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Asymptotic behavior of the linearized semigroup at space-periodic stationary solution of the compressible Navier-Stokes equation

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Abstract

The asymptotic behavior of the linearized semigroup at spatially periodic stationary solution of the compressible Navier-Stokes equation in a periodic layer of \mathbb{R}^n ($n = 2, 3$) is investigated. It is shown that if the Reynolds and Mach numbers are sufficiently small, then the linearized semigroup is decomposed into two parts; one behaves like a solution of an $n - 1$ dimensional linear heat equation as time goes to infinity and the other one decays exponentially.

Keywords: Compressible Navier-Stokes equation, space-periodic stationary solution, linearized semigroup, spectrum.

1 Introduction

This paper studies the stability of stationary solution to the compressible Navier-Stokes equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad (1.1)$$

$$\rho(\partial_t v + (v \cdot \nabla)v) - \mu \Delta v - (\mu + \mu') \nabla \operatorname{div} v + \nabla p(\rho) = \rho g \quad (1.2)$$

in a periodic layer Ω_* of \mathbb{R}^n with $n = 2, 3$:

$$\Omega_* = \{x = (x', x_n); x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \omega_{*,1}(x') < x_n < \omega_{*,2}(x')\},$$

where $\omega_{*,1}$ and $\omega_{*,2}$ are smooth Q_* -periodic functions in x' with the periodic cell $Q_* = \prod_{j=1}^{n-1} [-\frac{\pi}{\alpha_{*,j}}, \frac{\pi}{\alpha_{*,j}})$ for positive constants $\alpha_{*,j}$ ($j = 1, \dots, n-1$), namely, $\omega_{*,1}$ and $\omega_{*,2}$ are smooth functions satisfying $\omega_{*,j}(x' + \frac{2\pi}{\alpha_i} \mathbf{e}_i') = \omega_{*,j}(x')$ ($i = 1, \dots, n-1, j = 1, 2$) with $\mathbf{e}_i' = {}^\top(0, \dots, \overset{i}{1}, \dots, 0) \in \mathbb{R}^{n-1}$; $\rho = \rho(x, t)$ and $v = {}^\top(v^1(x, t), \dots, v^n(x, t))$ denote the unknown density and velocity, respectively, at $x \in \Omega_*$ and $t \geq 0$; $p = p(\rho)$ is the pressure that is assumed to be a smooth function of ρ ; $g = {}^\top(g^1(x), \dots, g^n(x))$ is a given external force; and, μ and μ' are the viscosity coefficients that are assumed to be constants. Here and in what follows the superscript $^\top$ stands for the transposition.

We assume that μ and μ' satisfy $\mu > 0$ and $\frac{2}{n}\mu + \mu' \geq 0$. Furthermore we assume that $\frac{\mu'}{\mu}$ satisfies

$$\frac{\mu'}{\mu} \leq \mu_1 \quad (1.3)$$

for a certain constant $\mu_1 > 0$. We also assume that p satisfies $p'(\rho_*) > 0$ for a given constant $\rho_* > 0$, and that $g = {}^\top(g^1(x), \dots, g^n(x))$ is Q_* -periodic in x' .

We consider (1.1)-(1.2) under the no-slip boundary condition on $\{x_n = \omega_{*,j}(x')\}$ ($j = 1, 2$) for v :

$$v|_{x_n=\omega_{*,j}(x')} = 0 \quad (x' \in \mathbb{R}^{n-1}, j = 1, 2). \quad (1.4)$$

One can see that if g is sufficiently small, the system (1.1)-(1.2) with (1.4) has a Q_* -periodic stationary solution $\bar{u}_s = {}^\top(\bar{\rho}_s, \bar{v}_s)$, whose components $\bar{\rho}_s$ and \bar{v}_s are in general non-uniform in x' and x_n .

The purpose of this paper is to study the large time behavior of solutions around \bar{u}_s . As a first step of the analysis, we here investigate spectral properties of the linearized semigroup around $\bar{u}_s = {}^\top(\bar{\rho}_s, \bar{v}_s)$.

Large time behavior of solutions of compressible Navier-Stokes equations in unbounded domains exhibit interesting phenomenon, and asymptotic behavior of solutions in large time have been obtained in detail. See, e.g., [7, 9, 14, 17, 18, 19, 20, 21, 22, 24] for the cases of multi-dimensional whole space, half space and exterior domains. In addition to these domains, the stability of flows in infinite layers and cylindrical domains has been paid much attention, e.g., the stability of parallel flows and periodic (in space or time) flow patterns, and so on, whose mathematical analysis is still under development in the case of compressible flows. A difficulty in the mathematical analysis appears due to the non-uniform velocity field of parallel flows and periodic flow patterns, which makes the hyperbolic aspect of the equations (1.1)-(1.2) stronger, and thus, the stability analysis is getting more difficult

compared with that of the motionless state (i.e., the flow with zero velocity field). As for the stability of stationary flow with non-zero velocity field on the whole space, we note that an interesting analysis was given by Shibata and Tanaka [24] to establish the decay rate of perturbations.

Concerning the large time behavior of solutions in the case of an n dimensional flat layer $\{x = (x', x_n); x' \in \mathbb{R}^{n-1}, 0 < x_n < 1\}$, (1.1)-(1.4) has a simple stationary solution $\bar{u}_s = {}^\top(\bar{\rho}_s(x_n), \bar{v}_s(x_n))$, called parallel flow, when g takes the form $g = {}^\top(g'(x_n), 0, \dots, 0, g^n(x_n))$. It was proved in [12, 13] that parallel flow is asymptotically stable for small initial perturbation if the Reynolds and Mach numbers are sufficiently small. (See also [5, 6] for time-periodic parallel flows.) Furthermore, the asymptotic behavior of the perturbation is described by an $n - 1$ dimensional linear heat equation in the case $n \geq 3$, and by a one-dimensional viscous Burgers equation in the case $n = 2$. The proof is based on the energy method and the spectral analysis of the linearized semigroup which is investigated by using the Fourier transform in x' variable. Similar results hold for the parallel flow in the case of cylindrical domain [1, 2, 3].

The analysis in [2, 6, 12] heavily depends on the structure of parallel flow that is uniform in x' variable. Due to this structure, the Fourier transform in x' is useful to investigate the stability of parallel flow, and one of the points in the analysis in [2, 6, 12] is that the *low frequency part* of the linearized operator can be regarded as a perturbation of the linearized operator around the motionless state which is sectorial. As for the Q_* -periodic stationary solution $\bar{u}_s = {}^\top(\bar{\rho}_s, \bar{v}_s)$ under consideration, the argument in [2, 6, 12] is formally applicable if one uses the Bloch transform instead of the Fourier transform. However, since the velocity field \bar{v}_s is in general non-uniform in both x' and x_n , due to the term $\bar{v}_s \cdot \nabla \phi$ in the linearized equation of (1.1), the *low frequency part* of the linearized operator is no longer regarded as a simple perturbation of the linearized one around the motionless state. Therefore, the argument in [2, 6, 12] cannot be directly applied to the problem under consideration. To overcome this difficulty we will make use of the solvability result on the stationary transport equation given by Heywood and Padula [10], which makes the formal argument a rigorous one.

Our main results of this paper is summarized as follows. After introducing suitable non-dimensional variables, the equations for the perturbation $u = {}^\top(\phi, w) = {}^\top(\gamma^2(\rho - \rho_s), v - v_s)$ takes the following form:

$$\partial_t \phi + \operatorname{div}(\phi v_s) + \gamma^2 \operatorname{div}(\rho_s w) = f^0, \quad (1.5)$$

$$\begin{aligned} \partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w + \nabla \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) \\ + \frac{1}{\gamma^2 \rho_s^2} (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) \phi + v_s \cdot \nabla w + w \cdot \nabla v_s = \tilde{f}, \end{aligned} \quad (1.6)$$

$$w|_{\partial\Omega} = 0, \quad (1.7)$$

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \quad (1.8)$$

Here $u_s = {}^\top(\rho_s, v_s)$ denotes the non-dimensionalization of $\bar{u}_s = {}^\top(\bar{\rho}_s, \bar{v}_s)$, and ν , $\tilde{\nu}$ and γ are non-dimensional parameters. The Reynolds number Re and Mach number Ma are given by $\operatorname{Re} = \frac{1}{\nu}$ and $\operatorname{Ma} = \frac{1}{\gamma}$, respectively. The terms f_0 and \tilde{f} on the right of (1.5) and (1.6) are nonlinear terms given by

$$f_0 = -\operatorname{div}(\phi w),$$

$$\begin{aligned} \tilde{f} = & -w \cdot \nabla w - \frac{\phi}{\rho_s(\gamma^2 \rho_s + \phi)} \left(\nu \Delta w + \tilde{\nu} \nabla \operatorname{div} w - \frac{\nu \phi}{\gamma^2 \rho_s} \Delta v_s - \frac{\tilde{\nu} \phi}{\gamma^2 \rho_s} \nabla \operatorname{div} v_s \right) \\ & + \frac{\phi}{\gamma^2 \rho_s^2} \nabla \left(P^{(1)} \left(\rho_s, \frac{\phi}{\gamma^2} \right) \frac{\phi}{\gamma^2} \right) + \frac{\phi^2}{\gamma^2 \rho_s^2 (\gamma^2 \rho_s + \phi)} \nabla \left(P \left(\rho_s + \frac{\phi}{\gamma^2} \right) \right) \\ & + \frac{1}{\rho_s} \nabla \left(P^{(2)} \left(\rho_s, \frac{\phi}{\gamma^2} \right) \frac{\phi^2}{\gamma^4} \right) \end{aligned}$$

with

$$P^{(1)}(\rho_s, \phi) = \int_0^1 P'(\rho_s + \theta \phi) d\theta, \quad P^{(2)}(\rho_s, \phi) = \int_0^1 (1 - \theta) P''(\rho_s + \theta \phi) d\theta.$$

The linearized problem for (1.5)-(1.8) is then formulated as

$$\partial_t u + Lu = 0, \quad u|_{t=0} = u_0, \quad (1.9)$$

where L is the operator of the form

$$\begin{aligned} L = & \begin{pmatrix} \operatorname{div}(\cdot v_s) & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla \left(\frac{P'(\rho)}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} \Delta - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ \frac{1}{\gamma^2 \rho_s^2} (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) & v_s \cdot \nabla + {}^\top \nabla v_s \end{pmatrix}. \end{aligned}$$

We consider the operator L as an operator on $L^2(\Omega)$ with domain $D(L) = \{u = {}^\top(\phi, w) \in L^2(\Omega); w \in H_0^1(\Omega), Lu \in L^2(\Omega)\}$. We will prove that there exist positive constants ν_0 and $\tilde{\gamma}_0$ such that if ν and $\frac{\gamma^2}{\nu}$ are sufficiently large, then the linearized semigroup e^{-tL} satisfies

$$\|\Pi e^{-tL} u_0\|_{L^2(\Omega)} \leq C(1+t)^{-\frac{n-1}{4}} \|u_0\|_{L^1(\Omega)}, \quad (1.10)$$

$$\|\Pi e^{-tL}u_0 - [e^{-tH}\sigma_0]u^{(0)}\|_{L^2(\Omega)} \leq Ct^{-\frac{n-1}{4}-\frac{1}{2}}\|u_0\|_{L^1(\Omega)}, \quad (1.11)$$

where Π is a bounded projection on $L^2(\Omega)$ with $\Pi L \subset L\Pi$, and the operator $H : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1})$ with domain $D(H) = H^2(\mathbb{R}^{n-1})$ is given by

$$H\sigma = -\frac{\gamma^2}{\nu} \sum_{j,k=1}^{n-1} a_{jk} \partial_{x_j} \partial_{x_k} \sigma + \sum_{j=1}^{n-1} a_j \partial_{x_j} \sigma \quad (\sigma \in D(H))$$

with $a_j \in \mathbb{R}$ and $a_{jk} \in \mathbb{R}$ satisfying $\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k \geq C|\eta'|^2$ ($\eta' \in \mathbb{R}^{n-1}$) and σ_0 is given by $\sigma_0(x') = \frac{|Q|}{|\Omega_{per}|} \int_{\omega_1(x')}^{\omega_2(x')} \phi_0(x', x_n) dx_n$. Here $Q = \Pi_{j=1}^{n-1} [-\frac{\pi}{\alpha_j}, \frac{\pi}{\alpha_j})$ is the non-dimensional periodic cell and Ω_{per} is the non-dimensional basic period domain defined by

$$\Omega_{per} = \{x = (x', x_n); |x_i| < \frac{\pi}{\alpha_i} \ (i = 1, \dots, n-1), \ \omega_1(x') < x_n < \omega_2(x')\}.$$

Moreover, $u^{(0)} = {}^\top(\phi^{(0)}, w^{(0)})$ is Q -periodic function in x' and it satisfies the following problem on the basic period domain Ω_{per} :

$$\begin{cases} \operatorname{div}(\phi^{(0)}v_s) + \gamma^2 \operatorname{div}(\rho_s w^{(0)}) = 0 & \text{in } \Omega_{per}, \\ \nabla \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} \right) + \frac{1}{\gamma^2 \rho_s^2} (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) \phi^{(0)} \\ \quad - \frac{\nu}{\rho_s} \Delta w^{(0)} - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w^{(0)} + v_s \cdot \nabla w^{(0)} + w^{(0)} \cdot \nabla v_s = 0 & \text{in } \Omega_{per}, \\ w^{(0)}|_{\{x_n=\omega_j(x')\}} = 0 \ (j = 1, 2), \\ \int_{\Omega_{per}} \phi^{(0)} = 1. \end{cases}$$

Concerning the $(I - \Pi)$ -part, as in [1, 2, 15], one can prove the exponential decay estimate:

$$\|e^{-tL}(I - \Pi)u_0\|_{H^1(\Omega)} \leq C\{e^{-at}\|u_0\|_{H^1 \times \tilde{H}^1} + t^{-\frac{1}{2}}\|w_0\|_{L^2}\} \quad (1.12)$$

for $u_0 = {}^\top(\phi_0, w_0) \in H^1(\Omega) \times \tilde{H}^1(\Omega)$ with some positive number a . Here $\tilde{H}^1(\Omega)$ denotes the set of all locally H^1 functions in Ω whose tangential derivatives near $\partial\Omega$ belong to $L^2(\Omega)$. See Theorem 3.3 for the details.

The proof of the main results are based on the method of the Bloch transformation as in [15] where the stability of the motionless state on a periodic layer was studied. By using the Bloch transformation one can reduce the linearized problem (1.9) on Ω to the problem $\partial_t u + L_{\eta'} u = 0$ on Ω_{per} under Q -periodic boundary conditions in x' . Here $L_{\eta'}$ is the linear operator obtained by replacing the partial derivatives ∂_{x_j} ($j = 1, \dots, n-1$) in L by $\partial_{x_j} + i\eta_j$ with parameter $\eta' = (\eta_1, \dots, \eta_{n-1}) \in Q'$, where Q' is the dual

cell defined by $Q' = \Pi_{j=1}^{n-1}[-\frac{\alpha_j}{2}, \frac{\alpha_j}{2}]$. In the case $|\eta'| \ll 1$ the operator $L_{\eta'}$ can be regarded as a perturbation of L_0 . By using the solvability results on stationary transport equations by Heywood-Padula [10], we investigate the eigenspace for the eigenvalue 0 of L_0 , which enables us to apply the analytic perturbation theory to $L_{\eta'}$ with $|\eta'| \ll 1$. We show that

$$\rho(-L_{\eta'}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > -\beta_0, |\lambda| \geq \frac{\beta_0}{2}\},$$

$$\sigma(-L_{\eta'}) \cap \{\lambda \in \mathbb{C}; |\lambda| < \frac{\beta_0}{2}\} = \{\lambda_{\eta'}\}$$

for some $\beta_0 > 0$, where $\lambda_{\eta'} = i \sum_{j=1}^{n-1} a_j \eta_j - \frac{\gamma^2}{\nu} \sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k + O(|\eta'|^3)$ as $\eta' \rightarrow 0$. It then follows that this part of e^{-tL} behaves as in (1.10) and (1.11). As for the remaining part of η' , one can establish the exponential decay estimate (1.12) as in the arguments given in [1, 2, 15].

This paper is organized as follows. In section 2 we introduce notations, function spaces and operators. In section 3 we state the main results of this paper. Section 4 is devoted to the proof of (1.10) and (1.11). In section 4.1 we formulate the linearized problem and reduce it to the ones on the basic period domain Ω_{per} by the Bloch transform. In section 4.2 we investigate the null space of L_0 . The spectrum of $L_{\eta'}$ with $|\eta'| \ll 1$ is then investigated in section 4.3. Section 4.4 is devoted to estimating the eigenprojection, and in section 4.5 the estimates (1.10) and (1.11) are established. In section 5 we give resolvent estimates for the principal part of L_0 by the Matsumura-Nishida energy method [22] and the result of Heywood-Padula [10].

2 Preliminaries

In this section we introduce notation used in this paper.

Let D be a domain. $L^p(D)$ ($1 \leq p \leq \infty$) denotes Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Similarly, $H^m(D)$ denotes the m -th order L^2 -Sobolev space on D and its norm is denoted by $\|\cdot\|_{H^m(D)}$. $C_0^m(D)$ stands for the set of C^m -functions having compact support on D . Furthermore, we denote the completion of $C_0^\infty(D)$ in $H^m(D)$ by $H_0^m(D)$, and the dual space of $H_0^m(D)$ is denoted by $H^{-m}(D)$. $\mathcal{S}(\mathbb{R}^k)$ denotes the Schwartz space on \mathbb{R}^k .

We simply write the set of all vector fields $w = {}^\top(w^1, \dots, w^n)$ on D with $w^j \in L^p(D)$ (resp. $H^m(D)$) ($j = 1, \dots, n$) as $L^p(D)$ (resp. $H^m(D)$), and its norm is denoted by $\|\cdot\|_{L^p(D)}$ (resp. $\|\cdot\|_{H^m(D)}$). We define $\|\cdot\|_{H^k(D) \times H^m(D)}$ by $\|u\|_{H^k(D) \times H^m(D)} = (\|\phi\|_{H^k(D)}^2 + \|w\|_{H^m(D)}^2)^{\frac{1}{2}}$ for $u = {}^\top(\phi, w)$ satisfying $\phi \in$

$H^k(D)$ and $w \in H^m(D)$. In particular, we write $\|u\|_{H^k(D) \times H^k(D)} = \|u\|_{H^k(D)}$ in the case $k = m$.

We next rewrite the problem (1.1)-(1.2) with (1.4) into the one in a non-dimensional form. We introduce the following non-dimensional variables:

$$x = l\tilde{x}, \quad t = \frac{l}{V}\tilde{t}, \quad v = V\tilde{v}, \quad \rho = \rho_*\tilde{\rho}, \quad p = \rho_*V^2P, \quad g = \frac{V^2}{l}\tilde{g}.$$

Here

$$l = |\Omega_{*,per}|^{\frac{1}{n}}, \quad V = \sqrt{l[g]_{H^2(\Omega_{*,per})}}, \quad [g]_{H^2(\Omega_{*,per})} = \sqrt{\sum_{s=0}^m l^{-n+2s} \|\partial_{\tilde{x}}^s g\|_{L^2(\Omega_{*,per})}^2}.$$

The problem (1.1)-(1.2) with (1.4) is transformed into the following non-dimensional problem in $\tilde{\Omega}$:

$$\partial_{\tilde{t}}\tilde{\rho} + \operatorname{div}_{\tilde{x}}(\tilde{\rho}\tilde{v}) = 0, \quad (2.1)$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v} + (\tilde{v} \cdot \nabla_{\tilde{x}})\tilde{v}) - \nu\Delta_{\tilde{x}}\tilde{v} - \tilde{\nu}\nabla_{\tilde{x}}\operatorname{div}_{\tilde{x}}\tilde{v} + \nabla_{\tilde{x}}P(\tilde{\rho}) = \tilde{\rho}\tilde{g}, \quad (2.2)$$

$$\tilde{v}|_{x_n=\omega_j(x')} = 0 \quad (j = 1, 2). \quad (2.3)$$

Here Ω is the non-dimensional periodic layer:

$$\Omega = \{x = (x', x_n); x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \omega_1(x') < x_n < \omega_2(x')\}$$

with ω_1 and ω_2 being smooth Q -periodic functions of x' with the period cell

$$Q = \Pi_{j=1}^{n-1}[-\frac{\pi}{\alpha_j}, \frac{\pi}{\alpha_j}),$$

where $\alpha_j = l\alpha_{*,j}$; and ν , $\tilde{\nu}$ and γ are non-dimensional parameters defined by

$$\nu = \frac{\mu}{\rho_*lV}, \quad \tilde{\nu} = \nu + \nu', \quad \nu' = \frac{\mu'}{\rho_*lV}, \quad \gamma = \frac{\sqrt{p'(\rho_*)}}{V}.$$

The assumption (1.3) is written as

$$\frac{\nu'}{\nu} \leq \nu_* \quad (2.4)$$

for some positive number ν_* . Note that under (2.4) we have

$$\nu \leq \nu + \tilde{\nu} \leq (\nu_* + 2)\nu. \quad (2.5)$$

In what follows, we omit tildes \sim of $\tilde{\rho}$, \tilde{v} , \tilde{x} , \tilde{t} , \tilde{p} and \tilde{g} for simplicity in notation.

