection sampling, compound Poisson, Gaussian approximation, infinite shot noise series, tempered stable distribution, characteristic function.

2010 Mathematics Subject Classification: 65C10, 68U20, 60E07, 60B10.

1 Introduction

The class of tempered stable distributions was first proposed by Tweedie [21]. Several featuring properties of tempered stable distributions and processes were revealed by Rosiński [18], such as a stable-like behavior over short intervals, the absolute continuity with respect to its short-range limiting stable process, an aggregational Gaussianity and an infinite shot noise series representation in closed form. Tempered stable distributions and processes have been used in a variety of applications, such as statistical physics [14, 15], mathematical finance [4], financial econometrics [3] and mathematical biology [16], to mention just a few. Simulation of the tempered stable distribution have thus been of great practical interest, in particular, for validation and estimation purposes.

This version: March 8, 2010.

*Email Address: reiichiro.kawai@gmail.com. Postal Address: Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK.

†Email Address: hiroki@math.kyushu-u.ac.jp. Postal Address: Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan.

On the one hand, it is well known that their increments with stability index smaller than one can be simulated exactly through, either single or double, acceptance-rejection sampling. Hence, we do not discuss the case of stability index smaller than one. To the best of our knowledge, on the other hand, there exist no *practically* exact simulation methods for tempered stable random variates with stability index greater than one. We will investigate various possible, existing and new, simulation techniques and discuss their advantages, limitations and trade-off between approximation error and computing effort, with a full view towards practical implementation.

The rest of this paper is organized as follows. After summarizing background material in brief on stable and tempered stable distributions in Section 2, we discuss acceptance-rejection sampling methods of Devroye [7] and Baeumer and Meerschaert [2]. The method of [7] yields an exact simulation in principle, but requires very time-consuming numerical integration for each fundamental quantity, while the approach of [2] provides an approximative, yet very handy and efficient, simulation method. In Section 4, we investigate simulation methods based on a suitable decomposition of the Lévy measure into small and large jump components. We apply the well known Gaussian approximation of Asmussen and Rosiński [1] to the small jump component. A trade-off issue between approximation error and computing effort has not been discussed. We also propose further compound Poisson extraction schemes on the small jump component. In Section 5, we discuss yet another simulation method based on infinite shot noise series representations. In principle, the infinite shot noise series only provides an exact simulation method for tempered stable Lévy *processes* since it simulates complete information of sample paths, that is, size, direction and timing of every single jump. A closed form of such a series representation is given in Rosiński [18] (first introduced in his discussion section of the article [3]). From a computational point of view, however, the form of infinite sum has raised important issues of finite truncation to be addressed. (See Imai and Kawai [10].)

2 Preliminaries

Let us begin this preliminary section with the notations which will be used throughout the paper. We denote by \mathbb{R} the one dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$. Let \mathbb{N} be the collection of positive integers with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $\stackrel{\mathcal{L}}{=}$ and $\stackrel{\mathcal{L}}{\rightarrow}$, respectively, identity and convergence in law. We write $f_L(z)$ for a smooth probability density function of a distribution L . We fix $(\Omega, \mathcal{F}, \mathbb{P})$ as our underlying probability space. We denote by $C_b^k(\mathbb{R}; \mathbb{R})$ by the class of k -time differentiable functions from \mathbb{R} to \mathbb{R} which, together with all their derivatives, are bounded. Finally, let us note that $\Gamma(-s) < 0$ for $s \in (0, 1)$, while $\Gamma(-s) > 0$ for $s \in (1, 2)$.

2.1 Spectrally Positive Stable Processes

Let $\{L_t^{(s)} : t \geq 0\}$ be a *totally positively skewed* stable (Lévy) process satisfying

$$\begin{aligned} \mathbb{E} \left[e^{iyL_t^{(s)}} \right] &= \exp \left[ta\Gamma(-\alpha) \cos\left(\frac{\pi\alpha}{2}\right) |y|^\alpha \left(1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn}(y)\right) \right] \\ &= \begin{cases} \exp \left[t \int_{\mathbb{R}_+} (e^{iyz} - 1) \frac{a}{z^{\alpha+1}} dz \right], & \text{if } \alpha \in (0, 1), \\ \exp \left[t \int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) \frac{a}{z^{\alpha+1}} dz \right], & \text{if } \alpha \in (1, 2), \end{cases} \end{aligned} \quad (2.1)$$

with some $a > 0$. Throughout this paper, we exclude the case $\alpha = 1$. We write $S(\alpha, a) := \mathcal{L}(L_1^{(s)})$. Note that the random variable $L_1^{(s)}$ takes values only in \mathbb{R}_+ if $\alpha \in (0, 1)$, while in \mathbb{R} if $\alpha \in (1, 2)$. It holds that for each $t > 0$, $\mathcal{L}(L_t^{(s)}) = S(\alpha, ta)$, and by the scaling property, $\mathcal{L}(t^{-1/\alpha} L_t^{(s)}) = S(\alpha, a)$. Note that the distribution $S(\alpha, ta)$ has density $t^{-1/\alpha} f_{S(\alpha, a)}(t^{-1/\alpha} x)$. The distribution $S(\alpha, a)$ can be simulated in the exact sense through the well known representation, due to Chambers et al. [5],

$$S(\alpha, a) \stackrel{\mathcal{L}}{=} (-a\Gamma(-\alpha) \cos(\pi\alpha/2))^{1/\alpha} \frac{\sin(\alpha U + \theta)}{(\cos U \cos \theta)^{1/\alpha}} \left(\frac{\cos((1-\alpha)U - \theta)}{E} \right)^{\frac{1-\alpha}{\alpha}}, \quad (2.2)$$

where $\theta := \arctan(\tan(\pi\alpha/2))$, U is a uniform random variable on $(-\pi/2, \pi/2)$ and E is a standard exponential random variable independent of U . See Zolotarev [22] for complete details on the stable distribution.

2.2 Spectrally Positive Tempered Stable Processes

Let $\{L_t^{(\text{ts})} : t \geq 0\}$ be a *centered* and *totally positively skewed* tempered stable (Lévy) process satisfying

$$\mathbb{E} \left[e^{iyL_t^{(\text{ts})}} \right] = \exp \left[t \int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) a \frac{e^{-bz}}{z^{\alpha+1}} dz \right] = \exp \left[ta\Gamma(-\alpha) ((b-iy)^\alpha - b^\alpha + iy\alpha b^{\alpha-1}) \right].$$

When $\alpha \in (0, 1)$, by adding back the centering term as $L_t^{(\text{ts})} + t\Gamma(1-\alpha)ab^{\alpha-1}$, we can recover the tempered stable subordinator. Throughout the paper, we will use the notations

$$TS(\alpha, a, b) := \mathcal{L} \left(L_1^{(\text{ts})} \right), \quad (2.3)$$

and

$$TS'(\alpha, a, b) := \mathcal{L} \left(L_1^{(\text{ts})} + \Gamma(1-\alpha)ab^{\alpha-1} \right). \quad (2.4)$$

It is known that

$$\frac{e^{-bz}}{\mathbb{E} \left[e^{-bL_1^{(s)}} \right]} f_{S(\alpha, a)}(z) = e^{-bz - a\Gamma(-\alpha)b^\alpha} f_{S(\alpha, a)}(z) = f_{TS'(\alpha, a, b)}(z), \quad (2.5)$$

and clearly

$$\begin{aligned} f_{TS(\alpha,a,b)}(z) &= f_{TS'(\alpha,a,b)}(z - \Gamma(1-\alpha)ab^{\alpha-1}) \\ &= e^{-bz-a(\alpha+1)\Gamma(-\alpha)b^\alpha} f_{S(\alpha,a)}(z - \Gamma(1-\alpha)ab^{\alpha-1}). \end{aligned} \quad (2.6)$$

Those hold for every $\alpha \in (0, 1) \cup (1, 2)$. Let us first focus on the case $\alpha \in (0, 1)$ with

$$\mathbb{E} \left[e^{iy(L_t^{(\text{ts})} + t\Gamma(1-\alpha)ab^{\alpha-1})} \right] = \exp \left[t \int_{\mathbb{R}_+} (e^{iyz} - 1) a \frac{e^{-bz}}{z^{\alpha+1}} dz \right] = \exp [ta\Gamma(-\alpha)((b-iy)^\alpha - b^\alpha)].$$

(Note that this never holds for $\alpha \in (1, 2)$.) Based upon this fact and the density function (2.5), it is well known (for example, [2, 8]) that when $\alpha \in (0, 1)$, the tempered stable distribution $TS'(\alpha, a, b)$ can be simulated *exactly* through acceptance-rejection sampling as follows.

Algorithm 0;

Step 1. Generate U as uniform $(0, 1)$ and V as $S(\alpha, a)$.

Step 2. If $U \leq e^{-bV}$, exit with V . Otherwise, return to Step 1.

For example, this exact acceptance-rejection sampling paves the way for efficient simulation of tempered stable Ornstein-Uhlenbeck processes. (See Kawai and Masuda [11, 12].) Moreover, another exact double rejection method is developed in [8] based on the Zolotarev integral representation [22] of the density function $f_{S(\alpha,a)}(z)$. In what follows, we concentrate on the case $\alpha \in (1, 2)$, for which there exist no exact simulation methods. Without loss of generality, we focus on a *centered* and *totally positively skewed* tempered stable random variate $X(\Delta)$ satisfying

$$\begin{aligned} \varphi_\Delta(y) &:= \mathbb{E} \left[e^{iyX(\Delta)} \right] = \exp \left[\Delta \int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) a \frac{e^{-bz}}{z^{\alpha+1}} dz \right] \\ &= \exp [\Delta a\Gamma(-\alpha)((b-iy)^\alpha - b^\alpha + iy\alpha b^{\alpha-1})]. \end{aligned} \quad (2.7)$$

Let us note that simulation of increments of general infinite-variation tempered stable Lévy processes (with bilateral jumps) is within our scope. This can be done simply through a convolution of two independent tempered stable random variables totally skewed in the opposite directions. (See Remark 3.1 for a related discussion.)

