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Abstract

This paper is concerned with large time behavior of solutions to the compressible Navier-Stokes equations in an infinite layer of \mathbb{R}^2 under slip boundary condition. It is shown that if the initial data is sufficiently small, the global solution uniquely exists and the large time behavior of the solution is described by a superposition of one-dimensional diffusion waves.

Keywords: Compressible Navier-Stokes equation, infinite layer, slip boundary condition, asymptotic behavior, diffusion waves.

1 Introduction

This paper studies large time behavior of solutions of the compressible Navier-Stokes equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad (1.1)$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \operatorname{div} v + \nabla P(\rho) = 0 \quad (1.2)$$

in an infinite layer Ω of \mathbb{R}^2 :

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2; x_1 \in \mathbb{R}, 0 < x_2 < 1\}$$

under the slip boundary condition

$$\partial_{x_2} v^1|_{x_2=0,1} = 0, \quad v^2|_{x_2=0,1} = 0. \quad (1.3)$$

Here $\rho = \rho(x, t) > 0$ and $v = {}^\top(v^1(x, t), v^2(x, t))$ denote the unknown density and velocity, respectively, at time $t \geq 0$ and position $x \in \Omega$; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$P'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are viscosity coefficients that are assumed to be constants and satisfy

$$\mu > 0, \quad \mu + \mu' \geq 0;$$

div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x . Here and in what follows ${}^\top \cdot$ means the transposition.

We impose the initial condition

$$\rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0. \quad (1.4)$$

Here $\rho_0 = \rho_0(x)$ and $v_0 = v_0(x)$ satisfy $\rho_0(x) \rightarrow \rho_*$ and $v_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

The aim of this paper is to investigate large time behavior of solutions to (1.1)-(1.4) around the motionless state $\rho = \rho_*$, $v = 0$. We rewrite (1.1)-(1.2) into the following equations for the perturbation

$$\partial_t \phi + \gamma \text{div} w = f^0(\phi, w), \quad (1.5)$$

$$\partial_t w - \nu \Delta w - \tilde{\nu} \nabla \text{div} w + \gamma \nabla \phi = \tilde{f}(\phi, w). \quad (1.6)$$

Here $u = {}^\top(\phi, w)$ with $\phi = \frac{1}{\rho_*}(\rho - \rho_*)$ and $w = \frac{1}{\gamma}v$ denotes the perturbation from $u_s = {}^\top(\rho_*, 0)$; ν , $\tilde{\nu}$ and γ are parameters given by

$$\nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu + \mu'}{\rho_*}, \quad \gamma = \sqrt{P'(\rho_*)};$$

and $f(\phi, w) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$ denote the nonlinear terms:

$$\begin{aligned} f^0(\phi, w) &= -\gamma \text{div}(\phi w), \\ \tilde{f}(\phi, w) &= -\gamma w \cdot \nabla w - \frac{\phi}{1 + \phi} \{ \nu \Delta w + \tilde{\nu} \nabla \text{div} w \} + \frac{\gamma \phi}{1 + \phi} \nabla \phi \\ &\quad - \frac{\rho_*}{\gamma(1 + \phi)} \nabla(P^{(2)}(\phi) \phi^2), \end{aligned}$$

where

$$P^{(2)}(\phi) = \int_0^1 (1 - \theta) P''(\rho_*(1 + \theta\phi)) d\theta.$$

The boundary condition (1.3) and initial condition (1.4) are transformed into

$$\partial_{x_2} w^1|_{x_2=0,1} = 0, \quad w^2|_{x_2=0,1} = 0 \quad (1.7)$$

and

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \quad (1.8)$$

Here u_0 satisfies $u_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

The large time behavior of solutions of the compressible Navier-Stokes equation (1.1)-(1.2) on the layer Ω was studied in [1, 2, 3, 4] under the non-slip boundary condition $v|_{x_2=0,1} = 0$. It was shown in [3] that the large time behavior of perturbations of the motionless state is described by a one-dimensional linear heat equation. In [4] stability of parallel flow was considered and it was proved that the large time behavior of perturbations of parallel flow is described by a one-dimensional viscous Burgers equation when the Reynolds and Mach numbers are sufficiently small. In the case of time-periodic parallel flow, the large time behavior of perturbations is also described by a one-dimensional diffusion equation ([1, 2]). In all cases of [1, 2, 3, 4], the asymptotic leading parts under the non-slip boundary condition exhibit purely diffusive phenomena. In this paper we show that the solution of (1.1)-(1.2) under the slip boundary condition (1.3) with (1.4) behaves like a superposition of one-dimensional diffusion waves as $t \rightarrow \infty$ as in the case of one-dimensional compressible Navier-Stokes equation [7, 10]. More precisely, consider problem (1.5)-(1.8) for u . We prove that, under appropriate conditions for u_0 , the solution $u(t)$ satisfies

$$\|\partial_x^k(u - \chi_+ \mathbf{a}_+ - \chi_- \mathbf{a}_-)(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1, \quad (1.9)$$

where $\mathbf{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given by

$$\chi_\pm(x_1, t) = z_\pm(x_1 \pm \gamma t, t). \quad (1.10)$$

Here $z_\pm = z_\pm(x_1, t)$ are the self-similar solutions of the viscous Burgers equations

$$\partial_t z_\pm - \frac{\nu + \tilde{\nu}}{2} \partial_{x_1}^2 z_\pm \mp c \partial_{x_1}(z_\pm^2) = 0 \quad (1.11)$$

satisfying

$$\int_{\mathbb{R}} z_\pm(x_1, t) dx_1 = \frac{1}{2} \int_{\Omega} (\phi_0(x) \pm (1 + \phi_0(x)) w_0^1(x)) dx \quad (1.12)$$

for some constant $c \in \mathbb{R}$. In contrast to the case of the non-slip boundary condition, we see that a hyperbolic aspect of (1.1)-(1.2) appears in the asymptotic leading part of the solution under the slip boundary condition.

To prove (1.9), we first establish the decay estimates for $u(t)$. We decompose the solution of (1.5)-(1.8) into its *low and high frequency* parts. The spectrum of the low-frequency part of the linearized semigroup is different from the one in the case of the non-slip boundary condition; it is the same as that in the case of the one-dimensional compressible Navier-Stokes equation. Therefore, the low-frequency part decays like one-dimensional heat kernel, namely, k th order derivative decays in the order $O(t^{-\frac{1}{4}-\frac{k}{2}})$ in the L^2 norm. For the high-frequency part (remainder part), we apply the Matsumura-Nishida energy method ([9]) to see that the high-frequency part decays in the order $O(t^{-\frac{5}{4}})$ in the H^2 norm. Based on the spectral properties of the low-frequency part of the linearized semigroup and the decay estimate for the high-frequency part, we deduce the asymptotic behavior (1.9) by applying the argument of Kawashima [7].

The paper is organized as follows. In section 2 we state the main results of this paper. In section 3 we study the spectral properties of the linearized operator, and in section 4 we rewrite (1.5)-(1.8) into a problem for a system of equations for the low and high frequency parts. Section 5 is devoted to estimating the low-frequency part, while the high-frequency part is estimated in section 6. In section 7 we give the estimates for the nonlinear terms. In section 8 we study the asymptotic behavior of the solution of (1.5)-(1.8).

2 Main Results

In this section we state the main results of this paper. We first introduce notation and function spaces which will be used throughout this paper.

For $1 \leq p \leq \infty$ we denote by $L^p(X)$ the usual Lebesgue space on a domain X and its norm is denoted by $\|\cdot\|_{L^p(X)}$. Let m be a nonnegative integer. The symbol $H^m(X)$ denotes the m -th order L^2 -Sobolev space on X with norm $\|\cdot\|_{H^m(X)}$. In particular, we write $\|\cdot\|_{L^2(X)}$ for $H^0(X)$.

We simply denote by $L^p(X)$ (resp., $H^m(X)$) the set of all vector fields $w = {}^\top(w^1, w^2)$ on X with $w^j \in L^p(X)$ (resp., $H^m(X)$), $j = 1, 2$, and its norm is also denoted by $\|\cdot\|_{L^p(X)}$ (resp., $\|\cdot\|_{H^m(X)}$). For $u = {}^\top(\phi, w)$ with $\phi \in H^k(X)$ and $w = {}^\top(w^1, w^2) \in H^m(X)$, we define $\|u\|_{H^k(X) \times H^m(X)}$ by $\|u\|_{H^k(X) \times H^m(X)} = \|\phi\|_{H^k(X)} + \|w\|_{H^m(X)}$. When $k = m$, we simply write $\|u\|_{H^k(X) \times H^k(X)} = \|u\|_{H^k(X)}$.

Partial derivatives of a function u in x , x_k ($k = 1, 2$) and t are denoted by $\partial_x u$, $\partial_{x_k} u$ and $\partial_t u$. We also write the higher order partial derivatives of u

in x as $\partial_x^l u = (\partial_x^\alpha u; |\alpha| = l)$.

In the case where $X = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^m(\Omega)$) as L^p (resp., H^m). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$. We denote the inner product of $L^2(\Omega)$ by

$$(f, g) = \int_{\Omega} f(x)g(x)dx, \quad f, g \in L^2(\Omega).$$

The average of a function f in x_2 on $(0, 1)$ is denoted by $\langle f \rangle$:

$$\langle f \rangle = \int_0^1 f(x_2)dx_2.$$

We set

$$H_*^2 = \{w = {}^\top(w^1, w^2) \in H^2(\Omega); \partial_{x_2} w^1|_{x_2=0,1} = 0, w^2|_{x_2=0,1} = 0\}.$$

For $\alpha \in \mathbb{R}$, we denote by $L_\alpha^1 = L_\alpha^1(\Omega)$ the weighted L^1 space with weight $(1 + |x_1|)^\alpha$, and its norm is denoted by

$$\|f\|_{L_\alpha^1} = \int_{\Omega} (1 + |x_1|)^\alpha |f(x)|dx.$$

We denote the Fourier transform of $f = f(x_1)$ ($x_1 \in \mathbb{R}$) by \hat{f} or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x_1)e^{-i\xi x_1}dx_1, \quad \xi \in \mathbb{R}.$$

The inverse Fourier transform is denoted by \mathcal{F}^{-1} :

$$\mathcal{F}^{-1}[f](x_1) = (2\pi)^{-1} \int_{\mathbb{R}} f(\xi)e^{i\xi x_1}d\xi, \quad x_1 \in \mathbb{R}.$$

For operators A, B , we denote the commutator of A and B by $[A, B]$:

$$[A, B]f = A(Bf) - B(Af).$$

We now state the main results of this paper. We have the following decay estimate of the L^2 norm of the solution u .

Theorem 2.1 *There exists a positive number ε_0 such that if $u_0 = {}^\top(\phi_0, w_0) \in (H^2 \times H_*^2) \cap L^1$ with $w_0 = {}^\top(w_0^1, w_0^2)$ satisfies $\|u_0\|_{H^2 \cap L^1} \leq \varepsilon_0$, then problem (1.5)-(1.8) has a unique global solution*

$$u(t) = {}^\top(\phi(t), w(t)) \in C([0, \infty); H^2 \times H_*^2)$$

and $u(t)$ satisfies

$$\|\partial_x^k u(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1}$$

for $t \geq 0$, $k = 0, 1, 2$.

We next consider the asymptotic behavior of solutions.

Theorem 2.2 *In addition to the assumption of Theorem 2.1, if $\phi_0, w_0^1 \in L_{1/2}^1$, then*

$$\|\partial_x^k(u - \chi_+ \mathbf{a}_+ - \chi_- \mathbf{a}_-)(t)\|_2 \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1.$$

Here $\mathbf{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given in (1.10)-(1.12).

The proof of Theorem 2.1 will be given in sections 3-7, and Theorem 2.2 will be proved in section 8.

3 Spectral Properties of Linearized Operator

We consider the linearized problem

$$\partial_t u + Lu = F, \quad u|_{t=0} = u_0, \quad (3.1)$$

where $u = {}^\top(\phi, w)$; $F = {}^\top(f^0, \tilde{f})$ with $\tilde{f} = {}^\top(f^1, f^2)$ is a given function, and L is an operator of the form

$$L = \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma \nabla & -\nu \Delta - \tilde{\nu} \nabla \operatorname{div} \end{pmatrix}$$

in $H^1 \times L^2$ with domain $D(L) = H^1 \times H_*^2$.