We next define Q' , $\Sigma_{j,\pm}(j = 1, \dots, n-1)$ and Σ_n by

$$Q' := \Pi_{i=1}^{n-1} \left[-\frac{\alpha_i}{2}, \frac{\alpha_i}{2} \right), \quad \Sigma_{j,\pm} := \{x \in \bar{\Omega}; x_j = \pm \frac{\pi}{\alpha_j}\} \quad (j = 1, \dots, n-1)$$

and

$$\Sigma_n := \{x \in \partial\Omega; x' \in Q, x_n = \omega_j(x'), j = 1, 2\}.$$

The inner product of $L^2(\Omega_{per})$ is defined by

$$(f, g) = \int_{\Omega_{per}} f(x) \overline{g(x)} dx$$

for $f, g \in L^2(\Omega_{per})$. Furthermore the mean value of $f \in L^2(\Omega_{per})$ over Ω_{per} is represented as $\langle \cdot \rangle$, i.e.,

$$\langle f \rangle = \int_{\Omega_{per}} f(x) dx.$$

Note that $|\Omega_{per}| = 1$ due to the non-dimensionalization.

We denote the $k \times k$ identity matrix by I_k . We define $(n+1) \times (n+1)$ diagonal matrices Q_0 and \tilde{Q} is defined by

$$Q_0 = \text{diag}(1, 0, \dots, 0), \quad \tilde{Q} = \text{diag}(0, 1, \dots, 1).$$

Note that $Q_0 u = {}^\top(\phi, 0)$ and $\tilde{Q} u = {}^\top(0, w)$ for $u = {}^\top(\phi, w)$.

The Fourier transform of $f = f(x')$ ($x' \in \mathbb{R}^{n-1}$) is denoted by \hat{f} or $\mathcal{F}[f]$:

$$\hat{f}(\xi') = \mathcal{F}[f](\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} dx', \quad \xi' \in \mathbb{R}^{n-1}.$$

We denote the inverse Fourier transform by \mathcal{F}^{-1} :

$$\mathcal{F}^{-1}[f](x') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} d\xi', \quad x' \in \mathbb{R}^{n-1}.$$

For an operator A , we denote by $\text{Ker}(A)$ and $\text{R}(A)$ the null space and the range of A , respectively. The resolvent set and the spectrum of A are denoted by $\rho(A)$ and $\sigma(A)$, respectively. We denote the commutator of operators A and B by $[A, B]$, i.e., $[A, B]f = A(Bf) - B(Af)$.

We introduce function spaces of periodic functions as follows:

$$\begin{aligned} L_{per}^2(\Omega_{per}) &:= \{u|_{\Omega_{per}}; u \in L_{loc}^2(\bar{\Omega}), u(x', x_n) \text{ is } Q\text{-periodic in } x'\}, \\ C_{per}^\infty(\bar{\Omega}_{per}) &:= \{u|_{\Omega_{per}}; u \in C^\infty(\bar{\Omega}), u(x', x_n) \text{ is } Q\text{-periodic in } x'\}, \end{aligned}$$

$$\begin{aligned}
C_{0,per}^\infty(\Omega_{per}) &:= \{u \in C_{per}^\infty(\overline{\Omega_{per}}); u = 0 \text{ in a neighborhood of } \partial\Omega\}, \\
H_{per}^l(\Omega_{per}) &:= \text{the closure of } C_{per}^\infty(\overline{\Omega_{per}}) \text{ in } H^l(\Omega_{per}), \\
H_{0,per}^l(\Omega_{per}) &:= \text{the closure of } C_{0,per}^\infty(\overline{\Omega_{per}}) \text{ in } H^l(\Omega_{per}).
\end{aligned}$$

We note that $L_{per}^2(\Omega_{per})$ can be identified with $L^2(\Omega_{per})$, and that $H_{per}^l(\Omega_{per})$ and $H_{0,per}^1(\Omega_{per})$ are given by

$$\begin{aligned}
H_{per}^l(\Omega_{per}) &= \{u \in H^l(\Omega_{per}); \partial_{x'}^\beta u|_{\Sigma_{j,-}} = \partial_{x'}^\beta u|_{\Sigma_{j,+}}, 1 \leq j \leq n-1, |\beta| \leq l-1\}, \\
H_{0,per}^1(\Omega_{per}) &= \{u \in H_{per}^1(\Omega_{per}); u|_{\Sigma_n} = 0\}.
\end{aligned}$$

Moreover, we define $L_{per,*}^2(\Omega_{per})$ by

$$L_{per,*}^2(\Omega_{per}) := \{f \in L_{per}^2(\Omega_{per}); \langle f \rangle = 0\},$$

and $H_{per,*}^l(\Omega_{per})$ by $H_{per,*}^l(\Omega_{per}) = H_{per}^l(\Omega_{per}) \cap L_{per,*}^2(\Omega_{per})$. We will abbreviate $L_{per}^2(\Omega_{per})$ to L_{per}^2 , and likewise, $H_{per}^l(\Omega_{per})$ to H_{per}^l , \dots , and so on. The norm $\|\cdot\|_{L^p(\Omega_{per})}$ is written as $\|\cdot\|_p$, and similarly, $\|\cdot\|_{H^l(\Omega_{per})}$ as $\|\cdot\|_{H^l}$.

We next introduce the Bloch transformation.

Definition 2.1. We define the operator T by $(T\varphi)(x', \eta')$ ($x' \in \mathbb{R}^{n-1}, \eta' \in \mathbb{R}^{n-1}$) for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$, where

$$\begin{aligned}
(T\varphi)(x', \eta') &= \frac{1}{(2\pi)^{\frac{n-1}{2}}|Q|^{\frac{1}{2}}} \sum_{(k_1, \dots, k_{n-1}) \in \mathbb{Z}^{n-1}} \hat{\varphi}(\eta' + \sum_{j=1}^{n-1} k_j \alpha_j \mathbf{e}'_j) e^{i \sum_{j=1}^{n-1} k_j \alpha_j x_j} \\
&= \frac{1}{|Q'|^{\frac{1}{2}}} \sum_{(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1}} \varphi(x' + \sum_{j=1}^{n-1} l_j \frac{2\pi}{\alpha_j} \mathbf{e}'_j) e^{-i \eta' \cdot (x' + \sum_{j=1}^{n-1} l_j \frac{2\pi}{\alpha_j} \mathbf{e}'_j)}.
\end{aligned}$$

We also define the operator U as follows. For a function $\varphi(x', \eta') \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ such that $\varphi(x', \eta')$ is Q -periodic in x' and $\varphi(x', \eta')e^{i\eta' \cdot x'}$ is Q' -periodic in η' , we define $(U\varphi)(x')$ ($x' \in \mathbb{R}^{n-1}$) by

$$(U\varphi)(x') = \frac{1}{|Q'|^{\frac{1}{2}}} \int_{Q'} \varphi(x', \eta') e^{ix' \cdot \eta'} d\eta'.$$

Note that $\varphi(x', \eta' + \alpha_j \mathbf{e}'_j) = \varphi(x', \eta') e^{-i\alpha_j \mathbf{e}'_j \cdot x'}$ ($j = 1, \dots, n-1$).

The operators T and U have the following properties. See, e.g., [23] for the details.

Proposition 2.2. (i) $(T\varphi)(x', \eta')$ is Q -periodic in x' and $(T\varphi)(x', \eta')e^{i\eta' \cdot x'}$ is Q' -periodic in η' .

(ii) T is uniquely extended to an isometric operator from $L^2(\mathbb{R}^{n-1})$ to $L^2(Q'; L^2(Q))$.

(iii) U is the inverse operator of T .

(iv) Let ψ be Q -periodic in x' . Then it holds that $T(\psi\varphi) = \psi T\varphi$.

(v) $T(\partial_{x_j}\varphi) = (\partial_{x_j} + i\eta_j)T\varphi$ ($j = 1, \dots, n-1$) and T defines an isomorphism from $H^l(\mathbb{R}^{n-1})$ to $L^2(Q'; H_{per}^l(Q))$. Here $H_{per}^l(Q)$ denotes the space of Q -periodic functions in $H^l(Q)$, as in the case of $H_{per}^l(\Omega_{per})$.

We introduce the following weighted inner product:

$$\langle u_1, u_2 \rangle = \int_{\Omega_{per}} \phi_1 \phi_2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} dx + \int_{\Omega_{per}} w_1 w_2 \rho_s dx$$

for $u_j = {}^\top(\phi_j, w_j)$ ($j = 1, 2$), where $\rho_s = \rho_s(x)$ is the density of the stationary solution. See Proposition 3.1 below. The associated norm $||| \cdot |||_2$ is defined by

$$|||u|||_2^2 \equiv \left(\left\| \sqrt{\frac{P'(\rho_s)}{\gamma^4 \rho_s}} \phi \right\|_2^2 + \|\sqrt{\rho_s} w\|_2^2 \right).$$

As we will see in Proposition 3.1, $\rho_s(x)$ and $\frac{P'(\rho_s(x))}{\rho_s(x)}$ are strictly positive in Ω_{per} if ν and $\frac{\gamma^2}{\nu}$ are suitably large, and $||| \cdot |||_2$ is equivalent to $\| \cdot \|_2$.

We introduce the Bogovskii lemma [4]. (See also [8, Chapter 3].)

Lemma 2.3. *There exists a bounded linear operator $\mathcal{B} : L_{per,*}^2 \rightarrow H_{per,0}^1$ such that*

$$\operatorname{div} \mathcal{B}g = g, \quad g \in L_{per,*}^2$$

$$\|\nabla \mathcal{B}g\|_2 \leq C\|g\|_2,$$

where C is a positive constant depending only on Ω_{per} . Furthermore, if $g = \operatorname{div} \mathbf{g}$ with $\mathbf{g} = {}^\top(g^1, \dots, g^n)$ satisfying $g^j|_{\Sigma_{j,-}} = g^j|_{\Sigma_{j,+}}$ ($j = 1, \dots, n-1$), $g^n|_{\Sigma_n} = 0$, then

$$\operatorname{div} \mathcal{B}(\operatorname{div} \mathbf{g}) = \operatorname{div} \mathbf{g},$$

$$\|\mathcal{B}(\operatorname{div} \mathbf{g})\|_2 \leq C\|\mathbf{g}\|_2.$$

For a given $\delta > 0$, we introduce the following notation $((\cdot, \cdot))$ and $\langle\langle \cdot, \cdot \rangle\rangle$:

$$((u_1, u_2)) := \frac{1}{\gamma^2}(\phi_1, \phi_2) + (w_1, w_2) - \delta\{(w_1, \mathcal{B}\phi_2) + (\mathcal{B}\phi_1, w_2)\},$$

$$\langle\langle u_1, u_2 \rangle\rangle := \langle u_1, u_2 \rangle - \delta\{(w_1, \mathcal{B}\phi_2) + (\mathcal{B}\phi_1, w_2)\}$$

for $u_j = {}^\top(\phi_j, w_j)$ ($j = 1, 2$). There exists a constant $c_0 > 0$ such that if $0 \leq \delta \leq \frac{c_0}{\gamma}$, then $((\cdot, \cdot))$ and $\langle\langle \cdot, \cdot \rangle\rangle$ define inner products on $L_{per,*}^2 \times L_{per}^2$ satisfying

$$\frac{1}{2}\|u\|_2^2 \leq ((u, u)) \leq \frac{3}{2}\|u\|_2^2$$

and

$$\frac{1}{2}|||u|||_2^2 \leq \langle\langle u, u \rangle\rangle \leq \frac{3}{2}|||u|||_2^2.$$

3 Main theorem

In this section we state the main theorem of this paper. We first mention the existence of stationary solution to the compressible Navier-Stokes equation. Let us consider the following stationary problem:

$$\operatorname{div}(\rho v) = 0, \quad (3.1)$$

$$\rho(v \cdot \nabla)v - \nu \Delta v - \tilde{\nu} \nabla \operatorname{div} v + \nabla P(\rho) = \rho g, \quad (3.2)$$

$$v|_{\Sigma_n} = 0, \quad \rho|_{\Sigma_{j,-}} = \rho|_{\Sigma_{j,+}}, \quad v|_{\Sigma_{j,-}} = v|_{\Sigma_{j,+}} \quad (j = 1, \dots, n-1), \quad (3.3)$$

$$\langle \rho \rangle = 1. \quad (3.4)$$

Then it holds the following assertion.

Proposition 3.1. *Let ν and $\tilde{\nu}$ satisfy (2.4). Then there exist constants $\nu_0 > 0$ and $\tilde{\gamma}_0 > 0$ such that if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, there exists a solution $u_s = {}^\top(\rho_s, v_s) = {}^\top(1 + \phi_s, v_s) \in H_{per}^3 \times (H_{per}^4 \cap H_{0,per}^1)$ to problem (3.1)-(3.4), and u_s satisfies*

$$\frac{\gamma^4}{\nu} \|\phi_s\|_{H^3}^2 + \nu \|v_s\|_{H^4}^2 \leq \frac{C_0}{\nu} \quad (3.5)$$

for some constant $C_0 > 0$ depending only on ν_* and Ω_{per} . Furthermore, solution of (3.1)-(3.4) satisfying (3.5) is unique.

The unique existence of stationary solution u_s in Proposition 3.1 was proved by Valli [25]. Estimate (3.5) can be obtained by looking carefully at the computations given in [25]. One can also prove Proposition 3.1 by applying Theorem 5.1 together with estimate (5.6)_k with $k = 3$.

In this paper we consider the linearized stability of the stationary solution $u_s = {}^\top(\rho_s, v_s)$ obtained in Proposition 3.1.

The linearized equation around the stationary solution u_s is written as

$$\partial_t u + Lu = 0 \quad (3.6)$$

for $u = {}^\top(\phi, w) \in D(L)$, where $L : L^2(\Omega) \rightarrow L^2(\Omega)$ is the operator defined by

$$L = \begin{pmatrix} \operatorname{div}(\cdot v_s) & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla \left(\frac{P'(\rho)}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} \Delta - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{\gamma^2 \rho_s^2} (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) & v_s \cdot \nabla + {}^\top(\nabla v_s) \end{pmatrix}$$

with domain

$$D(L) = \{u = {}^\top(\phi, w) \in L^2(\Omega); w \in H_0^1(\Omega), Lu \in L^2(\Omega)\}.$$

Here ∇v_s is $n \times n$ -matrix whose (j, k) component is $\partial_{x_k} v_s^j$.

As in [11], we see that $-L$ generates a C_0 -semigroup e^{-tL} . The semigroup e^{-tL} has the following properties.

Theorem 3.2. *There exist constants $\nu_0 > 0$ and $\tilde{\gamma}_0 > 0$ such that if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, there exists a bounded projection $\Pi : L^2(\Omega) \rightarrow L^2(\Omega)$ satisfying $\Pi L \subset L \Pi$ and $\Pi e^{-tL} = e^{-tL} \Pi$, where $e^{-tL} \Pi$ satisfies*

$$\|e^{-tL} \Pi u_0\|_{L^2(\Omega)} \leq C(1+t)^{-\frac{n-1}{4}} \|u_0\|_{L^1(\Omega)}, \quad (3.7)$$

$$\|e^{-tL} \Pi u_0 - [e^{-tH} \sigma_0] u^{(0)}\|_{L^2(\Omega)} \leq C t^{-\frac{n-1}{4} - \frac{1}{2}} \|u_0\|_{L^1(\Omega)} \quad (3.8)$$

for $u_0 \in L^1(\Omega) \cap L^2(\Omega)$, where $H : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1})$ is the linear operator defined by

$$D(H) = H^2(\mathbb{R}^{n-1}),$$

$$H\sigma = - \sum_{j,k=1}^{n-1} a_{jk} \partial_{x_j} \partial_{x_k} \sigma + \sum_{j=1}^{n-1} a_j \partial_{x_j} \sigma \quad (\sigma \in D(H)),$$

and $\sigma_0(x')$ is given by $\sigma_0(x') = \frac{|Q|}{|\Omega_{per}|} \int_{\omega_1(x')}^{\omega_2(x')} \phi_0(x', x_n) dx_n$. Here a_j, a_{jk} ($j, k = 1, \dots, n-1$) are real constants and the matrix $(a_{jk})_{j,k=1}^{n-1}$ satisfies $\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k \geq \frac{\kappa_0 \gamma^2}{\nu} |\eta'|^2$ for all $\eta' \in \mathbb{R}^{n-1}$, where κ_0 is a constant independent of $\nu, \tilde{\nu}$ and γ . Moreover, $u^{(0)} = {}^\top(\phi^{(0)}(x', x_n), w^{(0)}(x', x_n))$ is Q -periodic function in x' that satisfies

$$\begin{cases} \operatorname{div}(\phi^{(0)} v_s) + \gamma^2 \operatorname{div}(\rho_s w^{(0)}) = 0, & \text{in } \Omega_{per} \\ \nabla \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} \right) + \frac{1}{\gamma^2 \rho_s^2} (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) \phi^{(0)} \\ \quad - \frac{\nu}{\rho_s} \Delta w^{(0)} - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w^{(0)} + v_s \cdot \nabla w^{(0)} + w^{(0)} \cdot \nabla v_s = 0, & \text{in } \Omega_{per} \\ w^{(0)}|_{\Sigma_n} = 0, \\ \int_{\Omega_{per}} \phi^{(0)} = 1. \end{cases}$$

As for the $(I - \Pi)$ -part, we can prove the following exponential decay estimates in a similar manner to [1, 2, 15].

We denote by $\tilde{H}^1(\Omega)$ the set of all locally H^1 functions in Ω whose tangential derivatives near $\partial\Omega$ belong to $L^2(\Omega)$.

Theorem 3.3. (i) Let $u_0 = {}^\top(\phi_0, w_0) \in H^1(\Omega) \times \tilde{H}^1(\Omega)$. Suppose that ν , $\tilde{\nu}$ and γ satisfy the assumption of Theorem 3.2. Then

$$e^{-tL}(I - \Pi)u_0 \in C([0, \infty); H^1(\Omega) \times \tilde{H}^1(\Omega)) \cap C((0, \infty); H^1(\Omega) \times H_0^1(\Omega))$$

and

$$\|e^{-tL}(I - \Pi)u_0\|_{H^1(\Omega)}^2 \leq Ce^{-at} \{ \|u_0\|_{H^1(\Omega) \times \tilde{H}^1(\Omega)}^2 + t^{-\frac{1}{2}} \|w_0\|_{L^2(\Omega)} \} \quad (3.9)$$

for all $t \geq 0$.

(ii) Let $T > 0$. Suppose that $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$ and $f \in L^2(0, T; H^1(\Omega) \times L^2(\Omega))$. Then there exists a unique solution $u(t)$ of

$$\begin{aligned} \partial_t u + Lu &= (I - \Pi)f, & u|_{t=0} &= (I - \Pi)u_0, \\ u &\in C([0, T]; H^1(\Omega) \times H_0^1(\Omega)) \cap L^2(0, T; H^1(\Omega) \times H^2(\Omega)), \\ u(t) &\in (I - \Pi)L^2(\Omega) & \text{for } 0 \leq t \leq T, \end{aligned}$$

and $u(t)$ satisfies

$$\begin{aligned} \|u(t)\|_{H^1(\Omega)}^2 + \int_0^t e^{-a(t-\tau)} \|u(\tau)\|_{H^1(\Omega) \times H^2(\Omega)}^2 d\tau \\ \leq C \left\{ e^{-at} \|u_0\|_{H^1(\Omega)}^2 + \int_0^t e^{-a(t-\tau)} \|f\|_{H^1(\Omega) \times L^2(\Omega)}^2 d\tau \right\} \end{aligned} \quad (3.10)$$

for $t \in [0, T]$, where a and C are positive constants independent of T .

Theorem 3.3 can be proved in a similar way as in [1, 2, 15], so we omit the proof. In the remaining of this paper we give a proof of Theorem 3.2.

4 Proof of Theorem 3.2

In this section we give a proof of Theorem 3.2. We first reduce the problem to the one on Ω_{per} by the Bloch transformation. We then investigate the reduced problems depending on the values of Bloch parameter η' . In subsection 4.2 we consider the spectrum for the case $\eta' = 0$; we then develop a perturbation argument to investigate the spectrum for $|\eta'| \ll 1$ in subsection 4.3. In subsection 4.4 we establish estimates for the eigenprojections for the eigenvalues near the imaginary axis. We finally give a proof of Theorem 3.2.

Throughout this section we assume that ν and $\tilde{\nu}$ satisfy (2.4), and thus, we use relation (2.5).