Remark 2.1. Let $L_t^{(s)}$ and $L_t^{(\text{ts})} + t\Gamma(1-\alpha)ab^{\alpha-1}$ be random variables respectively with distributions $S(\alpha, ta)$ under the probability measure \mathbb{Q} and $TS'(\alpha, ta, b)$ under \mathbb{P} . It is a straightforward application of Theorem 33.3 of Sato [19] to evaluate an expected value related to tempered stable random variables by the density transform

$$\mathbb{E}_{\mathbb{P}} \left[\Phi \left(L_t^{(\text{ts})} \right) \right] = \mathbb{E}_{\mathbb{Q}} \left[\frac{d\mathbb{P}}{d\mathbb{Q}} \Big|_{\mathcal{G}} \Phi \left(L_t^{(s)} \right) \right], \quad (2.8)$$

with $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbb{E}_{\mathbb{P}}[|\Phi(L_t^{(\text{ts})})|] < +\infty$. Here, the Radon-Nykodym derivative is given in closed form $(d\mathbb{P}/d\mathbb{Q})|_{\mathcal{G}} = e^{-bL_t^{(s)}}/\mathbb{E}_{\mathbb{Q}}[e^{-bL_t^{(s)}}]$, \mathbb{Q} -a.s., where \mathcal{G} is the minimal σ -field generated

by the random variable $L_t^{(s)}$. The equality (2.8) is valid for every $\alpha \in (0, 1) \cup (1, 2)$. Evaluation of expectations based on (2.8) does not require simulation of $L_t^{(ts)}$, but only requires simulation of $L_t^{(s)}$, which is simple through the representation (2.2). This density transform formulation is found useful in the computation of Greeks under an asset price model driven by tempered stable processes. (See Kawai and Takeuchi [13] for details.) However, this formulation is not valid for simulation of replications but only valid for evaluation of expectations. \square

3 Acceptance-Rejection Sampling

In this section, we discuss two (one exact and the other approximative) acceptance-rejection sampling techniques for simulation of tempered stable random variables.

3.1 Exact Sampling Using Density Function

It holds by the well known result of Devroye [7] that for $z \in \mathbb{R}$,

$$f_{TS(\alpha, \Delta a, b)}(z) \leq \min \left[C_1(\Delta), \frac{C_2(\Delta)}{z^2} \right] := q_\Delta(z), \quad (3.1)$$

where

$$C_1(\Delta) := \frac{1}{2\pi} \int_{\mathbb{R}} |\phi_\Delta(y)| dy, \quad C_2(\Delta) := \frac{1}{2\pi} \int_{\mathbb{R}} |\phi_\Delta''(y)| dy, \quad (3.2)$$

and that

$$C_3(\Delta) := \int_{\mathbb{R}} q_\Delta(z) dz = 4\sqrt{C_1(\Delta)C_2(\Delta)}.$$

Let U_1 and U_2 be iid uniform random variables on $(-1, +1)$. It is also shown that the random variable V defined by

$$V := \sqrt{\frac{C_2(\Delta)}{C_1(\Delta)}} \frac{U_1}{U_2} \quad (3.3)$$

has density $(C_3(\Delta))^{-1} q_\Delta(z)$. Based on the above facts, we can employ a simulation method based on acceptance-rejection sampling as follows.

Algorithm 1;

Step 1. Generate V as (3.3) and U as $U(0, 1)$ independent of V . If $|V| < \sqrt{C_2(\Delta)/C_1(\Delta)}$, then go to Step 3.

Step 2. If

$$C_2(\Delta)U < f_{TS(\alpha, \Delta a, b)}(V)V^2,$$

then exit with $Y_1(\Delta) \leftarrow V$. Otherwise, go to Step 1.

Step 3. If

$$C_1(\Delta)U < f_{TS(\alpha, \Delta a, b)}(V),$$

then exit with $Y_1(\Delta) \leftarrow V$. Otherwise, go to Step 1.

This is an exact simulation algorithm, that is, $Y_1(\Delta) \stackrel{\mathcal{L}}{=} X(\Delta)$. The acceptance rate at Step 1 of Algorithm 1 is given by

$$p_1(\Delta) := \frac{1}{C_3(\Delta)},$$

and thus the expected number of times Step 1 is executed is $C_3(\Delta)$. As discussed in Remark 4 of [8], Algorithm 1 has already been enhanced in terms of constant shift in the sense that the tempered stable distribution is centered in our setting (2.7).

By recalling (2.7) and observing that

$$\varphi''_{\Delta}(y) = -\varphi_{\Delta}(y)\Delta a \left[\Delta a \Gamma(1-\alpha)^2 (b^{\alpha-1} - (b-iy)^{\alpha-1})^2 + \Gamma(2-\alpha)(b-iy)^{\alpha-2} \right],$$

it seems difficult to obtain $C_1(\Delta)$ and $C_2(\Delta)$ in closed form. We thus need to compute $C_1(\Delta)$ and $C_2(\Delta)$ based on (3.2) through some numerical integration techniques. Note that numerical integration here does not have to be extremely accurate, as long as the inequality (3.1) holds true.

The important point in question is how to prepare $f_{TS(\alpha, \Delta a, b)}(V)$ in Step 2 and 3 of Algorithm 1, where as mentioned, the density $f_{TS(\alpha, \Delta a, b)}(z)$ is unavailable in closed form for any $\alpha \in (1, 2)$. One straightforward approach is to compute the density by the Fourier inverse of the characteristic function (2.7), while the other is to compute the density $f_{S(\alpha, \Delta a)}(z)$ of the associated stable distribution in order to use the relation (2.6). It would be more sensible to take the latter route since some math tools provide a function that returns density values of the stable distribution, such as `dstable` in R language. With the help of such existing functions, we may either (i) compute $f_{S(\alpha, \Delta a)}(V - \Delta \Gamma(1-\alpha)ab^{\alpha-1})$ whenever required for each V , (ii) compute $f_{S(\alpha, \Delta a)}(v_k)$ at several pre-selected points $\{v_k\}_{k \in \mathbb{N}}$ and use interpolation for each replication, or could be (iii) a combination of them. Note that they are, strictly speaking, both approximative since numerical integration is used. Hence, the choice is, in principle, up to how many replications to be generated.

To discuss the efficiency of this numerical approach, the key quantities are $C_1(\Delta)$, $C_2(\Delta)$ and $C_3(\Delta)$. Observe that for each $y \in \mathbb{R}$, $\lim_{\Delta \downarrow 0} |\varphi_{\Delta}(y)| = 1$ and $\lim_{\Delta \downarrow 0} |\varphi''_{\Delta}(y)| = 0$. Hence, it holds that $\lim_{\Delta \downarrow 0} C_1(\Delta) = +\infty$ and $\lim_{\Delta \downarrow 0} C_2(\Delta) = 0$. Also, it is more expensive to compute $C_1(\Delta)$ with a smaller Δ , while $C_2(\Delta)$ with a larger Δ . Nevertheless, it seems difficult to discuss the behavior of $C_3(\Delta)$ with respect to Δ , which largely governs the efficiency of Algorithm 1.

We report in Table 1 numerical results of $C_3(\Delta)$ for various parameter settings. It seems safe to conclude that Algorithm 1 tends to be more efficient (i) with a larger Δ , (ii) with a larger α , and (iii) with a larger b . In other words, Algorithm 1 is more efficient when the tempered stable distribution is closer to a Gaussian distribution. (It is known that the tempered stable distribution approaches to a Gaussian distribution with larger Δ , b and α , while it is closer to a stable distribution with smaller Δ and b . See, for example, Section 3 of [18].) We also provide in Figure 1 comparisons of the density $f_{TS(\alpha, \Delta a, b)}(z)$ and its bounding function $q_{\Delta}(z)$ in the inequality (3.1). In conclusion, it seems sensible to employ this approach for simulation of increments over a longer time stepsize, but not for simulation of small increments, for example in approximation of stochastic differential equations.

b	α	$\Delta = 0.001$	$\Delta = 0.010$	$\Delta = 0.100$	$\Delta = 1.000$
0.1	1.2	0.280	0.317	0.382	0.483
	1.5	0.483	0.499	0.529	0.573
	1.8	0.615	0.618	0.624	0.631
1.0	1.2	0.328	0.400	0.505	0.597
	1.5	0.512	0.550	0.596	0.626
	1.8	0.623	0.629	0.634	0.636
2.0	1.2	0.350	0.435	0.544	0.615
	1.5	0.527	0.571	0.612	0.632
	1.8	0.627	0.632	0.635	0.637

Table 1: Numerical results of the acceptance rate $p_1(\Delta)$ of Algorithm 1 under various parameter settings. We fix $a = 1$ here.

Remark 3.1. For simulation of the bilateral tempered stable distribution, that is, with a characteristic function

$$y \mapsto \exp \left[\Delta \int_{\mathbb{R}_0} (e^{iyz} - 1 - iyz) \left(a_+ \frac{e^{-b_+z}}{z^{\alpha_++1}} \mathbb{1}_{\mathbb{R}_+}(z) + a_- \frac{e^{-b_-|z|}}{|z|^{\alpha_-+1}} \mathbb{1}_{\mathbb{R}_-}(z) \right) dz \right],$$

we need to implement Algorithm 1 at least twice; once for the positive component and the other for the negative. This is so because the simple relation (2.6) does not hold simultaneously for both the positive and negative components. \square

3.2 Approximative Sampling with Stable Proposal Distribution

The second acceptance-rejection sampling is an approximative method of [2]. Let us first state the algorithm.

Algorithm 2;

Step 0. Fix $c > 0$.