To investigate (3.1), we consider the Fourier transform of (3.1) in $x_1 \in \mathbb{R}$:

$$\partial_t \hat{\phi} + i\gamma \xi \hat{w}^1 + \gamma \partial_{x_2} \hat{w}^2 = \hat{f}^0, \quad (3.2)$$

$$\partial_t \hat{w}^1 + (\nu + \tilde{\nu}) \xi^2 \hat{w}^1 - \nu \partial_{x_2}^2 \hat{w}^1 - i\tilde{\nu} \xi \partial_{x_2} \hat{w}^2 + i\gamma \xi \hat{\phi} = \hat{f}^1, \quad (3.3)$$

$$\partial_t \hat{w}^2 + \nu \xi^2 \hat{w}^2 - (\nu + \tilde{\nu}) \partial_{x_2}^2 \hat{w}^2 - i\tilde{\nu} \xi \partial_{x_2} \hat{w}^1 + \gamma \partial_{x_2} \hat{\phi} = \hat{f}^2, \quad (3.4)$$

$$\partial_{x_2} \hat{w}^1|_{x_2=0,1} = \hat{w}^2|_{x_2=0,1} = 0, \quad (3.5)$$

$$\hat{u}|_{t=0} = \hat{u}_0 = {}^\top(\hat{\phi}_0, \hat{w}_0). \quad (3.6)$$

We thus arrive at the following problem

$$\partial_t \hat{u} + \hat{L}_\xi \hat{u} = \hat{F}, \quad \hat{u}|_{t=0} = \hat{u}_0, \quad (3.7)$$

with a parameter $\xi \in \mathbb{R}$. Here $\hat{u} = \hat{u}(\xi, x_2, t)$; \hat{L}_ξ is the operator

$$\hat{L}_\xi = \begin{pmatrix} 0 & i\gamma \xi & \gamma \partial_{x_2} \\ i\gamma \xi & (\nu + \tilde{\nu}) \xi^2 - \nu \partial_{x_2}^2 & -i\tilde{\nu} \xi \partial_{x_2} \\ \gamma \partial_{x_2} & -i\tilde{\nu} \xi \partial_{x_2} & \nu \xi^2 - (\nu + \tilde{\nu}) \partial_{x_2}^2 \end{pmatrix}$$

with domain $D(\hat{L}_\xi) = H^1(0, 1) \times H_*^2(0, 1)$, where $H_*^2(0, 1) = \{w = {}^\top(w^1, w^2) \in H^2(0, 1); \partial_{x_2} w^1|_{x_2=0,1} = w^2|_{x_2=0,1} = 0\}$. For $-\hat{L}_0$ we have the following result.

Lemma 3.1 (i) $\lambda = 0$ is a semisimple eigenvalue of $-\hat{L}_0$.
(ii) The eigenprojection Π for $\lambda = 0$ of $-\hat{L}_0$ is given by

$$\Pi u = \begin{pmatrix} \langle \phi \rangle \\ \langle w^1 \rangle \\ 0 \end{pmatrix}$$

for $u = {}^\top(\phi, w)$ with $w = {}^\top(w^1, w^2)$.

The proof of Lemma 3.1 is straightforward and we omit it.
We next expand \hat{u} and \hat{F} into the Fourier series:

$$\hat{\phi} = \sum_{k=0}^{\infty} \hat{\phi}_k \cos k\pi x_2, \hat{w}^1 = \sum_{k=0}^{\infty} \hat{w}_k^1 \cos k\pi x_2, \hat{w}^2 = \sum_{k=1}^{\infty} \hat{w}_k^2 \sin k\pi x_2, \quad (3.8)$$

$$\hat{f}^0 = \sum_{k=0}^{\infty} \hat{f}_k^0 \cos k\pi x_2, \hat{f}^1 = \sum_{k=0}^{\infty} \hat{f}_k^1 \cos k\pi x_2, \hat{f}^2 = \sum_{k=1}^{\infty} \hat{f}_k^2 \sin k\pi x_2. \quad (3.9)$$

It then follows that

$$\partial_t \hat{\phi}_k + i\gamma\xi \hat{w}_k^1 + \gamma \hat{w}_k^2 k\pi = \hat{f}_k^0, \quad (3.10)$$

$$\partial_t \hat{w}_k^1 + \nu(\xi^2 + k^2\pi^2) \hat{w}_k^1 + \tilde{\nu}\xi^2 \hat{w}_k^1 - i\tilde{\nu}k\pi\xi \hat{w}_k^2 + i\gamma\xi \hat{\phi}_k = \hat{f}_k^1, \quad (3.11)$$

$$\partial_t \hat{w}_k^2 + \nu(\xi^2 + k^2\pi^2) \hat{w}_k^2 + i\tilde{\nu}k\pi\xi \hat{w}_k^1 + \tilde{\nu}k^2\pi^2 \hat{w}_k^2 - \gamma k\pi \hat{\phi}_k = \hat{f}_k^2. \quad (3.12)$$

We rewrite it in the form

$$\partial_t \hat{u}_k + \hat{L}_{\xi,k} \hat{u}_k = \hat{F}_k, \quad (3.13)$$

where $\hat{u}_k = {}^\top(\hat{\phi}_k, \hat{w}_k^1, \hat{w}_k^2)$, $\hat{F}_k = {}^\top(\hat{f}_k^0, \hat{f}_k^1, \hat{f}_k^2)$ and

$$\hat{L}_{\xi,k} = \begin{pmatrix} 0 & i\gamma\xi & \gamma k\pi \\ i\gamma\xi & \nu(\xi^2 + k^2\pi^2) + \tilde{\nu}\xi^2 & -i\tilde{\nu}k\pi\xi \\ -\gamma k\pi & i\tilde{\nu}k\pi\xi & \nu(\xi^2 + k^2\pi^2) + \tilde{\nu}k^2\pi^2 \end{pmatrix}.$$

As for the the spectrum of $-\hat{L}_{\xi,k}$, we have the following lemma.

Lemma 3.2 (i) The eigenvalues $-\hat{L}_{\xi,k}$ are given by

$$\begin{aligned} \lambda_{0,k}(\xi) &= -\nu(\xi^2 + k^2\pi^2), \\ \lambda_{\pm,k}(\xi) &= -\frac{1}{2}(\nu + \tilde{\nu})(\xi^2 + k^2\pi^2) \\ &\quad \pm \frac{1}{2}\sqrt{(\nu + \tilde{\nu})^2(\xi^2 + k^2\pi^2)^2 - 4\gamma^2(\xi^2 + k^2\pi^2)}. \end{aligned} \quad (3.14)$$

(ii) The eigenprojections for $\lambda_{0,k}$ and $\lambda_{\pm,k}$ are given by the following $P_{0,k}$ and $P_{\pm,k}$, respectively:

$$P_{0,k} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - \frac{\xi^2}{\xi^2 + k^2 \pi^2} & \frac{ik\pi\xi}{\xi^2 + k^2 \pi^2} \\ 0 & -\frac{ik\pi\xi}{\xi^2 + k^2 \pi^2} & 1 - \frac{k^2 \pi^2}{\xi^2 + k^2 \pi^2} \end{pmatrix},$$

$$P_{+,k} = \frac{1}{\lambda_{+,k} - \lambda_{-,k}} \begin{pmatrix} -\lambda_{-,k} & i\gamma\xi & \gamma k\pi \\ i\gamma\xi & \frac{\xi^2 \lambda_{+,k}}{\xi^2 + k^2 \pi^2} & -\frac{ik\pi\xi \lambda_{+,k}}{\xi^2 + k^2 \pi^2} \\ -\gamma k\pi & \frac{ik\pi\xi \lambda_{+,k}}{\xi^2 + k^2 \pi^2} & \frac{k^2 \pi^2 \lambda_{+,k}}{\xi^2 + k^2 \pi^2} \end{pmatrix},$$

$$P_{-,k} = \frac{1}{\lambda_{+,k} - \lambda_{-,k}} \begin{pmatrix} \lambda_{+,k} & -i\gamma\xi & -\gamma k\pi \\ -i\gamma\xi & -\frac{\xi^2 \lambda_{-,k}}{\xi^2 + k^2 \pi^2} & \frac{ik\pi\xi \lambda_{-,k}}{\xi^2 + k^2 \pi^2} \\ \gamma k\pi & -\frac{ik\pi\xi \lambda_{-,k}}{\xi^2 + k^2 \pi^2} & -\frac{k^2 \pi^2 \lambda_{-,k}}{\xi^2 + k^2 \pi^2} \end{pmatrix}.$$

Lemma 3.2 can be proved by elementary computations.

4 Decay estimate: Proof of Theorem 2.1

We consider the nonlinear problem

$$\begin{cases} \partial_t u + Lu = F(u), \\ u|_{t=0} = u_0. \end{cases} \quad (4.1)$$

Here $u = {}^\top(\phi, w)$ and $F(u) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$.

One can prove the local solvability for (4.1) as in [5].

Proposition 4.1 *Assume that $u_0 = {}^\top(\phi_0, w_0) \in H^2 \times H_*^2$ and $\|\phi_0\|_\infty \leq \frac{1}{2}$. Then there exists $T_0 > 0$ depending on $\|u_0\|_{H^2}$ such that problem (4.1) has a unique solution $u = {}^\top(\phi, w)$ on $[0, T_0]$ satisfying $u \in C([0, T_0]; H^2 \times H_*^2) \cap C^1([0, T_0]; L^2)$ with $w \in L^2(0, T_0; H^3)$ and $\|\phi_0(t)\|_\infty \leq \frac{3}{4}$ for $t \in [0, T_0]$. Furthermore, the inequality*

$$\sup_{t \in [0, T_0]} \{\|u(t)\|_{H^2} + \|\partial_t u(t)\|_2\} + \int_0^{T_0} \|w\|_{H^3}^2 dt \leq C_0 \{1 + \|u_0\|_{H^2}^2\}^a \|u_0\|_{H^2}^2 \quad (4.2)$$

holds with some constants $C_0 > 0$ and $a > 0$.

The global existence of $u(t)$ follows in a standard manner from Proposition 4.1 and Proposition 4.5 below which provides the a priori bound $\|u(t)\|_{H^2} \leq C\|u_0\|_{H^2 \cap L^1}$ when $\|u_0\|_{H^2 \cap L^1}$ is sufficiently small.

We next consider the a priori estimates for $u(t)$. Let r_0 be a number satisfying $0 < r_0 \leq 1$. We introduce the cut-off function $\mathbf{1}_{\{|\xi| \leq r_0\}}$ defined by

$$\mathbf{1}_{\{|\xi| \leq r_0\}} = \begin{cases} 1 & (|\xi| < r_0), \\ 0 & (|\xi| \geq r_0). \end{cases} \quad (4.3)$$

We introduce the projections P_1 and P_∞ defined by

$$P_1 u = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} \Pi \mathcal{F} u, \quad P_\infty = I - P_1. \quad (4.4)$$

It follows from Lemma 3.2 that

$$\begin{aligned} P_1 e^{-tL} u_0 &= \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} [e^{\lambda_{+,0} t} P_{+,0} + e^{\lambda_{-,0} t} P_{-,0}] \Pi \hat{u}_0 \\ &= \mathcal{F}^{-1} \frac{\mathbf{1}_{\{|\xi| \leq r_0\}}}{\lambda_{+,0} - \lambda_{-,0}} \left[e^{\lambda_{+,0} t} \begin{pmatrix} -\lambda_{-,0} & i\gamma\xi & 0 \\ i\gamma\xi & \lambda_{+,0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\ &\quad \left. + e^{\lambda_{-,0} t} \begin{pmatrix} \lambda_{+,0} & -i\gamma\xi & 0 \\ -i\gamma\xi & -\lambda_{-,0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \langle \hat{\phi}_0 \rangle \\ \langle \hat{w}_0^1 \rangle \\ 0 \end{pmatrix}. \end{aligned} \quad (4.5)$$

for $u_0 = {}^\top(\phi_0, w_0^1, w_0^2)$. We also note that $P_1 u$ does not depend on x_2 , and so,

$$\partial_{x_2} P_1 u = 0.$$

We decompose $u = {}^\top(\phi, w)$ into

$$u = u_1 + u_\infty,$$

where

$$u_1 = P_1 u = {}^\top(\phi_1, w_1^1, w_1^2), \quad u_\infty = P_\infty u = {}^\top(\phi_\infty, w_\infty^1, w_\infty^2).$$

Remark 4.2 We see from the definition of P_1 that $u_1 = u_1(x_1, t)$ satisfies

$$\|\partial_{x_1}^{k+l} u_1\|_2 \leq \|\partial_{x_1}^l u_1\|_2$$

for arbitrary k and l . We also note that u_∞ satisfies

$$\|u_\infty\|_2 \leq C \|\partial_x u_\infty\|_2.$$

We will frequently make use of these properties in the subsequent arguments.