4.1 Formulation

Let us consider the resolvent problem for (3.6)

$$(\lambda + L)u = f, \quad u = {}^\top(\phi, w) \in D(L). \quad (4.1)$$

Here $f = {}^\top(f_0, \tilde{f})$, $\tilde{f} = {}^\top(f_1, \dots, f_n)$ and $\lambda \in \mathbb{C}$ is a resolvent parameter. As in [15], applying the Bloch transform T to (4.1), we have

$$(\lambda + L_{\eta'})Tu = Tf \quad \text{on } \Omega_{per} \quad (4.2)$$

with the dual parameter $\eta' \in Q'$, where $L_{\eta'}$ is the operator on L_{per}^2 of the form

$$\begin{aligned} L_{\eta'} = & \begin{pmatrix} {}^\top\nabla_{\eta'}(\cdot v_s) & \gamma^2 {}^\top\nabla_{\eta'}(\rho_s \cdot) \\ \nabla_{\eta'}\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot\right) & -\frac{\nu}{\rho_s} \Delta_{\eta'} - \frac{\tilde{\nu}}{\rho_s} \nabla_{\eta'} {}^\top\nabla_{\eta'} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ \frac{1}{\gamma^2 \rho_s^2}(\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) & v_s \cdot \nabla_{\eta'} + {}^\top\nabla v_s \end{pmatrix}. \end{aligned}$$

with domain $D(L_{\eta'})$

$$D(L_{\eta'}) = \{u = {}^\top(\phi, w) \in L_{per}^2; w \in H_{0,per}^1, L_{\eta'}u \in L_{per}^2\}.$$

Here we define $\nabla_{\eta'}$ and $\Delta_{\eta'}$ by

$$\nabla_{\eta'} = \nabla + i\tilde{\eta}', \quad \Delta_{\eta'} = \nabla_{\eta'} \cdot \nabla_{\eta'},$$

where $\tilde{\eta}' = (\eta', 0)$. It is not difficult to see that $D(L_{\eta'}) = D(L_0)$ for all $\eta' \in Q'$ and that $L_{\eta'}$ is a closed operator on L_{per}^2 .

We write $L_{\eta'}$ in the form

$$L_{\eta'} = L_0 + \sum_{j=1}^{n-1} \eta_j L_j^{(1)} + \sum_{j,k=1}^{n-1} \eta_j \eta_k L_{jk}^{(2)},$$

where

$$\begin{aligned} L_0 := & \begin{pmatrix} \operatorname{div}(\cdot v_s) & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot\right) & -\frac{\nu}{\rho_s} \Delta - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ \frac{1}{\gamma^2 \rho_s^2}(\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) & v_s \cdot \nabla + {}^\top\nabla v_s \end{pmatrix}, \\ L_j^{(1)} := & i \begin{pmatrix} v_s^j & \gamma^2 \rho_s {}^\top \mathbf{e}_j \\ \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot\right) \mathbf{e}_j & -\frac{1}{\rho_s} \left(2\nu \mathbf{e}_j \otimes \mathbf{e}_j \partial_{x_j} + \tilde{\nu} \mathbf{e}_j \operatorname{div} - \tilde{\nu} \nabla ({}^\top \mathbf{e}_j)\right) + v_s^j \end{pmatrix}, \end{aligned}$$

$$L_{jk}^{(2)} := \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\rho_s} \left(\nu \delta_{jk} I_n + \tilde{\nu} \mathbf{e}_j^\top \mathbf{e}_k \right) \end{pmatrix}.$$

We also set

$$M_{\eta'} = \sum_{j=1}^n \eta_j L_j^{(1)} + \sum_{j,k=1}^n \eta_j \eta_k L_{jk}^{(2)},$$

i.e.,

$$M_{\eta'} = \begin{pmatrix} i\tilde{\eta}' \cdot v_s & i\gamma^2 \rho_s^\top \tilde{\eta}' \\ i\frac{P'(\rho_s)}{\gamma^2 \rho_s} \tilde{\eta}' & \frac{\nu}{\rho_s} \left(|\eta'|^2 - 2i\tilde{\eta}' \cdot \nabla \right) I_n - i\frac{\tilde{\nu}}{\rho_s} \tilde{\eta}'^\top \nabla_{\eta'} - i\frac{\tilde{\nu}}{\rho_s} \nabla^\top \tilde{\eta}' + (v_s \cdot i\tilde{\eta}') I_n \end{pmatrix}.$$

By (4.2), if $\lambda \in \rho(-L_{\eta'})$ for all $\eta' \in Q^*$, then we have

$$u = U(\lambda + L_{\eta'})^{-1} T f \quad \text{in } \Omega.$$

Therefore, to investigate the resolvent of $-L$, we will consider the problem for $-L_{\eta'}$:

$$\lambda u + L_{\eta'} u = f, \quad u \in D(L_{\eta'}). \quad (4.3)$$

4.2 The null space of L_0

In this subsection we investigate the null space of L_0 . We first show that 0 is an eigenvalue and $\dim \text{Ker}(L_0) = 1$.

Proposition 4.1. *There exist constants $\nu_0 > 0$ and $\tilde{\gamma}_0 > 0$ such that if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, then there exists a unique $u^{(0)} = {}^\top(\phi^{(0)}, w^{(0)}) \in D(L_0)$ satisfying $L_0 u^{(0)} = 0$ and $\langle \phi^{(0)} \rangle = 1$. Furthermore, $u^{(0)} = {}^\top(\phi^{(0)}, w^{(0)})$ satisfies*

$$\|\phi^{(0)} - 1\|_{H^2}^2 + \nu^2 \|w^{(0)}\|_{H^3}^2 \leq \frac{C}{\gamma^4} \quad (4.4)$$

uniformly for ν , $\tilde{\nu}$ and γ with $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$.

Proof. We consider the problem $L_0 u = 0$ with $u = {}^\top(\phi, w) \in D(L_0)$. Suppose that $\phi = 0$. Then taking the inner product of $L_0 u = 0$ with u we have

$$\begin{aligned} \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\text{div} w\|_2^2 &= -\text{Re} \{ (v_s \cdot \nabla w + w \cdot \nabla v_s, \rho_s w) \} \\ &\leq C \|v_s\|_{C^1} \|\rho_s\|_\infty \|\nabla w\|_2^2. \end{aligned}$$

Therefore, if $\nu > C \|v_s\|_{C^1} \|\rho_s\|_\infty$, then $w = 0$, and hence, $u = 0$.

In what follows we assume that $\nu > C\|v_s\|_{C^1}\|\rho_s\|_\infty$. We will show that there exists a function $u = {}^\top(\phi, w)$ with $\phi \neq 0$ satisfying $L_0 u = 0$ if ν and $\frac{\gamma^2}{\nu}$ are sufficiently large. We may assume that $\langle \phi \rangle = 1$, and hence, we consider the problem

$$\begin{cases} L_0 u = 0, & u = {}^\top(\phi, w) \in D(L_0), \\ \langle \phi \rangle = 1. \end{cases} \quad (4.5)$$

To solve (4.5), we introduce the operator $\mathcal{A}[v] : H_{per,*}^2 \times H_{per}^1 \longrightarrow H_{per,*}^2 \times H_{per}^1$ defined by

$$\mathcal{A}[v] = \begin{pmatrix} \operatorname{div}(\cdot v) & \gamma^2 \operatorname{div} \\ \nabla & -\nu \Delta - \tilde{\nu} \nabla \operatorname{div} \end{pmatrix}$$

with domain

$$D(\mathcal{A}[v]) = \{u \in H_{per,*}^2 \times (H_{per}^3 \cap H_{0,per}^1); \mathcal{A}[v]u \in H_{per,*}^2 \times H_{per}^1\}.$$

We set $\phi = 1 + \tilde{\phi}$. Then (4.5) is written as

$$\mathcal{A}[v_s]\tilde{u} + B\tilde{u} = F. \quad (4.6)$$

Here $\tilde{u} = {}^\top(\tilde{\phi}, w) \in D(\mathcal{A}[v_s])$, $Bu = {}^\top((B\tilde{u})_1, (B\tilde{u})_2)$ and $F = {}^\top(F_0, \tilde{F})$, where

$$\begin{aligned} (B\tilde{u})_1 &= -\gamma^2 \operatorname{div}(\phi_s w), \\ (B\tilde{u})_2 &= -\frac{\phi_s}{1 + \phi_s}(\nu \Delta w + \tilde{\nu} \nabla \operatorname{div} w) - \nabla(\tilde{P}^{(1)}(\phi_s)\phi_s\tilde{\phi}) \\ &\quad - \frac{1}{\gamma^2} \left(\frac{1}{1 + \phi_s} \right)^2 (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s)\tilde{\phi} - v_s \cdot \nabla w - w \cdot \nabla v_s, \\ F_0 &= -\operatorname{div} v_s, \\ \tilde{F} &= -\nabla(\tilde{P}^{(1)}(\phi_s)\phi_s) - \frac{1}{\gamma^2} \left(\frac{1}{1 + \phi_s} \right)^2 (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s), \\ \tilde{P}^{(1)}(\phi_s) &= \int_0^1 \tilde{P}'(1 + \theta \phi_s) d\theta, \quad \tilde{P}(\rho) = \frac{P'(\rho)}{\gamma^2 \rho}. \end{aligned}$$

We set

$$\|\tilde{u}\|_{H^2 \times H^3}^* = \|\tilde{\phi}\|_{H^2} + \nu \|w\|_{H^3}$$

for $\tilde{u} = {}^\top(\tilde{\phi}, w) \in H_{per,*}^2 \times H_{per}^3$. It follows from Proposition 3.1 and (5.6)_k with $k = 2$ that if ν and $\frac{\gamma^2}{\nu}$ are sufficiently large, then

$$\|\mathcal{A}[v_s]^{-1}F\|_{H^2 \times H^3}^* \leq C \left\{ \frac{\nu}{\gamma^2} \|f^0\|_{H^2} + \|\tilde{f}\|_{H^1} \right\} \quad (4.7)$$

for $F = {}^\top(f^0, \tilde{f}) \in H_{per,*}^2 \times H_{per}^1$. By Proposition 3.1, we have

$$\begin{aligned} \|(B\tilde{u})_1\|_{H^2} &\leq C\|w\|_{H^3}, \\ \|(B\tilde{u})_2\|_{H^1} &\leq C\left\{\frac{1}{\gamma^2}\|\tilde{\phi}\|_{H^2} + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu}\right)\|w\|_{H^3}\right\} \end{aligned}$$

for $\tilde{u} = {}^\top(\tilde{\phi}, w) \in D(\mathcal{A}[v_s])$. Combining these estimates with (4.7), we see that there are positive numbers ν_0 and $\tilde{\gamma}_0$ such that if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, then $\|\mathcal{A}[v_s]^{-1}B\tilde{u}\|_{H^2 \times H^3}^* \leq \frac{1}{2}\|\tilde{u}\|_{H^2 \times H^3}^*$. Therefore, there exists a unique $\tilde{u} \in D(\mathcal{A}[v_s])$ satisfying $\tilde{u} = -\mathcal{A}[v_s]^{-1}B\tilde{u} + \mathcal{A}[v_s]^{-1}F$, that is, a unique solution $\tilde{u} \in D(\mathcal{A}[v_s])$ of (4.6), and \tilde{u} satisfies the estimate

$$\|\tilde{u}\|_{H^2 \times H^3}^* \leq C\left\{\frac{\nu}{\gamma^2}\|v_s\|_{H^4} + \|\phi_s\|_{H^3}\right\} \leq \frac{C}{\gamma^2}.$$

This completes the proof. \square

We next consider the eigenspace for the eigenvalue 0. We introduce the adjoint operator $L_{\eta'}^* : L_{per}^2 \rightarrow L_{per}^2$ defined by

$$\begin{aligned} L_{\eta'}^* = & \begin{pmatrix} -v_s \cdot \nabla_{\eta'} \left(\frac{P'(\rho_s)}{\rho_s} \cdot \right) \frac{\rho_s}{P'(\rho_s)} & -\gamma^2 \nabla_{\eta'} \cdot (\rho_s \cdot) \\ -\nabla_{\eta'} \left(\frac{\rho_s}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} \Delta_{\eta'} - \frac{\tilde{\nu}}{\rho_s} \nabla_{\eta'} ({}^\top \nabla_{\eta'}) \end{pmatrix} \\ & + \begin{pmatrix} 0 & \frac{\gamma^2}{P'(\rho_s)} (\nu \Delta^\top v_s + \tilde{\nu}^\top \nabla \operatorname{div} v_s) \\ 0 & -\operatorname{div} v_s - \frac{1}{\rho_s} v_s \cdot \nabla_{\eta'} (\rho_s \cdot) + \nabla_{\eta'} v_s \end{pmatrix} \end{aligned}$$

with domain

$$D(L_{\eta'}^*) = \{u = {}^\top(\phi, w) \in L_{per}^2; w \in H_{0,per}^1, L_{\eta'}^* u \in L_{per}^2\}.$$

Then it holds that $D(L_{\eta'}^*) = D(L_0^*)$ and $\langle L_{\eta'} u_1, u_2 \rangle = \langle u_1, L_{\eta'}^* u_2 \rangle$.

Concerning the eigenspace for eigenvalue 0, we have the following

Lemma 4.2. *Let $\Pi^{(0)}$ be defined by*

$$\Pi^{(0)} u = \langle u, u^{(0)*} \rangle u^{(0)} = \langle \phi \rangle u^{(0)}$$

for $u = {}^\top(\phi, w)$, where

$$u^{(0)*} = \gamma^2 \begin{pmatrix} \frac{\gamma^2 \rho_s}{P'(\rho_s)} \\ 0 \end{pmatrix}.$$

Then the following assertions hold.

(i) $u^{(0)*}$ satisfies $u^{(0)*} \in D(L_0^*)$, $L_0^* u^{(0)*} = 0$ and $\langle u^{(0)}, u^{(0)*} \rangle = 1$.

(ii) $\Pi^{(0)}$ is a bounded projection on L_{per}^2 satisfying

$$\Pi^{(0)} L_0 \subset L_0 \Pi^{(0)} = 0, \quad \Pi^{(0)} u^{(0)} = u^{(0)} \quad \text{and} \quad \Pi^{(0)} L_{per}^2 = \text{span}\{u^{(0)}\}.$$

The assertions of Lemma 4.2 are easily verified, and so we omit the proof. We introduce some notation. In what follows we set

$$X_0 = \Pi^{(0)} L_{per}^2, \quad X_1 = (I - \Pi^{(0)}) L_{per}^2.$$

Observe that $u = {}^\top(\phi, w) \in X_1$ if and only if $\langle \phi \rangle = 0$.

Lemma 4.3. *If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then $\rho(-L_0|_{X_1}) \supset \{\lambda; \text{Re} \lambda \geq -\beta_0\}$ for some positive constant β_0 . Furthermore, if $\text{Re} \lambda \geq -\beta_0$, then*

$$\|((\lambda + L_0)|_{X_1})^{-1} f\|_2 \leq \frac{C}{\text{Re} \lambda + 2\beta_0} \|f\|_2, \quad (4.8)$$

$$\|\nabla \tilde{Q}((\lambda + L_0)|_{X_1})^{-1} f\|_2 \leq C \frac{1}{(\text{Re} \lambda + 2\beta_0)^{\frac{1}{2}}} \|f\|_2. \quad (4.9)$$

Proof. We first establish estimates (4.8) and (4.9). For $f = {}^\top(f^0, \tilde{f}) \in X_1$, we consider

$$(\lambda + L_0)u = f, \quad u = {}^\top(\phi, w) \in D(L_0) \cap X_1. \quad (4.10)$$

Note that $\langle \phi \rangle = \langle f^0 \rangle = 0$. We compute

$$\begin{aligned} \text{Re} \langle (\lambda + L_0)u, u \rangle &= \text{Re} \lambda \langle u, u \rangle + \text{Re} \langle L_0 u, u \rangle \\ &\quad - \delta \text{Re} \{((L_0 u)_2, \mathcal{B}\phi) + (\mathcal{B}(L_0 u)_1, w)\}, \end{aligned}$$

where $(u)_1 = \phi$ and $(u)_2 = w$ for $u = {}^\top(\phi, w)$. Since

$$\begin{aligned} -\text{Re}((L_0 u)_2, \mathcal{B}\phi) &\geq \left\| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \phi \right\|_2^2 - C\{\nu \|\nabla w\|_2 + \tilde{\nu} \|\text{div} w\|_2\} \|\phi\|_2 \\ &\quad - C \frac{\nu}{\gamma^2} \|v_s\|_{H^4} \|\phi\|_2^2 - C \|v_s\|_{H^4} \|\phi\|_2 \|w\|_2, \\ -\text{Re}(\mathcal{B}(L_0 u)_1, w) &\geq -\|\mathcal{B}(\text{div}(\phi v_s + \gamma^2 \rho_s w))\|_2 \|w\|_2 \\ &\geq -C\{\|v_s\|_{H^4} \|\phi\|_2 \|w\|_2 + \gamma^2 \|w\|_2^2\}, \end{aligned}$$

we have

$$\begin{aligned}
\operatorname{Re} \langle (\lambda + L_0)u, u \rangle &\geq \operatorname{Re} \lambda \langle u, u \rangle + \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2 + \delta \left\| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \phi \right\|_2^2 \\
&\quad - C \|v_s\|_{H^4} \left\| \sqrt{\frac{P'(\rho_s)}{\gamma^4 \rho_s}} \phi \right\|_2^2 - \frac{C}{\gamma^2} \|v_s\|_{H^4} \|\phi\|_2^2 \\
&\quad - C \frac{\nu}{\gamma^2} \|v_s\|_{H^4} \|\phi\|_2 \|w\|_2 - C \|v_s\|_{H^4} \|w\|_2^2 \\
&\quad - C \delta \{ (\nu \|\nabla w\|_2 + \tilde{\nu} \|\operatorname{div} w\|_2) \|\phi\|_2 + \frac{\nu}{\gamma^2} \|v_s\|_{H^4} \|\phi\|_2^2 \\
&\quad + \|v_s\|_{H^4} \|\phi\|_2 \|w\|_2 + \gamma^2 \|w\|_2^2 \}.
\end{aligned}$$

Therefore, by using Proposition 3.1, we see that there exist constants $\nu_0 > 0$, $\tilde{\gamma}_0 > 0$ and $C_0 > 0$ such that if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, then it holds with $\delta = C_0 \min\{\frac{1}{\nu}, \frac{\nu}{\gamma^2}, \frac{1}{\gamma}\}$ that

$$\operatorname{Re} \langle (\lambda + L_0)u, u \rangle \geq \operatorname{Re} \lambda \langle u, u \rangle + \frac{\nu}{2} \|\nabla w\|_2^2 + \frac{\tilde{\nu}}{2} \|\operatorname{div} w\|_2^2 + \frac{\delta}{2} \|\phi\|_2^2.$$

Since $\operatorname{Re} \langle f, u \rangle \leq \|f\|_2 \|u\|_2 + \delta \|f\|_2 \|u\|_2 \leq C(1 + \delta) \|f\|_2 \|u\|_2$, we have

$$\operatorname{Re} \lambda \langle u, u \rangle + \frac{\nu}{2} \|\nabla w\|_2^2 + \frac{\tilde{\nu}}{2} \|\operatorname{div} w\|_2^2 + \frac{\delta}{2} \|\phi\|_2^2 \leq C(1 + \delta) \|f\|_2 \|u\|_2.$$

By the Poincaré inequality, we have

$$\operatorname{Re} \lambda \|u\|_2^2 + c_1(\nu \|w\|_2^2 + \delta \|\phi\|_2^2) + \frac{\nu}{4} \|\nabla w\|_2^2 \leq C(1 + \delta) \|f\|_2 \|u\|_2. \quad (4.11)$$

It follows that there exists $\beta_0 > 0$ with $\beta_0 = O(\min\{\nu, \frac{\nu}{\gamma^2}, \frac{1}{\nu}, \frac{1}{\gamma}\})$ such that

$$(\operatorname{Re} \lambda + 2\beta_0) \|u\|_2^2 + \nu \|\nabla w\|_2^2 \leq C \|f\|_2 \|u\|_2.$$

We thus have

$$\|u\|_2 \leq \frac{C}{\operatorname{Re} \lambda + 2\beta_0} \|f\|_2, \quad (4.12)$$

and so if $\operatorname{Re} \lambda \geq -\beta_0$, then

$$\|\nabla w\|_2 \leq \frac{C}{\nu^{\frac{1}{2}} (\operatorname{Re} \lambda + 2\beta_0)^{\frac{1}{2}}} \|f\|_2. \quad (4.13)$$

If we prove the existence of solution $u \in D(L_0) \cap X_1$ of (4.10) for all $f \in X_1$ with $\lambda \gg 1$, then we see from (4.12) and (4.13) that $\rho(-L_0|_{X_1}) \supset$

$\{\lambda; \operatorname{Re} \lambda \geq -\beta_0\}$ by a standard continuation argument. To prove the existence of solution of (4.10), we first consider the problem

$$\lambda \phi + \operatorname{div}(\phi v_s) = F^0, \quad F^0 = f^0 - \gamma^2 \operatorname{div}(\rho_s \tilde{w}) \quad (4.14)$$

for a given $\tilde{w} \in H_{0,per}^1$ and $F^0 \in L_{per,*}^2$. Applying [10, Theorem 1], we see that there exists a constant λ_0 satisfying $\lambda_0 \geq C\|v\|_{H^3}$ such that if $\lambda \geq \lambda_0$, there exists a solution ϕ of (4.14). Furthermore, ϕ satisfies

$$\|\phi\|_2 \leq \frac{C}{\lambda} \|F^0\|_2 \leq \frac{C}{\lambda} \{\|f^0\|_2 + \|\tilde{w}\|_{H^1}\}. \quad (4.15)$$

We denote this solution ϕ by $\phi(\tilde{w})$. We next consider the problem

$$\begin{cases} \lambda w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w = \tilde{F}, \\ w|_{\Sigma_{j,-}} = w|_{\Sigma_{j,+}}, \quad w|_{\Sigma_n} = 0. \end{cases} \quad (4.16)$$

where

$$\tilde{F} = \tilde{f} - \nabla \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) - \frac{1}{\gamma^2 \rho_s^2} (\nu \Delta v_s + \tilde{\nu} \operatorname{div} v_s) - v_s \cdot \nabla \tilde{w} - \tilde{w} \cdot \nabla v_s,$$

with $\phi = \phi(\tilde{w})$. Since $\operatorname{Re}(-\frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w, \rho_s w) = \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2$, there exists a unique solution $w \in H_{0,per}^1$ of (4.16) such that

$$\lambda \|w\|_2^2 + \|\nabla w\|_2^2 \leq C \|\tilde{F}\|_{H^{-1}}^2 \leq C \{\|\tilde{f}\|_2^2 + \frac{1}{\lambda} \|f^0\|_2^2 + \frac{1}{\lambda} \|\nabla \tilde{w}\|_2^2 + \|\tilde{w}\|_2^2\} \quad (4.17)$$

for $\lambda \geq \lambda_0$. We define the operator $\Gamma : H_{0,per}^1 \rightarrow H_{0,per}^1$ by the solution operator for (4.16), i.e., $\Gamma(\tilde{w}) = w$ for $\tilde{w} \in H_{0,per}^1$, with w being the solution of (4.16).