Step 1. Generate U as uniform $(0, 1)$ and $V(\Delta)$ as $S(\alpha, \Delta a)$.

Step 2. If $U \leq e^{-b(V(\Delta)+c)}$, exit with $Y_2(\Delta, c) \leftarrow V(\Delta) - \Delta\Gamma(1 - \alpha)ab^{\alpha-1}$. Otherwise, return to Step 1.

This is not an exact simulation algorithm, that is, $\mathcal{L}(Y_2(\Delta, c)) \neq TS(\alpha, \Delta a, b)$ for any $c \in \mathbb{R}_+$. The constant shift $-\Delta\Gamma(1 - \alpha)ab^{\alpha-1}$ in Step 2 accounts for the difference between (2.3) and (2.4). Basic properties of Algorithm 2 are discussed in [2]. The acceptance rate at Step 2 of Algorithm 2 is

$$p_2(\Delta, c) := \mathbb{E} \left[e^{-b(V(\Delta)+c)}; V(\Delta) > -c \right] + \mathbb{P}(V(\Delta) \leq -c).$$

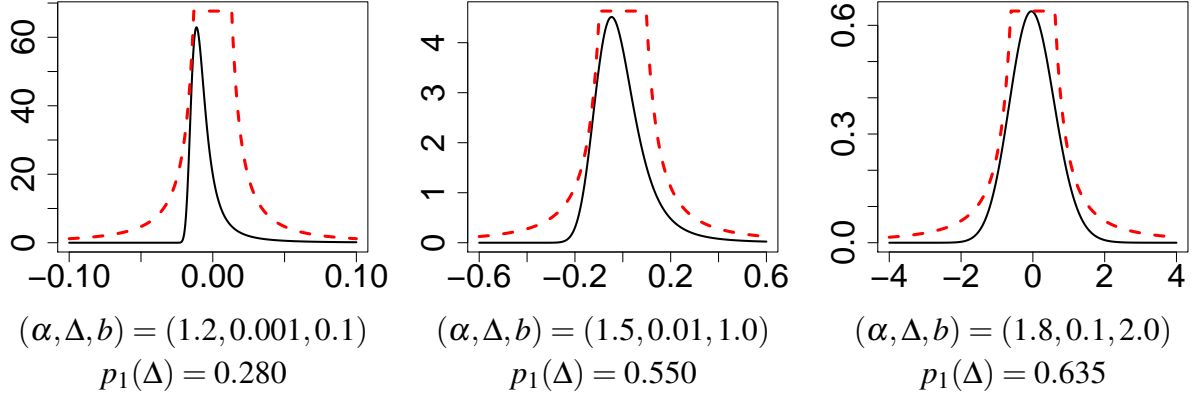


Figure 1: Comparison of $f_{TS(\alpha, \Delta, b)}(z)$ (solid line) and $q_\Delta(z)$ (dotted line) in the inequality (3.1). We fix $a = 1$ here.

The distribution function $\mathbb{P}(Y_2(\Delta, c) \leq z)$ and a density function $f_{\mathcal{L}(Y_2(\Delta, c))}(z)$ are given by

$$\mathbb{P}(Y_2(\Delta, c) \leq z) = \frac{1}{p_2(\Delta, c)} \left(\mathbb{P}(V(\Delta) \leq \min(x, -c)) + \int_{\min(x, -c)}^z e^{-b(y+c)} f_{S(\alpha, \Delta a)}(y) dy \right),$$

$$f_{\mathcal{L}(Y_2(\Delta, c))}(z) = \begin{cases} p_2(\Delta, c)^{-1} f_{S(\alpha, \Delta a)}(z), & \text{if } z \in (-\infty, -c], \\ p_2(\Delta, c)^{-1} e^{-b(z+c)} f_{S(\alpha, \Delta a)}(z), & \text{if } z \in (-c, +\infty). \end{cases}$$

The parameter c in Algorithm 2 acts as a truncation of the entire real line \mathbb{R} to the domain on which the exponential tempering e^{-bz} is performed. It is also proved in Theorem 8 [2] that the density $f_{\mathcal{L}(Y_2(\Delta, c))}(z)$ converges in $L^1(\mathbb{R})$ to its target density $f_{TS(\alpha, \Delta a, b)}(z)$ as $c \uparrow +\infty$, and as a consequence, the Kolmogorov-Smirnov distance $D_{KS}(\Delta, c) := D_{KS}(\mathcal{L}(Y_2(\Delta, c)), TS(\alpha, \Delta a, b))$ converges to zero as well. Nevertheless, it is not sensible to simply aim at a smaller distribution error by taking $c \uparrow +\infty$, since then Algorithm 3 becomes extremely inefficient due to the low acceptance rate, that is, for each $\Delta > 0$, $\lim_{c \uparrow +\infty} p_2(\Delta, c) = 0$. (Note also that for each $c > 0$, $\lim_{\Delta \downarrow 0} p_2(\Delta, c) = 1$.)

Concerning the computing effort, as before, we wish to find c maximizing $p_2(\Delta, c)$. Asymptotic behaviors of $p_2(\Delta, c)$ with respect to c are difficult to obtain in closed form. Next, it is not clear how to choose an appropriate criterion to measure the distribution error. Natural candidates include $L^1(\mathbb{R})$ - and $L^2(\mathbb{R})$ -distances between $f_{\mathcal{L}(Y_2(\Delta, c))}(z)$ and $f_{TS(\alpha, \Delta a, b)}(z)$, while the Kolmogorov-Smirnov distance $D_{KS}(\Delta, c) := D_{KS}(\mathcal{L}(Y_2(\Delta, c)), TS(\alpha, \Delta a, b))$ is certainly valid as well. None of them are tractable in an explicit manner. Let us present in Table 2 numerical results of the quantity $D_{KS}(\Delta, c)/p_2(\Delta, c)$ for different values of c . We only provide results for a single parameter set $(\alpha, a, b) = (1.5, 1.0, 1.0)$ and $\Delta = 0.1$ and $\Delta = 0.01$.

It can be observed that the Kolmogorov-Smirnov distance $D_{KS}(\Delta, c)$ decreases in c , while the acceptance rate $p_2(\Delta, c)$ has to decrease as well. The quantities $D_{KS}(\Delta, c)/p_2(\Delta, c)$ indicate that

$\Delta = 0.01$				$\Delta = 0.10$			
c	$D_{KS}(\Delta, c)$	$p_2(\Delta, c)$	$\frac{D_{KS}(\Delta, c)}{p_2(\Delta, c)}$	c	$D_{KS}(\Delta, c)$	$p_2(\Delta, c)$	$\frac{D_{KS}(\Delta, c)}{p_2(\Delta, c)}$
0.00	2.48E-2	0.954	2.60E-2	0.0	1.26E-1	0.878	1.43E-1
0.06	1.58E-2	0.931	1.69E-2	0.6	3.68E-2	0.661	5.57E-2
0.12	6.28E-3	0.896	7.02E-3	0.8	1.29E-2	0.560	2.31E-2
0.13	5.03E-3	0.889	5.66E-3	1.0	2.65E-3	0.464	5.71E-3
0.14	4.73E-3	0.881	5.37E-3	1.1	9.48E-4	0.421	2.25E-3
0.15	4.73E-3	0.874	5.42E-3	1.2	9.10E-4	0.381	2.39E-3
0.18	4.73E-3	0.850	5.57E-3	1.3	9.10E-4	0.345	2.64E-3

Table 2: Numerical results of distribution error and acceptance rate for different levels c .

choosing c greater than around 0.14 for $\Delta = 0.01$ and 1.1 for $\Delta = 0.1$ would not help in total, just as can be seen from the fact that $D_{KS}(\Delta, c)$ does not improve anymore, while the acceptance rate $p_2(\Delta, c)$ still gets worse. Let us however remind again that the Kolmogorov-Smirnov distance is simply one of various candidates as a measure of distribution error and the best choice of the parameter c may be different for different criteria. Finally, we draw in Figure 2 some resulting density functions $f_{\mathcal{L}(Y_2(\Delta, c))}(z)$ with different choices of c when $\Delta = 0.1$, together with the target tempered stable density function $f_{TS(\alpha, \Delta a, b)}(z)$. We do not provide figures for $\Delta = 0.01$, while two densities are almost indistinguishable even with a very small $c > 0$.

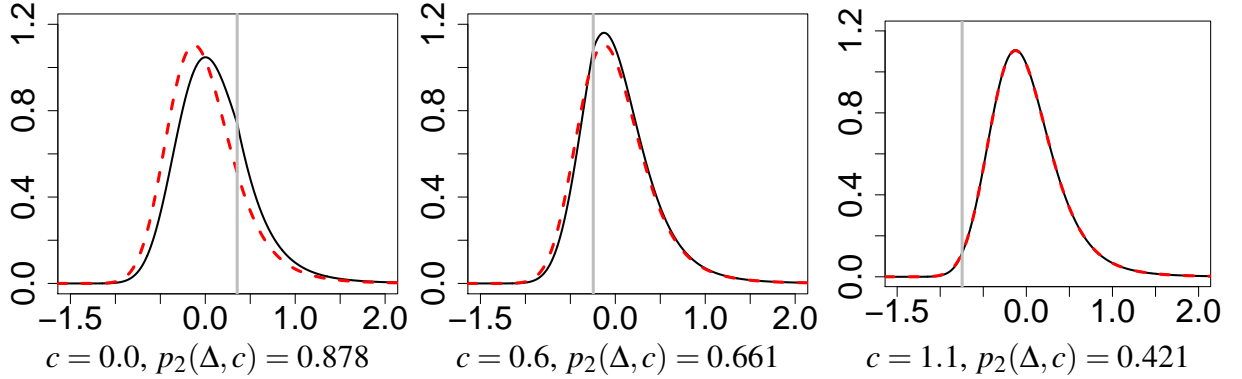


Figure 2: Comparison of two density functions $f_{\mathcal{L}(Y_2(\Delta, c))}(z)$ (solid line) and $f_{TS(\alpha, \Delta a, b)}(z)$ (dotted line) under $(\alpha, a, b) = (1.5, 1.0, 1.0)$ and $\Delta = 0.10$. The horizontal line indicates $x = -c - \Delta\Gamma(1 - \alpha)ab^{\alpha-1}$.