Proposition 4.3 *Let $u(t)$ be a solution of (4.1) on $[0, T]$. Assume that $u \in C([0, T]; H^2 \times H_*^2) \cap C^1([0, T]; L^2)$ with $w \in L^2(0, T; H^3)$. Then*

$$u_l = {}^\top(\phi_l, w_l) \in C^1([0, T]; H^l(\Omega)) \quad (\forall l = 0, 1, 2, \dots)$$

and

$$u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0, T]; H^2 \times H_*^2) \cap C^1([0, T]; L^2)$$

with $w_\infty \in L^2(0, T; H^3)$.

Furthermore, u_1 and u_∞ satisfy

$$u_1 = P_1 e^{-tL} u_0 + \int_0^t P_1 e^{-(t-\tau)L} F(u(\tau)) d\tau, \quad (4.6)$$

$$\partial_t u_\infty + L u_\infty = F_\infty, \quad u_\infty|_{t=0} = P_\infty u_0, \quad (4.7)$$

where $F_\infty = P_\infty F = {}^\top(f_\infty^0, \tilde{f}_\infty)$, $\tilde{f}_\infty = (f_\infty^1, f_\infty^2)$.

Proof. Since $P_j L \subset L P_j$ ($j = 1, \infty$), applying P_j to (4.1) we obtain the desired results. ■

We define $M(t) \geq 0$ by

$$M(t) = M_1(t) + M_\infty(t) \quad (t \in [0, T]). \quad (4.8)$$

Here $M_1(t)$ and $M_\infty(t)$ are define by

$$M_1(t) = \sup_{0 \leq \tau \leq t} \left\{ \sum_{k=0}^2 (1 + \tau)^{\frac{1}{4} + \frac{k}{2}} \|\partial_{x_1}^k u_1(\tau)\|_2 + (1 + \tau)^{\frac{3}{4}} \|\partial_\tau u_1(\tau)\|_2 \right\},$$

$$M_\infty(t) = \left(\sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{5}{2}} \{ \|u_\infty(\tau)\|_{H^2}^2 + \|\partial_\tau u_\infty(\tau)\|_2^2 \} \right)^{\frac{1}{2}}.$$

We note that, by the Gagliardo-Nirenberg-Sobolev inequality,

$$\|u_1(t)\|_\infty \leq C \|u_1(t)\|_2^{\frac{1}{2}} \|\partial_{x_1} u_1(t)\|_2^{\frac{1}{2}} \leq C (1 + t)^{-\frac{1}{2}} M_1(t),$$

$$\|u_\infty(t)\|_\infty \leq C \|u_\infty(t)\|_{H^2} \leq C (1 + t)^{-\frac{5}{4}} M_\infty(t).$$

We introduce the quantities $E_\infty(t)$ and $D_\infty(t)$ for $u_\infty(t) = {}^\top(\phi_\infty(t), w_\infty(t))$:

$$E_\infty(t) = \|u_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_2^2,$$

$$D_\infty(t) = \|\nabla \phi_\infty(t)\|_{H^1}^2 + \|\nabla w_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_{H^1}^2.$$

Proposition 4.4 *Let $u(t)$ be a solution of (4.1) on $[0, T]$. Then there exists $\varepsilon_1 > 0$ such that if $\|u(t)\|_{H^2} \leq \varepsilon_1$ and $M(t) \leq 1$ for $t \in [0, T]$, the estimates*

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\} \quad (4.9)$$

and

$$\begin{aligned} E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau \\ \leq C\{e^{-at} E_\infty(0) + (1+t)^{-\frac{5}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} \mathcal{R}(\tau) d\tau\} \end{aligned} \quad (4.10)$$

hold uniformly for $t \in [0, T]$ with $C > 0$ independent of T . Here $a = a(\nu, \tilde{\nu}, \gamma)$ is a positive constant; and $\mathcal{R}(t)$ is a function satisfying the estimate

$$\mathcal{R}(t) \leq C\{(1+t)^{-\frac{5}{2}} M(t)^3 + M(t) D_\infty(t)\}. \quad (4.11)$$

The estimate (4.9) will be proved in section 5, and the estimates (4.10) and (4.11) will be proved in sections 6 and 7.

From Proposition 4.4, one can show the following uniform estimate of $M(t)$ as in [4].

Proposition 4.5 *If $\|u_0\|_{H^2 \cap L^1}$ is sufficiently small, then*

$$M(t) \leq C\|u_0\|_{H^2 \cap L^1}. \quad (4.12)$$

Theorem 2.1 now follows from Propositions 4.1 and 4.5.

5 Estimates on $P_1 u$

In this section we estimate the low-frequency part $u_1 = P_1 u$ and prove estimate (4.9) in Proposition 4.4.

Proof of (4.9). We see from Lemma 3.2 and the definition of Π that

$$\begin{aligned} \|\partial_{x_1}^l e^{-tL} P_1 u_0\|_2 &\leq C \left(\int_{\mathbb{R}} |\xi|^{2l} e^{-c_0 |\xi|^2 t} \mathbf{1}_{\{|\xi| \leq r_0\}} |\Pi \hat{u}_0|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\mathbb{R}} |\xi|^{2l} e^{-c_0 |\xi|^2 t} \mathbf{1}_{\{|\xi| \leq r_0\}} d\xi \right)^{\frac{1}{2}} \|u_0\|_1 \\ &\leq C(1+t)^{-\frac{1}{4} - \frac{l}{2}} \|u_0\|_1 \end{aligned} \quad (5.1)$$

for $l \geq 0$, and hence, by (4.6), we have

$$\begin{aligned}\|\partial_{x_1}^k u_1(t)\|_2 &\leq \|\partial_{x_1}^k e^{-tL} P_1 u_0\|_2 + \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 F(u(\tau))\|_2 d\tau \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_1 + \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 F(u(\tau))\|_2 d\tau\end{aligned}$$

for $k = 0, 1, 2$.

Let us estimate $\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 F(u(\tau))\|_2 d\tau$. By the Sobolev inequality: $\|\phi\|_\infty \leq C\|\phi\|_{H^2}$, we see that there exists $\varepsilon_2 > 0$ such that if $\|u(t)\|_{H^2} \leq \varepsilon_2$ for $t \in [0, T]$, then $\|\phi(t)\|_\infty \leq \frac{1}{2}$ for $t \in [0, T]$, and hence $P_1 F(u)$ is written as

$$P_1 F(u) = P_1 \partial_{x_1} F_0(u) + P_1 \tilde{F}(u),$$

where

$$F_0(u) = \begin{pmatrix} -\gamma\phi_1 w_1^1 \\ -\frac{\gamma}{2}(w_1^1)^2 + (\nu + \tilde{\nu})(\phi_1 \partial_{x_1} w_1^1) + \frac{\gamma}{2}\phi_1^2 - \frac{\rho_*}{\gamma} P^{(2)}(\phi)\phi_1^2 \\ 0 \end{pmatrix}.$$

Here each term in $P_1 \tilde{F}(u)$ includes u_∞ , $O((\partial_{x_1} u_1)^2)$, $O(u_1^2 \partial_{x_1} \phi_1)$ or $O(u_1^2 \partial_{x_1}^l w_1)$ ($l = 1, 2$), and $P_1 \tilde{F}(u)$ is estimated as

$$\|P_1 \tilde{F}(u(\tau))\|_1 \leq C(1+\tau)^{-\frac{5}{4}} M(t)^2.$$

It then follows from (5.1) that

$$\begin{aligned}\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 \tilde{F}(u(\tau))\|_2 d\tau &\leq C \int_0^t (1+t-\tau)^{-\frac{1}{4}-\frac{k}{2}} (1+\tau)^{-\frac{5}{4}} d\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} M(t)^2.\end{aligned}$$

As for the estimates for $P_1 \partial_{x_1} F_0(u)$ part, we write it as

$$\begin{aligned}&\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 \partial_{x_1} F_0(u(\tau))\|_2 d\tau \\ &= \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \|\partial_{x_1}^k e^{-(t-\tau)L} P_1 \partial_{x_1} F_0(u(\tau))\|_2 d\tau \\ &=: I_1 + I_2.\end{aligned}$$

Since $\partial_{x_1} e^{-tL} P_1 = e^{-tL} P_1 \partial_{x_1}$, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{t}{2}} \|\partial_{x_1}^{k+1} e^{-(t-\tau)L} P_1 F_0(u(\tau))\|_2 d\tau \\ &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-\frac{1}{2}} d\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} M(t)^2, \quad k = 0, 1, 2. \end{aligned}$$

As for I_2 , we have, for $k = 0, 1$,

$$\begin{aligned} I_2 &\leq C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{4}-\frac{k}{2}} (1+\tau)^{-1} d\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} M(t)^2. \end{aligned}$$

For $k = 2$, we have

$$\begin{aligned} I_2 &\leq C \int_{\frac{t}{2}}^t \|\partial_{x_1} e^{-(t-\tau)L} P_1 \partial_{x_1}^2 F_0(u(\tau))\|_2 d\tau \\ &\leq C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}} (1+\tau)^{-\frac{3}{2}} d\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{5}{4}} M(t)^2. \end{aligned}$$

We thus obtain

$$\|\partial_{x_1}^k u_1(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \{\|u_0\|_1 + M(t)^2\}. \quad (5.2)$$

We next estimate the time derivative. We have

$$\begin{aligned} \|-Lu_1(t)\|_2 &\leq C\{\|\partial_{x_1}^2 w_1(t)\|_2 + \|\partial_{x_1} u_1(t)\|_2\} \\ &\leq C(1+t)^{-\frac{3}{4}} \{\|u_0\|_1 + M(t)^2\}, \end{aligned}$$

and

$$\|P_1 F(u(t))\|_2 \leq C(1+t)^{-\frac{5}{4}} M(t)^2.$$

Since

$$\partial_t u_1 = -Lu_1 + P_1 F(u),$$

we obtain

$$\begin{aligned} \|\partial_t u_1(t)\|_2 &\leq \|Lu_1(t)\|_2 + \|P_1 F(u(t))\|_2 \\ &\leq C(1+t)^{-\frac{3}{4}} \{\|u_0\|_1 + M(t)^2\}. \end{aligned} \quad (5.3)$$

By (5.2) and (5.3), we deduce the desired estimate. This completes the proof. \blacksquare

6 Estimates on $P_\infty u$

In this section we estimate the high-frequency part $u_\infty = P_\infty u$ by using the Matsumura-Nishida energy method to prove estimate (4.10) in Proposition 4.4.

We introduce the quantity $D[w]$ which is defined by

$$D[w] = \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2.$$

We also define operators \tilde{P}_1 and \tilde{P}_∞ by

$$\tilde{P}_1 \phi = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} \langle \mathcal{F} \phi \rangle, \quad \tilde{P}_\infty = I - \tilde{P}_1.$$

Note that $P_1 u = {}^\top(\tilde{P}_1 \phi, \tilde{P}_1 w^1, 0)$ and $P_\infty u = {}^\top(\tilde{P}_\infty \phi, \tilde{P}_\infty w^1, w^2)$ for $u = {}^\top(\phi, w^1, w^2)$.

To prove (4.10), we prepare some basic estimates.

