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Kagei, Yoshiyuki
Faculty of Mathematics, Kyushu University

Nishida, Takaaki
Department of Applied Complex System, Kyoto University

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Yoshiyuki Kagei
& Takaaki Nishida

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Institute of Mathematics for Industry
Graduate School of Mathematics
Kyushu University
Fukuoka, JAPAN

On Chorin's method for stationary solutions of the Oberbeck-Boussinesq equation

Yoshiyuki Kagei[1] and Takaaki Nishida[2]

[1] Faculty of Mathematics,
Kyushu University,
Nishi-ku, Motooka 744,
Fukuoka 819-0395, Japan

[2] Department of Applied Complex System,
Kyoto University,
Yoshida Honmachi, Sakyo-ku,
Kyoto, 606-8317, Japan

Abstract

Stability of stationary solutions of the Oberbeck-Boussinesq system (OB) and the corresponding artificial compressible system is considered. The latter system is obtained by adding the time derivative of the pressure with small parameter $\varepsilon > 0$ to the continuity equation of (OB), which was proposed by A. Chorin to find stationary solutions of (OB) numerically. Both systems have the same sets of stationary solutions and the system (OB) is obtained from the artificial compressible one as the limit $\varepsilon \to 0$ which is a singular limit. It is proved that if a stationary solution of the artificial compressible system is stable for sufficiently small $\varepsilon > 0$, then it is also stable as a solution of (OB). The converse is proved provided that the velocity field of the stationary solution satisfies some smallness condition.

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Keywords. Oberbeck-Boussinesq equation, artificial compressible system, Chorin's method, singular perturbation, stability, cellular convection patterns.

1 Introduction

This paper is concerned with the Oberbeck-Boussinesq equation

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, & (1.1)\\ \mathrm{Pr}^{-1}\left(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla \boldsymbol{v}\right) - \Delta \boldsymbol{v} + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 &= \mathbf{0}, & (1.2)\\ \partial_t \theta + \boldsymbol{v}\cdot\nabla\theta - \Delta\theta - \mathrm{Ra}\boldsymbol{v}\cdot\boldsymbol{e}_3 &= 0, & (1.3) \end{aligned}$$

and the artificial compressible system for (1.1)–(1.3):

$$\begin{aligned}
\varepsilon^2\partial_t p + \operatorname{div}\boldsymbol{v} &= 0, & (1.4)\\
\mathrm{Pr}^{-1}(\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) - \Delta\boldsymbol{v} + \nabla p - \mathrm{Ra}\theta\boldsymbol{e}_3 &= \mathbf{0}, & (1.5)\\
\partial_t\theta + \boldsymbol{v}\cdot\nabla\theta - \Delta\theta - \mathrm{Ra}\boldsymbol{v}\cdot\boldsymbol{e}_3 &= 0. & (1.6)
\end{aligned}$$

Here $\boldsymbol{v} = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, $p = p(x,t)$ and $\theta = \theta(x,t)$ denote the unknown velocity field, pressure and temperature deviation from the heat conductive state, respectively, at time $t > 0$ and position $x \in \mathbb{R}^3$; $\boldsymbol{e}_3 = {}^\top(0,0,1) \in \mathbb{R}^3$; $\mathrm{Pr} > 0$ and $\mathrm{Ra} > 0$ are non-dimensional parameters, called Prandtl and Rayleigh numbers, respectively; and $\varepsilon > 0$ is a small parameter, called artificial Mach number. Here and in what follows, the superscript ${}^\top\cdot$ stands for the transposition. The systems (1.1)–(1.3) and (1.4)–(1.6) are considered in the infinite layer Ω:

$$\Omega = \{x = (x', x_3); x' = (x_1, x_2) \in \mathbb{R}^2, 0 < x_3 < 1\}.$$

The Oberbeck-Boussinesq equation (1.1)–(1.3) is a system of equations which describes convection phenomena of viscous fluid occupying Ω heated from below (heated at $x_3 = 0$) under the gravitational force. It is well known ([1, 6, 7, 10]) that under the boundary condition

$$\boldsymbol{v}|_{x_3=0,1} = \mathbf{0}, \quad \theta|_{x_3=0,1} = 0, \tag{1.7}$$

there exists a critical number $\mathrm{Ra}_c > 0$ such that when $\mathrm{Ra} < \mathrm{Ra}_c$, the heat conductive state $\boldsymbol{v} = \mathbf{0}$, $\theta = 0$ is stable, while, when $\mathrm{Ra} > \mathrm{Ra}_c$, the heat conductive state is unstable and convective cellular stationary solutions bifurcate from the heat conductive state.

A. Chorin ([2, 3, 4]) proposed the artificial compressible system such as (1.4)–(1.6) to find stationary solutions of equations for viscous incompressible fluid numerically. In the context of the Oberbeck-Boussinesq equation (1.1)–(1.3), the idea is stated as follows. Obviously, the sets of stationary solutions of (1.1)–(1.3) and (1.4)–(1.6) are the same ones. If solutions of the artificial compressible system (1.4)–(1.6) converge to a function $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ as $t \to \infty$, then the limit u_s is a stationary solution of (1.4)–(1.6) which is thus a stationary solution of (1.1)–(1.3). By using this method, Chorin numerically obtained stationary cellular convection solutions of (1.1)–(1.3).

Since the limit u_s in Chorin's method described above is a large time limit of solutions of (1.4)–(1.6), u_s is stable as a solution of (1.4)–(1.6). It is of interest to consider whether u_s is stable as a solution of (1.1)–(1.3), in other words, whether u_s represents an observable stationary flow in the real world, and, conversely, what kind of stationary flows can be computed by Chorin's method. These questions are to be formulated as stability problem for stationary solutions of the systems (1.1)–(1.3) and (1.4)–(1.6). Since the system (1.1)–(1.3) is obtained from (1.4)–(1.6) as the limit $\varepsilon \to 0$, one could expect that solutions of (1.1)–(1.3) would be approximated by solutions of (1.4)–(1.6) when $\varepsilon \ll 1$. However, this limiting process is a singular limit, and hence, it is not straightforward to conclude that stability properties of u_s

as a solution of (1.1)–(1.3) are the same as those as a solution of (1.4)–(1.6) even when $0 < \varepsilon \ll 1$.

The purpose of this paper is to investigate stability relations of stationary solutions between the systems (1.1)–(1.3) and (1.4)–(1.6) when ε is sufficiently small. We investigate the spectra of the linearized operators around a stationary solution of (1.1)–(1.3) and (1.4)–(1.6) for $0 < \varepsilon \ll 1$.

We first show that if a stationary solution $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ of (1.4)–(1.6) is asymptotically stable for sufficiently small ε, then so is u_s as a stationary solution of (1.1)–(1.3). More precisely, we consider (1.1)–(1.3) and (1.4)–(1.6) under the boundary condition (1.7) and the periodicity condition

$$p,\ \boldsymbol{v} \text{ and } \theta \text{ are } \mathcal{Q}\text{-periodic in } (x_1, x_2), \tag{1.8}$$

where $\mathcal{Q} = [-\pi/\alpha_1, \pi/\alpha_1) \times [-\pi/\alpha_2, \pi/\alpha_2)$. Here α_j, $j = 1, 2$, are positive constants. We denote the basic period domain by $\Omega_{per} = \mathcal{Q} \times (0, 1)$. We introduce the linearized operators around u_s associated with (1.1)–(1.3) and (1.4)–(1.6). Let $L : L^2_{per,\sigma} \times L^2_{per} \to L^2_{per,\sigma} \times L^2_{per}$ be the operator acting on $\boldsymbol{U} = {}^\top(\boldsymbol{w}, \theta) \in D(L)$ defined by

$$L = \begin{pmatrix} -\mathrm{Pr}\mathbb{P}\Delta + \mathbb{P}(\boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s)) & -\mathrm{Pr}\mathrm{Ra}\mathbb{P}\boldsymbol{e}_3 \\ {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix}$$

with domain $D(L) = [(H^2_{per} \cap H^1_{0,per})^3 \cap L^2_{per,\sigma}] \times [H^2_{per} \cap H^1_{0,per}]$. Here L^2_{per}, H^k_{per}, $\cdots$, denote L^2, H^k, $\cdots$ spaces over Ω_{per} with periodicity condition in x', $\mathbb{P}$ denotes the Helmholtz projection from $(L^2_{per})^3$ to $L^2_{per,\sigma}$, where $L^2_{per,\sigma}$ denotes the set of all solenoidal vector fields $\boldsymbol{w} = {}^\top(w^1, w^2, w^3)$ in $(L^2_{per})^3$ with $w^3|_{x_3=0,1} = 0$. We define the operator $L_\varepsilon : H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per} \to H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}$, acting on $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon)$, by

$$L_\varepsilon = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ \mathrm{Pr}\nabla & -\mathrm{Pr}\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) & -\mathrm{Pr}\mathrm{Ra}\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix}$$

with domain $D(L_\varepsilon) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}]$. Here $H^1_{per,*} = \{p \in H^1_{per}; \int_{\Omega_{per}} p(x)\,dx = 0\}$.

We prove that if there exists a positive number b_0 such that $\rho(-L_{\varepsilon_n}) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0\}$ for some sequence $\varepsilon_n \to 0$ as $n \to \infty$, then there exists a constant $b_1 > 0$ such that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_1\}$. Therefore, a stationary solution obtained by Chorin's method with $0 < \varepsilon \ll 1$ is stable as a solution of the Oberbeck-Boussinesq system (1.1)–(1.3). Furthermore, we prove an instability result: if $\sigma(-L) \cap \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda > 0\} \neq \emptyset$, then $\sigma(-L_\varepsilon) \cap \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda > 0\} \neq \emptyset$ for $0 < \varepsilon \ll 1$. This shows that unstable stationary solutions of (1.1)–(1.3) cannot be obtained by Chorin's method with $0 < \varepsilon \ll 1$.

As a converse of the above result, we prove that if

$$\rho(-L) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0\} \tag{1.9}$$

for some constant $b_0 > 0$, then there exist constants $\delta_0 > 0$ and $b_1 > 0$ such that

$$\rho(-L_\varepsilon) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_1\} \tag{1.10}$$

for $0 < \varepsilon \ll 1$, provided that

$$\inf_{\boldsymbol{w}\in H^1_{0,per},\boldsymbol{w}\neq\boldsymbol{0}} \frac{\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w})_{L^2}}{\|\nabla\boldsymbol{w}\|^2_{L^2}} \geq -\delta_0. \tag{1.11}$$

This gives a sufficient condition for u_s to be computed by Chorin's method with $0 < \varepsilon \ll 1$. We note that no smallness condition for the temperature θ_s of u_s is required.

This result is applicable to stable bifurcating cellular convective patterns such as roll pattern, hexagonal pattern and etc., when $\mathrm{Ra} \sim \mathrm{Ra}_c$. In fact, the velocity fields of bifurcating convective patterns are small when $\mathrm{Ra} \sim \mathrm{Ra}_c$ since they bifurcate from $\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$ when Ra crosses Ra_c, and hence, condition (1.11) is satisfied.

To prove the first result, we show that an eigenvalue λ of $-L_\varepsilon$ with $\mathrm{Im}\,\lambda = O(1)$ can be obtained as a perturbation of an eigenvalue of $-L$ as $\varepsilon \to 0$. In fact, it is proved that if λ_0 is an eigenvalue of $-L$, then, for any neighborhood of λ_0, there exists an eigenvalue of $-L_\varepsilon$ for $0 < \varepsilon \ll 1$ and the total projection P_ε on the neighborhood satisfies $P_\varepsilon = P_0 + O(\varepsilon)$, where P_0 is a projection whose velocity-temperature part is the eigenprojection for the eigenvalue λ_0 of $-L$ (Theorem 5.2 (ii)). Based on this observation we show the first result by a contradiction argument. In particular, we find that if $-L$ has an eigenvalue with positive real part, then so does $-L_\varepsilon$ with $0 < \varepsilon \ll 1$, which gives the instability result.

To prove the converse, we investigate the resolvent $(\lambda + L_\varepsilon)^{-1}$ according to the cases for λ near the imaginary axis with $|\mathrm{Im}\,\lambda| \geq O(\varepsilon^{-1})$, $|\mathrm{Im}\,\lambda| \leq O(\varepsilon^{-1})$ and $|\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})$ under the condition (1.9). By a standard energy method one can show that $-L_\varepsilon$ is a sectorial operator and $\{\lambda; \mathrm{Re}\,\lambda \geq -b, |\mathrm{Im}\,\lambda| \geq O(\varepsilon^{-1})\} \subset \rho(-L_\varepsilon)$ with some $b > 0$ for $0 < \varepsilon \ll 1$. We show that the spectrum of $-L_\varepsilon$ near the imaginary axis with $|\mathrm{Im}\,\lambda| \leq O(\varepsilon^{-1})$ can be treated as a kind of regular perturbation of that of $-L$, which concludes that $\{\lambda; \mathrm{Re}\,\lambda \geq -b, |\mathrm{Im}\,\lambda| \leq O(\varepsilon^{-1})\} \subset \rho(-L_\varepsilon)$. The spectrum of $-L_\varepsilon$ in the region $|\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})$ stems from the "compressible" aspect of $-L_\varepsilon$ and does not appear in the spectrum of $-L$. We prove $\{\lambda; \mathrm{Re}\,\lambda \geq -b, |\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})\} \subset \rho(-L_\varepsilon)$ by an energy method provided that the condition (1.11) holds.

One more remark is added in order. Due to the translation invariance in x_1 and x_2 variables, 0 is an eigenvalue of $-L_\varepsilon$ whenever $\partial_{x_1} u_s \neq 0$ or $\partial_{x_2} u_s \neq 0$. In this case nonzero $\partial_{x_j} u_s$ are eigenfunctions for the eigenvalue 0. Correspondingly, 0 is an eigenvalue of $-L$ whenever $\partial_{x_1}\boldsymbol{U}_s \neq 0$ or $\partial_{x_2}\boldsymbol{U}_s \neq 0$. We formulate the main results of this paper by taking this translation invariance into account. We note that under the translation invariance one can show the orbital stability of $\boldsymbol{U}_s$ and u_s for the corresponding nonlinear problems.