The implementation of Algorithm 2 is very simple and requires no computation of a density function unlike in Algorithm 1. In particular, when Δ is small, the acceptance rate remains remarkably high while the distribution error is negligible. We may find an optimal parameter value of c instantaneously through a standard numerical approach, such as the Nelder-Mead direct search

method. In total, this algorithm would be a better choice than Algorithm 1 for simulation of the tempered stable distribution from a practical point of view.

Remark 3.2. The Zolotarev integral representation is known [22] even for $\alpha \in (1, 2)$, but has to be expressed separately on the positive and negative domains. It thus seems difficult to develop a double rejection method [8] of practical use. \square

4 Decomposition into Small and Large Jump Components

In this section, we consider approximative simulation methods based on decomposition into a small jump component and the remaining large jump component. To be more precise, write

$$g_\Delta(z) := \Delta a \frac{e^{-bz}}{z^{\alpha+1}}, \quad z \in \mathbb{R}_+,$$

and decompose the characteristic exponent $\int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) \Delta a z^{-\alpha-1} e^{-bz} dz$ into three independent components as

$$\begin{aligned} \ln \mathbb{E} \left[e^{iyX(\Delta)} \right] &= \int_{\mathbb{R}_+} (e^{iyz} - 1 - iyz) g_\Delta(z) dz \\ &= \int_0^\varepsilon (e^{iyz} - 1 - iyz) g_\Delta(z) dz + \int_\varepsilon^{+\infty} (e^{iyz} - 1) g_\Delta(z) dz - iy \int_\varepsilon^{+\infty} z g_\Delta(z) dz, \\ &=: h_{\varepsilon, \Delta}^{(1)}(y) + h_{\varepsilon, \Delta}^{(2)}(y) - h_{\varepsilon, \Delta}^{(3)}(y), \end{aligned}$$

for some $\varepsilon > 0$. First, the component $h_{\varepsilon, \Delta}^{(3)}(y)$ clearly corresponds to a constant as

$$h_{\varepsilon, \Delta}^{(3)}(y) = iy \frac{\Delta a}{\alpha - 1} \left(\varepsilon^{1-\alpha} e^{-b\varepsilon} - b^{\alpha-1} \Gamma(2 - \alpha, b\varepsilon) \right) =: iy\theta_{\varepsilon, \Delta}.$$

In what follows, we use the notations $H_{\varepsilon, \Delta}^{(1)}$ and $H_{\varepsilon, \Delta}^{(2)}$ for random variables satisfying

$$\ln \mathbb{E} \left[e^{iyH_{\varepsilon, \Delta}^{(k)}} \right] = h_{\varepsilon, \Delta}^{(k)}(y), \quad k = 1, 2,$$

and call $H_{\varepsilon, \Delta}^{(1)}$ the small jump component and $H_{\varepsilon, \Delta}^{(2)}$ the large jump component. It holds that for each $\varepsilon > 0$ and $\Delta > 0$,

$$X(\Delta) \stackrel{\mathcal{L}}{=} H_{\varepsilon, \Delta}^{(1)} + H_{\varepsilon, \Delta}^{(2)} - \theta_{\varepsilon, \Delta}.$$

4.1 Simulation of Large Jump Component

We first discuss simulation of the large jump component $H_{\varepsilon, \Delta}^{(2)}$. This component is compound Poisson with intensity

$$\xi_{\varepsilon, \Delta} := \int_\varepsilon^{+\infty} \Delta a \frac{e^{-bz}}{z^{\alpha+1}} dz = \frac{\Delta a}{\alpha(\alpha-1)} \left(\frac{(\alpha-1)e^{-b\varepsilon}}{\varepsilon^\alpha} - \frac{be^{-b\varepsilon}}{\varepsilon^{\alpha-1}} + b^\alpha \Gamma(2 - \alpha, b\varepsilon) \right). \quad (4.1)$$

4.1.1 Straightforward Compound Poisson Simulation

The straightforward method is based on the summation of iid suitable random variables through

$$H_{\varepsilon, \Delta}^{(2)} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{N_{\varepsilon, \Delta}} Y_k,$$

where $N_{\varepsilon, \Delta}$ is a Poisson random variate with intensity $\xi_{\varepsilon, \Delta}$ and $\{Y_k\}_{k \in \mathbb{N}}$ is a sequence of iid random variables with common probability density

$$\frac{1}{\xi_{\varepsilon, \Delta}} \Delta a \frac{e^{-bz}}{z^{\alpha+1}}, \quad z \in (\varepsilon, +\infty).$$

This concept is indeed straightforward, while never as handy as often claimed in the literature, for mainly two reasons. First, when the truncation ε is chosen too small (which is in principle desirable), the compound Poisson intensity $\xi_{\varepsilon, \Delta}$ explodes of order $\varepsilon^{-\alpha}$. Also, we need to deal with numerical integration and inversion of the common distribution of the random sequence $\{Y_k\}_{k \in \mathbb{N}}$.

4.1.2 Acceptance-Rejection Sampling with Pareto Proposal Distribution

We here present an exact simulation technique for the large jump component $H_{\varepsilon, \Delta}^{(2)}$. Notice that the Lévy density of the component is bounded from above as

$$\Delta a \frac{e^{-bz}}{z^{\alpha+1}} \leq \Delta a \frac{e^{-b\varepsilon}}{\alpha \varepsilon^\alpha} \left(\frac{\alpha \varepsilon^\alpha}{z^{\alpha+1}} \right) =: \xi_{\varepsilon, \Delta}^{(0)} \frac{\alpha \varepsilon^\alpha}{z^{\alpha+1}}, \quad z \in (\varepsilon, +\infty), \quad (4.2)$$

where $\alpha \varepsilon^\alpha z^{-\alpha-1}$ serves as a Pareto probability density function on $(\varepsilon, +\infty)$. This Pareto random variable can easily be simulated as $\varepsilon U^{-1/\alpha}$, where $U \sim U(0, 1)$. Suppose that ε and Δ are set such that $\xi_{\varepsilon, \Delta}^{(0)} \geq 1$. Then, we can employ acceptance-rejection sampling for simulating the compound Poisson component as follows.

Algorithm 3;

Step 1. Generate U_1 and U_2 as independent uniform $(0, 1)$ and let $V \leftarrow \varepsilon U_2^{-1/\alpha}$.

Step 2. If $U_1 \leq e^{-b(V-\varepsilon)}$, exit with $Y_3(\Delta) \leftarrow V$. Otherwise, return to Step 1.

The acceptance rate at Step 2 of Algorithm 3 is $1/\xi_{\varepsilon, \Delta}^{(0)}$. The expected number of times Step 1 is executed is thus $\xi_{\varepsilon, \Delta}^{(0)}$, while the expected number of times Algorithm 3 is executed for simulation of $H_{\varepsilon, \Delta}^{(2)}$ is $\xi_{\varepsilon, \Delta}$. Therefore, Step 1 will be executed $\xi_{\varepsilon, \Delta} \xi_{\varepsilon, \Delta}^{(0)}$ times on average. In what follows, we assume that the compound Poisson component $H_{\varepsilon, \Delta}^{(2)}$ is always simulated through Algorithm 3 and will say that the expected *time* required for simulation of $H_{\varepsilon, \Delta}^{(2)}$ is $\xi_{\varepsilon, \Delta} \xi_{\varepsilon, \Delta}^{(0)}$.

4.2 Simulation of Small Jump Component

We next discuss simulation of the small jump component $H_{\varepsilon,\Delta}^{(1)}$. Throughout, we will use the notation

$$\kappa_k(\varepsilon, \Delta) := \int_0^\varepsilon z^k g_\Delta(z) dz = \frac{\Delta a}{b^{k-\alpha}} \gamma(k-\alpha, b\varepsilon).$$

Clearly, since the random variable $H_{\varepsilon,\Delta}^{(1)}$ consists of infinitely many jumps, the compound Poisson simulation never applies. To investigate approximative simulation techniques for $H_{\varepsilon,\Delta}^{(1)}$, let us derive its first three moments

$$\begin{aligned} \mathbb{E} \left[H_{\varepsilon,\Delta}^{(1)} \right] &= 0, \\ \text{Var} \left(H_{\varepsilon,\Delta}^{(1)} \right) &= \int_0^\varepsilon z^2 g_\Delta(z) dz = \kappa_2(\varepsilon, \Delta) =: \sigma_{\varepsilon,\Delta}^2, \\ \mathbb{E} \left[\left(H_{\varepsilon,\Delta}^{(1)} - \mathbb{E} \left[H_{\varepsilon,\Delta}^{(1)} \right] \right)^3 \right] &= \int_0^\varepsilon z^3 g_\Delta(z) dz = \kappa_3(\varepsilon, \Delta). \end{aligned}$$

Note also that for each $\Delta > 0$, $\kappa_k(\varepsilon, \Delta) \sim \varepsilon^{k-\alpha} \Delta a / (k-\alpha)$, as $\varepsilon \downarrow 0$.