Proposition 6.1 *Let k and j be nonnegative integers satisfying $0 \leq 2k + j \leq 2$. Then*

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^k \partial_{x_1}^j u_\infty\|_2^2 + D[\partial_t^k \partial_{x_1}^j w_\infty] + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_t^k \partial_{x_1}^j \dot{\phi}_\infty\|_2^2 \leq C R_{j,k}^{(1)}, \quad (6.1)$$

where

$$\begin{aligned} \dot{\phi}_\infty &= \partial_t \phi_\infty + \gamma w \cdot \nabla \phi_\infty, \\ R_{j,k}^{(1)} &= \frac{\gamma}{2} (\operatorname{div} w, |\partial_t^k \partial_{x_1}^j \phi_\infty|^2) - \gamma ([\partial_t^k \partial_{x_1}^j, w] \cdot \nabla \phi_\infty, \partial_t^k \partial_{x_1}^j \phi_\infty) \\ &\quad + (\partial_t^k \partial_{x_1}^j \tilde{f}_\infty^0, \partial_t^k \partial_{x_1}^j \phi_\infty) + (\partial_t^k \partial_{x_1}^j \tilde{f}_\infty, \partial_t^k \partial_{x_1}^j w_\infty) + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_t^k \partial_{x_1}^j \tilde{f}_\infty^0\|_2^2. \end{aligned}$$

Here and in what follows, \tilde{f}_∞^0 denotes

$$\tilde{f}_\infty^0 = \gamma \tilde{P}_1(w \cdot \nabla \phi_\infty) - \gamma \tilde{P}_\infty(w \cdot \nabla \phi_1 + \phi \operatorname{div} w).$$

Proof. Equation (4.7) is written as

$$\partial_t \phi_\infty + \gamma w \cdot \nabla \phi_\infty + \gamma \operatorname{div} w_\infty = \tilde{f}_\infty^0, \quad (6.2)$$

$$\partial_t w_\infty - \nu \Delta w_\infty - \tilde{\nu} \nabla \operatorname{div} w_\infty + \gamma \nabla \phi_\infty = \tilde{f}_\infty. \quad (6.3)$$

We compute $(\partial_{x_1}^j (6.2), \partial_{x_1}^j \phi_\infty) + (\partial_{x_1}^j (6.3), \partial_{x_1}^j w_\infty)$ to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_{x_1}^j u_\infty\|_2^2 + D[\partial_{x_1}^j w_\infty] \\ &= -\gamma (w \cdot \nabla \partial_{x_1}^j \phi_\infty, \partial_{x_1}^j \phi_\infty) - \gamma ([\partial_{x_1}^j, w] \cdot \nabla \phi_\infty, \partial_{x_1}^j \phi_\infty) \\ &\quad + (\partial_{x_1}^j \tilde{f}_\infty^0, \partial_{x_1}^j \phi_\infty) + (\partial_{x_1}^j \tilde{f}_\infty, \partial_{x_1}^j w_\infty) \\ &= \frac{\gamma}{2} (\operatorname{div} w, |\partial_{x_1}^j \phi_\infty|^2) - \gamma ([\partial_{x_1}^j, w] \cdot \nabla \phi_\infty, \partial_{x_1}^j \phi_\infty) \\ &\quad + (\partial_{x_1}^j \tilde{f}_\infty^0, \partial_{x_1}^j \phi_\infty) + (\partial_{x_1}^j \tilde{f}_\infty, \partial_{x_1}^j w_\infty). \end{aligned} \quad (6.4)$$

We set $\dot{\phi} := \partial_t \phi + \gamma w \cdot \nabla \phi$. From (6.2), we have $\partial_{x_1}^j \dot{\phi}_\infty = -\gamma \operatorname{div} \partial_{x_1}^j w_\infty + \partial_{x_1}^j \tilde{f}_\infty^0$, and hence

$$\|\partial_{x_1}^j \dot{\phi}_\infty\|_2^2 \leq C(\gamma^2 \|\operatorname{div} \partial_{x_1}^j w_\infty\|_2^2 + \|\partial_{x_1}^j \tilde{f}_\infty^0\|_2^2).$$

We thus obtain

$$\begin{aligned} & \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \dot{\phi}_\infty\|_2^2 \\ & \leq C \left\{ \nu \|\nabla \partial_{x_1}^j w_\infty\|_2^2 + \tilde{\nu} \|\operatorname{div} \partial_{x_1}^j w_\infty\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \tilde{f}_\infty^0\|_2^2 \right\} \\ & = C \{ D[\partial_{x_1}^j w_\infty] + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \tilde{f}_\infty^0\|_2^2 \}. \end{aligned} \quad (6.5)$$

By (6.4) + $\frac{1}{2C} \times (6.5)$, we obtain

$$\frac{d}{dt} \|\partial_{x_1}^j u_\infty\|_2^2 + D[\partial_{x_1}^j w_\infty] + \frac{\nu + \tilde{\nu}}{C\gamma^2} \|\partial_{x_1}^j \dot{\phi}_\infty\|_2^2 \leq CR_{j,0}^{(1)}. \quad (6.6)$$

Replacing $\partial_{x_1}^j$ by ∂_t , we also have

$$\frac{d}{dt} \|\partial_t u_\infty\|_2^2 + D[\partial_t w_\infty] + \frac{\nu + \tilde{\nu}}{C\gamma^2} \|\partial_t \dot{\phi}_\infty\|_2^2 \leq CR_{0,1}^{(1)}. \quad (6.7)$$

This completes the proof. ■

Proposition 6.2 *It holds that*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \{ D[w_\infty] - 2\gamma(\phi_\infty, \operatorname{div} w_\infty) \} + \frac{1}{2} \|\partial_t u_\infty\|_2^2 \\ & \leq C \{ \gamma^2 \|\operatorname{div} w_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2 + \|w \cdot \nabla \phi_\infty\|_2^2 \}. \end{aligned} \quad (6.8)$$

Proof. We compute $((6.2), \partial_t \phi_\infty) + ((6.3), \partial_t w_\infty)$ to obtain

$$\begin{aligned} & \|\partial_t u_\infty\|_2^2 + \frac{1}{2} \frac{d}{dt} D[w_\infty] + \gamma \{ (\operatorname{div} w_\infty, \partial_t \phi_\infty) + (\nabla \phi_\infty, \partial_t w_\infty) \} \\ & = -\gamma \{ (w \cdot \nabla \phi_\infty, \partial_t \phi_\infty) + (\tilde{f}_\infty^0, \partial_t \phi_\infty) + (\tilde{f}_\infty, \partial_t w_\infty) \}. \end{aligned}$$

Since $(\nabla \phi_\infty, \partial_t w_\infty) = -(\phi_\infty, \operatorname{div} \partial_t w_\infty)$, we have

$$\begin{aligned} & \|\partial_t u_\infty\|_2^2 + \frac{1}{2} \frac{d}{dt} D[w_\infty] + \gamma \{ (\operatorname{div} w_\infty, \partial_t \phi_\infty) - (\phi_\infty, \operatorname{div} \partial_t w_\infty) \} \\ & = -\gamma \{ (w \cdot \nabla \phi_\infty, \partial_t \phi_\infty) + (\tilde{f}_\infty^0, \partial_t \phi_\infty) + (\tilde{f}_\infty, \partial_t w_\infty) \} \\ & \leq \frac{1}{4} \|\partial_t u_\infty\|_2^2 + C \{ \|w \cdot \nabla \phi_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2 \}. \end{aligned} \quad (6.9)$$

Adding $-2\gamma(\operatorname{div} w_\infty, \partial_t \phi_\infty)$ to both sides of (6.9), we obtain

$$\begin{aligned} & \|\partial_t u_\infty\|_2^2 + \frac{1}{2} \frac{d}{dt} D[w_\infty] - \gamma \frac{d}{dt} (\phi_\infty, \operatorname{div} w_\infty) \\ & \leq -2\gamma(\operatorname{div} w_\infty, \partial_t \phi_\infty) + \frac{1}{4} \|\partial_t u_\infty\|_2^2 + C\{\|w \cdot \nabla \phi_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2\} \\ & \leq \frac{1}{2} \|\partial_t u_\infty\|_2^2 + C\{\gamma^2 \|\operatorname{div} w_\infty\|_2^2 + \|w \cdot \nabla \phi_\infty\|_2^2 + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2\}, \end{aligned}$$

which gives the desired estimate. This completes the proof. \blacksquare

Proposition 6.3 *Let j and l be integers satisfying $0 \leq j + l \leq 1$. Then*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 \\ & \leq CR_{j,l}^{(2)} + C\left\{ \frac{1}{\nu + \tilde{\nu}} \|\partial_t \partial_{x_1}^j \partial_{x_2}^l w_\infty\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^{j+1} \partial_{x_2}^l \nabla w_\infty\|_2^2 \right\}, \end{aligned} \quad (6.10)$$

where

$$\begin{aligned} R_{j,l}^{(2)} &= \frac{\gamma}{2} (\operatorname{div} w, |\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty|^2) + C\{(\nu + \tilde{\nu}) \|[\partial_{x_1}^j \partial_{x_2}^{l+1}, w] \cdot \nabla \phi_\infty\|_2^2 \\ & \quad + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \partial_{x_2}^l h_\infty^0\|_2^2\}, \\ h_\infty^0 &= \partial_{x_2} \tilde{f}_\infty^0 + \frac{\gamma}{\nu + \tilde{\nu}} f_\infty^2. \end{aligned}$$

Proof. We compute $\partial_{x_2}(6.2) + \frac{\gamma}{\nu + \tilde{\nu}} \times$ (the second component of (6.3)) to obtain

$$\begin{aligned} & \partial_{x_2} \dot{\phi}_\infty + \gamma \partial_{x_2} \operatorname{div} w_\infty + \frac{\gamma}{\nu + \tilde{\nu}} \partial_t w_\infty^2 - \frac{\nu \gamma}{\nu + \tilde{\nu}} \Delta w_\infty^2 \\ & - \frac{\tilde{\nu} \gamma}{\nu + \tilde{\nu}} \partial_{x_2} \operatorname{div} w_\infty + \frac{\gamma^2}{\nu + \tilde{\nu}} \partial_{x_2} \phi_\infty = h_\infty^0. \end{aligned}$$

This gives

$$\partial_{x_2} \dot{\phi}_\infty + \frac{\gamma^2}{\nu + \tilde{\nu}} \partial_{x_2} \phi_\infty = h_\infty^0 - \frac{\gamma}{\nu + \tilde{\nu}} H(w_\infty), \quad (6.11)$$

where

$$H(w) = \partial_t w^2 - \nu \partial_{x_1}^2 w^2 + \nu \partial_{x_1} \partial_{x_2} w^1.$$

Applying $\partial_{x_1}^j \partial_{x_2}^l$ to (6.11) we have

$$\partial_{x_1}^j \partial_{x_2}^{l+1} \dot{\phi}_\infty + \frac{\gamma^2}{\nu + \tilde{\nu}} \partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty = \partial_{x_1}^j \partial_{x_2}^l h_\infty^0 - \frac{\gamma}{\nu + \tilde{\nu}} H(\partial_{x_1}^j \partial_{x_2}^l w_\infty). \quad (6.12)$$

We also write (6.12) as

$$\begin{aligned}
& \partial_t \partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty + \frac{\gamma^2}{\nu + \tilde{\nu}} \partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty \\
&= -\gamma w \cdot \nabla \partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty - \gamma [\partial_{x_1}^j \partial_{x_2}^{l+1}, w] \cdot \nabla \phi_\infty + \partial_{x_1}^j \partial_{x_2}^l h_\infty^0 \\
&\quad - \frac{\gamma}{\nu + \tilde{\nu}} H(\partial_{x_1}^j \partial_{x_2}^l w_\infty).
\end{aligned} \tag{6.13}$$

Taking the inner product of (6.13) with $\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty$ we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 \\
&= \frac{\gamma}{2} (\operatorname{div} w, |\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty|^2) + (-\gamma [\partial_{x_1}^j \partial_{x_2}^{l+1}, w] \cdot \nabla \phi_\infty \\
&\quad + \partial_{x_1}^j \partial_{x_2}^l h_\infty^0 - \frac{\gamma}{\nu + \tilde{\nu}} H(\partial_{x_1}^j \partial_{x_2}^l w_\infty), \partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty) \\
&\leq \frac{\gamma^2}{2(\nu + \tilde{\nu})} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{\gamma}{2} (\operatorname{div} w, |\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty|^2) \\
&\quad + C\{(\nu + \tilde{\nu}) \|[\partial_{x_1}^j \partial_{x_2}^{l+1}, w] \cdot \nabla \phi_\infty\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \partial_{x_2}^l h_\infty^0\|_2^2 \\
&\quad + \frac{1}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^l \partial_t w_\infty\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^{j+1} \partial_{x_2}^l \nabla w_\infty\|_2^2\} \\
&= \frac{\gamma^2}{2(\nu + \tilde{\nu})} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + R_{j,l}^{(2)} \\
&\quad + C\left\{ \frac{1}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^l \partial_t w_\infty\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^{j+1} \partial_{x_2}^l \nabla w_\infty\|_2^2 \right\}.
\end{aligned} \tag{6.14}$$

The desired estimate follows from (6.14). This completes the proof. \blacksquare

Proposition 6.4 *Let j and l be integers satisfying $0 \leq j + l \leq 1$. Then*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \dot{\phi}_\infty\|_2^2 \\
&\leq C R_{j,l}^{(2)} + C \left\{ \frac{1}{\nu + \tilde{\nu}} \|\partial_t \partial_{x_1}^j \partial_{x_2}^l w_\infty\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^{j+1} \partial_{x_2}^l \nabla w_\infty\|_2^2 \right\}.
\end{aligned} \tag{6.15}$$

Proof. From (6.12), we get

$$\begin{aligned}
& \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \dot{\phi}_\infty\|_2^2 \\
&\leq C \left\{ \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^l \partial_t w_\infty\|_2^2 \right. \\
&\quad \left. + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^{j+1} \partial_{x_2}^l \nabla w_\infty\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \partial_{x_2}^l h_\infty^0\|_2^2 \right\}.
\end{aligned} \tag{6.16}$$

By (6.10) + $\frac{1}{2C} \times (6.16)$, we obtain

$$\begin{aligned} & \frac{d}{dt} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{\gamma^2}{2(\nu + \tilde{\nu})} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \phi_\infty\|_2^2 + \frac{\nu + \tilde{\nu}}{2C\gamma^2} \|\partial_{x_1}^j \partial_{x_2}^{l+1} \dot{\phi}_\infty\|_2^2 \\ & \leq CR_{j,l}^{(2)} + C \left\{ \frac{1}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^l \partial_t w_\infty\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\nabla \partial_{x_1}^{j+1} \partial_{x_2}^l w_\infty\|_2^2 \right\}. \end{aligned} \quad (6.17)$$

This completes the proof. ■

Proposition 6.5 *Let j and l be integers satisfying $0 \leq j + l \leq 1$. Then*

$$\begin{aligned} & \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_x^{l+1} \phi_\infty\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \partial_x^{l+2} w_\infty\|_2^2 \\ & \leq C \left\{ \frac{1}{\nu + \tilde{\nu}} \|\partial_t \partial_{x_1}^j w_\infty\|_{H^l}^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \dot{\phi}\|_{H^{l+1}}^2 \right. \\ & \quad \left. + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \tilde{f}_\infty^0\|_{H^{l+1}}^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_{x_1}^j \tilde{f}_\infty\|_{H^l}^2 \right\}. \end{aligned} \quad (6.18)$$

To prove Proposition 6.5 we will apply the following lemma.