This paper is organized as follows. In section 2 we introduce notations used in this paper. In section 3 we state the main theorems of this paper. Section 4 is devoted to the study of the null spaces of L and L_ε for $0 < \varepsilon \ll 1$. We then prove the first results in section 5. The converse is proved in section 6; and we give the estimates for the derivatives of the pressure component of $(\lambda + L_\varepsilon)^{-1}$ in section 7.

2 Preliminaries

In this section we introduce notation used in this paper.

For given α_1, $\alpha_2 > 0$, we denote the basic period cell by

$$\mathcal{Q} = \left[-\tfrac{\pi}{\alpha_1}, \tfrac{\pi}{\alpha_1}\right) \times \left[-\tfrac{\pi}{\alpha_2}, \tfrac{\pi}{\alpha_2}\right).$$

We denote the basic period domain by Ω_{per}:

$$\Omega_{per} = \mathcal{Q} \times (0,1).$$

We denote by C^∞_{per} the space of restrictions of functions in $C^\infty(\overline{\Omega})$ which are $\mathcal{Q}$-periodic in $x' = (x_1, x_2)$. We also denote by $C^\infty_{0,per}$ the space of restrictions of functions in C^∞ which are $\mathcal{Q}$-periodic in $x' = (x_1, x_2)$ and vanish near $x_3 = 0, 1$. For $1 \le r \le \infty$ we denote by $L^r(\Omega_{per})$ the usual Lebesgue space over Ω_{per}, and its norm is denoted by $\|\cdot\|_r$. The k th order L^2 Sobolev space over Ω_{per} is denoted by $H^k(\Omega_{per})$, and its norm is denoted by $\|\cdot\|_{H^k}$.

We set

$$L^2_{per} = \text{the } L^2(\Omega_{per})\text{-closure of } C^\infty_{0,per},$$

$$H^k_{per} = \text{the } H^k(\Omega_{per})\text{-closure of } C^\infty_{per},$$

$$H^1_{0,per} = \text{the } H^1(\Omega_{per})\text{-closure of } C^\infty_{0,per}.$$

We note that if $f \in H^1_{0,per}$, then $f|_{x_j=-\pi/\alpha_j} = f|_{x_j=\pi/\alpha_j}$ and $f|_{x_3=0,1} = 0$. The inner product of $f_j \in L^2_{per}$ $(j = 1, 2)$ is denoted by

$$(f_1, f_2) = \int_{\Omega_{per}} f_1(x)\overline{f_2(x)}\, dx,$$

where $\overline{z}$ denotes the complex conjugate of z.

The mean value of a function $\phi(x)$ over Ω_{per} is denoted by $\langle \phi \rangle$:

$$\langle \phi \rangle = \frac{1}{|\Omega_{per}|} \int_{\Omega_{per}} \phi(x)\, dx.$$

The set of all $\phi \in L^2_{per}$ with $\langle \phi \rangle = 0$ is denoted by $L^2_{per,*}$, i.e.,

$$L^2_{per,*} = \{\phi \in L^2_{per} : \langle \phi \rangle = 0\}.$$

Furthermore, we set

$$H^k_{per,*} = H^k_{per} \cap L^2_{per,*}.$$

We denote by $C^\infty_{0,per,\sigma}$ the set of all vector fields $\boldsymbol{v}$ in $(C^\infty_{0,per})^3$ with $\operatorname{div} \boldsymbol{v} = 0$. We set

$$L^2_{per,\sigma} = \text{the } L^2(\Omega_{per})^3\text{-closure of } C^\infty_{0,per,\sigma}.$$

It is known that $(L^2_{per})^3 = L^2_{per,\sigma} \oplus G^2_{per}$, where $G^2_{per} = \{\nabla p;\, p \in H^1_{per}\}$ is the orthogonal complement of $L^2_{per,\sigma}$. The orthogonal projection $\mathbb{P}$ on $L^2_{per,\sigma}$ is called the Helmholtz projection. We define the projection $\boldsymbol{P}$ from $(L^2_{per})^3 \times L^2_{per}$ onto $L^2_{per,\sigma} \times L^2_{per}$ by

$$\boldsymbol{P} = \begin{pmatrix} \mathbb{P} & 0 \\ 0 & I \end{pmatrix}.$$

For simplicity the set of all vector fields in $(L^2_{per})^3$ (resp. $(H^1_{0,per})^3$, $(H^k_{per})^3$) are frequently denoted by L^2_{per} (resp. $H^1_{0,per}$, H^k_{per}) if no confusion will occur.

We also use notation L^2_{per} for the set of all $u = {}^\top(p, \boldsymbol{w}, \theta)$ with $p \in L^2_{per}$, $\boldsymbol{w} = {}^\top(w^1, w^2, w^3) \in L^2_{per}$ and $\theta \in L^2_{per}$ if no confusion will occur.

Let ε be a positive number. We introduce an inner product $\langle\langle u_1, u_2\rangle\rangle_\varepsilon$ for $u_j = {}^\top(p_j, \boldsymbol{w}_j, \theta_j)$ $(j = 1, 2)$ defined by

$$\langle\langle u_1, u_2\rangle\rangle_\varepsilon = \varepsilon^2 (p_1, p_2) + \mathrm{Pr}^{-1}(\boldsymbol{w}_1, \boldsymbol{w}_2) + (\theta_1, \theta_2).$$

We also define the inner product $\langle \boldsymbol{U}_1, \boldsymbol{U}_2\rangle$ for $\boldsymbol{U}_j = {}^\top(\boldsymbol{w}_j, \theta_j)$ $(j = 1, 2)$ by

$$\langle \boldsymbol{U}_1, \boldsymbol{U}_2\rangle = \mathrm{Pr}^{-1}(\boldsymbol{w}_1, \boldsymbol{w}_2) + (\theta_1, \theta_2).$$

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. The kernel and the range of A are denoted by $\mathrm{Ker}\,(A)$ and $R(A)$, respectively.

3 Main Results

In this section we state the main results of this paper. Let $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ be a stationary solution of (1.1)–(1.3) under the boundary conditions (1.7) and (1.8) satisfying $\int_{\Omega_{per}} p_s(x)\,dx = 0$. We consider the linearized problem for the Oberbeck-Boussinesq system (1.1)–(1.3) around $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$:

$$\mathrm{div}\, \boldsymbol{w} = 0, \tag{3.1}$$

$$\mathrm{Pr}^{-1}\partial_t \boldsymbol{w} - \Delta \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 = \boldsymbol{0}, \tag{3.2}$$

$$\partial_t \theta - \Delta \theta + \boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{w} \cdot \nabla \theta_s - \mathrm{Ra}\boldsymbol{w} \cdot \boldsymbol{e}_3 = 0 \tag{3.3}$$

under the boundary condition

$$\boldsymbol{w}|_{x_3=0,1} = \boldsymbol{0}, \quad \theta|_{x_3=0,1} = 0, \tag{3.4}$$

and the periodicity condition

$$p,\ \boldsymbol{w} \text{ and } \theta \text{ are } \mathcal{Q}\text{-periodic in } (x_1, x_2). \tag{3.5}$$

The linearized problem for the artificial compressible system is written as

$$\varepsilon^2 \partial_t p + \mathrm{div}\, \boldsymbol{w} = 0, \tag{3.6}$$

$$\mathrm{Pr}^{-1}\partial_t \boldsymbol{w} - \Delta \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 = \boldsymbol{0}, \tag{3.7}$$

$$\partial_t \theta - \Delta \theta + \boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{w} \cdot \nabla \theta_s - \mathrm{Ra}\boldsymbol{w} \cdot \boldsymbol{e}_3 = 0 \tag{3.8}$$

with the boundary conditions (3.4) and (3.5).

By applying the projection $\boldsymbol{P}$, problem (3.1)–(3.5) is written as

$$\begin{aligned} \mathrm{Pr}^{-1}\partial_t \boldsymbol{w} - \mathbb{P}\Delta \boldsymbol{w} + \mathrm{Pr}^{-1}\mathbb{P}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) - \mathrm{Ra}\mathbb{P}\theta \boldsymbol{e}_3 &= \boldsymbol{0}, \quad (3.9) \\ \partial_t \theta - \Delta\theta + \boldsymbol{v}_s \cdot \nabla\theta + \boldsymbol{w} \cdot \nabla\theta_s - \mathrm{Ra}\boldsymbol{w} \cdot \boldsymbol{e}_3 &= 0 \quad (3.10) \end{aligned}$$

with the boundary conditions (3.4) and (3.5). We introduce the linearized operator around $\boldsymbol{U}_s = {}^\top(\boldsymbol{v}_s, \theta_s)$ associated with problem (3.9)–(3.10) under (3.4) and (3.5). We define the operator $L : L^2_{per,\sigma} \times L^2_{per} \to L^2_{per,\sigma} \times L^2_{per}$ by

$$L = \begin{pmatrix} -\mathrm{Pr}\mathbb{P}\Delta + \mathbb{P}(\boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s)) & -\mathrm{Pr}\mathrm{Ra}\mathbb{P}\boldsymbol{e}_3 \\ {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix}$$

with domain $D(L) = [(H^2_{per} \cap H^1_{0,per})^3 \cap L^2_{per,\sigma}] \times [H^2_{per} \cap H^1_{0,per}]$.

We also introduce the linearized operator around $u_s = {}^\top(p_s, \boldsymbol{w}_s, \theta_s)$ associated with (3.6)–(3.8) under (3.4) and (3.5). We define the operator $L_\varepsilon : H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per} \to H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}$ by

$$L_\varepsilon = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ \mathrm{Pr}\nabla & -\mathrm{Pr}\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) & -\mathrm{Pr}\mathrm{Ra}\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix}$$

with domain $D(L_\varepsilon) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}]$.

Before going further we make one observation. Due to the translation invariance in x_1 and x_2 variables, 0 is an eigenvalue of $-L_\varepsilon$ whenever $\partial_{x_1} u_s \neq 0$ or $\partial_{x_2} u_s \neq 0$. In this case nonzero $\partial_{x_j} u_s$ are eigenfunctions for the eigenvalue 0. Correspondingly, 0 is an eigenvalue of $-L$ whenever $\partial_{x_1}\boldsymbol{U}_s \neq 0$ or $\partial_{x_2}\boldsymbol{U}_s \neq 0$. For simplicity we consider the case $\partial_{x_j} u_s \neq 0$ (and hence $\partial_{x_j}\boldsymbol{U}_s \neq 0$) for $j = 1, 2$.

Theorem 3.1. *Let $\partial_{x_j} u_s \neq 0$ for $j = 1, 2$. If there exists a positive number b_0 such that $\rho(-L_{\varepsilon_n}) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0\} \setminus \{0\}$ for some sequence $\varepsilon_n \to 0$ as $n \to \infty$ and 0 is a semisimple eigenvalue of $-L_{\varepsilon_n}$ with $\mathrm{Ker}\,(-L_{\varepsilon_n}) = \mathrm{span}\,\{\partial_{x_1} u_s, \partial_{x_2} u_s\}$, then there exists a constant $b_1 > 0$ such that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_1\} \setminus \{0\}$ and 0 is a semisimple eigenvalue of $-L$ with $\mathrm{Ker}\,(-L) = \mathrm{span}\,\{\partial_{x_1}\boldsymbol{U}_s, \partial_{x_2}\boldsymbol{U}_s\}$.*

Theorem 3.1 shows that if u_s is obtained by Chorin's method with $0 < \varepsilon \ll 1$, then it is stable as a solution of the Oberbeck-Boussinesq system. In particular, we have the following instability result.

Theorem 3.2. *Let $\partial_{x_j} u_s \neq 0$ for $j = 1, 2$. If $\sigma(-L) \cap \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda > 0\} \neq \emptyset$, then $\sigma(-L_\varepsilon) \cap \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda > 0\} \neq \emptyset$ for sufficiently small ε.*

By Theorem 3.2, we see that unstable stationary solutions of the Oberbeck-Boussinesq system cannot be obtained by Chorin's method with $0 < \varepsilon \ll 1$.

We next give a sufficient condition for u_s to be computed by Chorin's method with $0 < \varepsilon \ll 1$.

Theorem 3.3. *Let $\partial_{x_j}\boldsymbol{U}_s \neq \boldsymbol{0}$ for $j = 1, 2$. Suppose that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0\} \setminus \{0\}$ for some constant $b_0 > 0$ and 0 is a semisimple eigenvalue of $-L$ with* $\mathrm{Ker}\,(-L) = \mathrm{span}\,\{\partial_{x_1}\boldsymbol{U}_s, \partial_{x_2}\boldsymbol{U}_s\}$. *Then there exist constants $\varepsilon_0 > 0$, $\delta_0 > 0$ and $b_1 > 0$ such that if*

$$\inf_{\boldsymbol{w}\in H^1_{per,0}, \boldsymbol{w}\neq\boldsymbol{0}} \frac{\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w})}{\|\nabla\boldsymbol{w}\|_2^2} \geq -\delta_0, \tag{3.11}$$

then $\rho(-L_\varepsilon) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_1\} \setminus \{0\}$ for all $0 < \varepsilon \leq \varepsilon_0$ and 0 is a semisimple eigenvalue of $-L_\varepsilon$ with $\mathrm{Ker}\,(-L_\varepsilon) = \mathrm{span}\,\{\partial_{x_1}u_s, \partial_{x_2}u_s\}$.

In Theorem 3.3 we require smallness condition for the velocity field $\boldsymbol{v}_s$ only but not for the temperature θ_s.

Remark 3.4. *It is known that $-L$ is sectorial. Furthermore, $-L_\varepsilon$ is also sectorial for each $\varepsilon > 0$. (See Proposition 6.1 below.) Consequently, under the assumptions of Theorems 3.1 and 3.3, $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ is asymptotically (orbitally) stable as a solution of both problems. More precisely, a solution of the nonlinear problem near u_s converges to a stationary solution $u_s(\cdot + a', \cdot)$ for some $a' \in \mathbb{R}^2$ as $t \to \infty$.*

Remark 3.5. *Theorems 3.1 and 3.3 also hold with slight modifications when $\partial_{x_j}u_s = 0$ (and hence $\partial_{x_j}\boldsymbol{U}_s = 0$) for both $j = 1, 2$ or for one of $j = 1, 2$. For example, if $\partial_{x_1}\boldsymbol{U}_s \neq \boldsymbol{0}$ and $\partial_{x_2}\boldsymbol{U}_s = \boldsymbol{0}$, then Theorem 3.3 holds with* $\mathrm{Ker}\,(-L) = \mathrm{span}\,\{\partial_{x_1}\boldsymbol{U}_s, \partial_{x_2}\boldsymbol{U}_s\}$ *and* $\mathrm{Ker}\,(-L_\varepsilon) = \mathrm{span}\,\{\partial_{x_1}u_s, \partial_{x_2}u_s\}$ *replaced by* $\mathrm{Ker}\,(-L) = \mathrm{span}\,\{\partial_{x_1}\boldsymbol{U}_s\}$ *and* $\mathrm{Ker}\,(-L_\varepsilon) = \mathrm{span}\,\{\partial_{x_1}u_s\}$, *respectively. Likewise, the theorems hold for other cases with similar modifications.*

Since the velocity fields of cellular stationary convective patterns bifurcating from the heat conductive state are small when $\mathrm{Ra} \sim \mathrm{Ra}_c$, we apply Theorem 3.3 and Remark 3.5 to obtain the following corollary.