First, based on the zero mean, it would be a valid approximation to simply replace $H_{\varepsilon,\Delta}^{(1)}$ by the mean value 0. Let us adopt the framework of Signahl [20] for evaluation of weak approximation error. It holds by the Taylor theorem that for each $f \in C_b^\infty(\mathbb{R}; \mathbb{R})$,

$$\begin{aligned} \mathbb{E} [f(X(\Delta))] - \mathbb{E} \left[f \left(H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta} \right) \right] \\ = \frac{\sigma_{\varepsilon,\Delta}^2}{2} \mathbb{E} \left[f'' \left(H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta} \right) \right] + \frac{\kappa_3(\varepsilon, \Delta)}{6} \mathbb{E} \left[f''' \left(H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta} \right) \right] + \dots \end{aligned}$$

Hence, we get for each $\Delta > 0$,

$$\left| \mathbb{E} [f(X(\Delta))] - \mathbb{E} \left[f \left(H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta} \right) \right] \right| = O(\varepsilon^{2-\alpha}),$$

as $\varepsilon \downarrow 0$. In total, the expected time required for this approximative simulation of $X(\Delta)$ is same as the one required for simulation of $H_{\varepsilon,\Delta}^{(2)}$ and is thus $\xi_{\varepsilon,\Delta} \xi_{\varepsilon,\Delta}^{(0)}$.

Next, it would be better to replace $H_{\varepsilon,\Delta}^{(1)}$ by with a normal random variable $Z_{\varepsilon,\Delta}$, where $Z_{\varepsilon,\Delta} \sim \mathcal{N}(0, \sigma_{\varepsilon,\Delta}^2)$. Let us write $X^{(0)}(\varepsilon, \Delta) := Z_{\varepsilon,\Delta} + H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta}$. This Gaussian approximation was justified in [1, 6] and is true in this case since $\sigma_{\varepsilon,\Delta}^2 / \varepsilon^2 \sim \Delta a \varepsilon^{-\alpha} / (2-\alpha) \uparrow +\infty$, as $\varepsilon \downarrow 0$. By the addition of this Gaussian component, the expected time required for simulation of $X^{(0)}(\varepsilon, \Delta)$ is increased by 1 and is thus

$$\tau_{\varepsilon,\Delta}^{(0)} := 1 + \xi_{\varepsilon,\Delta} \xi_{\varepsilon,\Delta}^{(0)}. \quad (4.3)$$

Taking into account the undesirable limit $\lim_{\varepsilon \downarrow 0} \xi_{\varepsilon,\Delta} \xi_{\varepsilon,\Delta}^{(0)} = +\infty$, the addition of the Gaussian component is negligible in terms of computing effort. Now, it holds by the Taylor theorem that for each

$f \in C_b^\infty(\mathbb{R}; \mathbb{R})$,

$$\begin{aligned} & \mathbb{E} \left[f \left(X^{(0)}(\varepsilon, \Delta) \right) \right] - \mathbb{E} [f(X(\Delta))] \\ &= \mathbb{E} \left[\left(Z_{\varepsilon, \Delta} - H_{\varepsilon, \Delta}^{(1)} \right) f'(X(\Delta)) \right] + \frac{1}{2} \mathbb{E} \left[\left(Z_{\varepsilon, \Delta} - H_{\varepsilon, \Delta}^{(1)} \right)^2 f''(X(\Delta)) \right] + \dots \end{aligned}$$

By further Taylor expansions and using the knowledge of $\mathcal{L}(Z_{\varepsilon, \Delta})$, we get

$$\left| \mathbb{E} \left[f \left(X^{(0)}(\varepsilon, \Delta) \right) \right] - \mathbb{E} [f(X(\Delta))] \right| \sim \frac{\kappa_3(\varepsilon, \Delta)}{6} \left| \mathbb{E} \left[f''' \left(H_{\varepsilon, \Delta}^{(2)} - \theta_{\varepsilon, \Delta} \right) \right] \right|,$$

as $\varepsilon \downarrow 0$. Recall that the true distribution has characteristic function $\varphi_\Delta(y)$. Meanwhile, it is straightforward that the approximation has characteristic function

$$\begin{aligned} \varphi_{\varepsilon, \Delta}^{(0)}(y) &:= \mathbb{E} \left[e^{iyX^{(0)}(\varepsilon, \Delta)} \right] = \varphi_\Delta(y) \exp \left[- \int_0^\varepsilon \left(e^{iyz} - 1 - iyz + \frac{1}{2}y^2z^2 \right) g_\Delta(z) dz \right] \\ &= \varphi_\Delta(y) \exp \left[- \frac{|y|^3}{3!} \int_0^\varepsilon \eta(yz) z^3 g_\Delta(z) dz \right], \end{aligned}$$

where η is a function from \mathbb{R} to \mathbb{C} satisfying $|\eta(x)| \leq 1$ for $x \in \mathbb{R}$. Therefore, it holds by the Parseval theorem that

$$\begin{aligned} \rho_{\varepsilon, \Delta}^{(0)} &:= \int_{\mathbb{R}} \left| f_{TS(\alpha, \Delta a, b)}(z) - f_{\mathcal{L}(X^{(0)}(\varepsilon, \Delta))}(z) \right|^2 dz \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \varphi_\Delta(y) - \varphi_{\varepsilon, \Delta}^{(0)}(y) \right|^2 dy \\ &\sim \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi_\Delta(y)|^2 \left| \frac{|y|^3}{3!} \int_0^\varepsilon \eta(yz) z^3 g_\Delta(z) dz \right|^2 dy \\ &\leq \frac{\kappa_3(\varepsilon, \Delta)^2}{72\pi} \int_{\mathbb{R}} |y^3 \varphi_\Delta(y)|^2 dy = O\left(\varepsilon^{6-2\alpha}\right), \end{aligned} \tag{4.4}$$

where all the asymptotics hold when $\varepsilon \downarrow 0$.

4.2.1 Further Compound Poisson of Flat Density

The Gaussian approximation can be improved by decomposing the Lévy density $g_\Delta(z)$ over $(0, \varepsilon)$ into two independent components $g_{\varepsilon, \Delta}^{(a)}(z) := g_\Delta(\varepsilon)$ and $g_\Delta(z) - g_{\varepsilon, \Delta}^{(a)}(z)$. We write

$$H_{\varepsilon, \Delta}^{(1)} = J_{\varepsilon, \Delta}^{(a)} + K_{\varepsilon, \Delta}^{(a)}$$

for this decomposition. The former component $J_{\varepsilon, \Delta}^{(a)}$, corresponding to the Lévy density $g_{\varepsilon, \Delta}^{(a)}(\varepsilon)$, is clearly *centered* compound Poisson with intensity

$$\xi_{\varepsilon, \Delta}^{(a)} := \frac{\Delta a e^{-b\varepsilon}}{\varepsilon^\alpha}, \tag{4.5}$$

with iid density $U(0, \varepsilon)$. In principle, this compound Poisson component can be simulated in the exact sense by

$$J_{\varepsilon, \Delta}^{(a)} \leftarrow \sum_{k=1}^{N_{\varepsilon, \Delta}^{(a)}} \varepsilon U_k - \frac{\Delta a e^{-b\varepsilon}}{2\varepsilon^{\alpha-1}},$$

The remaining component $K_{\varepsilon, \Delta}^{(a)}$, corresponding to the Lévy density $g_{\Delta}(z) - g_{\varepsilon, \Delta}^{(a)}(z)$, is still centered, is of infinite activity and is thus approximated by a normal random variable. Here, we define for each $k \in \mathbb{N}$,

$$\begin{aligned} \kappa_k^{(a)}(\varepsilon, \Delta) &:= \int_0^\varepsilon z^k \left(g_{\Delta}(z) - g_{\varepsilon, \Delta}^{(a)}(z) \right) dz \\ &= \frac{\Delta a}{b^{k-\alpha}} \gamma(k - \alpha, b\varepsilon) - \frac{\Delta a e^{-b\varepsilon}}{(k+1)\varepsilon^{\alpha-k}} \\ &\sim \Delta a \varepsilon^{k-\alpha} \frac{1 + \alpha}{(k - \alpha)(k + 1)}, \end{aligned}$$

as $\varepsilon \downarrow 0$. The Gaussian approximation can easily be justified with a variance $\kappa_2^{(a)}(\varepsilon, \Delta)$. Let $Z_{\varepsilon, \Delta}^{(a)}$ be a normal random variable with mean zero and a variance $\kappa_2^{(a)}(\varepsilon, \Delta)$. In a similar manner to the previous case, it holds by the Taylor theorem that for each $f \in C_b^\infty(\mathbb{R}; \mathbb{R})$,

$$\begin{aligned} \mathbb{E} \left[f \left(X^{(a)}(\varepsilon, \Delta) \right) \right] - \mathbb{E} [f(X(\Delta))] \\ = \mathbb{E} \left[\left(Z_{\varepsilon, \Delta}^{(a)} - K_{\varepsilon, \Delta}^{(a)} \right) f'(X(\Delta)) \right] + \frac{1}{2} \mathbb{E} \left[\left(Z_{\varepsilon, \Delta}^{(a)} - K_{\varepsilon, \Delta}^{(a)} \right)^2 f''(X(\Delta)) \right] + \dots, \end{aligned}$$

where

$$X^{(a)}(\varepsilon, \Delta) := Z_{\varepsilon, \Delta}^{(a)} + J_{\varepsilon, \Delta}^{(a)} + H_{\varepsilon, \Delta}^{(2)} - \theta_{\varepsilon, \Delta}.$$

By further Taylor expansions of f' and f'' and using the fact that $Z_{\varepsilon, \Delta}^{(a)}$ is Gaussian, we get

$$\left| \mathbb{E} \left[f \left(X^{(a)}(\varepsilon, \Delta) \right) \right] - \mathbb{E} [f(X(\Delta))] \right| \sim \frac{\kappa_3^{(a)}(\varepsilon, \Delta)}{6} \left| \mathbb{E} \left[f''' \left(H_{\varepsilon, \Delta}^{(2)} - \theta_{\varepsilon, \Delta} \right) \right] \right|,$$

as $\varepsilon \downarrow 0$. Therefore, by further introducing this compound Poisson $J_{\varepsilon, \Delta}^{(a)}$, we can reduce the weak error by a factor of

$$\lim_{\varepsilon \downarrow 0} \frac{\kappa_3^{(a)}(\varepsilon, \Delta)}{\kappa_3(\varepsilon, \Delta)} = \frac{1 + \alpha}{4} \in \left(\frac{1}{2}, \frac{3}{4} \right).$$