Lemma 6.6 *Let $u = {}^\top(p, v)$ be a solution of the Stokes system*

$$\begin{cases} \operatorname{div} v = f, \\ -\Delta v + \nabla p = g, \\ \partial_{x_2} v^1|_{x_2=0,1} = v^2|_{x_2=0,1} = 0. \end{cases} \quad (6.19)$$

Then there exists a constant $C > 0$ such that, for $0 \leq j + l \leq 1$,

$$\|\partial_{x_1}^j \partial_x^{l+2} v\|_2 + \|\partial_{x_1}^j \partial_x^{l+1} p\|_2 \leq C \{ \|\partial_{x_1}^j \partial_x^{l+1} f\|_2 + \|\partial_{x_1}^j \partial_x^l g\|_2 \}.$$

It is not difficult to prove Lemma 6.6 by using the Fourier transform in x_1 and the Fourier series expansion in x_2 as in (3.8) and (3.9). More precisely, one can prove Lemma 6.6 for $l = 0$ by using Lemma 3.2 with $\nu = \gamma = 1, \tilde{\nu} = 0$; and for $l = 1$, in addition to Lemma 3.2, we also use the equations (6.19) to estimate $\|\partial_{x_2}^2 p\|_2$ and $\|\partial_{x_2}^3 v\|_2$. We omit the detail.

Proof of Proposition 6.5. We rewrite equation (6.2)-(6.3) in the following form:

$$\begin{cases} \operatorname{div} w_\infty = \frac{1}{\gamma} (\tilde{f}_\infty^0 - \dot{\phi}_\infty) \\ -\Delta w_\infty + \nabla \left(\frac{\gamma}{\nu} \phi_\infty \right) = \frac{1}{\nu} \{ \tilde{f}_\infty - (\partial_t w_\infty - \frac{\tilde{\nu}}{\gamma} \nabla (\tilde{f}_\infty^0 - \dot{\phi}_\infty)) \}. \end{cases} \quad (6.20)$$

It then follows from Lemma 6.6 that

$$\begin{aligned}
& \frac{\gamma^2}{\nu^2} \|\partial_{x_1}^j \partial_x^{l+1} \phi_\infty\|_2^2 + \|\partial_{x_1}^j \partial_x^{l+2} w_\infty\|_2^2 \\
& \leq C \left\{ \left\| \frac{1}{\gamma} \partial_{x_1}^j (\tilde{f}_\infty^0 - \dot{\phi}_\infty) \right\|_{H^{l+1}}^2 \right. \\
& \quad \left. + \left\| \frac{1}{\nu} \partial_{x_1}^j (\tilde{f}_\infty - \partial_t w_\infty + \frac{\tilde{\nu}}{\gamma} \nabla (\tilde{f}_\infty^0 - \dot{\phi}_\infty)) \right\|_{H^l}^2 \right\} \\
& \leq C \left\{ \frac{1}{\nu^2} \|\partial_t \partial_{x_1}^j w_\infty\|_{H^l}^2 + \frac{\nu^2 + \tilde{\nu}^2}{\gamma^2 \nu^2} \|\partial_{x_1}^j \dot{\phi}_\infty\|_{H^{l+1}}^2 \right. \\
& \quad \left. + \frac{\nu^2 + \tilde{\nu}^2}{\gamma^2 \nu^2} \|\partial_{x_1}^j \tilde{f}_\infty^0\|_{H^{l+1}}^2 + \frac{1}{\nu^2} \|\partial_{x_1}^j \tilde{f}_\infty\|_{H^l}^2 \right\}.
\end{aligned} \tag{6.21}$$

By $\frac{\nu^2}{\nu + \tilde{\nu}} \times (6.21)$ we obtain the desired result. ■

We are now in a position to prove (4.10).

Proof of (4.10). We compute $b_1 \times \{(6.1)_{j=k=0} + (6.1)_{j=1, k=0}\} + (6.8)$ with a positive number b_1 . Taking b_1 suitably large, we see that

$$E_0(t) = b_1 \|u_\infty(t)\|_2^2 + D[w_\infty(t)] - 2\gamma(\phi_\infty(t), \operatorname{div} w_\infty(t))$$

is equivalent to

$$\|u_\infty(t)\|_2^2 + D[w_\infty(t)],$$

and obtain

$$\frac{1}{2} \frac{d}{dt} E_1(t) + \frac{1}{2} D_1(t) \leq C N_1(t), \tag{6.22}$$

where

$$\begin{aligned}
E_1(t) &= E_0(t) + b_1 \|\partial_{x_1} u_\infty(t)\|_2^2, \\
D_1(t) &= b_1 \sum_{j=0}^1 \left(D[\partial_{x_1}^j w_\infty(t)] + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\partial_{x_1}^j \dot{\phi}_\infty(t)\|_2^2 \right) + \|\partial_t u_\infty(t)\|_2^2, \\
N_1(t) &= \sum_{j=0}^1 |R_{j,0}^{(1)}| + \|\tilde{f}_\infty^0\|_2^2 + \|\tilde{f}_\infty\|_2^2 + \|w \cdot \nabla \phi_\infty\|_2^2.
\end{aligned}$$

We next consider $b_2 \times (6.22) + (6.15)_{j=l=0}$. Then, with a suitably large $b_2 > 0$, we have

$$\frac{1}{2} \frac{d}{dt} E_2(t) + \frac{1}{2} D_2(t) \leq C N_2(t), \tag{6.23}$$

where

$$\begin{aligned} E_2(t) &= b_2 E_1(t) + \|\partial_{x_2} \phi_\infty(t)\|_2^2, \\ D_2(t) &= \frac{b_2}{2} D_1(t) + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_2} \phi_\infty(t)\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\dot{\phi}_\infty(t)\|_{H^1}^2, \\ N_2(t) &= b_2 N_1(t) + |R_{0,0}^{(2)}|. \end{aligned}$$

It then follows from $b_3 \times (6.23) + (6.18)_{j=l=0}$ with a suitably large $b_3 > 0$ that

$$\frac{1}{2} \frac{d}{dt} E_3(t) + \frac{1}{2} D_3(t) \leq C N_3(t), \quad (6.24)$$

where

$$\begin{aligned} E_3(t) &= b_3 E_2(t), \\ D_3(t) &= \frac{b_3}{2} D_2(t) + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_x \phi_\infty(t)\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^2 w_\infty(t)\|_2^2, \\ N_3(t) &= b_3 N_2(t) + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\tilde{f}_\infty^0\|_{H^1}^2. \end{aligned}$$

We next compute $(6.24) + b_4 \times \{(6.1)_{j=2,k=0} + (6.1)_{j=0,k=1}\} + (6.15)_{j=1,l=0}$. Taking $b_4 > 0$ suitably large, we have

$$\frac{1}{2} \frac{d}{dt} E_4(t) + \frac{1}{2} D_4(t) \leq C N_4(t), \quad (6.25)$$

where

$$\begin{aligned} E_4(t) &= E_3(t) + b_4 \{\|\partial_{x_1}^2 u_\infty(t)\|_2^2 + \|\partial_t u_\infty(t)\|_2^2\} + \|\partial_{x_1} \partial_{x_2} \phi_\infty(t)\|_2^2, \\ D_4(t) &= D_3(t) + b_4 \{D[\partial_{x_1}^2 w_\infty(t)] + D[\partial_t w_\infty(t)]\} + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_1} \partial_{x_2} \phi_\infty(t)\|_2^2 \\ &\quad + \frac{\nu + \tilde{\nu}}{\gamma^2} \{\|\partial_{x_1} \dot{\phi}_\infty(t)\|_{H^1}^2 + \|\partial_t \dot{\phi}_\infty(t)\|_2^2\}, \\ N_4(t) &= N_3(t) + |R_{2,0}^{(1)}(t)| + |R_{0,1}^{(1)}(t)| + |R_{1,0}^{(2)}(t)|. \end{aligned}$$

It then follows from $b_5 \times (6.25) + (6.18)_{j=1,l=0}$ with a suitably large $b_5 > 0$ that

$$\frac{1}{2} \frac{d}{dt} E_5(t) + \frac{1}{2} D_5(t) \leq C N_5(t), \quad (6.26)$$

where

$$\begin{aligned} E_5(t) &= b_5 E_4(t), \\ D_5(t) &= \frac{b_5}{2} D_4(t) + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_1} \partial_x \phi_\infty(t)\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_{x_1} \partial_x^2 w_\infty(t)\|_2^2, \\ N_5(t) &= N_4(t) + \|\partial_{x_1} \tilde{f}_\infty^0(t)\|_{H^1}^2 + \|\partial_{x_1} \tilde{f}_\infty(t)\|_2^2. \end{aligned}$$

We next consider $b_6 \times (6.26) + (6.15)_{j=0, l=1}$. Then, with $b_6 > 0$ suitably large, we have

$$\frac{1}{2} \frac{d}{dt} E_6(t) + \frac{1}{2} D_6(t) \leq C N_6(t), \quad (6.27)$$

where

$$\begin{aligned} E_6(t) &= b_6 E_5(t) + \|\partial_{x_2}^2 \phi_\infty(t)\|_2^2, \\ D_6(t) &= \frac{b_6}{2} D_5(t) + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_{x_2}^2 \phi_\infty(t)\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|\dot{\phi}_\infty(t)\|_{H^2}^2, \\ N_6(t) &= N_5(t) + |R_{0,1}^{(2)}(t)|. \end{aligned}$$

We then deduce from $b_7 \times (6.27) + (6.18)_{j=0, l=1}$ with a suitably large $b_7 > 0$ that

$$\frac{1}{2} \frac{d}{dt} E_7(t) + \frac{1}{2} D_7(t) \leq C N_7(t), \quad (6.28)$$

where

$$\begin{aligned} E_7(t) &= b_7 E_6(t), \\ D_7(t) &= \frac{b_7}{2} D_6(t) + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\partial_x^2 \phi_\infty(t)\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^3 w_\infty(t)\|_2^2, \\ N_7(t) &= N_6(t) + \|\tilde{f}_\infty^0(t)\|_{H^2}^2 + \|\tilde{f}_\infty(t)\|_{H^1}^2. \end{aligned}$$

By (6.2) we have

$$\|\partial_t \phi_\infty\|_{H^1}^2 \leq C \{ \gamma^2 \|\nabla w_\infty\|_{H^1}^2 + \gamma^2 \|w \cdot \nabla \phi_\infty\|_{H^1}^2 + \|\tilde{f}_\infty^0\|_{H^1}^2 \}. \quad (6.29)$$

We then deduce from $b_8 \times (6.28) + (6.29)$ with a suitably large $b_8 > 0$ that

$$\frac{1}{2} \frac{d}{dt} E_8(t) + \frac{1}{2} D_8(t) \leq C N_8(t), \quad (6.30)$$

where

$$\begin{aligned} E_8(t) &= b_8 E_7(t), \\ D_8(t) &= \frac{b_8}{2} D_7(t) + \|\partial_t \phi_\infty(t)\|_{H^1}^2, \\ N_8(t) &= N_7(t) + \|(w \cdot \nabla \phi_\infty)(t)\|_{H^1}^2. \end{aligned}$$

Note that $D_8(t)$ is equivalent to $D_\infty(t)$. By Remark 4.2, we have $D_8(t) \geq c_1 E_8(t)$ for some constant $c_1 > 0$, and hence,

$$\frac{d}{dt} E_8(t) + c_1 E_8(t) + D_8(t) \leq 2C N_8(t).$$

This implies

$$E_8(t) + \int_0^t e^{-c_1(t-\tau)} D_8(\tau) d\tau \leq e^{-c_1 t} E_8(0) + 2C \int_0^t e^{-c_1(t-\tau)} N_8(\tau) d\tau.$$

Since, by (6.3),

$$\begin{aligned} \nu \partial_{x_2}^2 w_\infty^1 &= \partial_t w_\infty^1 - \nu \partial_{x_1}^2 w_\infty^1 - \tilde{\nu} \partial_{x_1} (\partial_{x_1} w_\infty^1 + \partial_{x_2} w_\infty^2) + \gamma \partial_{x_1} \phi_\infty - f_\infty^1, \\ (\nu + \tilde{\nu}) \partial_{x_2}^2 w_\infty^2 &= \partial_t w_\infty^2 - \nu \partial_{x_1}^2 w_\infty^2 - \tilde{\nu} \partial_{x_2} \partial_{x_1} w_\infty^1 + \gamma \partial_{x_2} \phi_\infty - f_\infty^2, \end{aligned}$$

and

$$\|\partial_{x_1} \partial_{x_2} w_\infty(t)\|_2^2 = (\partial_{x_1}^2 w_\infty(t), \partial_{x_2}^2 w_\infty(t)) \leq \|\partial_{x_1}^2 w_\infty(t)\|_2 \|\partial_{x_2}^2 w_\infty(t)\|_2,$$

we have

$$\begin{aligned} \|\partial_{x_1} \partial_{x_2} w_\infty(t)\|_2^2 + \|\partial_{x_2}^2 w(t)\|_2^2 &\leq C\{E_8(t) + \|\tilde{f}_\infty(t)\|_2^2\} \\ &\leq C\{E_8(t) + (1+t)^{-\frac{5}{2}} M(t)^4\}. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau \\ \leq C\{e^{-at} E_\infty(0) + (1+t)^{-\frac{5}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} N_8(\tau) d\tau\}. \end{aligned}$$

holds uniformly for $t \in [0, T]$ with $C > 0$ independent of T . Here $a = a(\nu, \tilde{\nu}, \gamma)$ is a positive constant. We will see in section 7 that N_8 satisfies the estimate

$$N_8(t) \leq C\{(1+t)^{-\frac{5}{2}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D_\infty(t)\}.$$

We thus obtain estimate (4.10) with $\mathcal{R}(t) = N_8(t)$ that satisfies (4.11). This completes the proof. ■

7 Estimates on Nonlinearities

In this section we estimate the nonlinearities to establish (4.11) for $\mathcal{R}(t) = N_8(t)$.

By using the Gagliardo-Nirenberg-Sobolev inequality and Remark 4.2 we have the following estimates on $f^0 = -\gamma \operatorname{div}(\phi w)$.

Proposition 7.1 *The following estimates hold uniformly for $t \in [0, T]$ with $C > 0$ independent of T :*

$$\|\phi \operatorname{div} w\|_{H^2} \leq C \left\{ (1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t) \sqrt{D_\infty(t)} \right\}, \quad (7.1)$$

$$\|w \cdot \nabla \phi\|_{H^1} + \|w \cdot \nabla \phi_1\|_{H^2} \leq C (1+t)^{-\frac{5}{4}} M(t)^2, \quad (7.2)$$

$$|(\operatorname{div} w, |\partial_t^k \partial_{x_1}^j \phi_\infty|^2)| \leq C \left\{ (1+t)^{-\frac{5}{2}} M(t)^3 + (1+t)^{-\frac{5}{2}} M(t) D_\infty(t) \right\}, \quad (7.3)$$

$$\|[\partial_t^k \partial_x^j, w] \cdot \nabla \phi_\infty\|_2 \leq C \left\{ (1+t)^{-2} M(t)^2 + (1+t)^{-\frac{5}{4}} M(t) \sqrt{D_\infty(t)} \right\} \quad (7.4)$$

for $2k + j \leq 2$ and

$$\|\tilde{f}_\infty^0\|_{H^2} \leq C \left\{ (1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t) \sqrt{D_\infty(t)} \right\}, \quad (7.5)$$

$$\|\partial_t \tilde{f}_\infty^0\|_2 \leq C \left\{ (1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t) \sqrt{D_\infty(t)} \right\}. \quad (7.6)$$

We next consider the estimates for \tilde{f}_∞ . We recall that $\|\phi(t)\|_\infty \leq \frac{1}{2}$ whenever $\|u(t)\|_{H^2} \leq \varepsilon_2$, which follows from the Sobolev inequality $\|u(t)\|_\infty \leq C\|u(t)\|_{H^2}$.

Proposition 7.2 *If $\|u(t)\|_{H^2} \leq \varepsilon_2$ and $M(t) \leq 1$ for $t \in [0, T]$, then*

$$\|\tilde{f}_\infty\|_2 \leq C (1+t)^{-\frac{5}{4}} M(t)^2, \quad (7.7)$$

$$\|\tilde{f}_\infty\|_{H^1} \leq C \left\{ (1+t)^{-\frac{5}{4}} M(t)^2 + (1+t)^{-\frac{1}{2}} M(t) \sqrt{D_\infty(t)} \right\}, \quad (7.8)$$

$$|(\partial_t \tilde{f}_\infty, \partial_t w_\infty)| \leq C \left\{ (1+t)^{-\frac{5}{2}} M(t)^3 + M(t) D_\infty(t) \right\} \quad (7.9)$$

where $C > 0$ is a constant independent of T .

Proof. The estimates (7.7)-(7.9) can be proved by using the Gagliardo-Nirenberg-Sobolev inequality and Remark 4.2. We here estimate the term $(\partial_t(\frac{\phi}{1+\phi} \Delta w), \partial_t w_\infty)$ only, which appears in (7.9). We set $g(\phi) = \frac{\phi}{1+\phi}$. Then

$$\begin{aligned} & |(\partial_t(g(\phi) \Delta w), \partial_t w_\infty)| \\ &= |-(\partial_t(\nabla(g(\phi)) \nabla w), \partial_t w_\infty) - (\partial_t(g(\phi) \nabla w), \partial_t \nabla w_\infty)| \\ &\leq |(\partial_t(g'(\phi) \nabla \phi \nabla w), \partial_t w_\infty)| + |(\partial_t(g(\phi) \nabla w), \partial_t \nabla w_\infty)| \\ &=: \text{I} + \text{II}. \end{aligned}$$

The first term on the right is estimated as

$$\begin{aligned} \text{I} &\leq |(g''(\phi) \partial_t \phi \nabla \phi \nabla w, \partial_t w_\infty)| + |(g'(\phi) \partial_t \nabla \phi \nabla w, \partial_t w_\infty)| \\ &\quad + |(g'(\phi) \nabla \phi \partial_t \nabla w, \partial_t w_\infty)| \\ &\leq C \{ \|\partial_t \phi\|_4 \|\nabla \phi\|_4 \|\nabla w\|_\infty \|\partial_t w_\infty\|_2 + \|\partial_t \nabla \phi\|_2 \|\nabla w\|_\infty \|\partial_t w_\infty\|_2 \\ &\quad + \|\nabla \phi\|_4 \|\partial_t \nabla w\|_2 \|\partial_t w_\infty\|_4 \} \\ &\leq C \{ \|\partial_t \phi\|_{H^1} \|\nabla \phi\|_{H^1} \|\nabla w\|_{H^2} \|\partial_t w_\infty\|_2 + \|\partial_t \nabla \phi\|_2 \|\nabla w\|_{H^2} \|\partial_t w_\infty\|_2 \\ &\quad + \|\nabla \phi\|_{H^1} \|\partial_t \nabla w\|_2 \|\partial_t w_\infty\|_{H^1} \} \\ &\leq C \left\{ (1+t)^{-\frac{5}{2}} M(t)^3 + (1+t)^{-\frac{1}{2}} M(t) D_\infty(t) \right\}. \end{aligned}$$

As for II, we have

$$\begin{aligned}
\text{II} &\leq C\{\|g'(\phi)\partial_t\phi\nabla w\|_2 + \|g(\phi)\partial_t\nabla w\|_2\}\|\partial_t\nabla w_\infty\|_2 \\
&\leq C\{\|\partial_t\phi\|_4\|\nabla w\|_4 + \|g(\phi)\|_\infty\|\partial_t\nabla w\|_2\}\|\partial_t\nabla w_\infty\|_2 \\
&\leq C\{\|\partial_t\phi\|_{H^1}\|\nabla w\|_{H^1} + \|\phi\|_\infty\|\partial_t\nabla w\|_2\}\|\partial_t\nabla w_\infty\|_2 \\
&\leq C\{(1+t)^{-\frac{5}{2}}M(t)^3 + M(t)D_\infty(t)\}.
\end{aligned}$$

The other terms can be estimated similarly. This completes the proof. ■

The desired estimate for $\mathcal{R}(t) = N_8(t)$ follows from Propositions 7.1 and 7.2.

8 Asymptotic Behavior: Proof of Theorem 2.2

In this section we prove Theorem 2.2. To this end we rewrite (1.1)-(1.2) in the form of conservation laws.

We set

$$m = \rho v = \rho_*(1 + \phi)v.$$

Then (1.1)-(1.2) is written as

$$\begin{cases} \partial_t \rho + \operatorname{div} m = 0, \\ \partial_t m - \mu \Delta(\frac{m}{\rho}) - (\mu + \mu') \nabla \operatorname{div}(\frac{m}{\rho}) + \nabla P(\rho) + \operatorname{div}(\frac{m \otimes m}{\rho}) = 0, \end{cases} \quad (8.1)$$

and the boundary condition (1.3) is transformed into

$$\partial_{x_2}\left(\frac{m^1}{\rho}\right)\Big|_{x_2=0,1} = 0, \quad m^2\Big|_{x_2=0,1} = 0. \quad (8.2)$$

We note that, from the proof of Theorem 2.1,

$$\|m^2(t)\|_2 = \|\gamma\rho(t)w^2(t)\|_2 = O(t^{-\frac{5}{4}}) \quad \text{as } t \rightarrow \infty.$$

Therefore, to prove Theorem 2.2, it suffices to investigate the asymptotic behavior of ${}^\top(\phi, m^1)$.

We decompose ${}^\top(\phi, m^1)$ as

$$\begin{aligned}
\phi &= \Phi + \Phi_\infty, \quad \Phi = \phi_1 = \tilde{P}_1\phi, \quad \Phi_\infty = \phi_\infty = \tilde{P}_\infty\phi, \\
m^1 &= \rho_*\gamma(M + M_\infty), \quad M = \frac{1}{\rho_*\gamma}\tilde{P}_1m^1, \quad M_\infty = \frac{1}{\rho_*\gamma}\tilde{P}_\infty m^1.
\end{aligned}$$

Note that $w^1 = \frac{M+M_\infty}{1+\phi}$.

Applying P_1 to (8.1) and using (8.2), we have

$$\begin{cases} \partial_t \Phi + \gamma \partial_{x_1} M = 0, \\ \partial_t M - (\nu + \tilde{\nu}) \partial_{x_1}^2 M + \gamma \partial_{x_1} \Phi = \partial_{x_1} \tilde{P}_1 g(U) + \partial_{x_1} \tilde{P}_1 \tilde{g}. \end{cases} \quad (8.3)$$

Here $U = {}^\top(\Phi, M)$,

$$\begin{aligned} g(U) &= -\frac{\rho_* P''(\rho_*)}{2\gamma} \Phi^2 - \gamma M^2, \\ \tilde{g} = \tilde{g}(x, t) &= -(\nu + \tilde{\nu}) \partial_{x_1}(\phi w^1) - \frac{\rho_* P''(\rho_*)}{2\gamma} (2\Phi \Phi_\infty + \Phi_\infty^2) \\ &\quad - \gamma(2MM_\infty + M_\infty^2) + \gamma(\phi w^1(M + M_\infty)), \end{aligned}$$

where $\phi = \Phi + \Phi_\infty$, $w^1 = \frac{M+M_\infty}{1+\phi}$.

We write (8.3) in the form

$$\begin{cases} \partial_t U + L_0 U = \partial_{x_1} P_0 G(U) + \partial_{x_1} P_0 \tilde{G}, & U = P_0 U, \\ U|_{t=0} = P_0 U_0, \end{cases} \quad (8.4)$$

where $U_0 = {}^\top(\phi_0, \frac{1}{\rho_* \gamma} m_0^1) = {}^\top(\phi_0, (1 + \phi_0)w_0^1)$,

$$\begin{aligned} L_0 &= \mathcal{F}^{-1} \begin{pmatrix} 0 & \gamma \partial_{x_1} \\ \gamma \partial_{x_1} & -(\nu + \tilde{\nu}) \partial_{x_1}^2 \end{pmatrix}, \\ G(U) &= \begin{pmatrix} 0 \\ g(U) \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}, \end{aligned}$$

and P_0 denotes the projection defined by

$$P_0(U) = \begin{pmatrix} \tilde{P}_1 \Phi \\ \tilde{P}_1 M \end{pmatrix}$$

for $U = {}^\top(\Phi, M)$.

We see from Lemma 3.2 that

$$e^{-tL_0} = \mathcal{F}^{-1}(e^{\lambda_+ t} P_+ + e^{\lambda_- t} P_-) \mathcal{F},$$

where

$$\begin{aligned} \lambda_\pm &= \lambda_{\pm,0} = -\frac{1}{2}(\nu + \tilde{\nu})\xi^2 \pm \frac{1}{2}\sqrt{(\nu + \tilde{\nu})^2 \xi^4 - 4\gamma^2 \xi^2}, \\ P_\pm &= \pm \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_\mp & i\gamma\xi \\ i\gamma\xi & \lambda_\pm \end{pmatrix}. \end{aligned}$$

We observe that, for $|\xi| \ll 1$,

$$\begin{aligned}\lambda_{\pm} &= -\frac{\nu + \tilde{\nu}}{2}\xi^2 \pm i\gamma\xi + O(\xi^3), \\ P_{\pm} &= \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} (1 + O(\xi)).\end{aligned}$$

We define $S(t)$ and $S_{\pm}(t)$ by

$$\begin{aligned}S(t) &= S_+(t) + S_-(t), \\ S_{\pm}(t) &= \mathcal{F}^{-1} \hat{S}_{\pm}(t) \mathcal{F}, \\ \hat{S}_{\pm}(t) &= \frac{1}{2} e^{-\frac{\nu + \tilde{\nu}}{2}\xi^2 t \pm i\gamma\xi t} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.\end{aligned}$$

Clearly, $e^{-tL_0}P_0$ has the same estimate as that for $e^{-tL}P_1$ such as (5.1). Furthermore, $e^{-tL_0}P_0$ is approximated by $S(t)$ in the following way. We define Π_0 by

$$\Pi_0 U_0 = {}^{\top}(\langle \phi_0 \rangle, \langle M_0 \rangle) \quad \text{for } U_0 = {}^{\top}(\phi_0, M_0).$$

Note that $\Pi_0 P_0 = P_0 \Pi_0 = P_0$.

Lemma 8.1 *The following estimates hold uniformly for $t > 0$:*

- (i) $\|\partial_{x_1}^k e^{-tL_0} P_0 U_0\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|U_0\|_1,$
- (ii) $\|\partial_{x_1}^k S_{\pm}(t) P_0 U_0\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|U_0\|_1,$
 $\|\partial_{x_1}^k S_{\pm}(t) \Pi_0 (I - P_0) U_0\|_2 \leq C t^{-\frac{k}{2}} e^{-c_0 t} \|U_0\|_2, \quad c_0 = \frac{1}{2}(\nu + \tilde{\nu})r_0^2,$
- (iii) $\|\partial_{x_1}^k (e^{-tL_0} - S(t)) P_0 U_0\|_2 \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \|U_0\|_1.$

Proof. The estimates in (i), (ii) can be obtained by the same computation as in (5.1). As for (iii), since

$$\begin{aligned}& |e^{\lambda_{\pm} t} P_{\pm} - \hat{S}_{\pm}(t)| \\ &= C \left\{ |e^{-\frac{\nu + \tilde{\nu}}{2}\xi^2 t \pm i\gamma\xi t} (e^{\lambda_{\pm} t + \frac{\nu + \tilde{\nu}}{2}\xi^2 t \mp i\gamma\xi t} - 1)| + C|\xi| e^{Re\lambda_{\pm} t} \right\} \\ &\leq C(|\xi|^3 t + |\xi|) e^{-\frac{\nu + \tilde{\nu}}{4}\xi^2 t},\end{aligned}$$

we have the desired estimate. \blacksquare

We denote by $U^{(0)}(t) = {}^{\top}(\phi^{(0)}(x_1, t), M^{(0),1}(x_1, t))$ the solution of the following integral equation:

$$U^{(0)}(t) = S(t) \Pi_0 U_0 + \int_0^t S(t - \tau) \partial_{x_1} G(U^{(0)}(\tau)) d\tau. \quad (8.5)$$

We see from (8.4) that $U(t)$ is written as

$$U(t) = e^{-tL_0} P_0 U_0 + \int_0^t e^{-(t-\tau)L_0} P_0 \partial_{x_1} (G(U) + \tilde{G})(\tau) d\tau. \quad (8.6)$$

We will show that $U(t)$ is approximated by $U^{(0)}(t)$ as $t \rightarrow \infty$.

By Lemma 8.1, we have the following estimates for $U^{(0)}(t)$.

Proposition 8.2 *If $\|U_0\|_{H^2 \cap L^1} \ll 1$, then (8.5) has a unique solution $U^{(0)}(t)$ that satisfies*

$$\|\partial_{x_1}^k U^{(0)}(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|U_0\|_{H^2 \cap L^1}, \quad k = 0, 1, 2, \quad (8.7)$$

$$\|\partial_{x_1}^k U^{(0)}(t)\|_\infty \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|U_0\|_{H^2 \cap L^1}, \quad k = 0, 1. \quad (8.8)$$

We have the following estimate for $U(t) - U^{(0)}(t)$.

Theorem 8.3 *If $\|U_0\|_{H^2 \cap L^1} \ll 1$, then*

$$\|\partial_{x_1}^k (U(t) - U^{(0)}(t))\|_2 \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta} \|U_0\|_{H^2 \cap L^1}, \quad k = 0, 1,$$

for any $\delta > 0$.

Proof. We introduce $N(t)$ defined by

$$N(t) = \sup_{0 \leq \tau \leq t} \left\{ \sum_{k=0}^1 (1+\tau)^{\frac{3}{4}+\frac{k}{2}-\delta} \|\partial_{x_1}^k (U(\tau) - U^{(0)}(\tau))\|_2 \right\}.$$

It follows from (8.5)-(8.6) that $U(t) - U^{(0)}(t)$ is written as

$$U(t) - U^{(0)}(t) = \sum_{j=0}^4 I_j(t),$$

where

$$\begin{aligned} I_0(t) &= (e^{-tL_0} P_0 - S(t) \Pi_0) U_0, \\ I_1(t) &= \int_0^t S(t-\tau) P_0 \partial_{x_1} (G(U(\tau)) - G(U^{(0)}(\tau))) d\tau, \\ I_2(t) &= \int_0^t (e^{-(t-\tau)L_0} - S(t-\tau)) P_0 \partial_{x_1} G(U(\tau)) d\tau, \\ I_3(t) &= - \int_0^t S(t-\tau) (I - P_0) \partial_{x_1} G(U^{(0)}(\tau)) d\tau, \\ I_4(t) &= \int_0^t e^{-(t-\tau)L_0} P_0 \partial_{x_1} \tilde{G}(\tau) d\tau. \end{aligned}$$

As for $I_0(t)$, we see from Lemma 8.1 (ii), (iii) that

$$\begin{aligned}\|\partial_{x_1}^k I_0(t)\|_2 &\leq \|\partial_{x_1}^k (e^{-tL_0} - S(t))P_0 U_0\|_2 + \|S(t)\Pi_0(I - P_0)\partial_{x_1}^k U_0\|_2 \\ &\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_{H^1 \cap L^1}.\end{aligned}$$

As for I_1 , by Theorem 2.1 and Proposition 8.2, we have

$$\begin{aligned}\|G(U) - G(U^{(0)})\|_1 &\leq C\{\|\Phi^2 - (\phi^{(0)})^2\|_1 + \|M^2 - (M^{(0),1})^2\|_1\} \\ &\leq C\|U + U^{(0)}\|_2\|U - U^{(0)}\|_2 \\ &\leq C(1+t)^{-1+\delta}N(t)\|U_0\|_{H^2 \cap L^1},\end{aligned}$$

and similarly,

$$\begin{aligned}\|\partial_{x_1}(G(U) - G(U^{(0)}))\|_1 &\leq C\{\|\partial_{x_1}(U + U^{(0)})\|_2\|U - U^{(0)}\|_2 \\ &\quad + \|U + U^{(0)}\|_2\|\partial_{x_1}(U - U^{(0)})\|_2\} \\ &\leq C(1+t)^{-\frac{3}{2}+\delta}N(t)\|U_0\|_{H^2 \cap L^1}.\end{aligned}$$

It then follows from Lemma 8.1 (ii) that

$$\begin{aligned}\|\partial_{x_1}^k I_1(t)\|_2 &\leq C\left\{\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1+\delta}d\tau\right. \\ &\quad \left.+\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1-\frac{k}{2}+\delta}d\tau\right\}N(t)\|U_0\|_{H^2 \cap L^1} \\ &\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta}N(t)\|U_0\|_{H^2 \cap L^1}.\end{aligned}$$

We next estimate $\partial_{x_1}^k I_2(t)$. Since

$$\begin{aligned}\|\partial_{x_1} G(U)\|_1 &\leq C\|U\|_2\|\partial_{x_1} U\|_2 \leq C(1+t)^{-1}M(t)^2, \\ \|\partial_{x_1}^2 G(U)\|_1 &\leq C\{\|U\|_2\|\partial_{x_1}^2 U\|_2 + \|\partial_{x_1} U\|_2^2\} \leq C(1+t)^{-\frac{3}{2}}M(t)^2.\end{aligned}$$

We see from Lemma 8.1 (iii) that

$$\begin{aligned}\|\partial_{x_1}^k I_2(t)\|_2 &\leq C\left\{\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1}d\tau\right. \\ &\quad \left.+\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1-\frac{k}{2}}d\tau\right\}M(t)^2 \\ &\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta}\|U_0\|_{H^2 \cap L^1}.\end{aligned}$$

Concerning $I_3(t)$, we first observe that $\partial_{x_1} G(U^{(0)}) = \Pi_0 \partial_{x_1} G(U^{(0)})$ since $\partial_{x_1} G(U^{(0)})$ depends only on x_1 and t . Furthermore, we have

$$\begin{aligned}\|\partial_{x_1} G(U^{(0)})\|_2 &\leq C \|U^{(0)}\|_\infty \|\partial_{x_1} U^{(0)}\|_2 \leq C(1+t)^{-\frac{5}{4}} \|U_0\|_{H^2 \cap L^1}^2, \\ \|\partial_{x_1}^2 G(U^{(0)})\|_2 &\leq C \{ \|U^{(0)}\|_\infty \|\partial_{x_1}^2 U^{(0)}\|_2 + \|\partial_{x_1} U^{(0)}\|_\infty \|\partial_{x_1} U^{(0)}\|_2 \} \\ &\leq C(1+t)^{-\frac{7}{4}} \|U_0\|_{H^2 \cap L^1}^2.\end{aligned}$$

It then follows from Lemma 8.1 (ii) that

$$\begin{aligned}\|\partial_{x_1}^k I_3(t)\|_2 &\leq C \int_0^t e^{-c_0(t-\tau)} (1+\tau)^{-\frac{5}{4}-\frac{k}{2}} d\tau \|U_0\|_{H^2 \cap L^1}^2 \\ &\leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}} \|U_0\|_{H^2 \cap L^1}^2.\end{aligned}$$

As for $I_4(t)$, we have

$$\begin{aligned}\|\tilde{G}\|_1 &\leq C(1+t)^{-1} M(t)^2, \\ \|\partial_{x_1} \tilde{G}\|_1 &\leq C(1+t)^{-\frac{3}{2}} M(t)^2,\end{aligned}$$

and hence, similarly to the estimate for $\partial_{x_1}^k I_2(t)$,

$$\|\partial_{x_1}^k I_4(t)\|_2 \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta} \|U_0\|_{H^2 \cap L^1}.$$

This completes the proof. ■

Proof of Theorem 2.2. It suffices to show that $\|\partial_{x_1}^k (U^{(0)} - \chi_+ \mathbf{b}_+ - \chi_- \mathbf{b}_-)(t)\|_2$ for $k = 0, 1$, where $\mathbf{b}_\pm = {}^\top(1, \pm 1) \in \mathbb{R}^2$. Here $\chi_\pm = \chi_\pm(x_1, t)$ is the diffusion waves given in (1.10)-(1.12) with $c = \frac{1}{2}(a+b)$, $a = -\frac{\rho_* P''(\rho_*)}{2\gamma}$, $b = -\gamma$. We follow the arguments in [7, 6]. We write U_0 as

$$U_0 = U_{0+} + U_{0-},$$

where

$$U_{0\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \Pi_0 U_0 = \frac{1}{2} \langle \phi_0 \pm \frac{1}{\rho_* \gamma} m_0^1 \rangle \mathbf{b}_\pm.$$

It then follows that

$$U^{(0)}(t) = S_+(t)U_{0+} + S_-(t)U_{0-} + I_{1,+}(t) + I_{1,-}(t),$$

where

$$I_{1,\pm}(t) = \int_0^t S_\pm(t-\tau) \partial_{x_1} \begin{pmatrix} 0 \\ a(\phi^{(0)})^2 + b(M^{(0),1})^2 \end{pmatrix} d\tau.$$

We write $I_{1,\pm}(t)$ as

$$I_{1,\pm} = \pm \frac{1}{2} \int_0^t e^{-(t-\tau)L_{\pm}} \partial_{x_1} (a(\phi^{(0)})^2 + b(M^{(0),1})^2) d\tau \mathbf{b}_{\pm},$$

where

$$e^{-tL_{\pm}} u_0 = \mathcal{F}^{-1} [e^{(-\frac{\nu+\bar{\nu}}{2}\xi^2 \pm i\gamma\xi)t} \hat{u}_0].$$

We note that $e^{-tL_{\pm}}$ satisfies the same estimates as those for $S_{\pm}(t)$ in Lemma 8.1 (ii).

We define $V(t) = {}^{\top}(\eta(t), \zeta(t))$ by

$$\begin{aligned} U^{(0)}(t) &= \chi_+(t) \mathbf{b}_+ + \chi_-(t) \mathbf{b}_- + V(t) \\ &= \begin{pmatrix} \chi_+ + \chi_- + \eta \\ \chi_+ - \chi_- + \zeta \end{pmatrix}, \end{aligned}$$

and introduce

$$Y(t) = \sup_{0 \leq \tau \leq t} \{(1+\tau)^{\frac{1}{2}} \|V(\tau)\|_2 + (1+\tau) \|\partial_{x_1} V(\tau)\|_2\}.$$

We write

$$\begin{aligned} (\phi^{(0)})^2 &= (\chi_+ + \chi_- + \eta)(\chi_+ + \chi_- + \eta) \\ &= \chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + (\chi_+ + \chi_-)\eta + \eta(\chi_+ + \chi_- + \eta) \\ &= \chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + (\chi_+ + \chi_- + \phi^{(0)})\eta \\ &= \chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + \sigma_1\eta, \end{aligned}$$

and

$$\begin{aligned} (M^{(0),1})^2 &= \chi_+^2 + \chi_-^2 - 2\chi_+\chi_- + (\chi_+ - \chi_- + M^{(0),1})\zeta \\ &= \chi_+^2 + \chi_-^2 - 2\chi_+\chi_- + \sigma_2\zeta, \end{aligned}$$

where $\sigma_1 = \chi_+ + \chi_- + \phi^{(0)}$ and $\sigma_2 = \chi_+ - \chi_- + M^{(0),1}$. It then follows that $I_{1,\pm}(t)$ is written in the following forms

$$\begin{aligned} I_{1,\pm}(t) &= \pm \frac{1}{2} \int_0^t e^{-(t-\tau)L_{\pm}} \partial_{x_1} \left((a+b)(\chi_+^2 + \chi_-^2) + 2(a-b)\chi_+\chi_- \right. \\ &\quad \left. + a\sigma_1\eta + b\sigma_2\zeta \right) d\tau \mathbf{b}_{\pm}. \end{aligned}$$

Since χ_{\pm} satisfies

$$\chi_{\pm}(t) = e^{-tL_{\pm}} \chi_{0\pm} \pm \frac{a+b}{2} \int_0^t e^{-(t-\tau)L_{\pm}} \partial_{x_1} (\chi_{\pm}^2)(\tau) d\tau,$$

where $\chi_{0\pm} = \chi_{\pm}(0)$, we see that

$$\begin{aligned}
V(t) &= U^{(0)}(t) - \chi_+(t)\mathbf{b}_+ - \chi_-(t)\mathbf{b}_- \\
&= S_+(t)(U_{0+} - \chi_{0+}\mathbf{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\mathbf{b}_-) + I_{1,+} + I_{1,-} \\
&\quad - \frac{a+b}{2} \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\chi_+^2)(\tau) d\tau \mathbf{b}_+ + \frac{a+b}{2} \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\chi_-^2)(\tau) d\tau \mathbf{b}_- \\
&= S_+(t)(U_{0+} - \chi_{0+}\mathbf{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\mathbf{b}_-) \\
&\quad + \frac{1}{2}(a+b) \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\chi_-^2)(\tau) d\tau \mathbf{b}_+ \\
&\quad - \frac{1}{2}(a+b) \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\chi_+^2)(\tau) d\tau \mathbf{b}_- \\
&\quad + (a-b) \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\chi_+\chi_-)(\tau) d\tau \mathbf{b}_+ \\
&\quad - (a-b) \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\chi_+\chi_-)(\tau) d\tau \mathbf{b}_- \\
&\quad + \frac{1}{2}a \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_1\eta)(\tau) d\tau d\tau \mathbf{b}_+ \\
&\quad - \frac{1}{2}a \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_1\eta)(\tau) d\tau \mathbf{b}_- \\
&\quad + \frac{1}{2}b \int_0^t e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_2\zeta)(\tau) d\tau \mathbf{b}_+ \\
&\quad - \frac{1}{2}b \int_0^t e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_2\zeta)(\tau) d\tau \mathbf{b}_-.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|\partial_{x_1}^k V(t)\|_{L^2} &\leq \sum_{j=\pm} \|\partial_{x_1}^k S_j(t)(U_{0j} - \chi_{0j} \mathbf{b}_j)\|_2 \\
&\quad + C_1 (\|\partial_{x_1}^k w_+(t)\|_2 + \|\partial_{x_1}^k w_-(t)\|_2) \\
&\quad + C_2 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+} \partial_{x_1}(\chi_+ \chi_-)(\tau)\|_2 d\tau \\
&\quad + C_3 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-} \partial_{x_1}(\chi_+ \chi_-)(\tau)\|_2 d\tau \\
&\quad + C_4 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_1 \eta)(\tau)\|_2 d\tau \\
&\quad + C_5 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_1 \eta)(\tau)\|_2 d\tau \\
&\quad + C_6 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+} \partial_{x_1}(\sigma_2 \zeta)(\tau)\|_2 d\tau \\
&\quad + C_7 \int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-} \partial_{x_1}(\sigma_2 \zeta)(\tau)\|_2 d\tau \\
&=: \sum_{j=\pm} \|\partial_{x_1}^k S_j(t)(U_{0j} - \chi_{0j} \mathbf{b}_j)\|_2 + \sum_{j=1}^7 I_j.
\end{aligned}$$

where

$$w_{\pm}(t) = \int_0^t e^{-(t-\tau)L_{\pm}} \partial_{x_1}(\chi_{\mp}^2)(\tau) d\tau,$$

$$C_1 = \frac{1}{2}|a+b|, \quad C_2 = C_3 = |a-b|, \quad C_4 = C_5 = \frac{1}{2}|a|, \quad C_6 = C_7 = \frac{1}{2}|b|.$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}} (U_{0\pm} - \chi_{0\pm} \mathbf{b}_{\pm}) dx_1 \\
&= \left[\frac{1}{2} \int_{\Omega} \left(\phi^{(0)} \pm \frac{1}{\rho_* \gamma} m_0^1 \right) dx - \int_{\mathbb{R}} \chi_{0\pm} dx_1 \right] \mathbf{b}_{\pm} = 0,
\end{aligned}$$

we have

$$\|\partial_{x_1}^k S_{\pm}(t)(U_{0\pm} - \chi_{0\pm} \mathbf{b}_{\pm})\|_2 \leq C t^{-\frac{1}{2} - \frac{k}{2}} \|u_0\|_{L_{1/2}^1}.$$

As for I_1 , we apply the estimates for w_{\pm} by T.-P. Liu [8] (see also [6, Lemma 4.2]) to obtain

$$I_1 \leq C(1+t)^{-\frac{1}{2} - \frac{k}{2}} \|u_0\|_{H^2 \cap L^1}^2.$$

We next estimate I_2 . For $1 \leq p \leq \infty$ and $l \geq 0$, we have

$$\|\partial_x^l(\chi_+\chi_-)(t)\|_1 \leq Ce^{-ct}\|u_0\|_{H^2 \cap L^1}^2. \quad (8.9)$$

See [7, 6] for estimate (8.9). It then follows from (8.9) and Lemma 8.1 (ii) that

$$\begin{aligned} I_2 &\leq C \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (\|\chi_+\chi_-(\tau)\|_1 + \|\partial_{x_1}^{k+1}(\chi_+\chi_-)(\tau)\|_2) d\tau \\ &\leq C \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} e^{-c\tau} d\tau \|u_0\|_{H^2 \cap L^1}^2 \\ &\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1}^2. \end{aligned}$$

Similarly, we have $I_3 \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1}^2$.

We next estimate I_4 . By Lemma 8.1, we have

$$\begin{aligned} I_4 &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \|\sigma_1\eta(\tau)\|_1 d\tau \\ &\quad + C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}} \|\partial_{x_1}^k(\sigma_1\eta)(\tau)\|_2 d\tau \\ &\quad + C \int_0^t e^{-c_0(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_{x_1}^k(\sigma_1\eta)(\tau)\|_2 d\tau \\ &=: I_{41} + I_{42} + I_{43}. \end{aligned}$$

By applying Proposition 8.2 and the following estimate

$$\|\partial_{x_1}^k \chi_{\pm}(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_1, \quad (8.10)$$

we see that $\|\sigma_1(\tau)\|_2 \leq C(1+\tau)^{-\frac{1}{4}} \|u_0\|_{H^2 \cap L^1}$. Since $\|\sigma_1\eta\|_1 \leq \|\sigma_1\|_2 \|\eta\|_2$, we have

$$\begin{aligned} I_{41} &= CY(t) \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-\frac{3}{4}} d\tau \|u_0\|_{H^2 \cap L^1} \\ &\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t). \end{aligned}$$

Similarly,

$$\begin{aligned} I_{42} &= CY(t) \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}} (1+\tau)^{-\frac{3}{4}-\frac{k}{2}} d\tau \|u_0\|_{H^2 \cap L^1} \\ &\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t), \end{aligned}$$

and

$$\begin{aligned} I_{43} &= CY(t) \int_0^t e^{-c_0(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-1-\frac{k}{2}} d\tau \|u_0\|_{H^2 \cap L^1} \\ &\leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t). \end{aligned}$$

We thus obtain $I_4 \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} Y(t)$. We can obtain the estimates for I_5, I_6, I_7 in a similar manner. It then follows that if $\|u_0\|_{H^2 \cap L^1} \ll 1$, we have

$$\|\partial_x^k V(t)\|_2 \leq C(1+t)^{-\frac{1}{2}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1} \quad (8.11)$$

for $k = 0, 1$.

Since $w^1 = \frac{1}{\rho_* \gamma} m^1 - \phi w^1$, we have $w_1^1 = M - \tilde{P}_1(\phi w^1)$, and so,

$$u = \begin{pmatrix} \phi_1 \\ w_1^1 \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_\infty \\ w_\infty^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} \Phi \\ M \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_\infty \\ -\tilde{P}_1(\phi w^1) + w_\infty^1 \\ w^2 \end{pmatrix}.$$

The desired estimate in Theorem 2.2 thus follows from Theorem 9.3 and (8.11). This completes the proof. ■

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