Corollary 3.6. *If u_s is a stable convective pattern of* (1.1)–(1.3) *bifurcating from the heat conductive state, then it is also stable as a solution of* (1.4)–(1.6) *for $0 < \varepsilon \ll 1$ when* $\mathrm{Ra} \sim \mathrm{Ra}_c$.

To prove Theorems 3.1–3.3, we will first investigate the null spaces of L and L_ε in section 4. Theorems 3.1 and 3.2 will be proved in section 5. The proof of Theorem 3.3 will be given in sections 6 and 7.

4 The null spaces of L and L_ε

In this section we investigate the relation of the null spaces of L and L_ε. We introduce the adjoint operator $L^* : L^2_{per,\sigma} \times L^2_{per} \to L^2_{per,\sigma} \times L^2_{per}$ of L:

$$L^* = \begin{pmatrix} -\mathrm{Pr}\mathbb{P}\Delta + \mathbb{P}(-\boldsymbol{v}_s\cdot\nabla + (\nabla\boldsymbol{v}_s)) & \mathrm{Pr}\mathbb{P}((\nabla\theta_s) - \mathrm{Ra}\boldsymbol{e}_3) \\ -\mathrm{Ra}{}^\top\boldsymbol{e}_3 & -\Delta - \boldsymbol{v}_s\cdot\nabla \end{pmatrix}$$

with domain $D(L^*) = [(H^2_{per} \cap H^1_{0,per})^3 \cap L^2_{per,\sigma}] \times [H^2_{per} \cap H^1_{0,per}]$.

In what follows we assume that 0 is a semisimple eigenvalue of $-L$ with $\mathrm{Ker}\,(-L) = \mathrm{span}\,\{\partial_{x_1}\boldsymbol{U}_s, \partial_{x_2}\boldsymbol{U}_s\}$.

Since $-L$ is a sectorial operator with compact resolvent and 0 is a semisimple eigenvalue of $-L$, we have the following resolvent estimate for $-L$.

Proposition 4.1. *Set $\boldsymbol{U}_j^{(0)} = \partial_{x_j}\boldsymbol{U}_s$ for $j = 1,2$. Then the following assertions hold.*

(i) *There exist $\boldsymbol{U}_j^* = {}^\top(\boldsymbol{w}_j^*, \theta_j^*) \in D(L^*)$ such that $L^*\boldsymbol{U}_j^* = \boldsymbol{0}$ and $\langle \boldsymbol{U}_j^{(0)}, \boldsymbol{U}_j^* \rangle = \delta_{jk}$ for $j,k = 1,2$. Furthermore,*

$$L^2_{per,\sigma} \times L^2_{per} = X_0 \oplus X_1,$$

where $X_0 = \mathrm{Ker}\,(-L)$ and

$$X_1 = R(-L) = \{\boldsymbol{U} \in L^2_{per,\sigma} \times L^2_{per}; \langle \boldsymbol{U}, \boldsymbol{U}_j^* \rangle = 0, j = 1,2\}.$$

The eigenprojection $\boldsymbol{\Pi}_0$ for the eigenvalue 0 of $-L$ is given by

$$\boldsymbol{\Pi}_0\boldsymbol{U} = \langle \boldsymbol{U}, \boldsymbol{U}_1^* \rangle \boldsymbol{U}_1^{(0)} + \langle \boldsymbol{U}, \boldsymbol{U}_2^* \rangle \boldsymbol{U}_2^{(0)}.$$

(ii) *Set $\boldsymbol{\Pi}_0^c = I - \boldsymbol{\Pi}_0$. There exist constants $a_0 > 0$ and $c_0 \in \mathbb{R}$ such that*

$$\Sigma \setminus \{0\} \subset \rho(-L),$$

where

$$\Sigma := \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -a_0|\mathrm{Im}\,\lambda|^2 + c_0\},$$

and the estimates

$$\|(\lambda + L)^{-1}\boldsymbol{F}\|_2 \leq C\left\{\frac{1}{|\lambda|}\|\boldsymbol{\Pi}_0\boldsymbol{F}\|_2 + \frac{1}{|\lambda|+1}\|\boldsymbol{\Pi}_0^c\boldsymbol{F}\|_2\right\},$$

$$\|\partial_x^2(\lambda + L)^{-1}\boldsymbol{F}\|_2 \leq C\left\{\frac{1}{|\lambda|}\|\boldsymbol{\Pi}_0\boldsymbol{F}\|_2 + \|\boldsymbol{\Pi}_0^c\boldsymbol{F}\|_2\right\}$$

hold uniformly for $\lambda \in \Sigma \setminus \{0\}$. Furthermore, if $\boldsymbol{\Pi}_0\boldsymbol{F} = 0$, then $\boldsymbol{\Pi}_0(\lambda + L)^{-1}\boldsymbol{F} = 0$ and the above estimates hold for any $\lambda \in \Sigma$.

We next introduce an operator $\mathscr{L}_{\varepsilon,\lambda} : H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per} \to H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}$ defined by

$$D(\mathscr{L}_{\varepsilon,\lambda}) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}],$$

$$\mathscr{L}_{\varepsilon,\lambda} = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ \mathrm{Pr}\nabla & \lambda - \mathrm{Pr}\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) & -\mathrm{Pr}\mathrm{Ra}\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & \lambda - \Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix},$$

and its adjoint $\mathscr{L}^*_{\varepsilon,\lambda} : H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per} \to H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}$ by

$$D(\mathscr{L}^*_{\varepsilon,\lambda}) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}],$$

$$\mathscr{L}^*_{\varepsilon,\lambda} = \begin{pmatrix} 0 & -\frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ -\mathrm{Pr}\nabla & \overline{\lambda} - \mathrm{Pr}\Delta - \boldsymbol{v}_s \cdot \nabla + (\nabla \boldsymbol{v}_s) & \mathrm{Pr}(\nabla\theta_s) - \mathrm{PrRa}\boldsymbol{e}_3 \\ 0 & -\mathrm{Ra}^\top \boldsymbol{e}_3 & \overline{\lambda} - \Delta - \boldsymbol{v}_s \cdot \nabla \end{pmatrix}.$$

Note that

$$\mathscr{L}_{\varepsilon,0} = L_\varepsilon.$$

We also introduce the operators $\boldsymbol{A}_\lambda$ and $\mathscr{A}_\lambda$ by

$$\boldsymbol{A}_\lambda \boldsymbol{w} = \begin{pmatrix} \lambda - \mathrm{Pr}\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) \\ {}^\top(\nabla\theta_s) - \mathrm{Ra}^\top \boldsymbol{e}_3 \end{pmatrix} \boldsymbol{w},$$

and

$$\mathscr{A}_\lambda \boldsymbol{w} = \begin{pmatrix} 0 \\ \lambda - \mathrm{Pr}\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) \\ {}^\top(\nabla\theta_s) - \mathrm{Ra}^\top \boldsymbol{e}_3 \end{pmatrix} \boldsymbol{w}.$$

We first prepare the following lemma.

Lemma 4.2. *There exists a bounded linear operator $\boldsymbol{V} : H^1_{per,*} \to [H^2_{per} \cap H^1_{0,per}]^3$ such that*

$$\mathrm{div}\, \boldsymbol{V} f = f, \quad \|\boldsymbol{V} f\|_{H^2} \leq C\|f\|_{H^1}$$

for $f \in H^1_{per,*}$.

This lemma follows from the solvability for the nonhomogeneous Stokes problem $\mathrm{div}\, \boldsymbol{v} = f$, $-\Delta \boldsymbol{v} + \nabla p = 0$ under the boundary conditions $v|_{x_3=0,1} = \boldsymbol{0}$, and p, $\boldsymbol{v}$ being $\mathcal{Q}$-periodic in x'. See, e.g., [5].

In what follows we set

$$Y := H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}.$$

Proposition 4.3. *Let $\varepsilon > 0$.*

(i) *Let $u^{(0)}_j = \partial_{x_j} u_s$ $(j = 1, 2)$. Then*

$$\mathrm{Ker}\,(\mathscr{L}_{\varepsilon,0}) = \mathrm{Ker}\,(L_\varepsilon) = \mathrm{span}\,\{u^{(0)}_1, u^{(0)}_2\}.$$

(ii) *For each $j = 1, 2$, there exists a unique $p^*_j \in H^1_{per,*}$ such that $\mathscr{L}^*_{\varepsilon,0} u^*_j = 0$, $u^*_j = {}^\top(p^*_j, \boldsymbol{w}^*_j, \theta^*_j)$, where ${}^\top(\boldsymbol{w}^*_j, \theta^*_j)$ $(j = 1, 2)$ are the functions given in Proposition 4.1 (i). Furthermore,*

$$R(\mathscr{L}_{\varepsilon,0}) = R(L_\varepsilon) = \{u \in Y; \langle\langle u, u^*_j \rangle\rangle_\varepsilon = 0, j = 1, 2\}.$$

(iii) *There exists a positive number ε_1 such that if $|\varepsilon| \leq \varepsilon_1$, then*

$$Y = \mathrm{Ker}\,(L_\varepsilon) \oplus R(L_\varepsilon).$$

Therefore, 0 is a semisimple eigenvalue of L_ε. Furthermore, if $F = {}^\top(f, \boldsymbol{g}, h) \in R(L_\varepsilon)$, then there exists a unique solution $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon) \cap R(L_\varepsilon)$ of $L_\varepsilon u = F$ and u satisfies

$$\|u\|_{H^1 \times H^2 \times H^2} \leq C\{\varepsilon^2 \|f\|_{H^1} + \|\boldsymbol{F}\|_2\},$$

where $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$.

Proof. Let $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(\mathscr{L}_{\varepsilon,0})$ satisfy $\mathscr{L}_{\varepsilon,0}u = 0$. Then $\boldsymbol{U} = {}^\top(\boldsymbol{w}, \theta)$ satisfies $\boldsymbol{U} \in D(L)$ and $L\boldsymbol{U} = \boldsymbol{0}$. By assumption, there exist $a_j \in \mathbb{C}$ such that $\boldsymbol{U} = a_1\partial_{x_1}\boldsymbol{U}_s + a_2\partial_{x_2}\boldsymbol{U}_s$. On the other hand, $\mathscr{L}_{\varepsilon,0}(a_1\partial_{x_1}u_s + a_2\partial_{x_2}u_s) = 0$. We thus obtain

$$\nabla p = \nabla(a_1\partial_{x_1}p_s + a_2\partial_{x_2}p_s).$$

Since p and $a_1\partial_{x_1}p_s + a_2\partial_{x_2}p_s$ are in $H^1_{per,*}$, we have $p = a_1\partial_{x_1}p_s + a_2\partial_{x_2}p_s$, and hence, $u = a_1\partial_{x_1}u_s + a_2\partial_{x_2}u_s$. This shows $\mathrm{Ker}\,(\mathscr{L}_{\varepsilon,0}) = \mathrm{span}\,\{u_1^{(0)}, u_2^{(0)}\}$ and (i) is proved.

As for (ii), since $L^*\boldsymbol{U}_j^* = \boldsymbol{0}$, there exists a unique $p_j^* \in H^1_{per,*}$ such that $u_j^* = {}^\top(p_j^*, \boldsymbol{U}_j^*)$ satisfies $u_j \in D(\mathscr{L}^*_{\varepsilon,0})$ and $\mathscr{L}^*_{\varepsilon,0}u_j^* = 0$.

Let us prove $R(\mathscr{L}_{\varepsilon,0}) = \{u \in Y; \langle\langle u, u_j^*\rangle\rangle_\varepsilon = 0, j = 1, 2\}$. Set $\tilde{\boldsymbol{w}} = \boldsymbol{w} - \varepsilon^2\boldsymbol{V}f$. Then

$$\mathscr{L}_{\varepsilon,0}\tilde{u} = \tilde{F} - \varepsilon^2\mathscr{A}_0\boldsymbol{V}f, \tag{4.1}$$

where $\tilde{u} = {}^\top(p, \tilde{\boldsymbol{w}}, \theta)$ and $\tilde{F} = {}^\top(0, \boldsymbol{g}, h)$. Therefore, we have

$$L\tilde{\boldsymbol{U}} = \boldsymbol{P}[\boldsymbol{F} - \varepsilon^2\boldsymbol{A}_0\boldsymbol{V}f], \tag{4.2}$$

where $\tilde{\boldsymbol{U}} = {}^\top(\tilde{\boldsymbol{w}}, \theta)$ and $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$. Since $\langle\langle F, u_j^*\rangle\rangle_\varepsilon = 0$ for $j = 1, 2$, we have

$$\begin{aligned}
&\langle \boldsymbol{P}[\boldsymbol{F} - \varepsilon^2\boldsymbol{A}_0\boldsymbol{V}f], \boldsymbol{U}_j^*\rangle \\
&= \langle \boldsymbol{F}, \boldsymbol{U}_j^*\rangle - \varepsilon^2\mathrm{Pr}^{-1}(-\mathrm{Pr}\Delta\boldsymbol{V}f + \boldsymbol{v}_s\cdot\nabla\boldsymbol{V}f + \boldsymbol{V}f\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}_j^*) \\
&\quad -\varepsilon^2(\boldsymbol{V}f\cdot\nabla\theta_s - \mathrm{Ra}\boldsymbol{V}f\cdot\boldsymbol{e}_3, \theta_j^*) \\
&= \langle \boldsymbol{F}, \boldsymbol{U}_j^*\rangle \\
&\quad -\varepsilon^2\mathrm{Pr}^{-1}(\boldsymbol{V}f, -\mathrm{Pr}\Delta\boldsymbol{w}_j^* - \boldsymbol{v}_s\cdot\nabla\boldsymbol{w}_j^* + (\nabla\boldsymbol{v}_s)\boldsymbol{w}_j^* + \mathrm{Pr}(\nabla\theta_s)\theta_j^* - \mathrm{Pr}\mathrm{Ra}\theta_j^*\boldsymbol{e}_3) \\
&= \langle \boldsymbol{F}, \boldsymbol{U}_j^*\rangle - \varepsilon^2\mathrm{Pr}^{-1}(\boldsymbol{V}f, \mathrm{Pr}\nabla p_j^*) \\
&= \langle \boldsymbol{F}, \boldsymbol{U}_j^*\rangle + \varepsilon^2(\mathrm{div}\,\boldsymbol{V}f, p_j^*) \\
&= \langle\langle F, u_j^*\rangle\rangle_\varepsilon = 0.
\end{aligned}$$

Applying Proposition 4.1, we see that there exists a unique solution $\tilde{\boldsymbol{U}} = {}^\top(\tilde{\boldsymbol{w}}, \theta) \in D(L)\cap X_1$ of (4.2) with estimate

$$\|\tilde{\boldsymbol{U}}\|_{H^2} \le C\{\varepsilon^2\|f\|_{H^1} + \|\boldsymbol{F}\|_2\}.$$

It then follows that there exists a unique $p \in H^1_{per,*}$ such that $\tilde{u} = {}^\top(p, \tilde{\boldsymbol{w}}, \theta) \in D(\mathscr{L}_{\varepsilon,0})$ is a solution of (4.1). Setting $u = {}^\top(p, \tilde{\boldsymbol{w}} + \varepsilon^2\boldsymbol{V}f, \theta)$, we see that $u \in D(\mathscr{L}_{\varepsilon,0})$ and $\mathscr{L}_{\varepsilon,0}u = F$ with estimate

$$\|u\|_{H^1\times H^2\times H^2} \le C\{\varepsilon^2\|f\|_{H^1} + \|\boldsymbol{F}\|_2\}. \tag{4.3}$$

We thus conclude that $R(\mathscr{L}_{\varepsilon,0}) = \{u \in Y; \langle\langle u, u_j^*\rangle\rangle_\varepsilon = 0, j = 1, 2\}$. Note that $R(L_\varepsilon)$ is closed.

As for (iii), we first show that there exists $u_{j,\varepsilon}^{(0)} \in D(L_\varepsilon)$ $(j = 1, 2)$ such that

$$\mathrm{Ker}\,(L_\varepsilon) = \mathrm{span}\,\{u_{1,\varepsilon}^{(0)}, u_{2,\varepsilon}^{(0)}\}, \quad \langle\langle u_{j,\varepsilon}^{(0)}, u_k^*\rangle\rangle_\varepsilon = \delta_{jk} \ \ (j, k = 1, 2).$$

We look for $u_{j,\varepsilon}^{(0)}$ in the form $u_{j,\varepsilon}^{(0)} = a_{1j}u_1^{(0)} + a_{2j}u_2^{(0)}$ $(j = 1, 2)$. By the conditions $\langle\langle u_{j,\varepsilon}^{(0)}, u_k^*\rangle\rangle = \delta_{jk}$, we have

$$B_\varepsilon \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.4}$$

where

$$B_\varepsilon = \begin{pmatrix} \langle\langle u_1^{(0)}, u_1^*\rangle\rangle_\varepsilon & \langle\langle u_2^{(0)}, u_1^*\rangle\rangle_\varepsilon \\ \langle\langle u_1^{(0)}, u_2^*\rangle\rangle_\varepsilon & \langle\langle u_2^{(0)}, u_2^*\rangle\rangle_\varepsilon \end{pmatrix}.$$

Since $\langle \boldsymbol{U}_j^{(0)}, \boldsymbol{U}_k^*\rangle = \delta_{jk}$, we see that

$$B_\varepsilon = \begin{pmatrix} 1 + \varepsilon^2(p_1^{(0)}, p_1^*) & \varepsilon^2(p_2^{(0)}, p_1^*) \\ \varepsilon^2(p_1^{(0)}, p_2^*) & 1 + \varepsilon^2(p_2^{(0)}, p_2^*) \end{pmatrix}.$$

Therefore,

$$B_\varepsilon \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as $\varepsilon \to 0$, and hence, B_ε has its inverse B_ε^{-1} for $0 < \varepsilon \ll 1$. We thus obtain the desired $u_{j,\varepsilon}^{(0)}$ $(j = 1, 2)$.

Set

$$P_{0,\varepsilon}u = \sum_{j=1}^{2} \langle\langle u, u_j^*\rangle\rangle_\varepsilon u_{j,\varepsilon}^{(0)}.$$

Then $P_{0,\varepsilon}$ is the projection on $\mathrm{Ker}\,(L_\varepsilon)$, i.e., the eigenprojection for the eigenvalue 0 of $-L_\varepsilon$. It is easy to see that $R(I - P_{0,\varepsilon}) = \{u \in Y; \langle\langle u, u_j^*\rangle\rangle_\varepsilon = 0, j = 1, 2\}$. This, together with (ii), implies that $R(L_\varepsilon) = R(I - P_{0,\varepsilon})$, and hence, $Y = \mathrm{Ker}\,(L_\varepsilon) \oplus R(L_\varepsilon)$. This decomposition and (ii), together with (4.3), yield the unique existence of solution $u \in D(L_\varepsilon) \cap R(L_\varepsilon)$ of $L_\varepsilon u = F$ for $F \in R(L_\varepsilon)$ with the desired estimate. This completes the proof □

5 Proof of Theorems 3.1 and 3.2

In this section we prove Theorems 3.1 and 3.2. We introduce some operators. We define $\mathbb{L}_\lambda : (H^2_{per})^3 \times L^2_{per} \to (L^2_{per})^3$ by

$$\mathbb{L}_\lambda \boldsymbol{U} = \lambda \boldsymbol{w} - \mathrm{Pr}\Delta \boldsymbol{w} + \boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s - \mathrm{PrRa}\theta \boldsymbol{e}_3$$

for $\boldsymbol{U} = {}^\top(\boldsymbol{w}, \theta) \in (H^2_{per})^3 \times L^2_{per}$.

Let $\lambda \in \rho(-L)$. For any $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h) \in L^2_{per,\sigma} \times L^2_{per}$, there exists a unique $p \in H^1_{per,*}$ satisfying

$$\nabla p = \mathbb{I}\boldsymbol{F} - \mathbb{L}_\lambda(\lambda + L)^{-1}\boldsymbol{F}, \tag{5.1}$$

and

$$\|p\|_{H^1} \le C\{|\lambda| C_{0,\lambda} + C_{2,\lambda} + 1\} \|\boldsymbol{F}\|_2,$$

where $\mathbb{I}$ is the $n \times (n+1)$ matrix given by $\mathbb{I} = (I, \boldsymbol{0})$ with I being the $n \times n$ identity matrix, i.e., $\mathbb{I}\boldsymbol{F} = \boldsymbol{g}$, and

$$C_{k,\lambda} = \sup \left\{ \frac{\|(\lambda + L)^{-1}\boldsymbol{F}\|_{H^k \times H^k}}{\|\boldsymbol{F}\|_2}; \, \boldsymbol{F} \in L^2_{per,\sigma} \times L^2_{per}, \, \boldsymbol{F} \neq \boldsymbol{0} \right\} \quad (k = 0, 2).$$

We denote this p by $p_\lambda[\boldsymbol{F}]$. The correspondence $\boldsymbol{F} \in L^2_{per,\sigma} \times L^2_{per} \mapsto p_\lambda[\boldsymbol{F}] \in H^1_{per,*}$ is linear, and hence, defines a bounded linear operator.

We first give an expression of $\mathscr{L}^{-1}_{\varepsilon,\lambda}$ in terms of $(\lambda + L)^{-1}$.

Proposition 5.1. *Let $\lambda \in \rho(-L)$. Then*

$$\mathscr{L}^{-1}_{\varepsilon,\lambda} F = \begin{pmatrix} p_\lambda[\boldsymbol{P}(\boldsymbol{F} - \varepsilon^2 \boldsymbol{A}_\lambda \boldsymbol{V} f)] \\ (\lambda + L)^{-1}\boldsymbol{P}(\boldsymbol{F} - \varepsilon^2 \boldsymbol{A}_\lambda \boldsymbol{V} f) + \varepsilon^2 \boldsymbol{V} f \end{pmatrix}$$

for $F = {}^\top(f, \boldsymbol{g}, h) \in H^1_{per,*} \times L^2_{per} \times L^2_{per}$.

Proof. For a given $F = {}^\top(f, \boldsymbol{g}, h) \in H^1_{per,*} \times L^2_{per} \times L^2_{per}$ we consider the problem $\mathscr{L}_{\varepsilon,\lambda} u = F$. As in the proof of Proposition 4.3, we set $\tilde{u} = u - {}^\top(0, \varepsilon^2 \boldsymbol{V} f, 0)$. Then we have $\mathscr{L}_{\varepsilon,\lambda}\tilde{u} = \tilde{F} - \varepsilon^2 \mathscr{A}_\lambda \boldsymbol{V} f$ with $\tilde{F} = {}^\top(0, \boldsymbol{g}, h)$. We thus arrive at

$$(\lambda + L)\tilde{\boldsymbol{U}} = \boldsymbol{P}(\boldsymbol{F} - \varepsilon^2 \boldsymbol{A}_\lambda \boldsymbol{V} f),$$

where $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$. Since $\lambda \in \rho(-L)$ we have

$${}^\top(\tilde{\boldsymbol{w}}, \theta) := \tilde{\boldsymbol{U}} = (\lambda + L)^{-1}\boldsymbol{P}(\boldsymbol{F} - \varepsilon^2 \boldsymbol{A}_\lambda \boldsymbol{V} f).$$

It then follows that $\tilde{u} = {}^\top(p, \tilde{\boldsymbol{w}}, \theta)$ with

$$p = p_\lambda[\boldsymbol{P}(\boldsymbol{F} - \varepsilon^2 \boldsymbol{A}_\lambda \boldsymbol{V} f)]$$

satisfies $\mathscr{L}_{\varepsilon,\lambda}\tilde{u} = \tilde{F} - \varepsilon^2 \mathscr{A}_\lambda \boldsymbol{V} f$. Setting $u = {}^\top(p, \boldsymbol{w}, \theta)$ with $\boldsymbol{w} = \tilde{\boldsymbol{w}} + \varepsilon^2 \boldsymbol{V} f$ we have the desired expression. This completes the proof. $\square$

We next give a perturbation result of eigenvalues of $-L$.

Theorem 5.2. *Let $\lambda_0 \in \sigma(-L)$. Then λ_0 is an eigenvalue of $-L$ with finite multiplicity and the following assertions hold.*

(i) *There exist $C > 0$, $m \in \mathbb{N}$ and $r_1 > 0$ such that*

$$\{\lambda; 0 < |\lambda - \lambda_0| \le r_1\} \subset \rho(-L)$$

and

$$\|(\lambda + L)^{-1}\boldsymbol{g}\|_{H^2} \le \frac{C}{|\lambda - \lambda_0|^m} \|\boldsymbol{g}\|_2$$

for λ with $0 < |\lambda - \lambda_0| \le r_1$.

(ii) *For any* $r_0 \in (0, r_1]$*, there exists a positive number* $\varepsilon_0 = \varepsilon_0(r_0, r_1, m, |\lambda_0|)$ *such that* $\sigma(-L_\varepsilon) \cap \{\lambda; |\lambda - \lambda_0| < r_0\} \neq \emptyset$ *for* $0 < \varepsilon \leq \varepsilon_0$*, and if* $|\lambda - \lambda_0| = r_0$*, then* $\lambda \in \rho(-L_\varepsilon)$ *for* $0 < \varepsilon \leq \varepsilon_0$ *and the total projection* P_ε*, the sum of eigenprojections for eigenvalues of* $-L_\varepsilon$ *lying inside the circle* $|\lambda - \lambda_0| = r_0$*, satisfies*

$$P_\varepsilon F = \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=r_0} (\lambda + L_\varepsilon)^{-1} F \, d\lambda = P_0 F + O(\varepsilon^2) F$$

in $H^1_{per,*} \times L^2 \times L^2$ *as* $\varepsilon \to 0$*. Here* P_0 *is the projection defined by*

$$P_0 F = \begin{pmatrix} \mathit{\Pi}_{\lambda_0} \boldsymbol{P}\boldsymbol{F} \\ \mathbf{\Pi}_{\lambda_0} \boldsymbol{P}\boldsymbol{F} \end{pmatrix}$$

for $F = {}^\top(f, \boldsymbol{g}, h) \in H^1_{per,*} \times L^2 \times L^2$*, where* $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$*,* $\mathbf{\Pi}_{\lambda_0}$ *is the eigenprojection for the eigenvalue* λ_0 *of* $-L$ *given by*

$$\mathbf{\Pi}_{\lambda_0} \boldsymbol{P}\boldsymbol{F} = \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=r_0} (\lambda + L)^{-1} \boldsymbol{P}\boldsymbol{F} \, d\lambda$$

and

$$\mathit{\Pi}_{\lambda_0} \boldsymbol{P}\boldsymbol{F} = -\frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=r_0} p_\lambda[\boldsymbol{P}\boldsymbol{F}] \, d\lambda,$$

i.e., $\mathit{\Pi}_{\lambda_0} \boldsymbol{F} \in H^1_{per,*}$ *is the function satisfying*

$$\nabla \mathit{\Pi}_{\lambda_0} \boldsymbol{P}\boldsymbol{F} = -\frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=r_0} (\mathbb{I} - \mathbb{L}_\lambda (\lambda + L)^{-1}) \boldsymbol{P}\boldsymbol{F} \, d\lambda.$$

(iii) *It holds that* $\dim R(P_\varepsilon) = \dim R(P_0) = \dim R(\mathbf{\Pi}_{\lambda_0})$ *for* $0 < \varepsilon \leq \varepsilon_0$.

Remark 5.3. *Let* $\tilde{m}$ *be the algebraic multiplicity of* λ_0*. Then one can take a basis of* $R(\mathbf{\Pi}_{\lambda_0})$ *consisting of functions* $\boldsymbol{U}_j, (\lambda_0 + L)\boldsymbol{U}_j, \cdots, (\lambda_0 + L)^{m_j - 1}\boldsymbol{U}_j$*,* $j = 1, \cdots, r$*, with* $(\lambda_0 + L)^{m_j} \boldsymbol{U}_j = \boldsymbol{0}$ *and* $m_1 + \cdots + m_r = \tilde{m}$*. We denote by* $p_{j\ell} \in H^1_{per,*}$ *the pressure satisfying*

$$\mathbb{L}_{\lambda_0} (\lambda_0 + L)^{m_j - \ell - 1} \boldsymbol{U}_j = \mathbb{I}(\lambda_0 + L)^{m_j - \ell} \boldsymbol{U}_j - \nabla p_{j\ell}$$

for $1 \leq j \leq r$*,* $0 \leq \ell \leq m_j - 1$*. It then follows that a set of functions* ${}^\top(p_{j\ell}, (\lambda_0 + L)^{m_j - \ell} \boldsymbol{U}_j)$*,* $1 \leq j \leq r$*,* $0 \leq \ell \leq m_j - 1$*, forms a basis of* $R(P_0)$*.*

Proof. Since $-L$ has compact resolvent, its spectrum consists of discrete eigenvalues with finite multiplicities. Let the algebraic multiplicity of λ_0 be $\tilde{m}$. Then we see that there exist a $m \in \mathbb{N}$, $1 \leq m \leq \tilde{m}$, and a constant $r_1 > 0$ such that $\{\lambda; 0 < |\lambda - \lambda_0| \leq r_1\} \subset \rho(-L)$ and

$$\|(\lambda + L)^{-1} \boldsymbol{g}\|_{H^2} \leq \frac{C}{|\lambda - \lambda_0|^m} \|\boldsymbol{g}\|_2$$

for λ with $0 < |\lambda - \lambda_0| \le r_1$. This proves (i).

As for (ii), we set

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and write $\lambda + L_\varepsilon$ as

$$\lambda + L_\varepsilon = \mathscr{L}_{\varepsilon,\lambda} + \lambda J = \mathscr{L}_{\varepsilon,\lambda}(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J).$$

Since $JF = {}^\top(f, \mathbf{0}, 0)$ for $F = {}^\top(f, \boldsymbol{g}, h)$, we see from (i) and Proposition 5.1 that

$$\|\mathscr{L}_{\varepsilon,\lambda}^{-1} JF\|_{H^1 \times H^2 \times H^2} \le C\varepsilon^2 \frac{(|\lambda_0| + r_0^m + 1)^2}{r_0^m} \|f\|_{H^1}$$

for $|\lambda - \lambda_0| = r_0$. Therefore, if

$$\varepsilon^2 \le \frac{r_0^m}{4C(|\lambda_0| + r_0^m + 1)^2} \frac{1}{|\lambda_0| + r_0},$$

then

$$\|\lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} JF\|_{H^1 \times H^2 \times H^2} \le C\varepsilon^2 (|\lambda_0| + r_0) \frac{(|\lambda_0| + r_0^m + 1)^2}{r_0^m} \|f\|_{H^1} \le \frac{1}{2} \|F\|_{H^1 \times L^2 \times L^2}$$

for $|\lambda - \lambda_0| = r_0$. It then follows that $(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J)$ is boundedly invertible both on Y and $D(L_\varepsilon)$ with estimates

$$\|(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J)^{-1} F\|_{H^1 \times L^2 \times L^2} \le 2\|F\|_{H^1 \times L^2 \times L^2}$$

for $F \in Y$ and

$$\|(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J)^{-1} F\|_{H^1 \times H^2 \times H^2} \le 2\|F\|_{H^1 \times H^2 \times H^2}$$

for $F \in D(L_\varepsilon)$. We thus find that $\lambda + L_\varepsilon = \mathscr{L}_{\varepsilon,\lambda} + \lambda J$ has a bounded inverse $(\lambda + L_\varepsilon)^{-1} = (\mathscr{L}_{\varepsilon,\lambda} + \varepsilon^2 \lambda J)^{-1}$ on Y which satisfies

$$\begin{aligned}
(\lambda + L_\varepsilon)^{-1} &= (I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J)^{-1} \mathscr{L}_{\varepsilon,\lambda}^{-1} \\
&= \sum_{N=0}^{\infty} (-\lambda)^N (\mathscr{L}_{\varepsilon,\lambda}^{-1} J)^N \mathscr{L}_{\varepsilon,\lambda}^{-1} \\
&= \mathscr{L}_{\varepsilon,\lambda}^{-1} - \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J \sum_{N=0}^{\infty} (-\lambda)^N (\mathscr{L}_{\varepsilon,\lambda}^{-1} J)^N \mathscr{L}_{\varepsilon,\lambda}^{-1}
\end{aligned}$$

with estimates

$$\|(\lambda + L_\varepsilon)^{-1} F\|_{H^1 \times L^2 \times L^2} \le C\|F\|_{H^1 \times L^2 \times L^2}$$

and

$$\left\| \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J \sum_{N=0}^{\infty} (-\lambda)^N (\mathscr{L}_{\varepsilon,\lambda}^{-1} J)^N \mathscr{L}_{\varepsilon,\lambda}^{-1} F \right\|_{H^1 \times L^2 \times L^2} \le C\varepsilon^2 \|F\|_{H^1 \times L^2 \times L^2}.$$

Using these estimates with Proposition 5.1 and the assertion (i), we obtain the desired result (ii).

As for (iii), we first observe that P_0 is a projection, i.e., $P_0^2 = P_0$. By (ii) we have $P_\varepsilon \to P_0$ in the operator norm as $\varepsilon \to 0$. It then follows from [8, Chap. I, Lemma 4.10] that $\dim R(P_\varepsilon) = \dim R(P_0)$ for $0 < \varepsilon \ll 1$. On the other hand, $\dim R(P_0) = \dim R(\mathbf{\Pi}_{\lambda_0})$. We thus obtain the desired result. This completes the proof. $\square$

Proof of Theorems 3.1 and 3.2. Theorem 3.2 is an immediate consequence of Theorem 5.2 (ii). To prove Theorems 3.1, we first observe that $\dim \mathrm{Ker}\,(L) = \dim \mathrm{Ker}\,(\mathscr{L}_{\varepsilon,0})$. Suppose that 0 is not a semisimple eigenvalue of L. Then

$$\dim R(\mathbf{\Pi}_0) > \dim \mathrm{Ker}\,(L) = \dim \mathrm{Ker}\,(\mathscr{L}_{\varepsilon,0}) = \dim \mathrm{Ker}\,(L_\varepsilon). \tag{5.2}$$

Since 0 is a semisimple eigenvalue of L_ε, we have $\dim \mathrm{Ker}\,(L_\varepsilon) = \dim R(P_\varepsilon)$. This, together with (5.2), implies that $\dim R(\mathbf{\Pi}_0) > \dim R(P_\varepsilon)$, which contradicts to Theorem 5.2 (iii) with $\lambda_0 = 0$. We thus conclude that 0 is a semisimple eigenvalue of $-L$.

Suppose that there exists $\lambda_0 \in \sigma(-L)$ with $\lambda_0 \neq 0$ and $\mathrm{Re}\,\lambda_0 \geq 0$. Then, by Proposition 4.1 (ii), we have $\mathrm{Im}\,\lambda_0 = O(1)$. Let $r_0 = (\mathrm{Re}\,\lambda_0 + b_0)/2 > 0$. We see from Theorem 5.2 (ii) that there exist $\varepsilon_0 > 0$ such that $\{\lambda; |\lambda - \lambda_0| < r_0\} \cap \sigma(-L_\varepsilon) \neq \emptyset$ for all $0 < \varepsilon \leq \varepsilon_0$. But if $|\lambda - \lambda_0| < r_0$, then

$$\mathrm{Re}\,\lambda = \mathrm{Re}\,(\lambda - \lambda_0) + (\mathrm{Re}\,\lambda_0 + b_0) - b_0 \geq r_0 - b_0 > -b_0.$$

This contradicts to the assumption $\{\lambda; \mathrm{Re}\,\lambda \geq -b_0\} \setminus \{0\} \subset \rho(-L_{\varepsilon_n})$ with $\varepsilon_n \to 0$. The proof of Theorem 3.1 is complete. $\square$

6 Proof of Theorem 3.3

In this section we give a proof of Theorem 3.3. We consider the resolvent problem for $-L_\varepsilon$:

$$\lambda u + L_\varepsilon u = F, \tag{6.1}$$

where $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon)$ and $F = {}^\top(f, \boldsymbol{g}, h) \in Y$. Problem (6.1) is written as

$$\varepsilon^2 \lambda p + \mathrm{div}\,\boldsymbol{w} = \varepsilon^2 f, \tag{6.2}$$

$$\mathrm{Pr}^{-1}\lambda \boldsymbol{w} - \Delta \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 = \mathrm{Pr}^{-1}\boldsymbol{g}, \tag{6.3}$$

$$\lambda\theta - \Delta\theta + \boldsymbol{v}_s \cdot \nabla\theta + \boldsymbol{w} \cdot \nabla\theta_s - \mathrm{Ra}\boldsymbol{w} \cdot \boldsymbol{e}_3 = h, \tag{6.4}$$

and $u = {}^\top(p, \boldsymbol{w}, \theta)$ satisfies the boundary conditions (3.4) and (3.5).

Proposition 6.1. *There exist constants $a_\varepsilon = O(1) > 0$ and $b_\varepsilon = O(1) > 0$ as $\varepsilon \to 0$ such that $\{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -a_\varepsilon \varepsilon^2 |\mathrm{Im}\,\lambda|^2 + b_\varepsilon\} \subset \rho(-L_\varepsilon)$ for all $0 < \varepsilon \leq 1$.*

Proof. We set

$$|||u|||_2 = \left(\varepsilon^2\|p\|_2^2 + \mathrm{Pr}^{-1}\|\boldsymbol{w}\|_2^2 + \|\theta\|_2^2\right)^{\frac{1}{2}}$$

for $u = {}^\top(p, \boldsymbol{w}, \theta)$. Taking the inner product $\langle\langle\cdot,\cdot\rangle\rangle_\varepsilon$ of (6.1) with u, we have

$$\begin{aligned}
&\lambda|||u|||_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2 \\
&\quad = 2\mathrm{Ra}\mathrm{Re}\,(\theta, w^3) + 2i\mathrm{Im}\,(p, \operatorname{div}\boldsymbol{w}) - \mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w}, \boldsymbol{w}) - (\boldsymbol{v}_s\cdot\nabla\theta, \theta) \\
&\qquad -\mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) - (\boldsymbol{w}\cdot\nabla\theta_s, \theta) + \langle\langle F, u\rangle\rangle_\varepsilon.
\end{aligned}\tag{6.5}$$

Noting that $\mathrm{Re}\,(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w}, \boldsymbol{w}) = \mathrm{Re}\,(\boldsymbol{v}_s\cdot\nabla\theta, \theta) = 0$, we see from the real part of (6.5) that

$$(\mathrm{Re}\,\lambda - b_2)|||u|||_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2 \le |||F|||_2|||u|||_2, \tag{6.6}$$

where $b_2 = 3\{\|\nabla\boldsymbol{v}_s\|_\infty + (\mathrm{Pr}+1)(2\mathrm{Ra} + \|\nabla\theta_s\|_\infty)\}$. The imaginary part of (6.5) yields

$$\begin{aligned}
|\mathrm{Im}\,\lambda|\,|||u|||_2^2 &\le 2\|p\|_2\|\operatorname{div}\boldsymbol{w}\|_2 + \mathrm{Pr}^{-1}\|\boldsymbol{v}_s\|_\infty\|\nabla\boldsymbol{w}\|_2\|\boldsymbol{w}\|_2 \\
&\quad +\|\boldsymbol{v}_s\|_\infty\|\nabla\theta\|_2\|\theta\|_2 + \mathrm{Pr}^{-1}\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2^2 + \|\nabla\theta_s\|_\infty\|\boldsymbol{w}\|_2\|\theta\|_2 \\
&\quad +|||F|||_2|||u|||_2 \\
&\le \left\{\left(2\varepsilon^{-1} + \mathrm{Pr}^{-\frac{1}{2}}\|\boldsymbol{v}_s\|_\infty\right)\|\nabla\boldsymbol{w}\|_2 + \|\boldsymbol{v}_s\|_\infty\|\nabla\theta\|_2\right. \\
&\quad \left.+\mathrm{Pr}^{-\frac{1}{2}}\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2 + \mathrm{Pr}^{\frac{1}{2}}\|\nabla\theta_s\|_\infty\|\theta\|_2 + |||F|||_2\right\}|||u|||_2,
\end{aligned}$$

and hence,

$$\begin{aligned}
|\mathrm{Im}\,\lambda|^2|||u|||_2^2 &\le 36\left\{\left(4\varepsilon^{-2} + \mathrm{Pr}^{-1}\|\boldsymbol{v}_s\|_\infty^2\right)\|\nabla\boldsymbol{w}\|_2^2 + \|\boldsymbol{v}_s\|_\infty^2\|\nabla\theta\|_2^2\right. \\
&\quad \left.+\mathrm{Pr}^{-1}\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2 + \mathrm{Pr}\|\nabla\theta_s\|_\infty^2\|\theta\|_2^2 + |||F|||_2^2\right\}.
\end{aligned}\tag{6.7}$$

It then follows from (6.6) and (6.7) that

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + a_\varepsilon\varepsilon^2|\mathrm{Im}\,\lambda|^2 - b_\varepsilon)|||u|||_2^2 + \tfrac{1}{2}\left(\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2\right) \\
&\le (\delta^{-1} + 36a_\varepsilon\varepsilon^2)|||F|||_2^2
\end{aligned}\tag{6.8}$$

for all $\delta > 0$, where

$$a_\varepsilon = \frac{1}{72(4 + \varepsilon^2(\mathrm{Pr}^{-1}+1)\|\boldsymbol{v}_s\|_\infty^2)}$$

and

$$b_\varepsilon = \delta + b_2 + 36a_\varepsilon\varepsilon^2\left(\|\nabla\boldsymbol{v}_s\|_\infty^2 + \mathrm{Pr}\|\nabla\theta_s\|_\infty^2\right).$$

It remains to estimate $\|\partial_x p\|_2$. To this end we use the estimates in section 7. We introduce quantities $D(\boldsymbol{w}, \theta)$ and $M(\boldsymbol{w}, \theta)$ defined by

$$\begin{aligned}
D(\boldsymbol{w}, \theta) &= \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2, \\
M(\boldsymbol{w}, \theta) &= \|\boldsymbol{v}_s\|_\infty^2(\mathrm{Pr}^{-2}\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2) + (\mathrm{Pr}^{-2}\|\nabla\boldsymbol{w}_s\|_\infty^2 + \|\nabla\theta_s\|_\infty^2)\|\boldsymbol{w}\|_2^2 \\
&\quad +\mathrm{Ra}(1+\mathrm{Ra})(\|\boldsymbol{w}\|_2^2 + \|\theta\|_2^2).
\end{aligned}$$

We set

$$\Sigma_1 = \{\lambda \in ; \operatorname{Re}\lambda + a_\varepsilon \varepsilon^2 |\operatorname{Im}\lambda| - b_\varepsilon \geq 0\}$$

and

$$\Sigma_2 = \{\lambda \in ; \operatorname{dist}(\lambda, \Sigma_1) \geq 1\}.$$

In what follows we assume that $\lambda \in \Sigma_2$. It then follows that

$$M(\boldsymbol{w}, \theta) \leq C|||F|||_2^2.$$

Here C is a positive constant depending on Ra and Pr but not on ε.

We consider $(1+\varepsilon^{-2}) \times (6.8) + d_0 \times (7.8)$. Then taking $d_0 > 0$ suitably small, we see that

$$\begin{aligned}
&(\operatorname{Re}\lambda + a_\varepsilon \varepsilon^2 |\operatorname{Im}\lambda|^2 - b_\varepsilon)|||u|||_2^2 + (\operatorname{Re}\lambda + |\operatorname{Im}\lambda|)\, D(\boldsymbol{w}, \theta) \\
&\quad + (\operatorname{Re}\lambda + \varepsilon^2 |\operatorname{Im}\lambda|^2)\,(|||\partial_{x'} u|||_2^2 + \varepsilon^2 \|\partial_{x_3} p\|_2^2) \\
&\quad + b_0 \left(\left(1 + \tfrac{1}{\varepsilon^2}\right) D(\boldsymbol{w}, \theta) + \|\varepsilon^2 \lambda p\|_{H^1}^2 + \tfrac{1}{2}|\lambda|^2 |||u|||_2^2 \right) \\
&\quad + b_0 \left(\|\partial_x^2 \boldsymbol{w}\|_2^2 + \|\partial_x^2 \theta\|_2^2 + \|\partial_x p\|_2^2 \right) \\
&\leq C(1+\varepsilon^2) \left\{ \varepsilon^2 \|\partial_x f\|_2^2 + \left(1 + \tfrac{1}{\varepsilon^2}\right) |||F|||_2^2 \right\}
\end{aligned} \tag{6.9}$$

with some positive constants b_0 and C uniformly for $0 < \varepsilon \leq 1$. This completes the proof. $\square$

We next show that the spectrum of $-L_\varepsilon$ in a disc with radius $O(\varepsilon^{-1})$ can be viewed as a perturbation of the one of $-L$. From the assumption of Theorem 3.3, we see that one can take the constant c_0 in Proposition 4.1 (ii) so that $c_0 < 0$ by changing $a_0 > 0$ suitably. In what follows we fix these $a_0 > 0$ and $c_0 < 0$. Using Proposition 4.1 (ii), Lemma 4.2, (5.1) and Proposition 5.1, we have the following estimates for $\mathscr{L}_{\varepsilon,\lambda}^{-1}$.

Proposition 6.2. *Let $\varepsilon > 0$. If $\lambda \in \Sigma \setminus \{0\}$, then $\mathscr{L}_{\varepsilon,\lambda}$ has a bounded inverse $\mathscr{L}_{\varepsilon,\lambda}^{-1}$ and ${}^\top(p, \boldsymbol{v}, \theta) = \mathscr{L}_{\varepsilon,\lambda}^{-1} F$ for $F = {}^\top(f, \boldsymbol{g}, h) \in Y$ satisfies*

$$\|\boldsymbol{U}\|_2 \leq \frac{C}{|\lambda|} \sum_{j=1}^{2} \left| \langle\langle F, u_j^* \rangle\rangle_\varepsilon \right| + C \left\{ \varepsilon^2 \|f\|_{H^1} + \frac{1}{|\lambda|+1} \|\boldsymbol{F}\|_2 \right\},$$

$$\|\partial_x^2 \boldsymbol{U}\|_2 + \|\partial_x p\|_2 \leq \frac{C}{|\lambda|} \sum_{j=1}^{2} \left| \langle\langle F, u_j^* \rangle\rangle_\varepsilon \right| + C \left\{ \varepsilon^2 (|\lambda|+1) \|f\|_{H^1} + \|\boldsymbol{F}\|_2 \right\},$$

where $\boldsymbol{U} = {}^\top(\boldsymbol{w}, \theta)$ and $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$. Furthermore, if $F \in Y_{1,\varepsilon}$, then ${}^\top(p, \boldsymbol{v}, \theta) = \mathscr{L}_{\varepsilon,\lambda}^{-1} F$ exists uniquely in $Y_{1,\varepsilon}$ and the above estimates hold with $\lambda = 0$ and $\langle\langle F, u_j^* \rangle\rangle_\varepsilon = 0$ $(j = 1, 2)$.

Proof. As in the proof of Proposition 4.3 (ii), we see that

$$\boldsymbol{\Pi}_0 \boldsymbol{P}[\boldsymbol{F} - \varepsilon^2 \boldsymbol{A}_\lambda \boldsymbol{V} f] = \sum_{j=1}^{2} \left\{ \langle\langle F, u_j^* \rangle\rangle_\varepsilon - \frac{\varepsilon^2 \lambda}{\mathrm{Pr}} (\boldsymbol{V} f, \boldsymbol{w}_j^*) \right\} \boldsymbol{U}_j^{(0)}.$$

Therefore, applying Proposition 4.1 (ii), Lemma 4.2 and Proposition 5.1 together with (5.1), we have the desired result. This completes the proof. □

Proposition 6.3. *There exist positive numbers* ε_1 *and* a_1 *such that*

$$\Sigma \cap \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0, |\lambda| \leq a_1 \varepsilon^{-1}\} \subset \rho(-L_\varepsilon|_{Y_{1,\varepsilon}})$$

for all $0 < \varepsilon \leq \varepsilon_1$. *Here*

$$Y_{1,\varepsilon} = R(L_\varepsilon) = \{u \in Y; \langle\langle u, u_j^* \rangle\rangle_\varepsilon = 0, j = 1, 2\}.$$

Proof. As in the proof of Theorem 5.2, we write the resolvent problem

$$(\lambda + L_\varepsilon) u = F$$

on $Y_{1,\varepsilon}$ as

$$\mathscr{L}_{\varepsilon,\lambda} u + \lambda J u = F, \tag{6.10}$$

where $F = {}^\top(f, \boldsymbol{g}, h) \in Y_{1,\varepsilon}$. If $\lambda \in \Sigma$, then it follows from Proposition 6.2 that (6.10) is written as

$$\mathscr{L}_{\varepsilon,\lambda}(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J) u = F,$$

and, furthermore, we have

$$\|\mathscr{L}_{\varepsilon,\lambda}^{-1} J F\|_{H^1 \times H^2 \times H^2} \leq \varepsilon^2 C_1 (|\lambda| + 1) \|f\|_{H^1}$$

for all $F = {}^\top(f, \boldsymbol{g}, h) \in Y_{1,\varepsilon}$. It then follows that there exists $\varepsilon_1 > 0$ such that if $\lambda \in \Sigma$ and $|\lambda| \leq 1/(4\sqrt{C_1}\varepsilon)$, then $\mathscr{L}_{\varepsilon,\lambda}^{-1} J F \in D(\mathscr{L}_{\varepsilon,\lambda}) = D(L_\varepsilon)$ and $\|\lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J F\|_{H^1 \times H^2 \times H^2} \leq \frac{1}{2} \|F\|_{H^1 \times L^2 \times L^2}$ for $0 < \varepsilon \leq \varepsilon_1$. Therefore, $(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J)$ is boundedly invertible both on $Y_{1,\varepsilon}$ and $Y_{1,\varepsilon} \cap D(L_\varepsilon)$ with estimates

$$\|(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J)^{-1} F\|_{H^1 \times L^2 \times L^2} \leq 2 \|F\|_{H^1 \times L^2 \times L^2}$$

for $F \in Y_{1,\varepsilon}$ and

$$\|(I + \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J)^{-1} F\|_{H^1 \times H^2 \times H^2} \leq 2 \|F\|_{H^1 \times H^2 \times H^2}$$

for $F \in Y_{1,\varepsilon} \cap D(L_\varepsilon)$. We thus find that $\lambda + L_\varepsilon = \mathscr{L}_{\varepsilon,\lambda} + \lambda J$ has a bounded inverse $(\lambda + L_\varepsilon)^{-1} = (\mathscr{L}_{\varepsilon,\lambda} + \varepsilon^2 \lambda J)^{-1}$ on $Y_{1,\varepsilon}$ which satisfies

$$(\lambda + L_\varepsilon)^{-1} = \mathscr{L}_{\varepsilon,\lambda}^{-1} - \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J \sum_{N=0}^{\infty} (-\lambda)^N (\mathscr{L}_{\varepsilon,\lambda}^{-1} J)^N \mathscr{L}_{\varepsilon,\lambda}^{-1}$$

and

$$\begin{aligned}
\|(\lambda+L_\varepsilon)^{-1}F\|_{H^1\times L^2\times L^2} &\le 2C_1\left\{\varepsilon^2(|\lambda|+1)\|f\|_{H^1}+\|\boldsymbol{F}\|_2\right\}\\
&\le 2C_1\left\{\varepsilon\left(\frac{1}{4\sqrt{C_1}}+\varepsilon\right)\|f\|_{H^1}+\|\boldsymbol{F}\|_2\right\}
\end{aligned}$$

with $\boldsymbol{F}={}^\top(\boldsymbol{g},h)$. This completes the proof. □

Theorem 3.3 follows from Propositions 4.3, 6.1 and 6.3 if $\sqrt{b_\varepsilon/a_\varepsilon}<a_1$ for $0<\varepsilon\ll 1$. In the case $\sqrt{b_\varepsilon/a_\varepsilon}\ge a_1$, there is some range of λ near the imaginary axis with $|\mathrm{Im}\,\lambda|=O(\varepsilon^{-1})$ to be proved that it belongs to $\rho(-L_\varepsilon)$.

To prove Theorem 3.3 when $\sqrt{b_\varepsilon/a_\varepsilon}\ge a_1$, we first prepare an estimate for the θ-component. We recall that the Poincaré inequality

$$\|\nabla\theta\|_2\ge\beta\|\theta\|_2$$

holds for $\theta\in H^1_{0,per}$ with some positive constant β.

Proposition 6.4. *Let* ${}^\top(p,\boldsymbol{w},\theta)$ *be a solution of* (6.2)–(6.4) *under boundary conditions* (3.4) *and* (3.5). *Then if* $\mathrm{Re}\,\lambda\ge-\frac{\beta^2}{2}$, *the following estimates hold:*

$$\|\theta\|_2\le\frac{1}{|\mathrm{Im}\,\lambda|}\left(1+\frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\{(\|\nabla\theta_s\|_\infty+\mathrm{Ra})\|\boldsymbol{w}\|_2+\|h\|_2\},$$

$$\|\nabla\theta\|_2\le\frac{2}{\beta}\{(\|\nabla\theta_s\|_\infty+\mathrm{Ra})\|\boldsymbol{w}\|_2+\|h\|_2\}.$$

Proof. We take the inner product of (6.4) with θ to obtain

$$\begin{aligned}
&\lambda\|\theta\|_2^2+\|\nabla\theta\|_2^2\\
&=-\left(i\mathrm{Im}\,(\boldsymbol{v}_s\cdot\nabla\theta,\theta)+(\boldsymbol{w}\cdot\nabla\theta_s,\theta)-\mathrm{Ra}(w^3,\theta)\right)+(h,\theta).
\end{aligned}\tag{6.11}$$

Using the Poincaré inequality we see from the real part of (6.11) that

$$\left(\mathrm{Re}\,\lambda+\frac{\beta^2}{2}\right)\|\theta\|_2^2+\frac{1}{2}\|\nabla\theta\|_2^2\le\{(\|\nabla\theta_s\|_\infty+\mathrm{Ra})\,\|\boldsymbol{w}\|_2+\|h\|_2\}\,\|\theta\|_2.$$

This implies that if $\mathrm{Re}\,\lambda\ge-\frac{\beta^2}{2}$, then

$$\|\nabla\theta\|_2\le\frac{2}{\beta}\{(\|\nabla\theta_s\|_\infty+\mathrm{Ra})\|\boldsymbol{w}\|_2+\|h\|_2\}.\tag{6.12}$$

Furthermore, the imaginary part of (6.11), together with (6.12), implies that

$$\begin{aligned}
|\mathrm{Im}\,\lambda|\|\theta\|_2^2 &= \left|-\mathrm{Im}\,\left((\boldsymbol{v}_s\cdot\nabla\theta,\theta)+(\boldsymbol{w}\cdot\nabla\theta_s,\theta)-\mathrm{Ra}(w^3,\theta)\right)+\mathrm{Im}\,(h,\theta)\right|\\
&\le \{\|\boldsymbol{v}_s\|_\infty\|\nabla\theta\|_2+(\|\nabla\theta_s\|_\infty+\mathrm{Ra})\|\boldsymbol{w}\|_2+\|h\|_2\}\,\|\theta\|_2\\
&\le \left(1+\tfrac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\{(\|\nabla\theta_s\|_\infty+\mathrm{Ra})\|\boldsymbol{w}\|_2+\|h\|_2\}\,\|\theta\|_2,
\end{aligned}$$

from which we have the desired estimate for $\|\theta\|_2$. This completes the proof. □

We are now in a position to complete the proof of Theorem 3.3.

Proposition 6.5. *For given $\mu_* > 0$ and $\eta_* > 0$ there exist constants $\varepsilon_1 > 0$ and $c_2 > 0$ such that if*

$$\inf_{\boldsymbol{w}\in H^1_{0,per},\boldsymbol{w}\neq\boldsymbol{0}} \frac{\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w})}{\|\nabla\boldsymbol{w}\|_2^2} \geq -\frac{\mathrm{Pr}}{32},$$

then

$$\left\{\lambda = \mu + i\frac{\eta}{\varepsilon};\ -c_2 \leq \mu \leq \mu_*,\ |\eta| \geq \eta_*\right\} \subset \rho(-L_\varepsilon)$$

for all $0 < \varepsilon \leq \varepsilon_1$. Here ε_1 and c_2 are positive constants depending only on Pr, Ra, *$\|\boldsymbol{v}_s\|_{C^1}$, $\|\nabla\theta_s\|_\infty$, β, μ_* and η_**

Proof. We see from (6.2) that

$$p = -\frac{1}{\varepsilon^2\lambda}\mathrm{div}\,\boldsymbol{w} + \frac{1}{\lambda}f. \tag{6.13}$$

Substituting (6.13) into (6.3), we have

$$\frac{\varepsilon^2\lambda^2}{\mathrm{Pr}}\boldsymbol{w} - \varepsilon^2\lambda\Delta\boldsymbol{w} - \nabla\mathrm{div}\,\boldsymbol{w} + \frac{\varepsilon^2\lambda}{\mathrm{Pr}}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s) - \varepsilon^2\lambda\mathrm{Ra}\theta\boldsymbol{e}_3 = \varepsilon^2\boldsymbol{G}_\lambda, \tag{6.14}$$

where $\boldsymbol{G}_\lambda = \frac{\lambda}{\mathrm{Pr}}\boldsymbol{g} - \nabla f$.

Let $\lambda = \mu + \frac{i}{\varepsilon}\eta$ with $|\eta| \geq \eta_*(> 0)$. Without loss of generality we may assume $\eta \geq \eta_*$. Taking the inner product of (6.14) with $\boldsymbol{w}$, we have

$$\begin{aligned}
&\frac{\varepsilon^2\lambda^2}{\mathrm{Pr}}\|\boldsymbol{w}\|_2^2 + \varepsilon^2\lambda\|\nabla\boldsymbol{w}\|_2^2 + \|\mathrm{div}\,\boldsymbol{w}\|_2^2 \\
&= \ -\varepsilon^2\lambda\left(\mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w},\boldsymbol{w}) + \mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta,w^3)\right) + \varepsilon^2(\boldsymbol{G}_\lambda,\boldsymbol{w}).
\end{aligned} \tag{6.15}$$

Since $\lambda^2 = (\mu^2 - \varepsilon^{-2}\eta^2) + 2i\varepsilon^{-1}\mu\eta$ and $\mathrm{Re}\,(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w},\boldsymbol{w}) = 0$, the real and imaginary parts of (6.15) yield

$$\begin{aligned}
&\frac{1}{\mathrm{Pr}}(\varepsilon^2\mu^2 - \eta^2)\|\boldsymbol{w}\|_2^2 + \varepsilon^2\mu\|\nabla\boldsymbol{w}\|_2^2 + \|\mathrm{div}\,\boldsymbol{w}\|_2^2 \\
&= \ -\varepsilon^2\mu\mathrm{Re}\,\left(\mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta,w^3)\right) \\
&\quad +\varepsilon\eta\mathrm{Im}\,\left(\mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w},\boldsymbol{w}) + \mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta,w^3)\right) \\
&\quad +\varepsilon^2\mathrm{Re}\,(\boldsymbol{G}_\lambda,\boldsymbol{w})
\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}
&\frac{2\varepsilon\mu\eta}{\mathrm{Pr}}\|\boldsymbol{w}\|_2^2 + \varepsilon\eta\|\nabla\boldsymbol{w}\|_2^2 \\
&= \ -\varepsilon^2\mu\mathrm{Im}\,\left(\mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w},\boldsymbol{w}) + \mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta,w^3)\right) \\
&\quad -\varepsilon\eta\mathrm{Re}\,\left(\mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta,w^3)\right) \\
&\quad +\varepsilon^2\mathrm{Im}\,(\boldsymbol{G}_\lambda,\boldsymbol{w}).
\end{aligned} \tag{6.17}$$

Since $|\mathrm{Im}\,\lambda| = \frac{\eta}{\varepsilon}$, by Proposition 6.4, we have

$$|\mathrm{Ra}(\theta, w^3)| \leq \frac{\mathrm{Ra}\varepsilon}{\eta}\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\left\{(\|\nabla\theta_s\|_\infty + \mathrm{Ra})\|\boldsymbol{w}\|_2^2 + \|h\|_2\|\boldsymbol{w}\|_2\right\},$$

and hence, we see from (6.16) that

$$\begin{aligned}
&\frac{1}{\mathrm{Pr}}(\eta^2 - \varepsilon^2\mu^2)\|\boldsymbol{w}\|_2^2 \\
\leq\ & \left(\varepsilon^2\mu + \frac{\eta}{\eta_*} + \varepsilon\eta\right)\|\nabla\boldsymbol{w}\|_2^2 + \left((\varepsilon^2|\mu| + \varepsilon\eta)\mathrm{Pr}^{-1}\|\nabla\boldsymbol{v}_s\|_\infty + \varepsilon\eta\mathrm{Pr}^{-2}\|\boldsymbol{v}_s\|_\infty^2\right)\|\boldsymbol{w}\|_2^2 \\
&+(\varepsilon^2|\mu| + \varepsilon\eta)\frac{\mathrm{Ra}\varepsilon}{\eta}\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\left\{(\|\nabla\theta_s\|_\infty + \mathrm{Ra})\|\boldsymbol{w}\|_2^2 + \|h\|_2\|\boldsymbol{w}\|_2\right\} \\
&+\varepsilon^2\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.
\end{aligned} \tag{6.18}$$

Similarly, we see from (6.17) that

$$\begin{aligned}
&\frac{2\mu\eta}{\mathrm{Pr}}\|\boldsymbol{w}\|_2^2 + \frac{3\eta}{4}\|\nabla\boldsymbol{w}\|_2^2 \\
\leq\ & -\eta\mathrm{Pr}^{-1}\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) + \left(\frac{\varepsilon^2|\mu|^2\mathrm{Pr}^{-2}\|\boldsymbol{v}_s\|_\infty^2}{\eta} + \varepsilon|\mu|\mathrm{Pr}^{-1}\|\nabla\boldsymbol{v}_s\|_\infty\right)\|\boldsymbol{w}\|_2^2 \\
&+(\varepsilon|\mu| + \eta)\frac{\mathrm{Ra}\varepsilon}{\eta}\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\left\{(\|\nabla\theta_s\|_\infty + \mathrm{Ra})\|\boldsymbol{w}\|_2^2 + \|h\|_2\|\boldsymbol{w}\|_2\right\} \\
&+\varepsilon\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.
\end{aligned} \tag{6.19}$$

We set $M = \inf_{\boldsymbol{w}\in H^1_{0,per}, \boldsymbol{w}\neq\boldsymbol{0}} \frac{\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w})}{\|\nabla\boldsymbol{w}\|_2^2}$. It then follows from $\frac{\eta_*}{4}\times(6.18) + (6.19)$ that

$$\begin{aligned}
&\left(I_{\varepsilon,\mu,\eta} + \frac{\eta_*}{4\mathrm{Pr}}\eta^2\right)\|\boldsymbol{w}\|_2^2 + \left(\frac{\eta}{2} - \frac{\varepsilon^2\eta_*\mu}{4} - \frac{\varepsilon\eta_*\eta}{4}\right)\|\nabla\boldsymbol{w}\|_2^2 \\
&\leq -\eta\mathrm{Pr}^{-1}M\|\nabla\boldsymbol{w}\|_2^2 + C_{\varepsilon.\lambda}(\|F\|_{H^1\times(L^2)^2})\|\boldsymbol{w}\|_2,
\end{aligned} \tag{6.20}$$

where

$$\begin{aligned}
I_{\varepsilon,\mu,\eta} =\ & \frac{2\mu\eta}{\mathrm{Pr}} - \frac{\varepsilon^2\mu^2}{\mathrm{Pr}}\left(\frac{\eta_*}{4} + \frac{\|\boldsymbol{v}_s\|_\infty^2}{\eta\mathrm{Pr}}\right) - \frac{\varepsilon|\mu|}{\mathrm{Pr}}\left(1 + \frac{\varepsilon\eta_*}{4}\right)\|\nabla\boldsymbol{v}_s\|_\infty \\
&-\frac{\varepsilon\eta_*\eta}{4\mathrm{Pr}}\left(\|\nabla\boldsymbol{v}_s\|_\infty + \frac{\|\boldsymbol{v}_s\|_\infty^2}{\mathrm{Pr}}\right) \\
&-\varepsilon\mathrm{Ra}\left(1 + \frac{\varepsilon\eta_*}{4}\right)\left(\frac{\varepsilon|\mu|}{\eta} + 1\right)\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)(\|\nabla\boldsymbol{v}_s\|_\infty + \mathrm{Ra}),
\end{aligned}$$

$$C_{\varepsilon,\lambda}(\|F\|_{H^1\times(L^2)^2}) = \varepsilon\left(1 + \frac{\varepsilon\eta_*}{4}\right)\left[\mathrm{Ra}\left(\frac{\varepsilon|\mu|}{\eta} + 1\right)\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\|h\|_2 + \|\boldsymbol{G}_\lambda\|_2\right].$$

We take $\varepsilon > 0$ so small that $\varepsilon^2 \mu_* \leq \frac{1}{8}$ and $\varepsilon \eta_* \leq \frac{1}{8}$ in (6.20). Then by using the Poincaré inequality, we have

$$\left(I_{\varepsilon,\mu,\eta} + \frac{\eta_*}{4\mathrm{Pr}}\eta^2 + \frac{\beta^2}{8}\eta \right) \|\boldsymbol{w}\|_2^2 + \frac{\eta}{16}\|\nabla \boldsymbol{w}\|_2^2 \leq -\eta \mathrm{Pr}^{-1} M \|\nabla \boldsymbol{w}\|_2^2 + C_{\varepsilon,\lambda}(\|F\|_{H^1\times(L^2)^2})\|\boldsymbol{w}\|_2. \tag{6.21}$$

and hence, if

$$M \geq -\frac{\mathrm{Pr}}{32},$$

we find from (6.21) that

$$\left(I_{\varepsilon,\mu,\eta} + \frac{\beta^2}{8}\eta \right) \|\boldsymbol{w}\|_2^2 + \frac{\eta}{32}\|\nabla \boldsymbol{w}\|_2^2 \leq C_{\varepsilon,\lambda}(\|F\|_{H^1\times(L^2)^2})\|\boldsymbol{w}\|_2. \tag{6.22}$$

Therefore, if we show $I_{\varepsilon,\mu,\eta} + \frac{\beta^2}{8}\eta \geq \frac{\beta^2}{16}\eta$ for $-c_2 \leq \mu \leq \mu_*$ and $\eta \geq \eta_*$ with some $c_2 > 0$ when $0 < \varepsilon \ll 1$, then the proof of Proposition 6.6 is complete.

Let us prove $I_{\varepsilon,\mu,\eta} + \frac{\beta^2}{8}\eta \geq \frac{\beta^2}{16}\eta$. When $0 \leq \mu \leq \mu_*$, we have

$$\begin{aligned} I_{\varepsilon,\mu,\eta} \ &\geq\ \eta \left\{ \mu \left(\frac{2}{\mathrm{Pr}} - K_\varepsilon \right) - \frac{\varepsilon \eta_*}{4\mathrm{Pr}} \left(\|\nabla \boldsymbol{v}_s\|_\infty + \frac{\|\boldsymbol{v}_s\|_\infty^2}{\mathrm{Pr}} \right) \right. \\ &\qquad \left. - \frac{\varepsilon \mathrm{Ra}}{\eta_*}\left(1 + \frac{\varepsilon\eta_*}{4}\right)\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)(\|\nabla \boldsymbol{v}_s\|_\infty + \mathrm{Ra}) \right\}, \end{aligned}$$

where

$$\begin{aligned} K_\varepsilon \ &=\ \frac{\varepsilon^2\mu_*}{\mathrm{Pr}}\left(\frac{1}{4} + \frac{\|\boldsymbol{v}_s\|_\infty^2}{\eta_*^2 \mathrm{Pr}} \right) + \frac{\varepsilon}{\mathrm{Pr}}\left(\frac{\varepsilon}{4} + \frac{1}{\eta_*} \right)\|\nabla \boldsymbol{v}_s\|_\infty \\ &\quad + \frac{\varepsilon^2 \mathrm{Ra}}{\eta_*^2}\left(1 + \frac{\varepsilon\eta_*}{4}\right)\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)(\|\nabla \boldsymbol{v}_s\|_\infty + \mathrm{Ra}). \end{aligned}$$

Therefore, if we take $\varepsilon > 0$ so that

$$\frac{2}{\mathrm{Pr}} - K_\varepsilon \geq 0$$

and

$$\begin{aligned} &\frac{\beta^2}{16} - \frac{\varepsilon\eta_*}{4\mathrm{Pr}}\left(\|\nabla \boldsymbol{v}_s\|_\infty + \frac{\|\boldsymbol{v}_s\|_\infty^2}{\mathrm{Pr}} \right) \\ &\quad - \frac{\varepsilon \mathrm{Ra}}{\eta_*}\left(1 + \frac{\varepsilon\eta_*}{4}\right)\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)(\|\nabla \boldsymbol{v}_s\|_\infty + \mathrm{Ra}) \geq 0, \end{aligned}$$

then we have $I_{\varepsilon,\mu,\eta} + \frac{\beta^2}{8}\eta \geq \frac{\beta^2}{16}\eta$ for $0 \leq \mu \leq \mu_*$ and $\eta \geq \eta_*$, and hence,

$$\frac{\beta^2}{16}\eta\|\boldsymbol{w}\|_2^2 + \frac{\eta}{32}\|\nabla \boldsymbol{w}\|_2^2 \leq C_{\varepsilon,\lambda}(\|F\|_{H^1\times(L^2)^2})\|\boldsymbol{w}\|_2 \tag{6.23}$$

for $0 \leq \mu \leq \mu_*$ and $\eta \geq \eta_*$.

When $\mu < 0$, we assume that $|\mu| \leq \frac{\mathrm{Pr}}{64}\beta^2$. We write $I_{\varepsilon,\mu,\eta}$ as $I_{\varepsilon,\mu,\eta} = \eta \tilde{I}_\varepsilon(\mu,\eta)$. Then we see that $\tilde{I}_\varepsilon(\mu,\eta) \geq \tilde{I}_\varepsilon\left(-\frac{\mathrm{Pr}}{64}\beta^2, \eta_*\right) = -\frac{\beta^2}{32} + O(\varepsilon)$, and so we take $\varepsilon > 0$ in such a way that $\tilde{I}_\varepsilon\left(-\frac{\Pr}{64}\beta^2, \eta_*\right) + \frac{\beta^2}{16} \ge 0$. It then follows that $I_{\varepsilon,\mu,\eta} + \frac{\beta^2}{8}\eta \ge \frac{\beta^2}{16}\eta$, and hence, we have (6.23) for $-c_2 \le \mu < 0$ and $\eta \ge \eta_*$ with $c_2 = \frac{\Pr}{128}\beta^2$. Similarly to the proof of Proposition 6.1, we can derive the necessary estimate for $\|\partial_x p\|_2^2$ by combining estimate (6.23), Propositions 6.4 and 7.5. This completes the proof. $\square$

Theorem 3.3 now follows by taking $\eta_* = \frac{a_1}{2}$, $\mu_* = 2b_\varepsilon|_{\varepsilon=1,\delta=1}$ and $\varepsilon > 0$ sufficiently small.

7 Estimate for $\|\partial_x p\|_2$

In this section we give basic energy estimates which yield the estimate for $\|\partial_x p\|_2$ in the resolvent estimate. The following Propositions 7.1–7.4 can be obtained by the Matsumura-Nishida energy method [9]. We here give an outline of the proof.

Proposition 7.1. *Let* $u = {}^\top(p, \boldsymbol{w}, \theta)$ *be a solution of* (6.2)–(6.4) *satisfying boundary conditions* (3.4) *and* (3.5). *Then*

$$\begin{aligned}(\mathrm{Re}\,\lambda + \varepsilon^2|\mathrm{Im}\,\lambda|^2)|||\partial_{x'}u|||_2^2 + \tfrac{1}{4}D(\partial_{x'}\boldsymbol{w}, \partial_{x'}\theta) + \tfrac{1}{4}\|\varepsilon^2\lambda\partial_{x'}p\|_2^2 \\ \le C(1+\varepsilon^2)\left\{\delta\varepsilon^2\|\partial_{x'}p\|_2^2 + \delta^{-1}\varepsilon^2\|\partial_{x'}f\|_2^2 + |||F|||_2^2 + M(\boldsymbol{w},\theta)\right\}\end{aligned} \tag{7.1}$$

for any $\delta > 0$, *where* C *is a positive constant independent of* ε *and* δ.

Proposition 7.1 can be proved as in the proof of Proposition 6.1 by taking the inner product $\langle\langle\partial_{x'}(6.1), \partial_{x'}u\rangle\rangle_\varepsilon$ and using the relation $\varepsilon^2\lambda\partial_{x'}p = -\mathrm{div}\,\partial_{x'}\boldsymbol{w} + \varepsilon^2\partial_{x'}f$ which follows from (6.2).

We next compute the inner product $\langle\langle(6.1), \lambda u\rangle\rangle_\varepsilon$ to obtain the following estimate.

Proposition 7.2. *Let* $u = {}^\top(p, \boldsymbol{w}, \theta)$ *be a solution of* (6.2)–(6.4) *satisfying boundary conditions* (3.4) *and* (3.5). *Then*

$$(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|)D(\boldsymbol{w},\theta) + \frac{1}{2}|\lambda|^2|||u|||_2^2 \le C\left\{\frac{1}{\varepsilon^2}\|\mathrm{div}\,\boldsymbol{w}\|_2^2 + M(\boldsymbol{w},\theta)\right\}, \tag{7.2}$$

where C *is a positive constant independent of* ε.

As for the normal derivative of p, we have the following estimate.

Proposition 7.3. *Let* $u = {}^\top(p, \boldsymbol{w}, \theta)$ *be a solution of* (6.2)–(6.4) *satisfying boundary conditions* (3.4) *and* (3.5). *Then*

$$\begin{aligned}(\mathrm{Re}\,\lambda + \varepsilon^2|\mathrm{Im}\,\lambda|^2)\varepsilon^2\|\partial_{x_3}p\|_2^2 + \tfrac{1}{4}\|\partial_{x_3}p\|_2^2 + \tfrac{1}{4}\|\varepsilon^2\lambda\partial_{x_3}p\|_2^2 \\ \le C\left\{\varepsilon^2\|\partial_{x_3}f\|_2^2 + |||F|||_2^2 + M(\boldsymbol{w},\theta) + |\lambda|^2\|\boldsymbol{w}\|_2^2 + \|\nabla\partial_{x'}\boldsymbol{w}\|_2^2\right\},\end{aligned} \tag{7.3}$$

where C *is a positive constant depending on* Pr *but independent of* ε.

Outline of Proof. We denote $\boldsymbol{w}' = {}^\top(w^1, w^2)$, $\nabla' = {}^\top(\partial_{x_1}, \partial_{x_2})$ and $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$. Applying ∂_{x_3} to (6.2), we have

$$\varepsilon^2 \lambda \partial_{x_3} p + \nabla' \cdot \partial_{x_3} \boldsymbol{w}' + \partial_{x_3}^2 w^3 = \varepsilon^2 \partial_{x_3} f. \tag{7.4}$$

The third equation of (6.3) is written as

$$\mathrm{Pr}^{-1} \lambda w^3 - \Delta' w^3 - \partial_{x_3}^2 w^3 + \partial_{x_3} p = \tilde{g}^3, \tag{7.5}$$

where

$$\tilde{g}^3 = \mathrm{Pr}^{-1} g^3 - \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla w^3 + \boldsymbol{w} \cdot \nabla v_s^3) + \mathrm{Ra}\theta.$$

We compute (7.4)+(7.5) to obtain

$$\varepsilon^2 \lambda \partial_{x_3} p + \partial_{x_3} p = H, \tag{7.6}$$

where

$$H = \varepsilon^2 \partial_{x_3} f + \tilde{g}^3 - (\mathrm{Pr}^{-1} \lambda w^3 - \Delta' w^3 + \nabla' \cdot \partial_{x_3} \boldsymbol{w}').$$

Taking the L^2 inner product of (7.6) with $\partial_{x_3} p$, we have

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + \varepsilon^2 |\mathrm{Im}\,\lambda|^2)\varepsilon^2 \|\partial_{x_3} p\|_2^2 + \tfrac{1}{4}\|\partial_{x_3} p\|_2^2 + \tfrac{1}{4}\|\varepsilon^2 \lambda \partial_{x_3} p\|_2^2 \\
&\quad \le C\left\{\varepsilon^2 \|\partial_{x_3} f\|_2^2 + |||F|||_2^2 + M(\boldsymbol{w}, \theta) + |\lambda|^2 \|\boldsymbol{w}\|_2^2 + \|\nabla \partial_{x'} \boldsymbol{w}\|_2^2\right\}.
\end{aligned}$$

This, together with the relation $\varepsilon^2 \lambda \partial_{x_3} p = -\partial_{x_3} p + H$, gives the desired estimate. □

The elliptic estimates yield the following estimate for the second order derivatives of $\boldsymbol{w}$ and θ and the first order derivatives of p.

Proposition 7.4. *Let $u = {}^\top(p, \boldsymbol{w}, \theta)$ be a solution of* (6.2)–(6.4) *satisfying boundary conditions* (3.4) *and* (3.5). *Then*

$$\begin{aligned}
&\|\partial_x^2 \boldsymbol{w}\|_2^2 + \|\partial_x^2 \theta\|_2^2 + \|\partial_x p\|_2^2 \\
&\quad \le C\left\{\varepsilon^4 \|f\|_{H^1}^2 + |||F|||_2^2 + \|\varepsilon^2 \lambda p\|_{H^1}^2 + |\lambda|^2(\|\boldsymbol{w}\|_2^2 + \|\theta\|_2^2) + M(\boldsymbol{w}, \theta)\right\},
\end{aligned} \tag{7.7}$$

where C is a positive constant depending on Pr *but independent of ε.*

Outline of Proof. We rewrite (6.1) as

$$\begin{cases}
\operatorname{div} \boldsymbol{w} = \varepsilon^2 f - \varepsilon^2 \lambda p, \\
-\Delta \boldsymbol{w} + \nabla p = \mathrm{Pr}^{-1} \boldsymbol{g} - (\mathrm{Pr}^{-1} \lambda \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) - \mathrm{Ra}\theta \boldsymbol{e}_3), \\
-\Delta \theta = h - (\lambda \theta + \boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{w} \cdot \nabla \theta_s - \mathrm{Ra}\boldsymbol{w} \cdot \boldsymbol{e}_3).
\end{cases}$$

Applying the estimates for the Stokes system and the elliptic equation (see, e.g., [5, 11]), we obtain the desired estimate. □

Combining the estimates in Propositions 7.1–7.4, we have the following estimate for $\|\partial_x p\|_2$ and $\|\partial_x^2 w\|_2$.

Proposition 7.5. *Let* $0 < \varepsilon \le 1$ *and let* $u = {}^\top(p, \boldsymbol{w}, \theta)$ *be a solution of* (6.2)–(6.4) *satisfying boundary conditions* (3.4) *and* (3.5). *Then*

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|)\, D(\boldsymbol{w}, \theta) + (\mathrm{Re}\,\lambda + \varepsilon^2 |\mathrm{Im}\,\lambda|^2)\, (|||\partial_{x'} u|||_2^2 + \varepsilon^2 \|\partial_{x_3} p\|_2^2) \\
&\quad + b\left(\|\varepsilon^2 \lambda \partial_x p\|_{H^1}^2 + \tfrac{1}{2}|\lambda|^2 |||u|||_2^2 + \|\partial_x^2 \boldsymbol{w}\|_2^2 + \|\partial_x^2 \theta\|_2^2 + \|\partial_x p\|_2^2\right) \\
&\le C(1+\varepsilon^2)\left\{\varepsilon^2 \|\partial_x f\|_2^2 + \left(1 + \tfrac{1}{\varepsilon^2}\right)(\|\mathrm{div}\, \boldsymbol{w}\|_2^2 + M(\boldsymbol{w}, \theta) + |||F|||_2^2)\right\}
\end{aligned} \tag{7.8}$$

Proof of Proposition 7.5. Let $0 \le \varepsilon \le 1$. We see from (7.1) + (7.2) + $d_2 \times$ (7.3) with suitably small $d_2 > 0$ that

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|)\, D(\boldsymbol{w}, \theta) + (\mathrm{Re}\,\lambda + \varepsilon^2 |\mathrm{Im}\,\lambda|^2)\, (|||\partial_{x'} u|||_2^2 + \varepsilon^2 \|\partial_{x_3} p\|_2^2) \\
&\quad + b\left(D(\partial_{x'} \boldsymbol{w}, \partial_{x'} \theta) + \|\partial_{x_3} p\|_2^2 + \|\varepsilon^2 \lambda \partial_x p\|_2 + \tfrac{1}{2}|\lambda|^2 |||u|||_2^2\right) \\
&\le C(1+\varepsilon^2)\{\delta \varepsilon^2 \|\partial_{x'} p\|_2^2 + \delta^{-1} \varepsilon^2 \|\partial_x f\|_2^2 \\
&\quad + \left(1 + \tfrac{1}{\varepsilon^2}\right)(\|\mathrm{div}\, \boldsymbol{w}\|_2^2 + M(\boldsymbol{w}, \theta) + |||F|||_2^2)\right\}
\end{aligned} \tag{7.9}$$

for any $\delta > 0$ with positive constants b and C independent of δ and ϵ.

Adding $d_3 \times$ (7.3) to (7.9) and taking $d_3 > 0$ suitably small, we have

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|)\, D(\boldsymbol{w}, \theta) + (\mathrm{Re}\,\lambda + \varepsilon^2 |\mathrm{Im}\,\lambda|^2)\, (|||\partial_{x'} u|||_2^2 + \varepsilon^2 \|\partial_{x_3} p\|_2^2) \\
&\quad + b\left(\|\varepsilon^2 \lambda \partial_x p\|_2^2 + \tfrac{1}{2}|\lambda|^2 |||u|||_2^2 + \|\partial_x^2 \boldsymbol{w}\|_2^2 + \|\partial_x^2 \theta\|_2^2 + \|\partial_x p\|_2^2\right) \\
&\le C(1+\varepsilon^2)\{\delta \varepsilon^2 \|\partial_{x'} p\|_2^2 + \delta^{-1} \varepsilon^2 \|\partial_x f\|_2^2 \\
&\quad + \left(1 + \tfrac{1}{\varepsilon^2}\right)((\|\mathrm{div}\, \boldsymbol{w}\|_2^2 + M(\boldsymbol{w}, \theta) + |||F|||_2^2)\right\}
\end{aligned} \tag{7.10}$$

for any $\delta > 0$ with positive constants b and C independent of δ and $0 < \epsilon \le 1$. Taking $\delta > 0$ suitably small, we arrive at

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|)\, D(\boldsymbol{w}, \theta) + (\mathrm{Re}\,\lambda + \varepsilon^2 |\mathrm{Im}\,\lambda|^2)\, (|||\partial_{x'} u|||_2^2 + \varepsilon^2 \|\partial_{x_3} p\|_2^2) \\
&\quad + b\left(\|\varepsilon^2 \lambda \partial_x p\|_{H^1}^2 + \tfrac{1}{2}|\lambda|^2 |||u|||_2^2 + \|\partial_x^2 \boldsymbol{w}\|_2^2 + \|\partial_x^2 \theta\|_2^2 + \|\partial_x p\|_2^2\right) \\
&\le C(1+\varepsilon^2)\left\{\varepsilon^2 \|\partial_x f\|_2^2 + \left(1 + \tfrac{1}{\varepsilon^2}\right)((\|\mathrm{div}\, \boldsymbol{w}\|_2^2 + M(\boldsymbol{w}, \theta) + |||F|||_2^2)\right\}
\end{aligned} \tag{7.11}$$

This completes the proof. □

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