As before, it holds by the Parseval theorem that

$$\begin{aligned}
\rho_{\varepsilon,\Delta}^{(a)} &:= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \varphi_{\Delta}(y) - \varphi_{\varepsilon,\Delta}^{(a)}(y) \right|^2 dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi_{\Delta}(y)|^2 \left| \exp \left[- \int_0^{\varepsilon} (e^{iyz} - 1 - iyz) \left(g_{\Delta}(z) - g_{\varepsilon,\Delta}^{(a)}(z) \right) dz \right] - 1 \right|^2 dy \\
&\sim \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi_{\Delta}(y)|^2 \left| \frac{|y|^3}{3!} \int_0^{\varepsilon} \eta(yz) z^3 \left(g_{\Delta}(z) - g_{\varepsilon,\Delta}^{(a)}(z) \right) dz \right|^2 dy \\
&\leq \frac{\left(\kappa_3^{(a)}(\varepsilon, \Delta) \right)^2}{72\pi} \int_{\mathbb{R}} |y^3 \varphi_{\Delta}(y)|^2 dy,
\end{aligned} \tag{4.6}$$

where all the asymptotics hold when $\varepsilon \downarrow 0$. Clearly, the expected time required for simulation of $X^{(a)}(\varepsilon, \Delta)$ is given by

$$\tau_{\varepsilon,\Delta}^{(a)} := 1 + \xi_{\varepsilon,\Delta}^{(a)} + \xi_{\varepsilon,\Delta} \xi_{\varepsilon,\Delta}^{(0)}. \tag{4.7}$$

Remark 4.1. It is straightforward that the compound Poisson component $J_{\varepsilon,\Delta}^{(a)}$ and the large jump component $H_{\varepsilon,\Delta}^{(2)}$ of Section 4.1 can be simulated exactly as a single compound Poisson random variable. Define

$$l_{\varepsilon,\Delta}^{(a)} := \frac{\xi_{\varepsilon,\Delta}^{(a)}}{\xi_{\varepsilon,\Delta}^{(a)} + \xi_{\varepsilon,\Delta}}, \quad r_{\varepsilon,\Delta}^{(a)} := \frac{\xi_{\varepsilon,\Delta}}{\xi_{\varepsilon,\Delta}^{(a)} + \xi_{\varepsilon,\Delta}},$$

where $\xi_{\varepsilon,\Delta}^{(a)}$ and $\xi_{\varepsilon,\Delta}$ are defined, respectively, in (4.5) and (4.1). Clearly, $l_{\varepsilon,\Delta}^{(a)} + r_{\varepsilon,\Delta}^{(a)} = 1$. Then, it holds that

$$J_{\varepsilon,\Delta}^{(a)} + H_{\varepsilon,\Delta}^{(2)} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\tilde{N}_{\varepsilon,\Delta}^{(a)}} Y_k^{(a)} - \Delta a e^{-b\varepsilon} 2\varepsilon^{\alpha-1},$$

where $\tilde{N}_{\varepsilon,\Delta}^{(a)}$ is a Poisson random variable with intensity $\xi_{\varepsilon,\Delta}^{(a)} + \xi_{\varepsilon,\Delta}$ and $\{Y_k^{(a)}\}_{k \in \mathbb{N}}$ is a sequence of iid random variables with common distribution which can be simulated exactly as follows.

Algorithm 3^(a);

Step 1. Generate $U_1 \leftarrow U(0, 1)$.

Step 2. If $U_1 \leq l_{\varepsilon,\Delta}^{(a)}$, then exit with $\varepsilon U_1 / l_{\varepsilon,\Delta}^{(a)}$.

Step 3. Generate $U_2 \leftarrow U(0, 1)$ and let $V \leftarrow \varepsilon((U_1 - l_{\varepsilon,\Delta}^{(a)})/r_{\varepsilon,\Delta}^{(a)})^{-1/\alpha}$. If $U_1 \leq e^{-b(V-\varepsilon)}$, then exit with V . Otherwise, go to Step 1.

We can show that the expected total time (number of implementations of Step 1) for simulation of $J_{\varepsilon,\Delta}^{(a)} + H_{\varepsilon,\Delta}^{(2)}$ is

$$\left(\xi_{\varepsilon,\Delta}^{(a)} + \xi_{\varepsilon,\Delta} \xi_{\varepsilon,\Delta}^{(0)} \right) \frac{r_{\varepsilon,\Delta}^{(a)} \left(1 - 1/\xi_{\varepsilon,\Delta}^{(0)} \right)}{\left(1 - r_{\varepsilon,\Delta}^{(a)} \left(1 - 1/\xi_{\varepsilon,\Delta}^{(0)} \right) \right)},$$

while the expected total time for simulation of $J_{\varepsilon,\Delta}^{(a)}$ and $H_{\varepsilon,\Delta}^{(2)}$ separately is $\xi_{\varepsilon,\Delta}^{(a)} + \xi_{\varepsilon,\Delta} \xi_{\varepsilon,\Delta}^{(0)}$. We can also show that the use of Algorithm 3^(a) helps reduce computing effort if ε is sufficiently large, while increases it by a factor of 2 as $\varepsilon \downarrow 0$. We thus do not consider using this algorithm as our interest is mainly in a small ε . \square

4.2.2 Further Compound Poisson of Exploding but Integrable Density

We here try to further take more compound Poisson out of the small jump component (and then approximate the reminder by Gaussian). A straightforward approach is to extract the density

$$g_{\varepsilon,\Delta}^{(b)}(z) := \Delta a \frac{e^{-b\varepsilon}}{\varepsilon^{1+\delta}} \frac{1}{z^{\alpha-\delta}}, \quad (4.8)$$

for some $\delta \in (\alpha - 1, \alpha)$, from the Lévy density $g_{\Delta}(z)$ over $(0, \varepsilon)$. As before, we decompose as

$$H_{\varepsilon,\Delta}^{(1)} = J_{\varepsilon,\Delta}^{(b)} + K_{\varepsilon,\Delta}^{(b)},$$

where $J_{\varepsilon,\Delta}^{(b)}$ indicates the centered compound Poisson random variable corresponding to the density (4.8), while the $K_{\varepsilon,\Delta}^{(b)}$ is the remaining infinite activity component to be approximated by Gaussian. The compound Poisson intensity is given by

$$\xi_{\varepsilon,\Delta}^{(b)} := \int_0^\varepsilon g_{\varepsilon,\Delta}^{(b)}(z) dz = \frac{\Delta a e^{-b\varepsilon}}{(\delta - \alpha + 1) \varepsilon^\alpha}. \quad (4.9)$$

Note here that this is independent of δ in the sense of asymptotics of $\varepsilon \downarrow 0$. We can derive that $J_{\varepsilon,\Delta}^{(b)}$ can be simulated exactly as

$$J_{\varepsilon,\Delta}^{(b)} \leftarrow \sum_{k=1}^{N_{\varepsilon,\Delta}^{(b)}} (\varepsilon^\alpha U_k)^{\frac{1}{\delta-\alpha}} - \varepsilon^{-\alpha} \frac{\delta-\alpha-1}{\delta-\alpha} \frac{a e^{-b\varepsilon} (\delta-\alpha)}{(\delta-\alpha+1)^2},$$

where $N_{\varepsilon,\Delta}^{(b)}$ is a Poisson random variable with intensity $\xi_{\varepsilon,\Delta}^{(b)}$ and $\{U_k\}$ is a sequence of iid uniform random variables on $(0, 1)$ as before. Using

$$\begin{aligned} \kappa_k^{(b)}(\varepsilon, \Delta) &:= \int_0^\varepsilon z^k \left(g_{\Delta}(z) - g_{\varepsilon,\Delta}^{(b)}(z) \right) dz \\ &= \frac{\Delta a}{b^{k-\alpha}} \gamma(k-\alpha, b\varepsilon) - \frac{\Delta a e^{-b\varepsilon}}{(k+1+\delta-\alpha) \varepsilon^{\alpha-k}} \\ &\sim \frac{\Delta a \varepsilon^{k-\alpha}}{(k-\alpha)(k+1+\delta-\alpha)}, \end{aligned}$$

as $\varepsilon \downarrow 0$, we can derive

$$\left| \mathbb{E} \left[f \left(X^{(b)}(\varepsilon, \Delta) \right) \right] - \mathbb{E} [f(X(\Delta))] \right| \sim \frac{\kappa_3^{(b)}(\varepsilon, \Delta)}{6} \left| \mathbb{E} \left[f''' \left(H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta} \right) \right] \right|,$$

as $\varepsilon \downarrow 0$, where

$$X^{(b)}(\varepsilon, \Delta) := Z_{\varepsilon, \Delta}^{(b)} + J_{\varepsilon, \Delta}^{(b)} + H_{\varepsilon, \Delta}^{(2)} - \theta_{\varepsilon, \Delta}.$$

Therefore, by further introducing the compound Poisson $J_{\varepsilon, \Delta}^{(b)}$, we can reduce the weak error by a factor of

$$\lim_{\varepsilon \downarrow 0} \frac{\kappa_3^{(b)}(\varepsilon, \Delta)}{\kappa_3(\varepsilon, \Delta)} = \frac{1}{4 + \delta - \alpha}.$$

