Dissipative Structure and Asymptotic Behavior for Symmetric Hyperbolic Systems

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Dissipative Structure and Asymptotic Behavior for Symmetric Hyperbolic Systems

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Chapter 1

Introduction

The dissipative structure of symmetric hyperbolic systems

$$A^0(u)\,u_t + \sum_{j=1}^{n} A^j(u)\,u_{x_j} + L(u)\,u = 0,$$

which include the discrete velocity model of the Boltzmann equations, and symmetric hyperbolic-parabolic systems of conservation laws

$$A^0(u)\,u_t + \sum_{j=1}^{n} A^j(u)\,u_{x_j} + L(u)\,u = \sum_{j,\,k=1}^{n} B^{jk}(u)\,u_{x_j x_k},$$

which include the compressible Navier-Stokes equations, was completely characterized by the stability condition called the Shizuta-Kawashima condition (see [58, 67]), which gives us the asymptotic stability result and the explicit rate of convergence: if the degenerate dissipation matrix L is symmetric, Y. Shizuta and S. Kawashima designed the compensating matrix K to capture the dissipation of systems over the degenerate kernel space of L.

Now we are concerned with the dissipative structure and asymptotic behavior of these systems that abound in great interests.

In order to investigate the dissipative structure, which is the key concept to consider the stability for $t \to \infty$, we define *dissipativity* or *strict dissipativity* by using the eigenvalue $\lambda = \lambda(i\xi)$ of each system as follows:

Definition 1.0.1 (**Dissipative structures:** Key to the stability for $t \to \infty$)**.** *Let $\lambda = \lambda(i\xi)$ be the eigenvalue. Then "dissipativity" or "strict dissipativity" is defined by λ, which characterises the dissipative structures as follows, respectively:*

- *Dissipativity:* $\operatorname{Re}\lambda(i\xi) \le 0$ *for any ξ.*
- *Strict dissipativity:* $\operatorname{Re}\lambda(i\xi) < 0$ *for any $\xi \neq 0$.*

Moreover, in order to discuss *strict dissipativity* in detail, we characterize it by introducing the following inequality of *Type* (p, q):

Definition 1.0.2 (**Strict dissipativity of** *Type* (p,q))**.** *We define strict dissipativity as the following inequality of Type* (p,q)*:*

$$\operatorname{Re}\lambda(i\xi) \le -\frac{c|\xi|^{2\mathbf{p}}}{(1+|\xi|^2)^{\mathbf{q}}},$$

where c *is a constant.*

Y. Shizuta and S. Kawashima concluded that the dissipative structure of symmetric hyperbolic systems or symmetric hyperbolic-parabolic systems could be characterized as **Type** $(1,1)$ in their general theory.

Recentry, however, we found some physical models to which the general theory of dissipative structure could not be applicable. Moreover, it has been shown that the dissipative structures of these systems are very different from that obtained in the general theory as follows:

Dissipative structure	*Type* (p,q)	*Examples*
General theory	**p=1, q=1**	Boltzmann equations, Compressible Navier-Stokes equations, *etc.*
New dissipativity	p=1, q=2	Dissipstive Timoshenko system, Euler-Maxwell equations
	p=2, q=3	Timoshenko-Fourier system, Timoshenko-Cattaneo system, Timoshenko system with memory.

Inspired by recent concrete examples, such as dissipative Timoshenko systems and compressible Euler-Maxwell equations and so on, the matrix L has the skew-symmetric part and is not symmetric. In this case, the partial positivity on $\operatorname{Ker}(L_1)^\perp$ ($L_1 :=$ the symmetric part of L) is available only. For this problem Y. Ueda, R. Duan and S. Kawashima [64] found a real compensating matrix S to make up the full positivity on $\operatorname{Ker}(L)^\perp$. Consequently, they developed decay properties of Type $(1,2)$ with weaker dissipative mechanism. However, it has been an open problem to develop a general way to capture the dissipation of the systems with decay properties of Type $(2,3)$, which is of much weaker dissipativity also in the low frequency region.

Indeed, the new dissipative strudtures are very weak in the high frequency region, which causes regularity-loss in the dissipation terms of the energy estimate and the time decay estimate of the solutions to the linearized system: therefore we call the dissipative structure of Type $(1,2)$ or Type $(2,3)$ **regularity-loss type**.

The difficulty caused by this fact prevents us from establishing the new method to analyze symmetric hyperbolic systems and symmetric hyperbolic-parabolic systems generally.

1.1 Examples of regularity-loss type

Typical examples of the models which have the dissipative structures of regularity-loss type are the Timoshenko system, which describes the vibration of the beam, and the Euler-Maxwell system, which describes the plasma phenomenon. In this section, we give the short introductions for these systems.

1.1.1 Timoshenko system

The original Timoshenko system was first introduced by S.P. Timoshenko (see [62, 63]) as the model system which describes the vibration of the beam called the Timoshenko beam. The Timoshenko beam theory has the advantage of describing not only the transversal movement but also the shear deformation and the rotational inertia effects. He described it in the form

$$\begin{cases} \rho\,\varphi_{tt} = \{K\,(\varphi_x - \psi)\}_x & (t,x) \in \mathbb{R}^+ \times (0,L), \\ I_\rho\,\psi_{tt} = (EI\psi_x)_x + K\,(\varphi_x - \psi) & (t,x) \in \mathbb{R}^+ \times (0,L) \end{cases} \tag{1.1.1}$$

with the boundary conditions

$$EI\psi_x = 0, \quad K\,(\varphi_x - \psi) = 0 \qquad \text{on} \ \ x = 0,\ L. \tag{1.1.2}$$

Here t is the time variable and x is the spacial variable which denotes a point on the center line of the beam. The unknown functions $\varphi = \varphi(t,x)$ and $\psi = \psi(t,x)$ denote the transversal displacement of the beam from the equilibrium state and the rotation angle of the filament of the beam, respectively. Note that the term $\varphi_x - \psi$ denotes the shearing stress. The coefficients ρ, I_ρ, E, I and K denote the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus, respectively. We see that the system (1.1.1) with the boundary condition (1.1.2) is conservative so that the total energy of the beam remains constant in time. For a physical derivation of the system, we refer to [20].

In recent years, the subject of the stability of the Timoshenko-type systems has received lots of attention. Especially for bounded reference configurations, we have many papers not only for the linear systems but also the nonlinear systems with relaxation parameters or memory terms, concerning uniform and decay of energy; see [3, 52, 56]. For similar problems dealing with the stability theory for the Timoshenko system with thermal dissipation, we refer to [51]. All the above mentioned papers treated the Timoshenko system in a bounded domain in which the Poincarè inequality and the type of the boundary conditions play a decisive role.

1.1.2 Euler-Maxwell system

Compressible Euler-Maxwell equations appear in the mathematical modelling of semiconductor sciences. When semiconductor devices are operated under some high frequency conditions, such as photoconductive switches, electro-optics, semiconductor lasers and high-speed computers, *etc.*, the electron transport in devices interacts with the propagating electromagnetic waves. Consequently, the Euler-Maxwell equations

$$\begin{cases} \partial_t n + \nabla\cdot(nu) = 0 & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t(nu) + \nabla\cdot(nu\otimes u) + \nabla p(n) = -n\left(E + u\times B\right) - nu, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t E - \nabla\times B = nu & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t B + \nabla\times E = 0 & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3 \end{cases}$$

with constraints

$$\nabla\cdot E = n_\infty - n, \quad \nabla\cdot B = 0 \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3$$

which take the the form of Euler equations for the conservation laws of mass density, current density and energy density for electrons, coupled to Maxwell's equations for self-consistent electromagnetic field, are introduced to describe the transport process. Here the unknowns $n>0,\ u\in\mathbb{R}^3$ are the density and the velocity of electrons, and $E\in\mathbb{R}^3,\ B\in\mathbb{R}^3$ denote the electric field and magnetic field, respectively. The pressure $p(n)$ is a given smooth function of n satisfying $p'(n)>0$ for $n>0$ and n_∞ is assumed to be a positive constant, which stands for the density of positively charged background ions. The reader is also referred to [8, 36] for more explanation.

1.2 Aim & Abstract

Our goal of this sybject is, based on the former theory of the Shizuta-Kawashima condition, to clarify the new decay property of regularity-loss type represented by the Timoshenko system, and build up the general way to overcome the difficulty of the weak dissipation caused by the regularity-loss property and show the corresponding energy decay structures precisely.

To this end, we start with seeking the lowest regularity index for the optimal time decay rate of the solutions in L^2 to the compressible Euler-Maxwell equations with the weaker dissipative mechanism. In Chapter 2, we develop a new time-decay estimate of $L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$-$L^r(\mathbb{R}^n)$ type by using the low-frequency and high-frequency analysis in Fourier spaces, and apply it to the equations to show that the minimal decay regularity coincides with the critical regularity for the global-in-time solutions. Due to the dissipative structure of regularity-loss, extra higher regularity than that for the global-in-time existence is usually imposed to obtain the optimal time decay rates of classical solutions to dissipative systems. Consequently, a notion of minimal decay regularity for dissipative systems of regularity-loss is firstly proposed. Moreover, the recent decay property for symmetric hyperbolic systems with non-symmetric dissipation is also extended to be the L^p-version.

From Chapter 3, we investigate the decay property and the nonlinear stability of the Timoshenko system by introducing various dissipative mechanism.

In Chapter 3, we consider the nonlinear version of the Timoshenko system with a frictional damping as the dissipative mechanism. We prove the global-in-time existence and uniqueness of the solutions under smallness assumption on the initial data in the Sobolev space H^2 (with the critical regularity-index) by employing the energy method. To this end, we first make a refinement of the energy method in the Fourier space employed in Ide-Haramoto-Kawashima for the linearized system, which gives an improvement on the energy estimate in Ide-Haramoto-Kawashima. Moreover, for initial data in $H^2 \cap L^1$, we show that the solutions decay in L^2 at the the optimal time decay rate $t^{-1/4}$ for $t \to \infty$. The proof starts with obtaining the precise estimate of the nonlinear solutions by employing the energy method in the Fourier space. And then, we apply the refined time-decay estimate of $L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$-$L^r(\mathbb{R}^n)$ type developped in Chapter 2 to the linear part and the nonlinear part of the energy estimate of the solutions, respectively.

In Chapter 4 and Chapter 5, we first show the global-in-time existence and uniqueness, and optimal decay rates of the solutions to the dissipative Timoshenko system in the framework of Besov spaces.

In Chapter 4, we consider the case of equal wave speeds, and construct the global solutions to the Cauchy problem pertaining to data in the spatially Besov spaces. Furthermore, the dissipative structure enables us to give a new decay framework which pays less attention on the traditional spectral analysis. Consequently, the optimal decay estimates of the solution and its derivatives of fractional order are shown by time-weighted energy approaches in terms of low-frequency and high-frequency decompositions. As a by-product, the usual decay estimate of $L^1(\mathbb{R})$-$L^2(\mathbb{R})$ type is also shown.

In Chapter 5, as a continued work of Chapter 4, we are concerned with the dissipative Timoshenko system in the case of non-equal wave speeds. Firstly, with the modification of a priori estimates in Chapter 4, we construct global solutions to the Timoshenko system pertaining to data in the Besov space with the regularity $s = 3/2$. Owing to the weaker dissipative mechanism, extra higher regularity than that for the global-in-time existence is usually imposed to obtain the optimal decay rates of classical solutions, and therfore it is almost impossible to obtain the optimal decay rates in the critical space. To overcome the outstanding difficulty, we develop a new frequency-localization time-decay inequality, which captures the information related to the integrability at the high-frequency part. Furthermore, by the energy approach in terms of high-frequency and low-frequency decomposition, we show the optimal decay rate for Timoshenko system in critical Besov spaces, which improves previous works greatly.

In Chapter 6 we study the Timoshenko system with Fourier's type heat conduction in the one-dimensional whole space. Note that our Timoshenko-Fourier system does not include any mechanical damping. We observe that the dissipative structure of the system is of the regularity-loss type which is a little different from that of the dissipative Timoshenko system studied in Ide-Haramoto-Kawashima. Moreover, we show the optimal L^2 decay estimates of the solution in a general situation. The proof is based on the detailed pointwise estimates of the solution in the Fourier space. Since the dissipative structure of the system is of the regularity-loss type for $a \neq 1$, our energy method looks similar to the one for the previous system (the dissipative Timoshenko system) but is much more complicated due to its weak dissipativity.

In Chapter 7 we consider the Timoshenko system with Cattaneo's type heat conduction in the one-dimensional whole space. We investigate the dissipative structure of the system and derive the optimal L^2 decay estimate of the solution in a general situation. Our decay estimate is based on the detailed pointwise estimate of the solutions in the Fourier space. This decay property is a little different from that of the dissipative Timoshenko system ([24]) in the low frequency region. However, in the high frequency region, it is just the same as that of the Timoshenko-Fourier system in Chapter6 or the dissipative Timoshenko system in Chapter3, although the stability number is different. Finally, we study the decay property of the Timoshenko system with the thermal effect of memory-type by reducing it to the Timoshenko-Cattaneo system.

In Chapter 8, we consider the initial value problem for the Timoshenko system with a memory term in the one-dimensional whole space. The aim of this chapter is to show the global-in-time existence and uniqueness of the solution to the Cauchy problem under the lowest regularity assumption on the initial data. To this end, we remake the pointwise estimate of the solution to the linearized system developped in [32] in order to get the way to construct the Lyapunov function which minimizes the number of the dissipation terms of regularity-loss. Next, we characterize the dissipative structure of the system by the straight calculation of the asmnptotic expansions of the eigenvalues. This characterization confirms that our pointwise estimate is optimal. Finally, based on our linearized system results, we investigate the nonlinear system and obtain the global-in-time existence and uniqueness in the critical sobolev space H^2. That is, we show that the global-in-time existence and uniqueness of the system could be proved in the minimal regularity assumption on the initial data and no need to employ any time-weighted norm as Liu and Kawashima did in [33]. This implies that our refinement of the Lyapunov function and its application to the nonlinear system completely overcomes the difficulty caused by the weak dissipation due to the regularity-loss property of the Timoshenko system with a memory term.

Notations:

- The Fourier transform $\hat{f}$ (or $\mathcal{F}[f]$) of a function $f \in \mathcal{S}$ (the Schwarz class) is denoted by
$$\mathcal{F}[f] := \int_{\mathbb{R}^n} f(x)\, e^{-2\pi x\cdot\xi} dx.$$
The Fourier transform of a tempered function in $\mathcal{S}'$ is defined by the dual argument in the standard way. Moreover, $\mathcal{F}^{-1}[f]$ stands for the inverse Fourier transform in $\mathbb{R}^n$.

- For $1 \le p \le \infty$, we denote by $L^p = L^p(\mathbb{R})$ the usual Lebesgue space on $\mathbb{R}$ with the norm $\|\cdot\|_{L^p}$.

- Denote by $\mathcal{C}([0,T],X)$ (resp.,$\mathcal{C}^1([0,T],X)$) the space of continuous (resp., continuously differentiable) functions on $[0,T]$ with values in a Banach space X.

- Also, $\|(f,g,h)\|_X$ means $\|f\|_X + \|g\|_X + \|h\|_X$, where $f,g,h \in X$.

- Every positive constants are denoted by the same symbol C or c without confusion. Besides, $f \lesssim g$ means $f \le Cg$, and $f \approx g$ means $f \lesssim g$ and $g \lesssim f$ simultaneously.

Chapter 2

Compressible Euler-Maxwell equations

2.1 Introduction

Let us consider isentropic Euler-Maxwell equations where the energy equation is replaced with state equation of the pressure-density relation. Precisely,

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) = -n\,(E + u \times B) - nu, \\ \partial_t E - \nabla \times B = nu, \\ \partial_t B + \nabla \times E = 0, \end{cases} \tag{2.1.1}$$

with constraints

$$\nabla \cdot E = n_\infty - n, \quad \nabla \cdot B = 0 \tag{2.1.2}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^3$. Observe that, for any vector $B_\infty \in \mathbb{R}^3$, system (2.1.1) admits a constant equilibrium state of the form

$$(n_\infty,\, 0,\, 0,\, B_\infty), \tag{2.1.3}$$

which are regarded as vectors in $\mathbb{R}^{10}$. In this chapter, we are concerned with problem (2.1.1)-(2.1.2) with the initial data

$$(n,\, u,\, E,\, B)|_{t=0} = (n_0,\, u_0,\, E_0,\, B_0)(x), \qquad x \in \mathbb{R}^3. \tag{2.1.4}$$

Notice that the constraint condition (2.1.2) holds true for every $t > 0$, if it holds at time $t = 0$, namely we assume that

$$\nabla \cdot E_0 = n_\infty - n_0, \quad \nabla \cdot B_0 = 0, \qquad x \in \mathbb{R}^3. \tag{2.1.5}$$

2.1.1 Known results & Motivation

System (2.1.1) is partially dissipative due to the damping term in the momentum equations. In one space dimension, by using the Godunov scheme with the fractional step together with the compensated compactness theory, Chen, Jerome, and Wang [8] constructed the global existence of weak solutions. In several space dimensions, the question of global weak solution of (2.1.1) is open, and only the global existence and large-time behavior of smooth solutions have been studied. To state some known results, it is convenient to reformulate the system (2.1.1) as

$$\begin{cases} \partial_t n + u\cdot\nabla n + n\,\mathrm{div}u = 0, \\ \partial_t u + (u\cdot\nabla)\,u + a(n)\nabla n + E + u\times B + u = 0, \\ \partial_t E - \nabla\times B - nu = 0, \\ \partial_t B + \nabla\times E = 0, \end{cases} \tag{2.1.6}$$

where $a(n) := p'(n)/n$ is the enthalpy function. For simplicity, we set $w = (n, u, E, B)^\top$ ($\top$: transpose), which is a column vector in $\mathbb{R}^{10}$. Then (2.1.6) can be written in vector form

$$A^0(w)\,w_t + \sum_{j=1}^{3} A^j(w)\,w_{x_j} + L(w)\,w = 0, \tag{2.1.7}$$

where the coefficient matrices are given explicitly by

$$A^0(w) = \begin{pmatrix} a(n) & 0 & 0 & 0 \\ 0 & nI & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad L(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & n\,(I-\Omega_B) & nI & 0 \\ 0 & -nI & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sum_{j=1}^{3} A^j(w)\xi_j = \begin{pmatrix} a(n)(u\cdot\xi) & p'(n)\,\xi & 0 & 0 \\ p'(n)\,\xi^\top & n\,(u\cdot\xi)\,I & 0 & 0 \\ 0 & 0 & 0 & -\Omega_\xi \\ 0 & 0 & \Omega_\xi & 0 \end{pmatrix}.$$

Here I is the identity matrix of third order, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, and Ω_ξ is the skew-symmetric matrix defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

which implies that $\Omega_\xi E^\top = (\xi\times E)^\top$ (as a column vector in $\mathbb{R}^3$) for $E = (E_1, E_2, E_3) \in \mathbb{R}^3$. Let us mention that (2.1.7) is a symmetric hyperbolic system, since $A^0(w)$ is real symmetric and positive definite matrix and $A^j(w)(j=1,2,3)$ are real symmetric. It is easy to check that the dissipative matrix $L(w)$ is nonnegative definite, however, $L(w)$ *is not real symmetric*. Such partial dissipation forces (2.1.1) to go beyond the class of generally dissipative hyperbolic systems satisfying the Kawashima-Shizuta condition, which have been well studied by [6, 21, 27, 29, 30, 58, 67, 77] in Sobolev spaces, [70, 71] in critical Besov spaces, see also references therein.

So far there are a number of efforts on global smooth solutions for (2.1.1) by various authors, see [15, 17, 45, 46, 61, 66] on the framework of Sobolev spaces, [69, 75] in critical Besov spaces. In the present chapter, we pay attention to global smooth solutions constructed in Sobolev spaces. Here we adopt some notations introduced in [66]:

$$w_\infty=(n_\infty,0,0,B_\infty)^\top,\quad w_0=(n_0,u_0,E_0,B_0)^\top;$$

$$N_0(t):=\sup_{0\le\tau\le t}\|(w-w_\infty)(\tau)\|_{H^s};$$

$$D_0(t)^2:=\int_0^t\left(\|(n-n_\infty,u)(\tau)\|_{H^s}^2+\|E(\tau)\|_{H^{s-1}}^2+\|\nabla B(\tau)\|_{H^{s-2}}^2\right)d\tau.$$

The global existence of smooth solutions is drawn briefly as follows (see [66] for details):

$$\begin{cases}\text{The initial data } w_0-w_\infty\in H^s(s\ge 3) \text{ and (2.1.5)},\\ I_0:=\|w_0-w_\infty\|_{H^s}\le\varepsilon_0,\end{cases}\tag{2.1.8}$$

$$\Rightarrow\begin{cases}w-w_\infty\in C\left([0,\infty);H^s\right)\cap C^1\left([0,\infty);H^{s-1}\right),\\ N_0(t)^2+D_0(t)^2\le CI_0^2.\end{cases}\tag{2.1.9}$$

Remark 2.1.1. Due to the non-symmetric dissipation, there is an 1-regularity-loss phenomenon of dissipation rates from the electromagnetic part (E,B), which was first observed in [15]. The regularity index $s\ge 4$ was needed for global smooth solutions. Later, this result was improved in [66], by relaxing the regularity requirement to $s\ge 3$, which coincides with the regularity needed for the standard result of local in time existence of smooth solutions [26, 35]. Denote by s_c the critical regularity for smooth solutions to dissipative systems. Therefore, it follows from [66] that "$s_c=3$" for the Euler-Maxwell system in $\mathbb{R}^3$.

In this chapter, we focus on the time decay of classical solutions to (2.1.1). For this purpose, we rewrite (2.1.1) as the linearized perturbation form around the equilibrium state w_∞. Set $\upsilon=nu/n_\infty$. Then

$$\begin{cases}\partial_t n+n_\infty\mathrm{div}\upsilon=0,\\ \partial_t\upsilon+a_\infty\nabla n+E+\upsilon\times B_\infty+\upsilon=(\mathrm{div}q_2+r_2)/n_\infty,\\ \partial_t E-\nabla\times B-n_\infty\upsilon=0,\\ \partial_t B+\nabla\times E=0,\end{cases}\tag{2.1.10}$$

where $a_\infty = p'(n_\infty)/n_\infty$,

$$q_2 = -n_\infty^2 v \otimes v/n - [\, p(n) - p(n_\infty) - p'(n_\infty)(n - n_\infty)\,]\, I$$

and

$$r_2 = -(n - n_\infty)\, E - n_\infty v \times (B - B_\infty).$$

We put $z := (\rho, v, E, h)^\top$, where $\rho = n - n_\infty$ and $h = B - B_\infty$. The corresponding initial data are given by

$$z|_{t=0} = (\rho_0,\, v_0,\, E_0,\, h_0)^\top(x) \tag{2.1.11}$$

with $\rho_0 = n_0 - n_\infty,\ v_0 = n_0 u_0/n_\infty$ and $h_0 = B_0 - B_\infty$. System (2.1.10) is also rewritten in vector form as

$$A^0 z_t + \sum_{j=1}^{3} A^j z_{x_j} + Lz = \sum_{j=1}^{3} Q_{x_j} + R, \tag{2.1.12}$$

where A^0, A^j and L are the constant matrices given by (2.1.7) with $w = w_\infty$, $Q(z) = (0, q_2^j/n_\infty, 0, 0)^\top$ and $R(z) = (0, r_2/n_\infty, 0, 0)^\top$. Notice that $Q(z) = O(|(\rho, v)|^2)$ and $R(z) = O(\rho|E| + |v||h|)$, which will be useful in the following.

The linearized form of (2.1.12) reads as

$$A^0 \partial_t z_{\mathcal{L}} + \sum_{j=1}^{3} A^j \partial_{x_j} z_{\mathcal{L}} + L z_{\mathcal{L}} = 0, \tag{2.1.13}$$

and the corresponding initial data $z_0 := (\rho_0, v_0, E_0, h_0)^\top$ satisfy

$$\mathrm{div} E_0 = -\rho_0, \quad \mathrm{div} h_0 = 0. \tag{2.1.14}$$

Next, we apply the Fourier transform of (2.1.13) to get

$$A^0 \partial_t \widehat{z_{\mathcal{L}}} + i|\xi| A(\omega)\, \widehat{z_{\mathcal{L}}} + L \widehat{z_{\mathcal{L}}} = 0, \tag{2.1.15}$$

where $A(\omega) = \sum_{j=1}^{3} A^j \omega_j$, and $\omega = \xi/|\xi| \in \mathbb{S}^2$. Set

$$\hat{\Phi}(\xi) = (A^0)^{-1} \left(i|\xi| A(\omega) + L \right).$$

We define the Green matrix $\widehat{\mathcal{G}(t)z} := e^{-t\hat{\Phi}(\xi)} \hat{z}$ which is a mapping from X_ξ to X_ξ with $X_\xi = \{\hat{z} \in \mathbb{C}^{10} \,|\, i\xi \cdot \hat{E} = -\hat{\rho},\ i\xi \cdot \hat{h} = 0\} \subset \mathbb{C}^{10}$. Then the linearized solution $z_{\mathcal{L}}$ of (2.1.13)-(2.1.14) is given by $\mathcal{G}(t) z_0$.

As shown by [65], by using the energy method in Fourier spaces, the Fourier image of $z_{\mathcal{L}}$ satisfies the following pointwise estimate

$$|\widehat{z_{\mathcal{L}}}(t, \xi)| \lesssim e^{-c_0 \eta(\xi) t} |\hat{z}_0| \tag{2.1.16}$$

for any $t \geq 0$ and $\xi \in \mathbb{R}^3$, where the dissipative rate $\eta(\xi) = |\xi|^2/(1 + |\xi|^2)^2$ and $c_0 > 0$ is a constant. Furthermore, the decay property was achieved:

$$\|\partial_x^k z_{\mathcal{L}}\|_{L^2} \lesssim (1+t)^{-\frac{3}{4} - \frac{k}{2}} \|z_0\|_{L^1} + (1+t)^{-\frac{\ell}{2}} \|\partial_x^{k+\ell} z_0\|_{L^2}, \tag{2.1.17}$$

where k and ℓ are non-negative integers.

Remark 2.1.2. The decay (2.1.17) is of the regularity-loss type, since $(1+t)^{-\ell/2}$ is created by assuming the additional ℓ-th order regularity on the initial data. Consequently, for the nonlinear Euler-Maxwell system, extra higher regularity than that for global-in-time existence of classical solutions is imposed to obtain the optimal decay rates. Actually, similar phenomena also appear in the study of other dissipative systems, for instance, quasi-linear hyperbolic systems of viscoelasticity in [12, 14], hyperbolic-elliptic systems of radiating gas in [23], dissipative Timoshenko systems in [25, 33], Vlasov-Maxwell-Boltzmann system in [18], and a plate equation with rotational inertia effect in [60], *etc.*.

A natural question follows. Which index characterises the minimal regularity for the optimal time decay for dissipative systems of regularity-loss? This motivates the following

Definition 2.1.1. *If the optimal decay rate of $L^1(\mathbb{R}^n)$-$L^2(\mathbb{R}^n)$ type is achieved under the lowest regularity assumption, then the lowest index is called the minimal decay regularity index of dissipative systems of regularity-loss, which is labelled as s_D.*

According to dissipative systems of regularity loss mentioned in Remark 2.1.2, it is not difficult to see that $s_D > s_c$. For example, for compressible Euler-Maxwell equations (2.1.1), we see that $s_c = 3$, whereas $s_D = 6$ was shown by Duan and his collaborators [15, 17], Ueda and Kawashima [65] independently. If the regularity of initial data is imposed higher than $s_D = 6$, then more decay information of solutions can be available, *e.g.*, see [15, 17, 61]. However, this is beyond our primary interest.

2.1.2 Aim, Strategy & Main results

The interest of this chapter is to seek the minimal decay regularity for (2.1.1) such that $s_D = 3$, which is the same critical regularity as $s_c = 3$ for global classical solutions. Obviously, such regularity assumption on the initial data is reduced heavily in comparison with known results in [15, 17, 61, 65] and therein references. Due to the less regularity assumption, actually, previous techniques used are invalidated.

To do this, we first develop a time-decay inequality of L^p-L^q-L^r type, which is a new and crucial ingredient in the present chapter. As we known, Umeda, Kawashima and Shizuta initiated a decay inequality of L^2-L^q-L^2 type with $\eta(\xi) = |\xi|^2/(1+|\xi|^2)$ in the earlier work [67] for generally hyperbolic-parabolic systems, and the high frequency part yields an exponential decay. Subsequently, a number of dissipative systems were investigated, see [12, 14, 18, 23, 24, 25, 38, 60, 64, 65], where the dissipative rate is of the (a, b)-type: $\eta(\xi) = |\xi|^{2a}/(1 + |\xi|^2)^b$. The corresponding decay estimates are also of L^2-L^q-L^2 type, however, the high-frequency part usually admits a polynomial decay provided the initial data is imposed more regularity, for example, see (2.1.17). Anyway, the following general L^p-L^q-L^r time-decay estimate is not available in previous papers.

Theorem 2.1.1. *(L^p-L^q-L^r estimates) Let $\eta(\xi)$ be a positive, continuous and real-valued function in $\mathbb{R}^n$ satisfying*

$$\eta(\xi) \sim \begin{cases} |\xi|^{\sigma_1}, & |\xi| \to 0; \\ |\xi|^{-\sigma_2}, & |\xi| \to \infty; \end{cases} \tag{2.1.18}$$

for $\sigma_1, \sigma_2 > 0$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$, it holds that

$$\begin{aligned} &\|\mathcal{F}^{-1}[|\xi|^k e^{-\eta(\xi)t}|\hat{\phi}(\xi)|]\|_{L^p} \\ \leq\ & C\underbrace{(1+t)^{-\gamma_{\sigma_1}(q,p)-\frac{k-j}{\sigma_1}}\|\partial_x^j\phi\|_{L^q}}_{Low-frequency\ Estimate} + \underbrace{(1+t)^{-\frac{\ell}{\sigma_2}+\gamma_{\sigma_2}(r,p)}\|\partial_x^{k+\ell}\phi\|_{L^r}}_{High-frequency\ Estimate}, \end{aligned} \tag{2.1.19}$$

for $\ell > n(\frac{1}{r}-\frac{1}{p})$, $1 \leq q, r \leq 2 \leq p \leq \infty$ and $0 \leq j \leq k$, where $\gamma_\sigma(q,p) := \frac{n}{\sigma}(\frac{1}{q}-\frac{1}{p})$ $(\sigma > 0)$ and C is some positive constant.

From Theorem 2.1.1, we see that the function decays like a generalized heat kernel at the low-frequency part, and for the high-frequency part, it decays in time not only with algebraic rates of any order as long as the function is spatially regular enough, but also additional information related the integrability is captured in comparison with (2.1.17). In the present chapter, the case $\sigma_1 = \sigma_2 = 2$ in the L^p-L^q-L^r estimate will be applied directly, since system (2.1.1) admits the dissipative rate of $(1, 2)$-type.

In [65], the authors obtained (2.1.17) for the linearized Euler-Maxwell system. Then the nonlinear case is achieved straightforward by using the standard Duhamel principle. To overcome the difficulty arising from the weaker dissipative mechanism of regularity-loss, the norm

$$W^{\perp}(t) := \sup_{0\leq\tau\leq t}(1+\tau)\|(\rho, \upsilon, E)\|_{W^{1,\infty}}$$

was used in such semigroup approach, which led to an extra-regularity of initial data. Here, we have to skip the Duhamel principle. Consequently, the energy method in Fourier spaces for (2.1.10)-(2.1.11) with fixed nonlinear terms is mainly performed, which enables us to obtain an inequality something like "the square form of Duhamel principle", see Section 2.3. Fortunately, the inequality (2.1.19) plays the key role to get the optimal decay estimates for (2.1.10)-(2.1.11) in the regularity $s_D = 3$. More precisely, the high-frequency estimate is divided into two parts, and on each part, the advantage of (2.1.19) with respect to (2.1.17) is that different values (for example, $r = 1$ or $r = 2$) can be chosen to obtain desired decay estimates. See (2.3.16)-(2.3.17), (2.3.19) and (2.3.21)-(2.3.22) for more details. Finally, it should be pointed out that the degenerate structures of $Q(z)$ and $R(z)$ in (2.1.12) are also helpful to obtain the desired decay.

Now, we present the decay result for the Cauchy problem of (2.1.7).

Theorem 2.1.2. *Assume that the initial data satisfy $w_0 - w_\infty \in H^3 \cap L^1$ and (2.1.5). Set $I_1 := \|w_0 - w_\infty\|_{H^3\cap L^1}$. Then there exists a positive ε_1 such that if $I_1 \leq \varepsilon_1$, then the classical solution of the Cauchy problem of (2.1.7) admits the optimal decay estimate*

$$\|w - w_\infty\|_{L^2} \leq C\|w_0 - w_\infty\|_{H^3\cap L^1}(1+t)^{-\frac{3}{4}}, \tag{2.1.20}$$

where $C > 0$ is some constant.

We would like to remark that $\ell \geq 0$ in the case of $p = r = 2$.

Let us mention that the regularity required on the initial data to obtain the L^1-L^2 decay coincides with $s_c = 3$ for global solutions of (2.1.7). In other words, the extra-regularity of initial data was not necessary to obtain the optimal time decay, which improve all previous efforts for Euler-Maxwell equations, so the minimal decay regularity satisfies that $s_D = 3$.

Remark 2.1.3. From the point of view of harmonic analysis, J. Xu and S. Kawashima investigated (2.1.1) in the spatially critical Besov spaces, where the critical regularity has been improved to be $s_c = 5/2$ (see [69, 75]). On the other hand, they [71] gave a decay framework $L^2 \cap \dot{B}^{-s}_{2,\infty}(0 < s \leq n/2)$ for generally dissipative system satisfying the Kawashima-Shizuta condition. Is there the possibility such that $s_D = 5/2$ for Euler-Maxwell equations based on the new functional framework? Here, we draw down an open question.

The rest of this chapter unfolds as follows. In Section 2.2, we shall prove the L^p-L^q-L^r decay estimates. Section 2.3 is devoted to develop the energy method in Fourier spaces for (2.1.10) with fixed nonlinear terms. With aid of L^p-L^q-L^r estimates, the optimal decay rate for (2.1.10)-(2.1.11) under the minimal regularity assumption is further shown. In Appendix (Section 2.4), as another application of L^p-L^q-L^r estimates, the decay property for symmetric hyperbolic systems with non-symmetric dissipation can be extended to be the L^p-version.

At the end of the Introduction, we would like to give a outlook for forthcoming works.

Remark 2.1.4. As applications of L^p-L^q-L^r decay estimates, two research lines will begin in the near future. One is to employ L^p-L^q-L^r estimates in Theorem 2.1.1 related to dissipative rates of (a, b) type, and investigate more dissipative systems of regularity-loss. Another is to develop corresponding L^p-L^q-L^r estimates on the kinetic level, and then study the Vlasov-Maxwell-Boltzmann system of regularity-loss.

2.2 The proof of L^p-L^q-L^r decay estimates

In this section, we give the proof for the time-decay estimate of L^p-L^q-L^r type by using the high-frequency and low-frequency decomposition method.

The proof of Theorem 2.1.1. Firstly, it follows from Hausdorff-Young's inequality that

$$\|\mathcal{F}^{-1}[|\xi|^k e^{-\eta(\xi)t}|\hat{\phi}(\xi)|]\|_{L^p} \lesssim \||\xi|^k e^{-\eta(\xi)t}|\hat{\phi}(\xi)|\|_{L^{p'}}, \tag{2.2.1}$$

for $\frac{1}{p} + \frac{1}{p'} = 1,\ 2 \leq p \leq \infty$.

Secondly, we deal with the $L^{p'}$-norm on the right-hand side of (2.2.1) at the low-frequency and high-frequency, respectively. It follows from the assumption (2.1.18) that there exists a constant $R_0 > 0$ such that

$$\begin{aligned} & \||\xi|^k e^{-\eta(\xi)t}|\hat{\phi}(\xi)|\|_{L^{p'}} \\ \leq\ & \||\xi|^k e^{-c|\xi|^{\sigma_1}t}|\hat{\phi}(\xi)|\|_{L^{p'}(|\xi|\leq R_0)} + \||\xi|^k e^{-c|\xi|^{-\sigma_2}t}|\hat{\phi}(\xi)|\|_{L^{p'}(|\xi|\geq R_0)} \\ \triangleq\ & I_1 + I_2, \end{aligned} \tag{2.2.2}$$

for some constant $c > 0$.

For I_1, we are led to the estimate

$$
\begin{aligned}
I_1 &\leq \||\xi|^k e^{-c|\xi|^{\sigma_1}t}|\hat{\phi}(\xi)|\|_{L^{p'}(|\xi|\leq R_0)} \\
&= \||\xi|^j|\hat{\phi}(\xi)||\xi|^{(k-j)}e^{-c|\xi|^{\sigma_1}t}\|_{L^{p'}(|\xi|\leq R_0)} \quad (0 \leq j \leq k) \\
&\leq \||\xi|^j\hat{\phi}\|_{L^{q'}(|\xi|\leq R_0)}\||\xi|^{(k-j)}e^{-c|\xi|^{\sigma_1}t}\|_{L^{s_1}(|\xi|\leq R_0)} \quad \Big(\frac{1}{q'}+\frac{1}{s_1}=\frac{1}{p'},\quad q'\geq 2\Big) \\
&\lesssim \|\partial_x^j\phi\|_{L^q}(1+t)^{-\frac{n}{\sigma_1 s_1}-\frac{k-j}{\sigma_1}} \quad \Big(\frac{1}{q}+\frac{1}{q'}=1\Big) \\
&\lesssim (1+t)^{-\frac{n}{\sigma_1}(\frac{1}{q}-\frac{1}{p})-\frac{k-j}{\sigma_1}}\|\partial_x^j\phi\|_{L^q},
\end{aligned}
\tag{2.2.3}
$$

where Hölder inequality was used in the third line and Hausdorff-Young's inequality was used again in the fourth line.

For I_2, we arrive at

$$
\begin{aligned}
I_2 &\leq \||\xi|^k e^{-ct/|\xi|^{\sigma_2}}|\hat{\phi}(\xi)|\|_{L^{p'}(|\xi|\geq R_0)} \\
&= \Big\||\xi|^{k+\ell}|\hat{\phi}(\xi)|\frac{e^{-ct/|\xi|^{\sigma_2}}}{|\xi|^\ell}\Big\|_{L^{p'}(|\xi|\geq R_0)} \\
&\leq \||\xi|^{k+\ell}\hat{\phi}\|_{L^{r'}(|\xi|\geq R_0)}\Big\|\frac{e^{-ct/|\xi|^{\sigma_2}}}{|\xi|^\ell}\Big\|_{L^{s_2}(|\xi|\geq R_0)} \quad \Big(\frac{1}{r'}+\frac{1}{s_2}=\frac{1}{p'},\quad r'\geq 2\Big) \\
&\lesssim \|\partial_x^{k+\ell}\phi\|_{L^r}\Big\|\frac{e^{-ct/|\xi|^{\sigma_2}}}{|\xi|^\ell}\Big\|_{L^{s_2}(|\xi|\geq R_0)} \quad \Big(\frac{1}{r}+\frac{1}{r'}=1\Big),
\end{aligned}
\tag{2.2.4}
$$

where

$$
\begin{aligned}
\int_{|\xi|\geq R_0}\frac{e^{-ct/|\xi|^{\sigma_2}}}{|\xi|^{\ell s_2}}d\xi &= \int_{\varrho\geq R_0}\frac{e^{-ct/\varrho^{\sigma_2}}}{\varrho^{\sigma_2(\ell s_2/\sigma_2)}}\varrho^{n-1}d\varrho \\
&= \int_0^{\sqrt[\sigma_2]{t}/R_0} t^{-\frac{\ell s_2}{\sigma_2}}y^{\ell s_2}e^{-cy^{\sigma_2}}\frac{t^{\frac{n-1}{\sigma_2}}}{y^{n-1}}(t^{\frac{1}{\sigma_2}}y^{-2}dy) \\
&\lesssim (1+t)^{-\frac{\ell s_2}{\sigma_2}+\frac{n}{\sigma_2}} \quad (\ell s_2 > n).
\end{aligned}
\tag{2.2.5}
$$

Let us point out that the change of variables $\varrho = |\xi|$ and $y = \sqrt[\sigma_2]{t}/\varrho$ in the first and second lines of (2.2.5) were performed, respectively.

Together with (2.2.4)-(2.2.5), we obtain

$$
I_2 \lesssim (1+t)^{-\frac{\ell}{\sigma_2}+\frac{n}{\sigma_2}(\frac{1}{r}-\frac{1}{p})}\|\partial_x^{k+\ell}\phi\|_{L^r}, \tag{2.2.6}
$$

where the constraint $\ell s_2 > n$ leads to $\ell > n(\frac{1}{r}-\frac{1}{p})$. It should be noted that I_2 can be bounded by $(1+t)^{-\frac{\ell}{\sigma_2}}\|\partial_x^{k+\ell}\phi\|_{L^2}$ with $\ell \geq 0$ if $p=r=2$.

Hence, combining (2.2.2)-(2.2.3) and (2.2.6) together, the proof of Theorem 2.1.1 is complete immediately. □

2.3 The proof of Theorem 2.1.2

Based on the L^p-L^q-L^r estimate for $(1,2)$-type in Theorem 2.1.1, the main objective of this section is to show the optimal decay estimate of L^1-L^2 type for (2.1.10)-(2.1.11) under the minimal regularity assumption. For clarity, we separate the proof into two parts.

2.3.1 Energy method in Fourier spaces

Since earlier works [58, 67], the energy method in Fourier spaces have been well developed for hyperbolic systems of viscoelasticity, hyperbolic-elliptic systems of radiating gas, compressible Euler-Maxwell equations, Timoshenko systems and the plate equation with rotational inertia effect and so on, see [14, 23, 25, 60, 65] and therein references. The interested reader is also referred to [64] for generally hyperbolic systems with non-symmetric dissipation. Usually, the energy method in Fourier spaces is adapted to linearized systems. Here, we shall perform the nonlinear version in Fourier spaces for (2.1.10)-(2.1.11), see (2.3.1) below. Let us mention that the similar estimate was first given by S. Kawashima in [28] for the Boltzmann equation, then well developed in [30] for hyperbolic systems of balance laws.

Proposition 2.3.1. *Let $z=(\rho,v,E,h)^\top$ be the global classical solutions constructed in [66] (also see (2.1.8)-(2.1.9)). Then the Fourier image of classical solutions of (2.1.10)-(2.1.11) satisfies the following pointwise estimate*

$$|\hat{z}(\xi)|^2 \lesssim e^{-c_1\eta(\xi)t}|\hat{z}_0(\xi)|^2 + \int_0^t e^{-c_1\eta(\xi)(t-\tau)}\left(|\xi|^2|\hat{Q}(\tau,\xi)|^2+|\hat{R}(\tau,\xi)|^2\right)d\tau, \tag{2.3.1}$$

for any $t\geq 0$ and $\xi\in\mathbb{R}^3$, where the dissipative rate $\eta(\xi):=|\xi|^2/(1+|\xi|^2)^2$ and $c_1>0$ is a constant.

Proof. Indeed, it suffices to show the influence of nonlinear terms, since the proof follows from the energy method in Fourier spaces as in [65]. Applying the Fourier transform to (2.1.10) gives

$$\begin{cases} \partial_t\hat{\rho}+n_\infty i|\xi|\hat{v}\cdot\omega=0, \\ \partial_t\hat{v}+a_\infty i|\xi|\hat{\rho}\,\omega+\hat{E}+\hat{v}\times B_\infty+\hat{v}=\left(i|\xi|\hat{q}_2\cdot\omega+\hat{r}_2\right)/n_\infty, \\ \partial_t\hat{E}+i|\xi|\hat{h}\times\omega-n_\infty\hat{v}=0, \\ \partial_t\hat{h}-i|\xi|\hat{E}\times\omega=0. \end{cases} \tag{2.3.2}$$

Also, we have

$$i|\xi|\hat{E}\cdot\omega=-\hat{\rho},\quad i|\xi|\hat{h}\cdot\omega=0. \tag{2.3.3}$$

For clarity, we divide it into three steps.

Step 1. (**Estimate for dissipative term of** $\hat{\upsilon}$): Performing the inner product of (2.3.2) with $a_\infty\hat{\rho},\ n_\infty\hat{\upsilon},\hat{E}$ and $\hat{h}$, respectively, then adding the resulting equalities together. We take the real part to get

$$\frac{d}{dt}\mathcal{E}_0+c_2|\hat{\upsilon}|^2=\mathrm{Re}\,\langle i|\xi|\hat{q}_2\cdot\omega+\hat{r}_2,\hat{\upsilon}\rangle, \tag{2.3.4}$$

where $\mathcal{E}_0:=a_\infty|\hat{\rho}|^2+n_\infty|\hat{\upsilon}|^2+|\hat{E}|^2+|\hat{h}|^2\approx|\hat{z}|^2$ and $c_2=2n_\infty$. Here and below, Re$\hat{f}$ means the real part of $\hat{f}$. Recalling those definitions of Q and R, it follows from Young's inequality that

$$\frac{d}{dt}\mathcal{E}_0+\frac{c_2}{2}|\hat{\upsilon}|^2\lesssim\left(|\xi|^2|\hat{Q}|^2+|\hat{R}|^2\right). \tag{2.3.5}$$

Step 2. (**Estimate for dissipative term of** $(\hat{\rho},\hat{E})$): Performing the inner product of the second and the third equations of (2.3.2) with $a_\infty i|\xi|\hat{\rho}\,\omega+\hat{E}$ and $\hat{\upsilon}$, respectively, and then adding the resulting equalities implies

$$\begin{aligned}&\{\langle a_\infty i|\xi|\hat{\rho}_t\,\omega,\hat{\upsilon}\rangle+\langle\hat{\upsilon}_t,a_\infty i|\xi|\hat{\rho}\,\omega\rangle\}+\{\langle\hat{\upsilon}_t,\hat{E}\rangle+\langle\hat{E}_t,\hat{\upsilon}\rangle\}\\&\quad+|a_\infty i|\xi|\hat{\rho}\,\omega+\hat{E}|^2-n_\infty|\hat{\upsilon}|^2-n_\infty a_\infty|\xi|^2|\hat{\upsilon}\cdot\omega|^2\\&\qquad+\langle\hat{\upsilon}\times B_\infty+\hat{\upsilon},\,a_\infty i|\xi|\hat{\rho}\,\omega+\hat{E}\rangle+i\xi\langle\hat{h}\times\omega,\,\hat{\upsilon}\rangle\\&=\langle(i|\xi|\hat{q}_2\cdot\omega+\hat{r}_2)/n_\infty,\,a_\infty i|\xi|\hat{\rho}\,\omega+\hat{E}\rangle,\end{aligned} \tag{2.3.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in the complex vector value $\mathbb{C}^n$ $(n\geq 1)$.

Taking the real part of (2.3.6) and just following from the similar procedure as in [65], we arrive at

$$\begin{aligned}&\frac{d}{dt}\left(\mathcal{E}_1+a_\infty|\xi|\mathcal{E}_2\right)+c_3\left(1+|\xi|^2\right)|\hat{\rho}|^2+c_3|\hat{E}|^2\\\leq\ &\varepsilon\frac{|\xi|^2}{1+|\xi|^2}|\hat{h}|^2+C_\varepsilon\left(1+|\xi|^2\right)|\hat{\upsilon}|^2+C\left(|\xi|^2|\hat{Q}|^2+|\hat{R}|^2\right),\end{aligned} \tag{2.3.7}$$

for any $\varepsilon>0$, where $\mathcal{E}_1:=\mathrm{Re}\,\langle\hat{\upsilon},\hat{E}\rangle$, $\mathcal{E}_2:=\mathrm{Re}\,\langle i\hat{\rho}\,\omega,\hat{\upsilon}\rangle$ and c_3,C_ε (depending on ε) are some positive constants.

Step 3. (**Estimate for dissipative term of** $\hat{h}$): Performing the inner product of the third and fourth equations of (2.3.2) with $i|\xi|\hat{h}\times\omega$ and $i|\xi|\hat{E}\times\omega$, respectively, adding the resulting equalities together, and then taking the real part gives

$$\frac{d}{dt}\left(|\xi|\mathcal{E}_3\right)+|\xi|^2|\hat{h}\times\omega|^2=|\xi|^2|\hat{E}\times\omega|^2-\mathrm{Re}\,\langle n_\infty\hat{\upsilon},i|\xi|\hat{h}\times\omega\rangle, \tag{2.3.8}$$

where $\mathcal{E}_3 := \mathrm{Re}\,\langle \hat{E}, i\hat{h}\times\omega\rangle$. Due to (2.3.3), we have $|\hat{h}\times\omega| \approx |\hat{h}|$. Furthermore, it follows from Young's inequality that there exists $c_4 > 0$ such that

$$\frac{d}{dt}\left(|\xi|\mathcal{E}_3\right) + c_4|\xi|^2|\hat{h}|^2 \lesssim |\xi|^2|\hat{E}|^2 + C|\hat{v}|^2. \tag{2.3.9}$$

Together energy inequalities (2.3.5), (2.3.7) and (2.3.9), the next step is to make the suitable linear combination for them. Here, we feel free to skip them, see Page 262 in [65] for similar details. That is, the Euler-Maxwell system admits Lyapunov function

$$\mathcal{E}[\hat{z}] := \mathcal{E}_0 + \frac{\alpha_1}{1+|\xi|^2}\Big\{\mathcal{E}_1 + a_\infty|\xi|\mathcal{E}_2 + \frac{\alpha_2|\xi|}{1+|\xi|^2}\mathcal{E}_3\Big\}$$

such that the following differential inequality holds

$$\frac{d}{dt}\mathcal{E}[\hat{z}] + c_1\mathcal{D}[\hat{z}] \lesssim \left(|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2\right), \tag{2.3.10}$$

where

$$\mathcal{D}[\hat{z}] = |\hat{\rho}|^2 + |v|^2 + \frac{1}{1+|\xi|^2}|\hat{E}|^2 + \frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{h}|^2,$$

and α_1, α_2 are suitable small constants which ensure that $\mathcal{E}[\hat{z}] \approx |\hat{z}|^2$. It follows from (2.3.10) that

$$\frac{d}{dt}\mathcal{E}[\hat{z}] + c_1\eta(\xi)\,\mathcal{E}[\hat{z}] \lesssim \left(|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2\right), \tag{2.3.11}$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$. Finally, the inequality (2.3.1) is followed from Gronwall's inequality. □

2.3.2 Optimal decay rates

In what follows, with preparations of Theorem 2.1.1 and Proposition 2.3.1, we prove the optimal decay estimate for (2.1.10). To show the minimal decay regularity of classical solutions, we define new time-weighted energy functionals:

$$N(t) = \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{4}}\|z(\tau)\|_{L^2},$$

$$D(t)^2 = \int_0^t \left(\|(\rho,v)(\tau)\|^2_{H^3} + \|E(\tau)\|^2_{H^2} + \|\nabla h(\tau)\|^2_{H^1}\right)d\tau.$$

Furthermore, the degenerate structure of nonlinear terms $Q(z)$ and $R(z)$ enables us to deduce a nonlinear energy inequality, which is included in the following

Proposition 2.3.2. *Let $z = (\rho, v, E, h)^\top$ be the global classical solutions of (2.1.10)-(2.1.11) (see (2.1.8)-(2.1.9)). Additionally, if $z_0 \in L^1$, then*

$$N(t) \lesssim \|z_0\|_{H^3\cap L^1} + N(t)D(t) + N(t)^2. \tag{2.3.12}$$

Proof. Let us begin with (2.3.1):

$$\begin{aligned}\int_{\mathbb{R}^3}|\hat{z}(\xi)|^2\,d\xi &\lesssim \int_{\mathbb{R}^3} e^{-c\eta(\xi)t}|\hat{z}_0(\xi)|^2\,d\xi \\ &\quad + \int_{\mathbb{R}^3}\int_0^t e^{-c\eta(\xi)(t-\tau)}\Big(|\xi|^2|\hat{Q}(\tau,\xi)|^2+|\hat{R}(\tau,\xi)|^2\Big)\,d\tau d\xi \\ &\triangleq J_1+J_2+J_3. \end{aligned} \tag{2.3.13}$$

For J_1, by taking $\sigma_1=\sigma_2=2,\ p=2,\ k=j=0,\ q=1$ and $r=\ell=2$ in Theorem 2.1.1, we arrive at

$$J_1 \lesssim (1+t)^{-\frac{3}{2}}\|z_0\|_{L^1}^2+(1+t)^{-2}\|\partial_x^2 z_0\|_{L^2}^2. \tag{2.3.14}$$

Next, we begin to estimate nonlinear terms. For J_2, it is written as the sum of low-frequency and high-frequency

$$J_2 := J_{2L}+J_{2H}.$$

For J_{2L}, by taking $\sigma_1=\sigma_2=2,\ p=2$ and $k=1,\ j=0,\ q=1$ in Theorem 2.1.1, we have

$$\begin{aligned} J_{2L} &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}}\|Q(\tau)\|_{L^1}^2\,d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}}\|z(\tau)\|_{L^2}^4\,d\tau \\ &\lesssim N(t)^4\int_0^t (1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-3}\,d\tau \\ &\lesssim N(t)^4(1+t)^{-\frac{5}{2}}, \end{aligned} \tag{2.3.15}$$

where we have used the fact $Q(z)=O(|(\rho,\upsilon)|^2)$. For simplicity, we set $z^{\perp}:=(\rho,\upsilon)$.

For J_{2H}, more elaborate estimates are needed. For this purpose, we write

$$J_{2H}=\Big(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\Big)(\cdots)\,d\tau := J_{2H1}+J_{2H2}.$$

Taking $\sigma_1 = \sigma_2 = 2,\ p = 2$ and $k = 1,\ \ell = 2,\ r = 2$ in Theorem 2.1.1 gives

$$\begin{aligned}
J_{2H1} &\lesssim \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|\partial_x^3 Q(\tau)\|_{L^2}^2 \, d\tau \\
&\lesssim \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|z^\perp\|_{L^\infty}^2 \|\partial_x^3 z^\perp\|_{L^2}^2 \, d\tau \\
&\lesssim \sup_{0\le\tau\le\frac{t}{2}} \left\{(1+t-\tau)^{-2}\|z\|_{L^\infty}^2\right\} \int_0^{\frac{t}{2}} \|\partial_x^3 z^\perp\|_{L^2}^2 \, d\tau \\
&\lesssim (1+t)^{-2} N_0^2(t) D^2(t) \\
&\lesssim (1+t)^{-2} \|z_0\|_{H^3}^2, \qquad (2.3.16)
\end{aligned}$$

where we have used (2.1.9) (taking $s = 3$) and the fact $Q(z) = O(|z^\perp|^2)$.

On the other hand, by taking $\sigma_1 = \sigma_2 = 2,\ p = 2$ and $k = 1,\ \ell = 2,\ r = 1$ in Theorem 2.1.1, we get

$$J_{2H2} \lesssim \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^3 Q(\tau)\|_{L^1}^2 \, d\tau. \qquad (2.3.17)$$

It follows from the Gagliardo-Nirenberg interpolation inequality in [43] that

$$\|\partial_x f\|_{L^2} \lesssim \|f\|_{L^2}^{\frac{2}{3}} \|\partial_x^3 f\|_{L^2}^{\frac{1}{3}}, \quad \|\partial_x^2 f\|_{L^2} \lesssim \|f\|_{L^2}^{\frac{1}{3}} \|\partial_x^3 f\|_{L^2}^{\frac{2}{3}}. \qquad (2.3.18)$$

Furthermore, together with (2.3.17)-(2.3.18), we obtain

$$\begin{aligned}
J_{2H2} &\lesssim \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{2}} \|z^\perp\|_{L^2}^2 \|\partial_x^3 z^\perp\|_{L^2}^2 \, d\tau \\
&\lesssim N(t)^2 \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{3}{2}} \|\partial_x^3 z^\perp\|_{L^2}^2 \, d\tau \\
&\lesssim N(t)^2 \sup_{\frac{t}{2}\le\tau\le t} \left\{(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}\right\} \int_0^{t} \|\partial_x^3 z^\perp\|_{L^2}^2 \, d\tau \\
&\lesssim (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2. \qquad (2.3.19)
\end{aligned}$$

For J_3, we write

$$J_3 := J_{3L} + J_{3H}.$$

Note that $R(z) = O(\rho|E| + |v||h|)$, by taking $\sigma_1 = \sigma_2 = 2,\ p = 2$ and $k = j = 0,\ q = 1$ in

Theorem 2.1.1, we obtain

$$\begin{aligned}
J_{3L} &\lesssim \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|R(\tau)\|_{L^1}^2 \, d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|z(\tau)\|_{L^2}^4 \, d\tau \\
&\lesssim N(t)^4 \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-3} \, d\tau \\
&\lesssim N(t)^4 (1+t)^{-\frac{3}{2}}. \qquad (2.3.20)
\end{aligned}$$

Similarly, we separate the high-frequency part J_{3H} as follows

$$J_{3H} = \Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \Big) (\cdots) \, d\tau := J_{3H1} + J_{3H2}.$$

Taking $\sigma_1 = \sigma_2 = 2,\ p = 2$ and $k = 0,\ \ell = r = 2$ in Theorem 2.1.1 leads to

$$\begin{aligned}
J_{3H1} &\lesssim \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|\partial_x^2 R(\tau)\|_{L^2}^2 \, d\tau \\
&\lesssim \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|z\|_{L^\infty}^2 \|\partial_x^2 z\|_{L^2}^2 \, d\tau \\
&\lesssim \sup_{0 \le \tau \le \frac{t}{2}} \Big\{ (1+t-\tau)^{-2} \|z\|_{L^\infty}^2 \Big\} \int_0^{\frac{t}{2}} \|\partial_x^2 z\|_{L^2}^2 \, d\tau \\
&\lesssim (1+t)^{-2} N_0^2(t) D^2(t) \\
&\lesssim (1+t)^{-2} \|z_0\|_{H^3}^2, \qquad (2.3.21)
\end{aligned}$$

where we have used (2.1.9) (taking $s = 3$). On the other hand, by taking $\sigma_1 = \sigma_2 = 2,\ p = 2$

and $k=0,\ \ell=2,\ r=1$ in Theorem 2.1.1, we arrive at

$$\begin{aligned}
J_{3H2} &\lesssim \int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^2 R(\tau)\|_{L^1}^2\,d\tau \\
&\lesssim \int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\|z\|_{L^2}^2\|\partial_x^2 z\|_{L^2}^2\,d\tau \\
&\lesssim N(t)^2\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}\|\partial_x^2 z\|_{L^2}^2\,d\tau \\
&\lesssim N(t)^2\sup_{\frac{t}{2}\le\tau\le t}\left\{(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}\right\}\int_0^t\|\partial_x^2 z\|_{L^2}^2\,d\tau \\
&\lesssim (1+t)^{-\frac{3}{2}}N(t)^2D(t)^2, \qquad (2.3.22)
\end{aligned}$$

where the Gagliardo-Nirenberg inequality $\|\partial_x f\|_{L^2}\lesssim\|f\|_{L^2}^{\frac{1}{2}}\|\partial_x^2 f\|_{L^2}^{\frac{1}{2}}$ was used in the second line.

Therefore, combining above inequalities (2.3.14)-(2.3.16) and (2.3.19)-(2.3.22), it follows from Plancherel's theorem that

$$\begin{aligned}
\|z\|_{L^2}^2 &\lesssim (1+t)^{-\frac{3}{2}}\|z_0\|_{H^3\cap L^1}^2+(1+t)^{-\frac{3}{2}}N(t)^4 \\
&\quad +(1+t)^{-\frac{3}{2}}N(t)^2D(t)^2, \qquad (2.3.23)
\end{aligned}$$

which leads to (2.3.12) exactly. □

According to the energy inequality (2.1.9) (taking $s=3$), the dissipation norm $D(t)\lesssim\|z_0\|_{H^3}\lesssim\|z_0\|_{H^3\cap L^1}$. Thus, if $\|z_0\|_{H^3\cap L^1}$ is sufficient small, then it holds that

$$N(t)\lesssim\|z_0\|_{H^3\cap L^1}+N(t)^2 \qquad (2.3.24)$$

which implies that $N(t)\lesssim\|z_0\|_{H^3\cap L^1}$, provided that $\|z_0\|_{H^3\cap L^1}$ is sufficient small. Consequently, the optimal decay estimate in Theorem 2.1.2 is achieved.

2.4 Appendix

In the last section, as another application of Theorem 2.1.1, we generalize recent decay properties in [64] for linear symmetric hyperbolic systems with non-symmetric dissipation.

2.4.1 Symmetric hyperbolic systems

Consider the Cauchy problem for the first-order linearized symmetric hyperbolic system of equations with dissipation

$$\begin{cases} A^0 w_t + \sum_{j=1}^{n} A^j w_{x_j} + Lw = 0, \\ w|_{t=0} = w_0, \end{cases} \tag{2.4.1}$$

with $w(t,x) \in \mathbb{R}^m$ for $t > 0$ and $x \in \mathbb{R}^n$, where $A^j (j = 0, 1, \cdots, n)$ and L are $m \times m$ real constant matrices. It is assumed that all $A^j (j = 0, 1, \cdots, n)$ are symmetric, A_0 is positive definite and L is nonnegative definite with a nontrivial kernel.

If the degenerate dissipation matrix L is symmetric, Kawashima and Shizuta [58] first formulated the so-called Kawashima-Shizuta condition which designs the compensating matrix K to capture the dissipation of systems over the degenerate kernel space of L. Inspired by recent concrete examples, such as dissipative Timoshenko systems and compressible Euler-Maxwell equations and so on, the matrix L has the skew-symmetric part and is not symmetric. In this case, the partial positivity on $\mathrm{Ker}(L_1)^{\perp}$ ($L_1 :=$ the symmetric part of L) is available only. Recently, Ueda, Duan and Kawashima [64] found a real compensating matrix S to make up the full positivity on $\mathrm{Ker}(L)^{\perp}$. Consequently, they developed decay properties for (2.4.1) with the weaker dissipative mechanism. Here, we don't collect those structural conditions formulated by [64] for brevity, however, we would like to keep the same notations as in [64] for the convenience of reader.

Based on **conditions** (A), (K), (S) and $(\mathrm{S})_1$ in [64], by employing the energy method in Fourier spaces, they arrived at

$$\frac{d}{dt}\mathcal{E}(t,\xi) + c\eta(\xi)\,\mathcal{E}(t,\xi) \le 0, \tag{2.4.2}$$

which implies that $\mathcal{E}(t,\xi) \le e^{-c\eta(\xi)t}\mathcal{E}(0,\xi)$, where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$ and $\mathcal{E}(t,\xi) \approx |\hat{w}|^2$. Furthermore, by Theorem 2.1.1, we can generalize the decay property in [64].

Proposition 2.4.1. *Assume that* **conditions** (A), (K), (S) and $(\mathrm{S})_1$ *in [64] hold. If the initial data* $w_0 \in W^{l,r} \cap L^q$ *for* $l \ge 0$ *and* $1 \le q, r \le 2$*, then the solution* $w(t,x)$ *of (2.4.1) satisfies the decay estimate*

$$\|\partial_x^k w\|_{L^p} \lesssim (1+t)^{-\gamma_2(q,p)-\frac{k}{2}}\|w_0\|_{L^q} + (1+t)^{-\frac{\ell}{2}+\gamma_2(r,p)}\|\partial_x^{k+\ell} w_0\|_{L^r} \tag{2.4.3}$$

for $\ell > n(\frac{1}{r} - \frac{1}{p})$, $2 \le p \le \infty$ *and* $0 \le k+\ell \le l$.

Here $\ell \ge 0$ when $p = r = 2$.

2.4.2 Symmetric hyperbolic systems with constraints

Inspired by Euler-Maxwell equations, the system (2.4.1) equipped with a general constraint was also investigated in [64]:

$$\sum_{j=1}^{n} \mathcal{Q}^j w_{x_j} + \mathcal{R}w = 0, \tag{2.4.4}$$

where $\mathcal{Q}^j$ and $\mathcal{R}$ are $m_1 \times m$ real constant matrices with $m_1 < m$. Let Π_1 be the orthogonal projection from $\mathbb{C}^{m_1}$ onto Image$(\mathcal{R}) = \{\mathcal{R}\phi : \phi \in \mathbb{C}^m\} \subset \mathbb{C}^{m_1}$. Set $\Pi_2 = I - \Pi_1$. Noticing that Π_1 and Π_2 are $m_1 \times m_1$ real symmetric matrices. Using these projections, the condition (2.4.4) can be decomposed as

$$\sum_{j=1}^{n} \Pi_1 \mathcal{Q}^j w_{x_j} + \mathcal{R}w = 0, \quad \sum_{j=1}^{n} \Pi_2 \mathcal{Q}^j w_{x_j} = 0. \tag{2.4.5}$$

To ensure that (2.4.4) or (2.4.5) holds at an arbitrary time $t > 0$ if it satisfies initially, the extra structure **condition** (C) is posted, see [64] for details. Additionally, those conditions in Proposition 2.4.1 need to be revised a little in this case. For convenience, the same notations as in [64] are kept. As a consequence, the dissipative inequality (2.4.2) still holds for the solution of (2.4.1) along with (2.4.4). Furthermore, we have a similar decay property as stated in Proposition 2.4.1.

Proposition 2.4.2. *Assume that* **conditions** (A), (C), (S), $(S^*)_1$ and (K^*) *in [64] hold. If the initial data $w_0 \in W^{l,r} \cap L^q$ for $l \geq 0$ and $1 \leq q, r \leq 2$, then the solution $w(t,x)$ of (2.4.1) satisfies (2.4.4) for all $t > 0$. Moreover, the solution satisfies the decay estimate (2.4.3).*

Remark 2.4.1. Propositions 2.4.1-2.4.2 go back to Theorem 2.2 and Theorem 5.2 in [64], if one takes $p = r = 2$ and $q = 1$. Therefore, the current decay properties can be regarded as a general L^p-version.

Chapter 3

Dissipative Timoshenko system

3.1 Introduction

In Chapter 3 we consider the Cauchy problem of the dissipative Timoshenko system in the one-dimensional whole space. This model system introduces a mechanical damping, and takes the form

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \psi_{tt} - \sigma(\psi_x)_x - (\varphi_x - \psi) + \gamma\,\psi_t = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+. \end{cases} \tag{3.1.1}$$

The initial data is given by

$$(\varphi,\, \varphi_t,\, \psi,\, \psi_t)(x,0) = (\varphi_0,\, \varphi_1,\, \psi_0,\, \psi_1)(x).$$

Here $\sigma(\eta)$ in the nonlinear term is assumed to be a smooth function of η satisfying $\sigma'(\eta) > 0$ for any η under considerations. The coefficient γ in the damping term is a positive constant by definition.

The corresponding linearized system at $z = 0$ is

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \psi_{tt} - a^2\,\psi_{xx} - (\varphi_x - \psi) + \gamma\,\psi_t = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \end{cases} \tag{3.1.2}$$

with $a > 0$ being the wave speed defined by $a^2 = \sigma'(0)$.

3.1.1 Formulation & Problem

Based on the change of variable ([24])

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\,\psi_x, \quad y = \psi_t,$$

the system (3.1.1) is transformed into the following first order system

$$v_t - u_x + y = 0, \tag{3.1.3a}$$

$$y_t - \sigma\left(\frac{z}{a}\right)_x - v + \gamma\, y = 0, \tag{3.1.3b}$$

$$u_t - v_x = 0, \tag{3.1.3c}$$

$$z_t - a\, y_x = 0. \tag{3.1.3d}$$

The corresponding initial data is given by

$$(v,\, y,\, u,\, z)(x, 0) = (v_0,\, y_0,\, u_0,\, z_0)(x), \tag{3.1.4}$$

where $v_0 = \varphi_{0,x} - \psi_0$, $y_0 = \psi_1$, $u_0 = \varphi_1$, $z_0 = a\,\psi_{0,x}$. Remark that the nonlinearity of the system (3.1.3) depends on the component z only.

In vector notation, we can write the system (3.1.3) as

$$A^0(z)\, W_t + A(z)\, W_x + L\, W = 0, \tag{3.1.5}$$

where $W = (v, y, u, z)^T$, $A^0(z) = \mathrm{diag}\,(1, 1, 1, b(z)/a)$ with $b(z) = \sigma'(z/a)/a$, and

$$A(z) = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b(z) \\ 1 & 0 & 0 & 0 \\ 0 & b(z) & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, the linearized system at $z = 0$ is given by

$$W_t + A\, W_x + L\, W = 0, \tag{3.1.6}$$

where $A^0(0) = I$ and

$$A := A(0) = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}.$$

Explicitly,

$$v_t - u_x + y = 0, \tag{3.1.7a}$$

$$y_t - a\, z_x - v + \gamma\, y = 0, \tag{3.1.7b}$$

$$u_t - v_x = 0, \tag{3.1.7c}$$

$$z_t - a\, y_x = 0. \tag{3.1.7d}$$

Consequently, the system (3.1.3) is regarded as a symmetric hyperbolic system with non-symmetric relaxation. In fact, the relaxation matrix L is not symmetric such that $\ker L \neq \ker L_1$, where L_1 denotes the symmetric part of L, which means the general theory on the dissipative structure developed in [58, 67] can not be applicable to this system (3.1.1). Therefore, the new approach to show the dissipative structure and the asymptotic stability of the system (3.1.1) has to be implemented.

3.1.2 Known results & Aim

The decay property of the linear system (3.1.2) was first investigated by J.E. Muñoz Rivera and R. Racke in [50]. They considered the linear system (3.1.2) in a bounded region and with simple boundary conditions and showed that the energy of the solutions decays exponentially when $a = 1$, but polynomially when $a \neq 1$ as $t \to \infty$.

To explain this interesting decay property, K. Ide, K. Haramoto and S. Kawashima [24] considered the linear system (3.1.2) in one-dimensional whole space. They showed that the dissipative structure of the system (3.1.6) can be characterized as

$$\operatorname{Re}\lambda(i\xi) \leq -c\,\eta(\xi), \qquad \eta(\xi) = \begin{cases} \xi^2/(1+\xi^2) & \text{for } \ a = 1, \\ \xi^2/(1+\xi^2)^2 & \text{for } \ a \neq 1, \end{cases}$$

where $\lambda(i\xi)$ denotes the eigenvalues of the linear system (3.1.2) in the Fourier space, and c is a positive constant. We note that the dissipative structure for $a = 1$ is the same one shown in the general theory [58, 67]. On the other hand, the dissipative strudture for $a \neq 1$ is much weaker in the high frequency region, which causes regularity-loss in the decay estimate of the solutions.

In fact, by using the energy method in the Fourier space, the authors in [24] derived the following pointwise estimate of the solution $W = (v, y, u, z)^T$ to the linear system (3.1.6).

$$|\hat{W}(\xi, t)| \leq C\, e^{-c\,\eta(\xi)\,t} |\hat{W}_0(\xi)|,$$

where $W_0 = (v_0, y_0, u_0, z_0)^T$ is the corresponding initial data.

Moreover, based on this pointwise estimate, they showed the time decay estimates of this solution in L^2 norm.

$$\|\partial_x^k W(t)\|_{L^2} \leq C\,(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|W_0\|_{L^1} + \begin{cases} C\, e^{-ct} \|\partial_x^k W_0\|_{L^2} & \text{for } \ a = 1, \\ C\,(1+t)^{-\frac{\ell}{2}} \|\partial_x^{k+\ell} W_0\|_{L^2} & \text{for } \ a \neq 1, \end{cases}$$

where k and l are nonnegative integers, and C and c are positive constants. Note that when $a \neq 1$, in order to obtain the optimal decay rate $(1+t)^{-1/4-k/2}$ we have to assume the additional ℓ-th order regularity on the initial data to make the decay rate $(1+t)^{-\ell/2}$ better than $(1+t)^{-1/4-k/2}$. Therefore the regularity-loss can not be avoided for $a \neq 1$.

Based on these linear system results in [24], K. Ide and S. Kawashima [25] proved the global-in-time existence and uniqueness, and the optimal decay of the solutions to the non-linear system (3.1.5). To state these results, they introduced the following time-weighted norms $\tilde{E}(t)$ and $\tilde{D}(t)$:

$$\tilde{E}(t)^2 := \sum_{j=0}^{[\frac{s}{2}]} \sup_{0\le\tau\le t} (1+\tau)^{j-\frac{1}{2}} \|\partial_x^j W(\tau)\|_{H^{s-2j}}^2,$$

$$\tilde{D}(t)^2 := \sum_{j=0}^{[\frac{s}{2}]} \int_0^t (1+\tau)^{j-\frac{3}{2}} \|\partial_x^j W(\tau)\|_{H^{s-2j}}^2 \, d\tau$$
$$+ \sum_{j=0}^{[\frac{s}{2}]-1} \int_0^t (1+\tau)^{j-\frac{1}{2}} \|\partial_x^j v(\tau)\|_{H^{s-1-2j}}^2 \, d\tau + \sum_{j=0}^{[\frac{s}{2}]} \int_0^t (1+\tau)^{j-\frac{1}{2}} \|\partial_x^j y(\tau)\|_{H^{s-2j}}^2 \, d\tau.$$

Then the results in [25] are stated as follows.

Proposition 3.1.1 (Global existence & L^2 decay estimate [25])**.** *Assume that the initial data satisfies $W_0 \in H^s \cap L^1$ for $s \ge 6$ and put $\tilde{E}_1 := \|W_0\|_{H^s} + \|W_0\|_{L^1}$, where W_0 is the corresponding initial data. Then there exists a positive constant $\tilde{\delta}_1$ such that if $\tilde{E}_1 \le \tilde{\delta}_1$, the Cauchy problem* (3.1.5) *with W_0 has a unique global in time solution $W(t)$ with $W \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$. Moreover this solution $W(t)$ verifies the energy estimate*

$$\tilde{E}(t)^2 + \tilde{D}(t)^2 \le C\, \tilde{E}_1^2$$

and the optimal decay estimate for lower order derivatives

$$\|\partial_x^k W(t)\|_{L^2} \le C\, \tilde{E}_1 (1+t)^{-\frac{1}{4}-\frac{k}{2}},$$

where $0 \le k \le [s/2]-1$, and $C > 0$ is a constant.

Remark. The norms $\tilde{E}(t)$ and $\tilde{D}(t)$ contain the time-weights with negative exponents. Also, their results require the high spacial regularity $s \ge 6$ and L^1 property on the initial data. These devices in [25] are crucial to control the weak dissipation due to the nonlinearity and the regularity-loss property of the system (3.1.1).

S. Kawashima and his collaborators have found the diffusion phenomenon of the linear system (3.1.2) and the nonlinear system (3.1.1). In other words, they showed that the solutions to the systems (3.1.2) and (3.1.1) approaches the diffusion wave expressed in term of the superposition of the heat kernels as $t \to \infty$. However, to overcome the difficulty caused by the nonlinearity and the regularity-loss property, the suitably large spatial regularity $s \ge 6$ was needed for the analysis of the nonlinear problem (3.1.1).

Based on these results, in Chapter 3 we aim at showing the global-in-time existence and uniqueness of the solutions to the system (3.1.1) under the smallness condition on the initial data in the Sobolev space with the critical regularity-index H^2. Also, we will show the asymptotic decay of this solution in L^2 at the optimal decay rate $t^{-1/4}$ for $t \to \infty$ under the condition on the initial data in $H^2 \cap L^1$.

3.2 Main results

To state our main results of Chapter 3, we introduce the energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ by

$$E(t)^2 := \sup_{0\le\tau\le t} \|W(\tau)\|_{H^s}^2,$$

$$D(t)^2 := \int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2 + \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2 \, d\tau.$$

Notice that in the dissipation norm $D(t)$ we have 1 regularity-loss for (v, u) but no regularity-loss for (y, z). Our first result is then stated as follows.

Theorem 3.2.1 (Global existence)**.** *Assume that the initial data satisfy $W_0 \in H^s$ for $s \ge 2$ and put $E_0 := \|W_0\|_{H^s}$. Then there exists a positive constant δ_0 such that if $E_0 \le \delta_0$, the Cauchy problem* (3.1.3) *and* (3.1.4) *has a unique global in time solution $W(t)$ with $W \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$. Moreover this solution $W(t)$ verifies the energy estimate*

$$E(t)^2 + D(t)^2 \le CE_0^2,$$

where $C > 0$ is a constant.

Remark. Our global-in-time existence and uniqueness result holds true under less regularity assumption $s \ge 2$ and without L^1 property on the initial data. This refinement is based on the better Lyapunov function constructed in the next section 3.3, which is the improvement of the previous result of the linear system in [24]. Our Lyapunov function produces the optimal dissipation estimate for z without any regularity-loss (see $D(t)$), which enables us to control the nonlinearity depending only on z.

Next we state the result on the optimal time decay estimate.

Theorem 3.2.2 (L^2 decay estimate)**.** *Assume that the initial data satisfy $W_0 \in H^2 \cap L^1$ and put $E_1 := \|W_0\|_{H^2} + \|W_0\|_{L^1}$. Then there is a positive constant δ_1 such that if $E_1 \le \delta_1$, then the solution $W(t)$ obtained in Theorem 3.2.1 satisfies the following optimal L^2 decay estimate:*

$$\|W(t)\|_{L^2} \le CE_1(1+t)^{-\frac{1}{4}},$$

where $C > 0$ is a constant.

Remark. In order to show the above decay estimate, first, we estimate the nonlinear solution by using the energy method in the Fourier space, and then apply the refined decay estimate of L^p-L^q-L^r type which was established in Chapter 2. For the details, see Section 3.5.

3.3 Linear system

The linear dissipative Timoshenko system (3.1.2) was studied intensively in [24]. In this section, we first review some of the main results in [24] and then give a refinement of the energy method in the Fourier space which was employed in [24]. This refinement gives the optimal pointwise estimates of the solutions in the Fourier space.

3.3.1 Review on previous results

Decay estimate

The result on the decay estimate of the solutions to (3.1.6), which was obtained in [24], is stated as follows.

Theorem 3.3.1 (L^2 decay estimate [24])**.** *The solution W of the problem* (3.1.6) *with the initial data W_0 satisfies the following decay estimates for $t \geq 0$:*

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|W_0\|_{L^p} + Ce^{-ct}\|\partial_x^k W_0\|_{L^2}$$

for $a = 1$, and

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|W_0\|_{L^p} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} W_0\|_{L^2}$$

for $a \neq 1$, where $1 \leq p \leq 2$, k and l are nonnegative integers, and C and c are positive constants.

The above decay estimates follow from the corresponding pointwise estimates of the solutions in the Fourier space. To state the result precisely, we take the Fourier transform of (3.1.6) to obtain

$$\begin{cases} \hat{W}_t + (i\xi A + L)\hat{W} = 0, \\ \hat{W}(\xi, 0) = \hat{W}_0(\xi). \end{cases} \tag{3.3.1}$$

Note that the solution to (3.3.1) is given by $\hat{W}(\xi,t) = e^{t\hat{\Phi}(i\xi)}\hat{W}_0(\xi)$, where

$$\hat{\Phi}(\zeta) = -(L + \zeta A), \quad \zeta \in \mathbb{C}. \tag{3.3.2}$$

Also we consider the eigenvalue problem corresponding to (3.3.1):

$$\lambda\phi + (i\xi A + L)\,\phi = 0, \tag{3.3.3}$$

where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^4$. We denote by $\lambda = \lambda(i\xi)$ the eigenvalue of the problem (3.3.1), which satisfies (3.3.3) for $\phi \neq 0$.

Now we state the result on the pointwise estimate of the solutions to (3.3.1), which was shown in [24].

Lemma 3.3.2 (Pointwise estimate [24])**.** *The solution $\hat{W}$ to the problem* (3.3.1) *satisfies the following pointwise estimates for any $\xi \in \mathbb{R}$ and $t \geq 0$:*

$$|\hat{W}(\xi,t)| \leq Ce^{-c\eta_1(\xi)t}|\hat{W}_0(\xi)| \qquad \text{for} \quad a = 1,$$

$$|\hat{W}(\xi,t)| \leq Ce^{-c\eta_2(\xi)t}|\hat{W}_0(\xi)| \qquad \text{for} \quad a \neq 1,$$

where $\eta_1(\xi) = \xi^2/(1+\xi^2)$ and $\eta_2(\xi) = \xi^2/(1+\xi^2)^2$, and C and c are positive constants.

Remark. The corresponding eigenvalue $\lambda(i\xi)$ satisfies $\operatorname{Re}\lambda(i\xi) \leq -c\eta_1(\xi)$ for $a = 1$ and $\operatorname{Re}\lambda(i\xi) \leq -c\eta_2(\xi)$ for $a \neq 1$, where c is a positive constant. This gives the characterization of the dissipative structure of the dissipative Timoshenko system (3.1.6). When $a \neq 1$, the dissipative structure is very weak in the high frequency region and verifies $\operatorname{Re}\lambda(i\xi) \sim -c\xi^{-2}$ with a positive constant c for $|\xi| \to \infty$.

Asymptotic expansion of eigenvalues

Let $\lambda_j(\zeta)$ be the eigenvalues of the matrix $\hat{\Phi}(\zeta)$ in (3.3.2). We review the result on the asymptotic expansion of the eigenvalues $\lambda_j(\zeta)$ for $|\zeta| \to 0$ and $|\zeta| \to \infty$, which was given in [24].

The eigenvalues $\lambda_j(\zeta)$, $j = 1, 2, 3, 4$, are the solutions of the characteristic equation

$$\det(\lambda I - \hat{\Phi}(\zeta)) = \lambda^4 + \gamma\lambda^3 + \{1 - (a^2+1)\zeta^2\}\lambda^2 - \gamma\zeta^2\lambda + a^2\zeta^4 = 0.$$

(i) When $|\zeta| \to 0$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) = \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \cdots. \tag{3.3.4}$$

Here each coefficient $\lambda_j^{(k)}$ is given by direct computations as

$$\begin{aligned} &\lambda_j^{(0)} = \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \alpha_j, \quad \lambda_j^{(3)} = 0 \qquad \text{for} \quad j = 1, 2, \\ &\lambda_j^{(0)} = \beta_j, \quad \lambda_j^{(1)} = 0 \qquad \text{for} \quad j = 3, 4, \end{aligned}$$

where $\alpha_j = \frac{1}{2}\big(\gamma \pm \sqrt{\gamma^2 - 4a^2}\big)$ and $\beta_j = -\frac{1}{2}\big(\gamma \pm \sqrt{\gamma^2 - 4}\big)$. Notice that $\operatorname{Re}\alpha_j > 0$ and $\operatorname{Re}\beta_j < 0$. Consequently, for $|\xi| \to 0$, we have

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} -(\operatorname{Re}\alpha_j)\,\xi^2 + O(|\xi|^4) & \text{for} \quad j = 1, 2, \\ \operatorname{Re}\beta_j + O(|\xi|^2) & \text{for} \quad j = 3, 4. \end{cases} \tag{3.3.5}$$

(ii) When $|\zeta| \to \infty$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) = \mu_j^{(1)}\zeta + \mu_j^{(0)} + \mu_j^{(-1)}\zeta^{-1} + \mu_j^{(-2)}\zeta^{-2} + \cdots.$$

Each coefficient $\mu_j^{(k)}$ is given by direct computations as follows: For $j = 1, 2$, we have

$$\begin{aligned} &\mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = \delta_j \qquad \text{for} \quad a = 1, \\ &\mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = 0, \quad \mu_j^{(-1)} = \pm\frac{1}{2P}, \quad \mu_j^{(-2)} = \frac{\gamma}{P^2} \qquad \text{for} \quad a \neq 1, \end{aligned}$$

and for $j = 3, 4$, we have

$$\mu_j^{(1)} = \pm a, \quad \mu_j^{(0)} = -\frac{\gamma}{2}, \quad \mu_j^{(-1)} = \pm\frac{\gamma^2}{8a},$$

where $\delta_j = \frac{1}{4}\big(-\gamma \pm \sqrt{\gamma^2 - 4}\big)$ and $P = a^2 - 1$. Notice that $\operatorname{Re}\delta_j < 0$. Consequently, when $a = 1$, we have

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} \operatorname{Re}\delta_j + O(|\xi|^{-1}) & \text{for} \quad j = 1, 2, \\ -\dfrac{\gamma}{2} + O(|\xi|^{-2}) & \text{for} \quad j = 3, 4 \end{cases} \tag{3.3.6}$$

for $|\xi| \to \infty$; while in the case $a \neq 1$, we have

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} -\dfrac{\gamma}{P^2}\,\xi^{-2} + O(|\xi|^{-3}) & \text{for} \quad j = 1, 2, \\ -\dfrac{\gamma}{2} + O(|\xi|^{-2}) & \text{for} \quad j = 3, 4 \end{cases} \tag{3.3.7}$$

for $|\xi| \to \infty$. According to the expansion (3.3.7) for $|\xi| \to \infty$, when $a \neq 1$, two eigenvalues are of the standard type and satisfy $\operatorname{Re}\lambda(i\xi) \sim -c$, while the other two are not of the standard type and satisfy $\operatorname{Re}\lambda(i\xi) \sim -c\xi^{-2}$.

Energy method in the Fourier space

The pointwise estimates stated in Lemma 3.3.2 were obtained in [24] by using the energy method in the Fourier space. Here we review this energy method by showing the Lyapunov function constructed in [24].

The Lyapunov function E constructed in [24] is equivalent to $|\hat{W}|^2 = |\hat{v}|^2 + |\hat{y}|^2 + |\hat{u}|^2 + |\hat{z}|^2$ and satisfies the differential inequality

$$E_t + cF \leq 0, \tag{3.3.8}$$

where F is the corresponding dissipative term and c is a positive constant. The explicit expressions of E and F are given respectively as follows: When $a = 1$, we have

$$\begin{aligned} E &= \frac{1}{2}\,|\hat{W}|^2 + \alpha_2 \Big\{ E_1 + \frac{\alpha_1 \xi}{1+\xi^2}(E_2 + E_3) \Big\}, \\ F &= |\hat{v}|^2 + |\hat{y}|^2 + \frac{\xi^2}{1+\xi^2}(|\hat{u}|^2 + |\hat{z}|^2), \end{aligned} \tag{3.3.9}$$

and when $a \neq 1$, we have

$$\begin{aligned} E &= \frac{1}{2}\,|\hat{W}|^2 + \frac{\alpha_2}{1+\xi^2} \Big\{ E_1 + \frac{\alpha_1 \xi}{1+\xi^2}(E_2 + E_3) \Big\}, \\ F &= \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}(|\hat{u}|^2 + |\hat{z}|^2), \end{aligned} \tag{3.3.10}$$

where

$$E_1 = -\operatorname{Re}(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}}), \quad E_2 = \operatorname{Re}(i\hat{v}\bar{\hat{u}}), \quad E_3 = \operatorname{Re}(i\hat{y}\bar{\hat{z}}),$$

and α_1 and α_2 are suitably small positive constants.

When $a = 1$, we see that the energy inequality (3.3.8) with the dissipative term F in (3.3.9) matches with the eigenvalues in (3.3.5) and (3.3.6). On the other hand, when $a \neq 1$, we find from (3.3.5) and (3.3.7) that the dissipative term F in (3.3.10) is not the desired one. In fact, we have two eigenvalues of the non-standard type in (3.3.7), while the corresponding F contains three components of the regularity-loss type. This suggests that the energy method in the Fourier space employed in [24] should be improved for $a \neq 1$. This will be done in the next subsection 3.3.2.

3.3.2 Refinement of energy method

When $a \neq 1$, we make a modification of the Lyapunov function such that the corresponding dissipative term matches with the eigenvalues in (3.3.5) and (3.3.7). The desired modification can be given by

$$\begin{aligned} E &= \frac{1}{2}|\hat{W}|^2 + \frac{\alpha_2}{1+\xi^2}\left(E_1 + \frac{\alpha_1\xi}{1+\xi^2}\{E_2 + (1+\xi^2)E_3\}\right), \\ F &= \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{z}|^2, \end{aligned} \tag{3.3.11}$$

where E_1, E_2 and E_3 are given in the previous subsection, and α_1 and α_2 are suitably small positive constants. In fact, we have:

Proposition 3.3.3. *Let $a \neq 1$. Then, for suitably small positive constants α_1 and α_2, the Lyapunov function E in* (3.3.11) *is equivalent to $|\hat{W}|^2$ and satisfies the differential inequality* (3.3.8) *with the dissipative term F in* (3.3.11)*, where c is a positive constant.*

Remark. Our dissipative term F in (3.3.11) completely matches with the eigenvalues in (3.3.5) and (3.3.7) for $a \neq 1$. Moreover, the above proposition yields the same pointwise estimate in Lemma 3.3.2 for $a \neq 1$.

As a simple corollary of Proposition 3.3.3, we have the optimal energy estimate for (3.1.6) for $a \neq 1$. In fact, we integrate (3.3.8) with respect to t over $(0, t)$. Then we multiply the resultant inequality by $(1+\xi^2)^{2(s-k)}|\xi|^{2k}$ and integrate with respect to $\xi \in \mathbb{R}$, where $0 \le k \le s$. This yields the following optimal energy estimate.

Proposition 3.3.4. *Let $a \neq 1$, and let s and k be integers with $0 \le k \le s$. Then the solution W to the problem* (3.1.6) *with W_0 satisfies the following energy estimate for any $t \ge 0$:*

$$\begin{aligned} &\|\partial_x^k W(t)\|_{H^{s-k}}^2 + \int_0^t \big(\|\partial_x^k v(\tau)\|_{H^{s-k-1}}^2 + \|\partial_x^k y(\tau)\|_{H^{s-k}}^2 \\ &\quad + \|\partial_x^{k+1} u(\tau)\|_{H^{s-k-2}}^2 + \|\partial_x^{k+1} z(\tau)\|_{H^{s-k-1}}^2 \big)\, d\tau \le C\|\partial_x^k W_0\|_{H^{s-k}}^2, \end{aligned} \tag{3.3.12}$$

where C is a positive constant.

Remark. The energy estimate (3.3.12) completely matches with the eigenvalues given in (3.3.5) and (3.3.7) for $a \neq 1$, and hence it seems optimal. We note that in the dissipative term of (3.3.12) for $a \neq 1$, we have one regularity-loss for two components v and u but no regularity-loss for other two components y and z.

The proof of Proposition 3.3.3. First our system (3.1.6) in the Fourier space is written explicitly in the form

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{3.3.13a}$$

$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + \gamma\hat{y} = 0, \tag{3.3.13b}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{3.3.13c}$$

$$\hat{z}_t - ai\xi\hat{y} = 0. \tag{3.3.13d}$$

When $a \neq 1$, we want to construct our Lyapunov function E in (3.3.11) and prove the differential inequality (3.3.8) with the corresponding F given in (3.3.11). This will be done by modifying the energy method employed in [24]. Our proof below is divided into four steps. The first two steps of our proof are the same as in [24], while the other two steps require some modifications.

Step 1. We multiply (3.3.13a), (3.3.13b), (3.3.13c) and (3.3.13d) by $\bar{\hat{v}}$, $\bar{\hat{y}}$, $\bar{\hat{u}}$ and $\bar{\hat{z}}$, respectively. Then, adding the resultant equations and taking the real part, we get

$$\frac{1}{2}(|\hat{W}|^2)_t + \gamma|\hat{y}|^2 = 0, \tag{3.3.14}$$

where $|\hat{W}|^2 = |\hat{v}|^2 + |\hat{y}|^2 + |\hat{u}|^2 + |\hat{z}|^2$. This will give us the energy estimate for $(\hat{v}, \hat{y}, \hat{u}, \hat{z})$ as well as the dissipative estimate for $\hat{y}$.

Step 2. We create the dissipative estimte for $\hat{v}$. To this end, we first multiply (3.3.13b) and (3.3.13a) by $-\bar{\hat{v}}$ and $-\bar{\hat{y}}$, respectively, and add the resulting two equalities. Also, we multiply (3.3.13c) and (3.3.13d) by $-a\bar{\hat{z}}$ and $-a\bar{\hat{u}}$, respectively, and add the resulting two equalities. Then, adding the two equalities obtained above and taking the real part, we arrive at

$$\begin{aligned} E_{1,t} + |\hat{v}|^2 - |\hat{y}|^2 &= \gamma \operatorname{Re}(\bar{\hat{v}}\hat{y}) - \xi \operatorname{Re}\{i(\bar{\hat{y}}\hat{u} + a^2\bar{\hat{u}}\hat{y})\} \\ &= \gamma \operatorname{Re}(\bar{\hat{v}}\hat{y}) - (a^2-1)\xi \operatorname{Re}(i\bar{\hat{u}}\hat{y}), \end{aligned}$$

where $E_1 := -\operatorname{Re}(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}})$. Applying the Young inequality, we obtain

$$E_{1,t} + (1-\varepsilon)|\hat{v}|^2 \leq C_\varepsilon|\hat{y}|^2 + |a^2-1||\xi||\hat{u}||\hat{y}| \tag{3.3.15}$$

for any $\varepsilon \in (0,1)$, where C_ε is a constant depending on ε.

Step 3. We create the dissipative estimate for $(\hat{u}, \hat{z})$. First, we multiply (3.3.13a) and (3.3.13c) by $i\xi\bar{\hat{u}}$ and $i\xi\bar{\hat{v}}$, respectively, and subtract the resulting two equalities. From the real part, we have

$$\xi E_{2,t} + \xi^2(|\hat{u}|^2 - |\hat{v}|^2) + \xi \operatorname{Re}(i\bar{\hat{u}}\hat{y}) = 0, \tag{3.3.16}$$

where $E_2 := \operatorname{Re}(i\hat{v}\bar{\hat{u}})$. Here we used the fact that $\operatorname{Re}\{i(\bar{\hat{u}}\hat{v}_t - \bar{\hat{v}}\hat{u}_t)\} = \{\operatorname{Re}(i\hat{v}\bar{\hat{u}})\}_t$. Similarly, we multiply (3.3.13b) and (3.3.13d) by $i\xi\bar{\hat{z}}$ and $i\xi\bar{\hat{y}}$, respectively. Then, subtracting the resulting two equalities and taking the real part, we have

$$\xi E_{3,t} + a\xi^2(|\hat{z}|^2 - |\hat{y}|^2) - \xi \operatorname{Re}\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} = 0, \tag{3.3.17}$$

where $E_3 := \operatorname{Re}(i\hat{y}\bar{\hat{z}})$.

When $a \neq 1$, we multiply (3.3.17) by $1 + \xi^2$ and add the resultant equality to (3.3.16). This gives

$$\begin{aligned} &\xi\{E_2 + (1+\xi^2)E_3\}_t + \xi^2|\hat{u}|^2 + a(1+\xi^2)\xi^2|\hat{z}|^2 \\ &\quad = \xi^2|\hat{v}|^2 + a(1+\xi^2)\xi^2|\hat{y}|^2 + \xi \operatorname{Re}\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} - \xi \operatorname{Re}(i\bar{\hat{u}}\hat{y}). \end{aligned}$$

Using the Young inequality, we obtain

$$\begin{aligned}&\xi\{E_2+(1+\xi^2)E_3\}_t+(1-\varepsilon)\xi^2|\hat{u}|^2+a(1-\varepsilon)(1+\xi^2)\xi^2|\hat{z}|^2\\&\quad\le C_\varepsilon(1+\xi^2)|\hat{v}|^2+C_\varepsilon(1+\xi^2)^2|\hat{y}|^2\end{aligned}\tag{3.3.18}$$

for any $\varepsilon\in(0,1)$, where C_ε is a constant depending on ε. This will give us a good dissipative estimate for $(\hat{u},\hat{z})$ in terms of the corresponding estimate for $(\hat{v},\hat{y})$.

Step 4. We construct our Lyapunov function E for $a\neq 1$. To this end, letting $\alpha_1>0$, we first multiply (3.3.15) and (3.3.18) by $\frac{1}{1+\xi^2}$ and $\frac{\alpha_1}{(1+\xi^2)^2}$, respectively, and add the resultant inequalities. Applying the Young inequality, we have

$$\begin{aligned}&\frac{1}{1+\xi^2}\Big(E_1+\frac{\alpha_1\xi}{1+\xi^2}\{E_2+(1+\xi^2)E_3\}\Big)_t\\&\quad+(1-\varepsilon-\alpha_1C_\varepsilon)\,\frac{1}{1+\xi^2}|\hat{v}|^2+\alpha_1(1-2\varepsilon)\,\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2\\&\quad+\alpha_1a(1-\varepsilon)\,\frac{\xi^2}{1+\xi^2}|\hat{z}|^2\;\le\;C_{\varepsilon,\alpha_1}|\hat{y}|^2,\end{aligned}\tag{3.3.19}$$

where C_{ε,α_1} is a constant depending on (ε,α_1). Next, letting $\alpha_2>0$, we multiply (3.3.19) by α_2 and add the resultant inequality to (3.3.14). This yields

$$\begin{aligned}&\frac{1}{2}\,(|\hat{W}|^2)_t+\frac{\alpha_2}{1+\xi^2}\Big(E_1+\frac{\alpha_1\xi}{1+\xi^2}\{E_2+(1+\xi^2)E_3\}\Big)_t\\&\quad+\alpha_2(1-\varepsilon-\alpha_1C_\varepsilon)\,\frac{1}{1+\xi^2}|\hat{v}|^2+(\gamma-\alpha_2C_{\varepsilon,\alpha_1})|\hat{y}|^2\\&\quad+\alpha_2\alpha_1(1-2\varepsilon)\,\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2+\alpha_2\alpha_1a(1-\varepsilon)\,\frac{\xi^2}{1+\xi^2}|\hat{z}|^2\;\le\;0.\end{aligned}\tag{3.3.20}$$

Here we take $\varepsilon>0$ such that $1-2\varepsilon>0$, i.e., $0<\varepsilon<\frac{1}{2}$. For this choice of ε we choose $\alpha_1>0$ and $\alpha_2>0$ such that $1-\varepsilon-\alpha_1C_\varepsilon>0$ and $\gamma-\alpha_2C_{\varepsilon,\alpha_1}>0$. Then (3.3.20) becomes the desired differential inequality (3.3.8) for our E and F given in (3.3.11) and for a small positive constant c. This completes the proof of Proposition 3.3.3. □

3.4 Energy method

The aim of this section is to prove the global existence result in Theorem 3.2.1. Our global existence result can be shown by the combination of a local existence result and the desired a priori estimate. Since our system (3.1.5) is a symmetric hyperbolic system, it is not difficult to show a local existence result by the standard method, and we omit the details. To state our result on the a priori estimate, we consider a solution $W(t)$ of the problem (3.1.5) with the initial data W_0 satisfying $W \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ for $s \geq 2$ and

$$\sup_{0 \leq t \leq T} \|W(t)\|_{L^\infty} \leq \delta, \tag{3.4.1}$$

where δ is a fixed positive constant. Our a priori estimate is now given as follows.

Proposition 3.4.1 (A priori estimate)**.** *Suppose that $W_0 \in H^s$ for $s \geq 2$ and put $E_0 = \|W_0\|_{H^s}$. Let $T > 0$ and let $W(t)$ be a solution to the Cauchy problem* (3.1.5) *with the initial data W_0 satisfying* (3.4.1)*. Then there exists a positive constant δ_2 independent of T such that if $E_0 \leq \delta_2$, we have the a priori estimate*

$$E(t)^2 + D(t)^2 \leq CE_0^2, \qquad t \in [0,T], \tag{3.4.2}$$

where $C > 0$ is a constant independent of T.

To prove the above a priori estimate in Proposition 3.4.1, we need to show the following energy inequality by applying the energy method.

Proposition 3.4.2 (Energy inequality)**.** *Suppose that $W_0 \in H^s$ for $s \geq 2$ and put $E_0 = \|W_0\|_{H^s}$. Let $T > 0$ and let $W(t)$ be a solution to the Cauchy problem* (3.1.5) *with the initial data W_0 satisfying* (3.4.1)*. Then we have the following energy inequality:*

$$E(t)^2 + D(t)^2 \leq CE_0^2 + CE(t)D(t)^2, \qquad t \in [0,T], \tag{3.4.3}$$

where $C > 0$ is a constant independent of T.

We note that the desired a priori estimate (3.4.2) easily follows from the energy inequality (3.4.3), provided that E_0 is suitably small. Therefore it is sufficient to prove (3.4.3) for our purpose.

3.4.1 Proof of Proposition 3.4.2

In this subsection we prove the energy inequality (3.4.3) in Proposition 3.4.2 by using the energy method. Our energy method is based on the refined Lyapunov function constructed in Subsection 3.3.2 and gives the optimal dissipation estimate for z without any regularity-loss, which can control the nonlinearity of the system (3.1.5).

Proof. Our proof is divided into 4 steps.

Step 1. (Basic energy and dissipation for y): We calculate as (3.1.3a) $\times\, v$ + (3.1.3b) $\times\, y$ + (3.1.3c) $\times\, u$ + (3.1.3d) $\times \{\sigma(z/a) - \sigma(0)\}/a$. This yields

$$\frac{1}{2}(v^2 + y^2 + u^2 + S(z))_t - \{vu + (\sigma(z/a) - \sigma(0))y\}_x + \gamma y^2 = 0, \tag{3.4.4}$$

where $S(z) := 2\int_0^{z/a}(\sigma(\eta)-\sigma(0))d\eta$ is equivalent to $|z|^2$. Integrate (3.4.4) with respect to x to have

$$\frac{d}{dt}E_0^{(0)} + 2\gamma\|y\|_{L^2}^2 = 0, \tag{3.4.5}$$

where

$$E_0^{(0)} := \|(v,y,u)\|_{L^2}^2 + \int_{\mathbb{R}} S(z)dx.$$

Since $E_0^{(0)}$ is equivalent to $\|W\|_{L^2}^2$, by integrating (3.4.5) with respect to t, we obtain

$$\|W(t)\|_{L^2}^2 + \int_0^t \|y(\tau)\|_{L^2}^2 d\tau \le CE_0^2. \tag{3.4.6}$$

Next, we apply ∂_x^k to (3.1.3) and write $\partial_x^k(v,y,u,z) = (V,Y,U,Z)$ for simplicity. Then we have

$$V_t - U_x - Y = 0, \tag{3.4.7a}$$

$$Y_t - \sigma'(z/a)(Z/a)_x - V + \gamma Y = [\partial_x^k, \sigma'(z/a)](z/a)_x, \tag{3.4.7b}$$

$$U_t - V_x = 0, \tag{3.4.7c}$$

$$Z_t - aY_x = 0, \tag{3.4.7d}$$

where $[A,B] := AB - BA$. We compute as $(3.4.14)\times V + (3.4.7b)\times Y + (3.4.17)\times U + (3.4.20)\times \sigma'(z/a)Z/a^2$. This gives

$$\begin{aligned}&\frac{1}{2}\big(V^2+Y^2+U^2+\sigma'(z/a)(Z/a)^2\big)_t - \{VU + \sigma'(z/a)(Z/a)Y\}_x + \gamma Y^2\\ &= \frac{1}{2}\sigma'(z/a)_t(Z/a)^2 - \sigma'(z/a)_x(Z/a)Y + Y[\partial_x^k, \sigma'(z/a)](z/a)_x.\end{aligned} \tag{3.4.8}$$

Integrate (3.4.8) with respect to x to have

$$\frac{d}{dt}E_0^{(k)} + 2\gamma\|\partial_x^k y\|_{L^2}^2 \le CR_0^{(k)} \tag{3.4.9}$$

for $1 \le k \le s$, where

$$E_0^{(k)} := \|\partial_x^k(v,y,u)\|_{L^2}^2 + \int_{\mathbb{R}} \sigma'(z/a)|\partial_x^k(z/a)|^2dx,$$

$$R_0^{(k)} := \int_{\mathbb{R}} |y_x||\partial_x^k z|^2 + |z_x||\partial_x^k z||\partial_x^k y| + |[\partial_x^k, \sigma'(z/a)]z_x||\partial_x^k y|dx.$$

Here in the term $R_0^{(k)}$ we used the relation $z_t = ay_x$ from (3.1.3d). Now we integrate (3.4.9) with respect to t and add for k with $1 \le k \le s$. Since $E_0^{(k)}$ is equivalent to $\|\partial_x^k W\|_{L^2}^2$, we obtain

$$\|\partial_x W(t)\|_{H^{s-1}}^2 + \int_0^t \|\partial_x y(\tau)\|_{H^{s-1}}^2\, d\tau \le CE_0^2 + CE(t)D(t)^2. \tag{3.4.10}$$

Here we have used the following estimates for $R_0^{(k)}$:

$$R_0^{(k)} \le C\|\partial_x(y,z)\|_{L^\infty}\|\partial_x^k(y,z)\|_{L^2}^2, \quad \sum_{k=1}^{s}\int_0^t R_0^{(k)}(\tau)\,d\tau \le CE(t)D(t)^2.$$

Consequently, adding (3.4.6) and (3.4.10), we arrive at

$$E(t)^2 + \int_0^t \|y(\tau)\|_{H^s}^2\,d\tau \le CE_0^2 + CE(t)D(t)^2. \tag{3.4.11}$$

Step 2. (Dissipation for v): We rewrite the system (3.1.3) in the form

$$\begin{aligned} &v_t - u_x - y = 0,\\ &y_t - az_x - v + \gamma y = g(z)_x,\\ &u_t - v_x = 0,\\ &z_t - ay_x = 0, \end{aligned} \tag{3.4.12}$$

where $g(z) := \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$ as $z \to 0$. We apply ∂_x^k to (3.4.12). Letting $(V,Y,U,Z) = \partial_x^k(v,y,u,z)$ as before, we have

$$V_t - U_x - Y = 0, \tag{3.4.13a}$$

$$Y_t - aZ_x - V + \gamma Y = \partial_x^k g(z)_x, \tag{3.4.13b}$$

$$U_t - V_x = 0, \tag{3.4.13c}$$

$$Z_t - aY_x = 0. \tag{3.4.13d}$$

To create the dissipation term V^2, we compute as $(3.4.13b) \times (-V) + (3.4.13a) \times (-Y) + (3.4.13c) \times (-aZ) + (3.4.13d) \times (-aU)$. This gives

$$\begin{aligned} &-(VY + aUZ)_t + (aVZ + a^2YU)_x + V^2\\ &\quad = Y^2 + \gamma VY + (a^2-1)\,YU_x - V\partial_x^k g(z)_x. \end{aligned} \tag{3.4.14}$$

Integrate (3.4.14) with respect to x to obtain

$$\begin{aligned} \frac{d}{dt}E_1^{(k)} + \|\partial_x^k v\|_{L^2}^2 &\le \|\partial_x^k y\|_{L^2}^2 + \gamma\|\partial_x^k v\|_{L^2}\|\partial_x^k y\|_{L^2}\\ &\quad + (a^2-1)\int_{\mathbb{R}} \partial_x^k y\,\partial_x^k u_x\,dx + R_1^{(k)} \end{aligned} \tag{3.4.15}$$

for $0 \le k \le s-1$, where

$$\begin{aligned} E_1^{(k)} &:= -\int_{\mathbb{R}} \partial_x^k v\,\partial_x^k y\,dx - a\int_{\mathbb{R}} \partial_x^k u\,\partial_x^k z\,dx,\\ R_1^{(k)} &:= \int_{\mathbb{R}} |\partial_x^k v|\,|\partial_x^{k+1} g(z)|\,dx. \end{aligned}$$

Adding (3.4.15) with k and $k+1$ and integrating by parts, we have

$$\frac{d}{dt}\big(E_1^{(k)} + E_1^{(k+1)}\big) + \|\partial_x^k v\|_{H^1}^2 \le \|\partial_x^k y\|_{H^1}^2 + \gamma\|\partial_x^k v\|_{H^1}\|\partial_x^k y\|_{H^1}$$
$$+ (a^2-1)\int_{\mathbb{R}} (\partial_x^k y\, \partial_x^k u_x - \partial_x^{k+1} y_x\, \partial_x^{k+1} u)\, dx + R_1^{(k)} + R_1^{(k+1)}$$
$$\le \|\partial_x^k y\|_{H^1}^2 + \gamma\|\partial_x^k v\|_{H^1}\|\partial_x^k y\|_{H^1} + |a^2-1|\|\partial_x^k y\|_{H^2}\|\partial_x^{k+1} u\|_{L^2} + R_1^{(k)} + R_1^{(k+1)}$$

for $0 \le k \le s-2$. We integtate this inequality with respect to t and add for k with $0 \le k \le s-2$. Noting that $\sum_{k=0}^{s-1} |E_1^{(k)}| \le C\|W\|_{H^{s-1}}^2$ and using the Young inequality, we obtain

$$\int_0^t \|v(\tau)\|_{H^{s-1}}^2\, d\tau \le \varepsilon\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau + C_\varepsilon\int_0^t \|y(\tau)\|_{H^s}^2 d\tau$$
$$+ CE_0^2 + CE(t)^2 + CE(t)D(t)^2 \tag{3.4.16}$$

for any $\varepsilon > 0$, where C_ε is a constant depending on ε. Here we also used the following estimates for $R_1^{(k)}$:

$$R_1^{(k)} \le C\|z\|_{L^\infty}\|\partial_x^k v\|_{L^2}\|\partial_x^{k+1} z\|_{L^2}, \quad \sum_{k=0}^{s-1}\int_0^t R_1^{(k)}(\tau)\, d\tau \le CE(t)D(t)^2.$$

Step 3. (Dissipation for u and z): To get the dissipation term U_x^2, we compute as $(3.4.13a) \times (-U_x) + (3.4.13c) \times V_x$. This gives

$$-(VU_x)_t + (VU_t)_x + U_x^2 \;=\; V_x^2 + YU_x. \tag{3.4.17}$$

Integrating (3.4.17) with respect to x, we have

$$\frac{d}{dt}E_2^{(k)} + \|\partial_x^{k+1} u\|_{L^2}^2 \le \|\partial_x^{k+1} v\|_{L^2}^2 + \|\partial_x^k y\|_{L^2}\|\partial_x^{k+1} u\|_{L^2} \tag{3.4.18}$$

for $0 \le k \le s-2$, where $E_2^{(k)} := -\int_{\mathbb{R}} \partial_x^k v\, \partial_x^{k+1} u\, dx$. We integrate (3.4.18) with respect to t and add for k with $0 \le k \le s-2$. Then we easily get

$$\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \le C\int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^{s-2}}^2\, d\tau + CE_0^2 + CE(t)^2. \tag{3.4.19}$$

In order to create the dissipation term Z_x^2, we compute as $(3.4.13b) \times (-Z_x) + (3.4.13d) \times Y_x$. This yields

$$-(YZ_x)_t + (YZ_t)_x + aZ_x^2 \;=\; aY_x^2 - (V - \gamma Y)\, Z_x - Z_x\partial_x^k g(z)_x. \tag{3.4.20}$$

Integrating (3.4.20) with respect to t, we obtain

$$\frac{d}{dt}E_3^{(k)} + a\|\partial_x^{k+1} z\|_{L^2}^2 \le a\|\partial_x^{k+1} y\|_{L^2}^2 + \|\partial_x^k v - \gamma\partial_x^k y\|_{L^2}\|\partial_x^{k+1} z\|_{L^2} + R_3^{(k)} \tag{3.4.21}$$

for $0 \le k \le s-1$, where

$$E_3^{(k)} := -\int_{\mathbb{R}} \partial_x^k y \, \partial_x^{k+1} z \, dx, \quad R_3^{(k)} := \int_{\mathbb{R}} |\partial_x^{k+1} z| \, |\partial_x^{k+1} g(z)| \, dx.$$

We integrate (3.4.21) with respect to t and add for k with $0 \le k \le s-1$. This yields

$$\int_0^t \|\partial_x z(\tau)\|_{H^{s-1}}^2 \, d\tau \le C \int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2 \, d\tau + CE_0^2 + CE(t)^2 + CE(t)D(t)^2. \tag{3.4.22}$$

Here we have used the estimates

$$R_3^{(k)} \le C\|z\|_{L^\infty} \|\partial_x^{k+1} z\|_{L^2}^2, \quad \sum_{k=0}^{s-1} \int_0^t R_3^{(k)}(\tau) \, d\tau \le CE(t)D(t)^2.$$

Step 4. Finally, combining (3.4.16), (3.4.19) and (3.4.22), and then taking $\varepsilon > 0$ in (3.4.16) suitably small, we arrive at the estimate

$$\int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2 \, d\tau \le C \int_0^t \|y(\tau)\|_{H^s}^2 \, d\tau + CE_0^2 + CE(t)^2 + CE(t)D(t)^2.$$

This combined with the basic estimate (3.4.11) yields the desired inequality $E(t)^2 + D(t)^2 \le CE_0^2 + CE(t)D(t)^2$. Thus the proof of Proposition 3.4.2 is comptlete.

□

3.5 L^2 decay estimate

The aim of this section is to show the optimal decay estimate stated in Theorem 3.2.2. For this purpose we derive the pointwise estimate of solutions in the Fourier space. We recall that the system (3.1.3) is written in the form of (3.4.12) or in the vector notation as

$$W_t + AW_x + LW = G_x, \tag{3.5.1}$$

where $G = (0, g(z), 0, 0)^T$ with $g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$ for $z \to 0$; the coefficient matrices A and L are given in (3.1.6).

Proposition 3.5.1 (Pointwise estimate). *Let W be a solution of* (3.5.1) *with the initial data W_0. Then the Fourier image $\hat{W}$ satisfies the pointwise estimate*

$$|\hat{W}(\xi,t)|^2 \le Ce^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2 + C\int_0^t e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{G}(\xi,\tau)|^2 \, d\tau \tag{3.5.2}$$

for $\xi \in \mathbb{R}$ and $t \ge 0$, where $\rho(\xi) := \xi^2/(1+\xi^2)^2$, and C and c are positive constants.

Our optimal decay estimate will be obtained by applying the following decay estimate of L^2-L^q-L^r type, which was established in Chapter 2.

Lemma 3.5.2 (Decay estimate of L^2-L^q-L^r type)**.** *Let U be a function satisfying*

$$|\hat{U}(\xi,t)| \le C|\xi|^m e^{-c\rho(\xi)t}|\hat{U}_0(\xi)| \tag{3.5.3}$$

for $\xi \in \mathbb{R}$ and $t \ge 0$, where $\rho(\xi) = \xi^2/(1+\xi^2)^2$, $m \ge 0$, and U_0 is a given function. Then we have

$$\begin{aligned}\|\partial_x^k U(t)\|_{L^2} \le{}& C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+m}{2}}\|U_0\|_{L^q}\\ &+ C(1+t)^{-\frac{\ell}{2}+\frac{1}{2}(\frac{1}{r}-\frac{1}{2})}\|\partial_x^{k+m+\ell}U_0\|_{L^r},\end{aligned} \tag{3.5.4}$$

where $k \ge 0$, $1 \le q$, $r \le 2$, $\ell > \frac{1}{r} - \frac{1}{2}$ ($\ell \ge 0$ if $r = 2$).

Remark. The first (resp. the second) term on the right hand side of (3.5.4) is corresponding to the low frequency region $|\xi| \le 1$ (resp. high frequency region $|\xi| \ge 1$). When $m = 0$, $q = 1$ and $r = 2$, the estimate (3.5.4) is reduced to

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell}U_0\|_{L^2},$$

which is just the same decay estimate obtained in [24] for the linear system (3.1.6).

The outline of the proof of Lemma 3.5.2 is as follows. From the Plancherel theorem and (3.5.3), we have

$$\|\partial_x^k U(t)\|_{L^2}^2 = \int_{\mathbb{R}} \xi^{2k}|\hat{U}(\xi,t)|^2\,d\xi \le C\int_{\mathbb{R}} \xi^{2(k+m)}e^{-c\rho(\xi)t}|\hat{U}_0(\xi)|^2\,d\xi$$

We divide the last integral into two parts corresponding to $|\xi| \le 1$ and $|\xi| \ge 1$, respectively, and estimate each part by applying the Hölder inequality and the Hausdorff-Young inequality. This yields the desired estimate (3.5.4). We omit the details and refer to Chapter 2.

3.5.1 Proof of Proposition 3.5.1

Proof. Taking the Fourier transform of (3.4.12), we have

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{3.5.5a}$$

$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + \gamma\hat{y} = i\xi\hat{g}, \tag{3.5.5b}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{3.5.5c}$$

$$\hat{z}_t - ai\xi\hat{y} = 0, \tag{3.5.5d}$$

where $g = g(z)$. We construct a Lyapunov function of the system (3.5.5) in the Fourier space. The computations below are essentially the same as in Subsection 3.3.2 and correspond to those in the proof of Proposition 3.4.2. We divide the proof into 4 steps.

Step 1. (Basic energy and dissipation for $\hat{y}$): We compute as (3.5.5a) $\times\ \bar{\hat{v}}$ + (3.5.5b) $\times\ \bar{\hat{y}}$ + (3.5.5c) $\times\ \bar{\hat{u}}$ + (3.5.5d) $\times\ \bar{\hat{z}}$ and take the real part. This yields

$$\frac{1}{2}\,E_{0,t} + \gamma|\hat{y}|^2 = \mathrm{Re}\,(i\xi\bar{\hat{y}}\hat{g}),$$

where $E_0 := |\hat{W}|^2$. Applying the Young inequality, we have

$$E_{0,t} + \gamma|\hat{y}|^2 \le C\xi^2|\hat{g}|^2. \tag{3.5.6}$$

Step 2. (Dissipation for $\hat{v}$): To create the dissipation term for $\hat{v}$, we compute as (3.5.5b) $\times$ $(-\bar{\hat{v}})$ + (3.5.5a) $\times$ $(-\bar{\hat{y}})$ + (3.5.5c) $\times$ $(-a\bar{\hat{z}})$ + (3.5.5d) $\times$ $(-a\bar{\hat{u}})$ and take the real part. This gives

$$\begin{aligned} E_{1,t} + |\hat{v}|^2 - |\hat{y}|^2 &= \gamma\,\mathrm{Re}\,(\bar{\hat{v}}\hat{y}) - \ \mathrm{Re}\,\{i\xi\,(\bar{\hat{y}}\hat{u} + a^2\bar{\hat{u}}\hat{y})\} - \mathrm{Re}\,(i\xi\bar{\hat{v}}\hat{g}) \\ &= \gamma\,\mathrm{Re}\,(\bar{\hat{v}}\hat{y}) - (a^2-1)\,\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) - \xi\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{g}) \end{aligned}$$

where $E_1 := -\mathrm{Re}\,(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}})$. We multiply this equality by $1+\xi^2$. Then, using the Young inequality, we obtain

$$(1+\xi^2)\,E_{1,t} + c_1(1+\xi^2)|\hat{v}|^2 \le \varepsilon\xi^2|\hat{u}|^2 + C_\varepsilon(1+\xi^2)^2|\hat{y}|^2 + C(1+\xi^2)\,\xi^2|\hat{g}|^2 \tag{3.5.7}$$

for any $\varepsilon > 0$, where c_1 is a positive constant with $c_1 < 1$ and C_ε is a constant depending on ε.

Step 3. (Dissipation for $\hat{u}$ and $\hat{z}$): To create the dissipation term $|\hat{u}|^2$, we compute as (3.5.5a)$\times$ $i\xi\bar{\hat{u}}$ − (3.5.5c) $\times$ $i\xi\bar{\hat{v}}$ and take the real part. The result is

$$\xi E_{2,t} + \xi^2(|\hat{u}|^2 - |\hat{v}|^2) + \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) = 0, \tag{3.5.8}$$

where $E_2 := \mathrm{Re}\,(i\hat{v}\bar{\hat{u}})$. For the dissipation term $|\hat{z}|^2$, we compute as (3.5.5b)$\times i\xi\bar{\hat{z}}$−(3.5.5d)$\times$ $i\xi\bar{\hat{y}}$ and take the real part. Then we have

$$\xi E_{3,t} + a\xi^2(|\hat{z}|^2 - |\hat{y}|^2) - \xi\,\mathrm{Re}\,\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} = -\xi^2\,\mathrm{Re}\,(\bar{\hat{z}}\hat{g}), \tag{3.5.9}$$

where $E_3 := \mathrm{Re}\,(i\hat{y}\bar{\hat{z}})$. Now we combine (3.5.8) and (3.5.9) such that (3.5.8)+(3.5.9)$\times(1+\xi^2)$. This gives

$$\begin{aligned} &\xi\,\{E_2 + (1+\xi^2)\,E_3\}_t + \xi^2|\hat{u}|^2 + a\,(1+\xi^2)\,\xi^2|\hat{z}|^2 \\ &\quad = \xi^2|\hat{v}|^2 + a\,(1+\xi^2)\,\xi^2|\hat{y}|^2 + (1+\xi^2)\,\xi\,\mathrm{Re}\,\{i\bar{\hat{z}}(\hat{v} - \gamma\hat{y})\} \\ &\qquad - \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) - (1+\xi^2)\,\xi^2\,\mathrm{Re}\,\{\bar{\hat{z}}\hat{g}\}. \end{aligned}$$

Using the Young inequality, we get

$$\begin{aligned} &\xi\{E_2 + (1+\xi^2)E_3\}_t + c_1\xi^2|\hat{u}|^2 + c_2\,(1+\xi^2)\,\xi^2|\hat{z}|^2 \\ &\quad \le C(1+\xi^2)|\hat{v}|^2 + C(1+\xi^2)^2|\hat{y}|^2 + C(1+\xi^2)\,\xi^2|\hat{g}|^2, \end{aligned} \tag{3.5.10}$$

where c_1 and c_2 are positive constants satisfying $c_1 < 1$ and $c_2 < a$, respectively.

Step 4. (Lyapunov function): Letting $\alpha_1 > 0$, we combine (3.5.7) and (3.5.10) such that $(3.5.7) + (3.5.10) \times \alpha_1$. Then we have

$$\begin{aligned}&\big\{(1+\xi^2)\,E_1 + \alpha_1\xi\{E_2 + (1+\xi^2)\,E_3\}\big\}_t + (c_1 - \alpha_1 C)(1+\xi^2)|\hat{v}|^2\\&\quad + (\alpha_1 c_1 - \varepsilon)\,\xi^2|\hat{u}|^2 + \alpha_1 c_2\,(1+\xi^2)\,\xi^2|\hat{z}|^2\\&\leq C_{\varepsilon,\alpha_1}(1+\xi^2)^2|\hat{y}|^2 + C_{\alpha_1}(1+\xi^2)\,\xi^2|\hat{g}|^2,\end{aligned} \tag{3.5.11}$$

where C_{ε,α_1} and C_{α_1} are constants depending on (ε,α_1) and α_1, respectively. Also, letting $\alpha_2 > 0$, we combine (3.5.6) and (3.5.11) such that $(3.5.6) + (3.5.11) \times \frac{\alpha_2}{(1+\xi^2)^2}$. Then, putting

$$E := E_0 + \frac{\alpha_2}{1+\xi^2}\Big(E_1 + \frac{\alpha_1\xi}{1+\xi^2}\{E_2 + (1+\xi^2)\,E_3\}\Big), \tag{3.5.12}$$

we obtain

$$\begin{aligned}&E_t + \alpha_2(c_1 - \alpha_1 C)\,\frac{1}{1+\xi^2}|\hat{v}|^2 + (\gamma - \alpha_2 C_{\varepsilon,\alpha_1})|\hat{y}|^2\\&\quad + \alpha_2(\alpha_1 c_1 - \varepsilon)\,\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \alpha_2\alpha_1 c_2\frac{\xi^2}{1+\xi^2}|\hat{z}|^2 \leq C_{\alpha_1,\alpha_2}\xi^2|\hat{g}|^2,\end{aligned} \tag{3.5.13}$$

where C_{α_1,α_2} is a constant depending on (α_1,α_2). Here we see that there is a small positive constant α_0 such that if $\alpha_1,\ \alpha_2 \in (0,\alpha_0]$, then E in (3.5.12) is equivalent to $|\hat{W}|^2$, that is,

$$c_0|\hat{W}|^2 \leq E \leq C_0|\hat{W}|^2, \tag{3.5.14}$$

where c_0 and C_0 are positive constants. Futhermore, we choose $\alpha_1 \in (0,\alpha_0]$ such that $c_1 - \alpha_1 C > 0$ and take $\varepsilon > 0$ so small as $\alpha_1 c_1 - \varepsilon > 0$. Finally, we choose $\alpha_2 \in (0,\alpha_0]$ such that $\gamma - \alpha_2 C_{\varepsilon,\alpha_1} > 0$. Then (3.5.13) becomes to

$$E_t + cF \leq C\xi^2|\hat{g}|^2, \tag{3.5.15}$$

where

$$F := \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{z}|^2 \tag{3.5.16}$$

This suggests that E in (3.5.12) is the desired Lyapunov function of the system (3.5.5). Noting (3.5.14), we find that $F \geq c\rho(\xi)E$, where $\rho(\xi) = \xi^2/(1+\xi^2)^2$. Therefore (3.5.15) becomes to $E_t + c\rho(\xi)E \leq C\xi^2|\hat{g}|^2$. Solving this ordinary differential inequality for E and using (3.5.14), we arrive at the desired estimate (3.5.2) in the form

$$|\hat{W}(\xi,t)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2 + C\int_0^t e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2\,d\tau.$$

This completes the proof of Proposition 3.5.1.

□

3.5.2 Proof of Theorem 3.2.2

Proof. Let W be the solution to the problem (3.1.5) with W_0 obtained in Theorem 3.2.1. Then W satisfies (3.5.1). Therefore we have the pointwise estimate (3.5.2). We integrate (3.5.2) with respect to ξ. Applying the Plancherel theorem, we obtain

$$\begin{aligned}\|W(t)\|_{L^2}^2 &= \int_{\mathbb{R}} |\hat{W}(\xi,t)|^2\,d\xi \\ &\le C\int_{\mathbb{R}} e^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2\,d\xi + C\int_0^t\int_{\mathbb{R}} e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{g}(\xi,\tau)|^2\,d\xi d\tau =: I+J. \quad (3.5.17)\end{aligned}$$

We estimate the terms I and J by applying Lemma 3.5.2. For I, using (3.5.4) with $m=0$, we have

$$\begin{aligned}I &= C\int_{\mathbb{R}} e^{-c\rho(\xi)t}|\hat{W}_0(\xi)|^2\,d\xi \\ &\le \underbrace{C(1+t)^{-\frac{1}{2}}\|W_0\|_{L^1}^2}_{k=0,\ q=1} + \underbrace{C(1+t)^{-1}\|\partial_x W_0\|_{L^2}^2}_{k=0,\ \ell=1,\ r=2} \\ &\le CE_1^2(1+t)^{-\frac{1}{2}}, \quad (3.5.18)\end{aligned}$$

where $E_1 = \|W_0\|_{H^2} + \|W_0\|_{L^1}$. On the other hand, for J we use (3.5.4) with $m=1$. Then we obtain

$$\begin{aligned}J &= C\int_0^t\int_{\mathbb{R}} e^{-c\rho(\xi)(t-\tau)}\xi^2|\hat{G}(\xi,\tau)|^2\,d\tau d\xi \\ &\le C\int_0^t \underbrace{(1+t-\tau)^{-\frac{3}{2}}\|G(\tau)\|_{L^1}^2}_{k=0,\ q=1}\,d\tau + C\int_0^t \underbrace{(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^2 G(\tau)\|_{L^1}^2}_{k=0,\ \ell=1,\ r=1}\,d\tau \\ &=: J_1 + J_2.\end{aligned}$$

Here we introduce the norms $N(t)$ and $D(t)$ by

$$N(t) = \sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{4}}\|W(\tau)\|_{L^2}, \quad D(t)^2 = \int_0^t \|\partial_x z(\tau)\|_{H^1}^2\,d\tau.$$

We know from Theorem 3.2.1 that $D(t) \le CE_0 \le CE_1$. For the low frequency part J_1, since $\|G\|_{L^1} \le C\|z\|_{L^2}^2$, we have

$$\begin{aligned}J_1 &\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|z(\tau)\|_{L^2}^4\,d\tau \\ &\le CN(t)^4\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-1}\,d\tau \le CN(t)^4(1+t)^{-1}. \quad (3.5.19)\end{aligned}$$

For the high frequency part J_2, using $\|\partial_x^2 G\|_{L^1} \le C\|z\|_{L^2}\|\partial_x^2 z\|_{L^2}$, we have

$$\begin{aligned}
J_2 &\le C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|z(\tau)\|_{L^2}^2\|\partial_x^2 z(\tau)\|_{L^2}^2\,d\tau \\
&\le CN(t)^2\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}\|\partial_x^2 z(\tau)\|_{L^2}^2\,d\tau \\
&\le CN(t)^2D(t)^2\sup_{0\le\tau\le t}\left\{(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}\right\} \\
&\le CN(t)^2D(t)^2(1+t)^{-\frac{1}{2}}. \qquad (3.5.20)
\end{aligned}$$

Combining (3.5.18), (3.5.19) and (3.5.20) and using $D(t) \le CE_1$, we obtain

$$(1+t)^{\frac{1}{2}}\|W(t)\|_{L^2}^2 \le CE_1^2 + CN(t)^4 + CE_1^2N(t)^2.$$

Thus we have the inequality $N(t)^2 \le CE_1^2 + CN(t)^4 + CE_1^2N(t)^2$. This inequality can be solved as $N(t) \le CE_1$, provided that E_1 is suitably small. Thus we have proved the desired decay estimate $\|W(t)\|_{L^2} \le CE_1(1+t)^{-1/4}$. This completes the proof of Theorem 3.2.2.

□

Chapter 4

Application to Besov spaces (I)

4.1 Introduction

Consider the following Timoshenko system (see [62, 63]), which is a set of two coupled wave equations of the form

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - \sigma(\psi_x)_x - (\varphi_x - \psi) + \gamma\psi_t = 0, \end{cases} \tag{4.1.1}$$

and describes the transverse vibrations of a beam. Here $t \geq 0$ is the time variable, $x \in \mathbb{R}$ is the spacial variable which denotes the point on the center line of the beam, $\varphi(t, x)$ is the transversal displacement of the beam from an equilibrium state, and ψ is the rotation angle of the filament of the beam. The smooth function $\sigma(\eta)$ satisfies $\sigma'(\eta) > 0$ for any $\eta \in \mathbb{R}$, and γ is a positive constant. System (4.1.1) is supplemented with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1)(x). \tag{4.1.2}$$

The linearized system of (4.1.1) reads correspondingly as

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \gamma\psi_t = 0, \end{cases} \tag{4.1.3}$$

with $a > 0$ is the propagation velocity defined by $a^2 = \sigma'(0)$. The case $a = 1$ corresponds to the Timoshenko system with equal wave speeds.

4.1.1 Known results

In a bounded domain, it is known that (4.1.3) is exponentially stable if the damping term φ_t is also present on the left-hand side of the first equation of (4.1.3) (see, e.g., [48]). Soufyane [59] showed that (4.1.3) could not be exponentially stable by considering only the damping term of the form ψ_t, unless for the case of $a = 1$ (equal wave speeds). A similar result was obtained by Rivera and Racke [50] with an alternative proof. Moreover, Rivera and Racke

[49] extended those results in [48, 50] to the Timoshenko system where the heat conduction described by the classical Fourier law was additional considered.

In the whole space, Kawashima *et. al.* [24] introduced the following quantities

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \tag{4.1.4}$$

so that the system (4.1.3) can be rewritten as

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \gamma y = 0. \end{cases} \tag{4.1.5}$$

The initial data are given by

$$(v,\, u,\, z,\, y)(x,0) = (v_0,\, u_0,\, z_0,\, y_0)(x), \tag{4.1.6}$$

where $v_0 = \varphi_{0,x} - \psi_0$, $y_0 = \psi_1$, $u_0 = \varphi_1$ and $z_0 = a\psi_{0,x}$. Furthermore, it was shown by [24] that the dissipative structure of (4.1.5) is characterized by

$$\begin{cases} \operatorname{Re}\lambda(i\xi) \le -c\eta_1(\xi) \qquad \text{for} \quad a = 1, \\ \operatorname{Re}\lambda(i\xi) \le -c\eta_2(\xi) \qquad \text{for} \quad a \neq 1, \end{cases} \tag{4.1.7}$$

where $\lambda(i\xi)$ denotes the eigenvalues of the system (4.1.5) in the Fourier space, $\eta_1(\xi) = \xi^2/(1+\xi^2)$, $\eta_2(\xi) = \xi^2/(1+\xi^2)^2$, and c is a positive constant. As the consequence, the following decay properties are shown for $U = (v, u, z, y)^\top$ of (4.1.5):

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2} \tag{4.1.8}$$

for $a = 1$, and

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} U_0\|_{L^2} \tag{4.1.9}$$

for $a \neq 1$, where $U_0 := (v_0, z_0, u_0, y_0)$, k and l are nonnegative integers, and c and C are positive constants. However, the energy functionals in [24] are not optimal. Recently, by a careful analysis for asymptotic expansions of the eigenvalues, the author and Kawashima [38] (see Chapter 3) gave the optimal energy method in Fourier spaces, which is regarded as an improved version of that in [24]. With the additional assumption $\int_{\mathbb{R}} U_0\, dx = 0$, Racke and Said-Houari [53] strengthened those decay properties in [24] such that linearized solutions decay faster with a rate of $t^{-\gamma/2}$, by introducing the integral space $L^{1,\gamma}(\mathbb{R})$.

Other studies on the dissipative Timoshenko system can be found in the literature. We refer to [47, 48] for frictional dissipation case, [19, 54, 55] for thermal dissipation case, and [3, 4, 33, 34] for memory-type dissipation case.

4.1.2 Main results

The main aim of Chapter 4 is to establish the global existence and optimal decay estimates of solutions in spatially critical Besov spaces. To the best of our knowledge, so far there are no results available in this direction for the Timoshenko system, although the critical space has already been succeeded in the study of fluid dynamical equations, see [2, 15, 22, 44] for Navier-Stokes equations, [10, 74, 76] for Euler equations and related models. In [70], under the assumptions of dissipative entropy and Shizuta-Kawashima condition, Xu and Kawashima have already studied generally dissipative hyperbolic systems where the dissipation matrix is symmetric, however, the Timoshenko system has the non-symmetric dissipation. More precisely, with the aid of variable change (4.1.4) (with $a = 1$), it is convenient to rewrite (4.1.1)-(4.1.2) as a Cauchy problem for the hyperbolic system of first order

$$\begin{cases} U_t + A(U)\,U_x + LU = 0, \\ U(0,x) = U_0(x), \end{cases} \tag{4.1.10}$$

where

$$A(U) = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \sigma'(z) & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}. \tag{4.1.11}$$

Notice that $A(U)$ is a real symmetrizable matrix due to $\sigma'(z) > 0$, and the matrix L is nonnegative definite but not symmetric, so the Timoshenko system (4.1.10) is an example of hyperbolic systems with non-symmetric dissipation. Consequently, the general theory (see [70]) for hyperbolic systems with symmetric dissipation can not be applied directly, which is the main motivation of this chapter.

The partial damping term γy is a weak dissipation, which enables us to capture the dissipation from contributions of (y, v, u_x, z_x) only, and the dissipative rates for u, z themselves are absent. To overcome the difficulty in the derivation of a priori estimates, an elementary fact in Proposition 4.2.1 (also see [70]) that indicates the relation between homogeneous and inhomogeneous Chemin-Lerner spaces, will be used, see proofs of Lemmas 4.3.1-4.3.4 for more details. On the other hand, Xu and Kawashima gave a new decay framework for general dissipative system satisfying the Shizuta-Kawashima condition (see [71]), which allows to pay less attention on the traditional spectral analysis. Inspired by the dissipative structure for the Timoshenko system (see [24] or (4.1.7)), we hope that the new decay framework can be adapted to the Timoshenko system with equal wave speeds. However, those analysis remain valid only for the case of high dimensions ($n \geq 3$) due to interpolation techniques used. To overcome this obstruction, the degenerate space $\dot{B}^{-1/2}_{2,\infty}$ rather than the general form $\dot{B}^{-s}_{2,\infty}$ $(0 < s \leq 1/2)$ will be employed. Notice that $L^1(\mathbb{R}) \hookrightarrow \dot{B}^0_{1,\infty}(\mathbb{R}) \hookrightarrow \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$. Additionally, we involve new observations so as to achieve the optimal decay estimates at the low-frequency, see (4.4.23) and (4.4.26) for details.

In this chapter, we focus on the Timoshenko system with equal wave speeds ($a = 1$). Now, main results are stated as follows.

Theorem 4.1.1. *Suppose that $U_0 \in B^{3/2}_{2,1}(\mathbb{R})$. There exists a positive constant δ_0 such that if*

$$\|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})} \leq \delta_0,$$

then the Cauchy problem (4.1.10) has a unique global classical solution $U \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R})$ satisfying

$$U \in \widetilde{\mathcal{C}}(B^{3/2}_{2,1}(\mathbb{R})) \cap \widetilde{\mathcal{C}}^1(B^{1/2}_{2,1}(\mathbb{R}))$$

Moreover, the following energy inequality holds that

$$\begin{aligned} &\|U\|_{\widetilde{L}^\infty(B^{3/2}_{2,1}(\mathbb{R}))} + \Big(\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \|(v, z_x)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}\Big) \\ \leq\ & C_0\|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})}, \end{aligned}$$

where $C_0 > 0$ is a constant.

Remark 4.1.1. To the best of our knowledge, Theorem 4.1.1 exhibits the optimal critical regularity of global well-posedness for (4.1.10), which is the first result in this direction for the Timoshenko system. Observe that there is 1-regularity-loss phenomenon for the dissipation rates due to the nonlinear influence, which is totally different in comparison with the linearized system (4.1.5) with $a = 1$.

Based on the global-in-time existence of solutions, we further obtain the optimal decay estimates. Denote $\Lambda^\alpha f := \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F} f$ $(\alpha \in \mathbb{R})$.

Theorem 4.1.2. *Let $U(t,x) = (v,u,z,y)(t,x)$ be the global classical solution of Theorem 4.1.1. If further the initial data $U_0 \in \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$ and*

$$\mathcal{M}_0 := \|U_0\|_{B^{3/2}_{2,1}(\mathbb{R}) \cap \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})}$$

is sufficiently small. Then the classical solution $U(t,x)$ of (4.1.10) admits the following decay estimates

$$\|\Lambda^\ell U\|_{X_1(\mathbb{R})} \lesssim \mathcal{M}_0 (1+t)^{-\frac{1}{4}-\frac{\ell}{2}} \tag{4.1.12}$$

for $0 \leq \ell \leq 1/2$, where $X_1 := B^{1/2-\ell}_{2,1}$ if $0 \leq \ell < 1/2$ and $X_1 := \dot{B}^0_{2,1}$ if $\ell = 1/2$.

Note that the $L^1(\mathbb{R})$ embedding property in Lemma 4.2.3, as an immediate byproduct of Theorem 4.1.2, the usual optimal decay estimates of $L^1(\mathbb{R})$-$L^2(\mathbb{R})$ type are available.

Corollary 4.1.1. *Let $U(t,x) = (v,u,z,y)(t,x)$ be the global classical solutions of Theorem 4.1.1. If further the initial data $U_0 \in L^1(\mathbb{R})$ and*

$$\widetilde{\mathcal{M}}_0 := \|U_0\|_{B^{3/2}_{2,1}(\mathbb{R}) \cap L^1(\mathbb{R})}$$

is sufficiently small, then

$$\|\Lambda^\ell U\|_{L^2(\mathbb{R})} \lesssim \widetilde{\mathcal{M}}_0 (1+t)^{-\frac{1}{4}-\frac{\ell}{2}} \tag{4.1.13}$$

for $0 \leq \ell \leq 1/2$.

Remark 4.1.2. We can say that Theorem 4.1.2 and Corollary 4.1.1 exhibit various decay rates of solution and its derivatives of fractional order. In comparison with [25], here, the harmonic analysis allows to reduce significantly the regularity requirements on the initial data.

The rest of this chapter unfolds as follows. In Section 4.2, we present useful properties in Besov spaces, which will be used in the subsequence analysis. In Section 4.3, we construct the global-in-time solution by Fourier localization energy methods. Based on the dissipative structure, in Section 4.4, we develop the decay property for the linearized Timoshenko system (4.1.4)-(4.1.5) in the framework of Besov spaces. Then, by employing localized time-weighted energy approaches, we deduce the optimal decay estimates for (4.1.9).

We present those definitions of Besov spaces and Chemin-Lerner spaces in Section 5.6 (Appendix) in Chapter 5 for the convenience of readers.

4.2 Tools

In this section, we only present analysis properties in Besov spaces and Chemin-Lerner spaces in $\mathbb{R}^n(n \geq 1)$, which will be used in the sequence section. For convenience of reader, the Appendix in Section 5.6 in Chapter 5 devoted to those definitions for Besov spaces and Chemin-Lerner spaces.

Firstly, we give an improved Bernstein inequality (see, *e.g.*, [68]), which allows the case of fractional derivatives.

Lemma 4.2.1. *Let* $0 < R_1 < R_2$ *and* $1 \leq a \leq b \leq \infty$.

(*i*) *If* $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq R_1\lambda\}$, *then*

$$\|\Lambda^\alpha f\|_{L^b} \lesssim \lambda^{\alpha+n(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a} \quad \textit{for any} \quad \alpha \geq 0;$$

(*ii*) *If* $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : R_1\lambda \leq |\xi| \leq R_2\lambda\}$, *then*

$$\|\Lambda^\alpha f\|_{L^a} \approx \lambda^\alpha\|f\|_{L^a} \quad \textit{for any} \quad \alpha \in \mathbb{R}.$$

Besov spaces obey various inclusion relations. Precisely,

Lemma 4.2.2. *Let* $s \in \mathbb{R}$ *and* $1 \leq p, r \leq \infty$, *then*

(*i*) *If* $s > 0$, *then* $B^s_{p,r} = L^p \cap \dot{B}^s_{p,r}$;

(*ii*) *If* $\tilde{s} \leq s$, *then* $B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,r}$; *this inclusion relation is false for the homogeneous Besov spaces;*

(*iii*) *If* $1 \leq r \leq \tilde{r} \leq \infty$, *then* $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^s_{p,\tilde{r}}$ *and* $B^s_{p,r} \hookrightarrow B^s_{p,\tilde{r}}$;

(*iv*) *If* $1 \leq p \leq \tilde{p} \leq \infty$, *then* $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$ *and* $B^s_{p,r} \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$;

(v) $\dot{B}^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0, \quad B^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0 \quad (1 \leq p < \infty)$,

where $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

Lemma 4.2.3. *Suppose that $\varrho > 0$ and $1 \leq p < 2$. It holds that*

$$\|f\|_{\dot{B}^{-\varrho}_{r,\infty}} \lesssim \|f\|_{L^p}$$

with $1/p - 1/r = \varrho/n$. In particular, this holds with $\varrho = n/2,\ r = 2$ and $p = 1$.

The global existence depends on a key fact, which indicates the connection between homogeneous Chemin-Lerner spaces and inhomogeneous Chemin-Lerner spaces, see [70] for the proof. Precisely,

Proposition 4.2.1. *Let $s \in \mathbb{R}$ and $1 \leq \theta, p, r \leq \infty$.*

(i) *It holds that*

$$L^\theta_T(L^p) \cap \widetilde{L}^\theta_T(\dot{B}^s_{p,r}) \subset \widetilde{L}^\theta_T(B^s_{p,r});$$

(ii) *Furthermore, as $s > 0$ and $\theta \geq r$, it holds that*

$$L^\theta_T(L^p) \cap \widetilde{L}^\theta_T(\dot{B}^s_{p,r}) = \widetilde{L}^\theta_T(B^s_{p,r})$$

for any $T > 0$.

Let us state the Moser-type product estimates, which plays an important role in the estimate of bilinear terms.

Proposition 4.2.2. *Let $s > 0$ and $1 \leq p, r \leq \infty$. Then $\dot{B}^s_{p,r} \cap L^\infty$ is an algebra and*

$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}.$$

Let $s_1, s_2 \leq n/p$ such that $s_1 + s_2 > n\max\{0, \frac{2}{p} - 1\}$. Then one has

$$\|fg\|_{\dot{B}^{s_1+s_2-n/p}_{p,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,1}}.$$

In the sequel we also need a estimate for commutator.

Proposition 4.2.3. *Let $1 < p < \infty,\ 1 \leq \theta \leq \infty$ and $s \in (-\frac{n}{p} - 1,\ \frac{n}{p}]$. Then there exists a generic constant $C > 0$ depending only on $s,\ n$ such that*

$$\begin{cases} \|[f, \dot{\Delta}_q]g\|_{L^p} \leq Cc_q 2^{-q(s+1)}\|f\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|g\|_{\dot{B}^s_{p,1}}, \\ \|[f, \dot{\Delta}_q]g\|_{L^\theta_T(L^p)} \leq Cc_q 2^{-q(s+1)}\|f\|_{\widetilde{L}^{\theta_1}_T(\dot{B}^{\frac{n}{p}+1}_{p,1})}\|g\|_{\widetilde{L}^{\theta_2}_T(\dot{B}^s_{p,1})}, \end{cases}$$

with $1/\theta = 1/\theta_1 + 1/\theta_2$, where the commutator $[\cdot,\cdot]$ is defined by $[f,g] = fg - gf$ and $\{c_q\}$ denotes a sequence such that $\|(c_q)\|_{l^1} \leq 1$.

Finally, we state a continuity result for compositions (see, *e.g.*, [22]) to end this section.

Proposition 4.2.4. *Let $s > 0$, $1 \leq p, r, \theta \leq \infty$, $F \in W^{[s]+3,\infty}_{loc}(I;\mathbb{R})$ with $F(0) = 0$, $T \in (0, \infty]$ and $f \in \widetilde{L}^{\theta}_T(B^s_{p,r}) \cap L^{\infty}_T(L^{\infty})$. Then there exists a function C depending only on s, p, r, n, and F such that*

$$\begin{cases} \|F(f) - F'(0)f\|_{\dot{B}^s_{p,r}} \leq C\left(\|f\|_{L^\infty}\right)\|f\|^2_{\dot{B}^s_{p,r}}, \\ \|F(f) - F'(0)f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})} \leq C\left(\|f\|_{L^\infty_T(L^\infty)}\right)\|f\|^2_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})}. \end{cases}$$

4.3 Global-in-time existence

Recently, Xu and Kawashima [70] have already established a local existence theory for generally symmetric hyperbolic systems in spatially critical Besov spaces, which is viewed as the generalization of the basic theory of Kato and Majda [26, 35]. Fortunately, the new result can be applied to the current problem (4.1.10) directly, since the non-symmetric dissipation L has no influence on the local-in-time existence. Precisely,

Proposition 4.3.1. *Assume that $U_0 \in B^{3/2}_{2,1}$, then there exists a time $T_0 > 0$ (depending only on the initial data) such that*

- (*i*) *(Existence): system (4.1.10) has a unique solution $U(t,x) \in \mathcal{C}^1([0,T_0] \times \mathbb{R})$ satisfying $U \in \widetilde{\mathcal{C}}_{T_0}(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_{T_0}(B^{1/2}_{2,1})$;*
- (*ii*) *(Blow-up criterion): if the maximal time $T^*(> T_0)$ of existence of such a solution is finite, then*
$$\limsup_{t \to T^*} \|U(t,\cdot)\|_{B^{3/2}_{2,1}} = \infty$$
if and only if
$$\int_0^{T^*} \|\nabla U(t,\cdot)\|_{L^\infty} dt = \infty.$$

Furthermore, in order to show that classical solutions in Proposition 4.3.1 are globally defined, the next task is to construct a priori estimates according to the dissipative mechanism produced by the Timoshenko system. To this end, we define by $E(T)$ the energy functional and by $D(T)$ the corresponding dissipation functional:

$$E(T) := \|U\|_{\widetilde{L}^\infty_T(B^{3/2}_{2,1})}$$

and

$$D(T) := \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \|(v, z_x)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}$$

for any time $T > 0$.

The first lemma is related to the nonlinear a priori estimate for the dissipation for y.

Lemma 4.3.1 (The dissipation for y). *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (4.1.10) for any $T > 0$, then*

$$E(T) + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \lesssim \|U_0\|_{B^{3/2}_{2,1}} + \sqrt{E(T)}D(T). \tag{4.3.1}$$

Proof. Firstly, we perform the usual energy method. Multiplying the first equation in (4.1.10) by v, the second one by u, the third one by $\sigma(z) - \sigma(0)$ and the last one by y, respectively, and then adding the resulting equalities, we get

$$\frac{1}{2}\frac{d}{dt}(v^2 + y^2 + u^2 + S(z)) - \Big(vu + [\sigma(z) - \sigma(0)]\Big)_x + \gamma y^2 = 0, \tag{4.3.2}$$

where

$$S(z) = 2\int_0^z \Big(\sigma(\eta) - \sigma(0)\Big)d\eta.$$

Note that $S(z)$ is equivalent to z^2, due to the fact $\sigma'(\eta) > 0$ and the smallness assumption (4.3.40) below. Then we perform the integral to (4.3.2) with respect to x and obtain the basic energy equality

$$\frac{1}{2}\frac{d}{dt}E_0(U) + \gamma\|y\|^2_{L^2} = 0, \tag{4.3.3}$$

where the energy functional $E_0(U)$ is defined by

$$E_0(U) = \|(v, u, y)\|^2_{L^2} + \int_{\mathbb{R}} S(z)dx \approx \|U\|^2_{L^2}.$$

By integrating in $t \in [0, T]$ and taking the square-root of the resulting inequality, we arrive at

$$\|U\|_{L^\infty_T(L^2)} + \sqrt{2\gamma}\|y\|_{L^2_T(L^2)} \leq \|U_0\|_{L^2} \tag{4.3.4}$$

for any $T > 0$.

Next, we perform the frequency-localization estimate and get the dissipation rate from y in homogeneous Chemin-Lerner spaces. Applying the operator $\dot{\Delta}_q (q \in \mathbb{Z})$ to (4.1.10) gives

$$\begin{cases} \dot{\Delta}_q v_t - \dot{\Delta}_q u_x + \dot{\Delta}_q y = 0, \\ \dot{\Delta}_q u_t - \dot{\Delta}_q v_x = 0, \\ \dot{\Delta}_q z_t - \dot{\Delta}_q y_x = 0, \\ \dot{\Delta}_q y_t - \sigma'(z)\dot{\Delta}_q z_x - \dot{\Delta}_q v + \gamma\dot{\Delta}_q y = [\dot{\Delta}_q, \sigma'(z)]\, z_x, \end{cases} \tag{4.3.5}$$

where the commutator is defined by $[f, g] := fg - gf$. Multiplying (4.3.5) with $\dot{\Delta}_q v$, $\dot{\Delta}_q u$, $\sigma'(z)\dot{\Delta}_q z$ and $\dot{\Delta}_q y$, respectively, and then adding the resulting equalities, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(|\dot{\Delta}_q v|^2 + |\dot{\Delta}_q y|^2 + |\dot{\Delta}_q u|^2 + \sigma'(z)|\dot{\Delta}_q z|^2\Big) \\ &-\Big\{(\dot{\Delta}_q u\dot{\Delta}_q v)_x + \Big(\sigma'(z)\dot{\Delta}_q z\dot{\Delta}_q y\Big)_x\Big\} + \gamma|\dot{\Delta}_q y|^2 \\ &= \frac{1}{2}\sigma'(z)_t|\dot{\Delta}_q z|^2 - \sigma'(z)_x\dot{\Delta}_q z\dot{\Delta}_q y + [\dot{\Delta}_q, \sigma'(z)]\, z_x\dot{\Delta}_q y. \end{aligned} \tag{4.3.6}$$

Furthermore, by employing the integral with respect to x, with the aid of Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E_0[\dot{\Delta}_q U]+\gamma\|\dot{\Delta}_q y\|_{L^2}^2 \qquad (4.3.7)\\ \lesssim\ &\|\sigma'(z)_t\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}^2+\|\sigma'(z)_x\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}\|\dot{\Delta}_q y\|_{L^2}\\ &+\|[\dot{\Delta}_q,\sigma'(z)]z_x\|_{L^2}\|\dot{\Delta}_q y\|_{L^2},\end{aligned}$$

where

$$E_0[\dot{\Delta}_q U]:=\|(\dot{\Delta}_q v,\dot{\Delta}_q y,\dot{\Delta}_q u)\|_{L^2}^2+\int_{\mathbb{R}}\sigma'(z)|\dot{\Delta}_q z|\,dx\approx\|\dot{\Delta}_q U\|_{L^2}^2.$$

From (4.1.10) and a priori assumption (4.3.40) below, we have

$$\|\sigma'(z)_t\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}^2\lesssim\|z_t\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}^2\lesssim\|y_x\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}^2. \qquad (4.3.8)$$

Similarly,

$$\|\sigma'(z)_x\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}\|\dot{\Delta}_q y\|_{L^2}\lesssim\|z_x\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}\|\dot{\Delta}_q y\|_{L^2}. \qquad (4.3.9)$$

Together with (4.3.8)-(4.3.9), by integrating in $t\in[0,T]$, with the help of Young's inequality, we are led to

$$\begin{aligned}&\sqrt{E_0[\dot{\Delta}_q U]}+\sqrt{2\gamma}\|\dot{\Delta}_q y\|_{L^2_T(L^2)}\\ \lesssim\ &\sqrt{E_0[\dot{\Delta}_q U_0]}+\sqrt{\|(y_x,z_x)\|_{L^\infty_T(L^\infty)}}\Big(\|\dot{\Delta}_q y\|_{L^2_T(L^2)}+\|\dot{\Delta}_q z\|_{L^2_T(L^2)}\Big)\\ &+\sqrt{\|[\dot{\Delta}_q,\sigma'(z)]z_x\|_{L^2_T(L^2)}\|\dot{\Delta}_q y\|_{L^2_T(L^2)}}. \qquad (4.3.10)\end{aligned}$$

It follows from the commutator estimate in Proposition 4.2.3 that

$$\|[\dot{\Delta}_q,\sigma'(z)]z_x\|_{L^2_T(L^2)}\lesssim c_q 2^{-\frac{3q}{2}}\|z\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}, \qquad (4.3.11)$$

where $\{c_q\}$ denotes a sequence such that $\|c_q\|_{\ell^1}\leq 1$. Therefore, we obtain

$$\begin{aligned}&2^{\frac{3q}{2}}\|\dot{\Delta}_q U\|_{L^\infty_T(L^2)}+\sqrt{2\gamma}2^{\frac{3q}{2}}\|\dot{\Delta}_q y\|_{L^2_T(L^2)}\\ \lesssim\ &\|\dot{\Delta}_q U_0\|_{L^2}+c_q\sqrt{\|(y_x,z_x)\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})}}\Big(\|y\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}+\|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}\Big)\\ &+c_q\sqrt{\|z\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}}\Big(\|y\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}+\|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}\Big). \qquad (4.3.12)\end{aligned}$$

Here, we would like to point out each $\{c_q\}$ has a possibly different form in (4.3.12) or in sequent inequalities, however, the bound $\|c_q\|_{\ell^1}\leq 1$ is well satisfied. Hence, summing up on

$q \in \mathbb{Z}$, we arrive at

$$\begin{aligned} & \|U\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})} + \sqrt{2\gamma}\|y\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} \\ \lesssim\ & \|U_0\|_{\dot{B}^{3/2}_{2,1}} + \sqrt{\|(y,z)\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}}\Big(\|y\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} + \|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}\Big). \end{aligned} \tag{4.3.13}$$

Finally, combining (4.3.4) and (4.3.13), we conclude that from Proposition 4.2.1

$$E(T) + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \lesssim \|U_0\|_{B^{3/2}_{2,1}} + \sqrt{E(T)}D(T). \tag{4.3.14}$$

Therefore, the proof of Lemma 4.3.1 is complete. □

Lemma 4.3.2 (The dissipation for v)**.** *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (4.1.10) for any $T > 0$, then*

$$\|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \sqrt{E(T)}D(T). \tag{4.3.15}$$

Proof. To do this, it is convenient to rewrite the system (4.1.10) as follows:

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - y_x = 0, \\ y_t - z_x - v + \gamma y = g(z)_x, \end{cases} \tag{4.3.16}$$

where the smooth function $g(z)$ is defined by

$$g(z) = \sigma(z) - \sigma(0) - z = O(z^2)$$

satisfying $g(0) = 0$ and $g'(0) = 0$. By multiplying four equations in (4.3.16) by $-y, -z, -u$ and $-v$, respectively, we deduce that

$$\frac{d}{dt}E_1(U) + \|v\|^2_{L^2} \le \|y\|^2_{L^2} + \gamma\|y\|_{L^2}\|v\|_{L^2} + \|g(z)_x\|_{L^2}\|v\|_{L^2}, \tag{4.3.17}$$

where

$$E_1(U) := -\int_{\mathbb{R}}(vy + uz)\,dx.$$

It follows from Young's inequality that

$$\frac{d}{dt}E_1(U) + \frac{1}{2}\|v\|^2_{L^2} \lesssim \|y\|^2_{L^2} + \|z\|_{L^\infty}\|z_x\|_{L^2}\|v\|_{L^2}. \tag{4.3.18}$$

Integrating (4.3.18) in $t \in [0, T]$ gives

$$\begin{aligned} & \|v\|^2_{L^2_t(L^2)} \\ \lesssim\ & (|E_1(U)| + |E_1(U_0)|) + \|y\|^2_{L^2_t(L^2)} + \|z\|_{L^\infty_t(L^\infty)}\|z_x\|_{L^2_t(L^2)}\|v\|_{L^2_t(L^2)} \\ \lesssim\ & E(T)^2 + \|U_0\|^2_{B^{3/2}_{2,1}} + \|y\|^2_{L^2_T(L^2)} + E(T)D^2(T), \end{aligned} \tag{4.3.19}$$

for any $T > 0$, where we have used the embedding property in Lemma 4.2.2. Then, by Young's inequality again, we get

$$\|v\|_{L^2_T(L^2)} \lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{L^2_T(L^2)} + \sqrt{E(T)}D(T). \tag{4.3.20}$$

Next, we turn to the localization energy estimate. Applying the operator $\dot{\Delta}_q (q \in \mathbb{Z})$ to (4.3.16) implies

$$\begin{cases} \dot{\Delta}_q v_t - \dot{\Delta}_q u_x + \dot{\Delta}_q y = 0, \\ \dot{\Delta}_q u_t - \dot{\Delta}_q v_x = 0, \\ \dot{\Delta}_q z_t - \dot{\Delta}_q y_x = 0, \\ \dot{\Delta}_q y_t - \dot{\Delta}_q z_x - \dot{\Delta}_q v + \gamma \dot{\Delta}_q y = \dot{\Delta}_q g(z)_x. \end{cases} \tag{4.3.21}$$

Multiplying the first equation in (4.3.21) by $-\dot{\Delta}_q y$, the second one by $-\dot{\Delta}_q z$, the third one by $-\dot{\Delta}_q u$ and the fourth one by $-\dot{\Delta}_q v$, respectively, then adding the resulting equalities, we have

$$\begin{aligned} &-(\dot{\Delta}_q v \dot{\Delta}_q y + \dot{\Delta}_q u \dot{\Delta}_q z)_t + (\dot{\Delta}_q v \dot{\Delta}_q z + \dot{\Delta}_q u \dot{\Delta}_q y)_x + |\dot{\Delta}_q v|^2 \\ &\quad = |\dot{\Delta}_q y|^2 + \gamma \dot{\Delta}_q y \dot{\Delta}_q v - \dot{\Delta}_q g(z)_x \dot{\Delta}_q v. \end{aligned} \tag{4.3.22}$$

With the aid of Hölder and Young's inequalities, we obtain

$$\frac{d}{dt} E_1[\dot{\Delta}_q U] + \frac{1}{2}\|\dot{\Delta}_q v\|^2_{L^2} \lesssim \|\dot{\Delta}_q y\|^2_{L^2} + \|\dot{\Delta}_q g(z)_x\|_{L^2} \|\dot{\Delta}_q v\|_{L^2}, \tag{4.3.23}$$

where

$$E_1[\dot{\Delta}_q U] := -\int_{\mathbb{R}} (\dot{\Delta}_q v \dot{\Delta}_q y + \dot{\Delta}_q u \dot{\Delta}_q z) dx.$$

By performing the integral with respect to $t \in [0, T]$, we are led to

$$\begin{aligned} &\|\dot{\Delta}_q v\|^2_{L^2_t(L^2)} \\ \lesssim\ & \|\dot{\Delta}_q U\|^2_{L^\infty_T(L^2)} + \|\dot{\Delta}_q U_0\|^2_{L^2} + \|\dot{\Delta}_q y\|^2_{L^2_T(L^2)} \\ & + \|\dot{\Delta}_q v\|_{L^\infty_T(L^2)} \|\dot{\Delta}_q g(z)_x\|_{L^1_T(L^2)}. \end{aligned} \tag{4.3.24}$$

Furthermore, Young's inequality enables us to get

$$\begin{aligned} & 2^{\frac{q}{2}} \|\dot{\Delta}_q v\|_{L^2_T(L^2)} \\ \lesssim\ & c_q \|U\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})} + c_q \|U_0\|_{\dot{B}^{1/2}_{2,1}} \\ & + c_q \|y\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + c_q \sqrt{\|v\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}} \, \|g(z)_x\|^{\frac{1}{2}}_{\widetilde{L}^1_T(\dot{B}^{1/2}_{2,1})}, \end{aligned} \tag{4.3.25}$$

where the norm of $g(z)$ on the right-side of (4.3.25) can be estimated by Proposition 4.2.4

$$\begin{aligned}\|g(z)_x\|_{\widetilde{L}^1_T(\dot{B}^{1/2}_{2,1})} &\lesssim \int_0^T \|g(z)\|_{\dot{B}^{3/2}_{2,1}}\,dt \\ &\lesssim \int_0^T \|z\|^2_{\dot{B}^{3/2}_{2,1}}\,dt \;\lesssim\; \|z_x\|^2_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}. \end{aligned} \tag{4.3.26}$$

Therefore, together with (4.3.25)-(4.3.26), by summing up on $q \in \mathbb{Z}$, we arrive at

$$\begin{aligned}&\|v\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} \\ \lesssim\;& \|U\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})} + \|U_0\|_{\dot{B}^{1/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} \\ &+\sqrt{\|v\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}}\|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}. \end{aligned} \tag{4.3.27}$$

Finally, noticing (4.3.20) and (4.3.27), it follows from Proposition 4.2.1 that

$$\|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \sqrt{E(T)}D(T), \tag{4.3.28}$$

which is just the inequality (4.3.15). □

Lemma 4.3.3 (The dissipation for z_x). *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (4.1.10) for any $T > 0$, then*

$$\begin{aligned}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} &\lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \\ &\quad+\|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \sqrt{E(T)}D(T). \end{aligned} \tag{4.3.29}$$

Proof. Multiplying the third equation in (4.3.16) by y_x and the fourth one by $-z_x$, respectively, and then integrating the resulting equalities over $\mathbb{R}$, we arrive at

$$\begin{aligned}&\frac{d}{dt}E_2(U) + \|z_x\|^2_{L^2} \\ \lesssim\;& \|y_x\|^2_{L^2} + (\|v\|_{L^2} + \|y\|_{L^2})\|z_x\|_{L^2} + \|z\|_{L^\infty}\|z_x\|^2_{L^2}, \end{aligned} \tag{4.3.30}$$

where

$$E_2(U) := -\int_{\mathbb{R}} z_x y\,dx.$$

Similar to the procedure leading to (4.3.20), we arrive at

$$\begin{aligned}\|z_x\|_{L^2_T(L^2)} &\lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \\ &\quad+\|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \sqrt{E(T)}D(T). \end{aligned} \tag{4.3.31}$$

On the other hand, from (4.3.21), we have

$$\begin{cases} \dot{\Delta}_q z_t - \dot{\Delta}_q y_x = 0, \\ \dot{\Delta}_q y_t - \dot{\Delta}_q z_x - \dot{\Delta}_q v + \gamma \dot{\Delta}_q y = \dot{\Delta}_q g(z)_x. \end{cases} \tag{4.3.32}$$

Then, by multiplying the first equation in (4.3.32) by $\dot{\Delta}_q y_x$ and the second one by $-\dot{\Delta}_q z_x$, respectively, and then employing the energy estimates on each block, we are led to

$$\begin{aligned} & 2^{\frac{q}{2}} \|\dot{\Delta}_q z_x\|_{L^2_T(L^2)} \\ \lesssim\ & c_q \left(\|U\|_{\widetilde{L}^\infty_T(B^{3/2}_{2,1})} + \|U_0\|_{B^{3/2}_{2,1}} \right) + c_q \|y_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} \\ & + c_q \varepsilon \|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + c_q C_\varepsilon \left(\|v\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + \|y\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} \right) \\ & \quad + c_q \sqrt{\|z_x\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}} \, \|g(z)_x\|^{\frac{1}{2}}_{\widetilde{L}^1_T(\dot{B}^{1/2}_{2,1})}. \end{aligned} \tag{4.3.33}$$

Furthermore, similar to the estimates (4.3.26)-(4.3.27), we get

$$\begin{aligned} & \|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} \\ \lesssim\ & \|U\|_{\widetilde{L}^\infty_T(B^{3/2}_{2,1})} + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} \\ & + \|v\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + \|y\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + \sqrt{\|z\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}} \, \|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}, \end{aligned} \tag{4.3.34}$$

where we have chosen $0 < \varepsilon \leq 1/2$.

Finally, by combining (4.3.31) and (4.3.34), we arrive at (4.3.29). □

Lemma 4.3.4 (The dissipation for u_x)**.** *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (4.1.10) for any $T > 0$, then*

$$\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})}. \tag{4.3.35}$$

Proof. Applying the inhomogeneous operator $\Delta_q (q \geq -1)$ to the first equation and second one of (4.3.16) gives

$$\begin{cases} \Delta_q v_t - \Delta_q u_x + \Delta_q y = 0, \\ \Delta_q u_t - \Delta_q v_x = 0. \end{cases} \tag{4.3.36}$$

Multiplying the first equation in (4.3.36) by $-\Delta_q u_x$ and the second one by $\Delta_q v_x$, we can obtain

$$\frac{d}{dt} E_3[\Delta_q U] + \|\Delta_q u_x\|^2_{L^2} \leq \|\Delta_q v_x\|^2_{L^2} + \|\Delta_q u_x\|_{L^2} \|\Delta_q y\|_{L^2}, \tag{4.3.37}$$

where

$$E_3[\Delta_q U] := -\int_{\mathbb{R}} \Delta_q v \Delta_q u_x \, dx.$$

Then we integrate (4.3.37) with respect to $t \in [0, T]$ to get

$$\begin{aligned} \|\Delta_q u_x\|^2_{L^2_t(L^2)} \quad &\leq \quad \Big(|E_3[\Delta_q U]| + E_3[\Delta_q U_0]\Big) \\ &\quad + \|\Delta_q v_x\|^2_{L^2_t(L^2)} + \|\Delta_q u_x\|_{L^2_t(L^2)} \|\Delta_q y\|_{L^2_t(L^2)}. \end{aligned} \tag{4.3.38}$$

By using Young's inequality and embedding properties in Lemma 4.2.2, we are led to

$$\begin{aligned} & 2^{-q/2} \|\Delta_q u_x\|_{L^2_T(L^2)} \\ \lesssim \quad & c_q E(T) + c_q \|U_0\|_{B^{3/2}_{2,1}} + c_q \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \\ & + c_q \sqrt{\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})}}, \end{aligned} \tag{4.3.39}$$

which leads to (4.3.35) immediately. □

Having Lemmas 4.3.1-4.3.4, we obtain the following a priori estimate for solutions. For brevity, we feel free to skip the details.

Proposition 4.3.2. *Suppose $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (5.1.10) for $T > 0$. There exists $\delta_1 > 0$ such that if*

$$E(T) \leq \delta_1, \tag{4.3.40}$$

then the following estimate holds:

$$E(T) + D(T) \lesssim \|U_0\|_{B^{3/2}_{2,1}} + \sqrt{E(T)} D(T). \tag{4.3.41}$$

Furthermore, it holds that

$$E(T) + D(T) \lesssim \|U_0\|_{B^{3/2}_{2,1}}. \tag{4.3.42}$$

By using the standard boot-strap argument, for instance, see [40] (Theorem 7.1, p.100), Theorem 4.1.1 follows from the local existence result (Proposition 4.3.1) and a priori estimate (Proposition 4.3.2). Here, we give the outline for completeness.

The proof of Theorem 4.1.1. If the initial data satisfy $\|U_0\|_{B^{3/2}_{2,1}} \leq \frac{\delta_1}{2}$, by Proposition 4.3.1, then we determine a time $T_1 > 0$ ($T_1 \leq T_0$) such that the local solutions of (4.1.10) exists in $\widetilde{\mathcal{C}}_{T_1}(B^{3/2}_{2,1})$ and $\|U\|_{\widetilde{L}^\infty_{T_1}(B^{3/2}_{2,1})} \leq \delta_1$. Therefore from Proposition 4.3.2 the solutions satisfy the a priori estimate

$$\|U\|_{\widetilde{L}^\infty_{T_1}(B^{3/2}_{2,1})} \leq C_1 \|U_0\|_{B^{3/2}_{2,1}} \leq \frac{\delta_1}{2}$$

provided $\|U_0\|_{B^{\sigma}_{2,1}} \leq \frac{\delta_1}{2C_1}$. Thus by Proposition 4.3.1 the system (4.1.10) for $t \geq T_1$ with the initial data $U(T_1)$ has again a unique solution U satisfying $\|U\|_{\widetilde{L}^{\infty}_{(T_1,2T_1)}(B^{3/2}_{2,1})} \leq \delta_1$, further $\|U\|_{\widetilde{L}^{\infty}_{2T_1}(B^{3/2}_{2,1})} \leq \delta_1$. Then by Proposition 4.3.2 we have

$$\|U\|_{\widetilde{L}^{\infty}_{2T_1}(B^{3/2}_{2,1})} \leq C_1\|U_0\|_{B^{3/2}_{2,1}} \leq \frac{\delta_1}{2}.$$

Subsequently, we continuous the same process for $0 \leq t \leq nT_1,\ n = 3, 4, ...$ and finally get a global solution $U \in \widetilde{\mathcal{C}}(B^{\sigma}_{2,1})$ satisfying

$$\begin{aligned}
&\|U\|_{\widetilde{L}^{\infty}(B^{3/2}_{2,1})} + \Big(\|y\|_{\widetilde{L}^{2}_{T}(B^{3/2}_{2,1})} + \|(v, z_x)\|_{\widetilde{L}^{2}_{T}(B^{1/2}_{2,1})} + \|u_x\|_{\widetilde{L}^{2}_{T}(B^{-1/2}_{2,1})}\Big) \\
\leq\ & C_1\|U_0\|_{B^{3/2}_{2,1}} \ \leq\ \frac{\delta_1}{2}.
\end{aligned}$$

□

4.4 Optimal decay rates

By employing the energy method in Fourier spaces in [24, 38], it is well-known that the linearized system (4.1.5)-(4.1.6) admits the dissipative structure

$$\operatorname{Re}\lambda(i\xi) \leq -c\eta_1(\xi), \quad \text{for} \quad a = 1,$$

with $\eta_1(\xi) = \frac{\xi^2}{1+\xi^2}$, that is, the following differential inequality holds

$$\frac{d}{dt}E[\hat{U}] + c_1\eta_1(\xi)|\hat{U}|^2 \leq 0, \tag{4.4.1}$$

where $E[\hat{U}] \approx |\hat{U}|^2$. As a matter of fact, following from the derivation of (4.4.1) as in [24, 38], we can deduce frequency-localization differential inequalities

$$\frac{d}{dt}E[\widehat{\Delta_q U}] + c_1\eta_1(\xi)|\widehat{\Delta_q U}|^2 \leq 0, \tag{4.4.2}$$

for $q \geq -1$, and

$$\frac{d}{dt}E[\widehat{\dot{\Delta}_q U}] + c_1\eta_1(\xi)|\widehat{\dot{\Delta}_q U}|^2 \leq 0, \tag{4.4.3}$$

for $q \in \mathbb{Z}$.

4.4.1 Decay property for the linearized system

As shown by [71], we do the similar high-frequency and low-frequency analysis to achieve the following decay property for (4.1.5)-(4.1.6).

Proposition 4.4.1. *If $U_0 \in \dot{B}^{\sigma}_{2,1}(\mathbb{R}) \cap \dot{B}^{-s}_{2,\infty}(\mathbb{R})$ for $\sigma \geq 0$ and $s > 0$, then the solutions $U(t,x)$ of (4.1.5)-(4.1.6) has the decay estimate*

$$\|\Lambda^{\ell} U\|_{B^{\sigma-\ell}_{2,1}} \lesssim \|U_0\|_{\dot{B}^{\sigma}_{2,1} \cap \dot{B}^{-s}_{2,\infty}} (1+t)^{-\frac{\ell+s}{2}} \tag{4.4.4}$$

for $0 \leq \ell \leq \sigma$. In particular, if $U_0 \in \dot{B}^{\sigma}_{2,1}(\mathbb{R}) \cap L^p(\mathbb{R})$ $(1 \leq p < 2)$, one further has

$$\|\Lambda^{\ell} U\|_{B^{\sigma-\ell}_{2,1}} \lesssim \|U_0\|_{\dot{B}^{\sigma}_{2,1} \cap L^p} (1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{\ell}{2}} \tag{4.4.5}$$

for $0 \leq \ell \leq \sigma$.

Additionally, we also obtain the decay property on the framework of homogeneous Besov spaces, see [71] for the similar proof.

Proposition 4.4.2. *If $U_0 \in \dot{B}^{\sigma}_{2,1}(\mathbb{R}) \cap \dot{B}^{-s}_{2,\infty}(\mathbb{R})$ for $\sigma \in \mathbb{R},\ s \in \mathbb{R}$ satisfying $\sigma + s > 0$, then the solution $U(t,x)$ of (4.1.5)-(4.1.6) has the decay estimate*

$$\|U\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \|U_0\|_{\dot{B}^{\sigma}_{2,1} \cap \dot{B}^{-s}_{2,\infty}} (1+t)^{-\frac{\sigma+s}{2}}. \tag{4.4.6}$$

In particular, if $U_0 \in \dot{B}^{\sigma}_{2,1}(\mathbb{R}) \cap L^p(\mathbb{R})$ $(1 \leq p < 2)$, one further has

$$\|U\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \|U_0\|_{\dot{B}^{\sigma}_{2,1} \cap L^p} (1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{\sigma}{2}}. \tag{4.4.7}$$

4.4.2 Localized time-weighted energy approaches

Firstly, we denote by $\mathcal{G}(t)$ the Green matrix associated with the linearized system (4.1.5)-(4.1.6) as follows:

$$\mathcal{G}(t)f = \mathcal{F}^{-1}[e^{-t\hat{\Phi}(i\xi)} \mathcal{F} f], \tag{4.4.8}$$

with

$$\hat{\Phi}(i\xi) = (i\xi A + L),$$

where

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}.$$

Then, by the classical Duhamel principle, the solution to Cauchy problem of the nonlinear Timoshenko system

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - y_x = 0, \\ y_t - z_x - v + \gamma y = g(z)_x, \end{cases} \tag{4.4.9}$$

with

$$U|_{t=0} = U_0(x) \tag{4.4.10}$$

can be represented by

$$U(t,x) = \mathcal{G}(t)U_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{R}(\tau)\,d\tau, \tag{4.4.11}$$

where $\mathcal{R} := (0,0,0,g(z)_x)^\top$. Note that the smooth function $g(z) = O(z^2)$ satisfying $g(0) = 0$ and $g'(0) = 0$.

Additionally, from the definition of $\mathcal{G}(t)$, it is not difficult to obtain the frequency-localization Duhamel principle for (4.4.9)-(4.4.10).

Lemma 4.4.1. *Suppose that $U(t,x)$ is a solution of (4.4.9)-(4.4.10). Then*

$$\Delta_q\Lambda^\ell U(t,x) = \Delta_q\Lambda^\ell[\mathcal{G}(t)U_0] + \int_0^t \Delta_q\Lambda^\ell[\mathcal{G}(t-\tau)\mathcal{R}(\tau)]\,d\tau \tag{4.4.12}$$

for $q \geq -1$ and $\ell \in \mathbb{R}$, and

$$\dot{\Delta}_q\Lambda^\ell U(t) = \dot{\Delta}_q\Lambda^\ell[\mathcal{G}(t)U_0] + \int_0^t \dot{\Delta}_q\Lambda^\ell[\mathcal{G}(t-\tau)\mathcal{R}(\tau)]\,d\tau \tag{4.4.13}$$

for $q \in \mathbb{Z}$ and $\ell \in \mathbb{R}$.

Based on Lemma 4.4.1, we shall deduce the optimal decay estimate by developing time-weighted energy approaches as in [37] in terms of high-frequency and low-frequency decompositions. For this purpose, we first define some sup-norms as follows

$$\mathcal{E}_0(t) := \sup_{0\leq\tau\leq t} \|U(\tau)\|_{B^{3/2}_{2,1}};$$

$$\mathcal{E}_1(t) := \sup_{0\leq\ell<\frac{1}{2}} \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{4}+\frac{\ell}{2}} \|\Lambda^\ell U(\tau)\|_{B^{1/2-\ell}_{2,1}} + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{2}} \|\Lambda^{\frac{1}{2}} U(\tau)\|_{\dot{B}^0_{2,1}}.$$

As a consequence, we have

Proposition 4.4.3. *Let* $U = (v, u, z, y)^\top$ *be the global classical solution in the sense of Theorem 4.1.1. Suppose that* $U_0 \in B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}$ *and the norm* $\mathcal{M}_0 := \|U_0\|_{B^{3/2}_{2,1}\cap \dot{B}^{-1/2}_{2,\infty}}$ *is sufficiently small. Then it holds that*

$$\|\Lambda^\ell U(t)\|_{X_1} \lesssim \mathcal{M}_0(1+t)^{-\frac{1}{4}-\frac{\ell}{2}} \tag{4.4.14}$$

for $0 \leq \ell \leq 1/2$*, where* $X_1 := B^{1/2-\ell}_{2,1}$ *if* $0 \leq \ell < 1/2$ *and* $X_1 := \dot{B}^0_{2,1}$ *if* $\ell = 1/2$*.*

Actually, Proposition 4.4.3 depends on an energy inequality related to sup-norms $\mathcal{E}_0(t)$ and $\mathcal{E}_1(t)$, which is included in the following proposition.

Proposition 4.4.4. *Let* $U = (v, u, z, y)^\top$ *be the global classical solution in the sense of Theorem 4.1.1. Additional, if* $U_0 \in \dot{B}^{-1/2}_{2,\infty}$*, then*

$$\mathcal{E}_1(t) \lesssim \mathcal{M}_0 + \mathcal{E}_1^2(t) + \mathcal{E}_0(t)\mathcal{E}_1(t), \tag{4.4.15}$$

where $\mathcal{M}_0$ *is the same notation defined in Proposition 4.4.3.*

Proof. The proof consists of two steps.

Step 1: High-frequency estimate

Due to $\Delta_q f \equiv \dot{\Delta}_q f (q \geq 0)$, it suffices to show the inhomogeneous case. It follows from the high-frequency analysis for (4.1.5)-(4.1.6) (see, *e.g.*, [71]) that

$$\|\Delta_q \Lambda^\ell \mathcal{G}(t) U_0\|_{L^2} \lesssim e^{-c_2 t}\|\Delta_q \Lambda^\ell U_0\|_{L^2} \quad (c_2 > 0) \tag{4.4.16}$$

for all $q \geq 0$. Then by Lemma 4.4.1, we arrive at

$$\begin{aligned} &\|\Delta_q \Lambda^\ell U\|_{L^2} \\ \leq\ & \|\Delta_q \Lambda^\ell[\mathcal{G}(t)U_0]\|_{L^2} + \int_0^t \|\Delta_q \Lambda^\ell[\mathcal{G}(t-\tau)\mathcal{R}(\tau)]\|_{L^2}\, d\tau \\ \lesssim\ & e^{-c_2 t}\|\Delta_q \Lambda^\ell U_0\|_{L^2} + \int_0^t e^{-c_2(t-\tau)}\|\Delta_q \Lambda^\ell \mathcal{R}(\tau)\|_{L^2}\, d\tau \end{aligned} \tag{4.4.17}$$

which leads to

$$\sum_{q\geq 0} 2^{q(1/2-\ell)}\|\Delta_q \Lambda^\ell U\|_{L^2} \ \lesssim\ \|U_0\|_{B^{1/2}_{2,1}} e^{-c_1 t} + \int_0^t e^{-c_2(t-\tau)}\|\mathcal{R}(\tau)\|_{\dot{B}^{1/2}_{2,1}}\, d\tau \tag{4.4.18}$$

for $0 \leq \ell \leq 1/2$. Next, we turn to estimate the norm $\|\mathcal{R}(\tau)\|_{\dot{B}^{1/2}_{2,1}}$ as follows

$$\begin{aligned} \|\mathcal{R}(\tau)\|_{\dot{B}^{1/2}_{2,1}} &= \|g'(z)z_x\|_{\dot{B}^{1/2}_{2,1}} \leq \|z\|_{\dot{B}^{1/2}_{2,1}}\|z\|_{\dot{B}^{3/2}_{2,1}} \leq \|\Lambda^\ell z\|_{\dot{B}^{1/2-\ell}_{2,1}}\|z\|_{B^{3/2}_{2,1}} \\ &\lesssim (1+\tau)^{-\frac{1}{4}-\frac{\ell}{2}}\mathcal{E}_0(t)\mathcal{E}_1(t), \end{aligned} \tag{4.4.19}$$

where Lemma 4.2.1 and Proposition 4.2.2 have been used.

Therefore, together with (4.4.18)-(4.4.19), we obtain

$$\sum_{q\geq 0} 2^{q(1/2-\ell)}\|\Delta_q\Lambda^\ell U\|_{L^2} \lesssim \|U_0\|_{B^{1/2}_{2,1}}e^{-c_1 t} + (1+t)^{-\frac{1}{4}-\frac{\ell}{2}}\mathcal{E}_0(t)\mathcal{E}_1(t) \tag{4.4.20}$$

for $0\leq \ell\leq 1/2$.

<u>*Step 2: Low-frequency estimate*</u>

In the following, we proceed with the different low-frequency estimate in comparison with [71], since those analysis remains true for higher dimensions due to interpolation techniques. Here, the proof involves new observations, which is divided into two cases.

(i) In the case of $0\leq \ell<1/2$, we have the low-frequency estimate for (4.1.4)-(4.1.5):

$$\|\Delta_{-1}\Lambda^\ell[\mathcal{G}(t)U_0]\|_{L^2} \lesssim \|\tilde{w}_0\|_{\dot{B}^{-1/2}_{2,\infty}}(1+t)^{-\frac{1}{4}-\frac{\ell}{2}}. \tag{4.4.21}$$

Then it follows from Lemma 4.4.1 that

$$\begin{aligned} & \|\Delta_{-1}\Lambda^\ell U(t,x)\|_{L^2} \\ \leq\ & \|\Delta_{-1}\Lambda^\ell[\mathcal{G}(t)U_0]\|_{L^2} + \int_0^t \|\Delta_{-1}\Lambda^\ell[\mathcal{G}(t-\tau)\mathcal{R}(\tau)]\|_{L^2}\,d\tau \\ \lesssim\ & \|U_0\|_{\dot{B}^{-1/2}_{2,\infty}}(1+t)^{-\frac{1}{4}-\frac{\ell}{2}} + I_1 + I_2, \end{aligned} \tag{4.4.22}$$

where

$$I_1 = \int_0^{\frac{t}{2}} \|\Delta_{-1}\Lambda^\ell[\mathcal{G}(t-\tau)\mathcal{R}(\tau)]\|_{L^2}\,d\tau,$$

and

$$I_2 = \int_{\frac{t}{2}}^t \|\Delta_{-1}\Lambda^\ell[\mathcal{G}(t-\tau)\mathcal{R}(\tau)]\|_{L^2}\,d\tau.$$

For I_1, noticing that the form $\mathcal{R}(\tau) \approx g(z)_x$, we arrive at

$$\begin{aligned}
I_1 &\lesssim \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{4}-\frac{\ell+1}{2}} \|g(z)\|_{\dot{B}^{-1/2}_{2,\infty}}\, d\tau \\
&\lesssim \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{4}-\frac{\ell+1}{2}} \|g(z)\|_{L^1}\, d\tau \\
&\lesssim \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{4}-\frac{\ell+1}{2}} \|z\|^2_{L^2}\, d\tau \\
&\lesssim \mathcal{E}_1^2(t) \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{4}-\frac{\ell+1}{2}} (1+\tau)^{-\frac{1}{2}}\, d\tau \\
&\lesssim \mathcal{E}_1^2(t)(1+t)^{-\frac{1}{4}-\frac{\ell+1}{2}} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{1}{2}}\, d\tau \\
&\lesssim \mathcal{E}_1^2(t)(1+t)^{-\frac{1}{4}-\frac{\ell}{2}},
\end{aligned} \tag{4.4.23}$$

where we have used the fact that $g(z) = O(z^2)$ and the embeddings $L^1 \hookrightarrow \dot{B}^{-1/2}_{2,\infty}$ in Lemma 4.2.3 and $B^{1/2}_{2,1} \hookrightarrow L^2$.

On the other hand, for I_2, we have

$$I_2 \lesssim \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{4}-\frac{\ell}{2}} \|g(z)_x\|_{\dot{B}^{-1/2}_{2,\infty}} d\tau. \tag{4.4.24}$$

It follows from Lemma 4.2.1 and Proposition 4.2.4 that

$$\|g(z)_x\|_{\dot{B}^{-1/2}_{2,\infty}} \lesssim \|g(z)\|_{\dot{B}^{1/2}_{2,\infty}} \lesssim \|z\|^2_{\dot{B}^{1/2}_{2,\infty}} \lesssim \|z\|^2_{\dot{B}^{1/2}_{2,1}} \lesssim (1+\tau)^{-1}\mathcal{E}_1^2(t). \tag{4.4.25}$$

Hence, together with (4.4.24)-(4.4.25), we are led to the estimate

$$\begin{aligned}
I_2 &\lesssim \mathcal{E}_1^2(t) \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{4}-\frac{\ell}{2}} (1+\tau)^{-1}\, d\tau \\
&\lesssim \mathcal{E}_1^2(t)(1+t)^{-1} \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{4}-\frac{\ell}{2}}\, d\tau \\
&\lesssim \mathcal{E}_1^2(t)(1+t)^{-\frac{1}{4}-\frac{\ell}{2}}.
\end{aligned} \tag{4.4.26}$$

Finally, combing (4.4.22)-(4.4.23) and (4.4.26), we conclude that

$$\|\Delta_{-1}\Lambda^{\ell} U(t,x)\|_{L^2} \lesssim \|U_0\|_{\dot{B}^{-1/2}_{2,\infty}} (1+t)^{-\frac{1}{4}-\frac{\ell}{2}} + \mathcal{E}_1^2(t)(1+t)^{-\frac{1}{4}-\frac{\ell}{2}}. \tag{4.4.27}$$

(*ii*) In the case of $\ell = 1/2$, similar to the procedure leading to (4.4.27), we can deduce the corresponding nonlinear low-frequency estimate

$$\begin{aligned}
&\sum_{q<0} \|\dot{\Delta}_q \Lambda^{1/2} U\|_{L^2} \\
\leq\ & \sum_{q<0} \|\dot{\Delta}_q \Lambda^{1/2} [\mathcal{G}(t) U_0]\|_{L^2} + \int_0^t \sum_{q<0} \|\dot{\Delta}_q \Lambda^{1/2} [\mathcal{G}(t-\tau)\mathcal{R}(\tau)]\|_{L^2}\, d\tau \\
\lesssim\ & \|U_0\|_{\dot{B}^{-1/2}_{2,\infty}} (1+t)^{-\frac{1}{2}} + \mathcal{E}_1^2(t)(1+t)^{-\frac{1}{2}}. \qquad (4.4.28)
\end{aligned}$$

With these preparations (4.4.20) and (4.4.27)-(4.4.28) in hand, the desired inequality (4.4.15) is followed directly by the definitions of $\mathcal{E}_0(t)$ and $\mathcal{E}_1(t)$. □

The proof of Proposition 4.4.3. From Theorem 4.1.1, we see that

$$\mathcal{E}_0(t) \lesssim \|U_0\|_{B^{3/2}_{2,1}} \lesssim \mathcal{M}_0.$$

Thus, if $\mathcal{M}_0$ is sufficient small, it follows from (4.4.15) that

$$\mathcal{E}_1(t) \ \lesssim\ \mathcal{M}_0 + \mathcal{E}_1^2(t),$$

which implies that $\mathcal{E}(t) \lesssim \mathcal{M}_0$ by the standard method, provided that $\mathcal{M}_0$ is sufficient small. Consequently, we obtain the decay estimate in Proposition 4.4.3. □

Chapter 5

Application to Besov spaces (II)

5.1 Introduction

In this work, we are concerned with the following Timoshenko system (see [62, 63]), which is a set of two coupled wave equations of the form

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - \sigma(\psi_x)_x - (\varphi_x - \psi) + \gamma\psi_t = 0. \end{cases} \tag{5.1.1}$$

System (5.1.1) describes the transverse vibrations of a beam. Here, $t \geq 0$ is the time variable, $x \in \mathbb{R}$ is the spacial variable which denotes the point on the center line of the beam, $\varphi(t,x)$ is the transversal displacement of the beam from an equilibrium state, and $\psi(t,x)$ is the rotation angle of the filament of the beam. The smooth function $\sigma(\eta)$ satisfies $\sigma'(\eta) > 0$ for any $\eta \in \mathbb{R}$, and γ is a positive constant. We focus on the Cauchy problem of (5.1.1), so the initial data are supplemented as

$$(\varphi,\, \varphi_t,\, \psi,\, \psi_t)(x,0) = (\varphi_0,\, \varphi_1,\, \psi_0,\, \psi_1)(x). \tag{5.1.2}$$

Based on the change of variable introduced by Ide, Haramoto, and the third author [24]:

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \tag{5.1.3}$$

with $a > 0$ being the sound speed defined by $a^2 = \sigma'(0)$, it is convenient to rewrite (5.1.1)-(5.1.2) as a Cauchy problem for the first-order hyperbolic system of $U = (v,u,z,y)^\top$

$$\begin{cases} U_t + A(U)\, U_x + LU = 0, \\ U(x,0) = U_0(x) \end{cases} \tag{5.1.4}$$

with $U_0(x) = (v_0, u_0, z_0, y_0)(x)$, where $v_0 = \varphi_{0,x} - \psi_0$, $u_0 = \varphi_1$, $z_0 = a\psi_{0,x}$, $y_0 = \psi_1$ and

$$A(U) = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & \frac{\sigma'(z/a)}{a} & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}.$$

Note that $A(U)$ is a real symmetrizable matrix due to $\sigma'(z/a) > 0$, and the dissipative matrix L is nonnegative definite but not symmetric. Such degenerate dissipation forces (5.1.4) to go beyond the class of generally dissipative hyperbolic systems, so the recent global-in-time existence (see [70]) for hyperbolic systems with symmetric dissipation can not be applied directly, which is the main motivation on studying the Timoshenko system (5.1.1).

Let us review several known results on (5.1.1). In a bounded domain, it is known that (5.1.1) is exponentially stable if the damping term φ_t is also present on the left-hand side of the first equation of (5.1.3) (see, e.g., [48]). Soufyane [59] showed that (5.1.1) could not be exponentially stable by considering only the damping term of the form ψ_t, unless for the case of $a = 1$ (equal wave speeds). A similar result was obtained by Rivera and Racke [50] with an alternative proof. In addition, Rivera and Racke [49] also investigated the Timoshenko system with the heat conduction, which is described by the classical Fourier law. In the whole space, Kawashima and his collaborators [24] considered the corresponding linearized form of (5.1.4):

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \gamma y = 0, \\ (v, u, z, y)(x, 0) = (v_0, u_0, z_0, y_0)(x), \end{cases} \tag{5.1.5}$$

and showed that the dissipative structure could be characterized by

$$\begin{cases} \operatorname{Re}\lambda(i\xi) \leq -c\eta_1(\xi) & \text{for} \quad a = 1, \\ \operatorname{Re}\lambda(i\xi) \leq -c\eta_2(\xi) & \text{for} \quad a \neq 1, \end{cases} \tag{5.1.6}$$

where $\lambda(i\xi)$ denotes the eigenvalues of the system (5.1.5) in the Fourier space, $\eta_1(\xi) = \frac{\xi^2}{1+\xi^2}$, $\eta_2(\xi) = \frac{\xi^2}{(1+\xi^2)^2}$, and $c > 0$ is some constant. Consequently, the following decay properties were established for $U = (v, u, z, y)^\top$ of (5.1.5) (see [24] for details):

$$\|\partial_x^k U(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1} + e^{-ct}\|\partial_x^k U_0\|_{L^2} \tag{5.1.7}$$

for $a = 1$, and

$$\|\partial_x^k U(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1} + (1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} U_0\|_{L^2} \tag{5.1.8}$$

for $a \neq 1$. Recently, under the additional assumption $\int_{\mathbb{R}} U_0 dx = 0$, Racke and Said-Houari [53] strengthened (5.1.7)-(5.1.8) such that linearized solutions decay faster with a rate of $t^{-\gamma/2}$, by introducing the integral space $L^{1,\gamma}(\mathbb{R})$.

Remark 5.1.1. Clearly, the high frequency part of (5.1.7) yields an exponential decay, whereas the corresponding part of (5.1.8) is of the regularity-loss type, since $(1 + t)^{-\ell/2}$ is created by assuming the additional ℓ-th order regularity on the initial data. Consequently, extra higher regularity than that for global-in-time existence of classical solutions is imposed to obtain the optimal decay rates.

In [25], Ide and Kawashima performed the time-weighted approach to establish the global existence and asymptotic decay of solutions to the nonlinear problem (5.1.4). To overcome the difficulty caused by the regularity-loss property, the spatially regularity $s \geq 6$ was needed. Denote by s_c the critical regularity for global existence of classical solutions. Actually, the local-in-time existence theory of Kato and Majda [26, 35] implies that $s_c = 2$ for the Timoshenko system (5.1.4), actually, the extra regularity is used to take care of optimal decay estimates. Consequently, some natural questions follow. Is $s = 6$ the minimal decay regularity for (5.1.4) with the regularity-loss? If not, which index characterises the minimal decay regularity? This motivates the following general definition.

Definition 5.1.1. *If the optimal decay rate of $L^1(\mathbb{R}^n)$-$L^2(\mathbb{R}^n)$ type is achieved under the lowest regularity assumption, then the lowest index is called the minimal decay regularity index for dissipative systems of regularity-loss, which is labelled as s_D.*

Recently, we are concerned with the global existence and large-time behavior for (5.1.4) in spatially critical Besov spaces. To the best of our knowledge, there are few results available in this direction for the Timoshenko system, although the critical space has already been succeeded in the study of fluid dynamical equations, see [2, 15, 22, 44] for Navier-Stokes equations, [10, 74, 76, 78] for Euler equations and related models. In [70, 71], under the assumptions of dissipative entropy and Shizuta-Kawashima condition, Xu and Kawashima have already investigated generally dissipative systems, however, the Timoshenko system admits the non-symmetric dissipation and goes beyond the class. Hence, as a first step, we considered the easy case, that is, (5.1.4) with the equal wave speed ($a = 1$) in Chapter 4. By virtue of an elementary fact in Proposition 5.2.3 (also see Chapter 4) that indicates the relation between homogeneous and inhomogeneous Chemin-Lernerspaces, we first constructed global solutions pertaining to data in the Besov space $B^{3/2}_{2,1}(\mathbb{R})$. Furthermore, the optimal decay rates of solution and its derivatives are shown in the space $B^{3/2}_{2,1}(\mathbb{R}) \cap \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$ by the frequency-localization Duhamel principle and energy approach in terms of high-frequency and low-frequency decomposition.

In Chapter 5, we hope to establish similar results for (5.1.4) with non-equal wave speeds ($a \neq 1$) that has weaker dissipative mechanism. If done, we shall improve two regularity indices for Timoshenko system with regularity-loss: $s_c = 3/2$ for global-in-time existence and $s_D = 3/2$ for the optimal decay estimate, which lead to reduce significantly the regularity requirements on the initial data in comparison with [25].

Before main results, let us explain new technical points for (5.1.4) with $a \neq 1$ and the strategy to get round the obstruction. Firstly, as in Chapter 4, the degenerate non-symmetric damping enables us to capture the dissipation from contributions of (y, v, u_x, z_x), however, there is an additional norm related to u_x in the proof for the dissipation of v. Indeed, we need to carefully take care of the topological relation between $\|u_x\|_{\widetilde{L}^2_T(\dot{B}^{-1/2}_{2,1})}$ and $\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}$ as in Proposition 5.2.3. To do this, we localize (5.1.4) with inhomogeneous blocks rather than homogeneous blocks to obtain the dissipative estimate for v.

Secondly, due to the weaker mechanism of regularity-loss, it seems that there is no possibility to capture optimal decay rates in the critical space $B^{3/2}_{2,1}(\mathbb{R})$, since the polynomial decay at the high-frequency part comes from the fact that the initial data is imposed extra higher regularity (see (5.1.8)). To overcome the outstanding difficulty, there are new ingredients in comparison with the case of equal wave speeds in Chapter 4. Precisely, we develop a new frequency-localization time-decay inequality for the dissipative rate $\eta(\xi)=\frac{|\xi|^2}{(1+|\xi|^2)^2}$ in $\mathbb{R}^n$, see Proposition 5.3.1. At the formal level, we see that the high-frequency part decays in time not only with algebraic rates of any order as long as the function is spatially regular enough, but also additional information related the L^p-integrability is available. Consequently, the high-frequency estimate in energy approaches can be divided into two parts, and on each part, different values of p (for example, $p=1$ or $p=2$) are chosen to get desired decay estimates, see Lemma 5.5.1. Additionally, it should be worth noting that the energy approach is totally different from that in Chapter 4, where the frequency-localization Duhamel principle was used. Here, we shall employ somewhat "the square formula of the Duhamel principle" based on the Littlewood-Paley pointwise estimate in Fourier space for the linear system with right-hand side, see (5.5.3)-(5.5.4) for details.

Our main results focus on the Timoshenko system with non-equal wave speeds ($a\neq 1$), which are stated as follows.

Theorem 5.1.1. *Suppose that* $U_0\in B^{3/2}_{2,1}(\mathbb{R})$. *There exists a positive constant* δ_0 *such that if*

$$\|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})}\leq \delta_0,$$

then the Cauchy problem (5.1.4) has a unique global classical solution $U\in\mathcal{C}^1(\mathbb{R}^+\times\mathbb{R})$ *satisfying*

$$U\in\widetilde{\mathcal{C}}(B^{3/2}_{2,1}(\mathbb{R}))\cap\widetilde{\mathcal{C}}^1(B^{1/2}_{2,1}(\mathbb{R}))$$

Moreover, the following energy inequality holds that

$$\begin{aligned}&\|U\|_{\widetilde{L}^\infty(B^{3/2}_{2,1}(\mathbb{R}))}+\Big(\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})}+\|(v,z_x)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}+\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}\Big)\\ \leq\ & C_0\|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})},\end{aligned}\tag{5.1.9}$$

where $C_0>0$ *is a constant.*

Remark 5.1.2. Theorem 5.1.1 exhibits the optimal critical regularity ($s_c=3/2$) of global-in-time existence for (5.1.4), which was proved by the revised energy estimates in comparison with [42], along with the local-in-time existence result in Proposition 5.4.1. Observe that there is 1-regularity-loss phenomenon for the dissipation rate of (v,u_x).

Furthermore, with the aid of the new frequency-localization time-decay inequality in Proposition 5.3.1, we can obtain the the optimal decay estimates by using the time-weighted energy approach in terms of high-frequency and low-frequency decomposition.

Theorem 5.1.2. *Let* $U(t,x)=(v,u,z,y)(t,x)$ *be the global classical solution of Theorem 5.1.1. Assume that the initial data satisfy* $U_0\in B^{3/2}_{2,1}(\mathbb{R})\cap\dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$. *Set* $I_0:=\|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})\cap\dot{B}^{-1/2}_{2,\infty}(\mathbb{R})}$. *If* I_0 *is sufficiently small, then the classical solution* $U(t,x)$ *of (5.1.4) admits the optimal decay estimate*

$$\|U\|_{L^2}\lesssim I_0(1+t)^{-\frac{1}{4}}.\tag{5.1.10}$$

Note that the embedding $L^1(\mathbb{R}) \hookrightarrow \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$ in Lemma 5.2.3, as an immediate byproduct of Theorem 5.1.2, the usual optimal decay estimate of $L^1(\mathbb{R})$-$L^2(\mathbb{R})$ type is available.

Corollary 5.1.1. *Let $U(t,x) = (v,u,z,y)(t,x)$ be the global classical solutions of Theorem 5.1.1. If further the initial data $U_0 \in L^1(\mathbb{R})$ and $\widetilde{I}_0 := \|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})\cap L^1(\mathbb{R})}$ is sufficiently small, then*

$$\|U\|_{L^2} \lesssim \widetilde{I}_0(1+t)^{-\frac{1}{4}}. \tag{5.1.11}$$

Remark 5.1.3. Let us mention that Theorem 5.1.2 and Corollary 5.1.1 exhibit the optimal decay rate in the Besov space with $s_c = 3/2$, that is, $s_D = 3/2$, which implies that the minimal decay regularity coincides with the the critical regularity for global solutions, and the extra higher regularity is not necessary. In addition, it is worth noting that the present work opens a door for the study of dissipative systems of regularity-loss type, which encourages us to develop frequency-localization time-decay inequalities for other dissipative rates and investigate systems with the regularity-loss mechanism.

Finally, we would like to mention other studies on the dissipative Timoshenko system with different effects, see, e.g., [47, 48] for frictional dissipation case, [19, 54, 55] for thermal dissipation case, and [3, 4, 33, 34] for memory-type dissipation case.

The rest of this chapter unfolds as follows. In Section 5.2, we present useful properties in Besov spaces, which will be used in the subsequence analysis. In Section 5.3, we shall develop new time-decay inequality with using frequency-localization techniques. Section 5.4 is devoted to construct the global-in-time existence of classical solutions to (5.1.4). Furthermore, in Section 5.5, we deduce the optimal decay estimate for (5.1.4) by employing energy approaches in terms of high-frequency and low-frequency decomposition. In Appendix (Section 5.6), we present those definitions for Besov spaces and Chemin-Lerner spaces for the convenience of reader.

5.2 Tools

In this section, we only collect useful analysis properties in Besov spaces and Chemin-Lerner spaces in $\mathbb{R}^n (n \geq 1)$. For convenience of reader, those definitions for Besov spaces and Chemin-Lerner spaces are given in the Appendix. Firstly, we give an improved Bernstein inequality (see, *e.g.*, [68]), which allows the case of fractional derivatives.

Lemma 5.2.1. *Let $0 < R_1 < R_2$ and $1 \leq a \leq b \leq \infty$.*

(i) If $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq R_1\lambda\}$, *then*

$$\|\Lambda^\alpha f\|_{L^b} \lesssim \lambda^{\alpha+n(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a} \quad \textit{for any} \quad \alpha \geq 0;$$

(ii) If $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : R_1\lambda \leq |\xi| \leq R_2\lambda\}$, *then*

$$\|\Lambda^\alpha f\|_{L^a} \approx \lambda^\alpha\|f\|_{L^a} \quad \textit{for any} \quad \alpha \in \mathbb{R}.$$

Besov spaces obey various inclusion relations. Precisely,

Lemma 5.2.2. *Let $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, then*

(*i*) *If $s > 0$, then $B^s_{p,r} = L^p \cap \dot{B}^s_{p,r}$;*

(*ii*) *If $\tilde{s} \le s$, then $B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,r}$; this inclusion relation is false for the homogeneous Besov spaces;*

(*iii*) *If $1 \le r \le \tilde{r} \le \infty$, then $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^s_{p,\tilde{r}}$ and $B^s_{p,r} \hookrightarrow B^s_{p,\tilde{r}}$;*

(*iv*) *If $1 \le p \le \tilde{p} \le \infty$, then $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$ and $B^s_{p,r} \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$;*

(*v*) *$\dot{B}^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0$, $\quad B^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0 (1 \le p < \infty)$;*

where $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

Lemma 5.2.3. *Suppose that $\varrho > 0$ and $1 \le p < 2$. It holds that*

$$\|f\|_{\dot{B}^{-\varrho}_{r,\infty}} \lesssim \|f\|_{L^p}$$

with $1/p - 1/r = \varrho/n$. In particular, this holds with $\varrho = n/2, r = 2$ and $p = 1$.

Moser-type product estimates are stated as follows, which plays an important role in the estimate of bilinear terms.

Proposition 5.2.1. *Let $s > 0$ and $1 \le p, r \le \infty$. Then $\dot{B}^s_{p,r} \cap L^\infty$ is an algebra and*

$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^\infty} \|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^\infty} \|f\|_{\dot{B}^s_{p,r}}.$$

Let $s_1, s_2 \le n/p$ such that $s_1 + s_2 > n \max\{0, \frac{2}{p} - 1\}$. Then one has

$$\|fg\|_{\dot{B}^{s_1+s_2-n/p}_{p,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}} \|g\|_{\dot{B}^{s_2}_{p,1}}.$$

In the analysis of decay estimates, we also need the general form of Moser-type product estimates, which was shown by Yong in [78].

Proposition 5.2.2. *Let $s > 0$ and $1 \le p, r, p_1, p_2, p_3, p_4 \le \infty$. Assume that $f \in L^{p_1} \cap \dot{B}^s_{p_4,r}$ and $g \in L^{p_3} \cap \dot{B}^s_{p_2,r}$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then it holds that

$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^{p_1}} \|g\|_{\dot{B}^s_{p_2,r}} + \|g\|_{L^{p_3}} \|f\|_{\dot{B}^s_{p_4,r}}.$$

In [70], Xu and Kawashima established a key fact, which indicates the connection between homogeneous Chemin-Lerner spaces and inhomogeneous Chemin-Lerner spaces.

Proposition 5.2.3. *Let $s \in \mathbb{R}$ and $1 \le \theta, p, r \le \infty$.*

(1) *It holds that*

$$L^\theta_T(L^p) \cap \widetilde{L}^\theta_T(\dot{B}^s_{p,r}) \subset \widetilde{L}^\theta_T(B^s_{p,r});$$

(2) *Furthermore, as $s > 0$ and $\theta \ge r$, it holds that*

$$L^\theta_T(L^p) \cap \widetilde{L}^\theta_T(\dot{B}^s_{p,r}) = \widetilde{L}^\theta_T(B^s_{p,r})$$

for any $T > 0$.

The property of continuity for product in $\widetilde{L}^\theta_T(B^s_{p,r})$ is similar to in the stationary case (Proposition 5.2.1), whereas the time exponent θ behaves according to the Hölder inequality.

Proposition 5.2.4. *The following inequality holds:*

$$\|fg\|_{\widetilde{L}^\theta_T(B^s_{p,r})} \lesssim (\|f\|_{L^{\theta_1}_T(L^\infty)}\|g\|_{\widetilde{L}^{\theta_2}_T(B^s_{p,r})} + \|g\|_{L^{\theta_3}_T(L^\infty)}\|f\|_{\widetilde{L}^{\theta_4}_T(B^s_{p,r})})$$

whenever $s > 0,\ 1 \le p \le \infty,\ 1 \le \theta, \theta_1, \theta_2, \theta_3, \theta_4 \le \infty$ and

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} = \frac{1}{\theta_3} + \frac{1}{\theta_4}.$$

As a direct corollary, one has

$$\|fg\|_{\widetilde{L}^\theta_T(B^s_{p,r})} \lesssim \|f\|_{\widetilde{L}^{\theta_1}_T(B^s_{p,r})}\|g\|_{\widetilde{L}^{\theta_2}_T(B^s_{p,r})}$$

whenever $s \ge n/p,\ \frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$.

Finally, we state a continuity result for compositions (see [1]) to end this section.

Proposition 5.2.5. *Let $s > 0,\ 1 \le p, r, \rho \le \infty,\ F \in W^{[s]+1,\infty}_{loc}(I;\mathbb{R})$ with $F(0) = 0$, $T \in (0, \infty]$ and $v \in \widetilde{L}^\rho_T(B^s_{p,r}) \cap L^\infty_T(L^\infty)$. Then*

$$\|F(v)\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \lesssim (1 + \|v\|_{L^\infty_T(L^\infty)})^{[s]+1}\|v\|_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

5.3 Frequency-localization time-decay inequality

In the recent decade, harmonic analysis tools, especially for techniques based on Littlewood-Paley decomposition and paradifferential calculus have proved to be very efficient in the study of partial differential equations. It is well-known that the frequency-localization operator $\dot{\Delta}_q f$ (or $\Delta_q f$) has a smoothing effect on the function f, even though f is quite rough. Moreover, the L^p norm of $\dot{\Delta}_q f$ can be preserved provided $f \in L^p(\mathbb{R}^n)$. To the best of our knowledge, so far there are few efforts about the decay property related to the operator $\dot{\Delta}_q f$. Here, the difficulty of regularity-loss mechanism forces us to develop the frequency-localization time-decay inequality. Precisely,

Proposition 5.3.1. *Set* $\eta(\xi)=\frac{|\xi|^2}{(1+|\xi|^2)^2}$. *If* $f\in\dot{B}^{\sigma+\ell}_{2,r}(\mathbb{R}^n)\cap\dot{B}^{-s}_{2,\infty}(\mathbb{R}^n)$ *for* $\sigma\in\mathbb{R},\ s\in\mathbb{R}$ *and* $1\leq r\leq\infty$ *such that* $\sigma+s>0$, *then it holds that*

$$\begin{aligned}&\Big\|2^{q\sigma}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\Big\|_{l^r_q}\\ \lesssim\ &\underbrace{(1+t)^{-\frac{\sigma+s}{2}}\|f\|_{\dot{B}^{-s}_{2,\infty}}}_{Low-frequency\ Estimate}+\underbrace{(1+t)^{-\frac{\ell}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|f\|_{\dot{B}^{\sigma+\ell}_{p,r}}}_{High-frequency\ Estimate},\end{aligned}\tag{5.3.1}$$

for $\ell>n\,(\frac{1}{p}-\frac{1}{2})$ *with* $1\leq p\leq 2$.

Proof. For clarity, the proof is separated into high-frequency and low-frequency parts.

(1) If $q\geq 0$, then $|\xi|\sim 2^q\geq 1$, which leads to

$$\begin{aligned}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}&\leq\|\widehat{\dot{\Delta}_q f}e^{-c_0t|\xi|^{-2}}\|_{L^2(|\xi|\geq1)}\\ &=\Big\||\xi|^{\ell}|\widehat{\dot{\Delta}_q f}|\frac{e^{-c_0t|\xi|^{-2}}}{|\xi|^{\ell}}\Big\|_{L^2(|\xi|\geq1)}\\ &\leq\||\xi|^{\ell}\widehat{\dot{\Delta}_q f}\|_{L^{p'}}\Big\|\frac{e^{-c_0t|\xi|^{-2}}}{|\xi|^{\ell}}\Big\|_{L^s(|\xi|\geq1)}\quad\Big(\frac{1}{p'}+\frac{1}{s}=\frac{1}{2},\ p'\geq2\Big)\\ &\leq 2^{q\ell}\|\dot{\Delta}_q f\|_{L^p}\Big\|\frac{e^{-c_1t|\xi|^{-2}}}{|\xi|^{\ell}}\Big\|_{L^s(|\xi|\geq1)}\quad\Big(\frac{1}{p}+\frac{1}{p'}=1\Big),\end{aligned}\tag{5.3.2}$$

where $c_1>0$ and the Hausdorff-Young's inequality was used in the last line. By performing the change of variable as in [72], we arrive at

$$\Big\|\frac{e^{-c_1t|\xi|^{-2}}}{|\xi|^{\ell}}\Big\|_{L^s(|\xi|\geq1)}\lesssim(1+t)^{-\frac{\ell}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\tag{5.3.3}$$

for $\ell>n\,(\frac{1}{p}-\frac{1}{2})$. Besides, it can be also bounded by $(1+t)^{-\frac{\ell}{2}}$ for $\ell\geq0$ if $p=2$. Then it follows from (5.3.2)-(5.3.3) that

$$2^{q\sigma}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\lesssim 2^{q(\sigma+\ell)}(1+t)^{-\frac{\ell}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|\dot{\Delta}_q f\|_{L^p}.\tag{5.3.4}$$

(2) If $q<0$, then $|\xi|\sim 2^q<1$, which implies that

$$|\widehat{\dot{\Delta}_q f}|e^{-\eta(\xi)t}\leq|\widehat{\dot{\Delta}_q f}|e^{-c_2t|\xi|^2}\lesssim|\widehat{\dot{\Delta}_q f}|e^{-c_2(2^q\sqrt{t})^2}\tag{5.3.5}$$

for $c_2>0$. Furthermore, we can obtain

$$2^{q\sigma}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\lesssim\|f\|_{\dot{B}^{-s}_{2,\infty}}(1+t)^{-\frac{\sigma+s}{2}}[(2^q\sqrt{t})^{\sigma+s}e^{-c_2(2^q\sqrt{t})^2}]\tag{5.3.6}$$

Let us remark that $\ell\geq0$ in the case of $p=2$.

for $\sigma \in \mathbb{R},\ s \in \mathbb{R}$ such that $\sigma + s > 0$. Note that

$$\left\|(2^q\sqrt{t})^{\sigma+s}e^{-c_2(2^q\sqrt{t})^2}\right\|_{l^r_q} \lesssim 1, \tag{5.3.7}$$

for any $r \in [1, +\infty]$. Combining (5.3.4), (5.3.6)-(5.3.7), we conclude that

$$\begin{aligned} &\left\|2^{q\sigma}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\right\|_{l^r_q} \\ \lesssim\ & \|f\|_{\dot{B}^{-s}_{2,\infty}}(1+t)^{-\frac{\sigma+s}{2}} + \|f\|_{\dot{B}^{\sigma+\ell}_{p,r}}(1+t)^{-\frac{\ell}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}, \end{aligned} \tag{5.3.8}$$

which is just the inequality (5.3.1). □

5.4 Global-in-time existence

As shown in Chapter 4, the recent local existence theory in [70] for generally symmetric hyperbolic systems can be applied to (5.1.4) directly.

Proposition 5.4.1 (Chapter 4, [42]). *Assume that $U_0 \in B^{3/2}_{2,1}$, then there exists a time $T_0 > 0$ (depending only on the initial data) such that*

- (*i*) (*Existence*) : *system (5.1.4) has a unique solution $U(t,x) \in \mathcal{C}^1([0,T_0]\times\mathbb{R})$ satisfying $U \in \widetilde{\mathcal{C}}_{T_0}(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_{T_0}(B^{1/2}_{2,1})$;*
- (*ii*) (*Blow-up criterion*) : *if the maximal time $T^*(> T_0)$ of existence of such a solution is finite, then*

$$\limsup_{t\to T^*}\|U(t,\cdot)\|_{B^{3/2}_{2,1}} = \infty$$

if and only if

$$\int_0^{T^*}\|\nabla U(t,\cdot)\|_{L^\infty}dt = \infty.$$

Furthermore, in order to show that classical solutions in Proposition 5.3.1 are globally defined, we need to construct a priori estimates according to the dissipative mechanism produced by the Timoshenko system. For this purpose, define by $E(T)$ the energy functional and by $D(T)$ the corresponding dissipation functional:

$$E(T) := \|U\|_{\widetilde{L}^\infty_T(B^{3/2}_{2,1})}$$

and

$$D(T) := \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \|(v,z_x)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}$$

for any time $T > 0$. Hence, we have the following

Proposition 5.4.2. *Suppose $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (5.1.4) for $T > 0$. There exists $\delta_1 > 0$ such that if $E(T) \leq \delta_1$, then*

$$E(T) + D(T) \lesssim \|U_0\|_{B^{3/2}_{2,1}} + \Big(\sqrt{E(T)} + E(T)\Big)D(T). \tag{5.4.1}$$

Furthermore, it holds that

$$E(T) + D(T) \lesssim \|U_0\|_{B^{3/2}_{2,1}}. \tag{5.4.2}$$

Actually, in the case of non-equal wave speeds ($a \neq 1$), a priori estimates on the dissipations for y, z_x and u_x coincide with the case of equal wave speeds. For brevity, we present them as lemmas only, the interested reader is referred to Chapter 4 for proofs.

Lemma 5.4.1 (The dissipation for y)**.** *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (5.1.4) for any $T > 0$, then*

$$E(T) + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \lesssim \|U_0\|_{B^{3/2}_{2,1}} + \sqrt{E(T)}D(T). \tag{5.4.3}$$

Lemma 5.4.2 (The dissipation for z_x)**.** *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (5.1.4) for any $T > 0$, then*

$$\begin{aligned}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \lesssim\ & E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \\ & + \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \sqrt{E(T)}D(T).\end{aligned} \tag{5.4.4}$$

Lemma 5.4.3 (The dissipation for u_x)**.** *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (5.1.4) for any $T > 0$, then*

$$\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})}. \tag{5.4.5}$$

However, the calculation for the dissipation of v is a little different. We would like to give the proof as follows.

Lemma 5.4.4 (The dissipation for v)**.** *If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (5.1.4) for any $T > 0$, then*

$$\begin{aligned}\|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \lesssim\ & E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \varepsilon\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \\ & + (1 + C_\varepsilon)\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + E(T)D(T)\end{aligned} \tag{5.4.6}$$

for $\varepsilon > 0$, where C_ε is a position constant dependent on ε.

Proof. It is convenient to rewrite the system (5.1.4) as follows:

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - a y_x = 0, \\ y_t - a z_x - v + \gamma y = g(z)_x, \end{cases} \tag{5.4.7}$$

where the smooth function $g(z)$ is defined by

$$g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0) z/a = O(z^2)$$

satisfying $g(0) = 0$ and $g'(0) = 0$.

Firstly, applying the inhomogeneous frequency-localization operator $\Delta_q (q \geq -1)$ to (5.4.7) gives

$$\begin{cases} \Delta_q v_t - \Delta_q u_x + \Delta_q y = 0, \\ \Delta_q u_t - \Delta_q v_x = 0, \\ \Delta_q z_t - a\Delta_q y_x = 0, \\ \Delta_q y_t - a\Delta_q z_x - \Delta_q v + \gamma \Delta_q y = \Delta_q g(z)_x. \end{cases} \tag{5.4.8}$$

Next, multiplying the first equation in (5.4.8) by $-\Delta_q y$, the second one by $-a\Delta_q z$, the third one by $-a\Delta_q u$ and the fourth one by $-\Delta_q v$, respectively, then adding the resulting equalities, we have

$$\begin{aligned} & -(\Delta_q v \Delta_q y + a\Delta_q u \Delta_q z)_t + (a\Delta_q v \Delta_q z + a^2 \Delta_q u \Delta_q y)_x + |\Delta_q v|^2 \\ = \; & |\Delta_q y|^2 + (a^2 - 1)\Delta_q y \Delta_q u_x + \gamma \Delta_q y \Delta_q v - \Delta_q g(z)_x \Delta_q v. \end{aligned} \tag{5.4.9}$$

Integrating the equality (5.4.9) in $x \in \mathbb{R}$, with the aid of Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} E_1[\Delta_q U] + \frac{1}{2}\|\Delta_q v\|_{L^2}^2 \\ \lesssim \; & \|\Delta_q y\|_{L^2}^2 + |a^2 - 1| \|\Delta_q y\|_{L^2} \|\Delta_q u_x\|_{L^2} \\ & + \|\Delta_q g(z)_x\|_{L^2} \|\Delta_q v\|_{L^2}, \end{aligned} \tag{5.4.10}$$

where

$$E_1[\Delta_q U] := -\int_{\mathbb{R}} (\Delta_q v \Delta_q y + \Delta_q u \Delta_q z)\, dx.$$

By performing the integral with respect to $t \in [0, T]$, we are led to

$$\begin{aligned} & \|\Delta_q v\|_{L_t^2(L^2)}^2 \\ \lesssim \; & \|\Delta_q U\|_{L_T^\infty(L^2)}^2 + \|\Delta_q U_0\|_{L^2}^2 + \|\Delta_q y\|_{L_T^2(L^2)}^2 \\ & + \|\Delta_q y\|_{L_T^2(L^2)} \|\Delta_q u_x\|_{L_T^2(L^2)} + \|\Delta_q g(z)_x\|_{L_T^2(L^2)}^2, \end{aligned} \tag{5.4.11}$$

where we have noticed the case of $a \neq 1$. Furthermore, Young's inequality enables us to get

$$\begin{aligned} & 2^{\frac{q}{2}}\|\Delta_q v\|_{L^2_T(L^2)} \\ \lesssim\ & c_q\|U\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})} + c_q\|U_0\|_{B^{1/2}_{2,1}} + \varepsilon c_q\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \\ & + c_q(1+C_\varepsilon)\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + c_q\|g(z)_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \end{aligned} \tag{5.4.12}$$

for $\varepsilon > 0$, where C_ε is a position constant dependent on ε and each $\{c_q\}$ has a possibly different form in (5.4.12), however, the bound $\|c_q\|_{\ell^1} \leq 1$ is well satisfied.

Recalling the fact $g'(0) = 0$, it follows from Propositions 5.2.4-5.2.5 that

$$\begin{aligned} \|g(z)_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} &= \|g'(z)z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \\ &\lesssim \|g'(z)-g'(0)\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \\ &\lesssim \|z\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}. \end{aligned} \tag{5.4.13}$$

Hence, together with (5.4.12)-(5.4.13), by summing up on $q \geq -1$, we deduce that

$$\begin{aligned} & \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \\ \lesssim\ & \|U\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})} + \|U_0\|_{B^{1/2}_{2,1}} + \varepsilon\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \\ & + (1+C_\varepsilon)\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \|z\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}, \end{aligned} \tag{5.4.14}$$

which leads to the inequality (5.4.6) immediately. □

Having Lemmas 5.4.1-5.4.4, by taking sufficiently small $\varepsilon > 0$, we can achieve the proof of Proposition 5.4.2. For brevity, we feel free to skip the details. Furthermore, along with local existence result (Proposition 5.4.1) and a priori estimate (Proposition 5.4.2), Theorem 5.1.1 follows from the standard boot-strap argument directly, see Chapter 4 for similar details.

5.5 Optimal decay rates

Due to the better dissipative structure in the case of $a = 1$ (see Chapter 4), we performed the Littlewood-Paley pointwise estimates for the linearized problem (5.1.5) and develop decay properties in the framework of Besov spaces. Furthermore, with the help of the frequency-localization Duhamel principle, the optimal decay estimates of (5.1.4) are shown by localized time-weighted energy approaches. For the case of $a \neq 1$, if the standard Duhamel principle is used, we need to deal with the weak mechanism of regularity-loss in the price of extra higher regularity, so it is impossible to achieve $s_D = 3/2$. Hence, we involve new observations. Actually, we perform "the square formula of the Duhamel principle" based on the Littlewood-Paley pointwise estimate in Fourier space for the linear system with right-hand side, see

(5.5.3)-(5.5.4). Furthermore, we proceed the optimal decay estimate for (5.1.4) in terms of high-frequency and low-frequency decompositions, with the aid of the frequency-localization time-decay inequality developed in Section 5.3.

To do this, we define the following energy functionals:

$$\mathcal{N}(t) = \sup_{0\leq\tau\leq t}(1+\tau)^{\frac{1}{4}}\|U(\tau)\|_{L^2}, \quad \mathcal{D}(t) = \|z_x(\tau)\|_{L^2_t(\dot{B}^{1/2}_{2,1})}.$$

The optimal decay estimate lies in a nonlinear time-weighted energy inequality, which is include in the following

Lemma 5.5.1. *Let $U = (v, u, z, y)^\top$ be the global classical solutions in Theorem 5.1.1. Additionally, if $U_0 \in \dot{B}^{-1/2}_{2,\infty}$, then it holds that*

$$\mathcal{N}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1}\cap\dot{B}^{-1/2}_{2,\infty}} + \mathcal{N}(t)\mathcal{D}(t) + \mathcal{N}(t)^2. \tag{5.5.1}$$

Proof. As in [38], perform the energy method in Fourier spaces to get

$$\frac{d}{dt}E[\hat{U}] + c_3\eta_1(\xi)|\hat{U}|^2 \lesssim \xi^2|\hat{g}|^2, \tag{5.5.2}$$

with $\eta_1(\xi) = \frac{\xi^2}{(1+\xi^2)^2}$, where $E[\hat{U}] \approx |\hat{U}|^2$. As a matter of fact, following from the derivation of (5.5.2), we can obtain the corresponding Littlewood-Paley pointwise energy inequality

$$\frac{d}{dt}E[\widehat{\dot{\Delta}_qU}] + c_3\eta_1|\widehat{\dot{\Delta}_qU}|^2 \lesssim \xi^2|\widehat{\dot{\Delta}_qg}|^2, \tag{5.5.3}$$

where $E[\widehat{\dot{\Delta}_qU}] \approx |\widehat{\dot{\Delta}_qU}|^2$. Gronwall's inequality implies that

$$|\widehat{\dot{\Delta}_qU}|^2 \lesssim e^{-c_3\eta_1 t}|\widehat{\dot{\Delta}_qU_0}|^2 + \int_0^t e^{-c_3\eta_1(t-\tau)}\xi^2|\widehat{\dot{\Delta}_qg}|^2\,d\tau. \tag{5.5.4}$$

It follows from Fubini and Plancherel theorems that

$$\begin{aligned}
\|U\|^2_{L^2} &= \sum_{q\in\mathbb{Z}}\|\dot{\Delta}_qU\|^2_{L^2} \\
&\lesssim \sum_{q\in\mathbb{Z}}\|\widehat{\dot{\Delta}_qU_0}e^{-\frac{1}{2}c_3\eta_1(\xi)t}\|^2_{L^2} \\
&\quad + \int_0^t\sum_{q\in\mathbb{Z}}\||\xi|\widehat{\dot{\Delta}_qg}e^{-\frac{1}{2}c_3\eta_1(\xi)(t-\tau)}\|^2_{L^2}\,d\tau \\
&\triangleq I_1 + I_2.
\end{aligned} \tag{5.5.5}$$

For I_1, by taking $p = r = 2,\ \sigma = 0,\ s = 1/2$ and $\ell = 1$ in Proposition 5.3.1, we arrive at

$$\begin{aligned} I_1 &= \Big(\sum_{q<0} + \sum_{q\geq 0}\Big)\Big(\cdots\Big) \\ &\lesssim \|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty}}(1+t)^{-\frac{1}{2}} + \sum_{q\geq 0} 2^{2q}\,\|\dot{\Delta}_q U_0\|^2_{L^2}(1+t)^{-1} \\ &\lesssim \|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty}}(1+t)^{-\frac{1}{2}} + \|U_0\|^2_{\dot{B}^{1}_{2,2}}(1+t)^{-1} \\ &\lesssim \|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty}\cap B^{3/2}_{2,1}}(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{5.5.6}$$

Next, we begin to bound the nonlinear term on the right-hand side of (5.5.5), which is written as the sum of low-frequency and high-frequency

$$I_2 = \int_0^t \Big(\sum_{q<0} + \sum_{q\geq 0}\Big)\Big(\cdots\Big) \triangleq I_{2L} + I_{2H}. \tag{5.5.7}$$

For I_{2L}, by taking $r = 2,\ \sigma = 1$ and $s = 1/2$ in Proposition 5.3.1, we have

$$\begin{aligned} I_{2L} &\leq \int_0^t (1+t-\tau)^{-\frac{3}{2}}\|g(z)\|^2_{\dot{B}^{-1/2}_{2,\infty}}d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{3}{2}}\|g(z)\|^2_{L^1}d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{3}{2}}\|z(\tau)\|^4_{L^2}d\tau \\ &\lesssim \mathcal{N}^4(t)\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+t)^{-1}d\tau \\ &\lesssim \mathcal{N}^4(t)(1+t)^{-1}, \end{aligned} \tag{5.5.8}$$

where we used the embedding $L^1(\mathbb{R}) \hookrightarrow \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$ in Lemma 5.2.3 and the fact $g(z) = O(z^2)$. For the high-frequency part I_{2H}, more elaborate estimates are needed. For the purpose, we write

$$I_{2H} = \Big(\int_0^{t/2} + \int_{t/2}^t\Big)\Big(\cdots\Big) \triangleq I_{2H1} + I_{2H2}.$$

For I_{2H1}, taking $p = r = 2,\ \sigma = 1$ and $\ell = 1/2$ in Proposition 5.3.1 gives

$$\begin{aligned} I_{2H1} &= \int_0^{t/2}\sum_{q\geq 0} 2^{3q}\,\|\dot{\Delta}_q g(z)\|^2_{L^2}(1+t-\tau)^{-\frac{1}{2}}\,d\tau \\ &\leq \int_0^{t/2}(1+t-\tau)^{-\frac{1}{2}}\|g(z)\|^2_{\dot{B}^{3/2}_{2,2}}\,d\tau. \end{aligned} \tag{5.5.9}$$

On the other hand, recalling $g(z) = O(z^2)$, Proposition 5.2.1 and Lemmas 5.2.1-5.2.2 enable us to get

$$\|g(z)\|_{\dot{B}^{3/2}_{2,2}} \lesssim \|g(z)\|_{\dot{B}^{3/2}_{2,1}} \lesssim \|z\|_{L^\infty}\|z_x\|_{\dot{B}^{1/2}_{2,1}}. \tag{5.5.10}$$

Combine (5.5.9) and (5.5.10) to arrive at

$$\begin{aligned} I_{2H1} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{1}{2}}\|z(\tau)\|^2_{L^\infty}\|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}}\,d\tau \\ &\lesssim \sup_{0\le\tau\le t/2}\Big\{(1+t-\tau)^{-\frac{1}{2}}\|z(\tau)\|^2_{L^\infty}\Big\}\int_0^{t/2}\|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}}\,d\tau \\ &\lesssim (1+t)^{-\frac{1}{2}}\|U_0\|^2_{B^{3/2}_{2,1}}\mathcal{D}^2(t) \\ &\lesssim (1+t)^{-\frac{1}{2}}\|U_0\|^2_{B^{3/2}_{2,1}}. \end{aligned} \tag{5.5.11}$$

For the last step of (5.5.11), we would like to explain a little. It follows from Proposition 5.2.3 and Remark 5.6.1 that

$$\mathcal{D}(t) \lesssim \|z_x\|_{\widetilde{L}^2_t(\dot{B}^{1/2}_{2,1})} \lesssim \|z_x\|_{\widetilde{L}^2_t(B^{1/2}_{2,1})} \lesssim \|U_0\|_{B^{3/2}_{2,1}}, \tag{5.5.12}$$

where we used the energy inequality (5.1.9) in Theorem 5.1.1. By choosing $r = 2,\ p = \sigma = 1$ and $\ell = 1/2$ in Proposition 5.3.1, I_{2H2} is proceeded as

$$\begin{aligned} I_{2H2} &= \int_{t/2}^{t}\sum_{q\ge0}2^{3q}\|\dot{\Delta}_q g(z)\|^2_{L^1}d\tau \\ &\le \int_{t/2}^{t}\|g(z)\|^2_{\dot{B}^{3/2}_{1,2}}\,d\tau. \end{aligned} \tag{5.5.13}$$

Thanks to $g(z) = O(z^2)$, it follows from Proposition 5.2.2 that

$$\|g(z)\|_{\dot{B}^{3/2}_{1,2}} \le \|g(z)\|_{\dot{B}^{3/2}_{1,1}} \lesssim \|z\|_{L^2}\|z_x\|_{\dot{B}^{1/2}_{2,1}}. \tag{5.5.14}$$

Together with (5.5.13)-(5.5.14), we are led to

$$\begin{aligned} I_{2H2} &\lesssim \mathcal{N}^2(t)\int_{t/2}^{t}(1+\tau)^{-\frac{1}{2}}\|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}}\,d\tau \\ &\lesssim \mathcal{N}^2(t)\sup_{t/2\le\tau\le t}(1+\tau)^{-\frac{1}{2}}\int_{t/2}^{t}\|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}}\,d\tau \\ &\lesssim (1+t)^{-\frac{1}{2}}\mathcal{N}^2(t)\mathcal{D}^2(t). \end{aligned} \tag{5.5.15}$$

Combine (5.5.11) and (5.5.15) to get

$$I_{2H} \lesssim (1+t)^{-\frac{1}{2}}\|U_0\|^2_{B^{3/2}_{2,1}} + (1+t)^{-\frac{1}{2}}\mathcal{N}^2(t)\mathcal{D}^2(t). \tag{5.5.16}$$

Therefore, it follows from (5.5.8) and (5.5.16) that

$$\begin{aligned} I_2 \;\lesssim\;& (1+t)^{-1}\mathcal{N}^4(t) + (1+t)^{-\frac{1}{2}}\|U_0\|^2_{B^{3/2}_{2,1}} \\ &+(1+t)^{-\frac{1}{2}}\mathcal{N}^2(t)\mathcal{D}^2(t). \end{aligned} \tag{5.5.17}$$

Finally, noticing (5.5.5)-(5.5.6) and (5.5.17), we conclude that

$$\begin{aligned} \|U\|^2_{L^2} \;\lesssim\;& (1+t)^{-\frac{1}{2}}\|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty}\cap B^{3/2}_{2,1}} + (1+t)^{-\frac{1}{2}}\mathcal{N}^2(t)\mathcal{D}^2(t) \\ &+(1+t)^{-1}\mathcal{N}^4(t) \end{aligned} \tag{5.5.18}$$

which leads to (5.5.1) directly. □

Proof of Theorem 5.1.2. Note that (5.5.12), we arrive at

$$\mathcal{D}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1}} \lesssim \|U_0\|_{B^{3/2}_{2,1}\cap\dot{B}^{-1/2}_{2,\infty}}. \tag{5.5.19}$$

Thus, if the norm $\|U_0\|_{B^{3/2}_{2,1}\cap\dot{B}^{-1/2}_{2,\infty}}$ is sufficiently small, then we have

$$\mathcal{N}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1}\cap\dot{B}^{-1/2}_{2,\infty}} + \mathcal{N}(t)^2 \tag{5.5.20}$$

which implies that $\mathcal{N}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1}\cap\dot{B}^{-1/2}_{2,\infty}}$, provided that $\|U_0\|_{B^{3/2}_{2,1}\cap\dot{B}^{-1/2}_{2,\infty}}$ is sufficiently small. Consequently, the desired decay estimate in Theorem 5.1.2 follows

$$\|U\|_{L^2} \lesssim \|U_0\|_{B^{3/2}_{2,1}\cap\dot{B}^{-1/2}_{2,\infty}}(1+t)^{-\frac{1}{4}}. \tag{5.5.21}$$

Hence, the proof of Theorem 5.1.2 is complete eventually. □

5.6 Appendix

For convenience of reader, in this section, we review the Littlewood–Paley decomposition and definitions for Besov spaces and Chemin-Lerner spaces in $\mathbb{R}^n (n \geq 1)$, see [5] for more details.

Let (φ, χ) is a couple of smooth functions valued in [0,1] such that φ is supported in the shell $\mathbf{C}\left(0, \frac{3}{4}, \frac{8}{3}\right) = \left\{\xi \in \mathbb{R}^n \mid \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$, χ is supported in the ball $\mathbf{B}(0, \frac{4}{3}) = \{\xi \in \mathbb{R}^n \mid |\xi| \leq \frac{4}{3}\}$ satisfying

$$\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1, \qquad q \in \mathbb{N}, \quad \xi \in \mathbb{R}^n$$

and

$$\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \qquad k \in \mathbb{Z}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

For $f \in \mathcal{S}'$(the set of temperate distributions which is the dual of the Schwarz class $\mathcal{S}$), define

$$\Delta_{-1} f := \chi(D) f = \mathcal{F}^{-1}\left(\chi(\xi)\mathcal{F}f\right), \quad \Delta_q f := 0 \qquad \text{for} \ \ q \leq -2;$$

$$\Delta_q f := \varphi(2^{-q}D) f = \mathcal{F}^{-1}\left(\varphi(2^{-q}|\xi|)\,\mathcal{F}f\right) \qquad \text{for} \ \ q \geq 0;$$

$$\dot{\Delta}_q f := \varphi(2^{-q}D) f = \mathcal{F}^{-1}\left(\varphi(2^{-q}|\xi|)\,\mathcal{F}f\right) \qquad \text{for} \ \ q \in \mathbb{Z},$$

where $\mathcal{F}f$, $\mathcal{F}^{-1}f$ represent the Fourier transform and the inverse Fourier transform on f, respectively. Observe that the operator $\dot{\Delta}_q$ coincides with Δ_q for $q \geq 0$.

Denote by $\mathcal{S}'_0 := \mathcal{S}'/\mathcal{P}$ the tempered distributions modulo polynomials $\mathcal{P}$. We first give the definition of homogeneous Besov spaces.

Definition 5.6.1. *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the homogeneous Besov spaces $\dot{B}^s_{p,r}$ is defined by*

$$\dot{B}^s_{p,r} = \{f \in S'_0 : \|f\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} = \begin{cases} \left(\sum_{q \in \mathbb{Z}} (2^{qs}\|\dot{\Delta}_q f\|_{L^p})^r\right)^{1/r}, & r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs}\|\dot{\Delta}_q f\|_{L^p}, & r = \infty. \end{cases}$$

Similarly, the definition of inhomogeneous Besov spaces is stated as follows.

Definition 5.6.2. *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the inhomogeneous Besov spaces $B^s_{p,r}$ is defined by*

$$B^s_{p,r} = \{f \in S' : \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \Big(\sum_{q=-1}^{\infty}(2^{qs}\|\Delta_q f\|_{L^p})^r\Big)^{1/r}, & r<\infty,\\ \sup_{q\geq -1} 2^{qs}\|\Delta_q f\|_{L^p}, & r=\infty. \end{cases}$$

On the other hand, we also present the definition of Chemin-Lerner spaces first initialled by J.-Y. Chemin and N. Lerner [7], which are the refinement of the space-time mixed spaces $L^\theta_T(\dot{B}^s_{p,r})$ or $L^\theta_T(B^s_{p,r})$.

Definition 5.6.3. *For $T>0,\ s\in\mathbb{R},\ 1\leq r,\theta\leq\infty$, the homogeneous mixed Chemin-Lerner spaces $\widetilde{L}^\theta_T(\dot{B}^s_{p,r})$ is defined by*

$$\widetilde{L}^\theta_T(\dot{B}^s_{p,r}) := \{f\in L^\theta(0,T;\mathcal{S}'_0) : \|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})}<+\infty\},$$

where

$$\|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})} := \Big(\sum_{q\in\mathbb{Z}}(2^{qs}\|\dot{\Delta}_q f\|_{L^\theta_T(L^p)})^r\Big)^{\frac{1}{r}}$$

with the usual convention if $r=\infty$.

Definition 5.6.4. *For $T>0,\ s\in\mathbb{R},\ 1\leq r,\theta\leq\infty$, the inhomogeneous Chemin-Lerner spaces $\widetilde{L}^\theta_T(B^s_{p,r})$ is defined by*

$$\widetilde{L}^\theta_T(B^s_{p,r}) := \{f\in L^\theta(0,T;\mathcal{S}') : \|f\|_{\widetilde{L}^\theta_T(B^s_{p,r})}<+\infty\},$$

where

$$\|f\|_{\widetilde{L}^\theta_T(B^s_{p,r})} := \Big(\sum_{q\geq -1}(2^{qs}\|\Delta_q f\|_{L^\theta_T(L^p)})^r\Big)^{\frac{1}{r}}$$

with the usual convention if $r=\infty$.

We further define

$$\widetilde{\mathcal{C}}_T(B^s_{p,r}) := \widetilde{L}^\infty_T(B^s_{p,r})\cap\mathcal{C}([0,T],B^s_{p,r})$$

and

$$\widetilde{\mathcal{C}}^1_T(B^s_{p,r}) := \{f\in\mathcal{C}^1([0,T],B^s_{p,r})\mid \partial_t f\in\widetilde{L}^\infty_T(B^s_{p,r})\},$$

where the index T will be omitted when $T=+\infty$.

By Minkowski's inequality, Chemin-Lerner spaces can be linked with the usual space-time mixed spaces $L^\theta_T(X)$ with $X=B^s_{p,r}$ or $\dot{B}^s_{p,r}$.

Remark 5.6.1. It holds that

$$\|f\|_{\widetilde{L}^\theta_T(X)}\leq\|f\|_{L^\theta_T(X)}\quad\text{if}\ \ r\geq\theta;\qquad \|f\|_{\widetilde{L}^\theta_T(X)}\geq\|f\|_{L^\theta_T(X)}\quad\text{if}\ \ r\leq\theta.$$

Chapter 6

Timoshenko-Fourier system

6.1 Introduction

In the previous chapters (from Chapter 3 to Chapter 5), we have treated the so-called dissipative Timoshenko system

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \gamma\,\psi_t = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+. \end{cases} \tag{6.1.1}$$

The frictional damping is taken into account in this model. Our second model system includes the heat conduction satisfying the Fourier law, which is called *Timoshenko-Fourier system*, and written in the form

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\,\theta_x = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \theta_t + b\,\psi_{tx} = \kappa\,\theta_{xx} & (x,t) \in \mathbb{R} \times \mathbb{R}^+. \end{cases} \tag{6.1.2}$$

We note that our Timoshenko-Fourier system (6.1.2) doesn't contain any mechanical damping. Here the unknown function $\theta = \theta(t,x)$ denotes the temperature, and κ and b are positive constants; in the physically reasonable situation, we only have $b \neq 0$ but we assume $b > 0$ for simplicity.

The decay property of the systems (6.1.1) and (6.1.2) in the bounded region $0 < x < 1$ was studied in [49] and [50], respectively. It was shown in [49, 50] that the energy of the both systems decays exponentially as $t \to \infty$ if $a = 1$, while in the case $a \neq 1$, the energy decays polynomially as $t \to \infty$.

To explain this interesting decay property, we need to investigate the dissipative structure of the systems (6.1.1) and (6.1.2). This was done in [24] for the dissipative Timoshenko system (6.1.1). It was shown in [24] that the dissipative structure of (6.1.1) is characterized by the property

$$\begin{aligned} \operatorname{Re}\lambda(i\xi) &\le -c\eta_1(\xi) \qquad \text{for} \quad a = 1, \\ \operatorname{Re}\lambda(i\xi) &\le -c\eta_2(\xi) \qquad \text{for} \quad a \neq 1, \end{aligned} \tag{6.1.3}$$

where $\lambda(i\xi)$ denotes the eigenvalues of the system (6.1.1) in the Fourie space, $\eta_1(\xi) = \xi^2/(1+\xi^2)$, $\eta_2(\xi) = \xi^2/(1+\xi^2)^2$, and c is a positive constant. This dissipative structure for $a = 1$ is the same as in the general theory developed in [67, 58]. On the other hand, the dissipative structure for $a \neq 1$ is very weak in the high frequency region and satisfies $\operatorname{Re}\lambda(i\xi) \sim -c\xi^{-2}$ for $|\xi| \to \infty$, which causes the regularity-loss in the decay estimate (see [24] or Theorem 3.3.1) and also in the dissipative term of the energy estimate (see [24] or Proposition 3.3.4).

Here we remark that the energy estimate (3.3.12) for $a \neq 1$ given in Proposition 3.3.4 in Chapter 3 is the optimal improvement of the corresponding estimate obtained in [24]. To derive this optimal energy estimate, we need to make a refinement of the energy method employed in [24] in the Fourier space. This has been done in Chapter 3.

In this chapter, we study the dissipative structure of the Timoshenko system (6.1.2) with the heat conduction. We will show that the dissipative structure of the system (6.1.2) can be characterized by the property

$$\begin{aligned} &\operatorname{Re}\lambda(i\xi) \le -c\rho_1(\xi) \qquad \text{for} \quad a = 1, \\ &\operatorname{Re}\lambda(i\xi) \le -c\rho_2(\xi) \qquad \text{for} \quad a \neq 1, \end{aligned} \tag{6.1.4}$$

where $\lambda(i\xi)$ is the eigenvalues of the system (6.1.2) in the Fourie space, $\rho_1(\xi) = \xi^4/(1+\xi^2)^2$, $\rho_2(\xi) = \xi^4/(1+\xi^2)^3$, and c is a positive constant. We find that the dissipative structure of (6.1.2) is different from that of (6.1.1) in the low frequency region. In fact, for $|\xi| \to 0$, we see that $\lambda(\xi) \sim -c\xi^4$ for (6.1.2) and $\lambda(\xi) \sim -c\xi^2$ for (6.1.1). However, there is no difference in the high frequence region $|\xi| \to \infty$.

As the consequence, we can prove the following optimal decay estimate of the solution $U = (\varphi_x - \psi, \varphi_t, \psi_x, \psi_t, \theta)$ to the system (6.1.2) in the whole space:

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^p} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2} \tag{6.1.5}$$

for $a = 1$, and

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^p} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} U_0\|_{L^2} \tag{6.1.6}$$

for $a \neq 1$, where $1 \le p \le 2$, k and l are nonnegative integers, and U_0 is the corresponding initial data. This decay estimate is a little different from that of (6.1.1) (see Theorem 3.3.1), and this difference comes from the difference in the low frequency region.

The above decay estimate is based on the corresponding pointwise estimate of the solution in the Fourier space. Our pointwise estimate is optimal and can be derived by the energy method in the Fourier space. Our energy method employed in this paper is an improved version of the energy method developed in [24].

Other studies on the dissipative Timoshenko system can be found in literature. We refer to [47, 48, 57] for frictional dissipation case, [56, 55, 54] for thermal dissipation case, and [4, 3, 33, 34] for memory-type dissipation case.

Similar decay property of the regularity-loss type was found also for other interesting model systems. We refer to [60] for a plate equation with rotational inertia effect, [23] for a hyperbolic-elliptic system of radiating gas, [15, 65] for the compressible Euler-Maxwell system, and [16] for the Vlasov-Maxwell-Boltzmann system.

6.2 Main results

We consider the system (6.1.2) with the initial data

$$(\varphi,\, \varphi_t,\, \psi,\, \psi_t,\, \theta)(x,0) = (\varphi_0,\, \varphi_1,\, \psi_0,\, \psi_1,\, \theta_0)(x).$$

As in the previous chapters, we introduce the quantities $v = \varphi_x - \psi$, $u = \varphi_t$, $z = a\psi_x$ and $y = \psi_t$ and rewrite the system (6.1.2) in the form

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - az_x - v + b\theta_x = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ \theta_t + by_x = \kappa\theta_{xx}. \end{cases} \tag{6.2.1}$$

The corresponding initial data are given by

$$(v,\, y,\, u,\, z,\, \theta)(x,0) = (v_0,\, y_0,\, u_0,\, z_0,\, \theta_0)(x), \tag{6.2.2}$$

where $v_0 = \varphi_{0,x} - \psi_0$, $y_0 = \psi_1$, $u_0 = \varphi_1$, $z_0 = a\psi_{0,x}$. Putting $U = (v, y, u, z, \theta)^T$ and $U_0 = (v_0, y_0, u_0, z_0, \theta_0)^T$, we can rewrite this initial value problem (6.2.1), (6.2.2) by using vector notations as

$$\begin{cases} U_t + AU_x + LU = BU_{xx}, \\ U(x,0) = U_0(x), \end{cases} \tag{6.2.3}$$

where

$$A = -\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a & -b \\ 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa \end{pmatrix}.$$

We see that A is real symmetric, L is nonnegative definite (not real symmetric) and B is real symmetric and nonnegative definite. Therefore our system is regarded as a symmetric hyperbolic-parabolic coupled system with non-symmetric relaxation. Since L is not symmetric, the general theory on the dissipative structure developed in [67, 58] is not applicable to the above system.

We take the Fourier transform of (6.2.3) to obtain

$$\begin{cases} \hat{U}_t + (i\xi A + L + \xi^2 B)\,\hat{U} = 0, \\ \hat{U}(\xi,0) = \hat{U}_0(\xi). \end{cases} \tag{6.2.4}$$

The solution to (6.2.4) is given by $\hat{U}(\xi,t) = e^{t\hat{\Phi}(i\xi)}\hat{U}_0(\xi)$, where

$$\hat{\Phi}(\zeta) = -(L + \zeta A - \zeta^2 B), \qquad \zeta \in \mathbb{C}. \tag{6.2.5}$$

The eigenvalue problem corresponding (6.2.4) is

$$\lambda\phi + (i\xi A + L + \xi^2 B)\phi = 0, \tag{6.2.6}$$

where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^5$. The eigenvalue $\lambda = \lambda(i\xi)$ of the problem (6.2.4) is the value of λ satisfying (6.2.6) for $\phi \neq 0$.

Now we state the main result in this chapter, which is on the decay estimate of the solutions to the Cauchy problem (6.2.3).

Theorem 6.2.1 (Decay estimates)**.** *The solution U of the problem* (6.2.3) *satisfies the following decay estimates for $t \geq 0$:*

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^p} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2} \tag{6.2.7}$$

for $a = 1$, and

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^p} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} U_0\|_{L^2} \tag{6.2.8}$$

for $a \neq 1$, where $1 \leq p \leq 2$, k and l are nonnegative integers, and C and c are positive constants.

Remark. The above decay estimates are a little different from those for the dissipative Timoshenko system obtained in [24] (see Theorem 3.3.1).

The key to the proof of the above decay estimates (6.2.7) and (6.2.8) is to show the following pointwise estimates of the solution in the Fourier space.

Lemma 6.2.2 (Pointwise estimates in the Fourier space)**.** *The solution $\hat{U}$ to the problem* (6.2.4) *satisfies the following pointwise estimates for any $\xi \in \mathbb{R}$ and $t \geq 0$:*

$$|\hat{U}(\xi,t)| \leq Ce^{-c\rho_1(\xi)t}|\hat{U}_0(\xi)| \qquad \text{for} \quad a = 1, \tag{6.2.9}$$

$$|\hat{U}(\xi,t)| \leq Ce^{-c\rho_2(\xi)t}|\hat{U}_0(\xi)| \qquad \text{for} \quad a \neq 1, \tag{6.2.10}$$

where $\rho_1(\xi) = \frac{\xi^4}{(1+\xi^2)^2}$ and $\rho_2(\xi) = \frac{\xi^4}{(1+\xi^2)^3}$, and C and c are positive constants.

Remark. The corresponding eigenvalue $\lambda(i\xi)$ satisfies $\operatorname{Re}\lambda(i\xi) \le -c\rho_1(\xi)$ for $a = 1$ and $\operatorname{Re}\lambda(i\xi) \le -c\rho_2(\xi)$ for $a \ne 1$. Since $\rho_1(\xi)$ and $\rho_2(\xi)$ for the system (6.2.3) are different from $\eta_1(\xi)$ and $\eta_2(\xi)$ for (3.3.1), the dissipative structure of (6.2.3) is different from that of (3.3.1). This difference appears in the low frequency region. In fact, for $|\xi| \to 0$, we see that $\lambda(\xi) \sim -c\xi^4$ for (6.2.3) and $\lambda(\xi) \sim -c\xi^2$ for (3.3.1). However, there is no essential difference in the high frequency region $|\xi| \to \infty$.

We prove the above pointwise estimates by using the energy method in the Fourier space. As in the previous section, we need to construct a suitable Lyapunov function E for the problem (6.2.4) which is equivalent to $|\hat{U}|^2$ and satisfies the differential inequality

$$E_t + cF \le 0, \tag{6.2.11}$$

where F is the corresponding dissipative term and c is a positive constant. Our Lyapunov function E will be given by (6.4.25) for $a = 1$ and (6.4.28) for $a \ne 1$. The corresponding dissipative term F will be given as follows:

$$F = \frac{\xi^4}{(1+\xi^2)^2}(|\hat{v}|^2 + |\hat{y}|^2) + \frac{\xi^2}{1+\xi^2}(|\hat{u}|^2 + |\hat{z}|^2) + \xi^2|\hat{\theta}|^2 \tag{6.2.12}$$

for $a = 1$, and

$$F = \frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2 + \frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{z}|^2 + \xi^2|\hat{\theta}|^2 \tag{6.2.13}$$

for $a \ne 1$. Namely, we can prove the following result.

Proposition 6.2.3. *Let $a = 1$ (resp. $a \ne 1$). Then, for suitably small positive constants α_1 and α_2, the Lyapunov function E in* (6.4.25) *(resp.* (6.4.28)*) is equivalent to $|\hat{U}|^2$ and satisfies the differential inequality* (6.2.11) *with the dissipative term F in* (6.2.12) *(resp.* (6.2.13)*), where c is a positive constant.*

Remark. Our dissipative term F in (6.2.12) (resp. (6.2.13)) completely matches with the eigenvalues in (6.3.3) and (6.3.6) (resp. (6.3.3) and (6.3.7)) for $a = 1$ (resp. $a \ne 1$).

Lemma 6.2.2 easily follows from Proposition 6.2.3.

Proof. We only give the proof for $a \ne 1$. Since E is equivalent to $|\hat{U}|^2$, we find that F in (6.2.13) satisfies

$$F \ge c\rho_2(\xi)E$$

for a positive constant c, where $\rho_2(\xi) = \frac{\xi^4}{(1+\xi^2)^3}$. Substituting this inequality into (6.2.11), we have

$$E_t + c\rho_2(\xi)E \le 0.$$

This differential inequality can be solved as

$$E(\xi, t) \le e^{-c\rho_2(\xi)t}E(\xi, 0),$$

which gives the desired pointwise estimate (6.2.10). □

Also, as a simple corollary of Proposition 6.2.3, we have the following optimal energy estimates of solutions to the problem (6.2.3).

Proposition 6.2.4. *Let s and k be integers with $0 \le k \le s$. Then the solution U to the problem* (6.2.3) *satisfies the following energy estimates for any $t \ge 0$:*

$$\begin{aligned}
&\|\partial_x^k U(t)\|_{H^{s-k}}^2 + \int_0^t \big(\|\partial_x^{k+2} v(\tau)\|_{H^{s-k-2}}^2 + \|\partial_x^{k+2} y(\tau)\|_{H^{s-k-2}}^2 \\
&\qquad + \|\partial_x^{k+1} u(\tau)\|_{H^{s-k-1}}^2 + \|\partial_x^{k+1} z(\tau)\|_{H^{s-k-1}}^2 + \|\partial_x^{k+1} \theta(\tau)\|_{H^{s-k}}^2 \big)\, d\tau \\
&\le C\|\partial_x^k U_0\|_{H^{s-k}}^2
\end{aligned} \tag{6.2.14}$$

for $a = 1$, and

$$\begin{aligned}
&\|\partial_x^k U(t)\|_{H^{s-k}}^2 + \int_0^t \big(\|\partial_x^{k+2} v(\tau)\|_{H^{s-k-3}}^2 + \|\partial_x^{k+2} y(\tau)\|_{H^{s-k-2}}^2 \\
&\qquad + \|\partial_x^{k+1} u(\tau)\|_{H^{s-k-2}}^2 + \|\partial_x^{k+1} z(\tau)\|_{H^{s-k-1}}^2 + \|\partial_x^{k+1} \theta(\tau)\|_{H^{s-k}}^2 \big)\, d\tau \\
&\le C\|\partial_x^k U_0\|_{H^{s-k}}^2
\end{aligned} \tag{6.2.15}$$

for $a \neq 1$, where C is a positive constant.

Remark. The above energy estimates completely match with the eigenvalues given in (6.3.3) and (6.3.6) for $a = 1$ and (6.3.3) and (6.3.7) for $a \neq 1$. Therefore they seem optimal. We note that in the dissipative term of (6.2.15) for $a \neq 1$, we have one regularity-loss for two components v and u but no regularity-loss for other three components y, z and θ.

Finally in this subsection, we give the proof of Theorem 6.2.1.

Proof. Now we prove Theorem 6.2.1. First we consider the case where $a = 1$. Applying the Plancherel theorem and using the pointwise estimate (6.2.9), we have

$$\|\partial_x^k U(t)\|_{L^2}^2 = C\int_{\mathbb{R}} \xi^{2k} |\hat{U}(\xi,t)|^2\, d\xi \le C\int_{\mathbb{R}} \xi^{2k} e^{-\rho_1(\xi)t} |\hat{U}_0(\xi)|^2\, d\xi. \tag{6.2.16}$$

We divided the last integral into two parts I_1 and I_2 corresponding to the regions $|\xi| \le 1$ and $|\xi| \ge 1$, respectively. Here we see that $\rho_1(\xi) \ge c\xi^4$ for $|\xi| \le 1$ so that we have

$$I_1 = C\int_{|\xi|\le 1} \xi^{2k} e^{-c\rho_1(\xi)t} |\hat{U}_0(\xi)|^2\, d\xi \le C\int_{|\xi|\le 1} \xi^{2k} e^{-c\xi^4 t} |\hat{U}_0(\xi)|^2\, d\xi.$$

For $1 \le p \le 2$, we choose p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Also, we take r such that $\frac{1}{r} + \frac{2}{p'} = 1$. Then we see that $\frac{1}{2r} = \frac{1}{p} - \frac{1}{2}$. Applying the Hölder inequality and the Hausdorff-Young inequality, we can estimate I_1 as

$$\begin{aligned}
I_1 &\le C\|\xi^{2k} e^{-c\xi^4 t}\|_{L^r(|\xi|\le 1)} \|\hat{U}_0\|_{L^{p'}}^2 \\
&\le C(1+t)^{-\frac{1}{4r} - \frac{k}{2}} \|U_0\|_{L^p}^2 = C(1+t)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}} \|U_0\|_{L^p}^2.
\end{aligned}$$

On the other hand, in the high frequency region $|\xi| \geq 1$, we have $\rho_1(\xi) \geq c$. Therefore we can estimate I_2 as

$$\begin{aligned} I_2 &= C\int_{|\xi|\geq 1} \xi^{2k} e^{-c\rho_1(\xi)t} |\hat{U}_0(\xi)|^2 d\xi \\ &\leq Ce^{-ct}\int_{|\xi|\geq 1} \xi^{2k} |\hat{U}_0(\xi)|^2 d\xi \leq Ce^{-ct}\|\partial_x^k U_0\|_{L^2}^2. \end{aligned}$$

Substituting these estimates into (6.2.16) gives the desired estimate (6.2.7).

Next we consider the case where $a \neq 1$. Using (6.2.10), we have

$$\|\partial_x^k U(t)\|_{L^2}^2 \leq C\int_{\mathbb{R}} \xi^{2k} e^{-\rho_2(\xi)t} |\hat{U}_0(\xi)|^2\, d\xi. \tag{6.2.17}$$

We divided this integral into two parts J_1 and J_2 corresponding to the regions $|\xi| \leq 1$ and $|\xi| \geq 1$, respectively. Since, $\rho_2(\xi) \geq c\xi^4$ for $|\xi| \leq 1$, the low frequency part J_1 is estimated just in the same way as I_1 for $a = 1$, and we have $J_1 \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|U_0\|_{L^p}^2$. On the other hand, in the high frequency region $|\xi| \geq 1$, we see that $\rho_2(\xi) \geq c|\xi|^{-2}$. Thus we have

$$\begin{aligned} J_2 &= C\int_{|\xi|\geq 1} \xi^{2k} e^{-c\rho_2(\xi)t} |\hat{U}_0(\xi)|^2\, d\xi \leq C\int_{|\xi|\geq 1} \xi^{2k} e^{-c|\xi|^{-2}t} |\hat{U}_0(\xi)|^2\, d\xi \\ &\leq C\sup_{|\xi|\geq 1}\left\{|\xi|^{-2l} e^{-c|\xi|^{-2}t}\right\}\int_{|\xi|\geq 1} \xi^{2(k+l)} |\hat{U}_0(\xi)|^2\, d\xi \\ &\leq C(1+t)^{-l}\|\partial_x^{k+l} U_0\|_{L^2}^2. \end{aligned}$$

Substituting these estimates into (6.2.17) gives the desired estimate (6.2.8) for $a \neq 1$. This completes the proof of Theorem 6.2.1. □

6.3 Asymptotic expansion of eigenvalues

In order to see whether the pointwise estimates in Lemma 6.2.2 are optimal or not, we investigate the asymptotic expansion of the eigenvalues of (6.2.4) for $|\xi| \to 0$ and $|\xi| \to \infty$. We denote by $\lambda_j(\zeta)$, $j = 1, 2, 3, 4, 5$, the eigenvalues of the matrix $\hat{\Phi}(\zeta)$ in (6.2.4), which are the solutions to the characteristic equation

$$\begin{aligned} \det(\lambda I - \hat{\Phi}(\zeta)) &= \lambda^5 - \kappa\zeta^2\lambda^4 + \left\{1 - (1 + a^2 + b^2)\,\zeta^2\right\}\lambda^3 \\ &\quad - \kappa\left\{1 - (a^2 + 1)\,\zeta^2\right\}\zeta^2\lambda^2 + (a^2 + b^2)\,\zeta^4\lambda - a^2\kappa\zeta^6 = 0. \end{aligned} \tag{6.3.1}$$

(i) When $|\zeta| \to 0$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) = \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \cdots. \tag{6.3.2}$$

We substitute $\lambda = \lambda_j(\zeta)$ in (6.3.2) into the characteristic quation (6.3.1) and calculate the coefficients $\lambda_j^{(k)}$, $k = 0, 1, 2\cdots$, successively. We have

$$\lambda_j^{(0)} = \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \alpha_j \qquad \text{for} \quad j = 1, 2, 3,$$
$$\lambda_j^{(0)} = \pm i, \quad \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \mp \frac{1}{2}\, iY,$$
$$\lambda_j^{(3)} = 0, \quad \lambda_j^{(4)} = -\frac{1}{2}\, b^2\kappa \mp \frac{1}{2}\, iZ \qquad \text{for} \quad j = 4, 5,$$

where α_j are the solutions of the algebraic equation

$$X^3 - \kappa X^2 + (a^2 + b^2)\, X - a^2\kappa = 0,$$

and $Y = 1 + a^2 + b^2$ and $Z = Y^2/4 + Y - 1$. We claim that $\mathrm{Re}\,\alpha_j > 0$ for $j = 1, 2, 3$. To see this, we put $f(X) = X^3 - \kappa X^2 + (a^2 + b^2)\, X - a^2\kappa$. Then $f(X)$ is continuous and satisfies $f(0)f(\kappa) < 0$. This imples that there exists at least one solution $X = \alpha_1$ of $f(X) = 0$ in the interval $(0, \kappa)$. We write other two solutions as α_2 and α_3. Then we have $\mathrm{Re}\,\alpha_2 = \mathrm{Re}\,\alpha_3$. Also, from Newton's relations, we have $\alpha_1 + \alpha_2 + \alpha_3 = \kappa$, which shows that $\mathrm{Re}\,\alpha_2 = \mathrm{Re}\,\alpha_3 = (\kappa - \alpha_1)/2 > 0$. Moreover, we see that $\mathrm{Re}\,\lambda_j^{(4)} = -b^2\kappa/2 < 0$ for $j = 4, 5$. Consequently, for $|\xi| \to 0$, we have

$$\mathrm{Re}\,\lambda_j(i\xi) = \begin{cases} -(\mathrm{Re}\,\alpha_j)\,\xi^2 + O(|\xi|^3) & \text{for} \quad j = 1, 2, 3, \\ -\dfrac{1}{2} b^2\kappa\,\xi^4 + O(|\xi|^5) & \text{for} \quad j = 4, 5. \end{cases} \tag{6.3.3}$$

(ii) To derive the asymptotic expansion of the eigenvalues $\lambda_j(\zeta)$ for $|\zeta| \to \infty$, we define the matrix $\hat{\Psi}(\zeta^{-1}) = B - \zeta^{-1}A - \zeta^{-2}L$. Then we have the relation $\hat{\Phi}(\zeta) = \zeta^2\hat{\Psi}(\zeta^{-1})$. Let $\mu_j(\zeta^{-1})$ be the eigenvalues of the matrix $\hat{\Psi}(\zeta^{-1})$, which are the solutions to the characteristic equation

$$\begin{aligned} \det(\mu I - \hat{\Psi}(\zeta^{-1})) &= \mu^5 - \kappa\mu^4 - \left\{(1 + a^2 + b^2) - \zeta^{-2}\right\}\zeta^{-2}\mu^3 \\ &\quad + \kappa\left\{(a^2 + 1) - \zeta^{-2}\right\}\zeta^{-2}\mu^2 + (a^2 + b^2)\,\zeta^{-4}\mu - a^2\kappa\zeta^{-4} = 0. \end{aligned} \tag{6.3.4}$$

When $|\zeta|^{-1} \to 0$, we have the asymptotic expansion of $\mu_j(\zeta^{-1})$ in the form

$$\mu_j(\zeta^{-1}) = \mu_j^{(2)} + \mu_j^{(1)}\zeta^{-1} + \mu_j^{(0)}\zeta^{-2} + \mu_j^{(-1)}\zeta^{-3} + \cdots . \tag{6.3.5}$$

This together with the relation $\lambda_j(\zeta) = \zeta^2\mu_j(\zeta^{-1})$ gives the asymptotic expansion of $\lambda_j(\zeta)$ for $|\zeta| \to \infty$:

$$\lambda_j(\zeta) = \mu_j^{(2)}\zeta^2 + \mu_j^{(1)}\zeta + \mu_j^{(0)} + \mu_j^{(-1)}\zeta^{-1} + \mu_j^{(-2)}\zeta^{-2} + \cdots .$$

We substitute the asymptotic expansion of $\mu_j(\zeta^{-1})$ in (6.3.5) into the characteristic equation (6.3.4) and calculate the coefficients $\mu_j^{(k)}$, $k = 2, 1, 0, -1, -2, \cdots$, successively. For four eigenvalues we find the following expressions: When $a = 1$, we have

$$\mu_j^{(2)} = 0, \quad \mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = \beta_j \qquad \text{for} \quad j = 1, 2, 3, 4,$$

and when $a \neq 1$, we have

$$\mu_j^{(2)} = 0, \quad \mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = 0,$$
$$\mu_j^{(-1)} = \pm\frac{1}{2P}, \quad \mu_j^{(-2)} = \frac{b^2}{2\kappa P^2} \qquad \text{for} \quad j = 1, 2,$$
$$\mu_j^{(2)} = 0, \quad \mu_j^{(1)} = \pm a, \quad \mu_j^{(0)} = -\frac{b^2}{2\kappa}, \qquad \text{for} \quad j = 3, 4,$$

where β_j are the solutions of the algebraic equation $4\kappa X^2 + 2b^2 X + \kappa = 0$ and satisfy $\mathrm{Re}\,\beta_j < 0$, and $P = a^2 - 1$ (for $a \neq 1$). On the other hand, for the last eigenvalue, we have

$$\mu_j^{(2)} = \kappa, \quad \mu_j^{(1)} = 0 \qquad \text{for} \quad j = 5.$$

Consequently, when $a = 1$, we have

$$\mathrm{Re}\,\lambda_j(i\xi) = \begin{cases} -\mathrm{Re}\,\beta_j + O(|\xi|^{-1}) & \text{for} \quad j = 1, 2, 3, 4, \\ -\kappa\,\xi^2 + O(1) & \text{for} \quad j = 5 \end{cases} \tag{6.3.6}$$

for $|\xi| \to \infty$; while in the case $a \neq 1$, we have

$$\mathrm{Re}\,\lambda_j(i\xi) = \begin{cases} -\dfrac{b^2}{2\kappa P^2}\,\xi^{-2} + O(|\xi|^{-3}) & \text{for} \quad j = 1, 2, \\ -\dfrac{b^2}{2\kappa} + O(|\xi|^{-1}) & \text{for} \quad j = 3, 4, \\ -\kappa\,\xi^2 + O(1) & \text{for} \quad j = 5 \end{cases} \tag{6.3.7}$$

for $|\xi| \to \infty$. According to the expansion (6.3.7) for $|\xi| \to \infty$, when $a \neq 1$, one eigenvalue is of the standard type and satisfies $\mathrm{Re}\,\lambda(i\xi) \sim -c\xi^2$, two eigenvalues are of the standard type and satisfy $\mathrm{Re}\,\lambda(i\xi) \sim -c$, while the other two are not of the standard type and satisfy $\mathrm{Re}\,\lambda(i\xi) \sim -c\xi^{-2}$.

The asymptotic expansions (6.3.3), (6.3.6) and (6.3.7) imply that the energy inequality in Proposition 6.2.3 and hence the pointwise estimates (6.2.9) and (6.2.10) in Lemma 6.2.2 seem optimal.

6.4 Energy method in Fourier spaces

The aim of this subsection is to prove Proposition 6.2.3.

Proof. The system (6.2.4) is written explicitly in the form

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{6.4.1a}$$

$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + bi\xi\hat{\theta} = 0, \tag{6.4.1b}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{6.4.1c}$$

$$\hat{z}_t - ai\xi\hat{y} = 0, \tag{6.4.1d}$$

$$\hat{\theta}_t + bi\xi\hat{y} + \kappa\xi^2\hat{\theta} = 0. \tag{6.4.1e}$$

By using the energy method for this system in the Fourier space, we construct the Lyapunov function E and prove the differential inequality (6.2.11) with the desired dissipation term F. Our proof below is divided into six steps.

Step 1. We multiply (6.4.1a), (6.4.1b), (6.4.1c), (6.4.1d) and (6.4.1e) by $\bar{\hat{v}}$, $\bar{\hat{y}}$, $\bar{\hat{u}}$, $\bar{\hat{z}}$ and $\bar{\hat{\theta}}$, respectively. Then, adding the resultant equations and taking the real part, we get

$$\frac{1}{2}\,(|\hat{U}|^2)_t + \kappa\xi^2|\hat{\theta}|^2 = 0, \tag{6.4.2}$$

where $|\hat{U}|^2 = |\hat{v}|^2 + |\hat{y}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{\theta}|^2$. This gives us the energy estimate for $\hat{U}$ together with the dissipative estimate for $\hat{\theta}$.

Step 2. We construct the dissipative estimate for $(\hat{v}, \hat{y})$. We multiply (6.4.1c) and (6.4.1a) by $i\xi\bar{\hat{v}}$ and $-i\xi\bar{\hat{u}}$, respectively. Then, adding the resulting equalities and taking the real part, we get

$$\xi E_{1,t} + \xi^2|\hat{v}|^2 = \xi^2|\hat{u}|^2 + \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}), \tag{6.4.3}$$

where $E_1 = \mathrm{Re}\,(i\hat{v}\bar{\hat{u}})$. Also, we multiply (6.4.1d) and (6.4.1b) by $i\xi\bar{\hat{y}}$ and $-i\xi\bar{\hat{z}}$, respectively. Then, adding the resulting equalities and taking the real part, we get

$$\xi E_{2,t} + a\xi^2|\hat{y}|^2 = a\xi^2|\hat{z}|^2 - \xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{z}}) - b\xi^2\,\mathrm{Re}\,(\bar{\hat{z}}\hat{\theta}), \tag{6.4.4}$$

where $E_2 = \mathrm{Re}\,(i\hat{z}\bar{\hat{y}})$.

(i) When $a = 1$, adding (6.4.3) to (6.4.4) and using the Young inequality, we obtain

$$\begin{aligned} &\xi(E_1 + E_2)_t + (1-\varepsilon)\,\xi^2(|\hat{v}|^2 + |\hat{y}|^2) \\ &\quad \le C_\varepsilon(1+\xi^2)(|\hat{u}|^2 + |\hat{z}|^2) + C_\varepsilon\xi^2|\hat{\theta}|^2 \end{aligned} \tag{6.4.5}$$

for any $\varepsilon \in (0,1)$, where C_ε is a positive constant depending on ε.

(ii) When $a \neq 1$, multiplying (6.4.4) by $1+\xi^2$, adding the resultant equality to (6.4.3), and using the Young inequality, we obtain

$$\begin{aligned} &\xi\left\{E_1 + (1+\xi^2)\,E_2\right\}_t + (1-\varepsilon)\left\{\xi^2|\hat{v}|^2 + a(1+\xi^2)\,\xi^2|\hat{y}|^2\right\} \\ &\quad \le C_\varepsilon(1+\xi^2)|\hat{u}|^2 + C_\varepsilon(1+\xi^2)^2|\hat{z}|^2 + C_\varepsilon(1+\xi^2)\,\xi^2|\hat{\theta}|^2 \end{aligned} \tag{6.4.6}$$

for any $\varepsilon \in (0,1)$, where C_ε is a positive constant depending on ε.

Step 3. To obtain the dissipative estimate for $\hat{u}$, we use (6.4.3) in the form

$$-\xi E_{1,t} + \xi^2|\hat{u}|^2 = \xi^2|\hat{v}|^2 - \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}). \tag{6.4.7}$$

We want to eliminate both the terms $\xi^2|\hat{v}|^2$ and $-\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})$ on the right hand side of (6.4.7). First, to eliminate the term $\xi^2|\hat{v}|^2$, we multiply (6.4.1b) and (6.4.1a) by $-\bar{\hat{v}}$ and $-\bar{\hat{y}}$, respectively. Then, adding the resultant equalities and taking the real part, we get

$$E_{3,t} + |\hat{v}|^2 = |\hat{y}|^2 - a\xi\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{z}) + b\xi\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{\theta}) - \xi\,\mathrm{Re}\,(i\hat{u}\bar{\hat{y}}), \tag{6.4.8}$$

where $E_3 = -\mathrm{Re}\,(\hat{v}\bar{\hat{y}})$. Here, to eliminate the term $-a\xi\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{z})$ on the right hand side of (6.4.8), we multiply (6.4.1c) and (6.4.1d) by $-\bar{\hat{z}}$ and $-\bar{\hat{u}}$, respectively. Then, adding the resultant equalities and taking the real part, we get

$$E_{4,t} + \xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{z}}) = -a\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}), \tag{6.4.9}$$

where $E_4 = -\mathrm{Re}\,(\hat{u}\bar{\hat{z}})$. We multiply (6.4.9) by a and add the result to (6.4.8). This gives

$$(E_3 + aE_4)_t + |\hat{v}|^2 = |\hat{y}|^2 + b\xi\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{\theta}) - (a^2-1)\,\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}), \tag{6.4.10}$$

where we have used the equality $a^2\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) + \mathrm{Re}\,(i\hat{u}\bar{\hat{y}}) = (a^2-1)\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})$.

Next we try to eliminate the term $|\hat{y}|^2$ on the right hand side of (6.4.10). For this purpose, we multiply (6.4.1e) and (6.4.1b) by $-\xi\bar{\hat{y}}$ and $\xi\bar{\hat{\theta}}$, respectively. Then, adding the resultant equalities and taking the real part, we get

$$\xi E_{6,t} + b\xi^2|\hat{y}|^2 = b\xi^2|\hat{\theta}|^2 - a\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}}) + \xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{\theta}}) + \kappa\xi^3\,\mathrm{Re}\,(i\bar{\hat{y}}\hat{\theta}), \tag{6.4.11}$$

where $E_6 = \mathrm{Re}\,(i\hat{y}\bar{\hat{\theta}})$. Moreover, we try to eliminate the term $\xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{\theta}})$ on the right hand side of (6.4.11). To this end, we multiply (6.4.1c) and (6.4.1e) by $-\bar{\hat{\theta}}$ and $-\bar{\hat{u}}$, respectively. Then, adding the resultant equalities and taking the real part, we get

$$E_{5,t} - b\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}) = -\xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{\theta}}) + \kappa\xi^2\,\mathrm{Re}\,(\bar{\hat{u}}\hat{\theta}), \tag{6.4.12}$$

where $E_5 = -\mathrm{Re}\,(\hat{u}\bar{\hat{\theta}})$. We add (6.4.11) to (6.4.12) and eliminate the term $\xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{\theta}})$ to have

$$\begin{aligned}(\xi E_6 + E_5)_t + b\xi^2|\hat{y}|^2 &= b\xi^2|\hat{\theta}|^2 + b\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})\\ &\quad + \kappa\xi^2\,\mathrm{Re}(\bar{\hat{u}}\hat{\theta}) - a\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}}) + \kappa\xi^3\,\mathrm{Re}\,(i\bar{\hat{y}}\hat{\theta}).\end{aligned} \tag{6.4.13}$$

Now we combine (6.4.7), (6.4.10) and (6.4.13) such that $\{(6.4.7) + (6.4.10) \times \xi^2\} \times b +$ (6.4.13). This can eliminate both the terms $\xi^2|\hat{y}|^2$ and $-\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})$ in (6.4.7) and we have

$$\begin{aligned}&\left\{b\left(-\xi E_1 + \xi^2(E_3 + aE_4)\right) + (\xi E_6 + E_5)\right\}_t + b\xi^2|\hat{u}|^2\\ &= b\xi^2|\hat{\theta}|^2 + b^2\xi^3\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{\theta}) - b(a^2-1)\,\xi^3\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})\\ &\qquad + \kappa\xi^2\,\mathrm{Re}\,(\bar{\hat{u}}\hat{\theta}) - a\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}}) + \kappa\xi^3\,\mathrm{Re}\,(i\bar{\hat{y}}\hat{\theta}),\end{aligned} \tag{6.4.14}$$

which still contains the term $\xi^3\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})$. Finally, to eliminate this term, we multiply (6.4.12) by $(a^2-1)\xi^2$ and add the resultant equality to (6.4.14). This yields

$$\begin{aligned}
&\left\{b\left(-\xi E_1+\xi^2(E_3+aE_4)\right)+(\xi E_6+E_5)+\xi^2(a^2-1)\,E_5\right\}_t\\
&\quad+b\xi^2|\hat{u}|^2=b\xi^2|\hat{\theta}|^2-(a^2+b^2-1)\,\xi^3\,\mathrm{Re}\,(i\hat{v}\bar{\hat{\theta}})\\
&\qquad+\kappa\left\{1+(a^2-1)\,\xi^2\right\}\xi^2\,\mathrm{Re}\,(\bar{\hat{u}}\hat{\theta})-a\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}})+\kappa\xi^3\,\mathrm{Re}\,(i\bar{\hat{y}}\hat{\theta}).
\end{aligned}\tag{6.4.15}$$

After a simple computation we arrive at

$$\begin{aligned}
&\left(\xi^2\mathbf{E}_3+\xi\mathbf{E}_4+E_5\right)_t+b\xi^2|\hat{u}|^2\\
&\le b\xi^2|\hat{\theta}|^2+(|a^2-1|+b^2)|\xi|^3|\hat{v}||\hat{\theta}|+\kappa\xi^2|\hat{u}||\hat{\theta}|\\
&\quad+\kappa|a^2-1|\xi^4|\hat{u}||\hat{\theta}|+a\xi^2|\hat{z}||\hat{\theta}|+\kappa|\xi|^3|\hat{y}||\hat{\theta}|,
\end{aligned}\tag{6.4.16}$$

where $\mathbf{E}_3=bE_3+abE_4+(a^2-1)E_5$ and $\mathbf{E}_4=-bE_1+E_6$.

Step 4. To obtain the dissipative estimate for $\hat{z}$, we use (6.4.4) in the form

$$-\xi E_{2,t}+a\xi^2|\hat{z}|^2=a\xi^2|\hat{y}|^2+\xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{z}})+b\xi^2\,\mathrm{Re}\,(\bar{\hat{z}}\hat{\theta}).\tag{6.4.17}$$

We want to eliminate both the terms $a\xi^2|\hat{y}|^2$ and $\xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{z}})$ on the right hand side of (6.4.17). First, to eliminate the term $a\xi^2|\hat{y}|^2$, multiplying (6.4.17) by b and (6.4.13) by a, and adding the resultant equalities, we get

$$\begin{aligned}
&\left\{-b\xi E_2+a(\xi E_6+E_5)\right\}_t+ab\xi^2|\hat{z}|^2\\
&=ab\xi^2|\hat{\theta}|^2+b\xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{z}})+ab\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})\\
&\quad+a\kappa\xi^3\,\mathrm{Re}\,(i\bar{\hat{y}}\hat{\theta})+a\kappa\xi^2\,\mathrm{Re}\,(\bar{\hat{u}}\hat{\theta})+b^2\xi^2\,\mathrm{Re}\,(\bar{\hat{z}}\hat{\theta})-a^2\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}}).
\end{aligned}\tag{6.4.18}$$

We can eliminate both the terms $b\xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{z}})$ and $ab\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y})$ on the right hand side of (6.4.18) by using (6.4.9). In fact, multiplying (6.4.9) by b and adding the resultant equality to (6.4.18), we get

$$\begin{aligned}
&\left\{-b\xi E_2+a(\xi E_6+E_5)+bE_4\right\}_t+ab\xi^2|\hat{z}|^2=ab\xi^2|\hat{\theta}|^2\\
&\quad+a\kappa\xi^3\,\mathrm{Re}\,(i\bar{\hat{y}}\hat{\theta})+a\kappa\xi^2\,\mathrm{Re}\,(\bar{\hat{u}}\hat{\theta})+b^2\xi^2\,\mathrm{Re}\,(\bar{\hat{z}}\hat{\theta})-a^2\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}}).
\end{aligned}\tag{6.4.19}$$

After a simple computation we arrive at

$$\begin{aligned}
&(\xi\mathbf{E}_6+\mathbf{E}_7)_t+ab\xi^2|\hat{z}|^2\\
&\le ab\xi^2|\hat{\theta}|^2+a\kappa|\xi|^3|\hat{y}||\hat{\theta}|+a\kappa\xi^2|\hat{u}||\hat{\theta}|+|b^2-a^2|\,\xi^2|\hat{z}||\hat{\theta}|,
\end{aligned}\tag{6.4.20}$$

where $\mathbf{E}_6=-bE_2+aE_6$ and $\mathbf{E}_7=aE_5+bE_4$.

Now we combine (6.4.16) and (6.4.20) to show a good dissipative estimate for $(\hat{u}, \hat{z})$ in terms of the corresponding estimate for $(\hat{v}, \hat{y}, \hat{\theta})$.
(i) When $a = 1$, we add (6.4.16) to (6.4.20) and use the Young inequality to obtain

$$\begin{aligned}
&\big\{(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) + (\xi\mathbf{E}_6 + \mathbf{E}_7)\big\}_t + b(1-\varepsilon)\,\xi^2(|\hat{u}|^2 + |\hat{z}|^2) \\
&\le C_\varepsilon \xi^2|\hat{\theta}|^2 + C|\xi|^3(|\hat{v}||\hat{\theta}| + |\hat{y}||\hat{\theta}|) \\
&\le C_{\varepsilon,\alpha_1}(1+\xi^2)\,\xi^2|\hat{\theta}|^2 + \alpha_1\varepsilon\frac{\xi^4}{1+\xi^2}(|\hat{v}|^2 + |\hat{y}|^2)
\end{aligned} \tag{6.4.21}$$

for any $\varepsilon \in (0,1)$ and $\alpha_1 > 0$, where C_ε and C_{ε,α_1} are constants depending on ε and (ε, α_1), respectively.
(ii) When $a \neq 1$, we multiply (6.4.20) by $1 + \xi^2$ and add the resultant inequality to (6.4.16). Then, applying the Young inequality, we obtain

$$\begin{aligned}
&\big\{(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) + (1+\xi^2)(\xi\mathbf{E}_6 + \mathbf{E}_7)\big\}_t \\
&\qquad + b(1-\varepsilon)\big\{\xi^2|\hat{u}|^2 + a(1+\xi^2)\,\xi^2|\hat{z}|^2\big\} \\
&\le C_\epsilon(1+\xi^2)\,\xi^2|\hat{\theta}|^2 + C\big\{|\xi|^3|\hat{v}||\hat{\theta}| + (1+\xi^2)|\xi|^3|\hat{y}||\hat{\theta}|\big\} \\
&\le C_{\epsilon,\alpha_1}(1+\xi^2)^2\,\xi^2|\hat{\theta}|^2 + \alpha_1\varepsilon\Big(\frac{\xi^4}{1+\xi^2}|\hat{v}|^2 + a\xi^4|\hat{y}|^2\Big)
\end{aligned} \tag{6.4.22}$$

for any $\varepsilon \in (0,1)$ and $\alpha_1 > 0$, where C_ε and C_{ε,α_1} are constants depending on ε and (ε, α_1), respectively.

Step 5. We construct the Lyapunov function for $a = 1$. When $a = 1$, letting $\alpha_1 > 0$, we multiply (6.4.5) and (6.4.21) by $\alpha_1\xi^2$ and $1+\xi^2$, respectively and add the resultant inequalities to get

$$\begin{aligned}
&\alpha_1\xi^3(E_1 + E_2)_t + (1+\xi^2)\,\big\{(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) + (\xi\mathbf{E}_6 + \mathbf{E}_7)\big\}_t \\
&\qquad + \alpha_1(1-2\varepsilon)\,\xi^4(|\hat{v}|^2 + |\hat{y}|^2) + b(1-\varepsilon-\alpha_1 C_\varepsilon)(1+\xi^2)\,\xi^2(|\hat{u}|^2 + |\hat{z}|^2) \\
&\le\ C_{\varepsilon,\alpha_1}(1+\xi^2)^2\,\xi^2|\hat{\theta}|^2.
\end{aligned} \tag{6.4.23}$$

Then, letting $\alpha_2 > 0$, we multiply (6.4.23) by $\alpha_2/(1+\xi^2)^2$ and add the resultant inequality to (6.4.2). This yields

$$\begin{aligned}
&\frac{1}{2}\,(|\hat{U}|^2)_t + \alpha_2\alpha_1\frac{\xi^3}{(1+\xi^2)^2}\,(E_1 + E_2)_t \\
&\qquad + \alpha_2\frac{1}{1+\xi^2}\,\big\{(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) + (\xi\mathbf{E}_6 + \mathbf{E}_7)\big\}_t \\
&+ \alpha_2\alpha_1(1-2\varepsilon)\,\frac{\xi^4}{(1+\xi^2)^2}\,(|\hat{v}|^2 + |\hat{y}|^2) \\
&+ \alpha_2 b(1-\varepsilon-\alpha_1 C_\varepsilon)\,\frac{\xi^2}{1+\xi^2}\,(|\hat{u}|^2 + |\hat{z}|^2) + (\kappa - \alpha_2 C_{\varepsilon,\alpha_1})\,\xi^2|\hat{\theta}|^2\ \le\ 0.
\end{aligned} \tag{6.4.24}$$

Here we take $\varepsilon > 0$ as $1 - 2\varepsilon > 0$. For this choice of ε, we choose $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $1 - \varepsilon - \alpha_1 C_\varepsilon > 0$ and $\kappa - \alpha_2 C_{\varepsilon,\alpha_1} > 0$. Now we define our Lyapunov function E by

$$\begin{aligned} E = \frac{1}{2}|\hat{U}|^2 &+ \alpha_2\alpha_1 \frac{\xi^3}{(1+\xi^2)^2}(E_1 + E_2) \\ &+ \alpha_2 \frac{1}{1+\xi^2}\left\{(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) + (\xi\mathbf{E}_6 + \mathbf{E}_7)\right\}, \end{aligned} \tag{6.4.25}$$

which is equivalent to $|\hat{U}|^2$ for suitably small $\alpha_1, \alpha_2 > 0$. Then (**??**) becomes the desired differential inequality (6.2.11) with the dissipative term F given in (6.2.12). This completes the proof of Proposition 6.2.3 for $a = 1$.

Step 6. We construct the Lyapunov function for $a \neq 1$. When $a \neq 1$, letting $\alpha_1 > 0$, we multiply (6.4.6) and (6.4.22) by $\alpha_1\xi^2$ and $1 + \xi^2$, respectively, and add the resultant inequalities to get

$$\begin{aligned} &\alpha_1\xi^3\left\{E_1 + (1+\xi^2)E_2\right\}_t + (1+\xi^2)\big\{(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) \\ &\quad + (1+\xi^2)(\xi\mathbf{E}_6 + \mathbf{E}_7)\big\}_t + \alpha_1(1-2\varepsilon)\left\{\xi^4|\hat{v}|^2 + a(1+\xi^2)\,\xi^4|\hat{y}|^2\right\} \\ &\qquad + b(1-\varepsilon-\alpha_1 C_\varepsilon)\left\{(1+\xi^2)\,\xi^2|\hat{u}|^2 + a(1+\xi^2)^2\,\xi^2|\hat{z}|^2\right\} \\ &\le\ C_{\varepsilon,\alpha_1}(1+\xi^2)^3\,\xi^2|\hat{\theta}|^2. \end{aligned} \tag{6.4.26}$$

Then, letting $\alpha_2 > 0$, we multiply (6.4.26) by $\alpha_2/(1+\xi^2)^3$ and add the resultant inequality to (6.4.2) to get

$$\begin{aligned} &\frac{1}{2}(|\hat{U}|^2)_t + \alpha_2\alpha_1\Big\{\frac{\xi^3}{(1+\xi^2)^3}E_1 + \frac{\xi^3}{(1+\xi^2)^2}E_2\Big\}_t \\ &\qquad + \alpha_2\Big\{\frac{1}{(1+\xi^2)^2}(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) + \frac{1}{1+\xi^2}(\xi\mathbf{E}_6 + \mathbf{E}_7)\Big\}_t \\ &\quad + \alpha_2\alpha_1(1-2\varepsilon)\left\{\frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2 + a\frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2\right\} \\ &\quad + \alpha_2 b(1-\varepsilon-\alpha_1 C_\varepsilon)\left\{\frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + a\frac{\xi^2}{1+\xi^2}|\hat{z}|^2\right\} \\ &\quad + (\kappa - \alpha_2 C_{\varepsilon,\alpha_1})\,\xi^2|\hat{\theta}|^2\ \le\ 0. \end{aligned} \tag{6.4.27}$$

Here we choose $\varepsilon \in (0,1)$ and $\alpha_1, \alpha_2 > 0$ just in the same way as in the case of $a = 1$, and then define the Lyapunov function E by

$$\begin{aligned} E = \frac{1}{2}|\hat{U}|^2 &+ \alpha_2\alpha_1\Big\{\frac{\xi^3}{(1+\xi^2)^3}E_1 + \frac{\xi^3}{(1+\xi^2)^2}E_2\Big\} \\ &+ \alpha_2\Big\{\frac{1}{(1+\xi^2)^2}(\xi^2\mathbf{E}_3 + \xi\mathbf{E}_4 + E_5) + \frac{1}{1+\xi^2}(\xi\mathbf{E}_6 + \mathbf{E}_7)\Big\}. \end{aligned} \tag{6.4.28}$$

Then (6.4.27) becomes the differential inequality (6.2.11) with the dissipative term F given in (6.2.13). Thus the proof of Proposition 6.2.3 is complete for $a \neq 1$. $\square$

Chapter 7

Timoshenko-Cattaneo system

7.1 Introduction

In Chapter 7 we study the decay property of the Timoshenko system with the heat conduction described by the Cattaneo low. The system is written in the form

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\theta_x = 0, \\ \theta_t + \tilde{q}_x + b\psi_{tx} = 0, \\ \tau_0\tilde{q}_t + \tilde{q} + \kappa\theta_x = 0, \end{cases} \tag{7.1.1}$$

where a, b, κ and τ_0 are positive constants; we regard τ_0 as a parameter of our system (7.1.1) satisfying $\tau_0 \in (0, 1]$. The original Timoshenko system, which consists of the first two equations in (7.1.1) with $b = 0$, was first introduced by S.P. Timoshenko ([62, 63]) and describes the vibration of the beam called the Timoshenko beam, while the last two equations in (7.1.1) with $b = 0$ represent the heat conduction described by the Cattaneo law. Here in (7.1.1), $t \geq 0$ is the time variable, $x \in \mathbb{R}$ is the spacial variable which denotes the point on the center line of the beam, and φ, ψ, θ and $\tilde{q}$ are the unknown functions of $t \geq 0$ and $x \in \mathbb{R}$, which denote the transversal displacement, the rotation angle of the beam, the temperature and the heat flow, respectively.

We put $\tau_0 = 0$ formally in (7.1.1). Then we have the Fourier law $\tilde{q} = -\kappa\theta_x$ from the last equation in (7.1.1). This together with the other three equations in (7.1.1) yields the Timoshenko-Fourier system

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\theta_x = 0, \\ \theta_t + b\psi_{tx} = \kappa\theta_{xx}. \end{cases} \tag{7.1.2}$$

Namely, when we put $\tau_0 = 0$ formally, our Timoshenko-Cattaneo system (7.1.1) is reduced to the Timoshenko-Fourier system (7.1.2).

The decay property of the Timoshenko-Cattaneo system (7.1.1) in a bounded region $0 < x < 1$ was studied in [56, 54]. It was shown in [56] that the energy of the solution does not decay exponentially as $t \to \infty$ if $a = 1$. More detailed deccay property of (7.1.1) was investigated in [54] by introducing the stability number

$$P := \frac{\tau_0}{\kappa}(1 - a^2 - b^2) + (a^2 - 1). \tag{7.1.3}$$

It was proved in [54] that the energy of the system decays exponentially as $t \to \infty$ if the stability number satisfies $P = 0$, while in the case $P \neq 0$, the energy decays polynomially as $t \to \infty$. We note that the decay result in [56] for $a = 1$ corresponds to that in [54] for $P \neq 0$.

To explain this decay property, we investigate the dissipative structure of the Timoshenko-Cattaneo system (7.1.1) in the whole space. We will show in Subsection 7.2.2 that the dissipative structure of the system (7.1.1) can be characterized by the property

$$\begin{aligned} &\operatorname{Re}\lambda(i\xi) \leq -c\rho_1(\xi) \qquad \text{for} \quad P = 0, \\ &\operatorname{Re}\lambda(i\xi) \leq -c\rho_2(\xi) \qquad \text{for} \quad P \neq 0. \end{aligned} \tag{7.1.4}$$

Here $\lambda(i\xi)$ denotes the eigenvalues of the system (7.1.1) in the Fourier space, the exponents $\rho_1(\xi)$ and $\rho_2(\xi)$ are given by $\rho_1(\xi) = \xi^4/(1 + \xi^2)^2$ and $\rho_2(\xi) = \xi^4/(1 + \xi^2)^3$, respectively, and c is a positive constant independent of the parameter $\tau_0 \in (0, 1]$. We note that the dissipative structure (7.1.4) for $P \neq 0$ is very weak in the high frequency region and satisfies $\operatorname{Re}\lambda(i\xi) \sim -c\xi^{-2}$ for $|\xi| \to \infty$.

The above dissipative structure (7.1.4) of the Timoshenko-Cattaneo system (7.1.1) is very similar to that of the Timoshenko-Fourier system (7.1.2). In fact, it was observed in Chapter 6 (see also [38, 49]) that the dissipative structute of the Timoshenko-Fourier system (7.1.2) is characterized by

$$\begin{aligned} &\operatorname{Re}\lambda(i\xi) \leq -c\rho_1(\xi) \qquad \text{for} \quad a = 1, \\ &\operatorname{Re}\lambda(i\xi) \leq -c\rho_2(\xi) \qquad \text{for} \quad a \neq 1, \end{aligned} \tag{7.1.5}$$

where $\lambda(i\xi)$ is the eigenvalues of the system (7.1.2), $\rho_1(\xi)$ and $\rho_2(\xi)$ are the same as in (7.1.4), and c is a positive constant. We note that the dissipative structure (7.1.5) of the Timoshenko-Fourier system (7.1.2) is formally obtained from (7.1.4) of the Timoshenko-Cattaneo system (7.1.1) by putting $\tau_0 = 0$ because the stability number P becomes $P = a^2 - 1$ for $\tau_0 = 0$.

A similar dissipative structure was first found in [24] (see also [50]) for the dissipative Timoshenko system

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \gamma\psi_t = 0, \end{cases} \tag{7.1.6}$$

where a and γ are positive constants. It was shown in [24] that

$$\begin{aligned} &\operatorname{Re}\lambda(i\xi) \leq -c\eta_1(\xi) \qquad \text{for} \quad a = 1, \\ &\operatorname{Re}\lambda(i\xi) \leq -c\eta_2(\xi) \qquad \text{for} \quad a \neq 1, \end{aligned} \tag{7.1.7}$$

where $\lambda(i\xi)$ denotes the eigenvalues of the system (7.1.6), the exponents $\eta_1(\xi)$ and $\eta_2(\xi)$ are given respectively by $\eta_1(\xi) = \xi^2/(1+\xi^2)$ and $\eta_2(\xi) = \xi^2/(1+\xi^2)^2$, and c is a positive constant. We note that this dissipative structure (7.1.7) for $a = 1$ is the same as that in the general theory developed in [67, 58].

It is interesting to compare the above dissipative structures (7.1.4) (or (7.1.5)) and (7.1.7). In the low frequency region $|\xi| \to 0$, we see that $\mathrm{Re}\,\lambda(i\xi) \sim -c\xi^4$ in (7.1.4) (or (7.1.5)), while we have $\mathrm{Re}\,\lambda(i\xi) \sim -c\xi^2$ in (7.1.7). On the other hand, in the high frequency region $|\xi| \to \infty$, we have the common dissipative structure. In fact, we have $\mathrm{Re}\,\lambda(i\xi) \sim -c$ both in (7.1.4) for $P = 0$ (or (7.1.5) for $a = 1$) and (7.1.7) for $a = 1$. Also, we see that $\mathrm{Re}\,\lambda(i\xi) \sim -c\xi^{-2}$ both in (7.1.4) for $P \neq 0$ (or (7.1.5) for $a \neq 1$) and (7.1.7) for $a \neq 1$. The Timoshenko system with Cattaneo law and frictional damping can be regarded as the symmetric hyperbolic system with non-symmetric relaxation term. The regularity-loss structure for the symmetric hyperbolic systems which contain the Timoshenko system with frictional damping is studied by [64]. However, the regularity-loss structure for the symmetric hyperbolic systems which contain the Timoshenko system with Cattaneo law remains an open question.

The above dissipative structure $\mathrm{Re}\,\lambda(i\xi) \sim -c\xi^{-2}$ in the high frequency region $|\xi| \to \infty$ is very weak and causes the regularity-loss in the decay estimate. In fact, we show that the solution to the Timoshenko-Cattaneo system (7.1.1) in the whole space satisfies the following decay estimates: Put $V = (\varphi_x - \psi,\, \varphi_t,\, \psi_x,\, \psi_t,\, \theta,\, \sqrt{\tau_0}\,\tilde{q})$ and write $|V|^2 = |(\varphi_x - \psi,\, \varphi_t,\, \psi_x,\, \psi_t,\, \theta)|^2 + \tau_0|\tilde{q}|^2$. Then we have

$$\|\partial_x^k V(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|V_0\|_{L^p} + Ce^{-ct}\|\partial_x^k V_0\|_{L^2} \tag{7.1.8}$$

for $P = 0$, and

$$\|\partial_x^k V(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|V_0\|_{L^p} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} V_0\|_{L^2} \tag{7.1.9}$$

for $P \neq 0$. Here V_0 denotes the initial data corresponding to V, $1 \le p \le 2$, k and l are nonnegative integers, and C and c are positive constants independent of $\tau_0 \in (0, 1]$. We note that when $P \neq 0$, we have the decay rate $(1+t)^{-\frac{l}{2}}$ only by assuming the additional l-th order regularity on the initial data. Therefore the decay estimate (7.1.9) for $P \neq 0$ is of the regularity-loss type. For a similar decay estimate of the regularity-loss type for the system (7.1.6), we refer the readers to [24].

The key to the proof of the above decay estimate is to show the corresponding pointwise estimate of the solution in the Fourier space. Our pointwise estimate seems optimal and can be derived by the energy method in the Fourier space. Our energy method for the Timoshenko-Cattaneo system (7.1.1) is a generalized version of the previous one employed in Chapter 6 for the Timoshenko-Fourier system (7.1.2). In fact, the previous energy method in Chapter 6 can be simply obtained from the present energy method by putting $\tau_0 = 0$ formally.

Finally in Section 7.3, we study the Timoshenko system with the thermal effect of memory-type (cf. [11]):

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\theta_x = 0, \\ \theta_t + b\psi_{tx} - \kappa \int_0^t g(t-\tau)\,\theta_{xx}(\tau)\,d\tau = 0, \end{cases} \tag{7.1.10}$$

where a, b and κ are positive constants, and $g(t)$ is an exponentially decaying function satisfying $g(t) > 0$ and $\int_0^\infty g(t)\,dt = 1$. We restrict to the simplest case where

$$g(t) = \frac{1}{\tau_0} e^{-\frac{t}{\tau_0}}$$

with τ_0 being a positive constant. In this special case, we will observe that the system (7.1.10) can be reduced to the Timoshenko-Cattaneo system (7.1.1) and therefore verifies the dissipative structure (7.1.4) and the decay estimates (7.1.8) and (7.1.9). We expect that the same conclusion holds true even for a more general memory kernel $g(t)$ but it remains an open question.

There are many other works on the Timoshenko system with dissipation. We refer to [47, 48, 57] for frictional damping case, [11, 55, 54, 56] for thermal dissipation case, and [4, 3, 33, 34] for memory-type dissipation case. The decay property of the regularity-loss type which is similar to (7.1.9) is known also for other interesting model systems. We refer to [23] for a hyperbolic-elliptic system of radiating gas, [60] for a plate equation with rotational inertia effect, [15, 65] for the compressible Euler-Maxwell system, and [16] for the Vlasov-Maxwell-Boltzmann system.

7.2 Timoshenko-Cattaneo system

7.2.1 Main results

We consider the Timoshenko-Cattaneo system (7.1.1) with the initial data

$$(\varphi,\, \varphi_t,\, \psi,\, \psi_t,\, \theta,\, \tilde{q})(x,0) = (\varphi_0,\, \varphi_1,\, \psi_0,\, \psi_1,\, \theta_0,\, \tilde{q}_0)(x).$$

We introduce the quantities $v = \varphi_x - \psi$, $u = \varphi_t$, $z = a\psi_x$, $y = \psi_t$ and $q = \frac{1}{\sqrt{\kappa}}\tilde{q}$, and rewrite the system (7.1.1) in the form of the first order system

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - az_x + b\theta_x - v = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ \theta_t + by_x + \sqrt{\kappa} q_x = 0, \\ \tau_0 q_t + \sqrt{\kappa}\theta_x + q = 0. \end{cases} \tag{7.2.1}$$

The corresponding initial data are given by

$$(v,\, y,\, u,\, z,\, \theta, q)(x,0) = (v_0,\, y_0,\, u_0,\, z_0,\, \theta_0, q_0)(x), \tag{7.2.2}$$

where $v_0 = \varphi_{0,x} - \psi_0$, $y_0 = \psi_1$, $u_0 = \varphi_1$, $z_0 = a\psi_{0,x}$ and $q_0 = \frac{1}{\sqrt{\kappa}}\,\tilde{q}_0$. We rewrite this initial value problem (7.2.1), (7.2.2) for the Timoshenko-Cattaneo system by using vector notations. Put $U := (v, y, u, z, \theta, q)^T$ and $U_0 := (v_0, y_0, u_0, z_0, \theta_0, q_0)^T$. Then we have

$$A^0 U_t + AU_x + LU = 0, \qquad U(x,0) = U_0(x), \tag{7.2.3}$$

where the coefficient matrices are given by

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & b & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & \sqrt{\kappa} \\ 0 & 0 & 0 & 0 & \sqrt{\kappa} & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that A^0 is real symmetric (diagonal) and positive definite, A is real symmetric, and L is nonnegative definite (but is not real symmetric). Therefore our Timoshenko-Cattaneo system in (7.2.3) is regarded as the first order symmetric hyperbolic system with non-symmetric relaxation. Unfortunately, we conclude that the general theory on the dissipative structure developed in [67, 58] is not applicable to our Timoshenko-Cattaneo system, because the relaxation matrix L is not symmetric such that $\ker L \neq \ker L_1$, where L_1 denotes the symmetric part of L.

We take the Fourier transform of (7.2.3) to obtain

$$A^0 \hat{U}_t + (i\xi A + L)\hat{U} = 0, \qquad \hat{U}(\xi,0) = \hat{U}_0(\xi), \tag{7.2.4}$$

where $\xi \in \mathbb{R}$ is the Fourier variable. The eigenvalue problem associated with (7.2.4) is

$$\lambda A^0 \phi + (i\xi A + L)\phi = 0, \tag{7.2.5}$$

where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^6$. Namely, the eigenvalue of the problem (7.2.4) is the solution $\lambda = \lambda(i\xi) \in \mathbb{C}$ to the characteristic equation

$$\det\{\lambda A^0 + (i\xi A + L)\} = 0. \tag{7.2.6}$$

Now we state the result on the decay estimate of the solution to the problem (7.2.3) for the Timoshenko-Cattaneo system. To this end, for the given solution $U = (v, y, u, z, \theta, q)^T$ to the problem (7.2.3), we define the vector function V by $V = (v, y, u, z, \theta, \sqrt{\tau_0}\, q)^T$ and write $|V|^2 = |(v, y, u, z, \theta)|^2 + \tau_0 |q|^2$, so that

$$\|V\|_{L^2}^2 = \|(v, y, u, z, \theta)\|_{L^2}^2 + \tau_0 \|q\|_{L^2}^2.$$

With this notation, our decay result can be stated as follows.

Theorem 7.2.1. *Let V be the above function defined from the solution U to the problem (7.2.3). Then V satisfies the following decay estimates for $t \geq 0$:*

$$\|\partial_x^k V(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}} \|V_0\|_{L^p} + Ce^{-ct} \|\partial_x^k V_0\|_{L^2} \tag{7.2.7}$$

for $P = 0$, and

$$\|\partial_x^k V(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}} \|V_0\|_{L^p} + C(1+t)^{-\frac{l}{2}} \|\partial_x^{k+l} V_0\|_{L^2} \tag{7.2.8}$$

for $P \neq 0$. Here V_0 is the initial data for V, $1 \leq p \leq 2$, k and l are nonnegative integers, and C and c are positive constants independent of $\tau_0 \in (0, 1]$.

Remark. The above decay estimates (7.2.7) and (7.2.8) are the same as those for the Timoshenko-Fourier system (7.1.2) obtained in Chapter 6 (also see [38]), although the stability number is different.

We remark that the decay estimate (7.2.8) for $P \neq 0$ is of the regularity-loss type because we can get the decay rate $(1+t)^{-l/2}$ only by assuming the additional l-th order regularity on the initial data.

To prove the above decay estimates (7.2.7) and (7.2.8), we need to show the following pointwise estimates of the solution in the Fourier space.

Lemma 7.2.2. *Let $\hat{V}$ be the function corresponding to the solution $\hat{U}$ to the problem* (7.2.4) *in the Fourier space. Then $\hat{V}$ satisfies the following pointwise estimates for any $\xi \in \mathbb{R}$ and $t \geq 0$:*

$$|\hat{V}(\xi, t)| \leq Ce^{-c\rho_1(\xi)t} |\hat{V}_0(\xi)| \qquad \text{for} \quad P = 0, \tag{7.2.9}$$

$$|\hat{V}(\xi, t)| \leq Ce^{-c\rho_2(\xi)t} |\hat{V}_0(\xi)| \qquad \text{for} \quad P \neq 0, \tag{7.2.10}$$

where $\hat{V}_0$ is the initial data for $\hat{V}$, $\rho_1(\xi) = \frac{\xi^4}{(1+\xi^2)^2}$ and $\rho_2(\xi) = \frac{\xi^4}{(1+\xi^2)^3}$, and C and c are positive constants independent of $\tau_0 \in (0, 1]$.

These pointwise estimates are also the same as those for the Timoshenko-Fourier system (7.1.2) obtained in Chapter 6 (also see [38]). Once these pointwise estimates in Lemma 7.2.2 are shown, the decay estimates (7.2.7) and (7.2.8) can be proved just in the same way as Chapter 6 and therefore we omit the proof of Theorem 7.2.1. The pointwise estimates in

Lemma 7.2.2 will be proved in Subsection 7.2.3 and Subsection 7.2.4 by applying the energy method in the Fourier space.

Finally in this subsection, we state the result on the energy estimates of the solution to the problem (7.2.3), which will be proved at the end of Subsection 7.2.4.

Proposition 7.2.3. *Let $V=(v,y,u,z,\theta,\sqrt{\tau_0}\,q)$ be the function corresponding to the solution U to the problem (7.2.3). Let s and k be integers with $0\le k\le s$. Then V satisfies the following energy estimates for any $t\ge 0$:*

$$\begin{aligned}\|\partial_x^k V(t)\|^2_{H^{s-k}} + \int_0^t \big(\, \|\partial_x^{k+2}(v,y)(\tau)\|^2_{H^{s-k-2}} \\ + \|\partial_x^{k+1}(u,z,\theta)(\tau)\|^2_{H^{s-k-1}} + \|\partial_x^k q(\tau)\|^2_{H^{s-k}}\,\big)\, d\tau \le C\|\partial_x^k V_0\|^2_{H^{s-k}}\end{aligned} \tag{7.2.11}$$

for $P=0$, and

$$\begin{aligned}&\|\partial_x^k V(t)\|^2_{H^{s-k}} + \int_0^t \big(\, \|\partial_x^{k+2} v(\tau)\|^2_{H^{s-k-3}} + \|\partial_x^{k+2} y(\tau)\|^2_{H^{s-k-2}} \\ &\qquad + \|\partial_x^{k+1} u(\tau)\|^2_{H^{s-k-2}} + \|\partial_x^{k+1}(z,\theta)(\tau)\|^2_{H^{s-k-1}} + \|\partial_x^k q(\tau)\|^2_{H^{s-k}}\,\big)\, d\tau \\ &\le C\|\partial_x^k V_0\|^2_{H^{s-k}}\end{aligned} \tag{7.2.12}$$

for $P\neq 0$, where C is a positive constant independent of $\tau_0\in(0,1]$.

Remark 1. Here we used the Sobolev space H^m with negative m, which is defined as usual, namely, $H^m=\{u\ ;\ \int_{\mathbb{R}}(1+|\xi|^2)^m\,|\hat{u}(\xi)|^2<\infty\}$.

Remark 2. We note that in the dissipative term of (7.2.12) for $P\neq 0$, we have one regularity-loss for two components v and u but there is no regularity-loss for other four components y, z, θ and q. Also, we remark that the energy estimates in Chapter 6 (also in [38]) for the Timoshenko-Fourier system (7.1.2) are formally obtained from (7.2.11) and (7.2.12) by putting $\tau_0=0$.

7.2.2 Asymptotic expansion of eigenvalues

We denote by $\lambda=\lambda_j(i\xi)$, $j=1,2,3,4,5,6$, the eigenvalues of the probelm (7.2.4), which are the solutions to the characterestic equation (7.2.6). We investigate the asymptotic expansion of these eigenvalues for $|\xi|\to 0$ and $|\xi|\to\infty$. It is convenient to introduce the variable $\zeta\in\mathbb{C}$ and consider the eigenvalues in the form $\lambda=\lambda_j(\zeta)$. These $\lambda=\lambda_j(\zeta)$ are the solutions of the characteristic equation

$$\begin{aligned}\det\{\lambda A^0+(\zeta A+L)\} = \tau_0\lambda^6+\lambda^5+\big[\tau_0\left\{1-(1+a^2+b^2)\,\zeta^2\right\}-\kappa\zeta^2\big]\,\lambda^4 \\ +\left\{1-(1+a^2+b^2)\,\zeta^2\right\}\lambda^3 \\ +\big[\tau_0(a^2+b^2)\,\zeta^2-\kappa\left\{1-(a^2+1)\,\zeta^2\right\}\big]\,\zeta^2\lambda^2 \\ +(a^2+b^2)\,\zeta^4\lambda-a^2\kappa\zeta^6\ =\ 0.\end{aligned} \tag{7.2.13}$$

(i) When $|\zeta| \to 0$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) = \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \cdots . \tag{7.2.14}$$

We determine the coefficients $\lambda_j^{(k)}$ in (7.2.14) by direct computations. For the first three eigenvalues, we find that

$$\lambda_j^{(0)} = \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \alpha_j \qquad \text{for} \quad j = 1, 2, 3,$$

where α_j are the solutions of the algebraic equation $f(X) := X^3 - \kappa X^2 + (a^2+b^2)\,X - a^2\kappa = 0$. These α_j are the same as in Chapter 6 and we know that $\operatorname{Re}\alpha_j > 0$ for $j = 1, 2, 3$. This follows from the fact that $f(0)f(\kappa) < 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = \kappa$. On the other hand, for the next two eigenvalues, we find that

$$\lambda_j^{(0)} = \pm i, \quad \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \mp\frac{1}{2}\,Y i, \quad \lambda_j^{(3)} = 0,$$

$$\lambda_j^{(4)} = -\frac{1}{2}\,\frac{b^2\kappa}{1+\tau_0^2} \mp \frac{1}{2}\,Zi \qquad \text{for} \quad j = 4, 5,$$

where $Y := 1+a^2+b^2$ and $Z := \frac{1}{4}\,Y^2 + Y - 1 - \tau_0\frac{b^2\kappa}{1+\tau_0^2}$. We see that $\operatorname{Re}\lambda_j^{(4)} = -\frac{1}{2}\frac{b^2\kappa}{1+\tau_0^2} < 0$ for $j = 4, 5$. We remark that these coefficients are reduced to the corresponding coefficients in Chapter 6 if we put $\tau_0 = 0$ formally. Finally, for the last eigenvalue, we have

$$\lambda_j^{(0)} = -\frac{1}{\tau_0} \qquad \text{for} \quad j = 6.$$

This coefficient is singular for $\tau_0 \to 0$.

Consequently, for each fixed $\tau_0 \in (0, 1]$ and for $|\xi| \to 0$, we have

$$\operatorname{Re}\,\lambda_j(i\xi) = \begin{cases} -(\operatorname{Re}\alpha_j)\,\xi^2 + \mathcal{O}(|\xi|^3) & \text{for} \quad j = 1, 2, 3, \\ -\dfrac{1}{2}\dfrac{b^2\kappa}{1+\tau_0^2}\,\xi^4 + \mathcal{O}(|\xi|^5) & \text{for} \quad j = 4, 5, \\ -\dfrac{1}{\tau_0} + \mathcal{O}(|\xi|) & \text{for} \quad j = 6. \end{cases} \tag{7.2.15}$$

This expansion for $|\xi| \to 0$ shows that we have $\operatorname{Re}\lambda(i\xi) \sim -c\xi^2$ for three eigenvalues, $\operatorname{Re}\lambda(i\xi) \sim -c\xi^4$ for two eigenvalues, and $\operatorname{Re}\lambda(i\xi) \sim -c$ for one eigenvalue.

(ii) To derive the asymptotic expansion of the eigenvalues $\lambda_j(\zeta)$ for $|\zeta| \to \infty$, we consider the characteristic equation in the form

$$\det\{\lambda A^0 + (\zeta A + L)\} = \zeta^6 \det\{\mu A^0 + (A + \zeta^{-1}L)\} = 0,$$

where we put $\mu = \frac{\lambda}{\zeta}$. Let $\mu = \mu_j(\zeta^{-1})$ be the solutions of the modified characteristic equation $\det\{\mu A^0 + (A + \zeta^{-1}L)\} = 0$:

$$\begin{aligned}\det\{\mu A^0 + (A + \zeta^{-1}L)\} &= \tau_0\mu^6 + \zeta^{-1}\mu^5 + \left[\tau_0\left\{\zeta^{-2} - (1 + a^2 + b^2)\right\} - \kappa\right]\mu^4 \\ &\quad + \left\{\zeta^{-2} - (1 + a^2 + b^2)\right\}\zeta^{-1}\mu^3 \\ &\quad + \left[\tau_0(a^2 + b^2) - \kappa\left\{\zeta^{-2} - (a^2 + 1)\right\}\right]\mu^2 \\ &\quad + (a^2 + b^2)\,\zeta^{-1}\mu - a^2\kappa \;=\; 0. \end{aligned} \tag{7.2.16}$$

Then we have the relation $\lambda_j(\zeta) = \zeta\mu_j(\zeta^{-1})$. On the other hand, the asymptotic expansion of $\mu_j(\zeta^{-1})$ for $|\zeta|^{-1} \to 0$ has the form

$$\mu_j(\zeta^{-1}) = \mu_j^{(1)} + \mu_j^{(0)}\zeta^{-1} + \mu_j^{(-1)}\zeta^{-2} + \mu_j^{(-2)}\zeta^{-3} + \cdots . \tag{7.2.17}$$

Consequently, we have the following asymptotic expansion of $\lambda_j(\zeta)$ for $|\zeta| \to \infty$:

$$\lambda_j(\zeta) = \mu_j^{(1)}\zeta + \mu_j^{(0)} + \mu_j^{(-1)}\zeta^{-1} + \mu_j^{(-2)}\zeta^{-2} + \cdots .$$

We determine the coefficients $\mu_j^{(k)}$ in (7.2.17) by direct computations. For the first two eigenvalues, these coefficients are given as follows: When $P = 0$, we have

$$\mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = \beta_j \qquad \text{for} \quad j = 1, 2,$$

and when $P \neq 0$, we have

$$\mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = 0, \quad \mu_j^{(-1)} = \pm\frac{\kappa - \tau_0}{2\kappa P}, \quad \mu_j^{(-2)} = \frac{b^2}{2\kappa P^2} \qquad \text{for} \quad j = 1, 2,$$

where P denotes the stability number given in (7.1.3). Here β_j are the solutions of the algebraic equation $4(\tau_0 Q + R)\,X^2 + 2QX + R = 0$, where $Q := -(1 - a^2 - b^2)$ and $R := \kappa - \tau_0$. We see that

$$\begin{aligned} QR &= -(1 - a^2 - b^2)(\kappa - \tau_0) \\ &= \tau_0(1 - a^2 - b^2) + \kappa(a^2 - 1) + b^2\kappa = b^2\kappa > 0, \end{aligned}$$

where we used $P = 0$ in the last equality. Thus the signs of $\tau_0 Q + R$, Q and R coincide, and we conclude that $\operatorname{Re}\beta_j < 0$ for $j = 1, 2$. We remark that all these coefficients are reduced to the corresponding coefficients in Chapter 6 if we put $\tau_0 = 0$ formally.

On the other hand, for the other four eigenvalues, we have

$$\mu_j^{(1)} = \gamma_j, \quad \mu_j^{(0)} = -\frac{1}{2}\delta_j \qquad \text{for} \quad j = 3, 4, 5, 6,$$

where γ_j are the solutions of the algebraic equation $\tau_0 Y^4 - \{\tau_0(a^2+b^2)+\kappa\}\, Y^2 + a^2\kappa = 0$, and δ_j are given by

$$\delta_j := \frac{\gamma_j^2 - (a^2+b^2)}{\tau_0\{\gamma_j^2 - (a^2+b^2)\} + (\tau_0\gamma_j^2 - \kappa)}.$$

We see easily that $\gamma_j \in \mathbb{R}$ for $j = 3,4,5,6$. Also, we find that

$$\begin{aligned}\{\gamma_j^2 - (a^2+b^2)\}(\tau_0\gamma_j^2 - \kappa) &= \tau_0\gamma_j^4 - \{\tau_0(a^2+b^2)+\kappa\}\gamma_j^2 + (a^2+b^2)\kappa \\ &= b^2\kappa > 0,\end{aligned}$$

which shows that $\delta_j > 0$ for $j = 3,4,5,6$.

Consequently, for each fixed $\tau_0 \in (0,1]$, when $P = 0$, we have

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} \operatorname{Re}\beta_j + \mathcal{O}(|\xi|^{-1}) & \text{for} \quad j = 1,2, \\ -\dfrac{1}{2}\,\delta_j + \mathcal{O}(|\xi|^{-1}) & \text{for} \quad j = 3,4,5,6 \end{cases} \tag{7.2.18}$$

for $|\xi| \to \infty$, while in the case $P \neq 0$, we have

$$\operatorname{Re}\,\lambda_j(i\xi) = \begin{cases} -\dfrac{b^2}{2\kappa P^2}\,\xi^{-2} + \mathcal{O}(|\xi|^{-3}) & \text{for} \quad j = 1,2, \\ -\dfrac{1}{2}\,\delta_j + \mathcal{O}(|\xi|^{-1}) & \text{for} \quad j = 3,4,5,6 \end{cases} \tag{7.2.19}$$

for $|\xi| \to \infty$. When $P = 0$, we see from (7.2.18) that all the eigenvalues are of the standard type and satisfy $\operatorname{Re}\lambda(i\xi) \sim -c$ for $|\xi| \to \infty$. On the other hand, when $P \neq 0$, we know from (7.2.19) that four eigenvalues satisfy $\operatorname{Re}\lambda(i\xi) \sim -c$ for $|\xi| \to \infty$, while the other two are not of the standard type and satisfy $\operatorname{Re}\lambda(i\xi) \sim -c\xi^{-2}$ for $|\xi| \to \infty$.

We conclude from the asymptotic expansions (7.2.15) for $|\xi| \to 0$ and (7.2.18), (7.2.19) for $|\xi| \to \infty$ that the energy inequalities (7.2.11) and (7.2.12) in Proposition 7.2.3 completely match with the eigenvalues of our problem (7.2.4). Also, we remark that these asymptotic expansions suggest the optimality of the pointwise estimates (7.2.9) and (7.2.10) in Lemma 7.2.2.

7.2.3 Pointwise estimates in Fourier spaces

The main purpose of this subsection is to prove Lemma 7.2.2 on the pointwise estimates in the Fourier space. This will be done by employing the energy method in the Fourier space. Here in this subsection, we only give a short outline of that energy method and the detailed computations will be given in the next subsection,

In our energy method, we construct a Lyapunov function E for the problem (7.2.4) in the Fourier space. Our Lyapunov function E is equivalent to $|\hat{V}|^2$ in the sense that

$$c|\hat{V}|^2 \le E \le C|\hat{V}|^2 \tag{7.2.20}$$

with positive constants c and C not depneding on $\tau_0 \in (0,1]$, and satisfies the differential inequality

$$E_t + cF \leq 0, \tag{7.2.21}$$

where $\hat{V}$ is given in Lemma 7.2.2, F is the corresponding dissipative term, and c is a positive constant independent of $\tau_0 \in (0,1]$.

After the detailed computations in the next subsection, we find that our Lyapunov function E and the corresponding dissipative term F can be given as follows: When $P = 0$, we have

$$\begin{aligned} E = \frac{1}{2}|\hat{V}|^2 + \tau_0\alpha_3\frac{\xi}{1+\xi^2}E_0 + \alpha_3\alpha_2\alpha_1\frac{\xi^3}{(1+\xi^2)^2}(E_1+E_2) \\ + \alpha_3\alpha_2\frac{1}{1+\xi^2}(H_1 + \tau_0\xi^2\tilde{E}_7 + H_2), \end{aligned} \tag{7.2.22}$$

$$F = \frac{\xi^4}{(1+\xi^2)^2}(|\hat{v}|^2 + |\hat{y}|^2) + \frac{\xi^2}{1+\xi^2}(|\hat{u}|^2 + |\hat{z}|^2 + |\hat{\theta}|^2) + |\hat{q}|^2, \tag{7.2.23}$$

and when $P \neq 0$, we have

$$\begin{aligned} E = \frac{1}{2}|\hat{V}|^2 + \tau_0\alpha_3\frac{\xi}{1+\xi^2}E_0 + \alpha_3\alpha_2\alpha_1\frac{\xi^3}{(1+\xi^2)^3}\big\{E_1 + (1+\xi^2)E_2\big\} \\ + \alpha_3\alpha_2\frac{1}{(1+\xi^2)^2}\big\{(H_1 + \tau_0\xi^2\tilde{E}_7) + (1+\xi^2)H_2\big\}, \end{aligned} \tag{7.2.24}$$

$$\begin{aligned} F = \frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2 + \frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 \\ + \frac{\xi^2}{1+\xi^2}(|\hat{z}|^2 + |\hat{\theta}|^2) + |\hat{q}|^2. \end{aligned} \tag{7.2.25}$$

Here α_1, α_2 and α_3 are suitably small positive constants independent of $\tau_0 \in (0,1]$, and the artificial energies E_0, E_1, E_2, $\tilde{E}_7$, H_1 and H_2 are specified in the next subsection.

More presicely, we can prove the following result.

Proposition 7.2.4. *Let* $P = 0$ (*resp.* $P \neq 0$). *Then, for suitably small positive constants* α_1, α_2 *and* α_3 *independent of* $\tau_0 \in (0,1]$, *the Lyapunov function* E *in* (7.2.22) (*resp.* (7.2.24)) *is equivalent to* $|\hat{V}|^2$ *in the sense of* (7.2.20) *and satisfies the differential inequality* (7.2.21) *with the dissipative term* F *in* (7.2.23) (*resp.* (7.2.25)), *where* c *is a positive constant independent of* $\tau_0 \in (0,1]$.

Remark. If we put $\tau_0 = 0$ formally, then our Lyapunov function E and the dissipative term F in Proposition 7.2.4 are respectively reduced to the corresponding E and F in Chapter 6 for the Timoshenko-Fourier system (7.1.2).

Proof of Lemma 7.2.2. The lemma easily follows from Proposition 7.2.4. Here we only give the proof for $P \neq 0$. We recall that $\hat{V} = (\hat{v}, \hat{y}, \hat{u}, \hat{z}, \hat{\theta}, \sqrt{\tau_0}\,\hat{q})$ and $\hat{U} = (\hat{v}, \hat{y}, \hat{u}, \hat{z}, \hat{\theta}, \hat{q})$. Since E is equivalent to $|\hat{V}|^2$ in the sense of (7.2.20), we find that F in (7.2.25) satisfies

$$F \geq c\rho_2(\xi)|\hat{U}|^2 \geq c\rho_2(\xi)|\hat{V}|^2 \geq c\rho_2(\xi)E,$$

where c is a positive constant independent of $\tau_0 \in (0, 1]$ and $\rho_2(\xi) = \xi^4/(1+\xi^2)^3$. Substituting this inequality into (7.2.21), we have $E_t + c\rho_2(\xi)E \leq 0$. This differential inequality can be solved as $E(\xi, t) \leq e^{-c\rho_2(\xi)t}E(\xi, 0)$, which combined with (7.2.20) gives the desired pointwise estimate (7.2.10). This completes the proof of Lemma 7.2.2. □

Finally in this subsection, we give the proof of Proposition 7.2.3 on the energy estimates.

Proof of Proposition 7.2.3. We prove the proposition only for $P \neq 0$. We integrate the differential inequality (7.2.21) with respect to t over $(0, t)$. Then we multiply the resulting inequality by $(1 + \xi^2)^{s-k}\xi^{2k}$ and integrate with respect to $\xi \in \mathbb{R}$, where $0 \leq k \leq s$. Since E is equivalent to $|\hat{V}|^2$ in the sense of (7.2.20) and F is given by (7.2.25) for $P \neq 0$, this computation together with the Plancherel theorem yields the desired energy estimate (7.2.12) for $P \neq 0$. Thus the proof of Proposition 7.2.3 is complete. □

7.2.4 Energy method in Fourier spaces

In this subsection we prove Proposition 7.2.4 by applying the energy method in the Fourier space. In each case of $P = 0$ and $P \neq 0$, we need to construct a suitable Lyapunov function E for the problem (7.2.4) and prove the differential inequality (7.2.21) with the desired dissipation term F.

First, we note that the explicit form of the Timoshenko-Cattaneo system in (7.2.4) (in the Fourier space) is given as

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{7.2.26a}$$

$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + bi\xi\hat{\theta} = 0, \tag{7.2.26b}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{7.2.26c}$$

$$\hat{z}_t - ai\xi\hat{y} = 0, \tag{7.2.26d}$$

$$\hat{\theta}_t + \sqrt{\kappa}i\xi\hat{q} + bi\xi\hat{y} = 0, \tag{7.2.26e}$$

$$\tau_0\hat{q}_t + \hat{q} + \sqrt{\kappa}i\xi\hat{\theta} = 0. \tag{7.2.26f}$$

We apply the energy method in the Fourier space for this system. Our energy method below is an improved version of the previous one in Chapter 6 (also see [38]) for the Timoshenko-Fourier system and is divided into seven steps.

Step 1. (*Basic energy and estimate for* $\hat{q}$):
We multiply the equations (7.2.26a), (7.2.26b), (7.2.26c), (7.2.26d), (7.2.26e) and (7.2.26f)

by $\bar{\hat{v}}$, $\bar{\hat{y}}$, $\bar{\hat{u}}$, $\bar{\hat{z}}$, $\bar{\hat{\theta}}$ and $\bar{\hat{q}}$, respectively. Then, adding the resultant equations and taking the real part, we get

$$\frac{1}{2}\,(|\hat{V}|^2)_t + |\hat{q}|^2 = 0, \tag{7.2.27}$$

where we recall that $|\hat{V}|^2 = |(\hat{v},\, \hat{y},\, \hat{u},\, \hat{z},\, \hat{\theta})|^2 + \tau_0|\hat{q}|^2$. This is the basic (physical) energy equality which contains the dissipation $\hat{q}$.

Step 2. (*Estimate for* $\hat{\theta}$):
We create the estimate for the dissipation $\hat{\theta}$. To this end, we multiply (7.2.26e) and (7.2.26f) by $i\xi\tau_0\bar{\hat{q}}$ and $-i\xi\bar{\hat{\theta}}$, respectively, add the resulting equations, and take the real part. This yields

$$\tau_0\xi E_{0,t} + \sqrt{\kappa}\xi^2|\hat{\theta}|^2 = \tau_0\sqrt{\kappa}\xi^2|\hat{q}|^2 + \tau_0 b\xi^2\mathrm{Re}\,(\hat{y}\bar{\hat{q}}) + \xi\,\mathrm{Re}\,(i\bar{\hat{\theta}}\hat{q}),$$

where $E_0 = \mathrm{Re}\,(i\hat{\theta}\bar{\hat{q}})$; ξE_0 is an artificial energy which is not used in Chapter 6. This equality becomes trivial when $\tau_0 = 0$, because we have $\hat{q} = -\sqrt{\kappa}i\xi\hat{\theta}$ for $\tau_0 = 0$. We simply estimate this equality as

$$\tau_0\xi E_{0,t} + \sqrt{\kappa}\xi^2|\hat{\theta}|^2 \le \tau_0\sqrt{\kappa}\xi^2|\hat{q}|^2 + \tau_0 b\xi^2|\hat{y}||\hat{q}| + |\xi||\hat{\theta}||\hat{q}|. \tag{7.2.28}$$

Step 3. (*Estimate for* $(\hat{v}, \hat{y})$):
This step is just the same as in Chapter 6. We multiply (7.2.26c) and (7.2.26a) by $i\xi\bar{\hat{v}}$ and $-i\xi\bar{\hat{u}}$, respectively, and add the resulting equalities. Also, we multiply (7.2.26d) and (7.2.26b) by $i\xi\bar{\hat{y}}$ and $-i\xi\bar{\hat{z}}$, respectively, and add the resulting equalities. Then, taking the real part, we have

$$\xi E_{1,t} + \xi^2|\hat{v}|^2 = \xi^2|\hat{u}|^2 + \xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{y}), \tag{7.2.29}$$

$$\xi E_{2,t} + a\xi^2|\hat{y}|^2 = a\xi^2|\hat{z}|^2 - \xi\,\mathrm{Re}\,(i\hat{v}\bar{\hat{z}}) - b\xi^2\,\mathrm{Re}\,(\bar{\hat{z}}\hat{\theta}), \tag{7.2.30}$$

where $E_1 = \mathrm{Re}\,(i\hat{v}\bar{\hat{u}})$ and $E_2 = \mathrm{Re}\,(i\hat{z}\bar{\hat{y}})$; ξE_1 and ξE_2 are artificial energies.

Step 4. (*Estimate for* $\hat{u}$):
We create the dissipation for $\hat{u}$. First we compute essentially in the same way as in Chapter 6. Namely, we calculate $(7.2.26\mathrm{b})\times(-\bar{\hat{v}}) + (7.2.26\mathrm{a})\times(-\bar{\hat{y}})$, $(7.2.26\mathrm{c})\times(-\bar{\hat{z}}) + (7.2.26\mathrm{d})\times(-\bar{\hat{u}})$, $(7.2.26\mathrm{e})\times(-i\xi\bar{\hat{y}}) + (7.2.26\mathrm{b})\times(i\xi\bar{\hat{\theta}})$, and $(7.2.26\mathrm{c})\times(-\bar{\hat{\theta}}) + (7.2.26\mathrm{e})\times(-\bar{\hat{u}})$, and take the real part. This yields the differential equalities for the artificial energies E_3, E_4, ξE_6, and E_5, where $E_3 = -\mathrm{Re}\,(\hat{v}\bar{\hat{y}})$, $E_4 = -\mathrm{Re}\,(\hat{u}\bar{\hat{z}})$, $E_6 = \mathrm{Re}\,(i\hat{y}\bar{\hat{\theta}})$, and $E_5 = -\mathrm{Re}\,(\hat{u}\bar{\hat{\theta}})$. We combine these four equalities together with $(7.2.29)\times(-1)$ such that it produces the modified energy H_1:

$$H_1 := b\left\{(-\xi E_1) + \xi^2(E_3 + aE_4)\right\} + (\xi E_6 + E_5) + (a^2 - 1)\xi^2 E_5. \tag{7.2.31}$$

Consequently, we obtain

$$\begin{aligned} H_{1,t} + b\xi^2|\hat{u}|^2 &= b\xi^2|\hat{\theta}|^2 - (1 - a^2 - b^2)\,\xi^3\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{\theta}) - a\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}}) \\ &\quad + \sqrt{\kappa}(a^2 - 1)\,\xi^3\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{q}) + \sqrt{\kappa}\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{q}) - \sqrt{\kappa}\xi^2\,\mathrm{Re}\,(\bar{\hat{y}}\hat{q}), \end{aligned} \tag{7.2.32}$$

which is reduced to the equation (6.4.15) in Chapter 6 (or (3.39) in [38]) if we put $\tau_0 = 0$ formally.

Next, we eliminate the term $-(1-a^2-b^2)\,\xi^3\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{\theta})$ on the right hand side of (7.2.32). For this purpose, we multiply (7.2.26a) and (7.2.26f) by $\tau_0\bar{\hat{q}}$ and $\bar{\hat{v}}$, respectively. Then, adding the resultant equalities and taking the real part, we get

$$\tau_0 E_{7,t} + \sqrt{\kappa}\xi\,\mathrm{Re}\,(i\bar{\hat{v}}\hat{\theta}) = \tau_0\xi\,\mathrm{Re}\,(i\hat{u}\bar{\hat{q}}) - \tau_0\mathrm{Re}\,(\hat{y}\bar{\hat{q}}) - \mathrm{Re}\,(\bar{\hat{v}}\hat{q}), \tag{7.2.33}$$

where $E_7 = \mathrm{Re}\,(\hat{v}\bar{\hat{q}})$ is an artificial energy which is not used in Chapter 6. We multiply (7.2.33) by $-\frac{1}{\sqrt{\kappa}}(1-a^2-b^2)\,\xi^2$ and add the result to (7.2.32). This yields

$$\begin{aligned}
&\big(H_1 + \tau_0\xi^2\tilde{E}_7\big)_t + b\xi^2|\hat{u}|^2 \\
&= b\xi^2|\hat{\theta}|^2 - a\xi^2\,\mathrm{Re}\,(\hat{z}\bar{\hat{\theta}}) + \frac{1}{\sqrt{\kappa}}(1-a^2-b^2)\,\xi^2\,\mathrm{Re}\,(\bar{\hat{v}}\hat{q}) \\
&+ \Big\{\frac{\tau_0}{\sqrt{\kappa}}(1-a^2-b^2) - \sqrt{\kappa}\Big\}\xi^2\,\mathrm{Re}\,(\hat{y}\bar{\hat{q}}) + \sqrt{\kappa}P\xi^3\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{q}) + \sqrt{\kappa}\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{q}),
\end{aligned}$$

where $\tilde{E}_7 = -\frac{1}{\sqrt{\kappa}}(1-a^2-b^2)\,E_7$, and P is the stability number defined in (7.1.3). This equality is simply estimated as

$$\begin{aligned}
&\big(H_1 + \tau_0\xi^2\tilde{E}_7\big)_t + b\xi^2|\hat{u}|^2 \le C\xi^2|\hat{\theta}|^2 + C\xi^2|\hat{z}||\hat{\theta}| \\
&\quad + C\xi^2|\hat{v}||\hat{q}| + C(1+\tau_0)\,\xi^2|\hat{y}||\hat{q}| + C|P||\xi|^3|\hat{u}||\hat{q}| + C|\xi||\hat{u}||\hat{q}|,
\end{aligned} \tag{7.2.34}$$

where C is a positive constant independent of $\tau_0 \in (0,1]$.

Step 5. (*Estimate for* $\hat{z}$):
We create the dissipation for $\hat{z}$ by calculating essentially in the same way as in Chapter 6. Namely, we combine the three differential equalities for ξE_6, E_5 and E_4 (which are used in the previous step) together with (7.2.30) $\times\,(-1)$ such that it produces the modified energy H_2:

$$H_2 = b(-\xi E_2) + a(\xi E_6 + E_5) + bE_4. \tag{7.2.35}$$

This yields

$$\begin{aligned}
H_{2,t} + ab\xi^2|\hat{z}|^2 &= ab\xi^2|\hat{\theta}|^2 + (b^2-a^2)\,\xi^2\,\mathrm{Re}\,(\bar{\hat{z}}\hat{\theta}) \\
&\quad - a\sqrt{\kappa}\xi^2\,\mathrm{Re}\,(\bar{\hat{y}}\hat{q}) + a\sqrt{\kappa}\xi\,\mathrm{Re}\,(i\bar{\hat{u}}\hat{q}),
\end{aligned}$$

which can be reduced to the equality (6.4.19) in Chapter 6 (or (3.43) in [38]) if we put $\tau_0 = 0$ formally. This equality is simply estimated as

$$H_{2,t} + ab\xi^2|\hat{z}|^2 \le C\xi^2|\hat{\theta}|^2 + C\xi^2|\hat{z}||\hat{\theta}| + C\xi^2|\hat{y}||\hat{q}| + C|\xi||\hat{u}||\hat{q}|, \tag{7.2.36}$$

where C is a positive constant independent of $\tau_0 \in (0,1]$.

Step 6. (*Lyapunov function for $P = 0$*):
We construct the Lyapunov function E for $P = 0$. Let $P = 0$. First, in order to get a good dissipation for $(\hat{v}, \hat{y})$, we add (7.2.29) and (7.2.30). By applying the Young inequality, we get

$$\begin{aligned} &\xi(E_1 + E_2)_t + (1 - \varepsilon)\,\xi^2(|\hat{v}|^2 + a|\hat{y}|^2) \\ &\qquad \le C_\varepsilon(1 + \xi^2)(|\hat{u}|^2 + a|\hat{z}|^2) + C_\varepsilon \xi^2|\hat{\theta}|^2 \end{aligned} \tag{7.2.37}$$

for any $\varepsilon \in (0, 1)$, where C_ε is a positive constant depending on ε but independent of $\tau_0 \in (0, 1]$. Next, we add (7.2.34) with $P = 0$ and (7.2.36) to get a good dissipation for $(\hat{u}, \hat{z})$. Applying the Young inequality, we obtain

$$\begin{aligned} &\big(H_1 + \tau_0\xi^2\tilde{E}_7 + H_2\big)_t + b\xi^2|\hat{u}|^2 + ab(1 - \varepsilon)\,\xi^2|\hat{z}|^2 \\ &\qquad \le C_\varepsilon\xi^2|\hat{\theta}|^2 + C\xi^2|\hat{v}||\hat{q}| + C(1 + \tau_0)\xi^2|\hat{y}||\hat{q}| + C|\xi||\hat{u}||\hat{q}| \end{aligned} \tag{7.2.38}$$

for any $\varepsilon \in (0, 1)$, where C_ε denotes a constant depending on ε but independent of $\tau_0 \in (0, 1]$.

Now we combine the above two inequalities. Letting $\alpha_1 > 0$, we multiply (7.2.37) and (7.2.38) by $\alpha_1\xi^2$ and $1 + \xi^2$, respectively, and add the resultant inequalities. This yields

$$\begin{aligned} &\big\{\alpha_1\xi^3(E_1 + E_2) + (1 + \xi^2)(H_1 + \tau_0\xi^2\tilde{E}_7 + H_2)\big\}_t \\ &\quad + \alpha_1(1 - \varepsilon)\,\xi^4(|\hat{v}|^2 + a|\hat{y}|^2) + b(1 - \alpha_1 C_\varepsilon)(1 + \xi^2)\,\xi^2|\hat{u}|^2 \\ &\qquad + ab(1 - \varepsilon - \alpha_1 C_\varepsilon)(1 + \xi^2)\,\xi^2|\hat{z}|^2 \\ &\le C_{\varepsilon,\alpha_1}(1 + \xi^2)\,\xi^2|\hat{\theta}|^2 + C(1 + \xi^2)\,\xi^2|\hat{v}||\hat{q}| \\ &\quad + C(1 + \tau_0)(1 + \xi^2)\,\xi^2|\hat{y}||\hat{q}| + C(1 + \xi^2)|\xi||\hat{u}||\hat{q}|. \end{aligned} \tag{7.2.39}$$

Here C_{ε,α_1} denotes a constant depending on (ε, α_1) but independent of $\tau_0 \in (0, 1]$. Also, letting $\alpha_2 > 0$, we multiply (7.2.28) and (7.2.39) by $1 + \xi^2$ and α_2, respectively, and add the resultant inequalities. If we introduce the quantity $\mathcal{E}$ by

$$\begin{aligned} (1 + \xi^2)^2\mathcal{E} &= \tau_0(1 + \xi^2)\,\xi E_0 \\ &\quad + \alpha_2\big\{\alpha_1\xi^3(E_1 + E_2) + (1 + \xi^2)(H_1 + \tau_0\xi^2\tilde{E}_7 + H_2)\big\}, \end{aligned} \tag{7.2.40}$$

then the result is written as follows.

$$\begin{aligned} &(1 + \xi^2)^2\mathcal{E}_t + \alpha_2\alpha_1(1 - \varepsilon)\,\xi^4(|\hat{v}|^2 + a|\hat{y}|^2) + \alpha_2 b\,(1 - \alpha_1 C_\varepsilon)(1 + \xi^2)\,\xi^2|\hat{u}|^2 \\ &\quad + \alpha_2 ab\,(1 - \varepsilon - \alpha_1 C_\varepsilon)(1 + \xi^2)\,\xi^2|\hat{z}|^2 + \sqrt{\kappa}(1 - \alpha_2 C_{\varepsilon,\alpha_1})(1 + \xi^2)\,\xi^2|\hat{\theta}|^2 \\ &\le \tau_0 C(1 + \xi^2)\,\xi^2|\hat{q}|^2 + \alpha_2 C(1 + \xi^2)\,\xi^2|\hat{v}||\hat{q}| + \alpha_2 C(1 + \xi^2)|\xi||\hat{u}||\hat{q}| \\ &\quad + C_{\alpha_2}(1 + \tau_0)(1 + \xi^2)\,\xi^2|\hat{y}||\hat{q}| + (1 + \xi^2)|\xi||\hat{\theta}||\hat{q}|. \end{aligned}$$

Here C_{α_2} is a constant depending on α_2 but independent of $\tau_0 \in (0,1]$. Moreover, estimating the right hand side by using the Young inequality, we obtain

$$\begin{aligned}
&(1+\xi^2)^2\mathcal{E}_t + \alpha_2\alpha_1(1-2\varepsilon)\,\xi^4(|\hat{v}|^2 + a|\hat{y}|^2)\\
&\quad + \alpha_2 b\,(1-\varepsilon-\alpha_1 C_\varepsilon)(1+\xi^2)\,\xi^2(|\hat{u}|^2 + a|\hat{z}|^2)\\
&\quad + \sqrt{\kappa}(1-\varepsilon-\alpha_2 C_{\varepsilon,\alpha_1})(1+\xi^2)\,\xi^2|\hat{\theta}|^2\\
&\le C_{\varepsilon,\alpha_1,\alpha_2}(1+\tau_0)^2(1+\xi^2)^2|\hat{q}|^2,
\end{aligned} \tag{7.2.41}$$

where $C_{\varepsilon,\alpha_1,\alpha_2}$ is a constant depending on $(\varepsilon, \alpha_1, \alpha_2)$ but independent of $\tau_0 \in (0,1]$.

Finally, letting $\alpha_3 > 0$, we multiply (7.2.41) by $\alpha_3/(1+\xi^2)^2$ and add the result to the basic energy equality (7.2.27). This yields

$$\begin{aligned}
&E_t + \alpha_3\alpha_2\alpha_1(1-2\varepsilon)\,\frac{\xi^4}{(1+\xi^2)^2}\,(|\hat{v}|^2 + a|\hat{y}|^2)\\
&\quad + \alpha_3\alpha_2 b\,(1-\varepsilon-\alpha_1 C_\varepsilon)\,\frac{\xi^2}{1+\xi^2}\,(|\hat{u}|^2 + a|\hat{z}|^2)\\
&\quad + \alpha_3\sqrt{\kappa}(1-\varepsilon-\alpha_2 C_{\varepsilon,\alpha_1})\,\frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2\\
&\quad + \big\{1-\alpha_3 C_{\varepsilon,\alpha_1,\alpha_2}(1+\tau_0)^2\big\}|\hat{q}|^2 \;\le\; 0,
\end{aligned} \tag{7.2.42}$$

where we put $E = |\hat{V}|^2/2 + \alpha_3\mathcal{E}$. Since $\mathcal{E}$ is given in (7.2.40), we see that this E is written explicitly in the form of (7.2.22). Also we find that our E contains the component $\hat{q}$ only in the terms $\tau_0 E_0$ and $\tau_0\tilde{E}_7$ with the factor τ_0; see also (7.2.31) and (7.2.35). Thus we conclude that there is a small positive constant α_0 independent of $\tau_0 \in (0,1]$ such that if α_1, α_2, $\alpha_3 \in (0,\alpha_0]$, then our E is equivalent to $|\hat{V}|^2$ in the sense of (7.2.20). This E becomes our desired Lyapunov function for suitably chosen α_1, α_2, $\alpha_3 \in (0,\alpha_0]$. In fact, in (7.2.42), we take $\varepsilon > 0$ such that $1-2\varepsilon > 0$. For this choice of ε, we choose α_1, α_2, $\alpha_3 \in (0,\alpha_0]$ independent of $\tau_0 \in (0,1]$ so small that $1-\varepsilon-\alpha_1 C_\varepsilon > 0$, $1-\varepsilon-\alpha_2 C_{\varepsilon,\alpha_1} > 0$ and $1-4\alpha_3 C_{\varepsilon,\alpha_1,\alpha_2} > 0$. Thus we conclude that our (7.2.42) becomes the desired differential inequality (7.2.21) with the dissipative term F given in (7.2.23). This completes the proof of Proposition 7.2.4 for $P = 0$.

Step 7. (*Lyapunov function for $P \neq 0$*):
We construct the Lyapunov function E for $P \neq 0$. Let $P \neq 0$. In this case, to get a good dissipation for $(\hat{v}, \hat{y})$, we combine (7.2.29) and (7.2.30) such that $(7.2.29) + (7.2.30) \times (1+\xi^2)$. Then, applying the Young inequality, we obtain

$$\begin{aligned}
&\xi\,\big\{E_1 + (1+\xi^2)E_2\big\}_t + (1-\varepsilon)\,\xi^2\,\big\{|\hat{v}|^2 + a(1+\xi^2)|\hat{y}|^2\big\}\\
&\quad \le C_\varepsilon(1+\xi^2)|\hat{u}|^2 + C_\varepsilon(1+\xi^2)^2|\hat{z}|^2 + C_\varepsilon(1+\xi^2)\,\xi^2|\hat{\theta}|^2
\end{aligned} \tag{7.2.43}$$

for any $\varepsilon \in (0,1)$, where C_ε is a positive constant depending on ε but independent of $\tau_0 \in (0,1]$. Next we combine (7.2.34) and (7.2.36) to get a good dissipation for $(\hat{u}, \hat{z})$. We make the combination (7.2.36) $\times (1+\xi^2)+$ (7.2.34). Then, applying the Young inequality, we obtain

$$\big\{H_1 + \tau_0\xi^2\tilde{E}_7 + (1+\xi^2)H_2\big\}_t + b\xi^2|\hat{u}|^2 + ab\,(1-\varepsilon)(1+\xi^2)\,\xi^2|\hat{z}|^2$$
$$\le C_\varepsilon(1+\xi^2)\,\xi^2|\hat{\theta}|^2 + C\xi^2|\hat{v}||\hat{q}| + C(1+\tau_0)(1+\xi^2)\,\xi^2|\hat{y}||\hat{q}|$$
$$+ C|P||\xi|^3|\hat{u}||\hat{q}| + C(1+\xi^2)|\xi||\hat{u}||\hat{q}| \tag{7.2.44}$$

for any $\varepsilon \in (0,1)$, where C_ε is a constant depending on ε but independent of $\tau_0 \in (0,1]$.

Now we combine the above two estimates for $P \neq 0$. Letting $\alpha_1 > 0$, we multiply (7.2.43) and (7.2.44) by $\alpha_1\xi^2$ and $1+\xi^2$, respectively, and add the resultant inequalities to get

$$\big[\alpha_1\xi^3\{E_1 + (1+\xi^2)E_2\} + (1+\xi^2)\{H_1 + \tau_0\xi^2\tilde{E}_7 + (1+\xi^2)H_2\}\big]_t$$
$$+ \alpha_1(1-\varepsilon)\,\xi^4\left\{|\hat{v}|^2 + a(1+\xi^2)|\hat{y}|^2\right\}$$
$$+ b\,(1-\alpha_1C_\varepsilon)(1+\xi^2)\,\xi^2|\hat{u}|^2 + ab\,(1-\varepsilon-\alpha_1C_\varepsilon)(1+\xi^2)^2\,\xi^2|\hat{z}|^2$$
$$\le C_{\varepsilon,\alpha_1}(1+\xi^2)^2\,\xi^2|\hat{\theta}|^2 + C(1+\xi^2)\xi^2|\hat{v}||\hat{q}| + C(1+\tau_0)(1+\xi^2)^2\,\xi^2|\hat{y}||\hat{q}|$$
$$+ C|P|(1+\xi^2)|\xi|^3|\hat{u}||\hat{q}| + C(1+\xi^2)^2|\xi||\hat{u}||\hat{q}|, \tag{7.2.45}$$

where C_{ε,α_1} is a constant depending on (ε,α_1) but independent of $\tau_0 \in (0,1]$. Then, letting $\alpha_2 > 0$, we multiply (7.2.28) and (7.2.45) by $(1+\xi^2)^2$ and α_2, respectively, and add the resultant two inequalities. We introduce the quantity $\mathcal{E}$ by

$$(1+\xi^2)^3\mathcal{E} = \tau_0(1+\xi^2)^2\,\xi E_0 + \alpha_2\big[\alpha_1\xi^3\{E_1 + (1+\xi^2)E_2\}$$
$$+ (1+\xi^2)\{H_1 + \tau_0\xi^2\tilde{E}_7 + (1+\xi^2)H_2\}\big]. \tag{7.2.46}$$

Then the result is written as

$$(1+\xi^2)^3\mathcal{E}_t + \alpha_2\alpha_1(1-\varepsilon)\,\xi^4\left\{|\hat{v}|^2 + a(1+\xi^2)|\hat{y}|^2\right\}$$
$$+ \alpha_2 b\,(1-\alpha_1C_\varepsilon)(1+\xi^2)\,\xi^2|\hat{u}|^2 + \alpha_2 ab\,(1-\varepsilon-\alpha_1C_\varepsilon)(1+\xi^2)^2\,\xi^2|\hat{z}|^2$$
$$+ \sqrt{\kappa}(1-\alpha_2C_{\varepsilon,\alpha_1})(1+\xi^2)^2\,\xi^2|\hat{\theta}|^2$$
$$\le \tau_0C(1+\xi^2)\,\xi^2|\hat{q}|^2 + \alpha_2C(1+\xi^2)\,\xi^2|\hat{v}||\hat{q}| + C_{\alpha_2}(1+\tau_0)(1+\xi^2)^2\,\xi^2|\hat{y}||\hat{q}|$$
$$+ (1+\xi^2)^2|\xi||\hat{\theta}||\hat{q}| + \alpha_2C|P|(1+\xi^2)|\xi|^3|\hat{u}||\hat{q}| + \alpha_2C(1+\xi^2)^2|\xi||\hat{u}||\hat{q}|,$$

where C_{α_2} is a constant depending on α_2 but independent of $\tau_0 \in (0,1]$. By using the Young inequality, we arrive at

$$(1+\xi^2)^3\mathcal{E}_t + \alpha_2\alpha_1(1-2\varepsilon)\,\xi^4\left\{|\hat{v}|^2 + a\,(1+\xi^2)|\hat{y}|^2\right\}$$
$$+ \alpha_2 b\,(1-\varepsilon-\alpha_1C_\varepsilon)(1+\xi^2)\,\xi^2\left\{|\hat{u}|^2 + a\,(1+\xi^2)|\hat{z}|^2\right\}$$
$$+ \sqrt{\kappa}(1-\varepsilon-\alpha_2C_{\varepsilon,\alpha_1})(1+\xi^2)^2\,\xi^2|\hat{\theta}|^2$$
$$\le C_{\varepsilon,\alpha_1,\alpha_2}(1+\tau_0)^2(1+P^2)(1+\xi^2)^3|\hat{q}|^2, \tag{7.2.47}$$

where $C_{\varepsilon,\alpha_1,\alpha_2}$ is a constant depending on $(\varepsilon, \alpha_1, \alpha_2)$ but independent of $\tau_0 \in (0, 1]$. Finally, we combine this inequality with the basic energy equality (7.2.27). Namely, letting $\alpha_3 > 0$, we multiply (7.2.47) by $\alpha_3/(1+\xi^2)^3$ and add the resultant inequality to (7.2.27). This yeilds

$$\begin{aligned} E_t &+ \alpha_3\alpha_2\alpha_1(1-2\varepsilon)\,\frac{\xi^4}{(1+\xi^2)^3}\,\{|\hat{v}|^2 + a(1+\xi^2)|\hat{y}|^2\} \\ &+ \alpha_3\alpha_2 b\,(1-\varepsilon-\alpha_1 C_\varepsilon)\,\frac{\xi^2}{(1+\xi^2)^2}\,\{|\hat{u}|^2 + a\,(1+\xi^2)|\hat{z}|^2\} \\ &+ \alpha_3\sqrt{\kappa}\,(1-\varepsilon-\alpha_2 C_{\varepsilon,\alpha_1})\,\frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2 \\ &+ \{1-\alpha_3 C_{\varepsilon,\alpha_1,\alpha_2}(1+\tau_0)^2(1+P^2)\}\,|\hat{q}|^2 \,\le\, 0, \end{aligned} \tag{7.2.48}$$

where we put $E = |\hat{V}|^2/2 + \alpha_3\mathcal{E}$. In view of (7.2.46), we observe that this E is written explicitly in the form of (7.2.24). Here we choose $\varepsilon \in (0, 1)$ and α_1, α_2, $\alpha_3 > 0$ similarly as in the previous step for $P = 0$. Then our E is the desired Lyapunov function which is equivalent to $|\hat{V}|^2$ in the sense of (7.2.20), and (7.2.48) becomes the differential inequality (7.2.21) with the dissipative term F given in (7.2.25). Thus the proof of Proposition 7.2.4 is complete for $P \neq 0$. □

7.3 Timoshenko system with thermal effect of memory-type

We consider the Timoshenko system (7.1.10) with the thermal effect of memory-type:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\theta_x = 0, \\ \theta_t + b\psi_{tx} - \kappa\displaystyle\int_0^t g(t-\tau)\,\theta_{xx}(\tau)\,d\tau = 0 \end{cases} \tag{7.3.1}$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \theta)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)(x). \tag{7.3.2}$$

Here a, b and κ are positive constants, and $g(t)$ is an exponentially decaying function satisfying $g(t) > 0$ and $\int_0^\infty g(t)\,dt = 1$.

In this chapter we only treat the simplest case where the memory kernel $g(t)$ is given explicitly in the form

$$g(t) = \frac{1}{\tau_0}\,e^{-\frac{t}{\tau_0}} \tag{7.3.3}$$

with τ_0 being a given positive parameter. In this case we introduce the quantity $\tilde{q}$ by

$$\tilde{q} = -\kappa \int_0^t g(t-\tau)\,\theta_x(\tau)\,d\tau.$$

Since $g'(t) = -g(t)/\tau_0$ and $g(0) = 1/\tau_0$ by (7.3.3), we find that $\tilde{q}_t = -(\tilde{q} + \kappa\theta_x)/\tau_0$. Consequently, we can transform our system (7.3.1) into the Timoshenko-Cattaneo system (7.1.1). The corresponding initial data are given as follows.

$$(\varphi,\, \varphi_t,\, \psi,\, \psi_t,\, \theta)(x,\, 0) = (\varphi_0,\, \varphi_1,\, \psi_0,\, \psi_1,\, \theta_0)(x), \quad \tilde{q}(x,\, 0) = 0.$$

As in Section 7.2, by introducing the quantities

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad q = \frac{1}{\sqrt{\kappa}}\,\tilde{q},$$

we can rewrite the above Timoshenko-Cattaneo system (7.1.1) in the form of the first order symmetric hyperbolic system (7.2.1) for $(v,\, y,\, u,\, z,\, \theta,\, q)$. The corresponding initial data are given by $(v,\, y,\, u,\, z,\, \theta,\, q)(x,\, 0) = (v_0,\, y_0,\, u_0,\, z_0,\, \theta_0,\, q_0)(x)$, where

$$v_0 = \varphi_{0,x} - \psi_0, \quad y_0 = \psi_1, \quad u_0 = \varphi_1, \quad z_0 = a\psi_{0,x}, \quad q_0 = 0.$$

For this system (7.1.1) associated with (7.3.1), we can apply Theorem 7.2.1 and obtain the decay estimates (7.2.7) and (7.2.8) for $P = 0$ and $P \neq 0$, respectively, where P is the stability number in (7.1.3). To state this decay result for our system (7.3.1), we introduce

$$W = (\varphi_x - \psi,\, \varphi_t,\, \psi_x,\, \psi_t,\, \theta), \quad W_0 = (\varphi_{0,x} - \psi_0,\, \varphi_1,\, \psi_{0,x},\, \psi_1,\, \theta_0).$$

Then we have:

Theorem 7.3.1. *Assume that the memory kernel $g(t)$ is given by* (7.3.3)*. Let W be the above function defined from the solution to the initial value problem* (7.3.1), (7.3.2)*. Then W satisfies the following decay estimates for $t \geq 0$:*

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|W_0\|_{L^p} + Ce^{-ct}\|\partial_x^k W_0\|_{L^2} \tag{7.3.4}$$

for $P = 0$, and

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|W_0\|_{L^p} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} W_0\|_{L^2} \tag{7.3.5}$$

for $P \neq 0$, where P is the stability number in (7.1.3)*. Here W_0 is the initial data for W defined above, $1 \leq p \leq 2$, k and l are nonnegative integers, and C and c are positive constants independent of the parameter τ_0.*

Remark. By considering the asymptotic expansion of the eigenvalues obtained in Subsection 7.2.2, we think that the above decay estimates (7.3.4) and (7.3.5) are optimal. We expect that the same decay result holds true even for the system (7.3.1) with a general memory kernel $g(t)$, which decays exponentially as $t \to \infty$ and satisfies $g(t) > 0$ and $\int_0^\infty g(t)\,dt = 1$, but it remains an open question.

Chapter 8

Timoshenko system with memory

8.1 Introduction

In this chapter we consider the Cauchy problem of the Timoshenko system with a memory term

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ \psi_{tt} - \sigma(\psi_x)_x + (\varphi_x - \psi) + \gamma\, g * \psi_{xx} = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \end{cases} \tag{8.1.1}$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t)(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1)(x), \quad x \in \mathbb{R} \tag{8.1.2}$$

in one dimensional whole space $\mathbb{R}$. The two coupled wave equations in (8.1.1) with $\gamma = 0$ is the original Timoshenko system, which was first introduced by S. P. Timoshenko in [62, 63] as a model system which describes the vibration of the beam called Timoshenko beam. This beam thery has the advantage of describing not only the transversal movement but also the shear deformation and the rotational inertia effects. Here t is the time variable and x is the spacial variable which denotes a point on the center line of the beam. The unknown functions $\varphi = \varphi(t,x)$ and $\psi = \psi(t,x)$ denote the transversal displacement and the rotation angle of the beam, respectively. Note that the term $\varphi_x - \psi$ denotes the shearing stress. The function $\sigma(\eta)$ in the nonlinear term is assumed to be a smooth function of η such that $\sigma'(\eta) > 0$ for any η under considerations. The term

$$g * \psi_{xx} := \int_0^t g(t-\tau)\, \psi_{xx}(\tau)\, d\tau$$

corresponds to a memory-type damping. In this chapter the relaxation function g is assumed to satisfy the following conditions.

Assumption: The function g is assumed to be a smooth function of $t \geq 0$ such that

$$\begin{cases} g(t) > 0, \quad \int_{\mathbb{R}^+} g(t)\,dt = 1, \\ -c_0\, g(t) \leq g'(t) \leq -c_1\, g(t), \\ |g''(t)| \leq C g(t) \end{cases} \tag{8.1.3}$$

for $t \geq 0$, where c_0, c_1 and C are positive constants.

Under (8.1.3), it follows that

$$|g(t)| + |g'(t)| + |g''(t)| \leq C e^{-c_1 t}$$

for $t \geq 0$, where C is a positive constant.

Based on the change of variable, which was first introduced in [24],

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t$$

with $a > 0$, which denotes the wave speed defined by $a^2 = \sigma^{'}(0)$, we reduce the system (8.1.1) and (8.1.2) to the first order hyperbolic system

$$v_t - u_x + y = 0, \tag{8.1.4a}$$

$$y_t - \sigma\left(\frac{z}{a}\right)_x - v + +b\, g * z_x = 0, \tag{8.1.4b}$$

$$u_t - v_x = 0, \tag{8.1.4c}$$

$$z_t - a\, y_x = 0 \tag{8.1.4d}$$

with the initial data

$$(v, y, u, z)(x, 0) = (v_0, y_0, u_0, z_0)(x), \tag{8.1.5}$$

where $b := \dfrac{\gamma}{a} < a$, $v_0 = \varphi_{0,x} - \psi_0$, $y_0 = \psi_1$, $u_0 = \varphi_1$, $z_0 = a\psi_{0,x}$. Note that the nonlinearity depends on $\underline{z \text{ only}}$. In vector notation, the Cauchy problem (8.1.4) and (8.1.5) can be rewritten as

$$\begin{cases} W_t + F(W)_x + Bg * W_x + LW = 0, \\ W(x, 0) = W_0(x), \end{cases} \tag{8.1.6}$$

where $W = (v, y, u, z)^T$, $W_0 = (v_0, y_0, u_0, z_0)^T$, $F(W) = -(u, \sigma(z/a), v, ay)^T$ (the superscript T means the transpose),

$$B = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad L = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The corresponding linearized system is given by

$$\begin{cases} W_t + AW_x + Bg * W_x + LW = 0, \\ W(0,x) = W_0(x), \end{cases} \tag{8.1.7}$$

where

$$A = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -a \\ \hline -1 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{array} \right).$$

The system (8.1.6) can be seen as a particular case of the first order symmetric hyperbolic system with the non-symmetric relaxation term LW and memory-type dissipation term $Bg * W_x$. Because the matrix L is not symmetric, the general theory of the dissipative structure established in [67] and [58] is not applicable to the system (8.1.6). Therefore, in order to know the dissipative structure and the asymptotic stability of the system (8.1.6), new ideas have to be implemented.

8.1.1 Known results

In [33], Liu and Kawashima first investigated the Cauchy problem (8.1.1) in one dimensional whole space $\mathbb{R}$. Due to the regularity-loss property and the weak dissipation of the system, they had to assume the stronger nonlinearity (to make the initial data much smaller) than usual. By virtue of the semi-group arguments, they obtained the global-in-time existence and uniqueness, and optimal decay of the solution to the problem (8.1.6) under enough smallness and high regularity assumptions on the initial data. There, they employed a time-weighted L^2 energy method combined with the optimal L^2 decay of lower-order derivatives of the solution.

Here, in order to state their results precisely, we introduce the energy norm $\tilde{E}$ and the corresponding dissipation norm $\tilde{D}$ as follows:

$$\tilde{E}(t)^2 := \sum_{j=0}^{[\frac{s}{2}]} \sup_{0\le\tau\le t} (1+\tau)^{\frac{j}{2}-\varepsilon} \|\partial_x^j W(\tau)\|_{H^{s-2j}}^2,$$

$$\tilde{D}(t)^2 := \sum_{j=0}^{[\frac{s}{2}]} \int_0^t (1+\tau)^{\frac{j}{2}-1-\varepsilon} \|\partial_x^j W(\tau)\|_{H^{s-2j}}^2 \, d\tau$$

for $\varepsilon \in (0, \frac{1}{4}]$ and $s \ge 0$.

Proposition 8.1.1 (Global-in-time existence & L^2 decay estimate [33])**.** *Assume $\sigma'(\eta) = a^2 + O(\eta^2)$ as $\eta \to 0$. Let s be an integer satisfying $s \ge 3$ for $a = 1$ and $s \ge 5$ for $a \ne 1$. Suppose that the initial data $W_0 \in H^s \cap L_1^1$ and $\int_{\mathbb{R}} W_0(x)\,dx = 0$. Put $E_1 := \|W_0\|_{H^s} + \|W_0\|_{L_1^1}$.*

Then there exists a positive constant δ_1 *such that if* $E_1 \le \delta_1$*, the Cauchy problem* (8.1.6) *has a unique global-in-time solution* $W(t)$ *with*

$$W \in C([0,\infty]; H^s) \cap C^1([0,\infty]; H^{s-1}),$$

and the solution $W(t)$ *verifies the energy estimate*

$$\tilde{E}(t)^2 + \tilde{D}(t)^2 \le C E_1^2.$$

Moreover, for this solution $W(t)$*, the following optimal decay estimates for lower-order derivatives hold:*

(1) *if* $a = 1$*, then for* $0 \le k \le [\frac{s-1}{2}]$*,*

$$\|\partial_x^k W(t)\|_{H^{s-2k-1}} \;\le\; C E_1 (1+t)^{-\frac{3}{8}-\frac{k}{4}};$$

(2) *if* $a \ne 1$*, then for* $0 \le k \le [\frac{s-3}{2}]$*,*

$$\|\partial_x^k W(t)\|_{H^{s-2k-3}} \;\le\; C E_1 (1+t)^{-\frac{3}{8}-\frac{k}{4}}.$$

In Proposition 8.1.1 they assumed $W_0 \in L^1_1$ and $\int_{\mathbb{R}} W_0(x)\,dx = 0$ because it gives 1/4 more decay, and this is crucial to their proof. Their proof of the global-in-time existence and uniqueness results is essentially parallel to the computations of the energy estimate in the Fourier space for the linearized system (8.2.13), which they showed in [32].

In [32], based on the energy estimate in the Fourier space, they derived the pointwise estimate of the solution to the linearized system (8.2.13) in the Fourier space, which gives a sharp decay estimate of the solutionin L^2.

Proposition 8.1.2 (Pointwise estimate in Fourier space [32])**.** *Let* $\hat{W} = (\varphi_x + \psi, \psi_t, a\psi_x, \varphi_t)$ *be a solution and* $\hat{W}_0$ *be the corresponding initial data to the Cauchy problem* (8.2.13)*. Then* $\hat{W}$ *satisfies the following pointwise estimates in the Fourier space for any* $\xi \in \mathbb{R}$ *and* $t \ge 0$*:*

$$|\hat{W}(\xi,t)| \le C e^{-c\tilde{\rho}_1(\xi)t} |\hat{W}_0(\xi)| \qquad \text{for} \quad a = 1,$$

$$|\hat{W}(\xi,t)| \le C e^{-c\tilde{\rho}_2(\xi)t} |\hat{W}_0(\xi)| \qquad \text{for} \quad a \ne 1,$$

where $\tilde{\rho}_1(\xi) = \xi^4/(1+\xi^2)^2$*,* $\tilde{\rho}_2(\xi) = \xi^4/(1+\xi^2)^4$*.*

This result implies that the decay property of the system is of the regularity-loss type and is much weaker than that of the system with a normal damping term (so called the dissipative Timoshenko system or the classical Timoshenko system): Concerning the recent developments for the dissipative Timoshenko system, we refer to [38, 42, 72].

Finally, we would like to mention other studies on the Timoshenko system with different dissipative mechanism; see, e.g., [47, 48] for frictional dissipation case, [19, 54, 55] for thermal dissipation case, and [3, 4, 34] for memory-type dissipation case.

8.1.2 Aim

The aim of this chapter is to show the global-in-time existence and uniqueness of the solution to the Cauchy problem (8.1.6) under the lowest regularity assumption on the initial data.

To this end, in Section 8.2 we remake the pointwise estimate of the solution to the linearized system (8.2.13) in [32] in order to get the way to construct the Lyapunov function which minimizes the number of the dissipation terms of regularity-loss.

Next, we characterize the dissipative structure of the system 8.1.6 by the straight calculation of the asmnptotic expansions of the eigenvalues. This characterization confirms that our pointwise estimate is optimal.

Finally, in Section 8.3, based on our linearized system results in Section 8.2, we investigate the nonlinear system (8.1.6) and obtain the global-in-time existence and uniqueness in the critical sobolev space H^2. That is, we show that the global-in-time existence and uniqueness of the system (8.1.6) could be proved in the minimal regularity assumption on the initial data and no need to employ any time-weighted norm as Liu and Kawashima did in [33]. This implies that our refinement of the Lyapunov function in Section 8.2 and its application to the nonlinear system completely overcomes the difficulty caused by the weak dissipation due to the regularity-loss property of the Timoshenko system with a memory term (8.1.1).

8.2 Linear system

In Section 8.2, we consider the linearized Timoshenko system of memory type (8.2.13) in one dimensional whole space $\mathbb{R}$. In Subsection 8.2.2, we derive the pointwise estimate of the solution to the linearized system (8.2.13) in the Fourier space, which gives the optimal decay estimate of the solution in L^2 with the desired decay rate $t^{-\frac{1}{8}}$ in Section 8.2.1.

In Subsection 8.2.3, we characterize the dissipative structure of the Timoshenko system of memory type (8.1.6), based on the information of the eigenvalues of the linearized system (8.2.13). Note that our pointwise estimate in Subsection 8.2.2 completely matches with the dissipative structure shown in Subsection 8.2.3.

8.2.1 Decay estimate

In this subsection, we show the L^2-decay estimate of the solution to the linearized Timoshenko system of memory type (8.2.13).

Applying the Fourier transform to (8.2.13) to have

$$\begin{cases} \hat{W}_t + i\xi A\hat{W} + i\xi Bg * \hat{W} + L\hat{W} = 0, \\ \hat{W}(\xi, 0) = \hat{W}_0(\xi), \end{cases} \tag{8.2.1}$$

where $\hat{W} = (\hat{v}, \hat{y}, \hat{u}, \hat{z})^T$, $\hat{W}_0 = (\hat{v}_0, \hat{y}_0, \hat{u}_0, \hat{z}_0)^T$. Firstly, we give the pointwise estimate of the solution to the linearized system in the Fourier space (8.2.1).

Theorem 8.2.1 (Pointwise estimate in Fourier space)**.** *Let $\hat{W}$ be a solution and $\hat{W}_0$ be the corresponding initial data to the Cauchy problem* (8.2.1)*. Then $\hat{W}$ satisfies the following pointwise estimates in the Fourier space for any $\xi \in \mathbb{R}$ and $t \geq 0$:*

$$|\hat{W}(\xi,t)| \leq Ce^{-c\rho_1(\xi)t}|\hat{W}_0(\xi)| \qquad \text{for} \quad a = 1, \tag{8.2.2}$$

$$|\hat{W}(\xi,t)| \leq Ce^{-c\rho_2(\xi)t}|\hat{W}_0(\xi)| \qquad \text{for} \quad a \neq 1, \tag{8.2.3}$$

where $\rho_1(\xi) = \xi^4/(1+\xi^2)^2$, $\rho_2(\xi) = \xi^4/(1+\xi^2)^3$.

Remark 8.2.2. *In [32], Liu and Kawashima showed the same result as* (8.2.2) *in the case of $a = 1$. However, in the case of $a \neq 1$, their estimate with $\tilde{\rho}_2(\xi) = \xi^4/(1+\xi^2)^4$ is worse one than expected from our dissipative structure result in Proposition 8.2.9. Our results with $\rho_1(\xi) = \xi^4/(1+\xi^2)^2$ for $a = 1$ and $\rho_2(\xi) = \xi^4/(1+\xi^2)^3$ for $a \neq 1$ completely match with the dissipative structure in Proposition 8.2.9.*

The proof of Theorem 8.2.1 is given by the energy method in the Fourier space, and the optimal estimate (8.2.3) in the case of $a \neq 1$ requires our improvement: See Subsection 8.2.2.

Secondly, the above pointwise estimates give us the optimal L^2-decay estimates of the solution to the linearized system (8.2.13) stated as follows.

Theorem 8.2.3 (L^2-decay estimate)**.** *Let W be a solution and W_0 be the corresponding initial data to the Cauchy problem* (8.2.13)*. Then W satisfies the following decay estimates for any $t \geq 0$:*

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|W_0\|_{L^p} + Ce^{-ct}\|\partial_x^k W_0\|_{L^2}, \tag{8.2.4}$$

for $a = 1$, while

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|W_0\|_{L^p} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} W_0\|_{L^2}, \tag{8.2.5}$$

for $a \neq 1$, where $1 \leq p \leq 2$ and $k, \ell \geq 0$ are constants.

Remark 8.2.4. *Clearly, the high frequency part in* (8.2.4) *for $a = 1$ yields an exponential decay. On the other hand, the corresponding part in* (8.2.5) *for $a \neq 1$ is of the regularity-loss type, since $(1+t)^{-\ell/2}$ is created only by assuming the additional ℓ-th order regularity on the initial data.*

In the rest of this subsection, we give the proof of Theorem 8.2.3. This proof is essentially same as the proof first given by Ide, Haramoto and Kawashoma in [24] (also see [32, 38]).

Proof of Theorem 8.2.3 ([24, 32, 38])**.** First we consider the case of $a = 1$. Applying the Plancherel theorem and using the pointwise estimate (8.2.2), we have

$$\|\partial_x^k W(t)\|_{L^2}^2 = C\int_{\mathbb{R}} \xi^{2k}|\hat{W}(t,\xi)|^2\, d\xi \leq C\int_{\mathbb{R}} \xi^{2k}e^{-\rho_1(\xi)t}|\hat{W}_0(\xi)|^2\, d\xi. \tag{8.2.6}$$

We divide the last integral into two parts I_1 and I_2 corresponding to the regions $|\xi| \le 1$ and $|\xi| \ge 1$, respectively. Here we see that $\rho_1(\xi) \ge c\xi^4$ for $|\xi| \le 1$ so that we have

$$I_1 = C\int_{|\xi|\le 1} \xi^{2k} e^{-c\rho_1(\xi)t} |\hat{W}_0(\xi)|^2 \, d\xi \le C\int_{|\xi|\le 1} \xi^{2k} e^{-c\xi^4 t} |\hat{W}_0(\xi)|^2 \, d\xi.$$

For $1 \le p \le 2$, we choose p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Also, we take r such that $\frac{1}{r} + \frac{2}{p'} = 1$. Then we see that $\frac{1}{2r} = \frac{1}{p} - \frac{1}{2}$. Applying the Hölder inequality and the Hausdorff-Young inequality, we estimate I_1 as

$$\begin{aligned} I_1 &\le C\|\xi^{2k} e^{-c\xi^4 t}\|_{L^r(|\xi|\le 1)} \|\hat{W}_0\|_{L^{p'}}^2 \\ &\le C(1+t)^{-\frac{1}{4r}-\frac{k}{2}} \|W_0\|_{L^p}^2 = C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|W_0\|_{L^p}^2. \end{aligned}$$

On the other hand, in the high frequency region $|\xi| \ge 1$, we have $\rho_1(\xi) \ge c$. Therefore we estimate I_2 as

$$\begin{aligned} I_2 &= C\int_{|\xi|\ge 1} \xi^{2k} e^{-c\rho_1(\xi)t} |\hat{W}_0(\xi)|^2 \, d\xi \\ &\le Ce^{-ct} \int_{|\xi|\ge 1} \xi^{2k} |\hat{W}_0(\xi)|^2 \, d\xi \le Ce^{-ct} \|\partial_x^k W_0\|_{L^2}^2. \end{aligned}$$

Substituting these estimates into (8.2.6) gives the desired estimate (8.2.4).

Next we consider the case of $a \ne 1$. Using (8.2.3), we have

$$\|\partial_x^k W(t)\|_{L^2}^2 \le C\int_{\mathbb{R}} \xi^{2k} e^{-\rho_2(\xi)t} |\hat{W}_0(\xi)|^2 \, d\xi. \tag{8.2.7}$$

We divide this integral into two parts J_1 and J_2 corresponding to the regions $|\xi| \le 1$ and $|\xi| \ge 1$, respectively. Since, $\rho_2(\xi) \ge c\xi^4$ for $|\xi| \le 1$, the low frequency part J_1 is estimated just in the same way as I_1. That is, $J_1 \le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|W_0\|_{L^p}^2$. On the other hand, in the high frequency region $|\xi| \ge 1$, we see that $\rho_2(\xi) \ge c|\xi|^{-2}$. Thus,

$$\begin{aligned} J_2 &= C\int_{|\xi|\ge 1} \xi^{2k} e^{-c\rho_2(\xi)t} |\hat{W}_0(\xi)|^2 \, d\xi \le C\int_{|\xi|\ge 1} \xi^{2k} e^{-c|\xi|^{-2}t} |\hat{W}_0(\xi)|^2 \, d\xi \\ &\le C \sup_{|\xi|\ge 1} \left\{ |\xi|^{-2l} e^{-c|\xi|^{-2}t} \right\} \int_{|\xi|\ge 1} \xi^{2(k+l)} |\hat{W}_0(\xi)|^2 \, d\xi \\ &\le C(1+t)^{-l} \|\partial_x^{k+l} W_0\|_{L^2}^2. \end{aligned}$$

Substituting these estimates into (8.2.7) gives the desired estimate (8.2.5) for $a \ne 1$. This completes the proof of Theorem 8.2.3.

□

8.2.2 Energy method in Fourier spaces

In this subsection, we prove Theotem 8.2.1 by the energy method in the Fourier space. Before the proof, we prepare some notations related to the convolution operator with respect to t.

Let $k(t)$ be a real valued kernel function. For any complex valued function $\varphi(t)$ and $\psi(t)$, we define

$$\begin{aligned}(k*\varphi)(t) &= \int_0^t k(t-\tau)\,\varphi(\tau)\,d\tau,\\ (k\diamond\varphi)(t) &= \int_0^t k(t-\tau)(\varphi(t)-\varphi(\tau))\,d\tau,\\ k[\varphi,\psi](t) &= \int_0^t k(t-\tau)(\varphi(t)-\varphi(\tau))(\bar\psi(t)-\bar\psi(\tau))\,d\tau,\end{aligned}$$

where $\bar\psi$ denotes the complex conjugate of ψ. Here we see that

$$k*\varphi = K(t) - k\diamond\varphi,\quad (k*\varphi)_t = k(t)\varphi - k^{'}\diamond\varphi, \tag{8.2.8}$$

where $K(t) = \displaystyle\int_0^t k(s)\,ds$ and $k^{'}(t) = \dfrac{dk(t)}{dt}$. Moreover, applying the Hölder inequality, we have

$$|(k\diamond\varphi)(t)|^2 \le \left(\int_0^t |k(\tau)|\,d\tau\right)|k|[\varphi,\varphi](t). \tag{8.2.9}$$

Finally, the following special equality first given by Rivera and his collaboraters in [3] takes very much important role in our proof of Theorem 8.2.1.

Lemma 8.2.5 ([3, 13, 41])**.** *Let k be a real valued function. Then, for any complex valued function φ, we have the following equality:*

$$-2\mathrm{Re}\,\{(k*\varphi)\,\bar\varphi_t\} = \frac{d}{dt}\{k[\varphi,\varphi] - K(t)|\varphi|^2\} + \{k^{'}[\varphi,\varphi] - k(t)|\varphi|^2\}. \tag{8.2.10}$$

Proof of Theorem 8.2.1. In the case of $a=1$, the proof of (8.2.2) is already shown in [32]. Therefore, here we focus on the proof for $a\neq 1$, which is of our refinement to show (8.2.3).

First, we rewrite the linearized system (8.2.13) in the form

$$\hat v_t - i\xi\hat u - \hat y = 0, \tag{8.2.11a}$$

$$\hat y_t - a\,i\xi\hat z + \hat v + b\,i\xi g*\hat z = 0, \tag{8.2.11b}$$

$$\hat u_t - i\xi\hat v = 0, \tag{8.2.11c}$$

$$\hat z_t - a\,i\xi\hat y = 0. \tag{8.2.11d}$$

The proof is given by the energy method in the Fourier space, and is divided into 5 steps.

Step 1: Take the inner product in $\mathbb{C}^4$ of $\hat{W}_t + i\xi A\hat{W} + i\xi Bg * \hat{W} + L\hat{W} = 0$ with $\hat{W}$, then take the real part to have

$$\frac{1}{2}\,(|\hat{W}|^2)_t + \mathrm{Re}\,\{b\,i\xi(g * \hat{z})\,\bar{\hat{y}}\} = 0. \tag{8.2.12}$$

To eliminate the second term in (8.2.12), we make the following equality: Multiply the complex conjugate of (8.2.11d) by $-g * \hat{z}$ and take the real part. This gives

$$-\mathrm{Re}\,\{(g * \hat{z})\,\bar{\hat{z}}_t\} - \mathrm{Re}\,\{a\,i\xi(g * \hat{z})\,\bar{\hat{y}}\} = 0. \tag{8.2.13}$$

Then, make the combination $a \times (8.2.12) + b \times (8.2.13)$ and use the equality (8.2.10) to obtain the following basic energy equality:

$$\frac{1}{2}\,E_{1,t} + D_1 = 0, \tag{8.2.14}$$

where we put

$$E_1 = a\,|\hat{W}|^2 + b\,g[\hat{z}, \hat{z}] - b\,G(t)|\hat{z}|^2, \quad D_1 = \frac{b}{2}\,(-g^{'}[\hat{z}, \hat{z}] + b\,g(t)|\hat{z}|^2),$$

and $G(t) = \displaystyle\int_0^t g(s)\,ds$. Here by putting $M_0^2 = g[\hat{z}, \hat{z}] + g(t)|\hat{z}|^2$ and using the assumption (8.1.3) on g, E_1 and D_1 can be estimated as

$$c(|\hat{W}|^2 + M_0^2) \le E_1 \le C(|\hat{W}|^2 + M_0^2), \quad D_1 \ge cM_0^2. \tag{8.2.15}$$

Note that the basic energy equality (8.2.14) does not contain any dissipation term except D_1, which is directly due to the memory term.

Step 2: Next we create the dissipation term $\xi^2|\hat{y}|^2$ by making use of the memory effect. Multiply (8.2.11b) by $-i\xi(g * \bar{\hat{z}})_t$ and take the real part to have

$$\frac{1}{2}\,(b\,|\xi|^2|g * \hat{z}|^2)_t - \mathrm{Re}\,\{i\xi\hat{y}_t\,(g * \bar{\hat{z}})_t\} + \mathrm{Re}\,\{i\xi(a\,i\xi\hat{z} - \hat{v})(g * \bar{\hat{z}})_t\} = 0. \tag{8.2.16}$$

Here we rewrite the second term in (8.2.16) by using $(g * \bar{\hat{z}})_t = g(0)\,\bar{\hat{z}} + g^{'} * \bar{\hat{z}}$. This yields

$$\frac{1}{2}\,(1 + \xi^2)\,E_{2,t} + g(0)\,\mathrm{Re}\,(i\xi\hat{y}\bar{\hat{z}}_t) = R_2, \tag{8.2.17}$$

where

$$(1 + \xi^2)\,E_2 = b\,|\xi|^2|g * \bar{\hat{z}}|^2 - 2\mathrm{Re}\,\{i\xi(g * \bar{\hat{z}}_t)\},$$

$$R_2 = -\mathrm{Re}\,\{i\xi(a\,i\xi\hat{z} - \hat{v})(g * \bar{\hat{z}})_t + i\xi\hat{y}\,(g^{'} * \bar{\hat{z}})_t\}.$$

Finally, to eliminate the second term in (8.2.17), firstly multiply the complex conjugate of (8.2.11d) by $-i\xi\hat{y}$ and take the real part to get

$$-\text{Re}\,(i\xi\hat{y}\bar{\hat{z}}_t) + a\,\xi^2|\hat{y}|^2 = 0. \tag{8.2.18}$$

Secondly, multiply (8.2.18) by $g(0)$ and add the result to (8.2.17). This gives the desired equality with the dissipation term $\xi^2|\hat{y}|^2$:

$$\frac{1}{2}\,(1+\xi^2)\,E_{2,t} + a\,g(0)\xi^2|\hat{y}|^2 = R_2. \tag{8.2.19}$$

Moreover, by using (8.2.8), (8.2.9) and (8.1.3), we see that

$$\begin{cases} |g * \bar{\hat{z}}| = |G(t)\bar{\hat{z}} - g \diamond \bar{\hat{z}}| \le C(|\bar{\hat{z}}| + M_0), \\ |(g * \bar{\hat{z}})_t| = |g(t)\bar{\hat{z}} - g' \diamond \bar{\hat{z}}| \le CM_0, \\ |(g' * \bar{\hat{z}})_t| = |g'(t)\bar{\hat{z}} - g'' \diamond \bar{\hat{z}}| \le CM_0. \end{cases} \tag{8.2.20}$$

Therefore, by using (8.2.20), E_2 and R_2 in (8.2.19) can be estimated as

$$|E_2| \le C(|\hat{W}|^2 + M_0^2), \quad |R_2| \le C\xi^2|\hat{z}|M_0 + C|\xi|(|\hat{v}| + |\hat{y}|)M_0. \tag{8.2.21}$$

Step 3: In this step, we create the dissipation term $|\hat{v}|^2$ by using the relaxation part $L\hat{W}$ in (8.2.1). Make the combination (8.2.11b) $\times\ \bar{\hat{v}}$ + (8.2.11a) $\times\ \bar{\hat{y}}$ to have

$$(\hat{v}_t\bar{\hat{y}} + \bar{\hat{v}}\hat{y}_t) + |\hat{v}|^2 - |\hat{y}|^2 - i\xi(\hat{v}\bar{\hat{y}} + a\,\bar{\hat{v}}\hat{z}) + b\,i\xi\bar{\hat{v}}(g * \hat{z}) = 0. \tag{8.2.22}$$

Similarly, make the combination (8.2.11c) $\times\ \bar{\hat{z}}$ + (8.2.11d) $\times\ \bar{\hat{u}}$ to get

$$(\hat{u}_t\bar{\hat{z}} + \bar{\hat{u}}\hat{z}_t) - i\xi(\hat{v}\bar{\hat{z}} + a\,\hat{y}\bar{\hat{u}}) = 0. \tag{8.2.23}$$

To eliminate the last term in (8.2.22), we make the following equality: Multiply the complex conjugate of (8.2.11c) by $-g * \hat{z}$. This gives

$$-\{\bar{\hat{u}}\,(g * \hat{z})\}_t + \bar{\hat{u}}\,(g * \hat{z})_t - i\xi\bar{\hat{v}}\,(g * \hat{z}) = 0. \tag{8.2.24}$$

Consequently, make the combination (8.2.22) + a × (8.2.23) + b × (8.2.24) and take the real part to obtain

$$E_{3,t} + |\hat{v}|^2 = R_3, \tag{8.2.25}$$

where

$$E_3 = \text{Re}\,\{(\hat{v}\bar{\hat{y}} + a\,\bar{\hat{v}}\hat{z} - b\,\hat{u}\,(g * \bar{\hat{z}})\}, \quad R_3 = |\hat{y}|^2 - (a^2-1)\,\text{Re}\,(i\xi\bar{\hat{y}}\hat{u}) - b\,\text{Re}\,\{\hat{u}\,(g * \bar{\hat{z}})_t\}.$$

Therefore, by using (8.2.20), E_3 and R_3 in (8.2.25) can also be estimated as

$$|E_3| \le C(|\hat{W}|^2 + M_0^2), \quad |R_3| \le |\hat{y}|^2 + C|\xi||\hat{y}||\hat{u}| + C|\hat{u}|M_0. \tag{8.2.26}$$

Remark 8.2.6. *In the case of $a = 1$, R_3 is estimated as $|R_3| \le |\hat{y}|^2 + C|\hat{u}|M_0$.*

Step 4: Finally, we create the dissipation term $\xi^2|\hat{u}|^2+\xi^2(1+\xi^2)|\hat{z}|^2$ by using the hyperbolic part $i\xi A\hat{W}$. Note that our refinement lies in this step: We create the dissipation term $\xi^2|\hat{u}|^2 + \xi^2(1+\xi^2)|\hat{z}|^2$ instead of $\xi^2(|\hat{u}|^2 + |\hat{z}|^2)$ in [32]. Make the combination (8.2.11a) $\times$ $i\xi\bar{\hat{u}}$ $-$ (8.2.11c) $\times$ $i\xi\bar{\hat{v}}$ to have

$$i\xi(\hat{v}_t\bar{\hat{u}} - \bar{\hat{v}}\hat{u}_t) + \xi^2(|\hat{u}|^2 - |\hat{v}|^2) - i\xi\hat{y}\bar{\hat{u}} = 0. \tag{8.2.27}$$

Also, make the combination (8.2.11b) $\times$ $i\xi\bar{\hat{z}}$ $-$ (8.2.11d) $\times$ $i\xi\bar{\hat{y}}$ to get

$$i\xi(\hat{y}_t\bar{\hat{z}} - \bar{\hat{y}}\hat{z}_t) + a\,\xi^2(|\hat{z}|^2 - |\hat{y}|^2) + i\xi\hat{v}\bar{\hat{z}} - b\,\xi^2(g * \hat{z})\,\bar{\hat{z}} = 0. \tag{8.2.28}$$

In the case of $a \ne 1$, we make the combination (8.2.27) + (8.2.28) $\times$ $(1+\xi^2)$. Then, use (8.2.8) and take the real part to obtain

$$\xi\{E_4 + (1+\xi^2)\,E_5\}_t + \xi^2\{|\hat{u}|^2 + (a - b\,G(t))(1+\xi^2)|\hat{z}|^2\} = R_4, \tag{8.2.29}$$

where

$$\begin{aligned}
&E_4 = \text{Re}\,\{i\hat{v}\bar{\hat{u}}\}, \quad E_5 = \text{Re}\,\{i\hat{z}\bar{\hat{y}}\},\\
&R_4 = \xi^2\{|\hat{v}|^2 + a\,(1+\xi^2)|\hat{y}|^2\} + \text{Re}\,(i\xi\hat{y}\bar{\hat{u}})\\
&\quad - \text{Re}\,\{i\xi(1+\xi^2)\,\hat{v}\bar{\hat{z}}\} - b\,\xi^2(1+\xi^2)\,\text{Re}\,\{(g \diamond \hat{z})\,\bar{\hat{z}}\}.
\end{aligned}$$

Therefore, by using (8.2.20), E_4 and R_4 in (8.2.29) can be estimated as

$$\begin{cases}
|E_4| + (1+\xi^2)|E_5| \le (1+\xi^2)|\hat{W}|^2,\\
|R_4| \le C\xi^2\{|\hat{v}|^2 + (1+\xi^2)|\hat{y}|^2\}\\
\qquad + |\xi|\{|\hat{y}||\hat{u}| + (1+\xi^2)|\hat{v}||\hat{z}|\} + C\xi^2(1+\xi^2)|\hat{z}|M_0.
\end{cases} \tag{8.2.30}$$

Remark 8.2.7. *In the case of $a = 1$, we employ the combination* (8.2.27) + (8.2.28) *instead of* (8.2.27) + (8.2.28) $\times$ $(1+\xi^2)$.

Step 5: Now we construct the Lyapunov function for $a \ne 1$. Combine the equalities obtained from Step 2 to Step 4 as (8.2.19) $\times$ $(1+\xi^2)^2$ + $\{$(8.2.25) $\times$ $\xi^2(1+\xi^2)$ + (8.2.29) $\times$ $\alpha_1\xi^2\}$ $\times$ α_2 with suitably small positive constants $\alpha_1 > 0$ and $\alpha_2 > 0$. Then divide the result by $(1+\xi^2)^3$. This gives

$$\frac{1}{2}\,\mathcal{E}_t + \mathcal{D} = \mathcal{R}, \tag{8.2.31}$$

where

$$\mathcal{E} = E_2 + \frac{2\alpha_2\xi^2}{(1+\xi^2)^2}\left\{E_3 + \alpha_1\xi\left(\frac{E_4}{1+\xi^2} + E_5\right)\right\},$$

$$(1+\xi^2)^3\,\mathcal{D} = \alpha_2(1+\xi^2)\,\xi^2|\hat{v}|^2 + a\,g(0)(1+\xi^2)^2\,\xi^2|\hat{y}|^2$$

$$+\,\alpha_2\alpha_1\xi^4\{|\hat{u}|^2 + (a - b\,G(t))(1+\xi^2)|\hat{z}|^2\},$$

$$(1+\xi^2)^3\,\mathcal{R} = (1+\xi^2)^2\,R_2 + \alpha_2\{(1+\xi^2)\,\xi^2 R_3 + \alpha_1\xi^2 R_4\}.$$

We see $a - b\,G(t) \geq a - b > 0$. Besides, by using the inequalites (8.2.21), (8.2.26) and (8.2.30), we estimate $\mathcal{E}$, $\mathcal{D}$ and $\mathcal{R}$ by the Young inequality. Then we choose $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\begin{cases} |\mathcal{E}| \leq C(|\hat{W}|^2 + M_0^2), \\ (1+\xi^2)^3\,\mathcal{D} \geq c\,(1+\xi^2)\,\xi^2|\hat{v}|^2 + c\,(1+\xi^2)^2\,\xi^2|\hat{y}|^2 + c\,\xi^4|\hat{u}|^2 + c\,(1+\xi^2)\,\xi^4|\hat{z}|^2, \\ |\mathcal{R}| \geq \dfrac{1}{2}\,\mathcal{D} + CM_0^2. \end{cases} \tag{8.2.32}$$

Furthermore, make the combination (8.2.14) + (8.2.31) $\times\ \alpha_3$ with any $\alpha_3 > 0$ to have

$$\frac{1}{2}\,E_t + D = 0, \tag{8.2.33}$$

where

$$E = E_1 + \alpha_3\mathcal{E}, \quad D = D_1 + \alpha_3(\mathcal{D} - \mathcal{R}).$$

Therefore, for a suitably small positive constant α_3, E and D can be estimated by using (8.2.15) and (8.2.32) as

$$c\,(|\hat{W}|^2 + M_0^2) \leq E \leq C(|\hat{W}|^2 + M_0^2), \tag{8.2.34}$$

$$D \geq \frac{c\,\xi^2}{(1+\xi^2)^2}|\hat{v}|^2 + \frac{c\,\xi^2}{1+\xi^2}|\hat{y}|^2 + \frac{c\,\xi^4}{(1+\xi^2)^3}|\hat{u}|^2 + \frac{c\,\xi^4}{(1+\xi^2)^2}|\hat{z}|^2 + c\,M_0^2. \tag{8.2.35}$$

Especially, we put $\rho_2(\xi) = \xi^4/(1+\xi^2)^3$, then we have $D \geq c\,\rho_2(\xi)(|\hat{W}|^2 + M_0^2) \geq c\,\rho_2(\xi)E$. Therefore, from (8.2.33) we obtain $E_t + c\,\rho_2(\xi)E \leq 0$. Consequently, the differential inequality

$$|E(t,\xi)| \leq Ce^{-c\rho_2(\xi)t}|E(0,\xi)|$$

and the estimate of E (8.2.34) yield the desired pointwise estimate (8.2.3) for $a \neq 1$. □

Remark 8.2.8. *When $a \neq 1$, we see from* (8.2.35) *that the coefficients of the three components $|\hat{y}|$, $|\hat{z}|$ and M_0^2 become c for $|\xi| \to \infty$, while the other two $|\hat{v}|$ and $|\hat{u}|$ become $c\xi^{-2}$ for $|\xi| \to \infty$. These suggest that our way of constructing the dissipation term for $a \neq 1$ completely matche with the dissipative structure results in Subsection 8.2.3.*

8.2.3 Spectral property

In this subsection, we characterize the dissipative structure of the Timoshenko system of memory type (8.1.6) in order to confirm our pointwise estimates (8.2.2) and (8.2.3) are optimal. To this end, in the first place, we simplify the linearized system (8.2.13) to the first order hyperbolic system without any convolution operators to make the calculations easier.

Assume

$$g(t) = \mu e^{-\mu t} \tag{8.2.36}$$

and put $w = \int_0^t g(t-\tau)z(\tau)\,d\tau$. Moreover, let $\tilde{z} = \sqrt{a-b}\,z$ and $\tilde{w} = \sqrt{b}\,(w-z)$. Then the linearized system (8.2.13) can be reduced to

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - c_1\,\tilde{z}_x - v + c_2\,\tilde{w}_x = 0, \\ u_t - v_x = 0, \\ \tilde{z}_t - c_1\,y_x = 0, \\ \tilde{w}_t + c_2\,y_x + \mu\,\tilde{w} = 0, \end{cases} \tag{8.2.37}$$

where $c_1 = \sqrt{a-b}$, $c_2 = \sqrt{b}$. Next, applying the Fourier transform to (8.2.37), we have

$$\begin{cases} \hat{v}_t - i\xi\hat{u} - \hat{y} = 0, \\ \hat{y}_t - c_1\,i\xi\hat{\tilde{z}} + \hat{v} + c_2\,i\xi\hat{\tilde{w}} = 0, \\ \hat{u}_t - i\xi\hat{v} = 0, \\ \hat{\tilde{z}}_t - c_1\,i\xi\hat{y} = 0, \\ \hat{\tilde{w}}_t + c_2\,i\xi\hat{y} + \mu\,\hat{\tilde{w}} = 0, \end{cases} \tag{8.2.38}$$

where $\xi \in \mathbb{R}$ is the Fourier variable. In vector notation, the system (8.2.38) is written as

$$\hat{\tilde{W}}_t + (i\xi\tilde{A} + \tilde{L})\,\hat{\tilde{W}} = 0, \tag{8.2.39}$$

where

$$\hat{\tilde{W}} = \begin{pmatrix} \hat{v} \\ \hat{y} \\ \hat{u} \\ \hat{\tilde{z}} \\ \hat{\tilde{w}} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -c_1 & c_2 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix}.$$

Consequently, the eigenvalue problem associated with (8.2.38) is given by

$$\lambda\phi + (i\xi\tilde{A} + \tilde{L})\phi = 0, \tag{8.2.40}$$

where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^5$. Namely, the eigenvalue of the system (8.2.39) is the solution $\lambda = \lambda(i\xi) \in \mathbb{C}$ to the characteristic equation

$$\det\{\lambda\phi + (i\xi\tilde{A} + \tilde{L})\phi\} = 0. \tag{8.2.41}$$

We see that this λ for the most simplified system (8.2.37) can be checked to be the eigenvalue of the original linearized system (8.2.12). Now we state our results on the dissipative structure of the system (8.1.6).

Proposition 8.2.9 (Dissipative structure)**.** *The dissipative structure of the Timoshenko system of memory type* (8.1.6) *with* (8.2.36) *could be characterized by*

$$\operatorname{Re}\lambda(i\xi) \le -c\rho(\xi), \quad \rho(\xi) = \begin{cases} \xi^4/(1+\xi^2)^2 & \text{for} \quad a = 1, \\ \xi^4/(1+\xi^2)^3 & \text{for} \quad a \neq 1, \end{cases}$$

where $\lambda(i\xi)$ *denotes the eigenvalue of the system* (8.1.6) *with* (8.2.36)*.*

Remark 8.2.10. *We see that the dissipative structure of* (8.1.6) *is much weaker than that of the dissipative Timoshenko system in the low frequency region;* $\lambda(\xi) \sim -c\xi^4$ *for* $|\xi| \to 0$ *in the case of* (8.1.6)*, while* $\lambda(\xi) \sim -c\xi^2$ *for* $|\xi| \to 0$ *in the case of the dissipative Timoshenko system. However, there is no difference in the high frequence region* $|\xi| \to \infty$ *(see [24, 38]). On the other hand we note that the dissipative structure of* (8.1.6) *is just the same as that of the Timoshenko-Fourier system (see [38]). The dissipative structure of* (8.1.6) *is also the same as that of the Timoshenko-Cattaneo system, however the stability number is a little defferent (see [39]).*

Now we prove the dissipative structure result in Proposition 8.2.9. To this end, we calculate the asymptotic expansions of the eigenvalues of (8.2.37) for $|\xi| \to 0$ and $|\xi| \to \infty$, respectively. Put $i\xi = \zeta \in \mathbb{C}$ and denote by $\lambda_j(\zeta)$ $(j = 1, 2, 3, 4, 5)$ the eigenvalues of the matrix $-(\zeta\tilde{A} + \tilde{L})$ in (8.2.39), which are the solutions to the characteristic equation

$$\begin{aligned} \det\left\{\lambda I + \zeta\tilde{A} + \tilde{L}\right\} &= \lambda^5 + \mu\lambda^4 - \left\{\left(c_1^2 + c_2^2 + 1\right)\zeta^2 - 1\right\}\lambda^3 \\ &- \mu\left\{1 - \left(c_1^2 + 1\right)\zeta^2\right\}\lambda^2 + \left(c_1^2 + c_2^2\right)\zeta^4\lambda + c_1^2\mu\zeta^4 = 0. \end{aligned} \tag{8.2.42}$$

(i) When $|\zeta| \to 0$, $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) = \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \lambda_j^{(3)}\zeta^3 + \lambda_j^{(4)}\zeta^4 \cdots \tag{8.2.43}$$

for $j = 1, 2, 3, 4, 5$. We substitute $\lambda = \lambda_j(\zeta)$ in (8.2.43) into the characteristic quation (8.2.42) and calculate the coefficients $\lambda_j^{(k)}$ $(k = 0, 1, 2\cdots)$, successively. Then we have

$$\lambda_j^{(0)} = \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \pm c_1 i, \quad \lambda_j^{(3)} = 0, \quad \lambda_j^{(4)} = -\frac{c_2^2}{2\mu} \pm \frac{c_1\left(c_1^2 + 1\right)}{2} i \qquad \text{for} \quad j = 1, 2,$$

$$\lambda_j^{(0)} = \pm i, \quad \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \frac{\mu c_2^2}{2\left(1 + \mu^2\right)} \mp \frac{1}{2}\left(c_1^2 + 1 + \frac{c_2^2}{1 + \mu^2}\right) \qquad \text{for} \quad j = 3, 4,$$

$$\lambda_5^{(0)} = -\mu.$$

Consequently, for $|\xi| \to 0$, we obtain

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} -\dfrac{c_2^2}{2\mu}\,\xi^4 + O(|\xi|^5) & \text{for} \quad j = 1, 2, \\ -\dfrac{\mu c_2^2}{2\,(1+\mu^2)}\,\xi^2 + O(|\xi|^3) & \text{for} \quad j = 3, 4, \\ -\mu + O(|\xi|) & \text{for} \quad j = 5. \end{cases} \tag{8.2.44}$$

(ii) Next, we derive the asymptotic expansion of the eigenvalues $\lambda_j(\zeta)$ for $|\zeta| \to \infty$. Let $\eta_j(\zeta^{-1})$ $(j = 1, 2, 3, 4, 5)$ be the eigenvalues of the matrix $-(\tilde{A}+\zeta^{-1}\tilde{L})$, which are the solutions to the characteristic equation

$$\begin{aligned} \det(\eta I + \tilde{A} + \zeta^{-1}\tilde{L}) &= \eta^5 + \mu\zeta^{-1}\eta^4 - \left\{(c_1^2 + c_2^2 + 1) - \zeta^{-2}\right\}\eta^3 \\ &+ \mu\left\{\zeta^{-2} - (c_1^2+1)\right\}\zeta^{-1}\eta^2 + (c_1^2 + c_2^2)\,\eta + c_1^2\mu\zeta^{-1} = 0. \end{aligned} \tag{8.2.45}$$

When $|\zeta|^{-1} \to 0$, we have the asymptotic expansion of $\mu_j(\zeta^{-1})$ in the form

$$\eta_j(\zeta^{-1}) = \eta_j^{(2)}\zeta + \eta_j^{(1)} + \eta_j^{(0)}\zeta^{-1} + \eta_j^{(-1)}\zeta^{-2} + \eta_j^{(-2)}\zeta^{-3} + \cdots \tag{8.2.46}$$

for $j = 1, 2, 3, 4, 5$. This together with the relation $\lambda_j(\zeta) = \zeta\eta_j(\zeta^{-1})$ gives the desired asymptotic expansion of $\lambda_j(\zeta)$ for $|\zeta| \to \infty$:

$$\lambda_j(\zeta) = \eta_j^{(2)}\zeta^2 + \eta_j^{(1)}\zeta + \eta_j^{(0)} + \eta_j^{(-1)}\zeta^{-1} + \eta_j^{(-2)}\zeta^{-2} + \cdots .$$

We substitute the asymptotic expansion of $\eta_j(\zeta^{-1})$ in (8.2.46) into the characteristic equation (8.2.45) and calculate the coefficients $\eta_j^{(k)}$ $(k = 2, 1, 0, -1, -2, \cdots)$, successively. Then, for $j = 1$, we have

$$\eta_1^{(2)} = \eta_1^{(1)} = 0, \quad \eta_1^{(0)} = -\frac{c_1^2}{c_1^2 + c_2^2}\,\mu.$$

For $2, 3, 4, 5$, we calculate the coefficients $\eta_j^{(k)}$ in the case of $c_1^2 + c_2^2 = 1$ and $c_1^2 + c_2^2 \neq 1$, respectively: when $c_1^2 + c_2^2 = 1$, we have

$$\eta_j^{(2)} = 0, \quad \eta_j^{(1)} = \pm 1, \quad \eta_j^{(0)} = \frac{-\mu c_2^2 \pm \sqrt{\mu^2 c_2^4 - 4}}{4},$$

where we note that $\operatorname{Re}\left(\dfrac{-\mu c_2^2 \pm \sqrt{\mu^2 c_2^4 - 4}}{4}\right) < 0$, and when $c_1^2 + c_2^2 \neq 1$, we obtain

$$\begin{aligned} &\eta_j^{(2)} = 0, \quad \eta_j^{(1)} = \pm 1, \quad \eta_j^{(0)} = 0, \\ &\qquad \eta_j^{(-1)} = \pm\frac{1}{2\,(c_1^2 + c_2^2 - 1)}, \quad \eta_j^{(-2)} = \frac{c_2^2}{2\,(c_1^2 + c_2^2 - 1)^2}\,\mu \qquad \text{for} \quad j = 2, 3, \\ &\eta_j^{(2)} = 0, \quad \eta_j^{(1)} = \pm\sqrt{c_1^2 + c_2^2}, \quad \eta_j^{(0)} = -\frac{c_2^2}{2\,(c_1^2 + c_2^2)}\,\mu, \qquad \text{for} \quad j = 4, 5. \end{aligned}$$

Consequently, when $c_1^2 + c_2^2 = 1$, we have

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} -\dfrac{c_1^2}{c_1^2+c_2^2}\,\mu + O(|\xi|^{-1}) & \text{for} \quad j = 1, \\ \operatorname{Re}\left(\dfrac{-\mu c_2^2 \pm \sqrt{\mu^2 c_2^4 - 4}}{4}\right) + O(|\xi|^{-1}) & \text{for} \quad j = 2,3,4,5 \end{cases} \tag{8.2.47}$$

for $|\xi| \to \infty$; while in the case $c_1^2 + c_2^2 \neq 1$, we obtain

$$\operatorname{Re}\lambda_j(i\xi) = \begin{cases} -\dfrac{c_1^2}{c_1^2+c_2^2}\,\mu + O(|\xi|^{-1}) & \text{for} \quad j = 1, \\ -\dfrac{c_2^2}{2\,(c_1^2+c_2^2-1)^2}\,\mu\,\xi^{-2} + O(|\xi|^{-3}) & \text{for} \quad j = 2,3, \\ -\dfrac{c_2^2}{2\,(c_1^2+c_2^2)}\,\mu + O(|\xi|^{-1}) & \text{for} \quad j = 4,5 \end{cases} \tag{8.2.48}$$

for $|\xi| \to \infty$.

Remark 8.2.11. *According to the expansion* (8.2.48) *for* $|\xi| \to \infty$*, when* $c_1^2 + c_2^2 \neq 1$*, three eigenvalues satisfy* $\operatorname{Re}\lambda(i\xi) \sim -c$*, while the other two satisfy* $\operatorname{Re}\lambda(i\xi) \sim -c\xi^{-2}$*. Therefore, the asymptotic expansions* (8.2.44), (8.2.47) *and* (8.2.48) *imply that the pointwise estimate for* $a \neq 1$ *(note that* $c_1^2 + c_2^2$ *corresponds to* a*) shown by Liu and Kawashima in [32] seems not optimal, though the estimate for* $a = 1$ *seems optimal. On the other hand, our pointwise estimates, not only* (8.2.2) *for* $a = 1$*, but also* (8.2.3) *for* $a \neq 1$*, seem optimal. Moreover, the number of the components which cause the regularity-loss also matches with the number of the eigenvalues which satisfy* $\operatorname{Re}\lambda(i\xi) \sim -c\xi^{-2}$ *for* $|\xi| \to \infty$ *(see Subsection 8.2.2).*

8.3 Nonlinear system

The main purpose of Section 8.3 is to show the global-in-time existence and uniqueness of the nonlinear Timoshenko system of memory type (8.1.6) under the least regularity assumption on the initial data.

To state our results, we introduce the energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ by

$$E(t)^2 := \sup_{0\le\tau\le t}\left(\|W\|_{H^s}^2 + \sum_{k=0}^{s}\|M_0^{(k)}\|_{L^2}^2\right)(\tau),$$

$$D(t)^2 := \int_0^t \Big(\|\partial_x v\|_{H^{s-2}}^2 + \|\partial_x y\|_{H^{s-1}}^2 + \|\partial_x^2 u\|_{H^{s-3}}^2 + \|\partial_x^2 z\|_{H^{s-2}}^2 + \sum_{k=0}^{s}\|M_0^{(k)}\|_{L^2}^2\Big)(\tau)\,d\tau,$$

where $M_0^{(k)}(t) := \Big(g[\partial_x^k z, \partial_x^k z] + g(t)|\partial_x^k z|^2\Big)^{\frac{1}{2}}$. Notice that in the dissipation norm $D(t)$ we have 1 regularity-loss for (v, u) but no regularity-loss for (y, z, M_0).

Our main theorem is stated as follows.

Theorem 8.3.1 (Global-in-time existence & uniqueness)**.** *Assume $\sigma'(\eta) = a^2 + O(\eta^2)$ as $\eta \to 0$ and the initial data $W_0 \in H^s$ for $s \geq 2$, and put $E_0 := \|W_0\|_{H^s}$. Then there exists a positive constant δ_0 such that if $E_0 \leq \delta_0$, the Cauchy problem* (8.1.6) *has a unique global solution $W(t)$ with*

$$W \in C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1}).$$

Moreover this solution $W(t)$ verifies the energy estimate

$$E(t)^2 + D(t)^2 \leq C E_0^2.$$

Remark 8.3.2. *Here we used the Sobolev space H^m with negative m, which is defined as usual, namely, $H^m = \{u \ ; \ \int_{\mathbb{R}} (1 + |\xi|^2)^m \, |\hat{u}(\xi)|^2 < \infty\}$.*

Remark 8.3.3. *Our global-in-time existence and uniqueness result holds true with $s \geq 2$, and we do not need L^1_1 property and $\int_{\mathbb{R}} W_0(x)\, dx = 0$, which are required in the previous result [**?**]. This less regularity requirement is due to the refined Lyapunov function, which produces the optimal dissipative estimate for z without any regularity-loss (see $D(t)$). This improvement enables us to control the nonlinearity depending only on z.*

Our global-in-time existence and uniqueness result is shown by the combination of a local existence result and the following a priori estimate result.

Proposition 8.3.4 (A priori estimate)**.** *Assume $\sigma'(\eta) = a^2 + O(\eta^2)$ as $\eta \to 0$ and the initial data $W_0 \in H^s$ for $s \geq 2$, and put $T > 0$ and $\delta > 0$. Let $W(t)$ be the solution to the Cauchy problem* (8.1.6) *satisfying*

$$W \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$$

and

$$\sup_{0 \leq t \leq T} \|W(t)\|_{L^\infty} \leq \delta. \tag{8.3.1}$$

Then there exists a positive constant δ_1 independent of T such that if $E_0 \leq \delta_1$, we have a priori estimate

$$E(t)^2 + D(t)^2 \leq C E_0^2. \tag{8.3.2}$$

Since our system (8.1.6) is a symmetric hyperbolic system, it is not difficult to show the local existence by the standard method based on the successive approximation sequence (see [26]). Therefore we omit the details.

To prove the above a priori estimate in Proposition 8.3.4, we need to show the following energy inequality by applying the energy method.

Proposition 8.3.5 (Energy inequality)**.** *Suppose that* $W_0 \in H^s$ *for* $s \geq 2$ *and put* $E_0 = \|W_0\|_{H^s}$*. Let* $T > 0$ *and let* $W(t)$ *be a solution to the Cauchy problem* (8.1.6)*, satisfying* (8.3.1)*. Then we have the following energy inequality:*

$$E(t)^2 + D(t)^2 \leq CE_0^2 + CE(t)^2D(t)^2, \qquad t \in [0,T]. \tag{8.3.3}$$

We note that the desired a priori estimate (8.3.2) easily follows from the energy inequality (8.3.3), provided that E_0 is suitably small. Therefore it is sufficient to prove (8.3.3) for our purpose (see Subsection 8.3.1).

8.3.1 Energy method

In this subsection we prove the energy estimate (8.3.3) in Proposition 8.3.5 by using the energy method. This proof is essentially parallel of the proof of Theorem 8.2.1 in Subsection 8.2.2. The most crucial point is to estimate the dissipation z, paying attention to the special nonlinearity of the system (8.1.4). In the coming computations, the following sharp estimates for composite functions will be used.

Lemma 8.3.6 ([23, 31])**.** *Let* $1 \leq p, q, r \leq \infty$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$*. Then the following estimates hold:*

$$\|\partial_x^k(uv)\|_{L^p} \;\leq\; C(\|u\|_{L^q}\|\partial_x^k v\|_{L^r} + \|v\|_{L^q}\|\partial_x^k u\|_{L^r})$$

for $k \geq 0$*, and*

$$\|[\partial_x^k, y]\partial_x^k v\|_{L^p} \;\leq\; C(\|\partial_x u\|_{L^q}\|\partial_x^k v\|_{L^r} + \|\partial_x v\|_{L^q}\|\partial_x^k u\|_{L^r})$$

for $k \geq 1$*, where* $[A,B] = AB - BA$ *denotes the commutator and* $L^p = L^p(\mathbb{R})$*.*

Proof of Proposition 8.3.5. Our proof is devided into 5 steps.

Step 1: The goal of this step is to get the estimate (8.3.12) of the basic energy of the solution $W(t)$ in H^2 and the memory-type dissipation terms. First, make the combination $(8.3.19) \times v + (8.3.14) \times y + (8.1.4c) \times u + (8.1.4d) \times \dfrac{\sigma(\frac{z}{a}) - \sigma(0)}{a}$ to have

$$\frac{1}{2}\left\{v^2 + y^2 + u^2 + 2\int_0^{\frac{z}{a}} (\sigma(\eta) - \sigma(0))\, d\eta\right\}_t$$
$$- \{vu + y\,(\sigma\left(\frac{z}{a}\right) - \sigma(0)) - b\, y\, g * z\}_x, - \frac{b}{a}\, z_t\, g * z = 0, \tag{8.3.4}$$

where we used $z_t = a\, y_x$ in the last term. Then, we use the equality (8.2.10) to the last term in (8.3.4). This gives

$$\frac{1}{2}\left\{v^2 + y^2 + u^2 + S(z) - \frac{b}{a}\, G(t)\, z^2 + \frac{b}{a}\, g[z,z]\right\}_t$$
$$- \{vu + y\,(\sigma\left(\frac{z}{a}\right) - \sigma(0)) - b\, y\, g * z\}_x, + \frac{b}{2a}\, \{-g^{'}[z,z] + g(t)z^2\} = 0, \tag{8.3.5}$$

where $S(z) := 2\int_0^{\frac{z}{a}} (\sigma(\eta) - \sigma(0))\, d\eta$ and $G(t) := \int_0^t g(s)\, ds$. Here, we have

$$S(z) - \frac{b}{a}\, G(t)\, z^2 = \left(1 - \frac{b}{a}\, G(t)\right) z^2 + O(z^4)$$

as $z \to 0$ and $1 - \frac{b}{a}\, G(t) \geq 1 - \frac{b}{a} > 0$. Therefore, integrating (8.3.5) with respect to $x \in \mathbb{R}$ yields the following basic energy equality:

$$\frac{d}{dt}\, E_0^{(0)} + \frac{b}{a} \int_{\mathbb{R}} (-g'[z,z] + g(t) z^2)\, dx = 0, \tag{8.3.6}$$

where

$$E_0^{(0)} := \|(v,y,u)\|_{L^2}^2 + \int_{\mathbb{R}} \left(S(z) - \frac{b}{a}\, G(t)\, z^2 + \frac{b}{a}\, g[z,z]\right) dx.$$

Next, apply ∂_x^k $(k \geq 1)$ to (8.3.19), (8.3.14), (8.1.4c) and (8.1.4d), and write $\partial_x^k(v,y,u,z) = (V,Y,U,Z)$ for simplicity to have

$$V_t - U_x - Y = 0, \tag{8.3.7a}$$

$$Y_t - \sigma'\left(\frac{z}{a}\right)\left(\frac{Z}{a}\right)_x - V + bg * Z_x = \left[\partial_x^k, \sigma'\left(\frac{z}{a}\right)\right]\left(\frac{z}{a}\right)_x, \tag{8.3.7b}$$

$$U_t - V_x = 0, \tag{8.3.7c}$$

$$Z_t - aY_x = 0. \tag{8.3.7d}$$

Then, make the combination $(8.3.7a) \times V + (8.3.7b) \times Y + (8.3.7c) \times U + (8.3.7d) \times \sigma'\left(\frac{z}{a}\right) \frac{Z}{a^2}$, and use (8.2.10) to have

$$\begin{aligned} &\frac{1}{2}\left\{V^2 + Y^2 + U^2 + \sigma'\left(\frac{z}{a}\right)\left(\frac{Z}{a}\right)^2 - \frac{b}{a}\, G(t)\, Z^2 + \frac{b}{a}\, g[Z,Z]\right\}_t \\ &\quad - \left\{VU + Y\sigma'\left(\frac{z}{a}\right)\left(\frac{Z}{a}\right) - b\, y\, g * Z\right\}_x + \frac{b}{2a}\left\{-g'[Z,Z] + g(t)\, Z^2\right\} \\ &= \frac{1}{2}\left(\frac{z}{a}\right)_t\left(\frac{Z}{a}\right)^2 - Y\sigma'\left(\frac{z}{a}\right)_x\left(\frac{Z}{a}\right) + Y\left[\partial_x^k, \sigma'\left(\frac{z}{a}\right)\right]\left(\frac{z}{a}\right)_x. \end{aligned} \tag{8.3.8}$$

Here we note that

$$\sigma'\left(\frac{z}{a}\right)\left(\frac{Z}{a}\right)^2 - G(t)^2 = \left(1 - \frac{b}{a}\, G(t)\right) Z^2 + O(z^2) Z^2$$

as $z \to 0$ and $1 - \dfrac{b}{a}\, G(t) \geq 1 - \dfrac{b}{a} > 0$. Therefore, integrating (8.3.8) with respect to $x \in \mathbb{R}$ yields

$$\frac{d}{dt}\, E_0^{(k)} + \frac{b}{a} \int_{\mathbb{R}} (-g^{'}[\partial_x^k z, \partial_x^k z] + g(t)|\partial_x^k z|^2)\, dx \leq R_0^{(k)} \tag{8.3.9}$$

for $1 \leq k \leq s$, where

$$E_0^{(k)} := \|\partial_x^k (v, y, u)\|_{L^2}^2 + \int_{\mathbb{R}} \left(\sigma^{'}\left(\frac{z}{a}\right) \left(\frac{\partial_x^k z}{a}\right)^2 - \frac{b}{a}\, G(t)|\partial_x^k z|^2 + \frac{b}{a}\, g[\partial_x^k z, \partial_x^k z] \right) dx,$$

$$R_0^{(k)} := \int_{\mathbb{R}} \left| \sigma^{'}\left(\frac{z}{a}\right)_t \right| |\partial_x^k z|^2 + \left| \sigma^{'}\left(\frac{z}{a}\right)_x \right| |\partial_x^k y\, \partial_x^k z| + |\partial_x^k y| \left| \left[\partial_x^k, \sigma^{'}\left(\frac{z}{a}\right)\right] z_x \right| dx.$$

Here, denote

$$M_0^{(k)}(t) = \left(g[\partial_x^k z, \partial_x^k z] + g(t)|\partial_x^k z|^2 \right)^{\frac{1}{2}}$$

for $k \geq 0$, then the following estimates hold.

$$\begin{cases} |g * \partial_x^k z| \leq C(|\partial_x^k z| + M_0^{(k)}), \\ |(g * \partial_x^k z)_t| + |(g^{'} * \partial_x^k z)_t| \leq C M_0^{(k)}. \end{cases} \tag{8.3.10}$$

Next, integrate the combination (8.3.6) + (8.3.9) with respect to t to have

$$E_0^{(k)}(t) + \int_{\mathbb{R}} g[\partial_x^k z, \partial_x^k z]\, dx + \int_0^t \|M_0^{(k)}(\tau)\|_{L^2}^2\, d\tau \leq E_0^{(k)}(0) + C \int_0^t R_0^{(k)}(\tau) d\tau$$

for $0 \leq k \leq s$, where $R_0^0 = 0$. By virtue of the assumptions $\sigma^{'}(\eta) > 0$ and (8.3.1), we can regard $E_0^{(k)}$ as

$$\|\partial_x^k W\|_{L^2}^2 + \int_{\mathbb{R}} g[\partial_x^k z, \partial_x^k z]\, dx.$$

Therefore, we obtain

$$\begin{aligned} &\|\partial_x^k W(t)\|_{L^2}^2 + \int_{\mathbb{R}} g[\partial_x^k z, \partial_x^k z]\, dx + \int_0^t \|M_0^{(k)}(\tau)\|_{L^2}^2\, d\tau \\ &\qquad \leq C\|\partial_x^k W_0\|_{L^2}^2 + C \int_0^t R_0^{(k)}(\tau) d\tau \end{aligned} \tag{8.3.11}$$

for $0 \leq k \leq s$. Here, by using (8.3.1) and Lemma 8.3.6, $R_0^{(k)}$ can be estimated as

$$R_0^{(k)} \leq C\|y_x\|_{L^\infty}\|z_x\|_{L^\infty}(\|\partial_x^k y\|_{L^2}^2 + \|\partial_x^k z\|_{L^2}^2).$$

Thus, add (8.3.11) for k with $0 \le k \le s$ to have

$$\|W(t)\|_{H^s}^2 + \sum_{k=0}^{s} \int_{\mathbb{R}} g[\partial_x^k z, \partial_x^k z]\, dx + \sum_{k=0}^{s} \int_0^t \|M_0^{(k)}(\tau)\|_{L^2}^2\, d\tau$$
$$\le C\|W_0\|_{H^s}^2 + CE(t)^2 D(t)^2. \tag{8.3.12}$$

Step 2: In this step we make the dissipation estimate of y (8.3.18) below. Firstly, we apply the Fourier transform to the nonlinear system (8.1.4) and rewrite the result as

$$\hat{v}_t - i\xi\hat{u} - \hat{y} = 0, \tag{8.3.13a}$$

$$\hat{y}_t - a\, i\xi\hat{z} + \hat{v} + b\, i\xi g * \hat{z} = i\xi\widehat{f(z)}, \tag{8.3.13b}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{8.3.13c}$$

$$\hat{z}_t - a\, i\xi\hat{y} = 0, \tag{8.3.13d}$$

where $f(z) := \sigma\Big(\dfrac{z}{a}\Big) - \sigma(0) - \sigma'(0)\Big(\dfrac{z}{a}\Big) = O(z^3)$, which follows from $\sigma' = a^2 + O(\eta^2)$. Then, we make the dissipation estimate of y just in the same way as **Step 2** in the proof of Theorem 8.2.1. This gives

$$\frac{1}{2}\,(1+\xi^2)\, E_{2,t} + a\, g(0)\xi^2|\hat{y}|^2 = \xi^2\widehat{f(z)}(g * \bar{\hat{z}})_t + R_2 \tag{8.3.14}$$

with E_2 and R_2, which can be estimated as

$$|E_2| \le C(|\hat{W}|^2 + M_0^2), \quad |R_2| \le C\xi^2|\hat{z}|M_0 + C|\xi|(|\hat{v}| + |\hat{y}|)M_0.$$

Therefore, multiplying (8.3.14) by $1/(1+\xi^2)$, and by using (8.2.20) and the Young inequality, we obtain

$$\frac{1}{2}\, E_{2,t} + a\, g(0)(1-\varepsilon)\, \frac{\xi^2}{1+\xi^2}|\hat{y}|^2$$
$$\le C_\varepsilon \frac{\xi^2}{(1+\xi^2)^2}\, |\hat{v}|^2 + C_\varepsilon \frac{\xi^4}{(1+\xi^2)^2}\, |\hat{z}|^2 + C_\varepsilon M_0^2 + C_\varepsilon \frac{\xi^4}{(1+\xi^2)^2}\, |\widehat{f(z)}|^2 \tag{8.3.15}$$

for $\varepsilon > 0$. We note that we have estimated the term $\xi^2\widehat{f(z)}(g * \bar{\hat{z}})_t$ of the nonlinear part as follows:

$$\frac{\xi^2}{1+\xi^2}\, |\widehat{f(z)}||(g * \bar{\hat{z}})_t| \le C_\varepsilon \frac{\xi^4}{(1+\xi^2)^2}\, |\widehat{f(z)}|^2 + \varepsilon M_0^2.$$

Next, we integrate (8.3.15) with respect to t over $(0, t)$. Then we multiply the resultant inequality by $(1+\xi^2)^{2(s-k)}|\xi|^{2k}$ and integrate with respect to $\xi \in \mathbb{R}$, where $0 \le k \le s-1$.

This yields

$$\int_0^t \|\partial_x y(\tau)\|_{H^{s-1}}^2\, d\tau \;\le\; C\|W_0\|_{H^s}^2 + C\|W(t)\|_{H^s}^2 + C\sum_{k=0}^{s} \|M_0^{(k)}(t)\|_{L^2}^2$$
$$+\,C\int_0^t \left(\|\partial_x v\|_{H^{s-2}}^2 + \|\partial_x^2 z\|_{H^{s-2}}^2 + \sum_{k=0}^{s} \|M_0^{(k)}\|_{L^2}^2 + \sum_{k=0}^{s-2} \|\partial_x^{k+2} f(z)\|_{L^2}^2 \right)(\tau)\, d\tau \tag{8.3.16}$$

for $s \ge 2$. Here, by using $\sigma'(\eta) = a^2 + O(\eta^2)$ as $\eta \to 0$, (8.3.1), Lemma 8.3.6 and (8.3.10), we have

$$\|\partial_x^{k+2} f(z)\|_{L^2} \le C\|z\|_{L^\infty}^2 \|\partial_x^{k+2} z\|_{L^2} \qquad \text{for} \;\; 0 \le k \le s-2,$$

which implies

$$\begin{aligned}\int_0^t \sum_{k=0}^{s-2} \|\partial_x^{k+2} f(z)\|_{L^2}^2(\tau)\, d\tau &\le C \sup_{0\le\tau\le t} \left(\|z(\tau)\|_{L^\infty}^4\right) \int_0^t \|\partial_x^2 z(\tau)\|_{H^{s-2}}^2\, d\tau \\ &\le C \sup_{0\le\tau\le t} \left(\|W(\tau)\|_{H^2}^2\right) \int_0^t \|\partial_x^2 z(\tau)\|_{H^{s-2}}^2\, d\tau \\ &\le CE(t)^2 D(t)^2. \end{aligned} \tag{8.3.17}$$

Thus, by using (8.3.12), from the inequarity (8.3.16) we arrive at the dissipation estimate of y

$$\int_0^t \|\partial_x y(\tau)\|_{H^{s-1}}^2\, d\tau \;\le\; C\|W_0\|_{H^s}^2 + CE(t)^2 D(t)^2. \tag{8.3.18}$$

Step 3: We make the dissipation estimate of v just in the same way as **Step 3** in the proof of Theorem 8.2.1. This gives

$$E_{3,t} + |\hat{v}|^2 = i\xi\widehat{f(z)}\,\bar{\hat{v}} + R_3, \tag{8.3.19}$$

where, by using (8.2.20), E_3 and R_3 can also be estimated as

$$|E_3| \le C(|\hat{W}|^2 + M_0^2), \quad |R_3| \le |\hat{y}|^2 + C|\xi||\hat{y}||\hat{u}| + C|\hat{u}|M_0.$$

Therefore, multiplying (8.3.19) by $\xi^2/(1+\xi^2)^2$, and by using (8.2.20) and the Young inequality, we obtain

$$\frac{\xi^2}{(1+\xi^2)^2}\,E_{3,t} + (1-\varepsilon)\,\frac{\xi^2}{(1+\xi^2)^2}\,|\hat{v}|^2$$
$$\le C_\varepsilon \frac{\xi^2}{1+\xi^2}\,|\hat{y}|^2 + C_\varepsilon \frac{\xi^4}{(1+\xi^2)^3}\,|\hat{u}|^2 + C_\varepsilon M_0^2 + C_\varepsilon \frac{\xi^4}{(1+\xi^2)^2}\,|\widehat{f(z)}|^2 \tag{8.3.20}$$

for $\varepsilon > 0$. We note that we have estimated the term $i\xi\widehat{f(z)}\,\bar{\hat{v}}$ of the nonlinear part as follows:

$$\frac{|\xi|^3}{(1+\xi^2)^2}\,|\widehat{f(z)}||\hat{v}| \le C_\varepsilon \frac{\xi^4}{(1+\xi^2)^2}\,|\widehat{f(z)}|^2 + \varepsilon\frac{\xi^2}{(1+\xi^2)^2}\,|\hat{v}|^2.$$

Next, we integrate (8.3.20) with respect to t over $(0,t)$. Then we multiply the resultant inequality by $(1+\xi^2)^{2(s-k)}|\xi|^{2k}$ and integrate with respect to $\xi \in \mathbb{R}$, where $0 \le k \le s-1$. This yields

$$\begin{aligned}\int_0^t \|\partial_x v(\tau)\|^2_{H^{s-2}}\,d\tau \;\le\; & C\|W_0\|^2_{H^s} + C\|W(t)\|^2_{H^s} + C\sum_{k=0}^{s}\|M_0^{(k)}(t)\|^2_{L^2}\\ & + C\int_0^t\left(\|\partial_x y\|^2_{H^{s-1}} + \|\partial_x^2 u\|^2_{H^{s-3}} + \sum_{k=0}^{s}\|M_0^{(k)}\|^2_{L^2} + \sum_{k=0}^{s-2}\|\partial_x^{k+2}f(z)\|^2_{L^2}\right)(\tau)\,d\tau\end{aligned} \tag{8.3.21}$$

for $s \ge 2$. By using (8.3.12) and (8.3.17), from (8.3.21) we arrive at the dissipation estimate of v

$$\int_0^t \|\partial_x v(\tau)\|^2_{H^{s-2}}\,d\tau \;\le\; C\|W_0\|^2_{H^s} + CE(t)^2D(t)^2. \tag{8.3.22}$$

Step 4: We make the dissipation estimate of u and z just in the same way as **Step 4** in the proof of Theorem 8.2.1. This gives

$$\begin{aligned}&\xi\{E_4 + (1+\xi^2)\,E_5\}_t + \xi^2\{|\hat{u}|^2 + (a - b\,G(t))(1+\xi^2)|\hat{z}|^2\}\\ &\quad = -\xi^2(1+\xi^2)\widehat{f(z)}\,\bar{\hat{z}} + R_4,\end{aligned} \tag{8.3.23}$$

where, by using (8.2.20), E_4 and R_4 can be estimated as

$$\begin{cases} |E_4| + (1+\xi^2)|E_5| \le (1+\xi^2)|\hat{W}|^2,\\ |R_4| \le C\xi^2\{|\hat{v}|^2 + (1+\xi^2)|\hat{y}|^2\}\\ \qquad + |\xi|\{|\hat{y}||\hat{u}| + (1+\xi^2)|\hat{v}||\hat{z}|\} + C\xi^2(1+\xi^2)|\hat{z}|M_0.\end{cases}$$

Therefore, multiplying (8.3.23) by $\xi^2/(1+\xi^2)^3$, and by using (8.1.3), (8.2.20) and the Young inequality, we obtain

$$\begin{aligned}&\frac{\xi^3}{(1+\xi^2)^3}\{E_4 + (1+\xi^2)\,E_5\}_t + (1-\varepsilon)\,\frac{\xi^4}{(1+\xi^2)^3}\{|\hat{u}|^2 + (1+\xi^2)|\hat{z}|^2\}\\ &\quad \le C_\varepsilon\frac{\xi^2}{(1+\xi^2)^2}\,|\hat{v}|^2 + C_\varepsilon\frac{\xi^2}{1+\xi^2}\,|\hat{y}|^2 + C_\varepsilon M_0^2 + C_\varepsilon\frac{\xi^4}{(1+\xi^2)^2}\,|\widehat{f(z)}|^2\end{aligned} \tag{8.3.24}$$

for $\varepsilon > 0$. We note that we have estimated the term $-\xi^2(1+\xi^2)\widehat{f(z)}\,\bar{\hat{z}}$ of the nonlinear part as follows:

$$\frac{\xi^4}{(1+\xi^2)^2}\,|\widehat{f(z)}||\hat{z}| \le C_\varepsilon \frac{\xi^4}{(1+\xi^2)^2}\,|\widehat{f(z)}|^2 + \varepsilon \frac{\xi^4}{(1+\xi^2)^2}\,|\hat{z}|^2.$$

Next, we integrate (8.3.24) with respect to t over $(0,t)$. Then we multiply the resultant inequality by $(1+\xi^2)^{2(s-k)}|\xi|^{2k}$ and integrate with respect to $\xi \in \mathbb{R}$, where $0 \le k \le s-2$. This yields

$$\begin{aligned}\int_0^t \left(\|\partial_x^2 u\|^2_{H^{s-3}} + \|\partial_x^2 z\|^2_{H^{s-2}}\right)(\tau)\,d\tau \;\le\; C\|W_0\|^2_{H^s} + C\|W(t)\|^2_{H^s} + C\sum_{k=0}^{s}\|M_0^{(k)}(t)\|^2_{L^2} \\ + C\int_0^t \left(\|\partial_x v\|^2_{H^{s-2}} + \|\partial_x y\|^2_{H^{s-1}} + \sum_{k=0}^{s}\|M_0^{(k)}\|^2_{L^2} + \sum_{k=0}^{s-2}\|\partial_x^{k+2} f(z)\|^2_{L^2}\right)(\tau)\,d\tau\end{aligned} \tag{8.3.25}$$

for $s \ge 2$. By using (8.3.12) and (8.3.17), the desired estimate of u and z follows from (8.3.25):

$$\int_0^t \left(\|\partial_x^2 u\|^2_{H^{s-3}} + \|\partial_x^2 z\|^2_{H^{s-2}}\right)(\tau)\,d\tau \;\le\; C\|W_0\|^2_{H^s} + CE(t)^2 D(t)^2. \tag{8.3.26}$$

Step 5: Consequently, we add all the dissipation estimates (8.3.18), (8.3.22) and (8.3.26) to have

$$\begin{aligned}&\int_0^t \left(\|\partial_x v\|^2_{H^{s-2}} + \|\partial_x y(t)\|^2_{H^{s-1}} + \|\partial_x^2 u\|^2_{H^{s-3}} + \|\partial_x^2 z\|^2_{H^{s-2}}\right)(\tau)\,d\tau \\ &\le C\|W_0\|^2_{H^s} + CE(t)^2 D(t)^2.\end{aligned} \tag{8.3.27}$$

Finally, we add (8.3.27) to the estimate of the basic energy and the memory-type dissipation terms (8.3.12). This yields

$$\begin{aligned}&\|W(t)\|^2_{H^s} + \sum_{k=0}^{s}\int_{\mathbb{R}} g[\partial_x^k z, \partial_x^k z]\,dx + \sum_{k=0}^{s}\int_0^t \|M_0^{(k)}(\tau)\|^2_{L^2}\,d\tau \\ &\quad + \int_0^t \left(\|\partial_x v\|^2_{H^{s-2}} + \|\partial_x y(t)\|^2_{H^{s-1}} + \|\partial_x^2 u\|^2_{H^{s-3}} + \|\partial_x^2 z\|^2_{H^{s-2}}\right)(\tau)\,d\tau \\ &\le C\|W_0\|^2_{H^s} + CE(t)^2 D(t)^2.\end{aligned} \tag{8.3.28}$$

Thus, from (8.3.28), we arrive at the desired energy inequarity (8.3.3) in Proposition 8.3.5.

□

Bibliography

[1] H. Abidi, Equation de Navier-Stokes avec densité et viscosité variables dans l'espace critique, *Revista Matemática Iberoamericana*, **23** (2007) 537-586.

[2] H. Abidi, G. L. Gui and P. Zhang, On the wellposedness of three-dimensional inhomogeneous Navier-Stokes equations in the critical spaces, *Arch. Ration. Mech. Anal.*, **204** (2012) 189-230.

[3] F. Amar-Khodja, A. Benabdallah, J.E. Muñoz Rivera, and R. Racke, Energy decay for Timoshenko systems of memory type, *J. Differential Equations*, **194** (2003) 82-115.

[4] M. S. Alves, C. A Raposo, J. E. Muñoz Rivera, M. Sepulveda and O. Vera Villagran, Uniform stabilization for the transmission problem of the Timoshenko system with memory, *J. Math. Anal. Appl.*, **369** (2010) 323-345.

[5] H. Bahouri, J. Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, *Grundlehren der Mathematischen Wissenschaften*, Berlin: Springer-Verlag, 2011.

[6] S. Bianchini, B. Hanouzet, R. Natalini, Asymptotic behavior of smooth solutions for partially dissipative hyperoblic systems with a convex entropy, *Comm. Pure Appl. Math.*, **60** (2007) 1559-1622.

[7] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, *J. Differential Equations*, **121** (1995) 314-328.

[8] G. Q. Chen, J. W. Jermore and D. H. Wang, Compressible Euler-Maxwell equations, *Transp. Theory Statist. Phys.*, **29** (2000) 311-331.

[9] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Invent. Math.*, **141** (2000) 579-614.

[10] R. Danchin, On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework, *J. Differential Equations*, **248** (2010) 2130-2170.

[11] F. Dell'oro and V. Pata, On the stability of Timoshenko systems with Gurtin-Pipkin thermal law, *eprint* arXiv:1304.0985 (2013).

[12] P. M. N. Dharmawardane, Global solutions and decay property of regularity-loss type for quasi-linear hyperbolic systems with dissipation, *J. Hyper. Differ. Equ.*, **10** (2013) 37-76.

[13] P.M.N. Dharmawardane, J.E. Muñoz Rivera and S. Kawashima, Decay property for second order hyperbolic systems of viscoelastic materials, *J. Math. Anal. appl.*, **366** (2010) 621-635.

[14] P. M. N. Dharmawardane, T. Nakamura and S. Kawashima, Decay estimates of solutions for quasi-linear hyperbolic systems of viscoelasticity, *SIAM J. Math. Anal.*, **44** (2012) 1976-2001.

[15] R. Duan, Global smooth flows for the compressible Eluer-Maxwell system: Relaxation case, *J. Hyper. Differ. Equ.*, **8** (2011) 375-413.

[16] R. Duan, Dissipative property of the Vlasov-Maxwell-Boltzmann system with a uniform ionic background, *SIAM J. Math. Anal.*, **43** (2011) 2732-2757.

[17] R.-J. Duan, Q. Q. Liu, and C. J. Zhu, The Cauchy problem on the compressible two-fluids Euler-Maxwell equations, *SIAM J. Math. Anal.*, **44** (2012) 102-133.

[18] R.-J. Duan and R. M. Strain, Optimal large-Time behavior of the Vlasov-Maxwell-Boltzmann system in the whole space, *Commun. Pure Appl. Math.* LXIV (2011) 1497-1546.

[19] H. D. Fernández Sare and R. Racke, On the stability of damped Timoshenko systems: Cattaneo vs. Fourier law, *Arch. Rational Mech. Anal.*, **194** (2009) 221-251.

[20] K. Graff, Wave Motion in Elastic Solids, *Dover*, New York (1975).

[21] B. Hanouzet and R. Natalini, Global existence of smooth solutions for partially disipative hyperbolic systems with a convex entropy, *Arch. Rational Mech. Anal.*, **169** (2003) 89-117.

[22] B. Haspot, Existence of global strong solutions in critical spaces for barotropic viscous fluids, *Arch. Ration. Mech. Anal.*, **202** (2011) 427-460.

[23] T. Hosono and S. Kawashima, Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system, *Math. Models Methods Appl. Sci.*, **16** (2006) 1839-1859.

[24] K. Ide, K. Haramoto and S. Kawashima, Decay property of regularity-loss type for dissipative Timoshenko system, *Math. Models Meth. Appl. Sci.*, **18** (2008) 647-667.

[25] K. Ide and S. Kawashima, Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system, *Math. Models Meth. Appl. Sci.*, **18** (2008) 1001-1025.

[26] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Rational Mech. Anal.*, **58** (1975) 181-205.

[27] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, *Doctoral Thesis*, Kyoto University (1984) http://repository.kulib.kyoto-u.ac.jp/dspace/handle/2433/97887

[28] S. Kawashima, Global existence and stability of solutions for discrete velocity models of the Boltzmann equation, Recent Topics in Nonlinear PDE (M. Mimura and T. Nishida, eds.), Lecture Notes in *Num. Appl. Anal.*, **6** Kinokuniya (1983) 59-85.

[29] S. Kawashima, W.-A. Yong, Dissipative structure and entropy for hyperbolic systems of balance laws, *Arch. Rational Mech. Anal.*, **174** (2004) 345-364.

[30] S. Kawashima and W.-A. Yong, Decay estimates for hyperbolic balance laws, *J. Anal. Appl.*, **28** (2009) 1-33.

[31] Y. Liu and S. Kawashima, Global existence and asymptotic behabior of solutions for quasi-linear dissipative plate wquation, *Discrete Contin. Dyn. Syst.*, **29** (2011) 1113-1139.

[32] Y. Liu and S. Kawashima, Decay property for the Timoshenko system with memory-type dissipation, *Math. Models Meth. Appl. Sci.*, **22** (2012) 1-19.

[33] Y. Liu and S. Kawashima, Global existence and asymptotic decay of solutions to the nonlinear Timoshenko system with memory, *Nonlinear Analysis*, **84** (2013) 1-17.

[34] Z. Liu and C. Peng, Exponential stability of a viscoelastic Timoshenko beam, *Adv. Math. Sci. Appl.*, **8** (1998) 343-351.

[35] A. Majda, Compressible Fluid Flow and Conservation laws in Several Space Variables, *Springer-Verlag*: Berlin/New York (1984).

[36] P. A. Markowich, C. Ringhofer and C. Schmeiser, Semiconductor Equations , Vienna, *Springer-Verlag* (1990).

[37] A. Matsumura, An energy method for the equations of motion of compressible viscous and heat-conductive fluids, *MRC Technical Summary Report*, Univ. of Wisconsin-Masison, ♯ 2194 (1981)

[38] N. Mori and S. Kawashima, Decay property for the Timoshenko system with Fourier's type heat conduction, *J. Hyperbolic Differential Equations*, **11** (2014) 135-157.

[39] N. Mori, S. Kawashima, Decay property of the Timoshenko-Cattaneo system, *Analysis and Applications* (2015) DOI: 10.1142.

[40] A. Matsumura and T. Nishida, The initial value problem for the quations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.*, **20** (1980) 67-104.

[41] J.E. Muñoz Rivera, M.G. Naso and F.M. Vegni, Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory, *J. Math. Anal. appl.*, **286** (2003) 692-704.

[42] N. Mori, J. Xu, S. Kawashima, Global existence and optimal decay rates for the Timoshenko system: The case of equal wave speeds, *J. Differential Equations*, **258** (2015) 1494-1518.

[43] L. Nirenberg, On elliptic partial differential equation, *Ann. Scu. Norm. Sup. Pisa*, Ser. III **13** (1959) 115-162.

[44] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system, *J. Funct. Anal.*, **262** (2012) 3556-3584.

[45] Y.-J. Peng, Stability of non-constant equilibrium solutions for Euler-Maxwell equations, *J. Math. Pure Anal.*, **103** (2015) 39-67.

[46] Y.-J. Peng, S. Wang and G.L. Gu, Relaxation limit and global existence of smooth solutions of compressible Euler-Maxwell equations, *SIAM J. Math. Anal.*, **43** (2011) 944-970.

[47] C. A. Raposo, W. D. Bastos and M. L. Santos, A transmission problem for Timoshenko system, *Comput. Appl. Math.*, **26** (2007) 215-234.

[48] C. A. Raposo, J. Ferreira, M. L. Santos and N. N. O. Castro, Exponential stability for the Timoshenko system with two weak damping, *Appl. Math. Lett.*, **18** (2005) 535-541.

[49] J. E. Muñoz Rivera and R. Racke, Mildy dissipative nonlinear Timoshenko systems: Global existence and exponential stability, *J. Math. Anal. Appl.*, **276** (2002) 248-278.

[50] J.E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems, *Discrete and Continuous Dynamical Systems*, **9** (2003) 1625-1639.

[51] J.E. Muñoz Rivera and R. Racke, Timoshenko systems with indefinite damping, *J. Math. Anal. Appl.*, **341** (2008) 1068-1083.

[52] J.E. Muñoz Rivera and H.D. Fernández Sare, Stability of Timoshenko systems with past history, *J. Math. Anal. Appl.*, **339** (2008) 482-502.

[53] R. Racke and B. Said-Houari, Decay rates and global existence for semilinear dissipatie Timoshenko systems, *Quart. Appl. Math.*, **71** (2013) 229-266.

[54] M. L. Santos, D. S. Almeida Júnior and J. E. Muñoz Rivera, The stability number of the Timoshenko system with second sound, *J. Differential Equations*, **253** (2012) 2715-2733.

[55] B. Said-Houari and A. Kasimov, Decay property of Timoshenko system in thermoelasticity, *Math. Methods Appl. Sci.*, **35** (2012) 314-333.

[56] H.D. Fernández Sare and R. Racke, On the stability of damped timoshenko systems-cattaneo versus fourier's law, *Arch. Rational Mech. Anal.*, **194** (2009) 221-251.

[57] D. Shi, D. Feng and Q. Yan, Feedback stabilization of rotating Timoshenko beam with adaptive gain, *Internet. J. Control.*, **3** (2001) 239-251.

[58] Y. Shizuta and S. Kawashima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, *Hokkaido Math. J.*, **14** (1985) 249-275.

[59] A. Soufyane, Stabilisation de la poutre de Timoshenko, *C. R. Acad. Sci. Paris Sér. I Math.*, **328** (1999) 731-734.

[60] Y. Sugitani and S. Kawashima, Decay estimates of solutions to a semi-linear dissipative plate equation, *J. Hyper. Differ. Equ.*, **7** (2010) 471-501.

[61] Z.Tan, Y.J.Wang and Y.Wang, Decay estimates of solutions to the compressible Euler-Maxwell system in $\mathbb{R}^3$, *J. Differential Equations*, **257** (2014) 2846-2873.

[62] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philosophical Magazine*, **41** (1921) 744-746.

[63] S.P. Timoshenko, On the transverse vibrations of bars of uniform cross-section, *Philosophical Magazine*, **43** (1922) 125-131.

[64] Y. Ueda, R.-J., Duan and S. Kawashima, Decay structure for symmetric hyperbolic systems with non-symmetric relaxation and its application, *Arch. Rational Mech. Anal.*, **205** (2012) 239-266.

[65] Y. Ueda and S. Kawashima, Decay property of regularity-loss type for the Euler-Maxwell system, *Methods Appl. Anal.*, **18** (2011) 245-268.

[66] Y. Ueda, S. Wang, and S. Kawashima, Dissipative structure of the regularity-loss type and time asymptotic decay of solutions for the Euler-Maxwell system, *SIAM J. Math. Anal.*, **44** (2012) 2002-2017.

[67] T. Umeda, S. Kawashima and Y. Shizuta, On the decay of solutions to the linearized equations of electro-magneto-fluid dynamics, *Japan J. Appl. Math.*, **1** (1984) 435-457.

[68] J. H. Wu, Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces, *Commun. Math. Phys.*, **263** (2005) 803-831.

[69] J. Xu, Global classical solutions to the compressible Euler-Maxwell equations, *SIAM J. Math. Anal.*, **43** (2011) 2688-2718.

[70] J. Xu and S. Kawashima, Global classical solutions for partially dissipative hyperbolic system of balance laws, *Arch. Rational Mech. Anal.*, **211** (2014) 513-553.

[71] J. Xu and S. Kawashima, The optimal decay estimates on the framework of Besov spaces for generally dissipative systems, *Arch. Rational Mech. Anal.*, (2015) doi: 10.1007/s00205-015-0860-3.

[72] J. Xu, N. Mori, S. Kawashima, L^p-L^q-L^r estimates and minimal decay regularity for compressible Euler-Maxwell equations, *J. Math. Pures Appl.*, **104** (2015) 965-981.

[73] J. Xu, N. Mori, S. Kawashima, Global existence and minimal decay regularity for the Timoshenko system: The case of non-equal wave speeds, *J. Differential Equations*, **259** (2015) 5533-5553.

[74] J. Xu and Z. J. Wang, Relaxation limit in Besov spaces for compressible Euler equations, *J. Math. Pures Appl.*, **99** (2013) 43-61.

[75] J. Xu, J. Xiong and S. Kawashima, Global well-posedness in critical Besov spaces for two-fluid Euler-Maxwell equations, *SIAM J. Math. Anal.*, **45** (2013) 1422-1447.

[76] J.Xu and Wen-A.Yong, Relaxation-time limits of non-isentropic hydrodynamic models for semiconductors, *J. Differential Equations*, **247** (2009) 1777-1795.

[77] W.-A. Yong, Entropy and global existence for hyperbolic balance laws, *Arch. Rational Mech. Anal.*, **172** (2004) 247-266.

[78] Y.Zhou, Local well-posedness for the incompressible Euler equations in the critical Besov spaces, *Ann. Inst. Fourier*, **54** (2004) 773-786.

List of Papers

Refereed Conference Papers:

1. N. Mori, S. Kawashima, Decay property for the Timoshenko system with Fourier's type heat conduction, J. Hyperbolic Differential Equations, Vol. 11, No. 1, pp. 135- 157 (2014).

2. N. Mori, S. Kawashima, Decay property of the Timoshenko-Cattaneo system, Analysis and Applications, DOI: 10.1142 (2015).

3. N. Mori, J. Xu, S. Kawashima, Global existence and optimal decay rates for the Timoshenko system: The case of equal wave speeds, J. Differential Equations, Vol. 258, Issue 5, pp. 1494- 1518 (2015).

4. J. Xu, N. Mori, S. Kawashima, L^p-L^q-L^r estimates and minimal decay regularity for compressible Euler-Maxwell equations, Journal de Mathématiques Pures Appliquées, Vol. 104, Issue 5, pp. 965-981 (2015).

5. J. Xu, N. Mori, S. Kawashima, Global existence and minimal decay regularity for the Timoshenko system: The case of non-equal wave speeds, J. Differential Equations, Vol. 259, Issue 11, pp. 5533- 5553 (2015).

Preprint:

1. N. Mori, S. Kawashima, Global existence and nonlinear stability for the dissipative Timoshenko system, RIMS Workshop Mathematical Analysis of Viscous Incompressible Fluid

2. N. Mori, J. Xu, S. Kawashima, Global existence and optimal decay rates of solutions to the classical Timoshenko system in the framework of Besov spaces, RIMS Workshop Mathematical Analysis in Fluid and Gas Dynamics.

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