We introduce the norm $\|\cdot\|_{(\lambda)} = (\lambda \|\cdot\|_2^2 + \|\nabla \cdot\|_2^2)^{\frac{1}{2}}$ of $H_{0,per}^1$. By (4.2), Γ satisfies $\|\Gamma(\tilde{w}_1) - \Gamma(\tilde{w}_2)\|_{(\lambda)} \leq \frac{C}{\lambda} \|\tilde{w}_1 - \tilde{w}_2\|_{(\lambda)}$ for $\tilde{w}_1, \tilde{w}_2 \in H_{0,per}^1$. Let $\lambda \geq \lambda_1 = \max\{2C, \lambda_0\}$. It then follows that $\|\Gamma(\tilde{w}_1) - \Gamma(\tilde{w}_2)\|_{(\lambda)} \leq \frac{1}{2} \|\tilde{w}_1 - \tilde{w}_2\|_{(\lambda)}$, and Γ is a contraction mapping on $H_{0,per}^1$. Therefore, there exists a unique $w \in H_{0,per}^1$ satisfying $\Gamma(w) = w$. For this fixed point w , we set $\phi = \phi(w)$. It then follows that $u = {}^\top(\phi, w) \in D(L_0|_{X_1})$ is a solution of (4.10) with $\lambda \geq \lambda_1$. We thus conclude that $\rho(-L_0|_{X_1}) \supset \{\lambda; \operatorname{Re} \lambda > -\beta_0\}$. This completes the proof. \square

Theorem 4.4. *If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then*

- (i) $X_1 = \operatorname{R}(L_0)$ and $X_0 = \operatorname{Ker}(L_0)$,

$$(ii) \quad L_{per}^2 = \text{Ker}(L_0) \oplus \text{R}(L_0),$$

(iii) 0 is a simple eigenvalue of $-L_0$.

Proof. Let $f = {}^\top(f^0, \tilde{f}) \in \text{R}(L_0)$. There exists a function $u \in D(L_0)$ such that $L_0 u = f$. Applying $\Pi^{(0)}$, we have $0 = \Pi^{(0)} L_0 u = \Pi^{(0)} f = \langle f^0 \rangle u^{(0)}$. This implies $\langle f^0 \rangle = 0$, and hence, $f \in X_1$. This shows $\text{R}(L_0) \subset X_1$. Let $f = {}^\top(f^0, \tilde{f}) \in X_1$. By Lemma 4.3, there exists a unique function $u = {}^\top(\phi, w) \in D(L_0)$ satisfying $L_0 u = f$ and $\langle \phi \rangle = 0$. This shows $X_1 \subset \text{R}(L_0)$, and (i) is proved. (ii) follows from (i), and (iii) follows from and Lemma 4.2. This completes the proof. \square

Theorem 4.5. *If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then $\rho(-L_0) \supset \{\lambda; \text{Re} \lambda \geq -\beta_0\} \setminus \{0\}$. Furthermore, if $\text{Re} \lambda \geq -\beta_0$ and $\lambda \neq 0$, then*

$$\begin{aligned} (\lambda + L_0)^{-1} f &= \frac{1}{\lambda} \Pi^{(0)} f + ((\lambda + L_0)|_{X_1})^{-1} \Pi_1 f, \\ \|(\lambda + L_0)^{-1} f\|_2 &\leq C \left(\frac{1}{|\lambda|} + \frac{1}{\text{Re} \lambda + 2\beta_0} \right) \|f\|_2, \\ \|\nabla \tilde{Q}(\lambda + L_0)^{-1} f\|_2 &\leq C \left(\frac{1}{|\lambda|} + \frac{1}{(\text{Re} \lambda + 2\beta_0)^{\frac{1}{2}}} \right) \|f\|_2. \end{aligned}$$

Proof. By Theorem 4.4, problem $(\lambda + L_0)u = f$ is equivalent to

$$\begin{cases} \lambda \Pi^{(0)} u = \Pi^{(0)} f, \\ (\lambda + L_0)u_1 = f_1, \end{cases}$$

where $u_1 = (I - \Pi^{(0)})u \in D(L_0) \cap X_1$ and $f_1 = (I - \Pi^{(0)})f \in X_1$. Applying Lemma 4.3 we see that if $\text{Re} \lambda \geq -\beta_0$ and $\lambda \neq 0$, then

$$(\lambda + L_0)^{-1} f = \frac{1}{\lambda} \Pi^{(0)} f + ((\lambda + L_0)|_{X_1})^{-1} f_1,$$

and

$$\begin{aligned} \|(\lambda + L_0)^{-1} f\|_2 &\leq C \left(\frac{1}{|\lambda|} + \frac{1}{\text{Re} \lambda + 2\beta_0} \right) \|f\|_2, \\ \|\nabla \tilde{Q}(\lambda + L_0)^{-1} f\|_2 &\leq C \left(\frac{1}{|\lambda|} + \frac{1}{(\text{Re} \lambda + 2\beta_0)^{\frac{1}{2}}} \right) \|f\|_2. \end{aligned}$$

This completes the proof. \square

As for the spectrum of $-L_0^*$, we have the following

Proposition 4.6. Define $\Pi^{(0)*}$ by

$$\Pi^{(0)*}u = \langle u, u^{(0)} \rangle u^{(0)*}.$$

If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then the following assertions hold true.

- (i) $\rho(-L_0^*) \supset \{\lambda; \operatorname{Re}\lambda \geq -\beta_0\} \setminus \{0\}$, 0 is a simple eigenvalue of $-L_0^*$,
- (ii) $\Pi^{(0)*}L_0^* \subset L_0^*\Pi^{(0)*} = 0$, $\operatorname{Ker}(L_0^*) = \Pi^{(0)*}L_{per}^2 = \operatorname{span}\{u^{(0)*}\}$,
- (iii) $\operatorname{R}(L_0^*) = (I - \Pi^{(0)*})L_{per}^2$,
- (iv) $L_{per}^2 = \operatorname{Ker}(L_0^*) \oplus \operatorname{R}(L_0^*)$,

Moreover, if λ satisfying $\operatorname{Re}\lambda \geq -\beta_0$ and $\lambda \neq 0$, then

$$\begin{aligned} (\lambda + L_0^*)^{-1}f &= \frac{1}{\lambda}\Pi^{(0)*}f + ((\lambda + L_0^*)|_{(I-\Pi^{(0)*})L_{per}^2})^{-1}(I - \Pi^{(0)*})f, \\ \|(\lambda + L_0^*)^{-1}f\|_2 &\leq C\left(\frac{1}{|\lambda|} + \frac{1}{\operatorname{Re}\lambda + 2\beta_0}\right)\|f\|_2, \end{aligned} \quad (4.18)$$

$$\|\nabla \tilde{Q}(\lambda + L_0^*)^{-1}f\|_2 \leq C\left(\frac{1}{|\lambda|^{\frac{1}{2}}} + \frac{1}{|\lambda|} + \frac{1}{(\operatorname{Re}\lambda + 2\beta_0)^{\frac{1}{2}}} + \frac{1}{\operatorname{Re}\lambda + 2\beta_0}\right)\|f\|_2. \quad (4.19)$$

Proof. By the closed range theorem, we have (i)-(iv). Furthermore, if $\operatorname{Re}\lambda \geq -\beta_0$ and $\lambda \neq 0$, we see from Theorem 4.5 that

$$\begin{aligned} \langle (\lambda + L_0^*)^{-1}f, g \rangle &= \langle f, (\lambda + L_0)^{-1}g \rangle \\ &\leq \|f\|_2 \|(\lambda + L_0)^{-1}g\|_2 \\ &\leq C\left(\frac{1}{|\lambda|} + \frac{1}{\operatorname{Re}\lambda + 2\beta_0}\right)\|f\|_2 \|g\|_2, \end{aligned}$$

and hence,

$$\|(\lambda + L_0^*)^{-1}f\|_2 \leq C\left(\frac{1}{|\lambda|} + \frac{1}{\operatorname{Re}\lambda + 2\beta_0}\right)\|f\|_2. \quad (4.20)$$

Computing $\langle (\lambda + L_0^*)u, u \rangle$, we have

$$\operatorname{Re}\lambda \|u\|_2^2 - c\|u\|_2^2 + \nu\|\nabla w\|_2^2 \leq \|f\|_2 \|u\|_2,$$

where c is a positive constant, and hence, if $\operatorname{Re}\lambda \geq -\beta_0$ and $\lambda \neq 0$, then

$$\nu\|\nabla w\|_2^2 \leq C\left\{\left(\frac{1}{|\lambda|} + \frac{1}{\operatorname{Re}\lambda + 2\beta_0}\right) + \left(\frac{1}{|\lambda|} + \frac{1}{\operatorname{Re}\lambda + 2\beta_0}\right)^2\right\}\|f\|_2^2.$$

This completes the proof. \square

4.3 Perturbation argument

In this subsection we develop a perturbation argument to investigate the spectrum of $-L_{\eta'}$ for $|\eta'| \ll 1$.

Proposition 4.7. *If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then there exists a constant $r_0 > 0$ such that if $|\eta'| \leq r_0$, then*

$$\Sigma_1 := \left\{ \lambda; \operatorname{Re} \lambda > -\frac{\beta_0}{2} \right\} \setminus \left\{ \lambda; |\lambda| \leq \frac{\beta_0}{4} \right\} \subset \rho(-L_{\eta'}).$$

Furthermore,

$$\|(\lambda + L_{\eta'})^{-1} f\|_2 \leq \frac{C}{\operatorname{Re} \lambda + \beta_0} \|f\|_2, \quad (4.21)$$

$$\|\nabla \tilde{Q}(\lambda + L_{\eta'})^{-1} f\|_2 \leq \frac{C}{(\operatorname{Re} \lambda + \beta_0)^{\frac{1}{2}}} \|f\|_2 \quad (4.22)$$

for $\lambda \in \Sigma_1$.

Proof. Recall that $L_{\eta'} = L_0 + M_{\eta'}$. By Theorem 4.5, we have

$$\|(\lambda + L_0)^{-1} f\|_2 + \|\nabla \tilde{Q}(\lambda + L_0)^{-1} f\|_2 \leq C \|f\|_2$$

for $\lambda \in \Sigma_1$, and hence, since $\|M_{\eta'} u\|_2 \leq C|\eta'|(\|u\|_2 + \|\nabla w\|_2)$, it holds that $\|M_{\eta'}(\lambda + L_0)^{-1} f\|_2 \leq 2C|\eta'| \|f\|_2$. Therefore, if η' satisfies $|\eta'| < \frac{1}{2C}$, then $\lambda \in \rho(-L_{\eta'})$ and

$$\begin{aligned} (\lambda + L_{\eta'})^{-1} &= (\lambda + L_0)^{-1} \sum_{N=0}^{\infty} (-1)^N (M_{\eta'}(\lambda + L_0)^{-1})^N, \\ \|(\lambda + L_{\eta'})^{-1} f\|_2 &\leq C' \left(\frac{1}{|\lambda|} + \frac{1}{\operatorname{Re} \lambda + \beta_0} \right) \sum_{N=0}^{\infty} \|M_{\eta'}(\lambda + L_0)^{-1}\|_2^N \|f\|_2 \\ &\leq \frac{C}{\operatorname{Re} \lambda + \beta_0} \|f\|_2, \\ \|\nabla \tilde{Q}(\lambda + L_{\eta'})^{-1} f\|_2 &\leq C \left(\frac{1}{|\lambda|} + \frac{1}{(\operatorname{Re} \lambda + \beta_0)^{\frac{1}{2}}} \right) \sum_{N=0}^{\infty} \|M_{\eta'}(\lambda + L_0)^{-1}\|_2^N \|f\|_2 \\ &\leq \frac{C}{(\operatorname{Re} \lambda + \beta_0)^{\frac{1}{2}}} \|f\|_2. \end{aligned}$$

This completes the proof. \square

We next establish an asymptotic formula for eigenvalues of $-L_{\eta'}$ near the imaginary axis.

Proposition 4.8. *If $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ and $|\eta'| \leq r_0$ (by taking ν_0 , $\tilde{\gamma}_0$ and r_0^{-1} suitably larger if necessary), then*

$$\sigma(-L_{\eta'}) \cap \{|\lambda| < \frac{\beta_0}{2}\} = \{\lambda_{\eta'}\}.$$

Here $\lambda_{\eta'}$ is a simple eigenvalue that satisfies

$$\lambda_{\eta'} = -\kappa(\eta') + O(|\eta'|^3) \quad (4.23)$$

as $\eta' \rightarrow 0$. Here

$$\kappa(\eta') = i \sum_{j=1}^{n-1} a_j \eta_j + \sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k, \quad (4.24)$$

where $a_j \in \mathbb{R}$ and $a_{jk} \in \mathbb{R}$; a_{jk} , $j, k = 1, \dots, n-1$, satisfy

$$\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k \geq \frac{\kappa_0 \gamma^2}{\nu} |\eta'|^2 \quad (\eta' \in \mathbb{R}^{n-1}) \quad (4.25)$$

with some constant $\kappa_0 > 0$ independent of ν , $\tilde{\nu}$ and γ . As a consequence,

$$\operatorname{Re} \lambda_{\eta'} \leq -\frac{\kappa_0 \gamma^2}{2\nu} |\eta'|^2. \quad (4.26)$$

Proof. Since

$$\begin{aligned} \|L_j^{(1)} u\|_2 &\leq C\{\|u\|_2 + \|\nabla w\|_2\} \leq C\{\|L_0 u\|_2 + \|u\|_2\}, \\ \|L_{jk}^{(2)} u\|_2 &\leq C\|w\|_2 \leq C\{\|L_0 u\|_2 + \|u\|_2\}, \end{aligned}$$

in view of Theorem 4.5, we can apply the analytic perturbation theory to see that $\sigma(-L_{\eta'}) \cap \{|\lambda| < \frac{\beta_0}{2}\}$ consist of a simple eigenvalue, say $\lambda_{\eta'}$, for sufficiently small η' , and that $\lambda_{\eta'}$ is expanded as

$$\lambda_{\eta'} = \sum_{j=1}^{n-1} \lambda_j^{(1)} \eta_j + \sum_{j,k=1}^{n-1} \lambda_{jk}^{(2)} \eta_j \eta_k + O(|\eta'|^3)$$

as $\eta' \rightarrow 0$, where

$$\begin{aligned} \lambda_j^{(1)} &= -\langle L_j^{(1)} u^{(0)}, u^{(0)*} \rangle, \\ \lambda_{jk}^{(2)} &= -\frac{1}{2} \langle (L_{jk}^{(2)} + L_{kj}^{(2)}) u^{(0)}, u^{(0)*} \rangle + \frac{1}{2} \langle (L_j^{(1)} S L_k^{(1)} + L_k^{(1)} S L_j^{(1)}) u^{(0)}, u^{(0)*} \rangle. \end{aligned}$$

Here $S = [(I - \Pi^{(0)})L_0(I - \Pi^{(0)})]^{-1}$. See, e.g., [16, Chap. VII], [23, Chap. XII].

By the definition of $L_j^{(1)}$, $u^{(0)}$ and $u^{(0)*}$, we have

$$\lambda_j^{(1)} = -i\langle v_s^j \phi^{(0)} + \gamma^2 \rho_s w^{(0),j} \rangle.$$

As for $\lambda_{jk}^{(2)}$, since $\frac{1}{2}\langle (L_{jk}^{(2)} + L_{kj}^{(2)})u^{(0)}, u^{(0)*} \rangle = 0$, it holds that

$$\lambda_{jk}^{(2)} = \frac{1}{2}\langle (L_j^{(1)}SL_k^{(1)} + L_k^{(1)}SL_j^{(1)})u^{(0)}, u^{(0)*} \rangle$$

We set $u_1^{(k)} = {}^\top(\phi_1^{(k)}, w_1^{(k)}) = SL_k^{(1)}u^{(0)}$. Then $u_1^{(k)}$ is a solution of

$$\begin{cases} L_0 u_1^{(k)} = (I - \Pi^{(0)})L_k^{(1)}u^{(0)}, \\ \langle \phi_1^{(k)} \rangle = 0. \end{cases} \quad (4.27)$$

To prove (4.25), we prepare some estimates of $u_1^{(k)}$.

Lemma 4.9. *If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then*

$$\|\phi_1^{(k)}\|_{H^1} \leq C, \quad \|w_1^{(k)}\|_{H^2} \leq \frac{C}{\nu}. \quad (4.28)$$

Proof. Since $\|(I - \Pi^{(0)})L_k^{(1)}u^{(0)}\|_{L^2 \times H^1} \leq C$, it follows from Proposition 3.1 and (5.6)_k with $k = 1$ that

$$\frac{1}{\nu}\|\phi_1^{(k)}\|_{H^1}^2 + \nu\|w_1^{(k)}\|_{H^2}^2 \leq \frac{C}{\nu}\left\{1 + \left(\frac{1}{\nu^2} + \frac{\nu^2}{\gamma^4}\right)\|w_1^{(k)}\|_{H^2}^2 + \frac{1}{\gamma^4}\|\phi_1^{(k)}\|_{H^1}^2\right\}.$$

Therefore, if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, we have $\|\phi_1^{(k)}\|_{H^1} \leq C$ and $\|w_1^{(k)}\|_{H^2} \leq \frac{C}{\nu}$. This completes the proof. \square

To proceed further, we introduce the functions $U_1^{(k)} = {}^\top(\Phi_1^{(k)}, W_1^{(k)}) \in H_{per,*}^2 \times (H_{per}^3 \cap H_{0,per}^1)$ ($k = 1, \dots, n-1$), which are the unique solution of the following Stokes system

$$A_S U_1^{(k)} = F^{(k)}, \quad \langle \Phi_1^{(k)} \rangle = 0. \quad (4.29)$$

Here

$$A_S = \begin{pmatrix} 0 & \text{div} \\ \nabla & -\Delta \end{pmatrix}, \quad F^{(k)} = \begin{pmatrix} 0 \\ \mathbf{e}_k \end{pmatrix}.$$

Set $\check{\phi}_1^{(k)} = i\Phi_1^{(k)}$ and $\check{w}_1^{(k)} = \frac{i}{\nu}W_1^{(k)}$. Then $\check{u}_1^{(k)} = {}^\top(\check{\phi}_1^{(k)}, \check{w}_1^{(k)})$ is the solution of

$$\mathcal{A}\check{u}_1^{(k)} = iF^{(k)}, \quad \langle \check{\phi}_1^{(k)} \rangle = 0. \quad (4.30)$$

Here

$$\mathcal{A} = \begin{pmatrix} 0 & \gamma^2 \operatorname{div} \\ \nabla & -\nu \Delta - \tilde{\nu} \nabla \operatorname{div} \end{pmatrix}.$$

Furthermore, we have the following lemma.

Lemma 4.10 ([15]). *Let $\check{\kappa}(\eta')$ be defined by*

$$\check{\kappa}(\eta') = \sum_{j,k=1}^{n-1} \check{a}_{jk} \eta_j \eta_k \quad \text{for } \eta' \in \mathbb{R}^{n-1},$$

where

$$\check{a}_{jk} = \frac{\gamma^2}{\nu} \frac{1}{|\Omega_{per}|} (\nabla W_1^{(j)}, \nabla W_1^{(k)}) = \frac{\gamma^2}{\nu} \langle \mathbf{e}_j, W_1^{(k)} \rangle.$$

Then there exists a constant $\check{\kappa}_0 > 0$ independent of ν , $\tilde{\nu}$ and γ such that

$$\check{\kappa}(\eta') \geq \frac{\check{\kappa}_0 \gamma^2}{\nu} |\eta'|^2 \quad \text{for all } \eta' \in \mathbb{R}^{n-1}. \quad (4.31)$$

To prove (4.25) we will show that $\check{\kappa}(\eta')$ is the principal part of $\operatorname{Re} \kappa(\eta')$ when ν and $\frac{\gamma^2}{\nu}$ are large enough. We estimate the difference of $u_1^{(k)}$ and $\check{u}_1^{(k)}$.

Lemma 4.11. *If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then*

$$\|\phi_1^{(k)} - \check{\phi}_1^{(k)}\|_2 \leq C \left(\frac{1}{\gamma^2} + \frac{1}{\nu^2} \right), \quad \|w_1^{(k)} - \check{w}_1^{(k)}\|_2 \leq C \left(\frac{1}{\nu \gamma^2} + \frac{1}{\nu^3} \right).$$

Proof. We write $L_0 u_1^{(k)} = (I - \Pi^{(0)}) L_k^{(1)} u^{(0)}$ as

$$\mathcal{A} u_1^{(k)} = -\mathcal{C} u_1^{(k)} + (I - \Pi^{(0)}) L_k^{(1)} u^{(0)}.$$

Here \mathcal{C} is defined by

$$\begin{aligned} \mathcal{C} = L_0 - \mathcal{A} = & \begin{pmatrix} \operatorname{div}(\cdot v_s) & \gamma^2 \operatorname{div}(\phi_s \cdot) \\ \nabla(\tilde{P}^{(1)}(\phi_s) \phi_s \cdot) & \nu \frac{\phi_s}{1+\phi_s} \Delta + \tilde{\nu} \frac{\phi_s}{1+\phi_s} \nabla \operatorname{div} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ \frac{1}{\gamma^2 \rho_s^2} (\nu \Delta v_s + \tilde{\nu} \nabla \operatorname{div} v_s) & v_s \cdot \nabla + {}^\top \nabla v_s \end{pmatrix}. \end{aligned}$$

It then follows that

$$\mathcal{A}(u_1^{(k)} - \check{u}_1^{(k)}) = -\mathcal{C}u_1^{(k)} + (I - \Pi^{(0)})L_k^{(1)}u^{(0)} - iF^{(k)},$$

and hence,

$$((\mathcal{A}(u_1^{(k)} - \check{u}_1^{(k)}), u_1^{(k)} - \check{u}_1^{(k)})) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= -((\mathcal{C}u_1^{(k)}, u_1^{(k)} - \check{u}_1^{(k)})), \\ I_2 &= (((I - \Pi^{(0)})L_k^{(1)}u^{(0)} - iF^{(k)}, u_1^{(k)} - \check{u}_1^{(k)})). \end{aligned}$$

The left-hand side is estimated as

$$\begin{aligned} &((\mathcal{A}(u_1^{(k)} - \check{u}_1^{(k)}), u_1^{(k)} - \check{u}_1^{(k)})) \\ &\geq \nu \|\nabla(w_1^{(k)} - \check{w}_1^{(k)})\|_2^2 + \tilde{\nu} \|\operatorname{div}(w_1^{(k)} - \check{w}_1^{(k)})\|_2^2 + \frac{\delta}{2} \|\phi_1^{(k)} - \check{\phi}_1^{(k)}\|_2^2 \\ &\quad - C\delta \{\nu^2 \|\nabla(w_1^{(k)} - \check{w}_1^{(k)})\|_2^2 + \tilde{\nu}^2 \|\operatorname{div}(w_1^{(k)} - \check{w}_1^{(k)})\|_2^2 + \gamma^2 \|w_1^{(k)} - \check{u}_1^{(k)}\|_2^2\}. \end{aligned}$$

Taking $\delta = \frac{1}{2C} \min\{\frac{1}{\nu}, \frac{\nu}{\gamma^2}\}$, we have

$$\begin{aligned} &((\mathcal{A}(u_1^{(k)} - \check{u}_1^{(k)}), u_1^{(k)} - \check{u}_1^{(k)})) \\ &\geq \frac{\nu}{2} \|\nabla(w_1^{(k)} - \check{w}_1^{(k)})\|_2^2 + \frac{\tilde{\nu}}{2} \|\operatorname{div}(w_1^{(k)} - \check{w}_1^{(k)})\|_2^2 + \frac{\delta}{2} \|\phi_1^{(k)} - \check{\phi}_1^{(k)}\|_2^2. \end{aligned} \quad (4.32)$$

By Proposition 3.1 and Lemma 4.9, we see that I_1 is estimated as

$$|I_1| \leq C \left\{ \left(\frac{1}{\nu\gamma^2} + \frac{\delta}{\nu^2} \right) \|\phi_1^{(k)} - \check{\phi}_1^{(k)}\|_2 + \left(\frac{1}{\gamma^2} + \frac{1}{\nu^2} \right) \|w_1^{(k)} - \check{w}_1^{(k)}\|_2 \right\}. \quad (4.33)$$

As for I_2 , we observe that

$$(I - \Pi^{(0)})L_k^{(1)}u^{(0)} - iF^{(k)} = \tilde{L}_k^{(1)}u^{(0)} - \Pi^{(0)}L_k^{(1)}u^{(0)},$$

where

$$\begin{aligned} \tilde{L}_k^{(1)}u^{(0)} &= L_k^{(1)}u^{(0)} - iF^{(k)} \\ &= i \left(\begin{pmatrix} v_s^k \\ \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} - 1 \right) \mathbf{e}_k \end{pmatrix} - \frac{1}{\rho_s} \left(2\nu \mathbf{e}_k \otimes \mathbf{e}_k \partial_{x_k} + \tilde{\nu} \mathbf{e}_k \operatorname{div} - \tilde{\nu} \nabla(\top \mathbf{e}_k) \right) + v_s^k \right) u^{(0)} \end{aligned}$$

Therefore, by Proposition 3.1, Proposition 4.1 and Lemma 4.9, we have

$$|I_2| \leq C \left\{ \frac{1}{\nu\gamma^2} \|\phi_1^{(k)} - \check{\phi}_1^{(k)}\|_2 + \frac{1}{\gamma^2} \|w_1^{(k)} - \check{w}_1^{(k)}\|_2 \right\}. \quad (4.34)$$

From (4.32), (4.33) and (4.34), we conclude

$$\|\phi_1^{(k)} - \check{\phi}_1^{(k)}\|_2 \leq C\left(\frac{1}{\gamma^2} + \frac{1}{\nu^2}\right), \quad \|w_1^{(k)} - \check{w}_1^{(k)}\|_2 \leq C\left(\frac{1}{\nu\gamma^2} + \frac{1}{\nu^3}\right).$$

This completes the proof. \square

Proof of (4.25). We now turn to the proof of (4.25). Since $\rho_s = 1 + \phi_s$ and $\check{w}_1^{(k)} = \frac{i}{\nu} W_1^{(k)}$, by Lemma 4.10 and 4.11, we have

$$\begin{aligned} \sum_{j,k=1}^{n-1} \eta_j \eta_k \langle Q_0 L_j^{(1)} S L_k^{(1)} u^{(0)} \rangle &= i \sum_{j,k=1}^{n-1} \eta_j \eta_k \langle v_s^j \phi_1^{(k)} + \gamma^2 \rho_s \mathbf{e}_j \cdot w_1^{(k)} \rangle \\ &= - \sum_{j,k=1}^{n-1} \eta_j \eta_k \left(\frac{\gamma^2}{\nu} \langle \mathbf{e}_j \cdot W_1^{(k)} \rangle - i \gamma^2 \langle \mathbf{e}_j \cdot (w_1^{(k)} - \check{w}_1^{(k)}) \rangle \right) \\ &\quad + i \sum_{j,k=1}^{n-1} \eta_j \eta_k \langle v_s^j \phi_1^{(k)} + \gamma^2 \phi_s \mathbf{e}_j \cdot w_1^{(k)} \rangle \\ &\leq - \frac{\gamma^2}{\nu} \left\{ \frac{\check{\kappa}_0}{2} - C \left(\frac{1}{\nu} + \frac{1}{\gamma^2} \right) \right\} |\eta'|^2, \end{aligned}$$

and we have

$$\sum_{j,k=1}^{n-1} \lambda_{jk}^{(2)} \eta_j \eta_k \leq - \frac{\gamma^2}{\nu} \left\{ \frac{\check{\kappa}_0}{2} - C \left(\frac{1}{\nu} + \frac{1}{\gamma^2} \right) \right\} |\eta'|^2.$$

If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, it holds that

$$\sum_{j,k=1}^{n-1} \lambda_{jk}^{(2)} \eta_j \eta_k \leq - \frac{\check{\kappa}_0 \gamma^2}{4\nu} |\eta'|^2. \quad (4.35)$$

This completes the proof. \square

4.4 Boundedness of eigenprojections

In this subsection we estimate the eigenprojections $\Pi_{\eta'}$ for eigenvalues $\lambda_{\eta'}$ of $-L_{\eta'}$. By Proposition 4.6, we have the following resolvent estimates for $-L_{\eta'}^*$ with $|\eta'| \leq r_0$.

Proposition 4.12. *If $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then there exists a constant $r_0 > 0$ such that if $|\eta'| \leq r_0$, then $\Sigma_1 \subset \rho(-L_{\eta'}^*)$ and*

$$\|(\lambda + L_{\eta'}^*)^{-1} f\|_2 \leq \frac{C}{\operatorname{Re} \lambda + \beta_0} \|f\|_2, \quad (4.36)$$

$$\|\nabla \tilde{Q}(\lambda + L_{\eta'}^*)^{-1} f\|_2 \leq \frac{C}{(\operatorname{Re} \lambda + \beta_0)^{\frac{1}{2}}} \|f\|_2 \quad (4.37)$$

uniformly for $\lambda \in \Sigma_1$.

Proof. We define $M_{\eta'}^*$ by $M_{\eta'}^* = L_{\eta'}^* - L_0^*$:

$$\begin{aligned} M_{\eta'}^* = & \begin{pmatrix} -iv_s \cdot \tilde{\eta}' & -i\gamma^2 \tilde{\eta}' \cdot (\rho_s \cdot) \\ -i\tilde{\eta}' \left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} (|\eta'|^2 - 2i\tilde{\eta}' \cdot \nabla) I_n - i\frac{\tilde{\nu}}{\rho_s} \tilde{\eta}'^\top \nabla_{\eta'} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ 0 & -i\frac{1}{\rho_s} v_s \cdot \tilde{\eta}'(\rho_s \cdot) + i\tilde{\eta}'^\top v_s \end{pmatrix}. \end{aligned} \quad (4.38)$$

Since $\|M_{\eta'}^* u\|_2 \leq C|\eta'|(\|u\|_2 + \|\nabla w\|_2)$, Proposition 4.12 follows from Proposition 4.6 as in the proof of Proposition 4.7. \square

We next investigate the estimate for $(\lambda + L_0^*)^{-1} f$ in a higher order Sobolev space.

Proposition 4.13. *There exists a constant $\tilde{\beta}_0 > 0$ such that if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, then, for any $\beta_1 \in (0, \tilde{\beta}_0]$, the estimate*

$$\|(\lambda + L_0^*)^{-1} f\|_{H^2 \times H^3} \leq C \|f\|_{H^2 \times H^1}$$

holds uniformly for λ satisfying $\frac{\beta_1}{2} \leq |\lambda| \leq \beta_1$.

Proof. We write $(\lambda + L_0^*)u = f$ as

$$(\lambda + \tilde{L}_0^*)u + \tilde{M}^* u = f, \quad (4.39)$$

where $\tilde{L}_0^* : H_{per}^2 \times H_{per}^1 \longrightarrow H_{per}^2 \times H_{per}^1$ is defined by

$$\tilde{L}_0^* = \begin{pmatrix} -\operatorname{div}(v_s \cdot) & -\gamma^2 \operatorname{div} \\ -\nabla & -\nu \Delta - \tilde{\nu} \operatorname{div} \end{pmatrix}$$

with domain

$$D(\tilde{L}_0^*) = \{u \in H_{per}^2 \times (H_{per}^3 \cap H_{0,per}^1); \tilde{L}_0^* u \in H_{per}^2 \times H_{per}^1\}.$$

Here

$$\begin{aligned} \tilde{M}^* = & \begin{pmatrix} -v_s \cdot \nabla \left(\frac{P'(\rho_s)}{\rho_s} \right) \frac{\rho_s}{P'(\rho_s)} & -\gamma^2 \operatorname{div}(\phi_s \cdot) \\ -\nabla(\phi_s \tilde{P}^{(1)}(\phi_s) \cdot) & \frac{\phi_s}{\rho_s} (\nu \Delta + \tilde{\nu} \nabla \operatorname{div}) \end{pmatrix} \\ & + \begin{pmatrix} \operatorname{div} v_s & \frac{\gamma^2}{P'(\rho_s)} (\nu \Delta^\top v_s + \tilde{\nu}^\top \nabla \operatorname{div} v_s) \\ 0 & -\operatorname{div} v_s - \frac{1}{\rho_s} v_s \cdot \nabla(\rho_s \cdot) + \nabla^\top v_s \end{pmatrix}. \end{aligned}$$

To prove the proposition, we first show the invertibility of $\lambda + \tilde{L}_0^*$ for $|\lambda| \ll 1$ and then regard \tilde{M}^* as a small perturbation of $\lambda + \tilde{L}_0^*$. To this end, we introduce the operator $\mathcal{A}^*[v_s] : H_{per,*}^2 \times H_{per}^1 \rightarrow H_{per,*}^2 \times H_{per}^1$ with domain $D(\mathcal{A}^*[v_s]) = D(\tilde{L}_0^*) \cap (H_{per,*}^2 \times H_{per}^1)$ defined by $\mathcal{A}^*[v_s]u = \tilde{L}_0^*u$ for $u \in D(\mathcal{A}^*[v_s])$. We observe that there exists a unique solution $u \in D(\mathcal{A}^*[v_s])$ of $(\lambda + \mathcal{A}^*[v_s])u = F$ for any $F \in H_{per,*}^2 \times H_{per}^1$ if $\operatorname{Re}\lambda \geq -\Lambda$ for some constant $\Lambda > 0$ and u satisfies estimate (5.5)₂ and, in particular, (5.6)_k with $k = 2$ when $\lambda = 0$. This can be seen with obvious modification of the proof of Theorem 5.1.

We also introduce the norm $\|u\|_{H^2 \times H^1}^* = \|\phi\|_{H^2} + \nu\|w\|_{H^1}$ for $u = {}^\top(\phi, w)$. Recall that $\|u\|_{H^2 \times H^3}^* = \|\phi\|_{H^2} + \nu\|w\|_{H^3}$.

As in the proof of Proposition 4.1, we see from (5.6)_k with $k = 2$, that there exist $\hat{\nu}_0 > 0$ and $\hat{\gamma}_0 > 0$ such that if $\nu \geq \hat{\nu}_0$ and $\frac{\gamma^2}{\nu} \geq \hat{\gamma}_0$, then

$$\|(\mathcal{A}^*[v_s])^{-1}F\|_{H^2 \times H^3}^* \leq C_1 \left\{ \frac{\nu}{\gamma^2} \|F^0\|_{H^2} + \|\tilde{F}\|_{H^1} \right\} \quad (4.40)$$

for $F = {}^\top(F^0, \tilde{F}) \in H_{*,per}^2 \times H_{per}^1$. By (4.40), we have

$$\|\lambda(\mathcal{A}^*[v_s])^{-1}u\|_{H^2 \times H^3}^* \leq C_2|\lambda|\|u\|_{H^2 \times H^1}^* \leq C_2|\lambda|\|u\|_{H^2 \times H^3}^*,$$

where $C_2 = C_2(\hat{\nu}_0, \hat{\gamma}_0) > 0$. Therefore, if $|\lambda| \leq \frac{1}{2C_2}$, then there exists a unique $u \in D(\mathcal{A}^*[v_s])$ satisfying $u = -\lambda(\mathcal{A}^*[v_s])^{-1}u + (\mathcal{A}^*[v_s])^{-1}F$, i.e., $(\lambda + \mathcal{A}^*[v_s])u = F$, and $u = (\lambda + \mathcal{A}^*[v_s])^{-1}F$ satisfies

$$\|u\|_{H^2 \times H^3}^* \leq 2C_2\|F\|_{H^2 \times H^1}^*. \quad (4.41)$$

We next show the invertibility of $\lambda + \tilde{L}_0^*$. We set $\tilde{u}^{(0)} = {}^\top(1, 0)$ and define $\tilde{\Pi}^{(0)}$ by $\tilde{\Pi}^{(0)}u = \langle \phi \rangle \tilde{u}^{(0)}$ for $u = {}^\top(\phi, w)$. Then $\tilde{\Pi}^{(0)}$ is a projection. We also set $\tilde{\Pi}_1 = I - \tilde{\Pi}^{(0)}$.

Consider the problem

$$(\lambda + \tilde{L}_0^*)u = F, \quad u \in D(\tilde{L}_0^*). \quad (4.42)$$

Since $\tilde{\Pi}^{(0)}\tilde{L}_0^*u = 0$, (4.42) is decomposed into the following system:

$$\begin{cases} \lambda\tilde{\Pi}^{(0)}u = \tilde{\Pi}^{(0)}F, \\ \lambda\tilde{\Pi}_1u + \mathcal{A}^*[v_s]\tilde{\Pi}_1u + \tilde{\Pi}_1\mathcal{A}^*[v_s]\tilde{\Pi}^{(0)}u = \tilde{\Pi}_1F, \\ u \in D(\tilde{L}_0^*). \end{cases} \quad (4.43)$$

By the first equation of (4.43), if $\lambda \neq 0$, then

$$\tilde{\Pi}^{(0)}u = \frac{1}{\lambda}\tilde{\Pi}^{(0)}F. \quad (4.44)$$

We set $\tilde{\beta}_0 = \frac{1}{2C_2}$. Let λ satisfy $\frac{\beta_1}{2} \leq |\lambda| \leq \beta_1$ for $\beta_1 \in (0, \tilde{\beta}_0]$. Substituting (4.44) into the second equation of (4.43), we have

$$\lambda \tilde{\Pi}_1 u + \mathcal{A}^*[v_s] \tilde{\Pi}_1 u = \tilde{\Pi}_1 F - \frac{1}{\lambda} \tilde{\Pi}_1 \mathcal{A}^*[v_s] \tilde{\Pi}^{(0)} F. \quad (4.45)$$

We set $G = \tilde{\Pi}_1 F - \frac{1}{\lambda} \tilde{\Pi}_1 \mathcal{A}^*[v_s] \tilde{\Pi}^{(0)} F$. Then $G \in \tilde{\Pi}_1[H_{per}^2 \times H_{per}^1] \subset H_{per,*}^2 \times H_{per}^1$. If λ satisfies $\frac{\beta_1}{2} \leq |\lambda| \leq \beta_1$, there exists a function $u_1 = {}^\top(\phi_1, w_1) \in D(\mathcal{A}^*[v_s])$ satisfying $(\lambda + \mathcal{A}^*[v_s])u_1 = G$. Furthermore, by (4.41) and Proposition 3.1, u_1 satisfies

$$\|u_1\|_{H^2 \times H^3}^* \leq 2C_2 \|G\|_{H^2 \times H^1}^* \leq C_3 \left(1 + \frac{1}{\nu\beta_1}\right) \|F\|_{H^2 \times H^1}^* \quad (4.46)$$

for $\frac{\beta_1}{2} \leq |\lambda| \leq \beta_1$ with some $C_3 = C_3(\hat{\nu}_0, \hat{\gamma}_0) > 0$.

We set $u = \frac{1}{\lambda} \tilde{\Pi}^{(0)} F + u_1$. Since $\tilde{\Pi}^{(0)} u_1 = \langle \phi_1 \rangle \tilde{u}^{(0)} = 0$, it holds that $\tilde{\Pi}^{(0)} u = \frac{1}{\lambda} \tilde{\Pi}^{(0)} F$, and we have $\tilde{\Pi}_1 u = u_1$. Consequently, u is a solution of (4.42). By (4.44) and (4.46), $u = (\lambda + \tilde{L}_0^*)^{-1} F$ satisfies

$$\|u\|_{H^2 \times H^3}^* \leq \frac{1}{|\lambda|} \|\tilde{\Pi}^{(0)} F\|_{H^2 \times H^3}^* + \|u_1\|_{H^2 \times H^3}^* \leq C_4 \|F\|_{H^2 \times H^1}^* \quad (4.47)$$

for $\frac{\beta_1}{2} \leq |\lambda| \leq \beta_1$. Here $C_4 = C_4(\hat{\nu}_0, \hat{\gamma}_0, \beta_1) > 0$. This implies that $(\lambda + \tilde{L}_0^*)^{-1} : H_{per}^2 \times H_{per}^1 \rightarrow H_{per}^2 \times (H_{per}^3 \cap H_{0,per}^1)$ is bounded and $u = (\lambda + \tilde{L}_0^*)^{-1} F$ satisfies (4.47).

We finally prove the invertibility of $\lambda + L_0^* = \lambda + \tilde{L}_0^* + \tilde{M}^*$. By Proposition 3.1, we have

$$\|\tilde{M}^* u\|_{H^2 \times H^1}^* \leq C \left(\frac{1}{\nu} + \frac{1}{\gamma^2 \nu} + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \|u\|_{H^2 \times H^3}^*. \quad (4.48)$$

It then follows from (4.47) and (4.48) that

$$\|(\lambda + \tilde{L}_0^*)^{-1} \tilde{M}^* u\|_{H^2 \times H^3}^* \leq CC_4 \left(\frac{1}{\nu} + \frac{1}{\gamma^2 \nu} + \frac{\nu + \tilde{\nu}}{\gamma^2} \right) \|u\|_{H^2 \times H^3}^*.$$

Therefore, there exist $\nu_0 > 0$ and $\tilde{\gamma}_0 > 0$ such that if $\nu \geq \nu_0$ and $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$, there exists a unique $u \in D(\tilde{L}_0^*)$ satisfying $u = -(\lambda + \tilde{L}_0^*)^{-1} \tilde{M}^* u + (\lambda + \tilde{L}_0^*)^{-1} f$, that is, a unique solution $u \in D(\tilde{L}_0^*)$ of (4.39), and $u = (\lambda + \tilde{L}_0^* + \tilde{M}^*)^{-1} f$ satisfies the estimate $\|u\|_{H^2 \times H^3}^* \leq 2C_4 \|f\|_{H^2 \times H^1}^*$. Since $H_{per}^2 \times (H_{per}^3 \cap H_{0,per}^1) \subset D(L_0^*)$, we have $u = (\lambda + L_0^*)^{-1} f$. This completes the proof. \square

We are now in a position to estimate the eigenprojections.

Proposition 4.14. *If $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu} \geq \tilde{\gamma}_0$ and $|\eta'| \leq r_0$ (by taking ν_0 and $\tilde{\gamma}_0$ suitably larger if necessary), then*

$$\|\Pi_{\eta'} u\|_2 \leq C \|u\|_1, \quad (4.49)$$

$$\|(\Pi_{\eta'} - \Pi^{(0)})u\|_2 \leq C |\eta'| \|u\|_1. \quad (4.50)$$

Proof. By Proposition 4.7, we see that $\Pi_{\eta'}$ and $\Pi_{\eta'}^*$ are given by

$$\Pi_{\eta'} = \frac{1}{2\pi i} \int_{|\lambda|=\frac{\beta_0}{4}} (\lambda + L_{\eta'})^{-1} d\lambda, \quad \Pi_{\eta'}^* = \frac{1}{2\pi i} \int_{|\lambda|=\frac{\beta_0}{4}} (\lambda + L_{\eta'}^*)^{-1} d\lambda.$$

Then $u_{\eta'} = \Pi_{\eta'} u^{(0)}$ is an eigenfunction of $-L_{\eta'}$ for the eigenvalue $\lambda_{\eta'}$ and $u_{\eta'}^* = \Pi_{\eta'}^* u^{(0)*}$ is an eigenfunction of $-L_{\eta'}^*$ for $\lambda_{\eta'}^* = \bar{\lambda}_{\eta'}$. Since $\lambda_{\eta'}$ is a simple eigenvalue, the eigenprojection $\Pi_{\eta'}$ is represented as

$$\Pi_{\eta'} u = \frac{\langle u, u_{\eta'}^* \rangle}{\langle u_{\eta'}, u_{\eta'}^* \rangle} u_{\eta'}.$$

From the proof of Proposition 4.7, we see that $(\lambda + L_{\eta'})^{-1}$ is expanded as

$$(\lambda + L_{\eta'})^{-1} = (\lambda + L_0)^{-1} + R_{\eta'}(\lambda)$$

and $R_{\eta'}(\lambda)$ is estimated as

$$\|R_{\eta'}(\lambda)f\|_2 \leq C |\eta'| \|f\|_2, \quad \|\nabla \tilde{Q} R_{\eta'}(\lambda)f\|_2 \leq C |\eta'| \|f\|_2$$

uniformly for $|\eta'| \leq r_0$ and $|\lambda| = \frac{\beta_0}{4}$. We write $u_{\eta'}$ as

$$\begin{aligned} u_{\eta'} &= \frac{1}{2\pi i} \int_{|\lambda|=\frac{\beta_0}{4}} (\lambda + L_0)^{-1} u^{(0)} d\lambda + \frac{1}{2\pi i} \int_{|\lambda|=\frac{\beta_0}{4}} R_{\eta'}(\lambda) u^{(0)} d\lambda \\ &= u^{(0)} + u_{\eta'}^{(1)}, \end{aligned}$$

where $u_{\eta'}^{(1)} = \frac{1}{2\pi i} \int_{|\lambda|=\frac{\beta_0}{4}} R_{\eta'}(\lambda) u^{(0)} d\lambda$. Then $u_{\eta'}^{(1)}$ satisfies

$$\|u_{\eta'}^{(1)}\|_2 \leq C |\eta'|, \quad \|\nabla \tilde{Q} u_{\eta'}^{(1)}\|_2 \leq C |\eta'|.$$

Similarly, $u_{\eta'}^*$ is written as $u_{\eta'}^* = u^{(0)*} + u_{\eta'}^{(1)*}$, where $u_{\eta'}^{(1)*} = \frac{1}{2\pi i} \int_{|\lambda|=\frac{\beta_0}{4}} R_{\eta'}^*(\lambda) u^{(0)} d\lambda$.

By Proposition 4.13, $u_{\eta'}^{(1)*}$ is estimated as $\|u_{\eta'}^{(1)*}\|_{H^2} \leq C |\eta'|$. Thus, by a similar argument to that in [15, Theorem 4.11], we obtain the desired estimates (4.49) and (4.50). This completes the proof. \square

4.5 Estimates of the Π -part of the semigroup

We finally prove Theorem 3.2. We set

$$\Pi = U\chi_0\Pi_{\eta'}T, \quad \chi_0 = \begin{cases} 1 & |\eta'| \leq r_0, \\ 0 & |\eta'| \geq r_0. \end{cases}$$

We see from Proposition 2.2 that $\Pi^2 = \Pi$. Furthermore, by Proposition 4.8, we have

$$e^{-tL}\Pi u_0 = U\chi_0 e^{-tL_{\eta'}}\Pi_{\eta'}T u_0 = U\chi_0 e^{\lambda_{\eta'}t}\Pi_{\eta'}T u_0.$$

Then, in a similar manner to the proof of [15], by using Proposition 4.8 and 4.14, we obtain the desired result. This completes the proof. \square

5 Resolvent estimates for the principal part of L_0

In this section we give a unique existence of solution of the following problem which provides the principal part of L_0 . We consider the problem:

$$\lambda\phi + \gamma^2 \operatorname{div} w + \operatorname{div}(\phi v) = f^0, \quad (5.1)$$

$$\lambda w - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w + \nabla \phi = \tilde{f}, \quad (5.2)$$

under the condition

$$w|_{\Sigma_n} = 0, \quad \phi|_{\Sigma_{j,-}} = \phi|_{\Sigma_{j,+}}, \quad w|_{\Sigma_{j,-}} = w|_{\Sigma_{j,+}} \quad (j = 1, \dots, n-1), \quad (5.3)$$

$$\langle \phi \rangle = 0. \quad (5.4)$$

Here $\lambda \in \mathbb{C}$ is a parameter and $v = {}^\top(v^1, \dots, v^n) \in H_{per}^4$ is a given function.

Throughout this section we assume that ν and $\tilde{\nu}$ satisfy (2.4), and thus, we use relation (2.5). The letter C denotes positive constants depending only on ν_* and Ω_{per} .

We have the following a priori estimates.

Theorem 5.1. *Let $k = 0, 1, 2, 3$. There exists a constant $\Lambda > 0$ such that if $\operatorname{Re} \lambda \geq -\Lambda$ and $\|v\|_{H^{3+(k-2)+}} \leq C \min \left\{ \frac{\gamma^2}{\nu}, \nu, \gamma \right\}$, then problem (5.1)-(5.4) as a unique solution $u = {}^\top(\phi, w) \in H_{per,*}^k \times (H_{per}^{k+1} \cap H_{0,per}^1)$ and u satisfies the following estimates (5.5) $_j$, $0 \leq j \leq k$:*

$$\begin{aligned} & \operatorname{Re}(\lambda + \Lambda)(\gamma^{-2}\|\phi\|_2^2 + \|w\|_2^2) + \frac{\nu}{2}\|\nabla w\|_2^2 + \frac{\tilde{\nu}}{2}\|\operatorname{div} w\|_2^2 + \frac{\delta_1}{2}\|\phi\|_2^2 \\ & \leq C \left\{ \left(\frac{1}{\delta_1 \gamma^4} + \frac{\delta_1^2}{\nu} \right) \|f^0\|_2^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^{-1}}^2 \right\}, \end{aligned} \quad (5.5)_0$$

$$\begin{aligned} & \operatorname{Re}(\lambda + \Lambda)(\gamma^{-2}\|\phi\|_{H^1}^2 + \|w\|_{H^1}^2) + \frac{1}{\nu}\|\phi\|_{H^1}^2 + \nu\|w\|_{H^2}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 \\ & \leq C\left\{\left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^1}^2 + \left(\frac{1}{\nu} + \frac{\gamma^2}{\nu^3}\right)\|\tilde{f}\|_2^2\right\}, \end{aligned} \quad (5.5)_1$$

$$\begin{aligned} & \operatorname{Re}(\lambda + \Lambda)(\gamma^{-2}\|\phi\|_{H^2}^2 + \|w\|_{H^1}^2) + \frac{1}{\nu}\|\phi\|_{H^2}^2 + \nu\|w\|_{H^3}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 \\ & \leq C\left\{\frac{|\lambda|^2}{(\nu + \tilde{\nu})^2}\left(\left(\frac{1}{\delta_1\gamma^4} + \frac{\delta_1^2}{\nu}\right)\|f^0\|_2^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^{-1}}^2\right) \right. \\ & \quad \left. + \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^2}^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^1}^2 + \frac{\gamma^2}{\nu^3}\|\tilde{f}\|_2^2\right\}, \end{aligned} \quad (5.5)_2$$

$$\begin{aligned} & \operatorname{Re}(\lambda + \Lambda)(\gamma^{-2}\|\phi\|_{H^3}^2 + \|w\|_{H^2}^2) + \frac{1}{\nu}\|\phi\|_{H^3}^2 + \nu\|w\|_{H^4}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 \\ & \leq C\left\{\frac{|\lambda|^2}{(\nu + \tilde{\nu})^2}\left(\left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3} + \frac{1}{\delta_1\gamma^4} + \frac{\delta_1^2}{\nu}\right)\|f^0\|_2^2 + \left(\frac{1}{\nu} + \frac{\gamma^2}{\nu^3}\right)\|\tilde{f}\|_{H^{-1}}^2\right) \right. \\ & \quad \left. + \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^3}^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^2}^2 + \frac{\gamma^2}{\nu^3}\|\tilde{f}\|_2^2\right\}. \end{aligned} \quad (5.5)_3$$

Here δ_1 is any positive number satisfying $0 < \delta_1 \leq C \min\left\{\frac{1}{\gamma}, \frac{1}{\nu}, \frac{\nu}{\gamma^2}\right\}$.

In particular, when $\lambda = 0$, $u = {}^\top(\phi, w)$ satisfies

$$\frac{1}{\nu}\|\phi\|_{H^k}^2 + \nu\|w\|_{H^{k+1}}^2 \leq C\left\{\frac{\nu + \tilde{\nu}}{\gamma^4}\|f^0\|_{H^k}^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^{k-1}}^2\right\} \quad (5.6)_k$$

for $k = 1, 2, 3$.

To prove Theorem 5.1, we first establish a priori estimates (5.5)₀-(5.5)₃, and then using the a priori estimates and the result by Heywood-Padula [10], we will prove the existence of solutions.

a. A priori estimate

In this subsection we will prove the a priori estimate (5.5)₃ by the Matsumura-Nishida energy method [22].

Lemma 5.2. *Let (ϕ, w) be a solution of (5.1)-(5.4). Then*

$$\begin{aligned} & \operatorname{Re}\lambda(\gamma^{-2}\|\phi\|_2^2 + \|w\|_2^2) + \nu\|\nabla w\|_2^2 + (\nu + \tilde{\nu})\|\operatorname{div} w\|_2^2 \\ & \leq C\left\{\frac{1}{\varepsilon\gamma^4}\|f^0\|_2^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^{-1}}^2 + \frac{1}{\gamma^2}\|v\|_{H^3}\|\phi\|_2^2 + \varepsilon\|\phi\|_2^2\right\}, \end{aligned} \quad (5.7)$$

$$\begin{aligned}
& \operatorname{Re} \lambda (\gamma^{-2} \|\phi\|_2^2 + \|w\|_2^2) + \frac{\nu}{2} \|\nabla w\|_2^2 + \frac{\tilde{\nu}}{2} \|\operatorname{div} w\|_2^2 + \frac{\delta_1}{2} \|\phi\|_2^2 \\
& \leq C \left\{ \left(\frac{1}{\delta_1 \gamma^4} + \frac{\delta_1^2}{\nu} \right) \|f^0\|_2^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^{-1}}^2 \right\},
\end{aligned} \tag{5.8}$$

for $\varepsilon > 0$, where $\delta_1 = \frac{1}{2C} \min \left\{ \frac{1}{\gamma}, \frac{1}{\nu}, \frac{\nu}{\gamma^2} \right\}$.

Proof. Taking the inner products of (5.1) and (5.2) with $\frac{1}{\gamma^2} \phi$ and w , respectively, we have

$$\begin{aligned}
& \operatorname{Re} \lambda (\gamma^{-2} \|\phi\|_2^2 + \|w\|_2^2) + \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2 \\
& = \operatorname{Re} \left\{ -\frac{1}{\gamma^2} (\operatorname{div}(\phi v), \phi) + \frac{1}{\gamma^2} (f^0, \phi) + (\tilde{f}, w) \right\}.
\end{aligned}$$

Since $\operatorname{Re}(\operatorname{div}(\phi v), \phi) = \frac{1}{2} \operatorname{Re}(\operatorname{div} v, |\phi|^2) \leq \|\operatorname{div} v\|_\infty \|\phi\|_2^2 \leq \|v\|_{H^3} \|\phi\|_2^2$, we have

$$\begin{aligned}
& \operatorname{Re} \lambda (\gamma^{-2} \|\phi\|_2^2 + \|w\|_2^2) + \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2 \\
& \leq C \left\{ \frac{1}{\nu} \|\tilde{f}\|_{H^{-1}}^2 + \frac{1}{\varepsilon \gamma^4} \|f^0\|_2^2 + \frac{1}{\gamma^2} \|v\|_{H^3} \|\phi\|_2^2 + \varepsilon \|\phi\|_2^2 \right\}
\end{aligned}$$

for arbitrary $\varepsilon > 0$. We thus obtain (5.7).

We next prove (5.8). We have

$$\begin{aligned}
& \operatorname{Re} \lambda ((u, u)) + \frac{1}{\gamma^2} \operatorname{Re}(\operatorname{div}(\phi v) + \gamma^2 \operatorname{div} w, \phi) + \operatorname{Re}(\nabla \phi - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w, w) \\
& - \delta_1 \operatorname{Re}(\nabla \phi - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w, \mathcal{B} \phi) - \delta_1 \operatorname{Re}(\mathcal{B}(\gamma^2 \operatorname{div} w + \operatorname{div}(\phi v)), w) \\
& \geq \operatorname{Re} \lambda ((u, u)) + \nu \|\nabla w\|_2^2 + \tilde{\nu} \|\operatorname{div} w\|_2^2 - \frac{1}{\gamma^2} \|v\|_{H^3} \|\phi\|_2^2 \\
& - \delta_1 \operatorname{Re}(\nabla \phi - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w, \mathcal{B} \phi) - \delta_1 \operatorname{Re}(\mathcal{B}(\gamma^2 \operatorname{div} w + \operatorname{div}(\phi v)), w).
\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned}
& \operatorname{Re}(\nabla \phi - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w, \mathcal{B} \phi) \leq -\frac{1}{2} \|\phi\|_2^2 + C \{ \nu^2 \|\nabla w\|_2^2 + \tilde{\nu}^2 \|\operatorname{div} w\|_2^2 \}, \\
& \operatorname{Re}(\mathcal{B}(\gamma^2 \operatorname{div} w + \operatorname{div}(\phi v)), w) \leq \|\mathcal{B}(\gamma^2 \operatorname{div} w + \operatorname{div}(\phi v))\|_2 \|w\|_2 \\
& \leq C \{ \gamma^2 \|w\|_2 + \|\phi v\|_2 \} \|w\|_2 \\
& \leq C \{ \gamma^2 \|w\|_2 + \|v\|_{H^4} \|\phi\|_2 \} \|w\|_2.
\end{aligned}$$

We set $\delta_1 = \frac{1}{2C} \min \left\{ \frac{1}{\gamma}, \frac{1}{\nu}, \frac{\nu}{\gamma^2} \right\}$ and $M = \min \left\{ \frac{\gamma^2}{\nu}, \nu, \gamma \right\}$. Then if $\|v\|_{H^4} \leq CM$, we have

$$\begin{aligned}
& \operatorname{Re}(\lambda + \Lambda)(\gamma^{-2} \|\phi\|_2^2 + \|w\|_2^2) + \frac{\nu}{2} \|\nabla w\|_2^2 + \frac{\tilde{\nu}}{2} \|\operatorname{div} w\|_2^2 + \frac{\delta_1}{2} \|\phi\|_2^2 \\
& \leq C \left\{ \left(\frac{1}{\delta_1 \gamma^4} + \frac{\delta_1^2}{\nu} \right) \|f^0\|_2^2 + \frac{1}{\nu} \|\tilde{f}\|_2^2 \right\}.
\end{aligned}$$

□

Lemma 5.3. *Let $\chi_0 \in C_{0,per}^\infty(\Omega_{per})$. Then*

$$\begin{aligned} & \operatorname{Re} \lambda (\gamma^{-2} \|\chi_0 \partial_x^k \phi\|_2^2 + \|\chi_0 \partial_x^k w\|_2^2) + \nu \|\chi_0 \nabla \partial_x^k w\|_2^2 + (\nu + \tilde{\nu}) \|\chi_0 \operatorname{div} \partial_x^k w\|_2^2 \\ & \leq C \left\{ \left(\nu + \tilde{\nu} + \frac{1}{\delta} \right) \|\partial_x^k w\|_2^2 + \frac{1}{\delta \gamma^4} \|f^0\|_{H^k}^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^{k-1}}^2 \right. \\ & \quad \left. + \frac{1}{\gamma^2} \|v\|_{H^{3+(k-2)_+}} \|\phi\|_{H^k}^2 + \delta \|\phi\|_{H^k}^2 \right\} \end{aligned} \quad (5.9)$$

for $k = 1, 2, 3$ and arbitrary δ satisfying $0 < \delta < 1$.

Proof. Applying ∂_x to (5.1) and (5.2) and taking the inner product of the resulting equations with $\chi_0^2 \partial_x \phi$ and $\chi_0^2 \partial_x w$, respectively, we have

$$\begin{aligned} & \operatorname{Re} \lambda (\gamma^{-2} \|\chi_0 \partial_x \phi\|_2^2 + \|\chi_0 \partial_x w\|_{H^2}^2) + \nu \|\chi_0 \nabla \partial_x w\|_2^2 + \tilde{\nu} \|\chi_0 \operatorname{div} \partial_x w\|_2^2 \\ & = \operatorname{Re} \left\{ -\frac{1}{\gamma^2} (\partial_x \operatorname{div}(\phi v), \chi_0^2 \nabla \phi) + (-\nu \partial_x^2 w - \tilde{\nu} \partial_x \operatorname{div} w + \partial_x \phi, 2\chi_0 \nabla \chi_0 \partial_x w) \right. \\ & \quad \left. - (\tilde{f}, \chi_0^2 \partial_x^2 w + 2\chi_0 \nabla \chi_0 \partial_x w) + \frac{1}{\gamma^2} (\partial_x f^0, \chi_0^2 \nabla \phi) \right\}. \end{aligned}$$

Since $\operatorname{Re}(\partial_x \operatorname{div}(\phi v), \chi_0^2 \partial_x \phi) \leq C \|v\|_{H^3} \|\phi\|_{H^1}^2$, we have

$$\begin{aligned} & \operatorname{Re} \lambda (\gamma^{-2} \|\chi \partial_x \phi\|_2^2 + \|\chi \partial_x w\|_{H^2}^2) + \nu \|\chi_0 \nabla \partial_x w\|_2^2 + \tilde{\nu} \|\chi_0 \operatorname{div} \partial_x w\|_2^2 \\ & \leq C \left\{ \nu \|\nabla w\|_2^2 + \frac{1}{\delta} \|\nabla w\|_2^2 + \frac{1}{\nu} \|\tilde{f}\|_2^2 + \frac{1}{\gamma^2} \|v\|_{H^3} \|\phi\|_{H^1}^2 + \frac{1}{\gamma^4 \delta} \|f^0\|_{H^1}^2 + \delta \|\phi\|_{H^1}^2 \right\} \end{aligned}$$

for arbitrary δ satisfying $0 < \delta < 1$. We thus obtain (5.9). As for higher order derivatives, since $\operatorname{Re}(\partial_x^k \operatorname{div}(\phi v), \chi_0^2 \partial_x^k \phi) \leq C \|v\|_{H^{3+(k-2)_+}} \|\phi\|_{H^k}^2$ ($k = 2, 3$), we can prove (5.9) for $k = 2, 3$ in a similar manner. □

We next derive the estimates near the boundary. We introduce local coordinates $\Psi(y_1, \dots, y_{n-1}, y_n)$ near the boundary Σ_n . For each $\bar{x} \in \Sigma_n$, there exists an open neighborhood \mathcal{O} of \bar{x} such that each $x \in \mathcal{O}$ is written as

$$x = \Psi(y', y_n) \equiv \lambda(y') + y_n \mathbf{n}(\lambda(y')),$$

where $y' = (y_1, \dots, y_{n-1})$ and $\lambda(y')$ is local coordinates on $\Sigma_n \cap \mathcal{O}$ and \mathbf{n} is the outward unit normal vector to Σ_n . We set $y = (y', y_n)$ and denote the tangential and normal derivatives by $\partial = (\partial_{y_1}, \dots, \partial_{y_{n-1}})$ and $\partial_N = \partial_{y_n}$, respectively.

As for the tangential derivatives, we have the following estimates.

Lemma 5.4. *Let $\chi \in C_0^\infty(\mathcal{O})$. Then*

$$\begin{aligned} & \operatorname{Re} \lambda (\gamma^{-2} \|\chi \partial^k \phi\|_2^2 + \|\chi \partial^k w\|_2^2) + \nu \|\chi \partial^k \nabla w\|_2^2 + (\nu + \tilde{\nu}) \|\chi \partial^k \operatorname{div} w\|_2^2 \\ & \leq C \left\{ \left(\nu + \tilde{\nu} + \frac{1}{\delta} \right) \|\nabla w\|_{H^{k-1}}^2 + \frac{1}{\delta \gamma^4} \|f^0\|_{H^k}^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^{k-1}}^2 \right. \\ & \quad \left. + \frac{1}{\gamma^2} \|v\|_{H^{3+(k-2)_+}} \|\phi\|_{H^k}^2 + \delta \|\phi\|_{H^k}^2 \right\} \end{aligned} \quad (5.10)$$

for $k = 1, 2, 3$ and arbitrary δ satisfying $0 < \delta < 1$.

Proof. Using the local coordinates $\Psi(y)$, we rewrite (5.1)-(5.2) on $\mathcal{O} \cap \Omega_{per}$ as

$$\begin{cases} \lambda \tilde{\phi} + \gamma^2 a_{kj} \partial_{y_k} \tilde{w}^j + a_{kj} \partial_{y_k} (\tilde{\phi} \tilde{v}_s^j) = f^0 \circ \Psi, \\ \lambda \tilde{w}^j - \nu a_{ki} \partial_{y_k} (a_{si} \partial_{y_s} \tilde{w}^j) - \tilde{\nu} a_{kj} \partial_{y_k} (a_{si} \partial_{y_s} \tilde{w}^i) + a_{kj} \partial_{y_k} \tilde{\phi} = \tilde{f} \circ \Psi \end{cases} \quad (5.11)$$

on $\Psi^{-1}(\mathcal{O} \cap \Omega_{per})$, where $j = 1, \dots, n$, $\tilde{w}(y) = w(\Psi(y))$, $\tilde{\phi}(y) = \phi(\Psi(y))$, $\tilde{v}_s(y) = v_s(\Psi(y))$, $\tilde{\rho}_s(y) = \rho_s(\Psi(y))$, $\eta_n = 0$, $a_{kj} = a_{kj}(y)$ is the entry (k, j) of $(D\Psi)^{-1} = (D\Psi^{-1}) \circ \Psi$. $D\Psi = (\partial_{y_j} \Psi^k)$ is the Jacobian matrix of Ψ . Applying $\partial_{y'}^k$ to (5.11) and taking the inner product of the resulting equation with $\chi^2 \partial_{y'} \tilde{u}$, $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$, we have the desired estimate in a similar manner to the proof of Lemma 5.3. \square

Concerning the normal derivative of ϕ , we have the following estimates.

Lemma 5.5. *Let $\chi \in C_0^\infty(\mathcal{O})$. Then*

$$\begin{aligned} & \operatorname{Re} \lambda \frac{\nu + \tilde{\nu}}{\gamma^2} \|\chi \partial^k \partial_N^{l+1} \phi\|_2^2 + \|\chi \partial^k \partial_N^{l+1} \phi\|_2^2 \\ & \leq C \left\{ |\lambda|^2 \|w\|_{H^{k+l}}^2 + \nu^2 \|\chi \partial^{k+1} \partial_N^l \nabla w\|_2^2 + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} \|f^0\|_{H^{k+l+1}}^2 \right. \\ & \quad \left. + \|\tilde{f}\|_{H^{k+l}}^2 + \frac{\nu + \tilde{\nu}}{\gamma^2} \|v\|_{H^{3+(k+l-2)_+}} \|\phi\|_{H^{k+l+1}}^2 \right\} \end{aligned} \quad (5.12)$$

for $k + l = 0, 1, 2$.

Proof. We apply $\frac{\nu + \tilde{\nu}}{\gamma^2} \partial_N$ to (5.1) and take the inner product of (5.2) with \mathbf{n} , we have

$$\begin{aligned} & \lambda \frac{\nu + \tilde{\nu}}{\gamma^2} \partial_N \phi + (\nu + \tilde{\nu}) \partial_N \operatorname{div} w + \frac{\nu + \tilde{\nu}}{\gamma^2} \partial_N \operatorname{div}(\phi v) = \frac{\nu + \tilde{\nu}}{\gamma^2} \partial_N f^0, \\ & \lambda w \cdot \mathbf{n} - \nu \Delta w \cdot \mathbf{n} - \tilde{\nu} \partial_N \operatorname{div} w + \partial_N \phi = \tilde{f} \cdot \mathbf{n}. \end{aligned} \quad (5.13)$$

Adding these two equations, we obtain

$$\begin{aligned}
& \lambda \frac{\nu + \tilde{\nu}}{\gamma^2} \partial_N \phi + \partial_N \phi \\
&= -\lambda w \cdot \mathbf{n} + \nu(\Delta w \cdot \mathbf{n} - \partial_N \operatorname{div} w) + \tilde{f} \cdot \mathbf{n} + \frac{\nu + \tilde{\nu}}{\gamma^2} \partial_N f^0 - \frac{\nu + \tilde{\nu}}{\gamma^2} \partial_N \operatorname{div}(\phi v).
\end{aligned} \tag{5.14}$$

Applying $\partial^k \partial_N^l$ ($k + l = 0, 1, 2$) to (5.14), we take the inner product of the resulting equation with $\chi^2 \partial^k \partial_N^{l+1} \phi$ ($k + l = 0, 1, 2$). By integration by parts, we have the desired estimate. \square

Lemma 5.6. *Let $\chi \in C_0^\infty(\mathcal{O})$. Then*

$$\begin{aligned}
& (\nu + \tilde{\nu}) \|\chi \partial^k \partial_N^{l+1} \operatorname{div} w\|_2^2 \\
& \leq C \left\{ \frac{|\lambda|^2}{\nu + \tilde{\nu}} \|w\|_{H^{k+l}}^2 + \nu \|\chi \partial^{k+1} \partial_N^l \nabla w\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\chi \partial^k \partial_N^{l+1} \phi\|_2^2 \right. \\
& \quad \left. + \frac{\nu + \tilde{\nu}}{\gamma^4} \|f^0\|_{H^{k+l+1}}^2 + \frac{1}{\nu + \tilde{\nu}} \|\tilde{f}\|_{H^{k+l}}^2 + \frac{1}{\gamma^2} \|v\|_{H^{3+(k+l-2)_+}} \|\phi\|_{H^{k+l+1}}^2 \right\}
\end{aligned} \tag{5.15}$$

for $k + l = 0, 1, 2$.

Proof. Adding $(\nu + \tilde{\nu}) \partial_N \operatorname{div} w$ to (5.13), we have

$$(\nu + \tilde{\nu}) \partial_N \operatorname{div} w = \lambda w \cdot \mathbf{n} + \nabla \phi \cdot \mathbf{n} - \nu(\Delta w \cdot \mathbf{n} - \partial_N \operatorname{div} w) - \tilde{f} \cdot \mathbf{n}. \tag{5.16}$$

Multiplying (5.16) by $\chi^2 \partial_N \operatorname{div} w$ and integrating the resulting equation, we obtain

$$(\nu + \tilde{\nu}) \|\chi \partial_N \operatorname{div} w\|_2^2 \leq C \left\{ \frac{|\lambda|^2}{\nu + \tilde{\nu}} \|w\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\chi \partial_N \phi\|_2^2 + \nu \|\chi \partial \nabla w\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\tilde{f}\|_2^2 \right\}.$$

As for higher order derivatives we apply $\partial^k \partial_N^l$ to (5.16) and take the inner product of the resulting equation with $\chi^2 \partial^k \partial_N^{l+1} \operatorname{div} w$. Substituting (5.12) for the resulting equation, we have the desired estimate. \square

We next estimate $\|\chi \partial^k \partial_x^{l+2} w\|_2^2$ for $k + l = 0, 1$.

Lemma 5.7. *Let $\chi \in C_0^\infty(\mathcal{O})$. Then, for arbitrary δ satisfying $0 < \delta < 1$,*

$$\begin{aligned}
& \nu \|\chi \partial^k \partial_x^{l+2} w\|_2^2 + \frac{1}{\nu} \|\chi \partial^k \partial_x^{l+1} \phi\|_2^2 \\
& \leq C \left\{ \frac{\lambda^2}{\nu} \|w\|_{H^{k+l}}^2 + (\nu + \tilde{\nu}) \|\nabla w\|_{H^{k+l}}^2 + \frac{1}{\nu} \|\phi\|_{H^{k+l}}^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^{k+l}}^2 \right. \\
& \quad \left. + (\nu + \tilde{\nu}) \|\chi \partial^k \partial_x^{l+1} \operatorname{div} w\|_2^2 \right\}
\end{aligned} \tag{5.17}$$

for $k + l = 1, 2$, $k \geq 1$.

Proof. Applying $\chi\partial^k$ ($k = 1, 2$) to (5.1) and (5.2), it holds that

$$\begin{cases} \operatorname{div}(\chi\partial^k w) = F^0 & \text{in } \mathcal{O} \cap \Omega_{per}, \\ -\Delta(\chi\partial^k w) + \frac{1}{\nu}\nabla(\chi\partial^k \phi) = \tilde{F} & \text{in } \mathcal{O} \cap \Omega_{per}, \\ (\chi\partial^k w)|_{\Sigma_{j,-}} = (\chi\partial^k w)|_{\Sigma_{j,+}}, & (j = 1, \dots, n-1), \\ (\chi\partial^k w)|_{\Sigma_n} = 0, \end{cases} \quad (5.18)$$

where

$$\begin{aligned} F^0 &= [\nabla, \chi\partial^k]w - \lambda\chi\partial^k \phi - \frac{1}{\gamma^2}\chi\partial^k \operatorname{div}(\phi v) + \frac{1}{\gamma^2}\chi\partial^k f^0, \\ \tilde{F} &= [\Delta, \chi\partial^k]w + \frac{1}{\nu}[\nabla, \chi\partial^k]\phi - \frac{\lambda}{\nu}\chi\partial^k w - \frac{\tilde{\nu}}{\nu}\chi\partial^k \nabla \operatorname{div} w + \frac{1}{\nu}\chi\partial^k \tilde{f}. \end{aligned}$$

By using the estimates for the Stokes problem (see, e.g., [8]), we have the desired estimate. \square

Lemma 5.8. *Let $u = {}^\top(\phi, w)$ be a solution of (5.1)-(5.4). Then*

$$\begin{aligned} &\frac{|\lambda|^2}{2}\|w\|_2^2 + \operatorname{Re}\lambda\nu\|\nabla w\|_2^2 + \operatorname{Re}\lambda\tilde{\nu}\|\operatorname{div} w\|_2^2 \\ &\leq C\left\{\frac{1}{\gamma^2}\|f^0\|_2^2 + \|\tilde{f}\|_2^2 + \frac{1}{\gamma^2}\|v\|_{H^3}^2\|\phi\|_{H^1}^2 + \gamma^2\|\operatorname{div} w\|_2^2\right\}. \end{aligned} \quad (5.19)$$

Proof. We take the inner product of (5.2) with λw , we have

$$|\lambda|^2\|w\|_2^2 + \lambda\nu\|\nabla w\|_2^2 + \lambda\tilde{\nu}\|\operatorname{div} w\|_2^2 - (\bar{\lambda}\phi, \operatorname{div} w) = (\tilde{f}, \lambda w). \quad (5.20)$$

Substituting (5.1) for $\lambda\phi$ in (5.20), the forth term of left-hand side is written as $-(\bar{\lambda}\phi, \operatorname{div} w) = \frac{\bar{\lambda}}{\lambda}(\gamma^2\operatorname{div} w + \operatorname{div}(\phi v) - f^0, \operatorname{div} w)$. It then follows that

$$\begin{aligned} &\frac{|\lambda|^2}{2}\|w\|_2^2 + \operatorname{Re}\lambda\nu\|\nabla w\|_2^2 + \operatorname{Re}\lambda\tilde{\nu}\|\operatorname{div} w\|_2^2 \\ &\leq C\left\{\|\tilde{f}\|_2^2 + \frac{1}{\gamma^2}\|f^0\|_2^2 + \frac{1}{\gamma^2}\|v\|_{H^3}^2\|\phi\|_{H^1}^2 + \gamma^2\|\operatorname{div} w\|_2^2\right\}. \end{aligned}$$

\square

Lemma 5.9. *Let $u = {}^\top(\phi, w)$ be a solution of (5.1)-(5.4). Then*

$$\nu\|w\|_{H^{k+2}}^2 + \frac{1}{\nu}\|\phi\|_{H^{k+1}}^2 \leq C\left\{\frac{|\lambda|^2}{\nu}\|w\|_{H^k}^2 + (\nu + \tilde{\nu})\|\operatorname{div} w\|_{H^{k+1}}^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^k}^2\right\} \quad (5.21)$$

for $k = 0, 1, 2$.

Proof. We write (5.1)-(5.2) as

$$\begin{cases} \operatorname{div} w = -\lambda\phi - \frac{1}{\gamma^2}\operatorname{div}(\phi w) + \frac{1}{\gamma^2}f^0, & \text{in } \Omega_{per} \\ -\nu\Delta w + \nabla\phi = -\lambda w + \tilde{\nu}\nabla\operatorname{div} w + \tilde{f}, & \text{in } \Omega_{per} \\ w|_{\Sigma_n} = 0, \quad \phi|_{\Sigma_{j,-}} = \phi|_{\Sigma_{j,+}}, \quad w|_{\Sigma_{j,-}} = w|_{\Sigma_{j,+}} \quad (j = 1, \dots, n-1), & \text{on } \partial\Omega_{per}. \end{cases} \quad (5.22)$$

We set $\tilde{F} = \tilde{\nu}\nabla\operatorname{div} w + \tilde{f}$. Applying the estimate for Stokes problem, we have

$$\begin{aligned} \|w\|_{H^{k+2}}^2 + \frac{1}{\nu^2}\|\phi\|_{H^{k+1}}^2 &\leq C\left\{\frac{|\lambda|^2}{\nu^2}\|w\|_{H^k}^2 + \|\operatorname{div} w\|_{H^{k+1}}^2 + \frac{1}{\nu^2}\|\tilde{F}\|_{H^k}^2\right\} \\ &\leq C\left\{\frac{|\lambda|^2}{\nu^2}\|w\|_{H^k}^2 + \left(1 + \frac{\tilde{\nu}^2}{\nu^2}\right)\|\nabla\operatorname{div} w\|_{H^k}^2 + \frac{1}{\nu^2}\|\tilde{f}\|_{H^k}^2\right\} \\ &\leq C\left\{\frac{|\lambda|^2}{\nu^2}\|w\|_{H^k}^2 + \left(1 + \frac{\tilde{\nu}}{\nu}\right)\|\nabla\operatorname{div} w\|_{H^k}^2 + \frac{1}{\nu^2}\|\tilde{f}\|_{H^k}^2\right\}. \end{aligned} \quad (5.23)$$

Multiplying (5.23) by ν , we obtain (5.21). \square

Proof of Theorem 5.1. Estimate (5.5)₀ follows from (5.8).

Let $b_j^{(1)}$ ($j = 1, \dots, 4$) be positive numbers independent of ν , $\tilde{\nu}$ and γ and consider

$$\begin{aligned} &(5.7) + \frac{1}{\nu + \tilde{\nu}} \times (5.19) + b_1^{(1)} \times \{(5.9)_{k=1} + (5.10)_{k=1}\} \\ &+ \frac{b_2^{(1)}}{\nu} \times (5.12)_{k=l=0} + b_3^{(1)} \times (5.15)_{k=l=0} + b_4^{(1)} \times (5.21)_{k=0}. \end{aligned}$$

Taking $b_j^{(1)}$ ($j = 1, \dots, 4$) suitably small and $\varepsilon = \delta = \frac{1}{2C\nu}$, we have

$$\begin{aligned} &\operatorname{Re}\lambda(\gamma^{-2}\|\phi\|_{H^1}^2 + \|w\|_{H^1}^2 + \|\operatorname{div} w\|_2^2) + \frac{1}{2\nu}\|\phi\|_{H^1}^2 + \frac{1}{2}\nu\|w\|_{H^2}^2 \\ &+ \frac{1}{2}(\nu + \tilde{\nu})\|\operatorname{div} w\|_{H^1}^2 + \frac{1}{2}\frac{|\lambda|^2}{\nu + \tilde{\nu}}\|w\|_2^2 \\ &\leq C\left\{\nu\|\nabla w\|_2^2 + \frac{1}{\gamma^2}\|v\|_{H^3}\|\phi\|_{H^1}^2 + \frac{1}{\nu\gamma^2}\|v\|_{H^3}^2\|\phi\|_{H^1}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\operatorname{div} w\|_2^2 \right. \\ &\quad \left. + \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu\gamma^2}\right)\|f^0\|_{H^1}^2 + \frac{1}{\nu}\|\tilde{f}\|_2^2\right\}. \end{aligned} \quad (5.24)$$

Computing $\frac{1}{2C} \times (5.24) + (5.7)$ with $\varepsilon = \frac{1}{8C\nu} + (5.7)$ with $\varepsilon = \frac{\nu}{16C\gamma^2}$, we have

$$\begin{aligned}
& \operatorname{Re}\lambda(\gamma^{-2}\|\phi\|_{H^1}^2 + \|w\|_{H^1}^2 + \|\operatorname{div}w\|_2^2) + \frac{1}{8\nu}\|\phi\|_{H^1}^2 + \frac{1}{2}\nu\|w\|_{H^2}^2 \\
& + \frac{1}{2}(\nu + \tilde{\nu})\|\operatorname{div}w\|_{H^1}^2 + \frac{1}{2}\frac{|\lambda|^2}{\nu + \tilde{\nu}}\|w\|_2^2 + \frac{\operatorname{Re}\lambda}{(\nu + \tilde{\nu})^2}(\|\phi\|_2^2 + \gamma^2\|w\|_2^2) \\
& + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\operatorname{div}w\|_2^2 \\
& \leq C\left\{\left(\frac{1}{\nu^2} + \frac{1}{\gamma^2}\right)\|v\|_{H^3}\|\phi\|_{H^1}^2 + \frac{1}{\nu\gamma^2}\|v\|_{H^3}^2\|\phi\|_{H^1}^2\right. \\
& \quad \left.+ \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^1}^2 + \left(\frac{1}{\nu} + \frac{\gamma^2}{\nu^3}\right)\|\tilde{f}\|_2^2\right\}.
\end{aligned}$$

If v satisfies $\|v\|_{H^3} \leq \frac{1}{32C} \min\{\frac{\gamma^2}{\nu}, \nu, \gamma\}$, then

$$\begin{aligned}
& \operatorname{Re}\lambda(\gamma^{-2}\|\phi\|_{H^1}^2 + \|w\|_{H^1}^2 + \|\operatorname{div}w\|_2^2) + \frac{1}{16\nu}\|\phi\|_{H^1}^2 + \frac{1}{2}\nu\|w\|_{H^2}^2 \\
& + \frac{1}{2}(\nu + \tilde{\nu})\|\operatorname{div}w\|_{H^1}^2 + \frac{1}{2}\frac{|\lambda|^2}{\nu + \tilde{\nu}}\|w\|_2^2 + \frac{\operatorname{Re}\lambda}{(\nu + \tilde{\nu})^2}(\|\phi\|_2^2 + \gamma^2\|w\|_2^2) \\
& + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\operatorname{div}w\|_2^2 \\
& \leq C\left\{\left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^1}^2 + \left(\frac{1}{\nu} + \frac{\gamma^2}{\nu^3}\right)\|\tilde{f}\|_2^2\right\}.
\end{aligned} \tag{5.25}$$

Furthermore, if $\lambda \geq -\Lambda_1 := -\frac{1}{32} \min\{\frac{\gamma^2}{\nu}, \nu + \tilde{\nu}, \nu\}$, we see from (5.25) that

$$\begin{aligned}
& \operatorname{Re}(\lambda + \Lambda_1)(\gamma^{-2}\|\phi\|_{H^1}^2 + \|w\|_{H^1}^2) + \frac{1}{\nu}\|\phi\|_{H^1}^2 + \nu\|w\|_{H^2}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 \\
& \leq C\left\{\left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^1}^2 + \left(\frac{1}{\nu} + \frac{\gamma^2}{\nu^3}\right)\|\tilde{f}\|_2^2\right\},
\end{aligned} \tag{5.26}$$

and hence we have (5.5)₁ and

$$\nu\|w\|_{H^2}^2 \leq C\left\{\left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^1}^2 + \left(\frac{1}{\nu} + \frac{\gamma^2}{\nu^3}\right)\|\tilde{f}\|_2^2\right\}. \tag{5.27}$$

Let $b_j^{(2)}$ ($j = 1, 4, 5$) and $b_{j,kl}^{(2)}$ ($j = 2, 3$) be positive numbers independent of $\nu, \tilde{\nu}$ and γ and consider

$$\begin{aligned}
& (5.25) + \frac{1}{\nu} \frac{|\lambda|^2}{\nu + \tilde{\nu}} \times (5.8) + b_1^{(2)} \times \{(5.9)_{k=2} + (5.10)_{k=2}\} \\
& + \sum_{k+l=1} b_{2,kl}^{(2)} \times (5.12) + b_{3,kl}^{(2)} \times (5.15) + b_4^{(2)} \times (5.17)_{k=1,l=0} + b_5^{(2)} \times (5.21)_{k=1}.
\end{aligned}$$

Taking $b_j^{(2)}$ ($j = 1, 4, 5$) and $b_{j,kl}^{(2)}$ ($j = 2, 3$) suitably small and $\varepsilon = \frac{1}{2C\nu}$, we have

$$\begin{aligned}
& \operatorname{Re}\lambda(\gamma^{-2}\|\phi\|_{H^2}^2 + \|w\|_{H^1}^2 + \|\chi_0\partial^2 w\|_2^2 + \|\chi\partial^2 w\|_2^2) + \frac{1}{2\nu}\|\phi\|_{H^2}^2 + \frac{1}{2}\nu\|w\|_{H^3}^2 \\
& + \frac{1}{2}(\nu + \tilde{\nu})\|\operatorname{div} w\|_{H^2}^2 + \frac{1}{2}\frac{|\lambda|^2}{\nu + \tilde{\nu}}\|w\|_{H^1}^2 + \frac{|\lambda|^2}{\nu + \tilde{\nu}}\|\operatorname{div} w\|_2^2 \\
& + \operatorname{Re}\lambda\|\operatorname{div} w\|_2^2 + \frac{\operatorname{Re}\lambda}{(\nu + \tilde{\nu})^2}\left(\|\phi\|_2^2 + \gamma^2\|w\|_2^2\right) + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 \\
& + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\operatorname{div} w\|_2^2 + \frac{|\lambda|^2\operatorname{Re}\lambda}{(\nu + \tilde{\nu})^2}(\gamma^{-2}\|\phi\|_2^2 + \|w\|_2^2) + \frac{|\lambda|^2\delta_1}{(\nu + \tilde{\nu})^2}\|\phi\|_2^2 \\
& \leq C\left\{\nu\|\partial_x^2 w\|_2^2 + \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^2}^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^1}^2 + \frac{\gamma^2}{\nu^3}\|\tilde{f}\|_2^2\right. \\
& \quad \left.+ \frac{1}{\gamma^2}\|v\|_{H^3}\|\phi\|_{H^2}^2 + \frac{|\lambda|^2}{(\nu + \tilde{\nu})^2}\left(\left(\frac{1}{\delta_1\gamma^4} + \frac{\delta_1^2}{\nu}\right)\|f^0\|_2^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^{-1}}^2\right)\right\}.
\end{aligned} \tag{5.28}$$

Let v satisfy $\|v\|_{H^3} \leq \frac{1}{8C} \min\{\frac{\gamma^2}{\nu}, \gamma, \nu\}$. Computing $\frac{1}{2C} \times (5.28) + (5.25)$ we have

$$\begin{aligned}
& \operatorname{Re}\lambda(\gamma^{-2}\|\phi\|_{H^2}^2 + \|w\|_{H^1}^2 + \|\chi_0\partial^2 w\|_2^2 + \|\chi\partial^2 w\|_2^2) + \frac{1}{2\nu}\|\phi\|_{H^2}^2 + \frac{1}{2}\nu\|w\|_{H^3}^2 \\
& + \frac{1}{2}(\nu + \tilde{\nu})\|\operatorname{div} w\|_{H^2}^2 + \frac{1}{2}\frac{|\lambda|^2}{\nu + \tilde{\nu}}\|w\|_{H^1}^2 + \frac{|\lambda|^2}{\nu + \tilde{\nu}}\|\operatorname{div} w\|_2^2 \\
& + \operatorname{Re}\lambda\|\operatorname{div} w\|_2^2 + \frac{\operatorname{Re}\lambda}{(\nu + \tilde{\nu})^2}\left(\|\phi\|_2^2 + \gamma^2\|w\|_2^2\right) + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\nabla w\|_2^2 \\
& + \frac{\gamma^2}{\nu + \tilde{\nu}}\|\operatorname{div} w\|_2^2 + \frac{|\lambda|^2\operatorname{Re}\lambda}{(\nu + \tilde{\nu})^2}(\gamma^{-2}\|\phi\|_2^2 + \|w\|_2^2) + \frac{|\lambda|^2\delta_1}{(\nu + \tilde{\nu})^2}\|\phi\|_2^2 \\
& \leq C\left\{\frac{|\lambda|^2}{(\nu + \tilde{\nu})^2}\left(\left(\frac{1}{\delta_1\gamma^4} + \frac{\delta_1^2}{\nu}\right)\|f^0\|_2^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^{-1}}^2\right)\right. \\
& \quad \left.+ \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3}\right)\|f^0\|_{H^2}^2 + \frac{1}{\nu}\|\tilde{f}\|_{H^1}^2 + \frac{\gamma^2}{\nu^3}\|\tilde{f}\|_2^2\right\}.
\end{aligned}$$

If λ satisfies $\lambda \geq -\Lambda_2 := -\frac{1}{8} \min\{\nu, \delta_1 \gamma^2\}$, we have

$$\begin{aligned}
& \operatorname{Re} \lambda (\gamma^{-2} \|\phi\|_{H^2}^2 + \|w\|_{H^1}^2) + \frac{1}{\nu} \|\phi\|_{H^2}^2 + \nu \|w\|_{H^3}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\nabla w\|_2^2 \\
& + (\nu + \tilde{\nu}) \|\operatorname{div} w\|_{H^2}^2 + \frac{|\lambda|^2}{\nu + \tilde{\nu}} \|w\|_{H^1}^2 + \frac{|\lambda|^2}{\nu + \tilde{\nu}} \|\operatorname{div} w\|_2^2 \\
& \leq C \left\{ \frac{|\lambda|^2}{(\nu + \tilde{\nu})^2} \left(\left(\frac{1}{\delta_1 \gamma^4} + \frac{\delta_1^2}{\nu} \right) \|f^0\|_2^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^{-1}}^2 \right) \right. \\
& \quad \left. + \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3} \right) \|f^0\|_{H^2}^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^1}^2 + \frac{\gamma^2}{\nu^3} \|\tilde{f}\|_2^2 \right\}
\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}
& \operatorname{Re}(\lambda + \Lambda_2) (\gamma^{-2} \|\phi\|_{H^2}^2 + \|w\|_{H^1}^2) + \frac{1}{\nu} \|\phi\|_{H^2}^2 + \nu \|w\|_{H^3}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\nabla w\|_2^2 \\
& \leq C \left\{ \frac{|\lambda|^2}{(\nu + \tilde{\nu})^2} \left(\left(\frac{1}{\delta_1 \gamma^4} + \frac{\delta_1^2}{\nu} \right) \|f^0\|_2^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^{-1}}^2 \right) \right. \\
& \quad \left. + \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3} \right) \|f^0\|_{H^2}^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^1}^2 + \frac{\gamma^2}{\nu^3} \|\tilde{f}\|_2^2 \right\}.
\end{aligned}$$

Let $b_j^{(3)}$ ($j = 1, 5$) and $b_{j,kl}^{(3)}$ ($j = 2, 3, 4$) be positive numbers independent of $\nu, \tilde{\nu}$ and γ and consider

$$\begin{aligned}
& (5.29) + \frac{1}{\nu} \frac{\lambda^2}{\nu + \tilde{\nu}} \times (5.27) + b_1^{(3)} \times \{(5.9)_{k=3} + (5.10)_{k=3}\} \\
& + \sum_{k+l=2} \left\{ \frac{b_{2,kl}^{(3)}}{\nu} \times (5.12) + b_{3,kl}^{(3)} \times (5.15) \right\} \\
& + \sum_{\substack{k+l=2 \\ k \geq 1}} b_{4,kl}^{(3)} \times (5.17) + b_5^{(3)} \times (5.21)_{k=2}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
& \operatorname{Re}(\lambda + \Lambda_3) (\gamma^{-2} \|\phi\|_{H^3}^2 + \|w\|_{H^2}^2) + \frac{1}{\nu} \|\phi\|_{H^3}^2 + \nu \|w\|_{H^4}^2 + \frac{\gamma^2}{\nu + \tilde{\nu}} \|\nabla w\|_2^2 \\
& \leq C \left\{ \frac{|\lambda|^2}{(\nu + \tilde{\nu})^2} \left(\left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3} + \frac{1}{\delta_1 \gamma^4} + \frac{\delta_1^2}{\nu} \right) \|f^0\|_2^2 + \left(\frac{1}{\nu} + \frac{\gamma^2}{\nu^3} \right) \|\tilde{f}\|_{H^{-1}}^2 \right) \right. \\
& \quad \left. + \left(\frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{1}{\nu^3} \right) \|f^0\|_{H^3}^2 + \frac{1}{\nu} \|\tilde{f}\|_{H^2}^2 + \frac{\gamma^2}{\nu^3} \|\tilde{f}\|_2^2 \right\}.
\end{aligned}$$

Taking $\Lambda = \max\{\Lambda_1, \Lambda_2, \Lambda_3\}$, we have (5.5)₁-(5.5)₃.

When $\lambda = 0$, we can obtain the estimate $(5.6)_k$ for $k = 1, 2, 3$ in a similar manner to above. In fact, estimate $(5.6)_k$ with $k = 1$ is obtained as follows. We do not use (5.19) in the computation for (5.24) and obtain

$$\begin{aligned} & \frac{1}{2\nu} \|\phi\|_{H^1}^2 + \frac{1}{2} \nu \|w\|_{H^2}^2 + \frac{1}{2} (\nu + \tilde{\nu}) \|\operatorname{div} w\|_{H^1}^2 \\ & \leq C \left\{ \nu \|\nabla w\|_2^2 + \frac{1}{\gamma^2} \|v\|_{H^3} \|\phi\|_{H^1}^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} \|f^0\|_{H^1}^2 + \frac{1}{\nu} \|\tilde{f}\|_2^2 \right\}, \end{aligned} \quad (5.30)$$

instead of (5.24). Computing $\frac{1}{2C} \times (5.30) + (5.7)$ with $\varepsilon = \frac{1}{8C\nu}$, we have

$$\begin{aligned} & \frac{1}{2\nu} \|\phi\|_{H^1}^2 + \frac{1}{2} \nu \|w\|_{H^2}^2 + \frac{1}{2} (\nu + \tilde{\nu}) \|\operatorname{div} w\|_{H^1}^2 \\ & \leq C \left\{ \frac{1}{\gamma^2} \|v\|_{H^3} \|\phi\|_{H^1}^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} \|f^0\|_{H^1}^2 + \frac{1}{\nu} \|\tilde{f}\|_2^2 \right\}. \end{aligned} \quad (5.31)$$

The desired estimate $(5.6)_k$ with $k = 1$ follows from (5.31) by taking $\|v\|_{H^3}$ suitably small. One can prove $(5.6)_k$ for $k = 2, 3$ in a similar manner. \square

b. Existence of solution of (5.1)-(5.4)

In this subsection we prove the existence of a solution $u = {}^\top(\phi, w)$ of (5.1)-(5.4).

Let $k = 0, 1, 2, 3$. We write (5.1)-(5.2) in the form

$$(\lambda + \mathcal{A}_k[v])u = f \quad (5.32)$$

for $u = {}^\top(\phi, w) \in D(\mathcal{A}_k[v])$ and $f = {}^\top(f^0, \tilde{f}) \in H_{per}^k \times H_{per}^{k-1}$. Here $\mathcal{A}_k[v] : H_{per,*}^k \times H_{per}^{k-1} \longrightarrow H_{per,*}^k \times H_{per}^{k-1}$ is the operator defined by

$$\mathcal{A}_k[v] = \begin{pmatrix} \operatorname{div}(\cdot v) & \gamma^2 \operatorname{div} \\ \nabla & -\nu \Delta - \tilde{\nu} \nabla \operatorname{div} \end{pmatrix}$$

with domain

$$D(\mathcal{A}_k[v]) = \{u \in H_{per,*}^k \times (H_{per}^{k+1} \cap H_{0,per}^1); \mathcal{A}_k[v]u \in H_{per,*}^k \times H_{per}^{k-1}\}.$$

If we prove the existence of solution $u \in D(\mathcal{A}_k[v])$ of (5.32) for all $f \in H_{per,*}^k \times H_{per}^{k+1}$ with $\lambda \gg 1$, then we see from estimates (5.5)₀-(5.5)₃ that $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\Lambda\} \subset \rho(-\mathcal{A}_k[v])$ by a standard continuation argument.

The existence of solution of (5.32) for $\lambda \gg 1$ can be shown in a similar manner to the proof of Lemma 4.3. We first consider the problem

$$\lambda \phi + \operatorname{div}(\phi v) = F^0, \quad F^0 = f^0 - \gamma^2 \operatorname{div} \tilde{w} \quad (5.33)$$

for a given $\tilde{w} \in H_{per}^{k+1} \cap H_{0,per}^1$ and $f^0 \in H_{per,*}^k$. Applying the results of [10], we see that there exists a constant λ_0 satisfying $\lambda_0 \geq \|v\|_{H^3}$ such that if $\lambda \geq \lambda_0$, then the solution operator for (5.33) is an isomorphism on $H_{per,*}^k$ with operator norm bounded by $\frac{C}{\lambda}$ uniformly for $\lambda \gg 1$. We denote this solution operator by $\phi(\tilde{w})$. We next consider the problem

$$\begin{cases} \lambda w - \nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w = \tilde{F}, \\ w|_{\Sigma_{j,-}} = w|_{\Sigma_{j,+}}, \quad w|_{\Sigma_n} = 0, \end{cases} \quad (5.34)$$

where $\tilde{F} = \tilde{f} - \nabla \phi$ with $\phi = \phi(\tilde{w})$. Since $-\nu \Delta w - \tilde{\nu} \nabla \operatorname{div} w$ is strongly elliptic, the solution operator for (5.34) provides an isomorphism from H_{per}^{k-1} to $H_{per}^{k+1} \cap H_{0,per}^1$. One can then define the operator $\Gamma : H_{per}^{k+1} \cap H_{0,per}^1 \rightarrow H_{per}^{k+1} \cap H_{0,per}^1$ by the solution operator for (5.34), i.e., $\Gamma(\tilde{w}) = w$ for $\tilde{w} \in H_{per}^{k+1} \cap H_{0,per}^1$, where w is the solution of (5.34). In a similar manner to the proof of Lemma 4.3, one can prove that there exists a unique $w \in H_{per}^{k+1} \cap H_{0,per}^1$ satisfying $\Gamma(w) = w$ if $\lambda \gg 1$. For this fixed point w , we set $\phi = \phi(w)$. Then by (5.33)-(5.34), we see that $u = {}^\top(\phi, w) \in D(\mathcal{A}_k[v])$ is a solution of (5.32) with $\lambda \gg 1$. This completes the proof. \square

Remark 5.10. *The unique existence of (5.33) for $k = 0, 1, 2$ follows from [10, Theorems 1, 2, 9] respectively. As for the case $k = 3$, it was proved in [10, Theorem 11] under the assumption $v \in W^{m,p}$ for some $p > n$, where n is the space dimension. This condition for v can be replaced by $v \in H^4$ in our case $k = 3$. In fact, one can show the unique existence of solution $\phi \in H^3$ of (5.33) when $v \in H^4$ as in the proof of [10, Theorem 11]. The point is to guarantee $\phi \nabla \operatorname{div} v \in H^2$ for $\phi \in H^2$, which follows from $v \in H^4$.*

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References

- [1] Aoyama, R., Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a parallel flow in a cylindrical domain, *Kyushu J. Math.*, **69** (2015), pp. 293–343.
- [2] Aoyama, R. and Kagei, Y., Spectral properties of the semigroup for the linearized compressible Navier-Stokes equation around a parallel flow in a cylindrical domain, *Adv. Differential Equations*, **21** (2016), pp. 265–300.

- [3] Aoyama, R. and Kagei, Y., Large time behavior of solutions to the compressible Navier-Stokes equations around a parallel flow in a cylindrical domain, *Nonlinear Anal.*, **127** (2015), pp. 362–396.
- [4] Bogovskii, M. E., Solution of the first boundary value problem for an equation of continuity of an incompressible medium, *Soviet Math. Dokl.*, **20** (1979), pp. 1094–1098.
- [5] Březina, J., Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow, *SIAM J. Math. Anal.* **45** (2013), pp. 3514–3574.
- [6] Březina, J. and Kagei, Y., Spectral properties of the linearized compressible Navier-Stokes equation around time-periodic parallel flow, *J. Differential Equations* **255** (2013), pp. 1132–1195.
- [7] Deckelnick, K., Decay estimates for the compressible Navier-Stokes equations in unbounded domains, *Math. Z.* **209** (1992), pp. 115–130.
- [8] Galdi G. P., *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Vol. 1, Springer-Verlag, New York (1994).
- [9] Hoff, D. and Zumbrun, K., Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow, *Indiana Univ. Math. J.* **44** (1995), pp. 604–676.
- [10] Heywood, J.G. and Padula M., On the steady transport equation. in *Fundamental Directions in Mathematical Fluid Mechanics*, ed. by Galdi, G.P., Heywood, J.G., Rannacher, R. Birkhäuser, (2000), pp. 149–170.
- [11] Iooss, G. and Padula, M., Structure of the linearized problem for compressible parallel fluid flows, *Ann. Univ. Ferrara Sez.*, **VII 43** (1998), pp. 157–171.
- [12] Kagei, Y., Asymptotic behavior of solutions of the compressible Navier-Stokes equation around a parallel flow, *Arch. Rational Mech. Anal.*, **205** (2012), pp. 585–650.
- [13] Kagei Y., Nagafuchi Y. and Sudou T., Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow, *Journal of Math-for-Industry*, **2** (2010A), pp. 39–56. Correction to "Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow" in *J. Math-for-Ind.*, **2** (2010A), pp. 39–56, *J. Math-for-Ind.*, **2** (2010B), pp. 235.

- [14] Kagei, Y. and Kobayashi, T., Asymptotic Behavior of Solutions of the Compressible Navier-Stokes Equation on the Half Space, Arch. Rational Mech. Anal., **177** (2005), pp. 231–330.
- [15] Kagei, Y., Makio, N., Spectral properties of the linearized semigroup of the compressible Navier-Stokes equation on a periodic layer, Publ. Res. Inst. Math. Sci., **51**, (2015), pp. 337–372.
- [16] Kato, T., *Perturbation theory for linear operators*, Springer-Verlag, Berlin, Heidelberg, New York, (1980).
- [17] Kawashima, S., Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnethydrodynamics, Ph. D. Thesis, Kyoto University (1983).
- [18] Kobayashi, T., Some estimates of solutions for the equations of motion of compressible viscous fluid in an exterior domain in \mathbf{R}^3 , J. Differential Equations **184** (2002), pp. 587-619.
- [19] Kobayashi, T. and Shibata, Y., Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in \mathbf{R}^3 , Comm. Math. Phys. **200** (1999), pp. 621-659.
- [20] Matsumura, A., An energy method for the equations of motion of compressible viscous and heat-conductive fluids, University of Wisconsin-Madison, MRC Technical Summary Report # 2194 (1981), pp. 1-16.
- [21] Matsumura, A. and Nishida, T., The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A **55** (1979), pp. 337-342.
- [22] Matsumura, A. and Nishida T., Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, Commun. Math. Phys. **89** (1983), pp. 445-464.
- [23] Reed, M., Simon, B., *Methods of modern mathematical physics IV*. Academic Press, (1979).
- [24] Shibata, Y. and Tanaka, K., Rate of convergence of non-stationary flow to the steady flow of compressible viscous fluid, Comput. Math. Appl. **53** (2007), pp. 605-623.
- [25] Valli, A., On the existence of stationary solutions to compressible Navier-Stokes equations, Ann. Inst. H Poincaré, **4** (1987), pp. 99-113.

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