Since δ can be taken arbitrarily in $(\alpha - 1, \alpha)$, we can improve the approximation error down by a factor of $1/4$, compared to the simplest Gaussian approximation. Moreover, comparing with the one introduced in Section 4.2.1, we get

$$\lim_{\varepsilon \downarrow 0} \frac{\kappa_3^{(b)}(\varepsilon, \Delta)}{\kappa_3^{(a)}(\varepsilon, \Delta)} = \frac{4}{(1 + \alpha)(4 + \delta - \alpha)}.$$

Hence, we can further improve the approximation error by up to a factor of $1/(1 + \alpha)$. As before, it holds by the Parseval theorem that

$$\begin{aligned} \rho_{\varepsilon, \Delta}^{(b)} &:= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \varphi_{\Delta}(y) - \varphi_{\varepsilon, \Delta}^{(b)}(y) \right|^2 dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi_{\Delta}(y)|^2 \left| \exp \left[- \int_0^{\varepsilon} (e^{iyz} - 1 - iyz) \left(g_{\Delta}(z) - g_{\varepsilon, \Delta}^{(b)}(z) \right) dz \right] - 1 \right|^2 dy \\ &\sim \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi_{\Delta}(y)|^2 \left| \frac{|y|^3}{3!} \int_0^{\varepsilon} \eta(yz) z^3 \left(g_{\Delta}(z) - g_{\varepsilon, \Delta}^{(b)}(z) \right) dz \right|^2 dy \\ &\leq \frac{\left(\kappa_3^{(b)}(\varepsilon, \Delta) \right)^2}{72\pi} \int_{\mathbb{R}} |y^3 \varphi_{\Delta}(y)|^2 dy, \end{aligned} \tag{4.10}$$

where all the asymptotics hold when $\varepsilon \downarrow 0$. Clearly, the expected time required for simulation of $X^{(b)}(\varepsilon, \Delta)$ is given by

$$\tau_{\varepsilon, \Delta}^{(b)} := 1 + \xi_{\varepsilon, \Delta}^{(b)} + \xi_{\varepsilon, \Delta} \xi_{\varepsilon, \Delta}^{(0)}. \tag{4.11}$$

Let us close this section with some numerical results. From a practical point of view, we present in Table 3 the quantities $\rho_{\varepsilon, \Delta}^{(0)}$, $\rho_{\varepsilon, \Delta}^{(a)}$ and $\rho_{\varepsilon, \Delta}^{(b)}$, defined respectively by (4.4), (4.6) and (4.10), for the approximation error, and $\tau_{\varepsilon, \Delta}^{(0)}$, $\tau_{\varepsilon, \Delta}^{(a)}$ and $\tau_{\varepsilon, \Delta}^{(b)}$, defined respectively by (4.3), (4.7) and (4.11), for the required computing time. We computed $\rho_{\varepsilon, \Delta}^{(0)}$, $\rho_{\varepsilon, \Delta}^{(a)}$ and $\rho_{\varepsilon, \Delta}^{(b)}$ as precisely as possible by numerical integration of their definitions and did not use their asymptotic upper bounds. To check a relatively extreme setting for the technique of Section 4.2.2, we set $\delta = \alpha - 1 + 0.1$ in (4.8). In addition, to compare with Algorithm 2 of Section 3.2, we fix $(\alpha, a, b) = (1.5, 1.0, 1.0)$ and present the corresponding quantities $(\int_{\mathbb{R}} |f_{TS(\alpha, \Delta a, b)}(z) - f_{\mathcal{L}(Y_2(\Delta, c))}(z)|^2 dz)^{1/2}$ and $1/p_2(\Delta, c)$ of Algorithm 2. In short, in the Gaussian approximation framework, a lot of additional computing effort is required for improvement in approximation error, either by taking a smaller truncation ε or by introducing

the techniques of Section 4.2.1 and 4.2.2. To achieve a similar level of approximation error to the method of Section 3.2, the Gaussian approximation requires an extraordinary larger computing effort.

$\Delta = 0.01$									
ε	approximation error			computing time			c	Algorithm 2	
	$\rho_{\varepsilon,\Delta}^{(0)}$	$\rho_{\varepsilon,\Delta}^{(a)}$	$\rho_{\varepsilon,\Delta}^{(b)}$	$\tau_{\varepsilon,\Delta}^{(0)}$	$\tau_{\varepsilon,\Delta}^{(a)}$	$\tau_{\varepsilon,\Delta}^{(b)}$		error	time
0.033	3.48E-2	2.21E-2	7.79E-3	2.15E+0	3.77E+0	1.83E+1	0.00	8.83E-2	1.04E+0
0.030	3.03E-2	1.92E-2	6.74E-3	2.55E+0	4.41E+0	2.21E+1	0.12	2.55E-2	1.11E+0
0.020	1.68E-2	1.06E-2	3.65E-3	6.33E+0	9.80E+0	4.10E+1	0.14	1.66E-2	1.13E+0
0.010	6.01E-3	3.76E-3	1.28E-3	4.46E+1	5.45E+1	1.44E+2	0.18	5.26E-3	1.17E+0
0.005	2.13E-3	1.33E-3	4.52E-4	3.53E+2	3.81E+2	6.34E+2	0.20	2.54E-3	1.19E+0
0.003	9.91E-4	6.20E-4	2.10E-4	1.64E+3	1.70E+3	2.24E+3	0.24	4.88E-4	1.24E+0
0.001	1.91E-4	1.19E-4	4.02E-5	4.44E+4	4.47E+4	4.75E+4	0.30	2.02E-4	1.32E+0

$\Delta = 0.1$									
ε	approximation error			computing time			c	Algorithm 2	
	$\rho_{\varepsilon,\Delta}^{(0)}$	$\rho_{\varepsilon,\Delta}^{(a)}$	$\rho_{\varepsilon,\Delta}^{(b)}$	$\tau_{\varepsilon,\Delta}^{(0)}$	$\tau_{\varepsilon,\Delta}^{(a)}$	$\tau_{\varepsilon,\Delta}^{(b)}$		error	time
0.130	1.59E-2	1.03E-2	4.05E-3	2.50E+0	4.38E+0	2.12E+1	0.00	2.02E-1	1.14E+0
0.120	1.43E-2	9.22E-3	3.57E-3	2.96E+0	5.09E+0	2.43E+1	0.60	6.42E-2	1.51E+0
0.100	1.10E-2	7.08E-3	2.68E-3	4.55E+0	7.42E+0	3.32E+1	0.80	2.35E-2	1.78E+0
0.010	3.71E-4	2.33E-4	7.94E-5	4.36E+3	4.46E+3	5.35E+3	1.10	1.93E-3	2.37E+0
0.005	1.32E-4	8.24E-4	2.79E-5	3.52E+4	3.55E+4	3.80E+4	1.30	1.63E-4	2.90E+0
0.001	1.18E-5	7.40E-6	2.50E-6	4.44E+6	4.44E+6	4.47E+6	1.50	9.72E-6	3.54E+0
0.0005	4.20E-6	2.60E-6	9.00E-7	3.55E+7	3.55E+7	3.56E+7	1.80	5.71E-6	4.78E+0

Table 3: Numerical results of the approximation error and the required computing time

5 Infinite Shot Noise Series Representation with Finite Truncation

It is known that every infinitely divisible random variable admits a shot noise series representation, and that the series is infinite if Lévy measure is infinite. Here, we discuss such representations of the tempered stable distribution with a view towards simulation. Fix $(\lambda, \lambda_1, \lambda_2) \in \mathbb{R}_+^3$ and define $\gamma(\Delta) := (\Delta a / \alpha)^{-1/\alpha} \zeta(1/\alpha) - \Delta \Gamma(1 - \alpha) a b \alpha^{-1}$. We denote by $\{\Gamma_k\}_{k \in \mathbb{N}}$ arrival times of a standard Poisson process, by $\{U_k\}_{k \in \mathbb{N}}$ a sequence of iid uniform random variables on $[0, 1]$, by $\{E_k^{(1)}\}_{k \in \mathbb{N}}$ and $\{E_k^{(4)}\}_{k \in \mathbb{N}}$ sequences of iid standard exponential random variables, by $\{E_k^{(2)}\}_{k \in \mathbb{N}}$ a sequence of iid exponential random variables with rate $b\lambda_1$, and by $\{E_k^{(3)}\}_{k \in \mathbb{N}}$ a sequence of iid gamma random variables with shape λ_1 and scale $(b\lambda_2)^{-1}$. Then, the tempered stable random variable

$X(\Delta)$ can be written as

$$X(\Delta) - \gamma(\Delta) \stackrel{\mathcal{L}}{=} \sum_{k=1}^{+\infty} \left[\left[\left(\frac{\alpha \Gamma_k}{\Delta a} \right)^{-1/\alpha} \wedge \frac{E_k^{(1)} U_k^{1/\alpha}}{b} \right] - \left(\frac{\alpha k}{\Delta a} \right)^{-1/\alpha} \right] \quad (5.1)$$

$$\stackrel{\mathcal{L}}{=} \sum_{k=1}^{+\infty} \left[E_k^{(2)} \mathbb{1} \left(\Gamma_k \leq \frac{\Delta a}{\lambda b (E_k^{(2)})^{\alpha+1}} e^{-b(1-\lambda)E_k^{(2)}} \right) - \left(\frac{\alpha k}{\Delta a} \right)^{-1/\alpha} \right] \quad (5.2)$$

$$\stackrel{\mathcal{L}}{=} \sum_{k=1}^{+\infty} \left[E_k^{(3)} \mathbb{1} \left(\Gamma_k \leq \frac{\Delta a}{(b\lambda_2)^{\lambda_1} (E_k^{(3)})^{\alpha+\lambda_1}} e^{-b(1-\lambda_2)E_k^{(3)}} \right) - \left(\frac{\alpha k}{\Delta a} \right)^{-1/\alpha} \right] \quad (5.3)$$

$$\stackrel{\mathcal{L}}{=} \sum_{k=1}^{+\infty} \left[\left(\frac{\alpha \Gamma_k}{\Delta a} \right)^{-1/\alpha} \mathbb{1} \left(\left(\frac{\alpha \Gamma_k}{\Delta a} \right)^{-1/\alpha} \leq \frac{E_k^{(4)}}{b} \right) - \left(\frac{\alpha k}{\Delta a} \right)^{-1/\alpha} \right]. \quad (5.4)$$

