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Contents

1 Introduction

In this paper, we prove several relations of finite multiple zeta values. The multiple zeta values are multi-variate generalizations of the values of the Riemann zeta function at positive integers. These real numbers are known to be related to number theory, knot theory, quantum field theory, arithmetic geometry and so on. The research on multiple zeta values began in 18th century by C. Goldbach and L. Euler. Euler considered the multiple zeta values in the case of depth two, and in 1990's, D. Zagier and M. Hoffman treated general depth for the first time.

In recent years, two types of finite versions of multiple zeta values, $\mathcal{A}$-finite multiple zeta values and symmetrized multiple zeta values have been investigated. The $\mathcal{A}$-finite multiple zeta value is a collection of certain finite sums whose setting was given by Zagier. The symmetrized multiple zeta value was introduced by M. Kaneko and Zagier to establish a crucial bridge between the multiple zeta values and $\mathcal{A}$-finite multiple zeta values.

For these two types of finite multiple zeta values, Kaneko and Zagier conjectured that the algebra generated by $\mathcal{A}$-finite multiple zeta values and the algebra generated by symmetrized multiple zeta values are isomorphic. From this point of view, we expect exactly the same equations hold for $\mathcal{A}$-finite multiple zeta values and symmetrized multiple zeta values.

In this thesis, we mainly focus on the relations among finite multiple zeta values. We prove three theorems (symmetric formula, sum formula and height-one duality theorem) for symmetrized multiple zeta values. We also prove the derivation relations for both of the finite multiple zeta values. Moreover, we give an alternative proof of the original derivation relations due to K. Ihara, Kaneko and Zagier by using Kawashima's relations.

2 Multiple zeta values

2.1 Definition of MZVs

We first review the theory of multiple zeta values. For integers $k_1, \dots, k_d \in \mathbb{Z}_{\geq 1}$ with $k_1 \geq 2$, the multiple zeta value (MZV) and the multiple zeta star value (MZSV) are defined by

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 > \cdots > n_d \geq 1} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}},$$
$$\zeta^\star(k_1, \dots, k_d) = \sum_{n_1 \geq \cdots \geq n_d \geq 1} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}},$$

respectively. The MZVs are the convergent special values of the multiple zeta function at positive integers. For an index $\mathbf{k} = (k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 1})^d$, the integer $k_1 + \cdots + k_d$ is called the weight of $\mathbf{k}$ (denoted by $|\mathbf{k}|$) and the integer d is called the depth of $\mathbf{k}$ (denoted by $l(\mathbf{k})$). We say an index $\mathbf{k} = (k_1, \dots, k_d)$ is admissible if $k_1 \geq 2$. It is known that multiple zeta values can be written as a linear combination of multiple zeta star values and vice versa. For example,

$$\begin{aligned}
\zeta^\star(k_1, k_2) &= \zeta(k_1, k_2) + \zeta(k_1 + k_2), \\
\zeta^\star(k_1, k_2, k_3) &= \zeta(k_1, k_2, k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1 + k_2 + k_3), \\
\zeta^\star(k_1, k_2, k_3, k_4) &= \zeta(k_1, k_2, k_3, k_4) + \zeta(k_1 + k_2, k_3, k_4) + \zeta(k_1, k_2 + k_3, k_4) + \zeta(k_1, k_2, k_3 + k_4) \\
&\quad + \zeta(k_1 + k_2 + k_3, k_4) + \zeta(k_1 + k_2, k_3 + k_4) + \zeta(k_1, k_2 + k_3 + k_4) \\
&\quad + \zeta(k_1 + k_2 + k_3 + k_4), \\
&\cdots,
\end{aligned}$$

and

$$\begin{aligned}
\zeta(k_1, k_2) &= \zeta^\star(k_1, k_2) - \zeta^\star(k_1 + k_2), \\
\zeta(k_1, k_2, k_3) &= \zeta^\star(k_1, k_2, k_3) - \zeta^\star(k_1 + k_2, k_3) - \zeta^\star(k_1, k_2 + k_3) + \zeta^\star(k_1 + k_2 + k_3), \\
\zeta(k_1, k_2, k_3, k_4) &= \zeta^\star(k_1, k_2, k_3, k_4) - \zeta^\star(k_1 + k_2, k_3, k_4) - \zeta^\star(k_1, k_2 + k_3, k_4) - \zeta^\star(k_1, k_2, k_3 + k_4) \\
&\quad + \zeta^\star(k_1 + k_2 + k_3, k_4) + \zeta^\star(k_1 + k_2, k_3 + k_4) + \zeta^\star(k_1, k_2 + k_3 + k_4) \\
&\quad - \zeta^\star(k_1 + k_2 + k_3 + k_4), \\
&\cdots.
\end{aligned}$$

We make a table of multiple zeta values with lower weight and depth.

	wt $=2$	wt $=3$	wt $=4$	wt $=5$
dep $=1$	$\zeta(2)$	$\zeta(3)$	$\zeta(4)$	$\zeta(5)$
dep $=2$		$\zeta(2,1)$	$\zeta(3,1),\zeta(2,2)$	$\zeta(4,1),\zeta(3,2),\zeta(2,3)$
dep $=3$			$\zeta(2,1,1)$	$\zeta(3,1,1),\zeta(2,2,1),\zeta(2,1,2)$
dep $=4$				$\zeta(2,1,1,1)$

	wt $=6$
dep $=1$	$\zeta(6)$
dep $=2$	$\zeta(5,1),\zeta(4,2),\zeta(3,3),\zeta(2,4)$
dep $=3$	$\zeta(4,1,1),\zeta(3,2,1),\zeta(3,1,2),\zeta(2,3,1),\zeta(2,2,2),\zeta(2,1,3)$
dep $=4$	$\zeta(3,1,1,1),\zeta(2,2,1,1),\zeta(2,1,2,1),\zeta(2,1,1,2)$
dep $=5$	$\zeta(2,1,1,1,1)$

We note that the number of multiple zeta values of weight k and depth d is $\binom{k-2}{d-1}$, and of weight k is 2^{k-2}.

2.2 Vector space spanned by MZVs

We describe the vector space over $\mathbb{Q}$ spanned by MZVs.

Definition 1.

$$\begin{aligned}
\mathcal{Z}_0 &= \mathbb{Q}, \quad \mathcal{Z}_1 = \{0\},\\
\mathcal{Z}_k &= \sum_{\substack{\text{weight}=k,\\ d\geq 1}} \mathbb{Q}\cdot\zeta(k_1,\ldots,k_d) \quad (k_1\geq 2),\\
\mathcal{Z} &= \sum_{k=0}^{\infty}\mathcal{Z}_k.
\end{aligned}$$

The vector space $\mathcal{Z}$ becomes a $\mathbb{Q}$-algebra. Moreover, $\mathcal{Z}$ has two product rules, which are called the harmonic (stuffle) product and the shuffle product. These products yield many linear relations, which are called (generalized) double shuffle relations among MZVs. The former product is obtained by considering the product of two defining series of MZVs and the latter is by the iterated integral expression of MZVs. In general, the product of MZVs with weight k and k' is a linear combination of MZVs with weight $k+k'$, which implies that $\mathcal{Z}_k\cdot\mathcal{Z}_{k'}\subset\mathcal{Z}_{k+k'}$. These products will be explained in the following sections. If we replace $\zeta(k_1,\ldots,k_d)$ by $\zeta^\star(k_1,\ldots,k_d)$, the corresponding vector space is the same one since any MZSVs can be expressed as a linear combination of MZVs and vice versa.

The MZVs of weight 2 is only $\zeta(2)$. Then, we find $\mathcal{Z}_2 = \mathbb{Q}\cdot\zeta(2)$. As for the weight 3 and 4 cases, it is known that $\mathcal{Z}_3 = \mathbb{Q}\cdot\zeta(3)$ and $\mathcal{Z}_4 = \mathbb{Q}\cdot\zeta(4)$. This is because we have the following relations:

$$\zeta(2,1) = \zeta(3) \qquad \text{(in weight 3)},$$
$$\zeta(2,1,1) = \zeta(4),\ \zeta(3,1)+\zeta(2,2) = \zeta(4),\ 4\zeta(3,1) = \zeta(4) \qquad \text{(in weight 4)}$$

by the duality theorem, the sum formula and the double shuffle relations, which will be explained later. For the weight $k \geq 5$ cases, the basis of $\mathcal{Z}_k$ are not determined. In [33], D. Zagier made a conjecture on the dimension of the $\mathbb{Q}$-vector space generated by MZVs of weight k.

Conjecture 1 (Zagier [33]). *We have*

$$dim_{\mathbb{Q}}\mathcal{Z}_k = d_k,$$

where d_k is the non-negative integer satisfying the following recurrence relation.

$$d_k = d_{k-2} + d_{k-3}\ (k \geq 3),\ d_0 = 1,\ d_1 = 0,\ d_2 = 1.$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28
2^{k-2}	–	–	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192

A. Goncharov [10], P. Deligne–Goncharov [8] and T. Terasoma [31] proved that the number d_k gives an upper bound of the dimension of the space generated by MZVs of weight k. The number d_k is far smaller than the total number 2^{k-2} of indices of weight k, hence there should be a lot of relations among MZVs.

Theorem 2.1 (Goncharov [10], Deligne–Goncharov [8], Terasoma [31]). *The inequality*

$$dim_{\mathbb{Q}}\mathcal{Z}_k \leq d_k$$

holds.

As for the basis of MZVs, M. Hoffman made the following conjecture in [12], which has recently been solved by F. Brown in [6].

Theorem 2.2 (conjectured by Hoffman [12], proved by Brown [6]). *The space $\mathcal{Z}$ is generated by MZVs whose component is* 2 *or* 3.

Conjecture 1 is solved if all MZVs whose component is 2 or 3 are linearly independent over $\mathbb{Q}$. However, it would be hard to solve the problem, since no single example of k is known at present where $dim_{\mathbb{Q}}\mathcal{Z}_k \geq 2$.

2.3 Two products of MZVs

Before we give a precise definition of the harmonic and the shuffle products, we give some examples.

2.3.1 The series expressions and the harmonic product of MZVs

We first look at the examples of the product of MZVs by focusing on the series expressions. The product of depth 1 MZVs is

$$\begin{aligned}\zeta(p)\zeta(q) &= \left(\sum_{m>0}\frac{1}{m^p}\right)\left(\sum_{n>0}\frac{1}{n^q}\right) = \sum_{m,n>0}\frac{1}{m^p n^q} \\ &= \left(\sum_{m>n>0} + \sum_{n>m>0} + \sum_{m=n>0}\right)\frac{1}{m^p n^q} \\ &= \zeta(p,q) + \zeta(q,p) + \zeta(p+q).\end{aligned}$$

The product of depth 1 and 2 MZVs is

$$\begin{aligned}\zeta(p)\zeta(q,r) &= \left(\sum_{l>0}\frac{1}{l^p}\right)\left(\sum_{m>n>0}\frac{1}{m^q n^r}\right) = \sum_{\substack{l>0\\ m>n>0}}\frac{1}{l^p m^q n^r} \\ &= \left(\sum_{l>m>n>0} + \sum_{m>l>n>0} + \sum_{m>n>l>0} + \sum_{l=m>n>0} + \sum_{m>l=n>0}\right)\frac{1}{l^p m^q n^r} \\ &= \zeta(p,q,r) + \zeta(q,p,r) + \zeta(q,r,p) + \zeta(p+q,r) + \zeta(q,p+r).\end{aligned}$$

We note here that the weights on the right-hand side are the sums of weights on the left-hand side.

2.3.2 The iterated integral expressions and the shuffle product of MZVs

To investigate the other product of MZVs, we need the iterated integral expression of MZVs. We consider the following iterated integral.

$$\begin{aligned}I(\varepsilon_1,\cdots,\varepsilon_k) &= \int\cdots\int_{1>t_1>\cdots>t_k>0} A_{\varepsilon_1}(t_1)\cdots A_{\varepsilon_k}(t_k)\,dt_1\cdots dt_k \\ &= \int_0^1 A_{\varepsilon_1}(t_1)dt_1 \int_0^{t_1} A_{\varepsilon_2}(t_2)dt_2 \cdots \int_0^{t_{k-1}} A_{\varepsilon_k}(t_k)dt_k,\end{aligned}$$

where $\varepsilon_j = 0$ or 1 $(1 \le j \le k)$ with $\varepsilon_1 = 0$ and $\varepsilon_k = 1$, and

$$A_0(t) = \frac{1}{t}, \quad A_1(t) = \frac{1}{1-t}.$$

We note that the above integral converges and $\zeta(k_1,\ldots,k_d)$ is represented as the following form. According to Zagier [33], this was first noticed by Kontsevich.

Theorem 2.3 (Iterated integral expression)**.**

$$\zeta(k_1,\ldots,k_d) = I(\underbrace{0,\ldots,0}_{k_1-1},1\,\underbrace{0,\ldots,0}_{k_2-1},1,0,\ldots,1,\underbrace{0,\ldots,0}_{k_d-1},1).$$

Example 2.4.

$$\begin{aligned}
\zeta(2)^2 &= I(0,1)I(0,1)\\
&= \iint\limits_{1>t_1>t_2>0} \frac{dt_1}{t_1}\frac{dt_2}{1-t_2} \iint\limits_{1>s_1>s_2>0} \frac{ds_1}{s_1}\frac{ds_2}{1-s_2}\\
&= \int\cdots\int\limits_{1>t_1>t_2>s_1>s_2>0} \frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\frac{ds_1}{s_1}\frac{ds_2}{1-s_2} + \int\cdots\int\limits_{1>t_1>s_1>t_2>s_2>0} \frac{dt_1}{t_1}\frac{ds_1}{s_1}\frac{dt_2}{1-t_2}\frac{ds_2}{1-s_2}\\
&\quad + \int\cdots\int\limits_{1>t_1>s_1>s_2>t_2>0} \frac{dt_1}{t_1}\frac{ds_1}{s_1}\frac{ds_2}{1-s_2}\frac{dt_2}{1-t_2} + \int\cdots\int\limits_{1>s_1>t_1>t_2>s_2>0} \frac{ds_1}{s_1}\frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\frac{ds_2}{1-s_2}\\
&\quad + \int\cdots\int\limits_{1>s_1>t_1>s_2>t_2>0} \frac{ds_1}{s_1}\frac{dt_1}{t_1}\frac{ds_2}{1-s_2}\frac{dt_2}{1-t_2} + \int\cdots\int\limits_{1>s_1>s_2>t_1>t_2>0} \frac{ds_1}{s_1}\frac{ds_2}{1-s_2}\frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\\
&= 4\int\cdots\int\limits_{1>t_1>t_2>t_3>t_4>0} \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{1-t_3}\frac{dt_4}{1-t_4} + 2\int\cdots\int\limits_{1>t_1>t_2>t_3>t_4>0} \frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}\frac{dt_4}{1-t_4}\\
&= 4I(0,0,1,1) + 2I(0,1,0,1)\\
&= 4\zeta(3,1) + 2\zeta(2,2).
\end{aligned}$$

2.4 Algebraic setup of MZVs

We recall the algebraic setup introduced by Hoffman in [12] and explain some results obtained by K. Ihara, M. Kaneko and Zagier in [14]. Let $\mathfrak{H} = \mathbb{Q}\langle x,y\rangle$ be the non-commutative polynomial ring in two indeterminates x and y. We call x and y the letters of $\mathfrak{H}$ and monomial elements in $\mathfrak{H}$ the words. We also let $\mathfrak{H}^1$ and $\mathfrak{H}^0$ its subrings $\mathbb{Q}+\mathfrak{H}y$ and $\mathbb{Q}+x\mathfrak{H}y$. We set $z_k = x^{k-1}y$ $(k=1,2,3,\ldots)$. Then, we see $\mathfrak{H}^1$ is freely generated by $\{z_k\}_{k\geq 1}$. Let $|w|$ the total degree for any word w and $l(w)$ be the degree of w with respect to y.

We define the $\mathbb{Q}$-linear map (called evaluation map) $Z : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ by

$$Z(1) = 1, \quad Z(z_{k_1}\cdots z_{k_d}) = \zeta(k_1,\ldots,k_d).$$

2.4.1 Double shuffle relations

We define two products of words. The one is the harmonic product $*$ on $\mathfrak{H}^1$ defined by

$$1 * w = w * 1 = w,$$
$$z_k w_1 * z_l w_2 = z_k(w_1 * z_l w_2) + z_l(z_k w_1 * w_2) + z_{k+l}(w_1 * w_2)$$

($k, l \in \mathbb{Z}_{\geq 1}$ and w, w_1, w_2 are words in $\mathfrak{H}^1$), together with $\mathbb{Q}$-bilinearity. The harmonic product $*$ is commutative and associative, therefore $\mathfrak{H}^1$ is a $\mathbb{Q}$-commutative algebra with respect to $*$. (See also Hoffman [12].) We denote it by $\mathfrak{H}^1_*$. The subset $\mathfrak{H}^0$ is a subalgebra of $\mathfrak{H}^1$ with respect to $*$. We denote it by $\mathfrak{H}^0_*$. For this product, we have

$$Z(w_1 * w_2) = Z(w_1)Z(w_2) \tag{1}$$

for any $w_1, w_2 \in \mathfrak{H}^0$.

The other product is the shuffle product $ш$ on $\mathfrak{H}$ defined by

$$1 ш w = w ш 1 = w,$$
$$u_1 w_1 ш u_2 w_2 = u_1(w_1 ш u_2 w_2) + u_2(u_1 w_1 ш w_2)$$

($u_1, u_2 \in \{x, y\}$ and w, w_1, w_2 are words in $\mathfrak{H}$), together with $\mathbb{Q}$-bilinearity. The shuffle product $ш$ is also commutative and associative, therefore $\mathfrak{H}$ is a $\mathbb{Q}$-commutative algebra with respect to $ш$. We denote it by $\mathfrak{H}_ш$. The subsets $\mathfrak{H}^1$ and $\mathfrak{H}^0$ are subalgebras of $\mathfrak{H}$ with respect to $ш$ and we denote them by $\mathfrak{H}^1_ш$, $\mathfrak{H}^0_ш$, respectively. Then, we also have

$$Z(w_1 ш w_2) = Z(w_1)Z(w_2) \tag{2}$$

for any $w_1, w_2 \in \mathfrak{H}^0$. From eq.(1) and eq.(2), we have the following equation which is called the finite double shuffle relations,

$$Z(w_1 * w_2 - w_1 ш w_2) = 0 \quad (w_1, w_2 \in \mathfrak{H}^0).$$

Example 2.5. *When $w_1 = w_2 = xy$, we have*

$$2\zeta(2,2) + \zeta(4) - (4\zeta(3,1) + 2\zeta(2,2)) = 0,$$

and we find

$$4\zeta(3,1) = \zeta(4).$$

2.4.2 Regularized double shuffle relations

These finite double shuffle relations do not yield all relations. In weights $2, 3$ and 4, we have $1, 2$ and 4 MZVs, i.e.,

$$\begin{aligned}&\zeta(2) && \text{(in weight 2)},\\ &\zeta(3), \zeta(2,1) && \text{(in weight 3)},\\ &\zeta(4), \zeta(3,1), \zeta(2,2), \zeta(2,1,1) && \text{(in weight 4)}.\end{aligned}$$

By all the double shuffle relations with the weight less than or equal 4, we only have $4\zeta(3,1) = \zeta(4)$. Thus, we can reduce the dimensions from $1,2,4$ to $1,2,3$ by using the relation. However, the correct dimensions are $1,1,1$. Therefore, we need a larger supply of relations. One of such is given by so-called the regularized double shuffle relations.

In [14], Ihara, Kaneko and Zagier find the following proposition.

Proposition 2.6 (Ihara–Kaneko–Zagier [14]). *We have two algebra homomorphisms*

$$Z^{*} : \mathfrak{H}^1_{*} \longrightarrow \mathbb{R}[T] \quad \text{and} \quad Z^{ш} : \mathfrak{H}^1_{ш} \longrightarrow \mathbb{R}[T]$$

which are uniquely characterized by the properties that they both extend the evaluation map $Z : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ and send y to T.

Example 2.7. *We have*

$$\begin{aligned} y * xy &= x^2y + xy^2 + yxy, \\ y \,ш\, xy &= 2xy^2 + yxy. \end{aligned}$$

Thus,

$$\begin{aligned} Z^{*}(yxy) &= \zeta(2)T - \zeta(3) - \zeta(2,1), \\ Z^{ш}(yxy) &= \zeta(2)T - 2\zeta(2,1). \end{aligned}$$

Then, they find the regularized double shuffle relations of MZVs.

Theorem 2.8 (Ihara–Kaneko–Zagier [14]). *For any $w_1 \in \mathfrak{H}^1$ and $w_2 \in \mathfrak{H}^0$, we have*

$$Z^{*}(w_1 \,ш\, w_2 - w_1 * w_2) = 0, \tag{3}$$

$$Z^{ш}(w_1 \,ш\, w_2 - w_1 * w_2) = 0. \tag{4}$$

Example 2.9. *From the previous example, we have*

$$y * xy - y \,ш\, xy = x^2y - xy^2$$

and thus

$$\zeta(2,1) = \zeta(3).$$

Before closing this subsection, we give some folklore conjectures.

Conjecture 2. *There are no nontrivial $\mathbb{Q}$-linear relations among MZVs of different weights.*

Related to Theorem 2.8, the following is widely believed.

Conjecture 3. *Every linear relations among MZVs can be deduced from eq.(3) or eq.(4) in Theorem 2.8.*

2.5 Derivation relations for MZVs

2.5.1 Derivation relations for MZVs

A derivation ∂ on $\mathfrak{H}$ is a $\mathbb{Q}$-linear endomorphism of $\mathfrak{H}$ satisfying Leibniz's rule $\partial(ww') = \partial(w)w' + w\partial(w')$. Such a derivation is uniquely determined by its images of generators x and y. Set $z = x + y$. For each $l \geq 1$, we define the derivation $\partial_l : \mathfrak{H} \to \mathfrak{H}$ by

$$\partial_l(x) = xz^{l-1}y, \quad \partial_l(y) = -xz^{l-1}y.$$

We note that $\partial_l(1) = 0$ and $\partial_l(z) = 0$. In [14], Ihara, Kaneko and Zagier proved the derivation relations for MZVs.

Theorem 2.10 (Ihara–Kaneko–Zagier [14]). *For an integer $l \in \mathbb{Z}_{\geq 1}$, we have*

$$\partial_l(\mathfrak{H}^0) \subset \ker Z.$$

2.5.2 Kawashima's relations

To prove Theorem 2.10, we use the linear part of Kawashima's relations. We first state Kawashima's relations which was shown by Kawashima in [21]. He proved this relation by using some properties of certain Newton series.

We define the two automorphisms ϕ and α on $\mathfrak{H}$ by

$$\phi(x) = x + y, \quad \phi(y) = -y,$$

and

$$\alpha(x) = y, \quad \alpha(y) = x.$$

We also let $\tilde{\alpha}$ the $\mathbb{Q}$-linear map on $\mathfrak{H}y$ satisfying

$$\tilde{\alpha}(wy) = \alpha(w)y \quad (w \in \mathfrak{H}).$$

The following relation was proved by Kawashima in [21].

Theorem 2.11 (Kawashima [21]). *For an integer $m \in \mathbb{Z}_{\geq 1}$ and $w, w' \in \mathfrak{H}^1$, we have*

$$\sum_{\substack{p+q=m,\\ p,q\geq 1}} Z(\phi(w) \dot{*} y^p) Z(\phi(w') \dot{*} y^q) = Z(\phi(w * w') \dot{*} y^m),$$

where the product $\dot{*}$ is given by $z_k w \dot{*} z_l w' = z_{k+l}(w * w')$ and the bilinearity.

For any $w \in \mathfrak{H}$, we define the $\mathbb{Q}$-linear map L_w on $\mathfrak{H}$ by

$$L_w(w') = ww' \qquad (w' \in \mathfrak{H}).$$

We note that $w \dot{*} y = xw = L_x(w)$ $(w \in \mathfrak{H}^1)$. To make the statement simpler, we set

$$\mathfrak{H}y * \mathfrak{H}y = \{w * w' \mid w, w' \in \mathfrak{H}y\}.$$

Then, the case $m = 1$ in Theorem 2.11 becomes the following corollary, which is called the linear part of Kawashima's relations.

Corollary 2.12. *We have*

$$L_x(\phi(\mathfrak{H}y * \mathfrak{H}y)) \subset \ker Z.$$

We note that the above relation looks simple but contains many linearly independent relations among MZVs.

2.5.3 A proof of the derivation relations

Here, we prove derivation relations for MZVs (Theorem 2.10) by using the linear part of Kawashima's relations. We prove the following theorem, instead. The proof seems to be new.

Theorem 2.13. *For any index $(m_1,\ldots,m_s)\in(\mathbb{Z}_{\geq 1})^s$ $(s\geq 0)$ and integer $l\in\mathbb{Z}_{\geq 1}$, we have*

$$\begin{aligned} &Z(xz^{m_1-1}y\cdots z^{m_s-1}y\partial_l(w))\\ &=-Z(xz^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}yw)\\ &\quad+\sum_{i=1}^{s}Z(xz^{m_1-1}y\cdots z^{m_i-1}xz^{l-1}yz^{m_{i+1}-1}y\cdots z^{m_s-1}yw)\quad (w\in\mathfrak{H}^1). \end{aligned}$$

When $s=0$, we understand the left-hand side is $Z(x\partial_l(w))$ and the right-hand side is $-Z(xz^{l-1}yw)$.

Remark 2.14. *We note that Theorem 2.10 is equivalent to Theorem 2.13. Let $w=xw'$. Then, we see from Theorem 2.10 and by $\partial_l(xw')=xz^{l-1}yw'+x\partial_l(w')$, the statement of Theorem 2.10 is equivalent to*

$$Z(x\partial_l(w'))=-Z(xz^{l-1}yw')\quad (w'\in\mathfrak{H}^1). \tag{5}$$

Thus, Theorem 2.13 (the case $s=0$) implies Theorem 2.10. On the other hand, since

$$\begin{aligned} \partial_l(z^{m_1-1}y\cdots z^{m_s-1}yw)=&-z^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}yw\\ &-\quad\cdots\cdots\\ &-z^{m_1-1}y\cdots z^{m_s-1}xz^{l-1}yw\\ &+z^{m_1-1}y\cdots z^{m_s-1}y\partial_l(w) \end{aligned}$$

(note $\partial_l(z)=0$), we see Theorem 2.13 by putting $z^{m_1-1}y\cdots z^{m_s-1}yw$ for w' in eq.(5).

We prove Theorem 2.13 by induction on $n=|w|$.
(I) When $n=0$, i.e., $w=1$, we need to show

$$\begin{aligned} &-Z(xz^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}y)\\ &+\sum_{i=1}^{s}Z(xz^{m_1-1}y\cdots z^{m_i-1}xz^{l-1}yz^{m_{i+1}-1}y\cdots z^{m_s-1}y)=0. \end{aligned}$$

We see by Corollary 2.12,

$$\begin{aligned}
&-Z(xz^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}y)+\sum_{i=1}^{s}Z(xz^{m_1-1}y\cdots z^{m_i-1}xz^{l-1}yz^{m_{i+1}-1}y\cdots z^{m_s-1}y)\\
&=Z(-xz^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}y+xz^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}y\\
&\quad+\quad\cdots\cdots\quad+xz^{m_1-1}y\cdots z^{m_s-1}xz^{l-1}y)\\
&=(-1)^sZ(L_x(\phi(x^{l-1}yx^{m_1-1}y\cdots x^{m_s-1}y+x^{m_1-1}zx^{l-1}yx^{m_2-1}y\cdots x^{m_s-1}y\\
&\qquad+\quad\cdots\cdots\quad+x^{m_1-1}y\cdots x^{m_s-1}zx^{l-1}y)))\\
&=(-1)^sZ(L_x(\phi(x^{m_1-1}y\cdots x^{m_s-1}y*x^{l-1}y)))=0.
\end{aligned}$$

(II) We assume the identity holds for $|w|=0,\ldots,n-1$. Suppose w is of degree n. We may assume that w is of the form $w=z^{r-1}yw'$ with $1\le r\le n, w'\in\mathfrak{H}^1$, by replacing $x^{r-1}y$ by $(z-y)^{r-1}y$ if w starts with $x^{r-1}y$.

$$\begin{aligned}
\text{L.H.S.}&=Z(xz^{m_1-1}y\cdots z^{m_s-1}y\partial_l(z^{r-1}yw'))\\
&=Z(-xz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw'+xz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}y\partial_l(w')).
\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}
Z(xz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}y\partial_l(w'))=Z(&-xz^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}yw'\\
&+xz^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}yz^{r-1}yw'\\
&+\quad\cdots\cdots\\
&+xz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw').
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{L.H.S.}&=Z(-xz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw'-xz^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}yw'\\
&\quad+xz^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}yz^{r-1}yw'+\quad\cdots\cdots\\
&\quad+xz^{m_1-1}y\cdots z^{m_s-1}xz^{l-1}yz^{r-1}yw'+xz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw')\\
&=Z(-xz^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}yw'+xz^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}yz^{r-1}yw'\\
&\quad+\quad\cdots\cdots\quad+xz^{m_1-1}y\cdots z^{m_s-1}xz^{l-1}yz^{r-1}yw')\\
&=\text{R.H.S.},
\end{aligned}$$

and hence the identity holds for n and by induction, the proof is done.

2.6 Some other known results

There are a lot of known relations among MZVs most of which have been proved with a variety of techniques. We have seen that the regularized double shuffle relations give rise to many $\mathbb{Q}$-linear relations and conjecturally these imply all the other linear relations. Here, we introduce a few special families of relations.

2.6.1 Duality theorem and Sum formula for MZVs

Definition 2. *Let* $\mathbf{k} = (k_1, \ldots, k_d)$ *be an admissible index of weight* k*. We write*

$$\mathbf{k} = (a_1 + 1, \underbrace{1, \ldots, 1}_{b_1 - 1}, \ldots, a_s + 1, \underbrace{1, \ldots, 1}_{b_s - 1})$$

with $a_p, b_q \geq 1$*. Then, we define the dual index of* $\mathbf{k}$ *as*

$$\mathbf{k}^* = (b_s + 1, \underbrace{1, \ldots, 1}_{a_s - 1}, \ldots, b_1 + 1, \underbrace{1, \ldots, 1}_{a_1 - 1}).$$

We note that

$$l(\mathbf{k}) + l(\mathbf{k}^*) = |\mathbf{k}|.$$

Example 2.15.

$$\begin{aligned}(4, 1, 1, 1, 3, 1, 1)^* &= (3 + 1, \underbrace{1, \ldots, 1}_{4-1}, 2 + 1, \underbrace{1, \ldots, 1}_{3-1})^* \\ &= (3 + 1, \underbrace{1, \ldots, 1}_{2-1}, 4 + 1, \underbrace{1, \ldots, 1}_{3-1}) \\ &= (4, 1, 5, 1, 1).\end{aligned}$$

Example 2.16.

$$(a + 1, \underbrace{1, \ldots, 1}_{b-1})^* = (b + 1, \underbrace{1, \ldots, 1}_{a-1}).$$

The following result, which is a direct consequence of the iterated integral expression, provides the so-called duality theorem for MZVs.

Theorem 2.17. *For any admissible index* $\mathbf{k}$*, we have*

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}^*).$$

The following theorem is called the sum formula for MZVs.

Theorem 2.18. *For integers* k, d *with* $0 < d < k$*, we have*

$$\sum_{\substack{k_1 + \cdots + k_d = k \\ k_i \geq 1,\, k_1 \geq 2}} \zeta(k_1, \ldots, k_d) = \zeta(k).$$

2.6.2 Ohno's relations for MZVs

Ohno's relations is one of the most famous relations among MZVs. This relation is a generalization of both the sum formula and the duality theorem.

Theorem 2.19 (Ohno [26]). *Let $\mathbf{k}^* = (k'_1, \dots, k'_d)$ be the dual index of $\mathbf{k} = (k_1, \dots, k_d)$. Then, for any index $\mathbf{k}$ and integer $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_d=m \\ \varepsilon_1,\cdots,\varepsilon_d \geq 0}} \zeta(k_1 + \varepsilon_1, k_2 + \varepsilon_2, \cdots, k_d + \varepsilon_d) = \sum_{\substack{\varepsilon'_1+\cdots+\varepsilon'_{d'}=m \\ \varepsilon'_1,\cdots,\varepsilon'_{d'} \geq 0}} \zeta(k'_1 + \varepsilon'_1, k'_2 + \varepsilon'_2, \cdots, k'_{d'} + \varepsilon'_{d'}).$$

Ohno's proof is due to an iterated integral expression of a multi-variable generating function.

2.6.3 Hoffman's symmetric formula for MZ(S)Vs

To state Hoffman's symmetric formula, we need some notation. Let S_d be the symmetric group of degree d. For a partition $\Pi = \{P_1, \dots, P_l\}$ of the set $\{1, \dots, d\}$, we let

$$c(\Pi) = (\mathrm{Card}P_1 - 1)! \cdots (\mathrm{Card}P_l - 1)!.$$

For a given such Π and a d-tupple $\mathbf{i} = \{i_1, \dots, i_d\}$ of real numbers with $i_j > 1$, we also let

$$\zeta(\mathbf{i}, \Pi) = \prod_{s=1}^{l} \zeta\left(\sum_{j \in P_s} i_j\right).$$

In [11], Hoffman proved the following theorem.

Theorem 2.20 (Hoffman [11]). *For any real numbers $i_1 \dots, i_d > 1$, we have*

$$\sum_{\sigma \in S_d} \zeta^\star(i_{\sigma(1)}, \dots, i_{\sigma(d)}) = \sum_{\text{partitions } \Pi \text{ of } \{1,\dots,d\}} c(\Pi)\zeta(\mathbf{i}, \Pi).$$

Example 2.21.

$$\begin{aligned}
&\sum_{\sigma \in S_3} \zeta^\star(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) \\
&= \zeta(i_1)\zeta(i_2)\zeta(i_3) + \zeta(i_1 + i_2)\zeta(i_3) + \zeta(i_1)\zeta(i_2 + i_3) + \zeta(i_1 + i_3)\zeta(i_2) \\
&\quad + 2\zeta(i_1 + i_2 + i_3).
\end{aligned}$$

Example 2.22.

$$\begin{aligned}
&\sum_{\sigma \in S_4} \zeta^\star(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, i_{\sigma(4)}) \\
&= \zeta(i_1)\zeta(i_2)\zeta(i_3)\zeta(i_4) + \zeta(i_1 + i_2)\zeta(i_3)\zeta(i_4) + \zeta(i_1)\zeta(i_2 + i_3)\zeta(i_4) + \zeta(i_1)\zeta(i_2)\zeta(i_3 + i_4) \\
&\quad + 2\zeta(i_1 + i_2 + i_3)\zeta(i_4) + \zeta(i_1 + i_2)\zeta(i_3 + i_4) + 2\zeta(i_1)\zeta(i_2 + i_3 + i_4) \\
&\quad + 6\zeta(i_1 + i_2 + i_3 + i_4).
\end{aligned}$$

Hoffman states the analogous relations of the above theorem for ordinary MZVs. We need a bit of additional notation. For a partion $\Pi = \{P_1, \ldots, P_l\}$ of the set $\{1, \ldots, d\}$, we let

$$\tilde{c}(\Pi) = (-1)^{d-l}(\mathrm{Card}P_1 - 1)! \cdots (\mathrm{Card}P_l - 1)!.$$

Here, we note $\tilde{c}(\Pi) = (-1)^{d-l}c(\Pi)$.

Theorem 2.23 (Hoffman [11])**.** *For any real numbers $i_1 \ldots, i_d > 1$, we have*

$$\sum_{\sigma \in S_d} \zeta(i_{\sigma(1)}, \ldots, i_{\sigma(d)}) = \sum_{\text{partitions } \Pi \text{ of } \{1,\ldots,d\}} \tilde{c}(\Pi)\zeta(\mathbf{i}, \Pi).$$

Example 2.24.

$$\begin{aligned}
&\sum_{\sigma \in S_3} \zeta(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) \\
&= \zeta(i_1)\zeta(i_2)\zeta(i_3) - \zeta(i_1 + i_2)\zeta(i_3) - \zeta(i_1)\zeta(i_2 + i_3) - \zeta(i_1 + i_3)\zeta(i_2) \\
&\quad + 2\zeta(i_1 + i_2 + i_3).
\end{aligned}$$

Example 2.25.

$$\begin{aligned}
&\sum_{\sigma \in S_4} \zeta(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, i_{\sigma(4)}) \\
&= \zeta(i_1)\zeta(i_2)\zeta(i_3)\zeta(i_4) - \zeta(i_1 + i_2)\zeta(i_3)\zeta(i_4) - \zeta(i_1)\zeta(i_2 + i_3)\zeta(i_4) - \zeta(i_1)\zeta(i_2)\zeta(i_3 + i_4) \\
&\quad + 2\zeta(i_1 + i_2 + i_3)\zeta(i_4) + \zeta(i_1 + i_2)\zeta(i_3 + i_4) + 2\zeta(i_1)\zeta(i_2 + i_3 + i_4) \\
&\quad - 6\zeta(i_1 + i_2 + i_3 + i_4).
\end{aligned}$$

3 Finite multiple zeta values

3.1 $\mathcal{A}$-finite multiple zeta values

We consider the collection of truncated sums $\zeta_p(k_1,\ldots,k_d)=\sum_{p>n_1>\cdots>n_d\geq 1}\frac{1}{n_1^{k_1}\cdots n_d^{k_d}}$ modulo all primes p in the quotient ring

$$\mathcal{A}=(\prod_p \mathbb{Z}/p\mathbb{Z})/(\bigoplus_p \mathbb{Z}/p\mathbb{Z}),$$

which is a $\mathbb{Q}$-algebra. Elements of $\mathcal{A}$ are represented by $(a_p)_p$, where $a_p\in\mathbb{Z}/p\mathbb{Z}$, and two elements $(a_p)_p$ and $(b_p)_p$ are identified if and only if $a_p=b_p$ for all but finitely many primes p.

Definition 3. *For integers $k_1,\ldots,k_d\in\mathbb{Z}_{\geq 1}$, the $\mathcal{A}$-finite multiple zeta value ($\mathcal{A}$-FMZV) and the $\mathcal{A}$-finite multiple zeta star value ($\mathcal{A}$-FMZSV) are defined by*

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_d)=\left(\sum_{p>n_1>\cdots>n_d\geq 1}\frac{1}{n_1^{k_1}\cdots n_d^{k_d}} \bmod p\right)_p\in\mathcal{A},$$
$$\zeta^{\star}_{\mathcal{A}}(k_1,\ldots,k_d)=\left(\sum_{p>n_1\geq\cdots\geq n_d\geq 1}\frac{1}{n_1^{k_1}\cdots n_d^{k_d}} \bmod p\right)_p\in\mathcal{A}.$$

The $\mathcal{A}$-FMZ(S)Vs are similar to MZ(S)Vs in many aspects. However, there are some differences.

Example 3.1 (Hoffman [13], Zhao [35]). *For an integer $k\in\mathbb{Z}_{\geq 1}$, we have*

$$\zeta_{\mathcal{A}}(k)=\zeta^{\star}_{\mathcal{A}}(k)=0.$$

Proof. Let p be a prime number larger than $k+1$. By taking a primitive root a modulo p, we have

$$\sum_{n=1}^{p-1}\frac{1}{n^k}\equiv\sum_{i=0}^{p-2}\frac{1}{a^{ik}}\equiv\frac{1-a^{-k(p-1)}}{1-a^{-k}}\equiv 0 \pmod p.$$

Thus, we have

$$\zeta_{\mathcal{A}}(k)=\zeta^{\star}_{\mathcal{A}}(k)=0.$$

∎

Example 3.2 (Hoffman [13], Zhao [35]). *For integers $k_1,k_2\in\mathbb{Z}_{\geq 1}$, we have*

$$\zeta_{\mathcal{A}}(k_1,k_2)=\zeta^{\star}_{\mathcal{A}}(k_1,k_2)=(-1)^{k_1}\binom{k_1+k_2}{k_1}\left(\frac{B_{p-k_1-k_2}}{k_1+k_2} \bmod p\right)_p,$$

where B_n is the Bernoulli number defined by the following generating function:

$$\sum_{n\geq 0}B_n\frac{t^n}{n!}=\frac{te^t}{e^t-1}.$$

Proof. By Example 3.1, we see

$$\zeta_{\mathcal{A}}(k_1,k_2)=\zeta^{\star}_{\mathcal{A}}(k_1,k_2).$$

Thus, we only need to show

$$\zeta^{\star}_{\mathcal{A}}(k_1,k_2)=(-1)^{k_1}\binom{k_1+k_2}{k_1}\left(\frac{B_{p-k_1-k_2}}{k_1+k_2}\mod p\right)_p.$$

We use the standard identity expressing sums of powers in terms of Bernoulli numbers. (See T. Arakawa, T. Ibukiyama and Kaneko [1].)

$$\sum_{i=1}^{n} i^k=\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j}. \tag{6}$$

Using this equality and Fermat's little theorem, we have

$$\begin{aligned}
\zeta^{\star}_{\mathcal{A}}(k_1,k_2)&=\left(\sum_{n_1=1}^{p-1}\frac{1}{n_1^{k_1}}\sum_{n_2=1}^{n_1}\frac{1}{n_2^{k_2}}\mod p\right)_p\\
&=\left(\sum_{n_1=1}^{p-1}\frac{1}{n_1^{k_1}}\sum_{n_2=1}^{n_1}n_2^{p-1-k_2}\mod p\right)_p\\
&=\left(\sum_{n_1=1}^{p-1}\frac{1}{n_1^{k_1}}\frac{1}{p-k_2}\sum_{j=0}^{p-1-k_2}\binom{p-k_2}{j}B_j n_1^{p-k_2-j}\mod p\right)_p\\
&=\left(\frac{1}{p-k_2}\sum_{j=0}^{p-1-k_2}\binom{p-k_2}{j}B_j\sum_{n_1=1}^{p-1}n_1^{p-k_1-k_2-j}\mod p\right)_p.
\end{aligned}$$

Here, we note that

$$\sum_{n_1=1}^{p-1}n_1^{p-k_1-k_2-j}\equiv 0\pmod p$$

unless $j=p-k_1-k_2$. Since

$$\begin{aligned}
\frac{1}{k_2}\binom{p-k_2}{p-k_1-k_2}&=\frac{1}{k_2}\binom{p-k_2}{k_1}\\
&=\frac{1}{k_2}\cdot\frac{(p-k_2)\cdots(p-k_1-k_2+1)}{k_1!}\\
&\equiv(-1)^{k_1}\frac{(k_2+1)\cdots(k_1+k_2-1)}{k_1!}\\
&=(-1)^{k_1}\frac{1}{k_1+k_2}\binom{k_1+k_2}{k_1}\qquad\pmod p
\end{aligned}$$

for $p > k_1 + k_2 - 1$, we have

$$\begin{aligned}
\zeta^{\star}_{\mathcal{A}}(k_1, k_2) &= \left(\frac{1}{p-k_2}\binom{p-k_2}{p-k_1-k_2} B_{p-k_1-k_2}(p-1) \mod p\right)_p \\
&= \left(\frac{1}{k_2}\binom{p-k_2}{k_1} B_{p-k_1-k_2} \mod p\right)_p \\
&= (-1)^{k_1}\binom{k_1+k_2}{k_1}\left(\frac{B_{p-k_1-k_2}}{k_1+k_2} \mod p\right)_p .
\end{aligned}$$

∎

We introduce the vector space over $\mathbb{Q}$ spanned by $\mathcal{A}$-FMZVs.

Definition 4.

$$\begin{aligned}
\mathcal{Z}_{\mathcal{A},0} &= \mathbb{Q}, \quad \mathcal{Z}_{\mathcal{A},1} = \{0\}, \\
\mathcal{Z}_{\mathcal{A},k} &= \sum_{\substack{k_1+\cdots+k_d=k, \\ k_i \geq 1\,(1\leq i\leq d),\, d\geq 1}} \mathbb{Q}\cdot\zeta_{\mathcal{A}}(k_1,\ldots,k_d), \\
\mathcal{Z}_{\mathcal{A}} &= \sum_{k=0}^{\infty} \mathcal{Z}_{\mathcal{A},k}.
\end{aligned}$$

We notice that $\mathcal{Z}_{\mathcal{A}}$ is a $\mathbb{Q}$-algebra with the harmonic product.

3.2 Symmetrized multiple zeta values

The symmetrized multiple zeta values or finite real multiple zeta values, which were first introduced by Kaneko–Zagier [19, 20], are defined for any integers $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 1}$ as follows:

$$\begin{aligned}
\zeta^{*}_{\mathcal{S}}(k_1,\ldots,k_d) &= \sum_{i=0}^{d} (-1)^{k_1+\cdots+k_i} \zeta^{*}(k_i,\ldots,k_1)\zeta^{*}(k_{i+1},\ldots,k_d), \\
\zeta^{ш}_{\mathcal{S}}(k_1,\ldots,k_d) &= \sum_{i=0}^{d} (-1)^{k_1+\cdots+k_i} \zeta^{ш}(k_i,\ldots,k_1)\zeta^{ш}(k_{i+1},\ldots,k_d).
\end{aligned}$$

Here, the symbols ζ^{*} and $\zeta^{ш}$ on the right-hand sides stand for the regularized values coming from harmonic and shuffle regularizations respectively, i.e., real values obtained by taking constant terms of harmonic and shuffle regularizations as explained in section 2. In the sums, we understand $\zeta^{*}(\emptyset) = \zeta^{ш}(\emptyset) = 1$.

Example 3.3. *From Example 2.7,*

$$Z^*(yxy) = \zeta(2)T - \zeta(3) - \zeta(2,1),$$
$$Z^{ш}(yxy) = \zeta(2)T - 2\zeta(2,1).$$

Thus, we have

$$\zeta^*(1,2) = -\zeta(3) - \zeta(2,1),$$
$$\zeta^{ш}(1,2) = -2\zeta(2,1).$$

Example 3.4.

$$\begin{aligned}\zeta^*_{\mathcal{S}}(2,1,3) &= \zeta(2,1,3) + \zeta(2)\zeta^*(1,3) - \zeta^*(1,2)\zeta(3) + \zeta(3,1,2)\\ &= \zeta(2,1,3) + \zeta(2)(-\zeta(4) - \zeta(3,1)) - (-\zeta(3) - \zeta(2,1))\zeta(3) + \zeta(3,1,2)\\ &= \zeta(2,1,3) - \zeta(6) - \zeta(4,2) - \zeta(2,4) - \zeta(5,1) - \zeta(3,3) - \zeta(2,3,1) - \zeta(3,2,1) - \zeta(3,1,2)\\ &\quad + \zeta(6) + 2\zeta(3,3) + \zeta(5,1) + \zeta(2,4) + \zeta(3,2,1) + \zeta(2,3,1) + \zeta(2,1,3) + \zeta(3,1,2)\\ &= -\zeta(4,2) + \zeta(3,3) + 2\zeta(2,1,3).\end{aligned}$$

We recall that $\mathcal{Z}$ is the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by all MZVs. We also note that this is a $\mathbb{Q}$-algebra. In [19, 20], Kaneko and Zagier proved that the difference $\zeta^*_{\mathcal{S}}(k_1,\ldots,k_d) - \zeta^{ш}_{\mathcal{S}}(k_1,\ldots,k_d)$ is in the principal ideal of $\mathcal{Z}$ generated by $\zeta(2)$ (or π^2), in other words, that the congruence

$$\zeta^*_{\mathcal{S}}(k_1,\ldots,k_d) \equiv \zeta^{ш}_{\mathcal{S}}(k_1,\ldots,k_d) \pmod{\zeta(2)}$$

holds in $\mathcal{Z}$. They then defined the symmetrized multiple zeta value (SMZV) $\zeta_{\mathcal{S}}(k_1,\ldots,k_d)$ as an element in the quotient ring $\mathcal{Z}/\zeta(2)$ by

$$\zeta_{\mathcal{S}}(k_1,\ldots,k_d) = \zeta^*_{\mathcal{S}}(k_1,\ldots,k_d) \bmod \zeta(2).$$

We also refer to the values $\zeta^*_{\mathcal{S}}(k_1,\ldots,k_d)$ and $\zeta^{ш}_{\mathcal{S}}(k_1\ldots,k_d)$ as (harmonic and shuffle versions of) symmetrized multiple zeta values.

As for the star version of SMZVs, for integers $k_1,\ldots,k_d \in \mathbb{Z}_{\geq 1}$, let us define $\zeta^{\star,*}_{\mathcal{S}}$ by

$$\zeta^{\star,*}_{\mathcal{S}}(k_1,\ldots,k_d) = \sum_{\substack{\circ \text{ is either a comma `,'}\\ \text{or a plus `+'}}} \zeta^*_{\mathcal{S}}(k_1 \circ k_2 \circ \cdots \circ k_d).$$

Then, we have the following proposition.

Example 3.5 (Kaneko–Zagier [19, 20])**.** *For an integer $k \in \mathbb{Z}_{\geq 1}$, we have*

$$\zeta^*_{\mathcal{S}}(k) = \zeta^{\star,*}_{\mathcal{S}}(k) \equiv 0 \pmod{\zeta(2)}.$$

Proof. From the definition of SMZVs, we have

$$\zeta^*_{\mathcal{S}}(k) = \zeta^{\star,*}_{\mathcal{S}}(k) = \begin{cases} 2\zeta(k) & (k : \text{even}), \\ 0 & (k : \text{odd}). \end{cases}$$

Here, by Euler's theorem, the assertion holds. ∎

Example 3.6 (Kaneko–Zagier [19, 20])**.** *For integers $k_1, k_2 \in \mathbb{Z}_{\geq 1}$, we have*

$$\zeta^*_{\mathcal{S}}(k_1, k_2) \equiv \zeta^{\star,*}_{\mathcal{S}}(k_1, k_2) \equiv (-1)^{k_1} \binom{k_1 + k_2}{k_1} \zeta(k_1 + k_2) \pmod{\zeta(2)}.$$

To prove Example 3.6, we need the following theorem, which was proved by P. Cartier [7] and Zagier [34] independently.

Theorem 3.7 (Cartier, Zagier)**.** *For integers $k_1, k_2 \in \mathbb{Z}_{\geq 1}$ with $k_1 \geq 2$, we have*

$$\begin{aligned} \zeta(k_1, k_2) = \frac{1}{2}(1 + (-1)^{k_1})\zeta(k_1)\zeta(k_2) + \frac{1}{2}\left((-1)^{k_1}\binom{k_1 + k_2}{k_1} - 1\right)\zeta(k_1 + k_2) \\ - (-1)^{k_1} \sum_{i=1}^{(k_1+k_2-3)/2} \left(\binom{k_1 + k_2 - 2i - 1}{k_1 - 1} + \binom{k_1 + k_2 - 2i - 1}{k_2 - 1} \right) \zeta(2i)\zeta(k_1 + k_2 - 2i). \end{aligned}$$

Proof of Example 3.6. By Example 3.5, we see

$$\zeta_{\mathcal{S}}(k_1, k_2)^* \equiv \zeta^{\star,*}_{\mathcal{S}}(k_1, k_2) \pmod{\zeta(2)}.$$

Thus, we only need to show

$$\zeta^*_{\mathcal{S}}(k_1, k_2) \equiv (-1)^{k_1} \binom{k_1 + k_2}{k_1} \zeta(k_1 + k_2) \pmod{\zeta(2)}.$$

From the definition of SMZVs, we have

$$\zeta^*_{\mathcal{S}}(k_1, k_2) = \zeta^*(k_1, k_2) + (-1)^{k_1}\zeta^*(k_1)\zeta^*(k_2) + (-1)^{k_1+k_2}\zeta^*(k_2, k_1).$$

If $k_1 + k_2$ is even, then we easily see

$$\zeta^*_{\mathcal{S}}(k_1, k_2) \equiv 0 \pmod{\zeta(2)}$$

by using

$$\zeta^*(k_1)\zeta^*(k_2) = \zeta^*(k_1, k_2) + \zeta^*(k_2, k_1) + \zeta(k_1 + k_2).$$

If $k_1 + k_2$ is odd, we see by Theorem 3.7 that

$$\begin{aligned} \zeta^*_{\mathcal{S}}(k_1, k_2) &\equiv \frac{(-1)^{k_1}}{2} \left(\binom{k_1 + k_2}{k_1} + \binom{k_1 + k_2}{k_2} \right) \zeta(k_1 + k_2) \\ &= (-1)^{k_1} \binom{k_1 + k_2}{k_1} \zeta(k_1 + k_2) \pmod{\zeta(2)}. \end{aligned}$$

∎

3.3 Finite multiple zeta values

Kaneko and Zagier conjectured the following.

Conjecture 4 (Kaneko–Zagier)**.** *There exists an algebra isomorphism ϕ between $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}/\zeta(2)$ such that*

$$\begin{array}{cccc} & \mathcal{Z}_{\mathcal{A}} & \to & \mathcal{Z}/\zeta(2) \\ \phi: & \cup & & \cup \\ & \zeta_{\mathcal{A}}(k_1,\dots,k_d) & \mapsto & \zeta_{\mathcal{S}}(k_1,\dots,k_d). \end{array}$$

We define two $\mathbb{Q}$-linear maps $Z_{\mathcal{A}}\colon \mathfrak{H}^1 \to \mathcal{A}$ and $Z_{\mathcal{S}}\colon \mathfrak{H}^1 \to \mathcal{Z}/\zeta(2)$ by $Z_{\mathcal{A}}(1) = 1$ and $Z_{\mathcal{A}}(x^{k_1-1}y\cdots x^{k_d-1}y) = \zeta_{\mathcal{A}}(k_1,\dots,k_d)$, and $Z_{\mathcal{S}}(1) = 1$ and $Z_{\mathcal{S}}(x^{k_1-1}y\cdots x^{k_d-1}y) = \zeta_{\mathcal{S}}(k_1,\dots,k_d)$, respectively. In view of Conjecture 4, we shall call both $\mathcal{A}$-finite multiple zeta values and symmetrized multiple zeta values as finite multiple zeta values (FMZVs). In the following, the letter '$\mathcal{F}$' stands either for '$\mathcal{A}$' or '$\mathcal{S}$.'

Now we mention the harmonic and shuffle product rules and the duality theorem for FMZVs. The former is due to Hoffman [12] for $\mathcal{A}$-FMZVs, and Kaneko and Zagier [19, 20] for SMZVs.

Theorem 3.8 (Hoffman [12], Kaneko–Zagier [19, 20])**.** *For any words $w = z_{k_1}\cdots z_{k_r}, w' = z_{k'_1}\cdots z_{k'_s} \in \mathfrak{H}^1$, we have*

$$\begin{aligned} Z_{\mathcal{F}}(w * w') &= Z_{\mathcal{F}}(w)Z_{\mathcal{F}}(w'), \\ Z_{\mathcal{F}}(w \mathbin{ш} w') &= (-1)^{|w|}Z_{\mathcal{F}}(z_{k_r}\cdots z_{k_1}z_{k'_1}\cdots z_{k'_s}). \end{aligned}$$

The duality theorems for $\mathcal{A}$-finite and symmetrized versions are proved by Hoffman [13] and D. Jarossay [15], respectively. We recall the automorphism ϕ on $\mathfrak{H}$ defined by

$$\phi(x) = z := x + y, \quad \phi(y) = -y.$$

Theorem 3.9 (Hoffman, Jarrossay)**.** *For any word $w \in \mathfrak{H}^1$, we have*

$$Z_{\mathcal{F}}(w) = Z_{\mathcal{F}}(\phi(w)).$$

3.4 Theorems on FMZVs

In this subsection, we introduce the following theorems.

Theorem 3.10 (Symmetric formula)**.** *Let $(k_1,\dots,k_d) \in (\mathbb{Z}_{\geq 1})^d$ be any index and let S_d be the symmetric group of degree d. Then, we have*

$$\sum_{\sigma\in S_d} \zeta_{\mathcal{F}}(k_{\sigma(1)},k_{\sigma(2)},\dots,k_{\sigma(d)}) = 0.$$

Theorem 3.11 (Sum formula)**.** *For integers $k, d \in \mathbb{Z}_{\geq 1}$ and i with $k \geq 2, 1 \leq i \leq d \leq k-1$, we have*

$$\sum_{\substack{k_1+k_2+\cdots+k_d=k \\ k_i \geq 2}} \zeta_{\mathcal{F}}(k_1, \ldots, k_d) = (-1)^{i-1}\left(\binom{k-1}{i-1} + (-1)^d \binom{k-1}{d-i}\right)\mathfrak{Z}_{\mathcal{F}}(k),$$

where

$$\mathfrak{Z}_{\mathcal{F}}(k) = \begin{cases} \left(\frac{B_{p-k}}{k} \bmod p\right)_p & (\mathcal{F} = \mathcal{A}), \\ \zeta(k) \bmod \zeta(2) & (\mathcal{F} = \mathcal{S}). \end{cases}$$

Theorem 3.12 (Height-one duality theorem)**.** *For integers $k, d \in \mathbb{Z}_{\geq 1}$, we have the equality*

$$\zeta_{\mathcal{F}}(k, \underbrace{1, \ldots, 1}_{d-1}) = \zeta_{\mathcal{F}}(d, \underbrace{1, \ldots, 1}_{k-1}).$$

In the case of $\mathcal{F} = \mathcal{A}$, the proofs are known. (See Hoffman [13] for the symmetric formula and height-one duality theorem, and Saito–Wakabasyashi [28] for the sum formula.) Therefore, we will prove them for SMVZs in the following.

3.4.1 Proof of Theorem 3.10 for SMZVs

From Theorem 3.8, we see that the SMZVs $\zeta^*_{\mathcal{S}}(k_1, \ldots, k_d)$ satisfy the harmonic product rule. Theorem 2.23 states that any symmetric sum

$$\sum_{\sigma \in S_d} \zeta(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(d)})$$

is a polynomial in the Riemann zeta values $\zeta(k)$. His proof only uses the harmonic product rule of MZVs, and hence applies to our $\zeta^*_{\mathcal{S}}(\mathbf{k})$'s. Therefore, we conclude in similar manner as in [13] that the symmetric sum

$$\sum_{\sigma \in S_d} \zeta^*_{\mathcal{S}}(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(d)})$$

is a sum of products of $\zeta^*_{\mathcal{S}}(k) = (1 + (-1)^k)\zeta(k)$, which is 0 when k is odd and a multiple of $\zeta(2)$ when k is even.

Remark. One can also prove Theorem 3.10 for SMZVs directly by using the definition. For example, we compute

$$\begin{aligned}
&\sum_{\sigma \in S_3} \zeta^*_{\mathcal{S}}(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}) \\
&= (1 + (-1)^{k_1})(1 + (-1)^{k_2})(1 + (-1)^{k_3}) \sum_{\sigma \in S_3} \zeta^*(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}) \\
&+ ((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})(\zeta(k_1 + k_2, k_3) + \zeta^*(k_3, k_1 + k_2)) \\
&+ ((-1)^{k_1} + (-1)^{k_3})(1 + (-1)^{k_2})(\zeta(k_1 + k_3, k_2) + \zeta^*(k_2, k_1 + k_3)) \\
&+ ((-1)^{k_2} + (-1)^{k_3})(1 + (-1)^{k_1})(\zeta(k_2 + k_3, k_1) + \zeta^*(k_1, k_2 + k_3)).
\end{aligned}$$

When the weight k is odd, the coefficients $(1+(-1)^{k_1})(1+(-1)^{k_2})(1+(-1)^{k_3})$ and $((-1)^{k_1}+(-1)^{k_2})(1+(-1)^{k_3})$ etc., become 0. When k is even, the factor $(1+(-1)^{k_1})(1+(-1)^{k_2})(1+(-1)^{k_3})$ becomes 0 if at least one k_i is odd. When all k_i are even, then $\sum_{\sigma\in S_3}\zeta^*(k_{\sigma(1)},k_{\sigma(2)},k_{\sigma(3)}) = \zeta^*(k_1)\zeta^*(k_2)\zeta^*(k_3)-\zeta(k_1+k_2)\zeta^*(k_3)-\zeta(k_1+k_3)\zeta^*(k_2)-\zeta(k_2+k_3)\zeta^*(k_1)+2\zeta(k_1+k_2+k_3)$ is 0 modulo $\zeta(2)$. As for the term $((-1)^{k_1}+(-1)^{k_2})(1+(-1)^{k_3})(\zeta(k_1+k_2,k_3)+\zeta^*(k_3,k_1+k_2))$, etc., if we write this as $((-1)^{k_1}+(-1)^{k_2})(1+(-1)^{k_3})(\zeta(k_1+k_2)\zeta^*(k_3)-\zeta(k_1+k_2+k_3))$, we see that either $((-1)^{k_1}+(-1)^{k_2})(1+(-1)^{k_3})=0$ or $(\zeta(k_1+k_2)\zeta^*(k_3)-\zeta(k_1+k_2+k_3))$ is a multiple of $\zeta(2)$.

3.4.2 Proof of Theorem 3.11 for SMZVs

We can prove Theorem 3.11 in the same manner as in Saito–Wakabayashi [28]. Set

$$S_{k,d,i} = \sum_{\substack{k_1+k_2+\cdots+k_d=k\\ k_i\geq 2}} \zeta^*_{\mathcal{S}}(k_1,\ldots,k_d).$$

We notice that the harmonic version of the SMZVs satisfy the harmonic product rule. Thus, $S_{k,d,i}$ enjoys the recursion relation in the following lemma, which can be proved in the same way as in [28, Proposition 2.2].

Lemma 3.13. *For integers $k,d,i\in\mathbb{Z}_{\geq 1}$ with $2\leq i+1\leq d\leq k-1$, we have*

$$(d-i)S_{k,d,i}+iS_{k,d,i+1}+(k-d)S_{k,d-1,i}=0.$$

We prove Theorem 3.11 by backward induction on d. To do this, we need the initial value.

Lemma 3.14. *For integers $k,i\in\mathbb{Z}_{\geq 1}$ with $1\leq i\leq k-1$, we have*

$$S_{k,k-1,i}\equiv(-1)^{i-1}\binom{k}{i}\zeta(k) \pmod{\zeta(2)}.$$

Proof. Since $S_{k,k-1,i}=\zeta^*_{\mathcal{S}}(\underbrace{1,\ldots,1}_{i-1},2,\underbrace{1,\ldots,1}_{k-i-1})$, we compute $\zeta^{\text{ш}}_{\mathcal{S}}(\underbrace{1,\ldots,1}_{i-1},2,\underbrace{1,\ldots,1}_{k-i-1})$ instead. Because of the fact that $\zeta^{\text{ш}}_{\mathcal{S}}(1,\ldots,1)=0$, we have by definition

$$\begin{aligned}S_{k,k-1,i}&\equiv\zeta^{\text{ш}}_{\mathcal{S}}(\underbrace{1,\ldots,1}_{i-1},2,\underbrace{1,\ldots,1}_{k-i-1}) \pmod{\zeta(2)}\\ &=\zeta^{\text{ш}}(\underbrace{1,\ldots,1}_{i-1},2,\underbrace{1,\ldots,1}_{k-i-1})+(-1)^k\zeta^{\text{ш}}(\underbrace{1,\ldots,1}_{k-i-1},2,\underbrace{1,\ldots,1}_{i-1}).\end{aligned}$$

By using [14, eq.(5.2)] for $w_0=xy^l$, we have $\zeta^{\text{ш}}(\underbrace{1,\ldots,1}_{m},2,\underbrace{1,\ldots,1}_{l-1})=(-1)^m\binom{m+l}{m}\zeta(2,\underbrace{1,\ldots,1}_{m+l-1})$. Thus,

$$S_{k,k-1,i}\equiv(-1)^{i-1}\left(\binom{k-1}{i-1}+\binom{k-1}{i}\right)\zeta(2,\underbrace{1,\ldots,1}_{k-2})=(-1)^{i-1}\binom{k}{i}\zeta(k) \pmod{\zeta(2)}.$$

$\blacksquare$

Let us consider the case $d = k - 1$ of Theorem 3.11. If k is even, the identity holds from Lemma 3.14. If k is odd, then d is even and the identity again follows because

$$\binom{k-1}{i-1} + (-1)^d \binom{k-1}{d-i} = \binom{k-1}{i-1} + \binom{k-1}{i} = \binom{k}{i}.$$

We assume the identity holds for d. By Lemma 3.13,

$$\begin{aligned}
(d-k)S_{k,d-1,i} &= (d-i)S_{k,d,i} + iS_{k,d,i+1} \\
&= (d-i)(-1)^{i-1}\left(\binom{k-1}{i-1} + (-1)^d\binom{k-1}{d-i}\right)\zeta(k) \\
&\quad + i(-1)^i\left(\binom{k-1}{i} + (-1)^d\binom{k-1}{d-i-1}\right)\zeta(k) \\
&= (-1)^{i-1}\left((d-i)\binom{k-1}{i-1} + (k-d+i)(-1)^d\binom{k-1}{d-i-1}\right)\zeta(k) \\
&\quad + (-1)^i\left((k-i)\binom{k-1}{i-1} + i(-1)^d\binom{k-1}{d-i-1}\right)\zeta(k) \\
&= (d-k)(-1)^{i-1}\left(\binom{k-1}{i-1} + (-1)^{d-1}\binom{k-1}{d-i-1}\right)\zeta(k).
\end{aligned}$$

Thus, the identity holds for $d - 1$.

Remark 3.15. *We mention an analogue of Theorem 3.11 on SMZVs. Set*

$$S^{\star}_{k,d,i} = \sum_{\substack{k_1+k_2+\cdots+k_d=k \\ k_i \geq 2}} \zeta^{\star,*}_{\mathcal{S}}(k_1, \ldots, k_d).$$

Since these $\zeta^{\star,*}_{\mathcal{S}}(k_1, \ldots, k_d)$ satisfy the same harmonic product rule as $\zeta^{\star}(k_1, \ldots, k_d)$, $S^{\star}_{k,d,i}$ enjoys the same recursion relation as Saito–Wakabayashi [28, Proposition 2.2], that is,

$$(d-i)S^{\star}_{k,d,i} + iS^{\star}_{k,d,i+1} - (k-d)S^{\star}_{k,d-1,i} = 0.$$

Writing $\mathbf{k}_i \sqcup \mathbf{k}_j$ for juxtaposition of indices $\mathbf{k}_i$ and $\mathbf{k}_j$, we see from Hoffman [13, Theorem 3.1] that

$$\zeta^{\star,*}_{\mathcal{S}}(k_d, k_{d-1}, \ldots, k_1) = (-1)^d \sum_{\mathbf{k}_1 \sqcup \cdots \sqcup \mathbf{k}_l = (k_1, \ldots, k_d)} (-1)^l \zeta^*_{\mathcal{S}}(\mathbf{k}_1) \cdots \zeta^*_{\mathcal{S}}(\mathbf{k}_l).$$

Consider the case $(k_1, \ldots, k_d) = (\underbrace{1, \ldots, 1}_{k-i-1}, 2, \underbrace{1, \ldots, 1}_{i-1})$ in this equality. Since $\zeta^*_{\mathcal{S}}(1, \ldots, 1) \equiv \zeta^{\text{Ш}}_{\mathcal{S}}(1, \ldots, 1) = 0 \pmod{\zeta(2)}$, the right-hand side is equal modulo $\zeta(2)$ to $\zeta^{\text{Ш}}_{\mathcal{S}}(\underbrace{1, \ldots, 1}_{i-1}, 2, \underbrace{1, \ldots, 1}_{k-i-1})$. Thus, we find

$$S^{\star}_{k,k-1,i} \equiv S_{k,k-1,i} \equiv (-1)^{i-1}\binom{k}{i}\zeta(k) \pmod{\zeta(2)}.$$

In a similar way as for the proof of Theorem 3.11 (i.e., by backward induction on d), we obtain

$$\sum_{\substack{k_1+k_2+\cdots+k_d=k\\ k_i\geq 2}} \zeta_{\mathcal{S}}^{\star,*}(k_1,\ldots,k_d) \equiv (-1)^{i-1}\left((-1)^d\binom{k-1}{i-1}+\binom{k-1}{d-i}\right)\zeta(k) \pmod{\zeta(2)}.$$

3.4.3 Proof of Theorem 3.12 for SMZVs

For a given index $\mathbf{k}$, we call the number of its elements greater than 1 the height. With this terminology, we shall call

$$\zeta_{\mathcal{S}}(k,\underbrace{1,\cdots,1}_{d-1})$$

height-one SMZVs. Here, we prove Theorem 3.12. To do this, we need the following key lemma.

Lemma 3.16. *For integers $k,d\in\mathbb{Z}_{\geq 1}$ with $k\geq 2$, we have*

$$\zeta^{\sqcup\!\sqcup}(\underbrace{1,\ldots,1}_{d-1},k)=(-1)^{d-1}\zeta^{\star}(k,\underbrace{1,\ldots,1}_{d-1}).$$

Proof. We recall the $\mathbb{Q}$-linear map L_w defined by

$$L_w(w')=ww' \qquad (w'\in\mathfrak{H}).$$

By the regularization formula proved by Ihara–Kaneko–Zagier in [14, eq.(5.2)], we have

$$Z^{\sqcup\!\sqcup}(y^m w)=(-1)^m Z(L_x(y^m \sqcup\!\sqcup w'))$$

for any integer $m\in\mathbb{Z}_{\geq 1}$ and $w=xw'\in\mathfrak{H}^0$. Then, we find

$$\begin{aligned}
\zeta^{\sqcup\!\sqcup}(\underbrace{1,\ldots,1}_{d-1},k) &= Z^{\sqcup\!\sqcup}(y^{d-1}x^{k-1}y)\\
&= (-1)^{d-1}Z^{\sqcup\!\sqcup}(L_x(y^{d-1}\sqcup\!\sqcup x^{k-2}y))\\
&= (-1)^{d-1}\sum_{\substack{a_1+\cdots+a_k=d-1\\ a_i\geq 0\,(i=1,\ldots,k)}} Z(xy^{a_1}\cdots xy^{a_{k-2}}xy^{a_{k-1}+a_k+1})\\
&= (-1)^{d-1}\sum_{\substack{a_1+\cdots+a_{k-1}=d-1\\ a_i\geq 0\,(i=1,\ldots,k-1)}} (a_{k-1}+1)Z(xy^{a_1}\cdots xy^{a_{k-2}}xy^{a_{k-1}+1})\\
&= (-1)^{d-1}\sum_{\substack{a_1+\cdots+a_{k-1}=d-1\\ a_i\geq 0\,(i=1,\ldots,k-1)}} (a_{k-1}+1)Z(x^{a_{k-1}+1}yx^{a_{k-2}}y\cdots x^{a_1}y)\\
&= (-1)^{d-1}\sum_{\substack{a_1+\cdots+a_{k-1}=d-1\\ a_i\geq 0\,(i=1,\ldots,k-1)}} (a_{k-1}+1)\zeta(a_{k-1}+2,a_{k-2}+1,\ldots,a_1+1).
\end{aligned}$$

In the above equality, we have also used the duality theorem for MZVs (Theorem 2.17). Here, we note that Theorem 2.17 says

$$Z(x^{k_1-1}y\cdots x^{k_d-1}y) = Z(xy^{k_d-1}\cdots xy^{k_1-1}).$$

That the last sum equals $\zeta^\star(d,\underbrace{1,\ldots,1}_{k-1})$ is due to Ohno [26, Proof of Theorem 2], see also Kaneko [18, §3]. Thus

$$\zeta^{Ш}(\underbrace{1,\ldots,1}_{d-1},k) = (-1)^{d-1}\zeta^\star(k,\underbrace{1,\ldots,1}_{d-1}).$$

∎

Now, we prove Theorem 3.12. When either k or $d=1$, the theorem clearly holds. We consider the case when $k,d\geq 2$. From the above Lemma 3.16, we have

$$\begin{aligned}
&\zeta_{\mathcal{S}}^{Ш}(k,\underbrace{1,\ldots,1}_{d-1}) - \zeta_{\mathcal{S}}^{Ш}(d,\underbrace{1,\ldots,1}_{k-1})\\
&= \zeta(k,\underbrace{1,\ldots,1}_{d-1}) + (-1)^k\zeta^\star(k,\underbrace{1,\ldots,1}_{d-1}) - (\zeta(d,\underbrace{1,\ldots,1}_{k-1}) + (-1)^d\zeta^\star(d,\underbrace{1,\ldots,1}_{k-1})).
\end{aligned}$$

Let $\psi(X)=\frac{\Gamma'(X)}{\Gamma(X)}$. By using the well-known generating series

$$\begin{aligned}
1-\sum_{k,d\geq 1}\zeta(k+1,\underbrace{1,\ldots,1}_{d-1})X^kY^d &= \exp\left(\sum_{n\geq 2}\zeta(n)\frac{X^n+Y^n-(X+Y)^n}{n}\right)\\
&= \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}
\end{aligned}$$

(cf. Aomoto [4] and Drinfel'd [9]) and $\psi(1-X) = -\sum_{k\geq 2}\zeta(k)X^{k-1}-\gamma$ (γ is Euler's constant), we have

$$\begin{aligned}
&\sum_{k,d\geq 2}\left(\zeta(k,\underbrace{1,\ldots,1}_{d-1}) - \zeta(d,\underbrace{1,\ldots,1}_{k-1})\right)X^{k-1}Y^{d-1}\\
&= \left(\frac{1}{Y}-\frac{1}{X}\right)\left(1-\frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}\right)+\psi(1-X)-\psi(1-Y).
\end{aligned}$$

On the other hand, from Kaneko and Ohno [17, Theorem 2],

$$\begin{aligned}
&\sum_{k,d\geq 2}\left((-1)^k\zeta^\star(k,\underbrace{1,\ldots,1}_{d-1}) - (-1)^d\zeta^\star(d,\underbrace{1,\ldots,1}_{k-1})\right)X^{k-1}Y^{d-1}\\
&= -\psi(X)+\psi(Y)-\pi(\cot(\pi X)-\cot(\pi Y))\frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)},
\end{aligned}$$

From these, and by the well-known equalities:

$$\pi\cot(\pi X) = \frac{1}{X} + \psi(1-X) - \psi(1+X),$$
$$\psi(X) = \psi(1+X) - \frac{1}{X},$$

we have

$$\begin{aligned}
&\sum_{k,d\geq 2}\Bigg(\zeta_{\mathcal{S}}^{\text{III}}(k,\underbrace{1,\ldots,1}_{d-1}) - \zeta_{\mathcal{S}}^{\text{III}}(d,\underbrace{1,\ldots,1}_{k-1})\Bigg)X^{k-1}Y^{d-1}\\
&= \left(\frac{1}{Y} - \frac{1}{X}\right)\left(1 - \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}\right) + \psi(1-X) - \psi(1-Y)\\
&\quad - \psi(X) + \psi(Y) - \pi(\cot(\pi X) - \cot(\pi Y))\frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}\\
&= \left(1 - \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}\right)(\psi(1-X) - \psi(1+X) - \psi(1-Y) + \psi(1+Y))\\
&= -2\left(1 - \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}\right)\sum_{l\geq 1}\zeta(2l)(X^{2l-1} - Y^{2l-1}).
\end{aligned}$$

Since the coefficients of $\frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}$ belong to the $\mathbb{Q}$-algebra $\mathcal{Z}$, we have

$$\zeta_{\mathcal{S}}^{\text{III}}(k,\underbrace{1,\ldots,1}_{d-1}) \equiv \zeta_{\mathcal{S}}^{\text{III}}(d,\underbrace{1,\ldots,1}_{k-1}) \pmod{\zeta(2)}.$$

This proves Theorem 3.12.

3.5 Derivation relations for FMZVs

3.5.1 Derivation relations for FMZVs

Kojiro Oyama conjectured the following derivation relations for FMZVs.

Conjecture 5 (Oyama)**.** *For $l \in \mathbb{Z}_{\geq 1}$, we have*

$$Z_{\mathcal{F}}(\partial_l(w)) = -Z_{\mathcal{F}}(z^{l-1}yw) \quad (w \in \mathfrak{H}^1). \tag{7}$$

Oyama proved this for $l \leq 4$ and Mitsuki Kosaki extended the proof further to $l \leq 6$. The aim of this section is to prove this conjecture for all l. Actually, we prove the identity in the following form, which looks more general than the conjecture but in fact is equivalent to the conjecture. The proof of Theorem 3.17 will be given later.

Theorem 3.17. *For any index* $(m_1, \dots, m_s) \in (\mathbb{Z}_{\geq 1})^s$ $(s \geq 0)$ *and integer* $l \in \mathbb{Z}_{\geq 1}$*, we have*

$$\begin{aligned}&Z_{\mathcal{F}}(z^{m_1-1}y \cdots z^{m_s-1}y\partial_l(w))\\&= -Z_{\mathcal{F}}(z^{l-1}yz^{m_1-1}y \cdots z^{m_s-1}yw)\\&\quad + \sum_{i=1}^{s} Z_{\mathcal{F}}(z^{m_1-1}y \cdots z^{m_{i-1}-1}yz^{m_i-1}xz^{l-1}yz^{m_{i+1}-1}y \cdots z^{m_s-1}yw) \quad (w \in \mathfrak{H}^1).\end{aligned}$$

When $s = 0$*, we understand* $z^{m_1-1}y \cdots z^{m_s-1}y = 1$ *on the left, and the right-hand side is* $-Z_{\mathcal{F}}(z^{l-1}yw)$*, which yield Conjecture 5.*

Remark 3.18. *We see Conjecture 5 implies Theorem 3.17 by putting* $z^{m_1-1}y \cdots z^{m_s-1}yw$ *for* w *in eq.(7), because*

$$\begin{aligned}\partial_l(z^{m_1-1}y \cdots z^{m_s-1}yw) = &- z^{m_1-1}xz^{l-1}yz^{m_2-1}y \cdots z^{m_s-1}yw\\&- \quad \cdots\cdots\\&- z^{m_1-1}y \cdots z^{m_s-1}xz^{l-1}yw\\&+ z^{m_1-1}y \cdots z^{m_s-1}y\partial_l(w)\end{aligned}$$

by the definition of ∂_l *(note* $\partial_l(z) = 0$*), and*

$$Z_{\mathcal{F}}(\partial_l(z^{m_1-1}y \cdots z^{m_s-1}yw)) = -Z_{\mathcal{F}}(z^{l-1}yz^{m_1-1}y \cdots z^{m_s-1}yw)$$

by eq.(7).

Example 3.19. *When* $l = 3$ *and* $w = xy$ *in Conjecture 5, we have*

$$\begin{aligned}Z_{\mathcal{F}}(\partial_3(xy)) &= -Z_{\mathcal{F}}(z^2yxy)\\&= -Z_{\mathcal{F}}(x^2yxy + xy^2xy + yxyxy + y^3xy).\end{aligned}$$

Since

$$\begin{aligned}\partial_3(xy) &= xz^2y^2 - x^2z^2y\\&= xyxy^2 + xy^4 - x^4y - x^2yxy,\end{aligned}$$

we get

$$\zeta_{\mathcal{F}}(5) - \zeta_{\mathcal{F}}(2,2,1) - \zeta_{\mathcal{F}}(2,1,2) - \zeta_{\mathcal{F}}(1,2,2) - \zeta_{\mathcal{F}}(2,1,1,1) - \zeta_{\mathcal{F}}(1,1,1,2) = 0.$$

Example 3.20. *The case* $s = 2$ *in Theorem 3.17 gives*

$$\begin{aligned}Z_{\mathcal{F}}(z^{m_1-1}yz^{m_2-1}y\partial_l(w)) = &-Z_{\mathcal{F}}(z^{l-1}yz^{m_1-1}yz^{m_2-1}yw)\\&+ Z_{\mathcal{F}}(z^{m_1-1}xz^{l-1}yz^{m_2-1}yw)\\&+ Z_{\mathcal{F}}(z^{m_1-1}yz^{m_2-1}xz^{l-1}yw).\end{aligned}$$

When $m_1 = 2, m_2 = 1, l = 2$ *and* $w = y$*, we get*

$$\begin{aligned}&\zeta_{\mathcal{F}}(4,1,1) + \zeta_{\mathcal{F}}(2,3,1) + \zeta_{\mathcal{F}}(2,1,3) + \zeta_{\mathcal{F}}(3,1,1,1) + \zeta_{\mathcal{F}}(1,3,1,1) + \zeta_{\mathcal{F}}(1,1,3,1) + \zeta_{\mathcal{F}}(1,1,1,3)\\&+ \zeta_{\mathcal{F}}(2,1,2,1) - \zeta_{\mathcal{F}}(2,1,1,1,1) + \zeta_{\mathcal{F}}(1,2,1,1,1) + \zeta_{\mathcal{F}}(1,1,1,2,1) - \zeta_{\mathcal{F}}(1,1,1,1,1,1) = 0.\end{aligned}$$

Let S_n be the symmetric group of n symbols, which acts on any index $\mathbf{a} = (a_1, \dots, a_n)$ by $\sigma(\mathbf{a}) = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$. For an integer s with $1 \le s \le n$, let $S_n^{(s)}$ be the subset of S_n given by

$$S_n^{(s)} = \left\{ \sigma \in S_n \mid \sigma^{-1}(1) < \cdots < \sigma^{-1}(s) \right\}.$$

Under these notations, we have the following theorem, which is in fact an almost immediate consequence of Theorem 3.17. The proof will also be given in the next section.

Theorem 3.21. *For any index $\mathbf{m} = (m_1, \dots, m_s) \in (\mathbb{Z}_{\ge 1})^s$ $(s \ge 0)$ and $\mathbf{l} = (l_1, \dots, l_t) \in (\mathbb{Z}_{\ge 1})^t$ $(t \ge 1)$, we set $\mathbf{a} = (a_1, \dots, a_{s+t}) = (\mathbf{m}, \mathbf{l})$. Then, we have*

$$\begin{aligned} &Z_{\mathcal{F}}(z^{m_1-1}y \cdots z^{m_s-1}y\partial_{l_1} \cdots \partial_{l_t}(w)) \\ &= (-1)^s \sum_{\sigma \in S_{s+t}^{(s)}} Z_{\mathcal{F}}(z^{a_{\sigma(1)}-1}u_1^\sigma \cdots z^{a_{\sigma(s+t)}-1}u_{s+t}^\sigma w) \quad (w \in \mathfrak{H}^1). \end{aligned}$$

Here, we set $u_i^\sigma = x$ if '$\sigma(i) \le s$ and $\sigma(i+1) > s$' or '$\sigma(i) > s$ and $\sigma(i) < \sigma(i+1)$,' and $u_i^\sigma = -y$ otherwise.

Example 3.22. *When $r = 2, s = 1$ in Theorem 3.21, we have*

$$\begin{aligned} Z_{\mathcal{F}}(z^{m_1-1}y\partial_{l_1}\partial_{l_2}(w)) &= Z_{\mathcal{F}}(z^{m_1-1}xz^{l_1-1}xz^{l_2-1}yw) - Z_{\mathcal{F}}(z^{m_1-1}xz^{l_2-1}yz^{l_1-1}yw) \\ &\quad - Z_{\mathcal{F}}(z^{l_1-1}yz^{m_1-1}xz^{l_2-1}yw) - Z_{\mathcal{F}}(z^{l_2-1}yz^{m_1-1}xz^{l_1-1}yw) \\ &\quad - Z_{\mathcal{F}}(z^{l_1-1}xz^{l_2-1}yz^{m_1-1}yw) + Z_{\mathcal{F}}(z^{l_2-1}yz^{l_1-1}yz^{m_1-1}yw). \end{aligned}$$

By putting $l_1 = 2, l_2 = 1, m_1 = 2$ and $w = y$, we get

$$\begin{aligned} &\zeta_{\mathcal{F}}(5,1) - \zeta_{\mathcal{F}}(2,4) - \zeta_{\mathcal{F}}(3,2,1) - \zeta_{\mathcal{F}}(2,3,1) - \zeta_{\mathcal{F}}(1,1,4) \\ &- 2\zeta_{\mathcal{F}}(3,1,1,1) - \zeta_{\mathcal{F}}(1,3,1,1) - 2\zeta_{\mathcal{F}}(1,1,3,1) - \zeta_{\mathcal{F}}(2,1,2,1) + \zeta_{\mathcal{F}}(2,2,1,1) \\ &- \zeta_{\mathcal{F}}(1,2,1,1,1) + \zeta_{\mathcal{F}}(1,1,1,1,1,1) = 0. \end{aligned}$$

Remark 3.23. *For two indices $\mathbf{m}, \mathbf{m}'$, we say $\mathbf{m}'$ refines $\mathbf{m}$ (denoted $\mathbf{m}' \succeq \mathbf{m}$) if $\mathbf{m}$ can be obtained from $\mathbf{m}'$ by combining some of its adjacent parts. Then, we have*

$$\begin{aligned} &Z_{\mathcal{F}}(x^{m_1-1}y \cdots x^{m_s-1}y\partial_{l_1} \cdots \partial_{l_t}(w)) \\ &= (-1)^s \sum_{\mathbf{m}' \succeq \mathbf{m}} \sum_{\sigma \in S_{s'+t}^{(s')}} Z_{\mathcal{F}}(z^{a'_{\sigma(1)}-1}u_1^\sigma \cdots z^{a'_{\sigma(s'+t)}-1}u_{s'+t}^\sigma w) \quad (w \in \mathfrak{H}^1), \end{aligned} \tag{8}$$

where s' in the second sum on the right is the depth of $\mathbf{m}'$ and $\mathbf{a}' = (a'_1, \dots, a'_{s'+t}) = (\mathbf{m}', \mathbf{l})$. We note here that eq.(8) is equivalent to Theorem 3.21. Assume that Theorem 3.21 holds, we see by $x^{m_1-1}y \cdots x^{m_s-1}y = \sum_{\mathbf{m}' \succeq \mathbf{m}} (-1)^{s'-s} z^{m'_1-1}y \cdots z^{m'_{s'}-1}y$ that

$$\begin{aligned} &Z_{\mathcal{F}}(x^{m_1-1}y \cdots x^{m_s-1}y\partial_{l_1} \cdots \partial_{l_t}(w)) \\ &= \sum_{\mathbf{m}' \succeq \mathbf{m}} (-1)^{s'-s} Z_{\mathcal{F}}(z^{m'_1-1}y \cdots z^{m'_{s'}-1}y\partial_{l_1} \cdots \partial_{l_t}(w)) \\ &= (-1)^s \sum_{\mathbf{m}' \succeq \mathbf{m}} \sum_{\sigma \in S_{s'+t}^{(s')}} Z_{\mathcal{F}}(z^{a'_{\sigma(1)}-1}u_1^\sigma \cdots z^{a'_{\sigma(s'+t)}-1}u_{s'+t}^\sigma w). \end{aligned}$$

Conversely, assume that eq.(8) holds. Since

$$z^{m_1-1}y\cdots z^{m_s-1}y=\sum_{\mathbf{m}'\succeq\mathbf{m}}x^{m'_1-1}y\cdots x^{m'_{s'}-1}y,$$

and

$$\sum_{\mathbf{m}'\succeq\mathbf{m}}(-1)^{s'}\sum_{\substack{\mathbf{m}''\succeq\mathbf{m}'\\ \mathbf{m}''=(m''_1,\ldots,m''_{s''})}}(m''_1,\ldots,m''_{s''})=(-1)^s(m_1,\ldots,m_s),$$

(the second equality is an identity of formal sums of indices) we have

$$\begin{aligned}
&Z_{\mathcal{F}}(z^{m_1-1}y\cdots z^{m_s-1}y\partial_{l_1}\cdots\partial_{l_t}(w))\\
&=\sum_{\mathbf{m}'\succeq\mathbf{m}}Z_{\mathcal{F}}(x^{m'_1-1}y\cdots x^{m'_{s'}-1}y\partial_{l_1}\cdots\partial_{l_t}(w))\\
&=\sum_{\mathbf{m}'\succeq\mathbf{m}}(-1)^{s'}\sum_{\mathbf{m}''\succeq\mathbf{m}'}\sum_{\sigma\in S^{(s'')}_{s''+t}}Z_{\mathcal{F}}(z^{a''_{\sigma(1)}-1}u_1^{\sigma}\cdots z^{a''_{\sigma(s''+t)}-1}u^{\sigma}_{s''+t}w)\\
&=(-1)^s\sum_{\sigma\in S^{(s)}_{s+t}}Z_{\mathcal{F}}(z^{a_{\sigma(1)}-1}u_1^{\sigma}\cdots z^{a_{\sigma(s+t)}-1}u^{\sigma}_{s+t}w),
\end{aligned}$$

where s'' is the depth of $\mathbf{m}''$ and $\mathbf{a}''=(a''_1,\ldots,a''_{s''})$.

Before closing this subsection, we mention the maximal number of linearly independent relations supplied by Conjecture 5. In Table 1, the first line means the weight of FMZVs. The second line gives the number of linearly independent elements in $\mathfrak{H}$ among all $\partial_l(w)+z^{l-1}yw$ with $l\in\mathbb{Z}_{\geq 1}$ and $w\in\mathfrak{H}^1$ varying under the condition $l+|w|=$ weight. Computations are performed by Mathematica.

Table 1: Number of Independent Derivation Relations for FMZVs

weight	2	3	4	5	6	7	8	9	10	11	12	13
relations	1	2	5	10	22	44	90	181	363	727	1456	2912

An interesting fact is that the number of independent relations of derivation relations in Table 1 coincides with that of the original derivation relations in Table 2, except that the weight is shifted by one. The reason for this coincidence is seen as follows. Write an element $w\in\mathfrak{H}^0$ as $w=xw',w'\in\mathfrak{H}^1$. Then by $\partial_l(w)=xz^{l-1}yw'+x\partial_l(w')$, the original derivation relations $Z(\partial_l(w))=0$ can be written as

$$Z(x(\partial_l(w')+z^{l-1}yw'))=0.$$

Hence the relations $Z_{\mathcal{F}}(\partial_l(w')+z^{l-1}yw')=0$ in weight k exactly corresponds to the relations $Z(x(\partial_l(w')+z^{l-1}yw'))=Z(\partial_l(w))=0$ in weight $k+1$.

Table 2: Number of Independent Derivation Relations for MZVs

weight	3	4	5	6	7	8	9	10	11	12	13	14
relations	1	2	5	10	22	44	90	181	363	727	1456	2912

3.5.2 Proofs of Theorem 3.17 and Theorem 3.21

We prove Theorem 3.17 by induction on $n = |w|$.
(I) When $n = 0$, i.e., $w = 1$, we need to show

$$\begin{aligned}&- Z_{\mathcal{F}}(z^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}y)\\&+\sum_{i=1}^{s} Z_{\mathcal{F}}(z^{m_1-1}y\cdots z^{m_{i-1}-1}yz^{m_i-1}xz^{l-1}yz^{m_{i+1}-1}y\cdots z^{m_s-1}y)=0\end{aligned}$$

for every $s \geq 0$. When $s = 0$, by Theorem 3.9, we have

$$-Z_{\mathcal{F}}(z^{l-1}y) = Z_{\mathcal{F}}(x^{l-1}y) = 0.$$

Here, we note that $\zeta_{\mathcal{F}}(l) = 0$ for any $l \in \mathbb{Z}_{\geq 1}$. When $s \geq 1$, by Theorem 3.8 and Theorem 3.9,

$$\begin{aligned}&- Z_{\mathcal{F}}(z^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}y)\\&+\sum_{i=1}^{s} Z_{\mathcal{F}}(z^{m_1-1}y\cdots z^{m_{i-1}-1}yz^{m_i-1}xz^{l-1}yz^{m_{i+1}-1}y\cdots z^{m_s-1}y)\\&= Z_{\mathcal{F}}(-z^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}y + z^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}y\\&\qquad + \quad \cdots\cdots \quad + z^{m_1-1}y\cdots z^{m_s-1}xz^{l-1}y)\\&= (-1)^s Z_{\mathcal{F}}(x^{l-1}yx^{m_1-1}y\cdots x^{m_s-1}y + x^{m_1-1}zx^{l-1}yx^{m_2-1}y\cdots x^{m_s-1}y\\&\qquad\qquad + \quad \cdots\cdots \quad + x^{m_1-1}y\cdots x^{m_s-1}zx^{l-1}y)\\&= (-1)^s Z_{\mathcal{F}}(x^{m_1-1}y\cdots x^{m_s-1}y * x^{l-1}y)\\&= (-1)^s Z_{\mathcal{F}}(x^{m_1-1}y\cdots x^{m_s-1}y)Z_{\mathcal{F}}(x^{l-1}y) = 0.\end{aligned}$$

(II) We assume the identity holds for $|w| = 0, \ldots, n-1$ and for every $s \geq 0$. Suppose w is of degree n. We may assume that w is of the form $w = z^{r-1}yw'$ with $1 \leq r \leq n, w' \in \mathfrak{H}^1$, by replacing $x^{r-1}y$ by $(z-y)^{r-1}y$ if w starts with $x^{r-1}y$.

$$\begin{aligned}\text{L.H.S.} &= Z_{\mathcal{F}}(z^{m_1-1}y\cdots z^{m_s-1}y\partial_l(z^{r-1}yw'))\\&= Z_{\mathcal{F}}(-z^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw' + z^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}y\partial_l(w')).\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}Z_{\mathcal{F}}(z^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}y\partial_l(w')) = Z_{\mathcal{F}}(&-z^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}yw'\\&+ z^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}yz^{r-1}yw'\\&+ \quad \cdots\cdots\\&+ z^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw').\end{aligned}$$

Thus,

$$\begin{aligned}
\text{L.H.S.} &= Z_{\mathcal{F}}(-z^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw' - z^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}yw' \\
&\qquad + z^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}yz^{r-1}yw' + \quad \cdots\cdots \\
&\qquad + z^{m_1-1}y\cdots z^{m_s-1}xz^{l-1}yz^{r-1}yw' + z^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}xz^{l-1}yw') \\
&= Z_{\mathcal{F}}(-z^{l-1}yz^{m_1-1}y\cdots z^{m_s-1}yz^{r-1}yw' + z^{m_1-1}xz^{l-1}yz^{m_2-1}y\cdots z^{m_s-1}yz^{r-1}yw' \\
&\qquad + \quad \cdots\cdots \quad + z^{m_1-1}y\cdots z^{m_s-1}xz^{l-1}yz^{r-1}yw') \\
&= \text{R.H.S.},
\end{aligned}$$

and hence the identity holds for n and by induction, the proof is done.

Now, we prove Theorem 3.21 by induction on t. We have proved the case $t = 1$. We assume the identity holds when the number of derivations on the left is less than t.

$$\begin{aligned}
&Z_{\mathcal{F}}(z^{m_1-1}y\cdots z^{m_s-1}y\partial_{l_1}\cdots\partial_{l_t}(w)) \\
&= Z_{\mathcal{F}}(-z^{l_1-1}yz^{m_1-1}y\cdots z^{m_s-1}y\partial_{l_2}\cdots\partial_{l_t}(w) \\
&\qquad + z^{m_1-1}xz^{l_1-1}yz^{m_2-1}y\cdots z^{m_s-1}y\partial_{l_2}\cdots\partial_{l_t}(w) \\
&\qquad + \quad \cdots\cdots \\
&\qquad + z^{m_1-1}yz\cdots z^{m_s-1}xz^{l_1-1}y\partial_{l_2}\cdots\partial_{l_t}(w)) \\
&= Z_{\mathcal{F}}(-z^{l_1-1}yz^{m_1-1}y\cdots z^{m_s-1}y\partial_{l_2}\cdots\partial_{l_t}(w) \\
&\qquad + z^{m_1-1}(z-y)z^{l_1-1}yz^{m_2-1}y\cdots z^{m_s-1}y\partial_{l_2}\cdots\partial_{l_t}(w) \\
&\qquad + \quad \cdots\cdots \\
&\qquad + z^{m_1-1}yz\cdots z^{m_s-1}(z-y)z^{l_1-1}y\partial_{l_2}\cdots\partial_{l_t}(w)) \\
&= (-1)^s\sum_{i=0}^{s}\sum_{\sigma\in S_{s+t}^{(s+1)}} Z_{\mathcal{F}}(z^{a'_{i,\sigma(1)}-1}u_1^{\sigma}\cdots z^{a'_{i,\sigma(s+t)}-1}u_{s+t}^{\sigma}w) \\
&\quad + (-1)^s\sum_{i=1}^{s}\sum_{\sigma\in S_{s+t-1}^{(s)}} Z_{\mathcal{F}}(z^{a''_{i,\sigma(1)}-1}u_1^{\sigma}\cdots z^{a''_{i,\sigma(s+t-1)}-1}u_{s+t-1}^{\sigma}w),
\end{aligned}$$

where $\mathbf{a}'_i = (a'_{i,1},\ldots,a'_{i,s+t}) = (m_1,\ldots,m_i,l_1,m_{i+1},\ldots,m_s,l_2,\ldots,l_t)$ and $\mathbf{a}''_i = (a''_{i,1},\ldots,a''_{i,s+t-1}) = (m_1,\ldots,m_{i-1},m_i+l_1,m_{i+1},\ldots,m_s,l_2,\ldots,l_t)$ in the last summation. We let

$$\begin{aligned}
L &:= \sum_{i=0}^{s}\sum_{\sigma\in S_{s+t}^{(s+1)}} Z_{\mathcal{F}}(z^{a'_{i,\sigma(1)}-1}u_1^{\sigma}\cdots z^{a'_{i,\sigma(s+t)}-1}u_{s+t}^{\sigma}w), \\
M &:= \sum_{i=1}^{s}\sum_{\sigma\in S_{s+t-1}^{(s)}} Z_{\mathcal{F}}(z^{a''_{i,\sigma(1)}-1}u_1^{\sigma}\cdots z^{a''_{i,\sigma(s+t-1)}-1}u_{s+t-1}^{\sigma}w), \\
N &:= \sum_{\sigma\in S_{s+t}^{(s)}} Z_{\mathcal{F}}(z^{a_{\sigma(1)}-1}u_1^{\sigma}\cdots z^{a_{\sigma(s+t)}-1}u_{s+t}^{\sigma}w).
\end{aligned}$$

For each element in L, there exists a unique element in N such that they are corresponding to each other except for one letter u_i between z^{m_i-1} and z^{l_1-1}, which is $-y$ in L and x in N. Similarly, by understanding $z^{m_i+l_1-1} = z^{m_i-1}\cdot z\cdot z^{l_1-1}$, there is one-to-one correspondence between the elements in M and N such that they are corresponding to each other except for u_i between z^{m_i-1} and z^{l_1-1}, which is z in M and x in N. Since $x = -y + z$, we have

$$\begin{aligned}&Z_{\mathcal{F}}(z^{m_1-1}y\cdots z^{m_s-1}y\partial_{l_1}\cdots\partial_{l_t}(w))\\&=(-1)^s\sum_{\sigma\in S_{s+t}^{(s)}} Z_{\mathcal{F}}(z^{a_{\sigma(1)}-1}u_1^{\sigma}\cdots z^{a_{\sigma(s+t)}-1}u_{s+t}^{\sigma}w)\quad (w\in\mathfrak{H}^1).\end{aligned}$$

Then, we find the identity holds for t.

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