The representation (5.1) is due to [18] and is derived with the generalized shot noise method, while the others (5.2)-(5.4) are due to Imai and Kawai [10] and are derived with the thinning method [17] for (5.2) and (5.3) and the rejection method [17] for (5.4). (There is yet another representation based on the so-called inverse Lévy measure method. We omit that here since the representation is not given in closed form and is thus requires a numerical approach. It is shown in Imai and Kawai [9, 10] that the representation achieves a much faster convergence than the above four representations (5.1)-(5.4) with respect to the common finite truncation $\{k \in \mathbb{N} : \Gamma_k \leq n\}$

It is obviously insensible to generate the above infinite sum for simulation of each increment $X(\Delta)$, while it may make sense when many iid replications of $X(\Delta)$ are required. Suppose we wish to generate N of them, $N = 100000$ say. Let $T = N\Delta$ and let $\{T_k\}_{k \in \mathbb{N}}$ be a sequence of iid uniform random variables on $[0, T]$. Then, we can form a sequence $\{X_{k\Delta} - X_{(k-1)\Delta}\}_{k=1, \dots, N}$ of iid increments with $\mathcal{L}(X_\Delta - X_0) = \mathcal{L}(X(\Delta))$ by generating *one* sample path of the tempered stable Lévy process $\{L_t^{(\text{ts})} : t \in [0, T]\}$ with the following series representation (based on (5.1), for example)

$$\left\{ L_t^{(\text{ts})} : t \in [0, T] \right\} \stackrel{\mathcal{L}}{=} \left\{ \sum_{k=1}^{+\infty} \left[\left[\left(\frac{\alpha \Gamma_k}{T a} \right)^{-1/\alpha} \wedge \frac{E_k^{(1)} U_k^{1/\alpha}}{b} \right] \mathbb{1}_{[0, t]}(T_k) - \frac{t}{T} \left(\frac{\alpha k}{T a} \right)^{-1/\alpha} \right] + \frac{t}{T} \gamma(T) : t \in [0, T] \right\}.$$

This procedure requires four infinite random sequences (with some finite truncation, of course) and an index search for the first component of the summands based on $\mathbb{1}_{((k-1)\Delta, k\Delta]}(\cdot)$. Clearly, the performance is mainly up to the convergence rate of the infinite sum, while we need to pay a lot for those factors especially when T is very large since then the convergence is extremely slow. Note that this approach makes sense only when generating many iid replications and it seems very difficult to analyze a trade-off between truncation error and required computing effort.

6 Concluding Remarks

In this paper, we have investigated various, existing and new, simulation methods of the tempered stable law with stability index greater than one, with primal interest in simulation of increments $X(\Delta)$ over a very short stepsize $\Delta > 0$: a suitable setting for approximation of stochastic differential equations through the Euler scheme. From a practical point of view, we have sought a simulation recipe of a good balance between computational load and approximation error, together with implementation ease. Results can be summarized as follows. The model-free acceptance-rejection sampling method of [8] provides an exact simulation method, in principle, but requires a lot of computing effort for computing density values. This method exhibits quite low acceptance rate when Δ is small and the stability index α is close to 2, that is, when the target is close to Gaussian. The acceptance-rejection sampling of [2] is approximative yet very handy with both very small computing time and approximation error. Finding an optimal value of the tuning parameter is relatively straightforward and is required only once in advance. The Gaussian approximation of [1] provides a different route to approximative simulation. We have shown that in this framework, the approximation error can be made very small by either simulating more large jump component or simulating more mass of the small jump component as compound Poisson random variables, while an extraordinary large amount of computing effort is additionally required for an improvement in approximation error. Infinite shot noise series representations of the tempered stable law can be used for sample paths simulation. In general, we need to pay a large computing effort due to the infinite shot noise series. Moreover, there exists a complex trade-off issue between approximation error and computing time through a finite truncation of infinite series. In conclusion, with a given computing budget, the approximative acceptance-rejection sampling of [2] is both most efficient and handiest based on numerical assessment of accuracy and can be used for validation and estimation purposes.

Acknowledgement

Research of HM is supported in part by Grant-in-Aid for Young Scientists (B), Japan.

References

- [1] Asmussen, S., Rosiński, J. (2001) Approximation of small jumps of Lévy processes with a view towards simulation, *Journal of Applied Probability*, **38**(2) 482-493.
- [2] Baeumer, B., Meerschaert, M.M. (2009) Tempered stable Lévy motion and transit superdiffusion, *Journal of Computational and Applied Mathematics*, **223**(10) 2438-2448.
- [3] Barndorff-Nielsen, O.E., Shephard, N. (2001) Non-Gaussian Ornstein-Uhlenbeck-based

- models and some of their uses in financial economics (with discussion), *J. R. Statist. Soc. B*, **63**(2) 167-241.
- [4] Carr, P., Geman, H., Madan, D.B., Yor, M. (2002) The fine structure of asset returns: An empirical investigation, *Journal of Business*, **75**, 303-325.
 - [5] Chambers, J.M., Mallows, C.L., Stuck, B.W. (1976) A method for simulating stable random variables, *Journal of the American Statistical Association*, **71**(354) 340-344.
 - [6] Cohen, S., Rosiński, J. (2007) Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes, *Bernoulli*, **13**(1) 195-210.
 - [7] Devroye, L. (1981) On the computer generation of random variables with a given characteristic function, *Computers and Mathematics with Applications*, **7**(6) 547-552.
 - [8] Devroye, L. (2009) Random variate generation for exponential and polynomially tilted stable distributions, *ACM Transactions on Modeling and Computer Simulation*, **19**(4) Article No. 18.
 - [9] Imai, J., Kawai, R. (2010) Numerical inverse Lévy measure method for infinite shot noise series representation, available at <http://sites.google.com/site/reiichirokawai/>
 - [10] Imai, J., Kawai, R. (2010) On finite truncation of infinite shot noise series representation of tempered stable law, available at <http://sites.google.com/site/reiichirokawai/>
 - [11] Kawai, R., Masuda, H. (2009) Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes, available at <http://hdl.handle.net/2324/15635>
 - [12] Kawai, R., Masuda, H. (2010) Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations, available at <http://hdl.handle.net/2324/16248>
 - [13] Kawai, R., Takeuchi, A. (2009) Computation of Greeks for asset price dynamics driven by stable and tempered stable processes, *under review*, available at <https://sites.google.com/site/reiichirokawai/>
 - [14] Mantegna, R.N., Stanley, H.E. (1994) Stochastic process with ultraslow convergence to a Gaussian: The truncated Lévy flights, *Physical Review Letters*, **73**, 2946-2949.
 - [15] Novikov, E.A. (1994) Infinitely divisible distributions in turbulence, *Physical Review E*, **50**, R3303-R3305.
 - [16] Palmer, K.J., Ridout, M.S., Morgan, B.J.T. (2008) Modelling cell generation times by using the tempered stable distribution, *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, **57**(4) 379-397.

- [17] Rosiński, J. (2001) Series representations of Lévy processes from the perspective of point processes, In: *Lévy Processes - Theory and Applications*, Eds. Barndorff-Nielsen, O.-E., Mikosch, T., Resnick, S.I., Birkhäuser, 401-415.
- [18] Rosiński, J. (2007) Tempering stable processes, *Stochastic Processes and their Applications*, **117**(6) 677-707.
- [19] Sato, K. (1999) *Lévy processes and infinitely divisible distributions*, Cambridge University Press.
- [20] Signahl, M. (2003) On error rates in normal approximations and simulation schemes for Lévy processes, *Stochastic Models*, **19**(3) 287-298.
- [21] Tweedie, M.C.K. (1984) An index which distinguishes between some important exponential families, In: *Statistics: Applications and New Directions: Proc. Indian Statistical Institute Golden Jubilee International Conference* (eds. J. Ghosh and J. Roy) 579-604.
- [22] Zolotarev, V.M. (1986) *One-Dimensional Stable Distributions*, American Mathematical Society, Providence, RI.

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

- MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata
- MI2008-2 Eiji ONODERA
The initial value problem for a third-order dispersive flow into compact almost Hermitian manifolds
- MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space
- MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme
- MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in infinite extensions over a p -adic field
- MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields
- MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited
- MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition
- MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

- MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE
- MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds
- MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator
- MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials
- MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality
- MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions
- MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings
- MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors
- MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization
- MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions
- MI2009-5 Toshiro HIRANOUCI & Yuichiro TAGUCHI
Flat modules and Groebner bases over truncated discrete valuation rings

- MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations
- MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow
- MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization
- MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity
- MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption
- MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve
- MI2009-12 Tetsu MASUDA
Hypergeometric q -functions of the q -Painlevé system of type $E_8^{(1)}$
- MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination
- MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications
- MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

- MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring
- MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions
- MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force
- MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application
- MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions
- MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations
- MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces
- MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions
- MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map
- MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2 -projection

- MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine's property (Pm)
- MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space
- MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme
- MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance
- MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis
- MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke's zeros and higher depth determinants
- MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type
- MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect
- MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type $(A_2 + A_1)^{(1)}$
- MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

- MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes
- MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model
- MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes
- MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations
- MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models
- MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons
- MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine
- MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization
- MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation
- MI2010-8 Takahiro ITO
Abstract collision systems on groups
- MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

- MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models
- MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems
- MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow
- MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates