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Ph.D. Thesis

Time periodic problem for equations for the compressible fluids

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Abstract

Time periodic problem for equations for viscous compressible fluids is considered on the whole space. When the space dimension is greater than or equal to 3, the existence of a time periodic solution to the compressible Navier-Stokes equation is proved for sufficiently small time periodic external force with some symmetry condition. The stability of the time periodic solution and the time decay estimate of the perturbation are also shown. Furthermore, for small time periodic external force without the symmetry condition the existence of a time periodic solution is stated. The stability of the time periodic solution and the decay of L^∞ norm of the perturbation are also stated. The existence of time periodic solution to the compressible Navier-Stokes-Korteweg system is also shown for small time periodic external force. The time periodic solution obtained here is asymptotically stable and the decay of L^∞ norm of perturbation is obtained. When the space dimension is equal to two, for the compressible Navier-Stokes equation the existence of a time periodic solution is proved under small time periodic external force with antisymmetry condition.

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Results in Chapter 1 were obtained in a joint research with Yoshiyuki Kagei and published in [19]. Results in Chapter 2 were published in [28]. Results in Chapter 3 are to be published in [29].

Contents

Introduction

This thesis studies time periodic problem for equations for viscous compressible fluids. The motion of barotropic flow of such fluids in $\mathbb{R}^n$ $(n \geq 2)$ is described by the following compressible Navier-Stokes equation:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \rho(\partial_t v + (v \cdot \nabla)v) - \mu \Delta v - (\mu + \mu')\nabla(\nabla \cdot v) + \nabla p(\rho) = \rho g. \end{cases} \tag{0.0.1}$$

Here $\rho = \rho(x,t)$ and $v = (v_1(x,t), \cdots, v_n(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t \geq 0$ and position $x \in \mathbb{R}^n$; $p = p(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$p'(\rho_*) > 0,$$

for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying

$$\mu > 0, \quad \frac{2}{n}\mu + \mu' \geq 0;$$

and $g = g(x,t)$ is a given external force.

In this thesis we assume that $g = g(x,t)$ satisfies the condition

$$g(x, t+T) \;=\; g(x,t) \quad (x \in \mathbb{R}^n,\, t \in \mathbb{R}), \tag{0.0.2}$$

for some constant $T > 0$.

We also consider time periodic problem for the compressible Navier-Stokes-Korteweg system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t \rho + \operatorname{div} M = 0, & (0.0.3) \\ \partial_t M + \operatorname{div}\Big(\dfrac{M \otimes M}{\rho}\Big) = \operatorname{div}\Big(\mathcal{S}(\dfrac{M}{\rho}) + \mathcal{K}(\rho)\Big) + \rho g, & (0.0.4) \\ \partial_t(\rho E) + \operatorname{div}(ME) + \operatorname{div}\Big(P(\rho,\theta)\dfrac{M}{\rho}\Big) & \\ \qquad = \tilde{\alpha}\Delta\theta + \operatorname{div}\Big(\Big(\mathcal{S}(\dfrac{M}{\rho}) + \mathcal{K}(\rho)\Big)\dfrac{M}{\rho}\Big) + Mg. & (0.0.5) \end{cases}$$

Here $\rho = \rho(x,t)$, $M = (M_1(x,t), M_2(x,t), M_3(x,t))$ and $E = E(x,t) > 0$ denote the unknown density, momentum, and total energy respectively, at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^3$; θ denotes the absolute temperature of the fluid satisfying

$$E = C_v\theta + \frac{1}{2}\frac{|M|^2}{\rho^2},$$

where C_v denotes the heat capacity at the constant volume that is assumed to be a positive constant; $\mathcal{S}$ and $\mathcal{K}$ denote the viscous stress tensor and the Korteweg stress tensor that are given by

$$\begin{cases} \mathcal{S}(\frac{M}{\rho}) = \Big(\mu' \mathrm{div}\, \frac{M}{\rho}\Big)\delta_{i,j} + 2\mu d_{ij}\Big(\frac{M}{\rho}\Big), \\ \mathcal{K}(\rho) = \frac{\kappa}{2}(\Delta\rho^2 - |\nabla\rho|^2)\delta_{i,j} - \kappa\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}, \end{cases} \tag{0.0.6}$$

where $d_{ij}\Big(\frac{M}{\rho}\Big) = \frac{1}{2}\left(\frac{\partial}{\partial x_i}\Big(\frac{M}{\rho}\Big)_j + \frac{\partial}{\partial x_j}\Big(\frac{M}{\rho}\Big)_i\right)$; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying

$$\mu > 0, \quad \frac{2}{3}\mu + \mu' \geq 0;$$

$P = P(\rho,\theta)$ is the pressure that is assumed to be a smooth function of ρ and θ satisfying

$$P_\rho(\rho_*, \theta_*) > 0, \quad P_\theta(\rho_*, \theta_*) > 0,$$

where ρ_* and θ_* are given positive constants; κ and $\tilde{\alpha}$ denote the capillary constant and the heat conductivity coefficient respectively, that are assumed to be positive constants; and $g = g(x,t)$ is a given external force.

We assume that $g = g(x,t)$ satisfies the condition

$$g(x, t+T) \;=\; g(x,t) \quad (x \in \mathbb{R}^3,\, t \in \mathbb{R}) \tag{0.0.7}$$

for some constant $T > 0$.

The system (0.0.3)-(0.0.5) is known to be a model system for two phase flow with phase transition between liquid and vapor in compressible fluid. In deriving (0.0.3)-(0.0.5), phase transition boundary is regarded as a diffuse interface. The diffuse interface model displays the phase boundary as a narrow transition layer. So (0.0.3)-(0.0.5) describes fluid state by the changes of the density. (Cf., [6, 14, 22] for the derivation of (0.0.3)-(0.0.5).) Note that if we assume that $\kappa = 0$, then we obtain the compressible Navier-Stokes equation.

Time periodic flow is one of the basic phenomena in fluid mechanics, and thus, time periodic problems for fluid dynamical equations have been extensively studied. We refer, e.g., to [10, 20, 24, 30, 36] for the incompressible Navier-Stokes case, and to [1, 7, 8, 25, 33] for the compressible case. Concerning the time periodic problem for the compressible Navier-Stokes equation for barotropic flow, Valli ([33]) proved the existence and (exponential) stability of time periodic solutions on a bounded domain of $\mathbb{R}^3$ for sufficiently small time periodic external forces. On the other hand, for large external forces, the existence of time periodic solutions on a bounded domain of $\mathbb{R}^3$ was proved in the framework of weak solutions by Feireisl, Matušu-Necasová, Petzeltová and Straškrava ([7]) for the system for isentropic flow and by Feireisl, Mucha, Novotný and Pokorný ([8]) for the Navier-Stokes-Fourier system for heat conductive flow under some dissipative heat flux boundary condition. As for the time periodic problem on unbounded domains, Ma, Ukai, and Yang [25] proved the existence and stability of time periodic solutions on the whole space $\mathbb{R}^n$. The authors of [25] showed that if $n \geq 5$, there exists a time periodic solution (ρ_{per}, v_{per}) around $(\rho_*, 0)$ for a sufficiently small $g \in C^0(\mathbb{R}; H^{N-1} \cap L^1)$ with $g(x, t+T) = g(x,t)$, where $N \in \mathbb{Z}$ satisfying $N \geq n+2$. Furthermore, the time periodic solution is stable under sufficiently small perturbations and there holds the estimate

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{H^{N-1}} \leq C(1+t)^{-\frac{n}{4}} \|(\rho_0, v_0) - (\rho_{per}(t_0), v_{per}(t_0))\|_{H^{N-1}\cap L^1},$$

where t_0 is a certain initial time and $(\rho, v)|_{t=t_0} = (\rho_0, v_0)$. Here H^k denotes the L^2-Sobolev space on $\mathbb{R}^n$ of order k.

As for the mathematical analysis for (0.0.3)-(0.0.5), most of literatures treated the system in terms of the density ρ, velocity $v = M/\rho$ and absolute temperature θ:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, & (0.0.8) \\ \rho(\partial_t v + (v\cdot\nabla)v) + \nabla P(\rho,\theta) = \mu\Delta v + (\mu+\mu')\nabla \operatorname{div} v + \kappa\rho\nabla\Delta\rho + \rho g, & (0.0.9) \\ \rho C_v(\theta_t + (v\cdot\nabla)\theta) + \theta P_\theta(\rho,\theta)\operatorname{div} v = \tilde{\alpha}\Delta\theta + \Psi(v) + \tilde{\Phi}(\rho, v), & (0.0.10) \end{cases}$$

where $\Psi(v)$ and $\tilde{\Phi}(\rho, v)$ are given by

$$\begin{cases} \Psi(v) = \mu'(\operatorname{div} v)^2 + 2\mu \mathbb{D}v : \mathbb{D}v, \quad \mathbb{D}v = (d_{ij}(v))_{i,j=1}^3, \\ \tilde{\Phi}(\rho, v) = \kappa\left(\frac{|\nabla\rho|^2}{2} + \rho\Delta\rho\right)\operatorname{div} v - \kappa(\nabla\rho\otimes\nabla\rho) : \nabla v. \end{cases}$$

Chen and Zhao ([4]) considered the stationary problem (0.0.8)-(0.0.10) for g of the form $g(x) = \operatorname{div} g_1(x) + g_2(x)$ around $(\rho_*, 0, \theta_*)$. It was shown in [4] that if g satisfies

$$\sum_{k=1}^{3} \|(1+|x|)^{k+1}\nabla^k g\|_{L^2} + \sum_{k=0}^{1} \|(1+|x|)^{3+k}\nabla^k g\|_{L^\infty} + \|(1+|x|)^2 g_1\|_{L^\infty} + \|(1+|x|)^{-1} g_2\|_{L^1} \ll 1, \qquad (0.0.11)$$

then there exists a stationary solution for problem (0.0.8)-(0.0.10) in the weighted $L^\infty \cap L^2$ space. The stability of the stationary solution was also considered in [4]. It was shown in [4] that if g satisfies (0.0.11), then the stationary solution (ρ^*, v^*, θ^*) is asymptotically stable under sufficiently small initial perturbations, and the perturbation satisfies

$$\|(\rho(t), v(t), \theta(t)) - (\rho^*, v^*, \theta^*)\|_{L^\infty} \to 0$$

as $t \to \infty$. Chen, Xiao and Zhao ([3]) and Cai, Tan and Xu ([2]) then considered time periodic problem for the barotropic and non-barotropic system of (0.0.8)-(0.0.10), respectively, on $\mathbb{R}^n$ with $n \geq 5$. They proved that there exists a time periodic solution $(\rho_{per}, v_{per}, \theta_{per})$ around $(\rho_*, 0, \theta_*)$ for a sufficiently small $g \in C^0(\mathbb{R}; H^{N-1} \cap L^1)$ satisfying

(0.0.2), where $N \in \mathbb{Z}$ satisfying $N \geq n+2$. Furthermore, the time periodic solution is stable under sufficiently small perturbations and it holds that

$$\|(\rho(t)-\rho_{per}(t), v(t)-v_{per}(t), \theta(t)-\theta_{per}(t))\|_{L^\infty} \to 0 \quad (t \to \infty).$$

In Chapter 1 of this thesis, we assume that the external force g satisfies the following oddness condition

$$g(-x,t) = -g(x,t) \quad (x \in \mathbb{R}^n,\ t \in \mathbb{R}). \tag{0.0.12}$$

We will show that for $n \geq 3$ if g satisfies (0.0.2) and (0.0.12) and g is small enough in some weighted Sobolev space, then (0.0.1) has a time periodic solution (ρ_{per}, v_{per}) and $u_{per}(t) = (\rho_{per}(t)-\rho_*, v_{per}(t))$ satisfies

$$\begin{aligned}&\sup_{t\in[0,T]} (\|u_{per}(t)\|_{L^2} + \||x|\nabla u_{per}(t)\|_{L^2}) \\ &\quad \leq C\{\|(1+|x|)g\|_{C([0,T];L^1\cap L^2)} + \|(1+|x|)g\|_{L^2(0,T;H^{m-1})}\}. \end{aligned} \tag{0.0.13}$$

Here m is an integer satisfying $m \geq [\frac{n}{2}]+1$. In addition, we will prove that the time periodic solution is stable under sufficiently small initial perturbation, and that the perturbation satisfies

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^2} = O(t^{-\frac{n}{4}}) \text{ as } t \to \infty. \tag{0.0.14}$$

The precise statements are given in Theorem 1.2.1 and Theorem 1.2.2 below.

The proof of the existence of a time periodic solution is given by an iteration argument by using the time-T-map associated with the linearized problem around $(\rho_*, 0)$. Substituting $\phi = \frac{\rho-\rho_*}{\rho_*}$ and $w = \frac{v}{\gamma}$ with $\gamma = \sqrt{p'(\rho_*)}$ into (0.0.1), we see that (0.0.1) is rewritten as

$$\partial_t u + Au = -B[u]u + G(u,g), \tag{0.0.15}$$

where

$$A = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma\nabla & -\nu\triangle - \tilde{\nu}\nabla\mathrm{div}\end{pmatrix}, \quad \nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu+\mu'}{\rho_*}, \tag{0.0.16}$$

$$B[\tilde{u}]u = \gamma \begin{pmatrix} \tilde{w}\cdot\nabla\phi \\ 0 \end{pmatrix} \text{ for } u = {}^\top(\phi, w),\ \tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}) \tag{0.0.17}$$

and

$$G(u,g) = \begin{pmatrix} f^0(u) \\ \tilde{f}(u,g) \end{pmatrix}, \tag{0.0.18}$$

$$f^0(u) = -\gamma\phi\mathrm{div} w, \tag{0.0.19}$$

$$\tilde{f}(u,g) = -\gamma(1+\phi)(w\cdot\nabla w) - \phi\partial_t w - \nabla(p^{(1)}(\phi)\phi^2) + \frac{1+\phi}{\gamma}g, \tag{0.0.20}$$

$$p^{(1)}(\phi) \;=\; \frac{\rho_*}{\gamma}\int_0^1 (1-\theta)p''(\rho_*(1+\theta\phi))d\theta.$$

To solve the time periodic problem for (0.0.15), we decompose u into a low frequency part u_1 and a high frequency part u_∞. Then u_1 and u_∞ satisfy

$$\partial_t u_1 + Au_1 = F_1(u,g), \tag{0.0.21}$$

$$\partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty(u,g), \tag{0.0.22}$$

where

$$F_1(u,g) = P_1[-B[\tilde{u}]u + G(u,g)],$$
$$F_\infty(u,g) = P_\infty[-B[\tilde{u}]u_1 + G(u,g)]$$

and

$$\tilde{u} = u = u_1 + u_\infty, \quad u_j = P_j u \quad (j = 1, \infty).$$

Here P_1 and P_∞ are bounded linear operators from L^2 into a low frequency part and a high frequency part, respectively, satisfying $P_1 + P_\infty = I$. (See sections 3 and 4 in Chapter 1 for the definitions and properties of P_1 and P_∞.)

We rewrite (0.0.21)-(0.0.22) as

$$u_1(t) = S_1(t)u_{01} + \mathscr{S}_1(t)F_1(u,g), \tag{0.0.23}$$

$$u_\infty(t) = S_{\infty,\tilde{u}}(t)u_{0\infty} + \mathscr{S}_{\infty,\tilde{u}}(t)F_\infty(u,g), \tag{0.0.24}$$

where

$$u_{01} = (I - S_1(T))^{-1}\mathscr{S}_1(T)F_1(u,g), \tag{0.0.25}$$

$$u_{0\infty} = (I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)F_\infty(u,g) \tag{0.0.26}$$

with

$$\tilde{u} = u = u_1 + u_\infty. \tag{0.0.27}$$

Here $S_1(t)$ is the solution operator for the linear initial value problem for (0.0.21) with the inhomogeneous term $F_1(u,g) \equiv 0$ under the initial condition $u_1|_{t=0} = u_{01}$; $\mathscr{S}_1(t)$ is the one for (0.0.21) with a given inhomogeneous term $F_1(u,g)$ under the initial condition $u_1|_{t=0} = 0$; and $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ are similarly defined by the solution operators for the linear initial value problem for (0.0.22). We will investigate properties of $S_1(t)$, $\mathscr{S}_1(t)$, $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ in weighted Sobolev spaces. The necessary estimates for $S_1(t)$ and $\mathscr{S}_1(t)$ will be obtained by the explicit formulas for these operators through the Fourier transform, while those for $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ will be established by a weighted energy method. One of the points in the proof is to establish boundedness of operators $(I - S_1(T))^{-1}$ and $(I - S_{\infty,\tilde{u}}(T))^{-1}$ in some weighted spaces. As for the low frequency part, due to the symmetric assumption on g in (0.0.12), one can consider problem (0.0.1) in a function space with a symmetry, which enables us to show the boundedness of $(I - S_1(T))^{-1}$ from $L^1((1+|x|)dx)$ to the weighted space with norm $\|u\|_{L^2} + \||x|\nabla u\|_{L^2}$. Concerning the high frequency part, the weighted energy method shows that the spectral radius of $S_{\infty,\tilde{u}}(T)$ is strictly less than 1 in the weighted Sobolev space $H^m((1+|x|^2)dx)$ with an integer $m \geq [\frac{n}{2}] + 1$ for sufficiently small $\tilde{u}$, which leads to the desired boundedness of $(I - S_{\infty,\tilde{u}}(T))^{-1}$. We note that, due to the spatial decay of the time periodic solution obtained under the symmetric assumption on g in (0.0.12), one can show the asymptotic stability of the time periodic solution, together with the decay estimate of L^2 norm of the perturbation as $t \to \infty$.

The stability of the time periodic solution will be also shown by a decomposition method associated with the spectral properties of the linearized operator which, in this case, is a decomposition into low and high frequency parts (cf., [15, 27]). Based on the estimate (0.0.13) for $u_{per}(t) = {}^\top(\rho_{per}(t) - \rho_*, v_{per}(t))$, we can apply the Hardy inequality to show the stability of the time periodic solution ${}^\top(\rho_{per}(t), v_{per}(t))$ under sufficiently small initial perturbations and the decay estimate (0.0.14) in a similar manner to [27]. In contrast to the problem in [27], the terms $v_{per} \cdot \nabla\phi + \phi \mathrm{div} v_{per}$ appear in the transport equation for the perturbation. These terms can be handled by using the energy method and the boundedness properties of the projection onto the low frequency part as in [15], together with the Hardy inequality. (See also [1]).

In Chapter 2 of this thesis, we will show the existence of a time periodic solution for (0.0.1) without assuming the oddness condition (0.0.12) for $n \geq 3$. It will be proved that if $n \geq 3$, g satisfies (0.0.2) and

$$\|g\|_{C([0,T];L^1)} + \|(1+|x|^n)g\|_{C([0,T];L^\infty)} + \|(1+|x|^{n-1})g\|_{L^2(0,T;H^{s-1})} \ll 1,$$

with an integer $s \geq [n/2] + 1$, then there exists a time periodic solution $(\rho_{per}, v_{per}) \in C([0,T]; H^s)$ with period T for (0.0.1), and $u_{per}(t) = (\rho_{per}(t) - \rho_*, v_{per}(t))$ satisfies

$$\begin{aligned} &\sup_{t\in[0,T]} (\|(1+|x|^{n-1})\rho_{per}(t)\|_{L^\infty} + \sum_{j=0}^{1} \|(1+|x|^{n-2+j})\partial_x^j v_{per}(t)\|_{L^\infty}) \\ &\leq C(\|g\|_{C([0,T];L^1)} + \|(1+|x|^n)g\|_{L^\infty(0,T;L^\infty)} + \|(1+|x|^{n-1})g\|_{L^2(0,T;H^{s-1})}). \quad (0.0.28) \end{aligned}$$

Furthermore, if g satisfies

$$\|g\|_{C([0,T];L^1)} + \|(1+|x|^n)g\|_{C([0,T];L^\infty)} + \|(1+|x|^{n-1})g\|_{L^2(0,T;H^s)} \ll 1,$$

then the time periodic solution (ρ_{per}, v_{per}) is asymptotically stable under sufficiently small initial perturbations, and the perturbation satisfies

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^\infty} \to 0$$

as $t \to \infty$. We expect that the decay estimate such as (0.0.14) would also hold for this case and it would be desirable to derive the optimal decay estimate of L^2 norm for the perturbations. The precise statements of our existence and stability results are given in Theorem 2.2.1 and Theorem 2.2.2 below.

We will prove the existence of a time periodic solution around $(\rho_*, 0)$ by an iteration argument by using the time-T-map associated with the linearized problem at $(\rho_*, 0)$. As in Chapter 1 we formulate the time periodic problem as a system of equations for low frequency part and high frequency part of the solution. In the proof of the existence of a time periodic solution without assuming the oddness condition (0.0.12), there are two key observations. One is concerned with the spectrum of the time-T-map for the low frequency part. Another one is concerned with the convection term $v \cdot \nabla v$. As for the former matter, as in Chapter 1, we need to investigate $(I - S_1(T))^{-1}$, where $S_1(T) = e^{-TA}$ with A being the linearized operator around $(\rho_*, 0)$ which acts on functions whose Fourier transforms have their supports in $\{\xi \in \mathbb{R}^n; |\xi| \leq r_\infty\}$ for some $r_\infty > 0$. (See (2.3.21) and (2.3.22) bellow.) We will show that the leading part of $(I - S_1(T))^{-1}$ coincides with the solution operator for the linearized stationary problem used by Shibata-Tanaka in [32]. In fact, the Fourier transform of $(I - S_1(T))^{-1}F$ takes the form $(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}$, where $\hat{F}$ is the Fourier transform of F and

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top \xi \\ i\gamma\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top \xi \end{pmatrix}.$$

By using the spectral resolution, we see that

$$(I - e^{-T\hat{A}_\xi})^{-1} \sim -\frac{1}{T} \begin{pmatrix} \frac{\nu+\tilde{\nu}}{\gamma^2} & -\frac{i^\top \xi}{\gamma|\xi|^2} \\ -\frac{i\xi}{\gamma|\xi|^2} & \frac{1}{\nu|\xi|^2}\left(I_n - \frac{\xi^\top \xi}{|\xi|^2}\right) \end{pmatrix} \quad \text{as} \quad \xi \to 0.$$

The right-hand side is the solution operator for the linearized stationary problem in the Fourier space. This motivates us to introduce a weighted L^∞ space for the low frequency part employed in the study of the stationary problem in [32].

As for the high frequency part, we will employ the weighted energy estimates established in Chapter 1.

Another point in our analysis is concerned with the convection term $v \cdot \nabla v$. Due to the slow decay of $v(x,t)$ as $|x| \to \infty$, there appears some difficulty in estimating $v \cdot \nabla v$. To overcome this, we will use the momentum formulation for the low frequency part, which takes the form of a conservation law, and the velocity formulation for the high frequency part, for which the energy method works well. We also note that, in estimating the high frequency part of $v \cdot \nabla v$, we will use the fact that a Poincaré type inequality $\|f\|_{L^2} \leq C\|\nabla f\|_{L^2}$ holds for the high frequency part.

The asymptotic stability of the time periodic solution (ρ_{per}, v_{per}) can be proved as in Kagei and Kawashima [16] by using the Hardy inequality.

In Chapter 3 of this thesis we consider time periodic problem for (0.0.3)-(0.0.5). We will show the existence of a time periodic solution for (0.0.3)-(0.0.5) around $(\rho_*, 0, E_*)$ on $\mathbb{R}^3$ with $E_* = C_v\theta_*$. It will be proved that if g satisfies (0.0.7) and

$$\|g\|_{C([0,T];L^1)} + \|(1+|x|^3)g\|_{C([0,T];L^\infty)} + \|(1+|x|^2)g\|_{L^2(0,T;H^{s-1})} \ll 1$$

for an integer $s \geq 2$, then there exists a time periodic solution $(\rho_{per} - \rho_*, M_{per}, E_{per} - E_*) \in C([0,T]; H^s)$ with period T for (0.0.3)-(0.0.5), and $(\rho_{per} - \rho_*, M_{per}, E_{per} - E_*)$ satisfies the estimate

$$\begin{aligned}&\sup_{t\in[0,T]}\Big\{\sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j(\rho_{per}-\rho_*)(t)\|_{L^\infty}+\sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j M_{per}(t)\|_{L^\infty}\\&\quad+\sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j(E_{per}-E_*)(t)\|_{L^\infty}\Big\}\\&\quad\le C(\|g\|_{C([0,T];L^1)}+\|(1+|x|^3)g\|_{C(0,T;L^\infty)}+\|(1+|x|^2)g\|_{L^2(0,T;H^{s-1})}).\end{aligned}\qquad(0.0.29)$$

Furthermore, the time periodic solution $(\rho_{per}, M_{per}, E_{per})$ for (0.0.3)-(0.0.5) is asymptotically stable under sufficiently small initial perturbations and the perturbation satisfies

$$\|(\rho(t), M(t), E(t)) - (\rho_{per}(t), M_{per}(t), E_{per}(t))\|_{L^\infty} \to 0 \quad (t\to\infty).$$

The precise statements of our results are given in Theorem 3.2.1 and Theorem 3.2.2 below.

The existence of time periodic solution is proved by using the time-T-map for the linearized semigroup at $(\rho_*, 0, E_*)$. We will employ a function space of hybrid type which, roughly speaking, consists of functions whose low frequency parts belong to a weighted $L^\infty \cap L^2$ space and high frequency parts belong to a weighted L^2-Sobolev space. For the low frequency part we introduce a function space similar to that employed in the study of the stationary problem in [4], that is, a set of periodic functions with values in a weighted $L^\infty \cap L^2$ space similar to (0.0.11). We investigate the spatial decay properties of the integral kernel of the time-T-map, and establish the estimates for the low frequency part by a potential theoretic method. Due to the conservation form of momentum and total energy we can estimate the nonlinear terms for the low frequency part. If we use (0.0.8)-(0.0.10) instead of (0.0.3)-(0.0.5), the slow decay of $\rho(x,t)$, $v(x,t)$ and $\theta(x,t)$ as $|x|\to\infty$ prevents us from obtaining the estimates of the terms $(v\cdot\nabla)v$, $(v\cdot\nabla)\theta$ and $\theta P_\theta(\rho,\theta)\mathrm{div}\, v$ in (0.0.9) and (0.0.10) for the low frequency part. As for the high frequency part, we employ the weighted energy method to obtain the a priori estimates.

The proof of the existence of time periodic solution is similar to the argument in Chapter 2. The main difference from Chapter 2 is as follows. In Chapter 2, a coupled system for the low frequency part and the high frequency part was used in the proof of the existence of the time periodic solution to avoid the derivative loss for the high frequency part due to the term $v\cdot\nabla\rho$. In this paper we do not use any coupled system as in Chapter 2 and directly treat (0.0.3)-(0.0.5) by making use of the smoothing effect for ρ due to the term $\kappa\nabla\Delta\rho$ arising in the Korteweg tensor. A key point in the proof of the existence of time periodic solution is to control the decay properties of solution as $|x|\to\infty$, which is similar to the case of the stationary problem. In [3] stationary solution was obtained in some function space where functions decay like $\rho(x)-\rho_* = O(|x|^{-2})$ and $(v(x), \theta(x)-\theta_*) = O(|x|^{-1})$ as $|x|\to\infty$. In this paper we require $\rho(x,t)-\rho_*$ to decay only in the order $O(|x|^{-1})$ as $|x|\to\infty$. The faster decay of $\rho(x)-\rho_*$ in [4] was obtained from the fact that $\rho(x)-\rho_*$ can be represented by the Bessel potential due to the Korteweg tensor. On the other hand, in the time dependent case, the method in [4] does not work well, and $\rho(x,t)-\rho_*$ is represented by the Newton potential which leads to the slower decay than the stationary case. We also note that since we consider the non-barotropic system, the decay of $\rho(x,t)-\rho_*$ is slower than that in Chapter 2 in the low frequency part.

The asymptotic stability of the time periodic solution $(\rho_{per}, M_{per}, E_{per})$ is proved by the energy method using the Hardy inequality as in [4, 16].

In Chapter 4 of this thesis we consider the existence of a time periodic solution for (0.0.1) on $\mathbb{R}^2$ under the following antisymmetry condition:

$$\begin{cases} g_1(-x_1,x_2,t) = -g_1(x_1,x_2,t), \quad g_1(x_1,-x_2,t) = g_1(x_1,x_2,t), \\ g_2(-x_1,x_2,t) = g_2(x_1,x_2,t), \quad g_2(x_1,-x_2,t) = -g_2(x_1,x_2,t), \\ g_1(x_2,x_1,t) = g_2(x_1,x_2,t), \quad g_2(x_2,x_1,t) = g_1(x_1,x_2,t). \end{cases} \tag{0.0.30}$$

It will be proved that if g satisfies (0.0.2), (0.0.30) and the estimate

$$\|(1+|x|)g\|_{C([0,T];L^1)} + \|(1+|x|^3)g\|_{C([0,T];L^\infty)} + \|(1+|x|^2)g\|_{L^2(0,T;H^{s-1})} \ll 1,$$

for an integer $s \geq 3$, then there exists a time periodic solution $u_{per} = (\rho_{per}-\rho_*, v_{per}) \in C(\mathbb{R};L^\infty)$ with $\nabla u_{per} \in C(\mathbb{R};H^{s-1})$ having time period T for (0.0.1), and u_{per} satisfies the estimate

$$\begin{aligned} &\sup_{t\in[0,T]} \Big\{ \sum_{j=0}^{1} \|(1+|x|^{1+j})\partial_x^j(\rho_{per}-\rho_*)(t)\|_{L^\infty} + \sum_{j=0}^{1} \|(1+|x|^{1+j})\partial_x^j v_{per}(t)\|_{L^\infty} \Big\} \\ &\quad \leq C\|g\|_{C([0,T];L^1_1)} + \|(1+|x|^3)g\|_{C([0,T];L^\infty)} + \|(1+|x|^2)g\|_{L^2(0,T;H^{s-1})}. \end{aligned}$$

The existence of a time periodic solution is shown by an iteration argument using time-T-map concerned with the linearized problem around the constant state. We use a system of equations decomposed by a low frequency part and high frequency part of solution as in Chapter 1. Concerning the low frequency part, we apply the potential theoretic method which control spatial decay properties for a solution. The same method was used in the study of the stationary problem [32] and the time periodic problem in Chapter 2 of this thesis for the space dimension $n \geq 3$. The main difference between this study and Chapter 2 is stated as follows. We denote by A_1 the linearized operator around $(\rho_*, 0)$ on the low frequency part. Then we estimate $(I - S_1(T))^{-1}$ in some weighted L^∞ space, generated by A_1. In contrast to [28], since we consider on $\mathbb{R}^2$, $(I-S_1(T))^{-1}$ has the worse order as $\log|x|$ at $x \to \infty$, which is the same order as the fundamental solution of the Laplace equation. More preciously, it follows from the spectral resolution that

$$\mathcal{F}(I-S_1(T))^{-1} \sim -\frac{1}{T} \begin{pmatrix} \frac{\nu+\tilde{\nu}}{\gamma^2} & -\frac{i^\top\xi}{\gamma|\xi|^2} \\ -\frac{i\xi}{\gamma|\xi|^2} & \frac{1}{\nu|\xi|^2}\left(I_2 - \frac{\xi^\top\xi}{|\xi|^2}\right) \end{pmatrix} \quad \text{as} \quad \xi \to 0, \tag{0.0.31}$$

where $\mathcal{F}$ denotes the Fourier transform. Then the order $\log|x|$ appears from the Stokes inverse in the right hand side of (0.0.31). This prevents us from controlling spatial decay properties for the convection term and the external force. To overcome this difficulty, since the slowly decaying order appears from the Stokes inverse, we introduce the antisymmetry condition which was used in the stationary problem for imcompressible fluid on $\mathbb{R}^2$ ([37]). Moreover, the following two key observations are used.

The one is concerned with the estimate for the convection term. Due to the slow decay of v at spatial infinity, we formulate the equations by not only using the conservation form with the momentum as in Chapter 2 but also rewriting the convection term as

$$\partial_{x_2}\begin{pmatrix} v_1v_2 \\ (v_2)^2-(v_1)^2 \end{pmatrix} + \partial_{x_1}\begin{pmatrix} 0 \\ v_2v_1 \end{pmatrix} + \nabla(v_1)^2$$

to make use of the antisymmetry condition effectively for the low frequency part. (Cf., Remark 4.3.5 bellow.) Furthermore, we establish an estimate for convolution under antisymmetry condition in the weighted L^∞ space stated in Lemma 4.3.8 bellow. Combining these, we obtain the estimate for the convection term in the weighted L^∞ space.

Another key observation is concerned with the estimate for the external force. We state a Poincaré type inequality in the weighted L^∞ space with the antisymmetry condition for the low frequency part. (Cf., Lemma (4.3.9) bellow). Using this the inequality, we can estimate a convolution related to the external force since the integral kernel has the same order as the first order derivative of the fundamental solution of the Laplace equation. If we would not use the inequality, the integral kernel would be obstructive for the estimate which has the order $\log|x|$ at spatial infinity.

As for the high frequency part, we use the velocity formulation to avoid some derivative loss by using the energy method as in chapters 1 and 2.

Note that we use a coupled system of the conservation form of the momentum and the velocity formulation, but not vorticity equation; and we do not need to assume that g is a derivative form of some scalar potential function as in [37].

This thesis is organized as follows. In Chapter 1, we show the existence of a time periodic solution to (0.0.1) for sufficiently small time periodic external force satisfying (0.0.12) when the space dimension is greater than or equal to 3. We also show the stability of the time periodic solution and the time decay estimate of the perturbation. In Chapter 2, it is proved that if time periodic external force g is sufficiently small without the assumption (0.0.12), then we have the existence of a time periodic solution of (0.0.1) on $\mathbb{R}^n$ for $n \geq 3$. The time periodic solution is shown to be asymptotically stable under sufficiently small initial perturbations and the L^∞ norm of the perturbation decays as time goes to infinity. In Chapter 3, as for (0.0.3)-(0.0.5), the existence of a time periodic solution is proved for a sufficiently small periodic external force on $\mathbb{R}^3$. The stability of the time periodic solution is proved for sufficiently small initial perturbations. It is also shown that the L^∞ norm of the perturbation decays as time goes to infinity. In Chapter 4, the existence of a time periodic solution to (0.0.1) is stated for sufficiently small time periodic external force satisfying (0.0.30).

In each section, notation is introduced which is used throughout the chapter and the main results are stated. Continuously, the proofs of the main results are given respectively.

Chapter 1

The existence and stability of time periodic solution to the compressible Navier-Stokes equation with symmetry

Time periodic problem for (0.0.1) on the whole space is studied. The existence of a time periodic solution is proved for sufficiently small time periodic external force with some symmetry when the space dimension is greater than or equal to 3. The proof is based on the spectral properties of the time-T-map associated with the linearized problem around the motionless state with constant density in some weighted Sobolev space. The stability of the time periodic solution is also proved and the decay estimate of the perturbation is established.

1.1 Preliminaries

In this section we first introduce some notations which will be used throughout this chapter. We then introduce some auxiliary lemmas which will be useful in the proof of the main results.

For a given Banach space X, the norm on X is denoted by $\|\cdot\|_X$.

Let $1 \leqq p \leqq \infty$. L^p stands for the usual L^p space over $\mathbb{R}^n$. The inner product of L^2 is denoted by $(\cdot,\cdot)$. For a nonnegative integer k, H^k stands for the usual L^2-Sobolev space of order k. (As usual, $H^0 = L^2$.)

The set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in L^p$ $(j = 1, \cdots, n)$, i.e., $(L^p)^n$, is simply denoted by L^p; and the norm $\|\cdot\|_{(L^p)^n}$ on it is denoted by $\|\cdot\|_{L^p}$ if no confusion will occur. Similarly, for a function space X, the set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in X$ $(j = 1, \cdots, n)$, i.e., X^n, is simply denoted by X; and the norm $\|\cdot\|_{X^n}$ on it is denoted by $\|\cdot\|_X$ if no confusion will occur. (For example, $(H^k)^n$ is simply denoted by H^k and the norm $\|\cdot\|_{(H^k)^n}$ is denoted by $\|\cdot\|_{H^k}$.)

For $u = {}^\top(\phi, w)$ with $\phi \in H^k$ and $w = {}^\top(w_1, \cdots, w_n) \in H^m$, we define the norm

$\|u\|_{H^k\times H^m}$ of u on $H^k\times H^m$ by

$$\|u\|_{H^k\times H^m} = \left(\|\phi\|_{H^k}^2 + \|w\|_{H^m}^2\right)^{\frac{1}{2}}.$$

When $m=k$, we simply write $H^k\times (H^k)^n$ as H^k, and, also, $\|u\|_{H^k\times (H^k)^n}$ as $\|u\|_{H^k}$ if no confusion will occur :

$$H^k := H^k\times (H^k)^n, \quad \|u\|_{H^k} := \|u\|_{H^k\times (H^k)^n} \quad (u = {}^\top(\phi, w)).$$

Similarly, when $u = {}^\top(\phi, w)\in X\times Y$ with $w = {}^\top(w_1,\cdots,w_n)$ for function spaces X and Y, we denote its norm $\|u\|_{X\times Y}$ by

$$\|u\|_{X\times Y} = \left(\|\phi\|_X^2 + \|w\|_Y^2\right)^{\frac{1}{2}} \quad (u = {}^\top(\phi, w)).$$

When $Y = X^n$, we simply write $X\times X^n$ as X, and also its norm $\|u\|_{X\times X^n}$ as $\|u\|_X$:

$$X := X\times X^n, \quad \|u\|_X := \|u\|_{X\times X^n} \quad (u = {}^\top(\phi, w)).$$

Let $\alpha = (\alpha_1,\cdots,\alpha_n)$ be a multi-index. We use the following notation

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}, \qquad |\alpha| = \sum_{j=1}^n \alpha_j.$$

For any integer $l\geq 0$, $\nabla^l f$ denotes x-derivatives of order l of a function f.

For $1\leq p<\infty$, L_1^p stands for the weighted L^p space over $\mathbb{R}^n$ defined by

$$L_1^p = \{f\in L^p; \|f\|_{L_1^p} := \|(1+|x|)f\|_{L^p} < +\infty\}.$$

For a nonnegative integer k, we define the space H_1^k by

$$H_1^k = \{f\in H^k; \|f\|_{H_1^k} := \|(1+|x|)f\|_{H^k} < +\infty\}.$$

We next introduce function spaces associated with low and high frequency parts. We denote by $\hat{f}$ or $\mathcal{F}[f]$ the Fourier transform of f :

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \quad (\xi\in\mathbb{R}^n).$$

The inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](x) = (2\pi)^{-n}\int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}d\xi \quad (x\in\mathbb{R}^n).$$

For a nonnegative integer k and positive constants r_1 and r_∞ with $r_1 < r_\infty$, $H^k_{(\infty)}$ denotes the set of all $f\in H^k$ satisfying supp $\hat{f}\subset\{|\xi|\geq r_1\}$, and $L^2_{(1)}$ denotes the set of all $f\in L^2$ satisfying supp $\hat{f}\subset\{|\xi|\leq r_\infty\}$. Note that $H^k\cap L^2_{(1)} = L^2_{(1)}$ for any nonnegative integer k. (Cf., Lemma 1.3.3 (ii) bellow).

Concerning weighted spaces for high frequency part, we define the space $H^k_{(\infty),1}$ by

$$H^k_{(\infty),1} = \{f \in H^k_{(\infty)}; \|f\|_{H^k_1} < +\infty\}.$$

As for the low frequency part, we define the space $\mathscr{H}^1_{(1),1}$ by

$$\mathscr{H}^1_{(1),1} = \{f \in L^2_{(1)}; \|f\|_{\mathscr{H}^1_{(1),1}} := \big(\|f\|^2_{L^2} + \||x|\nabla f\|^2_{L^2}\big)^{\frac{1}{2}} < +\infty\}.$$

We will consider the time periodic problem in function spaces with some symmetry. We define Γ by

$$(\Gamma u)(x) = {}^\top(\phi(-x), -w(-x)) \quad (u(x) = {}^\top(\phi(x), w(x)), \quad x \in \mathbb{R}^n).$$

We indicate function spaces satisfying the symmetric condition $\Gamma u = u$ by the subscript $\cdot_{sym}$. More precisely, We denote by X_{sym} the set of all $u = {}^\top(\phi, w) \in X$ satisfying the symmetric conditions $\Gamma u = u$, i.e., $\phi(-x) = \phi(x)$ and $w(-x) = -w(x)$ $(x \in \mathbb{R}^n)$:

$$X_{sym} = \{u = {}^\top(\phi, w) \in X; \Gamma u = u\}.$$

Let $-\infty \leq a < b \leq \infty$. We denote by $C^k([a,b]; X)$ the set of all C^k functions on $[a,b]$ with values in X. The Bochner space on (a,b) is denoted by $L^p(a,b;X)$ and the L^2-Bochner-Sobolev space of order k is denoted by $H^k(a,b;X)$.

As for the high frequency part, we will work in the space $\mathscr{Y}^k_\infty(a,b)$:

$$\mathscr{Y}^k_\infty(a,b) = \big\{u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([a,b]; (H^k_{(\infty),1})_{sym}); w_\infty \in L^2(a,b; H^{k+1}_{(\infty),1}) \cap H^1(a,b; H^{k-1}_{(\infty),1})\}$$

equipped with the norm

$$\|u_\infty\|_{\mathscr{Y}^k_\infty(a,b)} = \left(\|u_\infty\|^2_{C([a,b]; H^k_{(\infty),1})} + \|w_\infty\|^2_{L^2(a,b; H^{k+1}_{(\infty),1}) \cap H^1(a,b; H^{k-1}_{(\infty),1})}\right)^{\frac{1}{2}}.$$

Here $k = m - 1$ or m with an integer m satisfying $m \geq [\frac{n}{2}] + 1$.

We will look for the low frequency part $u_1 = {}^\top(\phi_1, w_1)$ in the space $H^1(0,T; (\mathscr{H}^1_{(1),1})_{sym})$. It then follows from the equation that $\partial_t w_1$ also belongs to $L^2(0,T; L^2_1)$. Since the nonlinearity includes $\phi \partial_t w$, it is convenient to work in the space $H^1(0,T; (\mathscr{H}^1_{(1),1})_{sym})$ incorporate with the norm $\|\partial_t w_1\|_{L^2(0,T; L^2_1)}$ in the iteration argument. We thus introduce the following function space for the low frequency part:

$$\mathscr{Y}_1(a,b) = \big\{u_1 = {}^\top(\phi_1, w_1) \in H^1(a,b; (\mathscr{H}^1_{(1),1})_{sym}); \partial_t w_1 \in L^2(a,b; L^2_1)\}$$

equipped with the norm

$$\|u_1\|_{\mathscr{Y}_1(a,b)} = \left(\|u_1\|^2_{H^1(a,b; (\mathscr{H}^1_{(1),1})_{sym})} + \|\partial_t w_1\|^2_{L^2(a,b; L^2_1)}\right)^{\frac{1}{2}}.$$

Note that $H^1(a,b; (\mathscr{H}^1_{(1),1})_{sym}) \subset C([a,b]; (\mathscr{H}^1_{(1),1})_{sym})$, where the imbedding is continuous.

We define the space $\mathscr{X}^k(a,b)$ by

$$\mathscr{X}^k(a,b) = \mathscr{Y}_1(a,b) \times \mathscr{Y}^k_\infty(a,b)$$

equipped with the norm

$$\|\{u_1, u_\infty\}\|_{\mathscr{X}^k(a,b)} = \left(\|u_1\|^2_{\mathscr{Y}_1(a,b)} + \|u_\infty\|^2_{\mathscr{Y}^k_\infty(a,b)} \right)^{\frac{1}{2}}.$$

We also introduce function spaces of T-periodic functions in t. $C_{per}(\mathbb{R}; X)$ denotes the set of all T-periodic continuous functions with values in X equipped with the norm $\|\cdot\|_{C([0,T];X)}$; and $L^2_{per}(\mathbb{R}; X)$ denotes the set of all T-periodic locally square integrable functions with values in X equipped with the norm $\|\cdot\|_{L^2(0,T;X)}$. Similarly, $H^1_{per}(\mathbb{R}; X)$ and $\mathscr{X}^k_{per}(\mathbb{R})$, and so on, are defined.

Let X be a Banach space and let L be a bounded linear operator on X. We denote by $r_X(L)$ the spectral radius of L.

For operators L_1 and L_2, $[L_1, L_2]$ denotes the commutator of L_1 and L_2:

$$[L_1, L_2]f = L_1(L_2 f) - L_2(L_1 f).$$

We next state some lemmas which will be used in the proof of the main results. These lemma are also used in chapters 2-4.

Lemma 1.1.1. *Let $n \geq 2$ and let $m \geq \left[\frac{n}{2}\right] + 1$. Then there holds the inequality*

$$\|f\|_{L^\infty} \leq C\|\nabla f\|_{H^{m-1}}$$

for $f \in H^m$.

Lemma 1.1.1 is proved as follows. Let $n \geq 2$ and set $2^* := \frac{2n}{n-2}$. Since $m \geq \left[\frac{n}{2}\right] + 1$, we see that $m - 1 \geq \frac{n}{2^*}$. It then follows from the Sobolev inequalities that

$$\|f\|_{L^\infty} \leq C\|f\|_{W^{m,2^*}} \leq C\|\nabla f\|_{H^{m-1}},$$

which shows Lemma 1.1.1.

Lemma 1.1.2. *Assume $n \geq 2$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right]+1$. Let m_j and μ_j ($j = 1, \cdots, \ell$) satisfy $0 \leq |\mu_j| \leq m_j \leq m + |\mu_j|$, $\mu = \mu_1 + \cdots + \mu_\ell$, $m = m_1 + \cdots + m_\ell \geq (\ell - 1)m + |\mu|$. Then there holds*

$$\| \partial_x^{\mu_1} f_1 \cdots \partial_x^{\mu_\ell} f_\ell \|_{L^2} \leq C \prod_{1 \leq j \leq \ell} \| f_j \|_{H^{m_j}} .$$

See, e.g., [18], for the proof of Lemma 1.1.2.

Lemma 1.1.3. *Let $n \geq 2$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right] + 1$. Suppose that F is a smooth function on I, where I is a compact interval of $\mathbb{R}$. Then for a multi-index α with $1 \leq |\alpha| \leq m$, there hold the estimates*

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{m-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^{m-1}} \|f_2\|_{H^{|\alpha|}}$$

for $f_1 \in H^m$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^n$ and $f_2 \in H^{|\alpha|}$; and

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{m-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^m} \|f_2\|_{H^{|\alpha|-1}}.$$

for $f_1 \in H^{m+1}$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^n$ and $f_2 \in H^{|\alpha|-1}$.

See, e.g., [16], for the proof of Lemma 1.1.3.

1.2 Main results of Chapter 1

In this section we state our results on the existence and stability of a time periodic solution for system (0.0.1).

We begin with the existence of a time periodic solution. To state the existence result, we introduce operators which decompose a function into its low and high frequency parts. We define operators P_1 and P_∞ on L^2 by

$$P_j f = \mathcal{F}^{-1}\hat{\chi}_j \mathcal{F}[f] \quad (f \in L^2, j = 1, \infty),$$

where

$$\begin{aligned}
&\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^n) \quad (j = 1, \infty), \quad 0 \leq \hat{\chi}_j \leq 1 \quad (j = 1, \infty),\\
&\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \leq r_1), \\ 0 & (|\xi| \geq r_\infty), \end{cases}\\
&\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi),\\
&0 < r_1 < r_\infty.
\end{aligned}$$

We fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (1.4.3) in Lemma 1.4.5 below holds for $|\xi| \leq r_\infty$.

Theorem 1.2.1. *Let $n \geq 3$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies (0.0.2) and $g \in C_{per}(\mathbb{R}; L_1^1 \cap L_1^2) \cap L_{per}^2(\mathbb{R}; H_1^{m-1})$. Set*

$$[g]_m = \|g\|_{C([0,T]; L_1^1 \cap L_1^2)} + \|g\|_{L^2(0,T; H_1^{m-1})}.$$

Then there exist constants $\delta_0 > 0$ and $C_0 > 0$ such that if $[g]_m \leq \delta_0$, then the system (0.0.1) has a time periodic solution ${}^\top(\rho_{per}, v_{per})$ with period T that satisfies $\{u_1, u_\infty\} \in \mathscr{X}_{per}^m(\mathbb{R})$ with $\|\{u_1, u_\infty\}\|_{\mathscr{X}^m(0,T)} \leq C_0[g]_m$ where $u_j = {}^\top(P_j(\rho_{per} - \rho_*), P_j v_{per})$ $(j = 1, \infty)$. Furthermore, the uniqueness of time periodic solutions of (0.0.1) holds in the class $\{{}^\top(\rho, v)\,;\, u = {}^\top(\rho - \rho_*, v)$ satisfies $\{P_1 u, P_\infty u\} \in \mathscr{X}_{per}^m(\mathbb{R})$, $\|\{P_1 u, P_\infty u\}\|_{\mathscr{X}^m(0,T)} \leq C_0\delta_0\}$.

Our next issue is to study the stability of the time periodic solution obtained in Theorem 1.2.1.

Let ${}^\top(\rho_{per}, v_{per})$ be the time periodic solution given in Theorem 3.2.1. We denote the perturbation by $u = {}^\top(\phi, w)$, where $\phi = \rho - \rho_{per}, w = v - v_{per}$. Substituting $\rho = \phi + \rho_{per}$ and $v = w + v_{per}$ into (0.0.1), we see that the perturbation $u = {}^\top(\phi, w)$ is governed by

$$\left\{ \begin{array}{l} \partial_t \phi + v_{per} \cdot \nabla \phi + \phi \mathrm{div} v_{per} + \rho_{per} \mathrm{div} w + w \cdot \nabla \rho_{per} = F^0, \\ \partial_t w + v_{per} \cdot \nabla w + w \cdot \nabla v_{per} - \frac{\mu}{\rho_{per}} \triangle w - \frac{\mu+\mu'}{\rho_{per}} \nabla \mathrm{div} w \\ \quad + \frac{\phi}{\rho_{per}^2} (\mu \triangle v_{per} + (\mu+\mu') \nabla \mathrm{div} v_{per}) + \nabla (\frac{p'(\rho_{per})}{\rho_{per}} \phi) = \tilde{F}, \end{array} \right. \tag{1.2.1}$$

where

$$\begin{aligned} F^0 &= -\mathrm{div}(\phi w), \\ \tilde{F} &= -w \cdot \nabla w - \frac{\phi}{\rho_{per}(\rho_{per} + \phi)} (\mu \triangle w + (\mu + \mu') \nabla \mathrm{div} w) \\ &\quad + \frac{\phi}{\rho_{per}(\rho_{per} + \phi)} (\frac{\phi}{\rho_{per}} \mu \triangle v_{per} + \frac{\phi}{\rho_{per}} (\mu + \mu') \nabla \mathrm{div} v_{per}) \\ &\quad + \frac{\phi}{\rho_{per}^2} \nabla (p^{(2)}(\rho_{per}, \phi) \phi) + \frac{\phi^2}{\rho_{per}^2 (\rho_{per} + \phi)} \nabla (p(\rho_{per} + \phi)) + \frac{1}{\rho_{per}} \nabla (p^{(3)}(\rho_{per}, \phi) \phi^2), \end{aligned}$$

$$p^{(2)}(\rho_{per}, \phi) = \int_0^1 p'(\rho_{per} + \theta \phi) d\theta, \quad p^{(3)}(\rho_{per}, \phi) = \int_0^1 (1 - \theta) p''(\rho_{per} + \theta \phi) d\theta.$$

We consider the initial value problem for (1.2.1) under the initial condition

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{1.2.2}$$

Our result on the stability of the time periodic solution is stated as follows.

Theorem 1.2.2. *Let $n \geq 3$ and let m be an integer satisfying $m \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies* (0.0.2) *and $g \in C_{per}(\mathbb{R}; L_1^1 \cap L_1^2) \cap L_{per}^2(\mathbb{R}; H_1^m)$. Let ${}^\top(\rho_{per}, v_{per})$ be the time periodic solution obtained in Theorem 3.2.1 and let $u_0 = {}^\top(\phi_0, w_0) \in H^m \cap L^1$. Then there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that if*

$$[g]_{m+1} \leq \epsilon_1, \quad \|u_0\|_{H^m \cap L^1} \leq \epsilon_2,$$

there exists a unique global solution $u = {}^\top(\phi, w) \in C([0, \infty); H^m)$ of (1.2.1)-(1.2.2) *and u satisfies*

$$\|\nabla^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4} - \frac{k}{2}} \qquad (t \in [0, +\infty), \quad k = 0, 1).$$

Theorem 1.2.2 follows from the same argument as that in [27]; and we omit the details. In contrast to the problem in [27], several linear terms with coefficients including v_{per} appear in the equations for the perturbation. In the transport equation for the perturbation, there appear the terms $v_{per} \cdot \nabla \phi + \phi \mathrm{div} v_{per}$ and these terms can be handled by using the energy method and the boundedness properties of the projection onto the low frequency part as in [15, 27], together with the Hardy inequality; the linear terms including v_{per} in the equation of motion for the perturbation can be handled by using the Hardy inequality. (See also [1]).

1.3 Reformulation of the problem

In this section we reformulate the time periodic problem for (0.0.1). To prove Theorem 1.2.1, it suffices to show the existence of a time periodic solution of (0.0.15). We decompose u into a low frequency part u_1 and a high frequency part u_∞; and we rewrite the problem into a system of equations (0.0.23)-(0.0.27) for u_1 and u_∞.

Let u satisfy (0.0.15) and set $u_1 = P_1 u$, $u_\infty = P_\infty u$. Then u_1 and u_∞ satisfy (0.0.21) and (0.0.22). Suppose that (0.0.21) and (0.0.22) are satisfied by some functions u_1 and u_∞. Then, since $P_1 + P_\infty = I$, by adding (0.0.21) to (0.0.22), we obtain

$$\begin{aligned}\partial_t(u_1+u_\infty)+A(u_1+u_\infty) &= -P_\infty(B[u_1+u_\infty]u_\infty)+(P_1+P_\infty)F(u_1+u_\infty,g)\\ &= -B[u_1+u_\infty](u_1+u_\infty)+G(u_1+u_\infty,g).\end{aligned}$$

Set $u = u_1 + u_\infty$, then we have

$$\partial_t u + Au + B[u]u = G(u,g).$$

Consequently, if we show the existence of a pair of functions $\{u_1, u_\infty\}$ satisfying (0.0.21)-(0.0.22), then we can obtain a solution u of (0.0.15). Therefore, we will consider (0.0.21)-(0.0.22) to solve the time periodic problem for (0.0.15).

The following two lemmas are concerned with symmetry of (0.0.15) and (0.0.21)-(0.0.22). We recall that Γ is defined by

$$(\Gamma u)(x) = {}^\top(\phi(-x), -w(-x)) \quad (u(x) = {}^\top(\phi(x), w(x)), \quad x \in \mathbb{R}^n).$$

Lemma 1.3.1. *Set $\boldsymbol{g}(x,t) = {}^\top(0, g(x,t))$ and assume that $(\Gamma \boldsymbol{g})(x,t) = \boldsymbol{g}(x,t)\,(x \in \mathbb{R}^n,\ t \in \mathbb{R})$.*

(i) *If $u = {}^\top(\phi, w)$ is a solution of (0.0.15), then Γu is also a solution of (0.0.15).*

(ii) *If $\{u_1, u_\infty\}$ is a solution of (0.0.21)-(0.0.22), then $\{\Gamma u_1, \Gamma u_\infty\}$ is also a solution of (0.0.21)-(0.0.22).*

Lemma 1.3.2. *Assume that $(\Gamma \boldsymbol{g})(x,t) = \boldsymbol{g}(x,t)$ $(x \in \mathbb{R}^n,\ t \in \mathbb{R})$.*

(i) *If $(\Gamma u)(x,t) = u(x,t)$ $(x \in \mathbb{R}^n,\ t \in \mathbb{R})$, then*

$$[\Gamma(\partial_t u + Au + B[u]u - G(u,g))](x,t) = [\partial_t u + Au + B[u]u - G(u,g)](x,t)$$

for $x \in \mathbb{R}^n, t \in \mathbb{R}$.

(ii) *If $\{\Gamma u_1(x,t), \Gamma u_\infty(x,t)\} = \{u_1(x,t), u_\infty(x,t)\}$ $(x \in \mathbb{R}^n,\ t \in \mathbb{R})$, then*

$$[\Gamma(\partial_t u_1 + Au_1 - F_1(u_1+u_\infty, g))](x,t) = [\partial_t u_1 + Au_1 - F_1(u_1+u_\infty, g)](x,t)$$

and

$$\begin{aligned}&[\Gamma(\partial_t u_\infty + Au_\infty + P_\infty(B[u_1+u_\infty]u_\infty) - F_\infty(u_1+u_\infty,g))](x,t)\\ &\quad = [\partial_t u_\infty + Au_\infty + P_\infty(B[u_1+u_\infty]u_\infty) - F_\infty(u_1+u_\infty,g)](x,t)\end{aligned}$$

for $x \in \mathbb{R}^n, t \in \mathbb{R}$.

Lemma 1.3.1 (i) and Lemma 1.3.2 (i) can be verified by direct computations. As for Lemma 1.3.1 (ii) and Lemma 1.3.2 (ii), by using the facts $\hat{f}(-\xi) = \widehat{f(-\cdot)}(\xi)$ and $\hat{\chi}_j(-\xi) = \hat{\chi}_j(\xi)$ $(j = 1, \infty)$, we see that $\Gamma P_j = P_j \Gamma$ $(j = 1, \infty)$. Based on these relations, Lemma 1.3.1 (ii) and Lemma 1.3.2 (ii) can be proved by a straightforward computation.

By Lemma 1.3.1 and Lemma 1.3.2, one can consider (0.0.21)-(0.0.22) in space of functions satisfying $\{\Gamma u_1, \Gamma u_\infty\} = \{u_1, u_\infty\}$, i.e., $u_j = {}^\top(\phi_j(x,t), w_j(x,t)) = {}^\top(\phi_j(-x,t), -w_j(-x,t))$ $(j = 1, \infty)$.

We look for a time periodic solution $\{u_1, u_\infty\}$ for the system (0.0.21)-(0.0.22). To solve the time periodic problem for (0.0.21)-(0.0.22), we introduce solution operators for the following linear problems:

$$\begin{cases} \partial_t u_1 + Au_1 = F_1, \\ u|_{t=0} = u_{01}, \end{cases} \tag{1.3.1}$$

and

$$\begin{cases} \partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \\ u|_{t=0} = u_{0\infty}, \end{cases} \tag{1.3.2}$$

where $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), u_{01}, u_{0\infty}, F_1$ and F_∞ are given functions.

To formulate the time periodic problem, we denote by $S_1(t)$ the solution operator for (1.3.1) with $F_1 = 0$, and by $\mathscr{S}_1(t)$ the solution operator for (1.3.1) with $u_{01} = 0$. We also denote by $S_{\infty,\tilde{u}}(t)$ the solution operator for (1.3.2) with $F_\infty = 0$ and by $\mathscr{S}_{\infty,\tilde{u}}(t)$ the solution operator for (1.3.2) with $u_{0\infty} = 0$. (The precise definition of these operators will be given later.)

If $\{u_1, u_\infty\}$ satisfies (0.0.21)-(0.0.22), then $u_1(t)$ and $u_\infty(t)$ are written as

$$u_1(t) = S_1(t)u_1(0) + \mathscr{S}_1(t)[F_1(u,g)], \tag{1.3.3}$$

$$u_\infty(t) = S_{\infty,u}u_\infty(0) + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)] \tag{1.3.4}$$

with $u = u_1 + u_\infty$.

Suppose that $\{u_1, u_\infty\}$ is a T-time periodic solution of (1.3.3)-(1.3.4). Then, since $u_1(T) = u_1(0)$ and $u_\infty(T) = u_\infty(0)$, we see that

$$\begin{cases} (I - S_1(T))u_1(0) = \mathscr{S}_1(T)[F_1(u,g)], \\ (I - S_{\infty,u}(T))u_\infty(0) = \mathscr{S}_{\infty,u}(T)[F_\infty(u,g)], \\ u = u_1 + u_\infty. \end{cases}$$

Therefore if $(I - S_1(T))$ and $(I - S_{\infty,u}(T))$ are invertible in a suitable sense, then one obtains (0.0.23)-(0.0.27). Therefore, to obtain a T-time periodic solution of (0.0.21)-(0.0.22), we look for a pair of functions $\{u_1, u_\infty\}$ satisfying (0.0.23)-(0.0.27). We will investigate the solution operators $S_1(t), \mathscr{S}_1(t), S_{\infty,u}(t)$ and $\mathscr{S}_{\infty,u}(t)$ in sections 5 and 6.

Next, we introduce some lemmas which will be used in the proof of Theorem 3.2.1. We first derive some inequalities for the low frequency part.

Lemma 1.3.3. (i) *Let k be a nonnegative integer. Then P_1 is a bounded linear operator from L^2 to H^k. In fact, it holds that*

$$\|\nabla^k P_1 f\|_{L^2} \le C\|f\|_{L^2} \qquad (f \in L^2).$$

As a result, for any p satisfying $2 \le p \le \infty$, P_1 is bounded from L^2 to L^p.

(ii) *Let k be a nonnegative integer and let $2 \le p \le \infty$. Then there hold the estimates*

$$\|\nabla^k f_1\|_{L^2} + \|f_1\|_{L^p} \le C\|f_1\|_{L^2} \quad (f \in L^2_{(1)}),$$

$$\begin{aligned} \|f_1\|_{H^k_1} &\le C\|f_1\|_{L^2_1} \quad (f \in L^2_{(1)} \cap L^2_1), \\ \|\nabla f_1\|_{H^k_1} &\le C\|f_1\|_{\mathscr{H}^1_{(1),1}} \quad (f \in \mathscr{H}^1_{(1),1}), \\ \|f_1\|_{L^2_1} + \|f_1\|_{\mathscr{H}^1_{(1),1}} &\le C\|f_1\|_{L^1_1} \quad (f \in L^2_{(1)} \cap L^1_1). \end{aligned}$$

Proof. The boundedness of P_1 from L^2 to H^k can be easily verified by using the Plancherel theorem, since $\operatorname{supp}\widehat{P_1 f} \subset \{\xi; |\xi| \le r_\infty\}$; and, then, the boundedness of P_1 from L^2 to L^p with $2 \le p \le \infty$ follows from the Sobolev inequality.

As for (ii), the first inequality can be obtained as in the same reason for (i). The second inequality is obtained by (i) and the following computation. For $0 \le |\alpha| \le k$ and $f_1 \in L^2_{(1),1}$, we see that

$$\begin{aligned} \||x|\partial_x^\alpha f_1\|_{L^2} &= (2\pi)^{-\frac{n}{2}}\|\partial_\xi(\xi^\alpha \hat{f}_1)\|_{L^2(|\xi|\le r_\infty)} \\ &\le C\big\{\||\xi|^{(|\alpha|-1)_+}\hat{f}_1\|_{L^2(|\xi|\le r_\infty)} + \||\xi|^{|\alpha|}\partial_\xi \hat{f}_1\|_{L^2(|\xi|\le r_\infty)}\big\} \\ &\le C\big\{\|\hat{f}_1\|_{L^2(|\xi|\le r_\infty)} + \|\partial_\xi \hat{f}_1\|_{L^2(|\xi|\le r_\infty)}\big\} \le C\|f_1\|_{L^2_1}. \end{aligned}$$

The third inequality follows from the second inequality with f_1 replaced by ∇f_1, since, by the first inequality, we have $\|\nabla f_1\|_{L^2_1} \le C\|f_1\|_{\mathscr{H}^1_{(1),1}}$. As for the last inequality, we have

$$\begin{aligned} \|f_1\|^2_{L^2_1} &= (2\pi)^{-n}\big\{\|\hat{f}_1\|^2_{L^2(|\xi|\le r_\infty)} + \|\partial_\xi \hat{f}_1\|^2_{L^2(|\xi|\le r_\infty)}\big\} \\ &\le C\big\{\sup_{|\xi|\le r_\infty}(|\hat{f}_1(\xi)| + |\partial_\xi \hat{f}_1(\xi)|)\big\}^2 \le C\|f_1\|^2_{L^1_1}, \end{aligned}$$

and, likewise, we can obtain $\|f_1\|_{\mathscr{H}^1_{(1),1}} \le C\|f_1\|_{L^1_1}$. This completes the proof. □

As for the high frequency part, we have the following inequalities.

Lemma 1.3.4. (i) *Let k be a nonnegative integer. Then P_∞ is a bounded linear operator on H^k.*

(ii) *There hold the inequalities*

$$\begin{aligned} \|P_\infty f\|_{L^2} &\le C\|\nabla f\|_{L^2} \quad (f \in H^1), \\ \|f_\infty\|_{L^2} &\le C\|\nabla f_\infty\|_{L^2} \quad (f_\infty \in H^1_{(\infty)}). \end{aligned}$$

Lemma 1.3.4 (i) immediately follows from the definition of P_∞ by using the Plancherel theorem; and, similarly, inequalities in (ii) can be easily seen since supp $\widehat{P_\infty f} \subset \{\xi; |\xi| \geq r_1\}$ and supp $\hat{f}_\infty \subset \{\xi; |\xi| \geq r_1\}$ for $f_\infty \in H^1_{(\infty)}$. We omit the proof.

Lemma 1.3.5. *Let χ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then there holds the estimate*

$$\||x|(\chi * f)\|_{L^2} \leq C\{\||x|\chi\|_{L^1}\|f\|_{L^2} + \|\chi\|_{L^1}\||x|f\|_{L^2}\} \qquad (f \in L^2_1).$$

Proof. Let χ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then

$$\begin{aligned} ||x|(\chi * f)| &\leq |x| \int_{\mathbb{R}^n} |\chi(x-y)f(y)|dy \\ &\leq C\int_{\mathbb{R}^n} |x-y||\chi(x-y)||f(y)|dy + C\int_{\mathbb{R}^n} |\chi(x-y)||y||f(y)|dy. \end{aligned}$$

Therefore, the Young inequality gives

$$\||x|(\chi * f)\|_{L^2} \leq C\{\||x|\chi\|_{L^1}\|f\|_{L^2} + \|\chi\|_{L^1}\||x|f\|_{L^2}\} \qquad (f \in L^2_1).$$

This completes the proof. □

Lemma 1.3.6. *Let $f_\infty \in H^1_{(\infty),1}$. Then there exists a positive constant C independent of f_∞ such that*

$$\||x|\nabla f_\infty\|^2_{L^2} \geq \frac{r_1^2}{2}\||x|f_\infty\|^2_{L^2} - C\|f_\infty\|^2_{L^2}.$$

Proof. Since supp $\hat{f}_\infty \subset \{|\xi| \geq r_1\}$, by the Plancherel theorem, we have

$$\begin{aligned} \||x|\nabla f_\infty\|^2_{L^2} &\geq \frac{1}{2}\sum_{j=1}^n \|\nabla(x_j f_\infty)\|^2_{L^2} - C\|f_\infty\|^2_{L^2} \\ &= \frac{1}{2}(2\pi)^{-n}\sum_{j=1}^n \|\xi(\partial_{\xi_j}\hat{f}_\infty)\|^2_{L^2} - C\|f_\infty\|^2_{L^2}. \\ &\geq \frac{r_1^2}{2}(2\pi)^{-n}\sum_{j=1}^n \|\xi(\partial_{\xi_j}\hat{f}_\infty)\|^2_{L^2} - C\|f_\infty\|^2_{L^2} \\ &\geq \frac{r_1^2}{2}\||x|f_\infty\|^2_{L^2} - C\|f_\infty\|^2_{L^2}. \end{aligned}$$

This completes the proof. □

1.4 Properties of $S_1(t)$ and $\mathscr{S}_1(t)$

In this section we investigate $S_1(t)$ and $\mathscr{S}_1(t)$ and establish an estimate for a solution u_1 of (1.3.1) satisfying $u_1(0) = u_1(T)$.

We consider the restriction of A on $L^2_{(1)}$. By Lemma 1.3.3 (ii), we see that $\|Au_1\|_{L^2} \leq C\|u_1\|_{L^2}$ for $u_1 \in L^2_{(1)}$.

Let

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top \xi \\ i\gamma\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top \xi \end{pmatrix} \quad (\xi \in \mathbb{R}^n).$$

Then, since $Au_1 = \mathcal{F}^{-1}\hat{A}_\xi \hat{u}_1$, we see that $\operatorname{supp} \hat{A}_\xi \hat{u}_1 \subset \{\xi; |\xi| \leq r_\infty\}$ for $u_1 \in L^2_{(1)}$. Therefore, the restriction of A on $L^2_{(1)}$ is a bounded linear operator on $L^2_{(1)}$.

We denote by A_1 the restriction of A on $L^2_{(1)}$. Then A_1 is a bounded linear operator on $L^2_{(1)}$ and it satisfies $\|A_1 u_1\|_{L^2} \leq C\|u_1\|_{L^2}$ for $u_1 \in L^2_{(1)}$ and

$$A_1 u_1 = \mathcal{F}^{-1}\hat{A}_\xi \mathcal{F} u_1 \quad (u_1 \in L^2_{(1)}).$$

Furthermore, $-A_1$ generates a uniformly continuous semigroup $S_1(t) = e^{-tA_1}$ that is given by

$$S_1(t)u_1 = \mathcal{F}^{-1} e^{-t\hat{A}_\xi}\mathcal{F}u_1 \quad (u_1 \in L^2_{(1)});$$

and it holds that $S_1(t)$ satisfies $S_1(\cdot)u_1 \in C^1([0,\infty); L^2_{(1)})$ for each $u_1 \in L^2$ and

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 \,(= -AS_1(t)u_1), \ S_1(0)u_1 = u_1 \ \text{ for } \ u_1 \in L^2_{(1)},$$

$$\|\partial_t^k S_1(t)u_1\|_{L^2} \leq \|A_1\|^k \|u_1\|_{L^2} \ \text{ for } \ u_1 \in L^2_{(1)}, \ t \geq 0, \ k = 0, 1,$$

where $\|A_1\|$ denotes the operator norm of A_1. The estimates can be obtained by the energy method based on the relation

$$(Au, u) = \nu\|\nabla u\|^2_{L^2} + \tilde{\nu}\|\nabla \cdot u\|^2_{L^2}.$$

We also define the operator $\mathscr{S}_1(t)$ by

$$\mathscr{S}_1(t)[F_1] = \int_0^t S_1(t-\tau)F_1(\tau)\,d\tau$$

for $F_1 \in L^2(0,T; L^2_{(1)})$. It follows that

$$\mathscr{S}_1(t)[F_1] = \mathcal{F}^{-1}\Big[\int_0^t e^{-(t-\tau)\hat{A}_\xi}\hat{F}_1(\tau)\,d\tau\Big],$$

$\mathscr{S}_1(\cdot)[F_1] \in H^1(0,T; L^2_{(1)})$ for each $F_1 \in L^2(0,T; L^2_{(1)})$ and

$$\partial_t \mathscr{S}_1(t)[F_1] + A_1\mathscr{S}_1(t)[F_1] = F_1(t) \ (\text{a.e.}\, t), \ \mathscr{S}_1(0)[F_1] = 0,$$

$$\|\mathscr{S}_1(\cdot)[F_1]\|_{H^1(0,T;L^2)} \leq C\|F_1\|_{L^2(0,T;L^2)},$$

where C is a positive constant depending on T.

We next show that A_1 has similar properties on $\mathscr{H}^1_{(1),1}$.

Proposition 1.4.1. (i) *A_1 is a bounded linear operator on $\mathscr{H}^1_{(1),1}$ and $S_1(t) = e^{-tA_1}$ is a uniformly continuous semigroup on $\mathscr{H}^1_{(1),1}$. Furthermore, it holds that $S_1(\cdot)u_1 \in C^1([0,T']; \mathscr{H}^1_{(1),1})$, $\partial_t S_1(\cdot)u_1 \in C([0,T']; L^2_1)$ for each $u_1 \in \mathscr{H}^1_{(1),1}$ and all $T' > 0$,*

$$\|\partial_t^k S_1(t)u_1\|_{\mathscr{H}^1_{(1),1}} \le C\|u_1\|_{\mathscr{H}^1_{(1),1}} \quad \text{for} \ \ u_1 \in \mathscr{H}^1_{(1),1}, \ \ t \in [0,T'], \ \ k = 0,1,$$

and

$$\|\partial_t S_1(t)u_1\|_{L^2_1} \le C\|u_1\|_{\mathscr{H}^1_{(1),1}} \quad \text{for} \ \ u_1 \in \mathscr{H}^1_{(1),1}, \ \ t \in [0,T'],$$

where T' is any given positive number and C is a positive constant depending on T'.

(ii) *$\mathscr{S}_1(\cdot)$ satisfies that $\mathscr{S}_1(\cdot)[F_1] \in H^1(0,T; \mathscr{H}^1_{(1),1})$ for each $F_1 \in L^2(0,T; \mathscr{H}^1_{(1),1})$ and*

$$\|\mathscr{S}_1(\cdot)[F_1]\|_{H^1(0,T;\mathscr{H}^1_{(1),1})} \le C\|F_1\|_{L^2(0,T;\mathscr{H}^1_{(1),1})} \ \ \text{for} \ \ F_1 \in L^2(0,T; \mathscr{H}^1_{(1),1}),$$

where C is a positive constant depending on T. If, in addition, $F_1 \in L^2(0,T; L^2_1)$, then $\partial_t \mathscr{S}_1(\cdot)[F_1] \in L^2(0,T; L^2_1)$ and

$$\|\partial_t \mathscr{S}_1(\cdot)[F_1]\|_{L^2(0,T;L^2_1)} \le C\|F_1\|_{L^2(0,T;L^2_1)} \ \ \text{for } F_1 \in L^2(0,T; L^2_1),$$

where C is a positive constant depending on T.

(iii) *It holds that*

$$S_1(t)\mathscr{S}_1(t')[F_1] = \mathscr{S}_1(t')[S_1(t)F_1]$$

for any $t \ge 0$, $t' \in [0,T]$ and $F_1 \in L^2(0,T; X)$, where $X = L^2_{(1)}, \mathscr{H}^1_{(1),1}$.

(iv) *It holds that $\Gamma S_1(t) = S_1(t)\Gamma$ and $\Gamma \mathscr{S}_1(t) = \mathscr{S}_1(t)\Gamma$. Consequently, the assertions* (i)–(iii) *above hold with function spaces $L^2_{(1)}$, $\mathscr{H}^1_{(1),1}$ and L^2_1 replaced by $(L^2_{(1)})_{sym}$, $(\mathscr{H}^1_{(1),1})_{sym}$ and $(L^2_1)_{sym}$, respectively.*

The proof of Proposition 1.4.1 will be given later.

We next investigate invertibility of $I - S_1(T)$.

Proposition 1.4.2. *Let $F_1 = {}^\top(F_1^0(x), \tilde{F}_1(x)) \in L^2_{(1)} \cap L^1_1$ and suppose that $\tilde{F}_1(-x) = -\tilde{F}_1(x)$ for $x \in \mathbb{R}^n$. Then there uniquely exists $u \in \mathscr{H}^1_{(1),1}$ that satisfies*

$$(I - S_1(T))u = F_1 \ \ \text{and} \ \ \|u\|_{\mathscr{H}^1_{(1),1}} \le C\|F_1\|_{L^1_1}. \tag{1.4.1}$$

Furthermore, if $\Gamma F_1 = F_1$, then $\Gamma u = u$.

The proof of Proposition 1.4.2 will be given later.

In view of Proposition 1.4.2, $I - S_1(T)$ has a bounded inverse $(I - S_1(T))^{-1}$: $(L^2_{(1)} \cap L^1_1)_{sym} \to (\mathscr{H}^1_{(1),1})_{sym}$ and it holds that

$$\|(I - S_1(T))^{-1}F_1\|_{\mathscr{H}^1_{(1),1}} \le C\|F_1\|_{L^1_1}.$$

Using Proposition 1.4.1 (ii) and Proposition 1.4.2, we can obtain the following estimate for $\mathscr{S}_1(T)(I - S_1(T))^{-1}$.

Proposition 1.4.3. *For* $F_1 \in L^2(0,T;(L^2_{(1)}\cap L^1_1)_{sym})$*, it holds that* $\mathscr{S}_1(T)[(I-S_1(T))^{-1}F_1] \in (\mathscr{H}^1_{(1),1})_{sym}$ *and*

$$\|\mathscr{S}_1(T)[(I-S_1(T))^{-1}F_1]\|_{\mathscr{H}^1_{(1),1}} \le C\|F_1\|_{L^2(0,T;L^1_1)}.$$

We are now in a position to give an estimate for a solution of (1.3.1) satisfying $u_1(0) = u_1(T)$.

Proposition 1.4.4. *Set*

$$u_1(t) = S_1(t)\mathscr{S}_1(T)[(I-S_1(T))^{-1}F_1] + \mathscr{S}_1(t)[F_1] \tag{1.4.2}$$

for $F_1 = {}^\top(F_1^0(x,t),\tilde{F}_1(x,t)) \in L^2(0,T;(L^2_{(1)}\cap L^1_1)_{sym})$*. Then* u_1 *is a solution of* (1.3.1) *in* $\mathscr{Y}_1(0,T)$ *satisfying* $u_1(0) = u_1(T)$ *and*

$$\|u_1\|_{\mathscr{Y}_1(0,T)} \le C\|F_1\|_{L^2(0,T;L^1_1)}.$$

Proof. We find from Proposition 1.4.1 (iii) and Proposition 1.4.2 that $u_1(0) = u_1(T)$. As for the estimate for u_1, the first term on the right-hand side of (1.4.2) is estimated by using Proposition 1.4.1 (i) and Proposition 1.4.3. The second term on the right-hand side of (1.4.2) is estimated by using Proposition 1.4.1 (ii) and Lemma 1.3.3 (ii). Hence, we obtain the desired estimate. This completes the proof. □

In the rest of this section we will give proofs of Proposition 1.4.1 and Proposition 1.4.2.

Lemma 1.4.5. ([26]) (i) *The set of all eigenvalues of* $-\hat{A}_\xi$ *consists of* $\lambda_j(\xi)\,(j=1,\pm)$*, where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2, \\ \lambda_\pm(\xi) = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu+\tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}. \end{cases}$$

If $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$*, then*

$$\operatorname{Re}\lambda_\pm = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2, \quad \operatorname{Im}\lambda_\pm = \pm\gamma|\xi|\sqrt{1-\frac{(\nu+\tilde{\nu})^2}{4\gamma^2}|\xi|^2}.$$

(ii) *If* $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$*, then* $e^{-t\hat{A}_\xi}$ *has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1,\pm} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ *is eigenprojections for* $\lambda_j(\xi)\,(j=1,\pm)$*, and* $\Pi_j(\xi)\,(j=1,\pm)$ *satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_n - \frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix}, \quad \Pi_\pm(\xi) = \pm\frac{1}{\lambda_+-\lambda_-}\begin{pmatrix} -\lambda_\mp & -i\gamma^\top\xi \\ -i\gamma\xi & \lambda_\pm\frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix}.$$

Furthermore, if $0 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$*, then there exist a constant* $C>0$ *such that the estimates*

$$\|\Pi_j(\xi)\| \le C\,(j=1,\pm) \tag{1.4.3}$$

hold for $|\xi| \le r_\infty$*.*

Hereafter we fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (1.4.3) in Lemma 1.4.5 holds for $|\xi| \le r_\infty$.

Lemma 1.4.6. *Let α be a multi-index. Then the following estimates hold true uniformly for ξ with $|\xi| \le r_\infty$ and $t \in [0, T]$.*

(i) $|\partial_\xi^\alpha \lambda_1| \le C|\xi|^{2-|\alpha|}$, $|\partial_\xi^\alpha \lambda_\pm| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 0)$.

(ii) $|(\partial_\xi^\alpha \Pi_1)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|$, $|(\partial_\xi^\alpha \Pi_\pm)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{F}_1|$ $(|\alpha| \ge 0)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(iii) $|\partial_\xi^\alpha (e^{\lambda_1 t})| \le C|\xi|^{2-|\alpha|}$ $(|\alpha| \ge 1)$.

(iv) $|\partial_\xi^\alpha (e^{\lambda_\pm t})| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 1)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}_1| \le C(|\xi|^{1-|\alpha|}|\hat{F}_1^0| + |\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|)$ $(|\alpha| \ge 1)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(vi) $|\partial_\xi^\alpha (I - e^{\lambda_1 t})^{-1}| \le C|\xi|^{-2-|\alpha|}$ $(|\alpha| \ge 0)$.

(vii) $|\partial_\xi^\alpha (I - e^{\lambda_\pm t})^{-1}| \le C|\xi|^{-1-|\alpha|}$ $(|\alpha| \ge 0)$.

Lemma 1.4.6 can be verified by direct computations based on Lemma 1.4.5.

Let us prove Proposition 1.4.1.

Proof of Proposition 1.4.1. We see from Lemma 1.3.3 (ii) that

$$\|A_1 u_1\|_{\mathscr{H}^1_{(1),1}} \le C\|\nabla u_1\|_{H^1_1} \le C\|u_1\|_{\mathscr{H}^1_{(1),1}} \quad (u_1 \in \mathscr{H}^1_{(1),1}),$$

and so, A_1 is bounded on $\mathscr{H}^1_{(1),1}$. It then follows that $S_1(\cdot)u_1 \in C^1([0, T']; \mathscr{H}^1_{(1),1})$ for each $u_1 \in \mathscr{H}^1_{(1),1}$ and

$$\|\partial_t^k S_1(t)u_1\|_{\mathscr{H}^1_{(1),1}} \le C\|u_1\|_{\mathscr{H}^1_{(1),1}} \quad \text{for} \quad u_1 \in \mathscr{H}^1_{(1),1}, \ \ t \in [0, T'], \ \ k = 0, 1,$$

where $T' > 0$ is any given positive number and C is a positive constant depending on T'. Since $\|A_1 u_1\|_{L^2_1} \le C\|\nabla u_1\|_{H^1_1} \le C\|u_1\|_{\mathscr{H}^1_{(1),1}}$ for $u_1 \in \mathscr{H}^1_{(1),1}$ by Lemma 1.3.3 (ii), we see from the relation $\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1$ that $\partial_t S_1(\cdot)u_1 \in C([0, T']; L^2_1)$ and

$$\|\partial_t S_1(t)u_1\|_{L^2_1} \le \|S_1(t)u_1\|_{\mathscr{H}^1_{(1),1}} \le C\|u_1\|_{\mathscr{H}^1_{(1),1}}.$$

The assertion (ii) follows from (i) and the relation $\partial_t \mathscr{S}_1(t)[F_1] = -A_1 \mathscr{S}_1(t)[F_1] + F_1(t)$. The assertion (iii) easily follows from the definitions of $S_1(t)$ and $\mathscr{S}_1(t)$. As for (iv), we observe that $\Gamma A_1 = A_1 \Gamma$, from which we find that $\Gamma S_1(t) = S_1(t)\Gamma$, and hence, $\Gamma \mathscr{S}_1(t) = \mathscr{S}_1(t)\Gamma$. This completes the proof. □

Let us finally prove Proposition 1.4.2.

Proof of Proposition 1.4.2. We define a function u

$$u = \mathcal{F}^{-1}(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1$$

for $F_1 = {}^\top(F_1^0, \tilde{F}_1)$. It suffices to show that $\|u\|_{\mathscr{H}^1_{(1),1}} \le C\|F_1\|_{L^1_1}$. By the Plancherel theorem, we see that

$$\begin{aligned}\|u\|_{L^2_{(1)}} &= (2\pi)^{-\frac{n}{2}}\|(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} \\ &\le (2\pi)^{-\frac{n}{2}}\{\|(I - e^{T\lambda_1})^{-1}\Pi_1\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} + \|(I - e^{T\lambda_+})^{-1}\Pi_+\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} \\ &\quad + \|(I - e^{T\lambda_-})^{-1}\Pi_-\hat{F}_1\|_{L^2(|\xi|\le r_\infty)}\} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Observe that $\Pi_1\hat{F}_1$ depends only on $\hat{\tilde{F}}_1$ but not on $\hat{F}_1^0$.

By using Lemma 1.4.5, Lemma 1.4.6 and the fact $\hat{\tilde{F}}_1(0) = 0$, we see that

$$I_1 \le C\left\|\frac{1}{|\xi|^2}\hat{\tilde{F}}_1\right\|_{L^2(|\xi|\le r_\infty)} \le C\left\|\frac{1}{|\xi|}\right\|_{L^2(|\xi|\le r_\infty)}\||x|\tilde{F}_1\|_{L^1}.$$

Since $\left\|\frac{1}{|\xi|}\right\|_{L^2(|\xi|\le r_\infty)} < +\infty$ for $n \ge 3$, we find that

$$I_1 \le C\||x|\tilde{F}_1\|_{L^1}.$$

Similarly, we can obtain $I_2 + I_3 \le C\|F_1\|_{L^1}$, and hence, we see that

$$\|u\|_{L^2_{(1)}} \le C\{\|F_1\|_{L^1} + \||x|\tilde{F}_1\|_{L^1}\}. \tag{1.4.4}$$

Next, by the Plancherel theorem, it follows that

$$\begin{aligned}\||x|\nabla u\|_{L^2_{(1)}} &= (2\pi)^{-\frac{n}{2}}\left\|(i\partial_\xi)\left(i\xi(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1\right)\right\|_{L^2(|\xi|\le r_\infty)} \\ &\le C\Big\{\|(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} + \left\|i\xi\partial_\xi\left((I - e^{-T\hat{A}_\xi})^{-1}\right)\hat{F}_1\right\|_{L^2(|\xi|\le r_\infty)} \\ &\quad + \|i\xi(I - e^{-T\hat{A}_\xi})^{-1}\partial_\xi\hat{F}_1\|_{L^2(|\xi|\le r_\infty)}\Big\}.\end{aligned}$$

The first term on right-hand side has already been estimated and it is bounded by the right-hand side of (1.4.4). As for the second and third terms on the right-hand side, similarly to above, one can find from Lemma 1.4.6 that

$$\left\|i\xi\partial_\xi\left((I - e^{-T\hat{A}_\xi})^{-1}\right)\hat{F}_1\right\|_{L^2(|\xi|\le r_\infty)} + \|i\xi(I - e^{-T\hat{A}_\xi})^{-1}\partial_\xi\hat{F}_1\|_{L^2(|\xi|\le r_\infty)} \le C\|F_1\|_{L^1_1}.$$

We thus obtain

$$\||x|\nabla u\|_{L^2_{(1)}} \le C\|F_1\|_{L^1_1}.$$

Finally, we see from Proposition 1.4.1 (iv) that if $\Gamma F_1 = F_1$, then $\Gamma u = u$. This completes the proof. $\square$

1.5 Properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$

In this section we investigate $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$.

We begin with the solvability of (1.3.2). Let us first consider the following system:

$$\begin{cases} \partial_t \phi + \gamma(\tilde{w} \cdot \nabla \phi) = f^0, \\ \phi|_{t=0} = \phi_0. \end{cases} \tag{1.5.1}$$

Lemma 1.5.1. ([17, Theorem 4.1].) Let $n \geq 3$ and let m be an integer satisfying $m \geq [\frac{n}{2}] + 1$. Set $k = m - 1$ or m. Assume that $\tilde{w} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1})$, $f^0 \in L^2(0, T'; H^k)$ and $\phi_0 \in H^k$. Here T' is a given positive number. Then (1.5.1) has a unique solution $\phi \in C([0, T']; H^k)$ and ϕ satisfies

$$\|\phi(t)\|_{H^k}^2 \leq C \left\{ \|\phi_0\|_{H^k}^2 + \int_0^t \|\tilde{w}\|_{H^{m+1}} \|\phi\|_{H^k}^2 ds + \int_0^t \|f^0\|_{H^k} \|\phi\|_{H^k} ds \right\}$$

and

$$\|\phi(t)\|_{H^k}^2 \leq C e^{C \int_0^t (1 + \|\tilde{w}\|_{H^{m+1}}) ds} \left\{ \|\phi_0\|_{H^k}^2 + \int_0^t \|f^0\|_{H^k}^2 ds \right\}$$

for $t \in [0, T']$. Moreover, the solution is unique in $C([0, T']; H^1)$.

We next consider the following system:

$$\begin{cases} \partial_t \phi_\infty + \gamma P_\infty(\tilde{w} \cdot \nabla \phi_\infty) = F_\infty^0, \\ \phi_\infty|_{t=0} = \phi_{0\infty}. \end{cases} \tag{1.5.2}$$

Note that (1.5.2) is rewritten as

$$\partial_t \phi_\infty + \gamma(\tilde{w} \cdot \nabla \phi_\infty) = F_\infty^0 + \gamma P_1(\tilde{w} \cdot \nabla \phi_\infty).$$

As for the solvability of (1.5.2), we have the following lemma.

Lemma 1.5.2. *Let $n \geq 3$ and let m be an integer satisfying $m \geq [\frac{n}{2}] + 1$. Set $k = m - 1$ or m. Assume that $\tilde{w} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1})$, $F_\infty^0 \in L^2(0, T'; H_{(\infty)}^k)$ and $\phi_{0\infty} \in H_{(\infty)}^k$. Here T' is a given positive number. Then (1.5.2) has a unique solution $\phi_\infty \in C([0, T']; H_{(\infty)}^k)$ and ϕ_∞ satisfies*

$$\begin{aligned} \|\phi_\infty(t)\|_{H^k}^2 \leq\ & C \Big\{ \|\phi_{0\infty}\|_{H^k}^2 + \int_0^t (\|\tilde{w}\|_{H^{m+1}} + \|\tilde{w}\|_{H^m}^2) \|\phi_\infty\|_{H^k}^2 ds \\ & + \int_0^t \|F_\infty^0\|_{H^k} \|\phi\|_{H^k} ds \Big\} \end{aligned}$$

and

$$\|\phi_\infty(t)\|_{H^k}^2 \leq C e^{C \int_0^t (1 + \|\tilde{w}\|_{H^{m+1}} + \|\tilde{w}\|_{H^m}^2) ds} \left\{ \|\phi_{0\infty}\|_{H^k}^2 + \int_0^t \|F_\infty^0\|_{H^k}^2 ds \right\}$$

for $t \in [0, T']$.

Proof. We define $\{\phi_\infty^{(\ell)}\}_{\ell=0}^\infty$ as follows. For $\ell = 0$, $\phi_\infty^{(0)}$ is the solution of

$$\begin{cases} \partial_t \phi_\infty^{(0)} + \gamma(\tilde{w} \cdot \nabla \phi_\infty^{(0)}) = F_\infty^0, \\ \phi_\infty^{(0)}|_{t=0} = \phi_{0\infty}. \end{cases} \tag{1.5.3}$$

For $\ell \geq 1$, $\phi_\infty^{(\ell)}$ is the solution of

$$\begin{cases} \partial_t \phi_\infty^{(\ell)} + \gamma(\tilde{w} \cdot \nabla \phi_\infty^{(\ell)}) = F_\infty^0 + \gamma P_1(\tilde{w} \cdot \nabla \phi_\infty^{(\ell-1)}), \\ \phi_\infty^{(\ell)}|_{t=0} = \phi_{0\infty}. \end{cases} \tag{1.5.4}$$

By Lemma 1.3.3 (i), we have

$$\|P_1(\tilde{w} \cdot \nabla \phi_\infty)\|_{H^m} \leq C\|\tilde{w}\|_{L^\infty}\|\nabla \phi_\infty\|_{L^2} \leq C\|\tilde{w}\|_{H^m}\|\phi_\infty\|_{H^k} \tag{1.5.5}$$

since $m \geq [\frac{n}{2}] + 1 \geq 2$. In view of Lemma 1.5.1 and (1.5.5), we find by a standard argument that

$$\|\phi_\infty^{(\ell+1)}(t) - \phi_\infty^{(\ell)}(t)\|_{H^k}^2 \leq M_0 \frac{(M_1 t)^{\ell+1}}{(\ell+1)!} \quad (\ell \geq 0),$$

where

$$M_0 = Ce^{C\int_0^{T'}(1+\|\tilde{w}\|_{H^{m+1}}+\|\tilde{w}\|_{H^m}^2)\,d\tau}\Big\{\|\phi_0\|_{H^k}^2 + \int_0^{T'} \|F_\infty^0\|_{H^k}^2\, d\tau\Big\},$$

$$M_1 = C\|\tilde{w}\|_{C([0,T'];H^m)}^2 e^{C\int_0^{T'}(1+\|\tilde{w}\|_{H^{m+1}})\,d\tau}.$$

Therefore, one can see that $\phi_\infty^{(\ell)}$ converges in $C([0,T'];H^k)$ to a function $\phi_\infty \in C([0,T'];H^k)$ that satisfies

$$\begin{cases} \partial_t \phi_\infty + \gamma(\tilde{w} \cdot \nabla \phi_\infty) = F_\infty^0 + \gamma P_1(\tilde{w} \cdot \nabla \phi_\infty), \\ \phi_\infty|_{t=0} = \phi_{0\infty}, \end{cases} \tag{1.5.6}$$

hence, ϕ_∞ is a solution of (1.5.2). The estimates for ϕ_∞ follows from Lemma 1.5.1 and (1.5.5).

It remains to prove supp $\hat{\phi}_\infty(t) \subset \{|\xi| \geq r_1\}$ for $t \in [0,T']$. Let $\tilde{\chi}_\infty \in C_0^\infty(\mathbb{R}^n)$ with supp $\tilde{\chi}_\infty \subset \{|\xi| < r_1\}$. Let us consider the Fourier transform of (1.5.2):

$$\partial_t \hat{\phi}_\infty + \gamma \tilde{\chi}_\infty \widehat{(\tilde{w} \cdot \nabla \phi_\infty)} = \hat{F}_\infty^0, \quad \hat{\phi}_\infty|_{t=0} = \hat{\phi}_{0\infty}.$$

Taking the inner product of this equation with $\tilde{\chi}_\infty^2 \hat{\phi}_\infty$, we have $\frac{d}{dt}\|\tilde{\chi}_\infty \hat{\phi}_\infty\|_{L^2}^2 = 0$. We thus deduce that $\|\tilde{\chi}_\infty \hat{\phi}_\infty(t)\|_{L^2}^2 = \|\tilde{\chi}_\infty \hat{\phi}_{0\infty}\|_{L^2}^2 = 0$ for $t \in [0,T']$. It then follows that supp $\hat{\phi}_\infty(t) \subset \{|\xi| \geq r_1\}$ for $t \in [0,T']$. This completes the proof. □

We next consider the following system:

$$\begin{cases} \partial_t w_\infty - \nu \triangle w_\infty - \tilde{\nu} \nabla \mathrm{div} w_\infty = \tilde{F}_\infty, \\ w_\infty|_{t=0} = w_{0\infty}. \end{cases} \tag{1.5.7}$$

Lemma 1.5.3. (i) *Let $n \geq 3$ and let m be an integer satisfying $m \geq [\frac{n}{2}] + 1$. Set $k = m - 1$ or m. Assume that $\tilde{F}_\infty \in L^2(0, T'; H^{k-1})$ and $w_{0\infty} \in H^k$. Here T' is a given positive number. Then* (1.5.7) *has a unique solution $w_\infty \in C([0, T']; H^k) \cap L^2(0, T'; H^{k+1}) \cap H^1(0, T'; H^{k-1})$ and*

$$\|w_\infty(t)\|_{H^k}^2 + \int_0^t \|w_\infty\|_{H^{k+1}}^2 + \|\partial_\tau w_\infty\|_{H^{k-1}}^2 \, d\tau \leq C \left\{ \|w_{0\infty}\|_{H^k}^2 + \int_0^t \|\tilde{F}_\infty\|_{H^{k-1}}^2 ds \right\}$$

for $t \in [0, T']$ with a positive constant C depending on T'.

(ii) *Assume, further, that $\tilde{F}_\infty \in L^2(0, T'; H^{k-1}_{(\infty)})$ and $w_{0\infty} \in H^k_{(\infty)}$. Then the solution w_∞ satisfies*

$$w_\infty \in C([0, T']; H^k_{(\infty)}) \cap L^2(0, T'; H^{k+1}_{(\infty)}) \cap H^1(0, T'; H^{k-1}_{(\infty)}).$$

Lemma 1.5.3 (i) follows from standard theory of parabolic equation. The assertion (ii) can be proved in a similar manner to the proof of Lemma 1.5.2. We omit the details.

By using Lemma 1.5.2 and Lemma 1.5.3, we show the solvability of (1.3.2).

Proposition 1.5.4. *Let $n \geq 3$ and let m be an integer satisfying $m \geq [\frac{n}{2}] + 1$. Set $k = m - 1$ or m. Assume that*

$$\begin{aligned}
&\tilde{w} \in C([0, T']; H^m) \cap L^2(0, T'; H^{m+1}), \\
&u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)}, \\
&F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0, T'; H^k_{(\infty)} \times H^{k-1}_{(\infty)}).
\end{aligned}$$

Here T' is a given positive number. Then there exists a unique solution $u_\infty = {}^\top(\phi_\infty, w_\infty)$ of (1.3.2) *satisfying*

$$\phi_\infty \in C([0, T']; H^k_{(\infty)}), \; w_\infty \in C([0, T']; H^k_{(\infty)}) \cap L^2(0, T'; H^{k+1}_{(\infty)}) \cap H^1(0, T'; H^{k-1}_{(\infty)}).$$

Remark 1.5.5. Concerning the condition for $\tilde{w}$, it is assumed in Proposition 1.5.4 that $\tilde{w} \in C([0, T']; H^s) \cap L^2(0, T'; H^{s+1})$. However, by taking a look at the proof bellow, it can be replaced by the condition that $\nabla \tilde{w} \in C([0, T']; H^{s-1}) \cap L^2(0, T'; H^s)$.

Proof. We define $u^{(\ell)}_\infty = {}^\top(\phi^{(\ell)}_\infty, w^{(\ell)}_\infty)$ ($\ell = 0, 1, \cdots$) as follows. For $\ell = 0$, $w^{(0)}_\infty = 0$ and $\phi^{(0)}_\infty$ is the solution of

$$\begin{cases} \partial_t \phi^{(0)}_\infty + \gamma P_\infty(\tilde{w} \cdot \nabla \phi^{(0)}_\infty) = F^0_\infty, \\ \phi^{(0)}_\infty|_{t=0} = \phi_{0\infty}. \end{cases} \tag{1.5.8}$$

For $\ell \geq 1$, $w^{(\ell)}_\infty$ is the solution of

$$\begin{cases} \partial_t w^{(\ell)}_\infty - \nu \triangle w^{(\ell)}_\infty - \tilde{\nu} \nabla \mathrm{div} w^{(\ell)}_\infty = -\gamma \nabla \phi^{(\ell-1)}_\infty + \tilde{F}_\infty, \\ w^{(\ell)}_\infty|_{t=0} = w_{0\infty}, \end{cases} \tag{1.5.9}$$

and $\phi_\infty^{(\ell)}$ is the solution of

$$\begin{cases} \partial_t \phi_\infty^{(\ell)} + \gamma P_\infty(\tilde{w} \cdot \nabla \phi_\infty^{(\ell)}) = -\gamma \mathrm{div} w_\infty^{(\ell)} + F_\infty^0, \\ \phi_\infty^{(\ell)}|_{t=0} = \phi_{0\infty}. \end{cases} \tag{1.5.10}$$

As in the proof of Lemma 1.5.2, by using Lemma 1.5.2 and Lemma 1.5.3, one can show that $u_\infty^{(\ell)} = {}^\top(\phi_\infty^{(\ell)}, w_\infty^{(\ell)})$ converges to a pair of function $u_\infty = {}^\top(\phi_\infty, w_\infty)$ in $C([0,T']; H^k_{(\infty)}) \times \left[C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)})\right]$. It is not difficult to see that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ is a unique solution of (1.3.2). This completes the proof. $\square$

We now define $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ formally introduced in section 4.

In the remaining of this section we fix an integer m satisfying $m \geq [\frac{n}{2}] + 1$ and a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfying

$$\tilde{\phi} \in C_{per}(\mathbb{R}; H^m), \;\; \tilde{w} \in C_{per}(\mathbb{R}; H^m) \cap L^2_{per}(\mathbb{R}; H^{m+1}) \tag{1.5.11}$$

In view of Proposition 1.5.4, we define $S_{\infty,\tilde{u}}(t)$ $(t \geq 0)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ $(t \in [0,T])$ as follows.

Let $k = m-1$ or m. The operator $S_{\infty,\tilde{u}}(t) : H^k_{(\infty)} \longrightarrow H^k_{(\infty)}$ $(t \geq 0)$ is defined by

$$u_\infty(t) = S_{\infty,\tilde{u}}(t) u_{0\infty} \;\; \text{for} \;\; u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},$$

where $u_\infty(t)$ is the solution of (1.3.2) with $F_\infty = 0$; and the operator $\mathscr{S}_{\infty,\tilde{u}}(t) : L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}) \longrightarrow H^k_{(\infty)}$ $(t \in [0,T])$ is defined by

$$u_\infty(t) = \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \;\; \text{for} \;\; F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}),$$

where $u_\infty(t)$ is the solution of (1.3.2) with $u_{0\infty} = 0$.

The operators $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ have the following properties in weighted Sobolev spaces.

Proposition 1.5.6. *Let $n \geq 3$ and let m be a nonnegative integer satisfying $m \geq [\frac{n}{2}] + 1$. Let $k = m-1$ or m and let ℓ be a nonnegative integer. Assume that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies* (1.5.11). *Then there exists a constant $\delta > 0$ such that if $\|\tilde{w}\|_{C([0,T]; H^m) \cap L^2(0,T; H^{m+1})} \leq \delta$, the following assertions hold true.*

(i) *It holds that $S_{\infty,\tilde{u}}(\cdot) u_{0\infty} \in C([0,\infty); H^k_{(\infty),\ell})$ for each $u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),\ell}$ and there exists a constant $a > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|S_{\infty,\tilde{u}}(t) u_{0\infty}\|_{H^k_{(\infty),\ell}} \leq C e^{-at} \|u_{0\infty}\|_{H^k_{(\infty),\ell}}$$

for all $t \geq 0$ and $u_{0\infty} \in H^k_{(\infty),\ell}$ with a constant $C = C(T) > 0$.

(ii) *It holds that* $\mathscr{S}_{\infty,\tilde{u}}(\cdot)F_\infty \in C([0,T]; H^k_{(\infty),\ell})$ *for each* $F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ *and* $\mathscr{S}_{\infty,\tilde{u}}(t)$ *satisfies the estimate*

$$\|\mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty]\|_{H^k_{(\infty),\ell}} \le C\left\{\int_0^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^k_{(\infty),\ell}\times H^{k-1}_{(\infty),\ell}}\, d\tau\right\}^{\frac{1}{2}}$$

for $t \in [0,T]$ *and* $F_\infty \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ *with* $C = C(T) > 0$.

(iii) *It holds that* $r_{H^k_{(\infty),\ell}}(S_{\infty,\tilde{u}}(T)) < 1$.

(iv) $I - S_{\infty,\tilde{u}}(T)$ *has a bounded inverse* $(I - S_{\infty,\tilde{u}}(T))^{-1}$ *on* $H^k_{(\infty),\ell}$ *and* $(I - S_{\infty,\tilde{u}}(T))^{-1}$ *satisfies*

$$\|(I - S_{\infty,\tilde{u}}(T))^{-1}u\|_{H^k_{(\infty),\ell}} \le C\|u\|_{H^k_{(\infty),\ell}} \quad \textit{for} \quad u \in H^k_{(\infty),\ell}.$$

(v) *If* $\Gamma\tilde{u} = \tilde{u}$, *then* $\Gamma S_{\infty,\tilde{u}}(t) = S_{\infty,\tilde{u}}(t)\Gamma$ *and* $\Gamma\mathscr{S}_{\infty,\tilde{u}}(t) = \mathscr{S}_{\infty,\tilde{u}}(t)\Gamma$. *Consequently, if* $\Gamma\tilde{u} = \tilde{u}$, *then the assertions* (i)–(iv) *above hold with function spaces* $H^k_{\infty,\ell}$ *and* $H^k_{\infty,\ell} \times H^{k-1}_{\infty,\ell}$ *replaced by* $(H^k_{\infty,\ell})_{sym}$ *and* $(H^k_{\infty,\ell} \times H^{k-1}_{\infty,\ell})_{sym}$, *respectively.*

Remark 1.5.7. In Proposition 1.5.6, it is assumed that

$$\|\tilde{w}\|_{C([0,T];H^s)\cap L^2(0,T;H^{s+1})} \le \delta.$$

However, by taking a look at the proof of Proposition 1.5.8 bellow, it can be replaced by the condition

$$\|\nabla\tilde{w}\|_{C([0,T];H^{s-1})\cap L^2(0,T;H^s)} \le \delta.$$

Proposition 1.5.6 will be proved by the weighted energy method. In fact, Proposition 1.5.6 follows from the weighted energy estimate in the following proposition.

Proposition 1.5.8. *Let* $n \ge 3$ *and let* m *be a nonnegative integer satisfying* $m \ge [\frac{n}{2}] + 1$. *Let* $k = m - 1$ *or* m *and let* ℓ *be a nonnegative integer. Assume that*

$$u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),\ell},$$
$$F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T'; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$$

for all $T' > 0$ *and that* $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ *satisfies* (1.5.11). *Assume also that* $u_\infty = {}^\top(\phi_\infty, w_\infty)$ *is the solution of* (1.3.2) *satisfying*

$$\phi_\infty \in C([0,T']; H^k_{(\infty)}),\ w_\infty \in C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)})$$

for all $T' > 0$.

Then there exist a positive constant δ *and an energy functional* $\mathcal{E}^k[u_\infty]$ *such that if*

$$\|\tilde{w}\|_{C([0,T];H^m)\cap L^2(0,T;H^{m+1})} \le \delta,$$

there holds the estimate

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}^k[u_\infty](t) + d(\|\phi_\infty(t)\|^2_{H^k_\ell} + \|w_\infty(t)\|^2_{H^{k+1}_\ell}) \\
&\leq \; C\{\|F_\infty(t)\|^2_{H^k_\ell\times H^{k-1}_\ell} + (\|\nabla\tilde{w}(t)\|_{H^m} + \|\tilde{w}(t)\|^2_{H^m})\|\phi_\infty(t)\|^2_{H^k_\ell}\}
\end{aligned} \tag{1.5.12}$$

on $(0,T')$ *for all* $T'>0$*. Here* d *is a positive constant;* C *is a positive constant depending on* T *but not on* T'*;* $\mathcal{E}^k[u_\infty]$ *is equivalent to* $\|u_\infty\|^2_{H^k_\ell}$*, i.e,*

$$C^{-1}\|u_\infty\|^2_{H^k_1} \leq \mathcal{E}^k[u_\infty] \leq C\|u_\infty\|^2_{H^k_1};$$

and $\mathcal{E}^k[u_\infty](t)$ *is absolutely continuous in* $t\in[0,T']$ *for all* $T'>0$*.*

The proof of Proposition 1.5.8 will be given in section 1.6.

By using Proposition 1.5.8, we prove Proposition 1.5.6.

Proof of Proposition 1.5.6. Set

$$\begin{aligned}
\omega &= \frac{1}{T}\int_0^T (\|\nabla\tilde{w}(t)\|_{H^m} + \|\tilde{w}(t)\|^2_{H^m})\,dt, \\
z(t) &= (\|\nabla\tilde{w}(t)\|_{H^m} + \|\tilde{w}(t)\|^2_{H^m}) - \omega, \\
Z(t) &= \int_0^t z(\tau)d\tau.
\end{aligned}$$

Observe that $Z(t)$ satisfies $Z(t+T)=Z(t)$ for any $t\in\mathbb{R}$, and so it holds that

$$\sup_{t\in\mathbb{R}}|Z(t)| \leq \sup_{\tau\in[0,T]}|Z(\tau)| \leq C(1+\|\tilde{w}\|^2_{L^2(0,T;H^{m+1})}),$$

where $C=C(T)>0$.

By Proposition 1.5.8 with $F_\infty=0$, we see that there exists a positive constant d_1 such that

$$\frac{d}{dt}\mathcal{E}^k_\ell[u_\infty](t) + d_1\mathcal{E}^k_\ell[u_\infty](t) \leq C\omega\mathcal{E}^k_\ell[u_\infty](t) + Cz(t)\mathcal{E}^k_\ell[u_\infty](t) \quad (t\geq 0). \tag{1.5.13}$$

If $\omega \leq \frac{d_1}{2C}$, then we find from (1.5.13) that

$$\frac{d}{dt}\mathcal{E}^k_\ell[u_\infty](t) + \frac{d_1}{2}\mathcal{E}^k_\ell[u_\infty](t) \leq Cz(t)\mathcal{E}^k_\ell[u_\infty](t) \quad (t\geq 0).$$

We thus obtain

$$\frac{d}{dt}\left(e^{\frac{d_1}{2}t}e^{-CZ(t)}\mathcal{E}^k_\ell[u_\infty](t)\right) \leq 0 \quad (t\geq 0),$$

and hence,

$$\mathcal{E}^k_\ell[u_\infty](t) \leq \mathcal{E}^k_\ell[u_\infty](0)e^{-\frac{d_1}{2}t}e^{CZ(t)} \leq e^{C(1+\|\tilde{w}\|^2_{L^2(0,T;H^{m+1})})}\mathcal{E}^k_\ell[u_\infty](0)e^{-\frac{d_1}{2}t} \quad (t\geq 0).$$

Consequently, we have

$$\|S_{\infty,\tilde{u}}(t)u_{0\infty}\|_{H^k_{(\infty),\ell}} \le Ce^{-\frac{d_1}{4}t}\|u_{0\infty}\|_{H^k_{(\infty),\ell}} \quad (t \ge 0).$$

This proves (i). The assertion (ii) is proved similarly; and we omit the proof.

As for (iii), since $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}) \in C_{per}(\mathbb{R}; H^m)$, it follows from (i) that, for each $j \in \mathbb{N}$,

$$\|(S_{\infty,\tilde{u}}(T))^j u\|_{H^k_{(\infty),\ell}} = \|S_{\infty,\tilde{u}}(jT)u\|_{H^k_{(\infty),\ell}} \le Ce^{-d_2 jT}\|u\|_{H^k_{(\infty),\ell}},$$

where $d_2 = \frac{d_1}{4} > 0$. Hence, we have

$$\|(S_{\infty,\tilde{u}}(T))^j\| \le Ce^{-d_2 jT}.$$

We thus obtain

$$\lim_{j\to\infty} \|(S_{\infty,\tilde{u}}(T))^j\|^{\frac{1}{j}} \le \lim_{j\to\infty} C^{\frac{1}{j}}e^{-d_2 T} = e^{-d_2 T} < 1.$$

This shows (iii). The assertion (iv) is an immediate consequence of (iii).

As for (v), we see that if $\Gamma\tilde{u} = \tilde{u}$, then $\Gamma P_\infty(B[\tilde{u}]u_\infty) = P_\infty(B[\tilde{u}]\Gamma u_\infty)$, and so,

$$\Gamma(\partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty)) = \partial_t \Gamma u_\infty + A\Gamma u_\infty + P_\infty(B[\tilde{u}]\Gamma u_\infty).$$

It then follows from the uniqueness of solutions of (1.3.2) that $\Gamma S_{\infty,\tilde{u}}(t) = S_{\infty,\tilde{u}}(t)\Gamma$ and $\Gamma\mathscr{S}_{\infty,\tilde{u}}(t) = \mathscr{S}_{\infty,\tilde{u}}(t)\Gamma$. This completes the proof. □

We conclude this section with the estimate for a solution u_∞ of (1.3.2) satisfying $u_\infty(0) = u_\infty(T)$.

Proposition 1.5.9. *Let $n \ge 3$ and let m be a nonnegative integer satisfying $m \ge [\frac{n}{2}]+1$. Assume that*

$$F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; (H^k_{(\infty),1} \times H^{k-1}_{(\infty),1})_{sym})$$

with $k = m-1$ or m. Assume also that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (1.5.11) and $\Gamma\tilde{u} = \tilde{u}$. Then there exists a positive constant δ such that if

$$\|\tilde{w}\|_{C([0,T];H^m)\cap L^2(0,T;H^{m+1})} \le \delta,$$

the following assertion holds true.

The function

$$u_\infty(t) := S_{\infty,\tilde{u}}(t)(I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)[F_\infty] + \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \qquad (1.5.14)$$

is a solution of (1.3.2) in $\mathscr{Y}^k_\infty(0,T)$ satisfying $u_\infty(0) = u_\infty(T)$ and the estimate

$$\|u_\infty\|_{\mathscr{Y}^k_\infty(0,T)} \le C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}.$$

Proof. By Proposition 1.5.8 and Proposition 1.5.6, we see that

$$\begin{aligned}
&\|u_\infty(t)\|_{H_1^k}^2 + \|w_\infty\|_{L^2(0,t;H_1^{k+1})}^2 \\
&\le C\Big\{\|(I - S_{\infty,\tilde u}(T))^{-1}\mathscr{S}_{\infty,\tilde u}(T)[F_\infty]\|_{H_1^k}^2 + \|F_\infty\|_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}^2 \\
&\quad + \int_0^T (\|\nabla\tilde w\|_{H^m} + \|\tilde w\|_{H^m}^2)\|\phi_\infty\|_{H_1^k}^2\, ds\Big\} \\
&\le C\left\{\|F_\infty\|_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}^2 + \delta\|\phi_\infty\|_{C([0,T];H_1^k)}^2\right\}
\end{aligned}$$

for $t \in [0,T]$. Therefore, if δ is so small that $C\delta \le \frac{1}{2}$, then we obtain

$$\|u_\infty\|_{C([0,T];H_1^k)}^2 + \|w_\infty\|_{L^2(0,T;H_1^{k+1})}^2 \le C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}^2. \tag{1.5.15}$$

Next, since $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfies (1.3.2), we obtain

$$\|\partial_t w_\infty\|_{H^{k-1}_{(\infty),1}} \le C\{\|w_\infty\|_{H^{k+1}_{(\infty),1}} + \|\phi_\infty\|_{H^k_{(\infty),1}} + \|\tilde F_\infty\|_{H^{k-1}_{(\infty),1}}\}.$$

Hence, it follows from (1.5.15) that

$$\|\partial_t w_\infty\|_{L^2(0,T;H^{k-1}_{(\infty),1})} \le C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}. \tag{1.5.16}$$

Consequently, we see from (1.5.15) and (1.5.16) that

$$\|u_\infty\|_{\mathscr{Y}^k_\infty(0,T)} \le C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),1}\times H^{k-1}_{(\infty),1})}.$$

This completes the proof. □

1.6 Weighted energy estimates for P_∞ part

In this section we prove Proposition 1.5.8 by a weighted energy method.

We first consider the following equation.

$$\begin{cases} \partial_t u_\infty + Au_\infty + B[\tilde u]u_\infty = F_\infty, \\ u|_{t=0} = u_{0\infty}, \end{cases} \tag{1.6.1}$$

where

$$F_\infty = \begin{pmatrix} F^0_\infty \\ \tilde F_\infty \end{pmatrix}, B[\tilde u]u = \begin{pmatrix} \gamma\tilde w\cdot\nabla\phi \\ 0 \end{pmatrix}, u = \begin{pmatrix} \phi \\ w \end{pmatrix}, \tilde u = \begin{pmatrix} \tilde\phi \\ \tilde w \end{pmatrix}.$$

We introduce some notations. For nonnegative integers k and ℓ, we define $E^k_\ell[u_\infty]$ by

$$E^k_\ell[u_\infty] = \kappa(|\phi_\infty|_{H^k_\ell}^2 + |w_\infty|_{H^k_\ell}^2) + \sum_{|\alpha|\le k-1}(\partial_x^\alpha w_\infty, |x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty).$$

Here κ is a positive constant to be determined later.

Note that there exists a constant $\kappa_0 > 0$ such that if $\kappa \geq \kappa_0$, then $E^k_\ell[u_\infty]$ is equivalent to $|u_\infty|^2_{H^k_\ell}$, i.e.,

$$C^{-1}|u_\infty|^2_{H^k_\ell} \leq E^k_\ell[u_\infty] \leq C|u_\infty|^2_{H^k_\ell}$$

for some constant $C > 0$.

We also define $D^k_\ell[u_\infty]$ for integers $k \geq 1$ and $\ell \geq 0$ by

$$D^k_\ell[u_\infty] = |\nabla\phi_\infty|^2_{H^{k-1}_\ell} + |\nabla w_\infty|^2_{H^k_\ell}.$$

Proposition 1.6.1. *Let m be a nonnegative integer satisfying $m \geq [\frac{n}{2}]+1$ and let ℓ be a nonnegative integer. Assume that*

$$\begin{aligned} &u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k, \\ &F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T';H^k \times H^{k-1}) \end{aligned}$$

for $k = m-1$ or $k = m$. Here T' is a given positive number. Assume also that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ is the solution of (1.6.1) with $\tilde{w} \in C([0,T'];H^m) \cap L^2(0,T';H^{m+1})$ and that $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfies

$$\phi_\infty \in C([0,T'];H^k),\ w_\infty \in C([0,T'];H^k) \cap L^2(0,T';H^{k+1}).$$

Then there exist positive constants $\kappa \geq \kappa_0$ and $d > 0$ such that the estimate

$$\begin{aligned} &\frac{d}{dt}E^k_\ell[\zeta_R u_\infty] + dD^k_\ell[\zeta_R u_\infty] \\ &\leq\ C\Big\{\epsilon|\zeta_R u_\infty|^2_{L^2_\ell} + \Big(\big(1+\frac{\ell^2}{\epsilon}\big)\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}\Big)\|\zeta_R\phi_\infty\|^2_{H^k_\ell} \\ &\quad + \Big(1+\frac{1}{\epsilon}\Big)|\zeta_R F_\infty|^2_{H^k_\ell \times H^{k-1}_\ell} \\ &\quad + \ell^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|^2_{H^m}\right)|\zeta_R u_\infty|^2_{H^k_{\ell-1}} \\ &\quad + \Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|^2_{H^m(N_R)}\right)|u_\infty|^2_{H^k_{\ell-1}(N_R)\times H^{k+1}_{\ell-1}(N_R)}\Big\} \end{aligned} \tag{1.6.2}$$

holds on $(0,T')$, where ϵ is any positive number; C is a positive constant independent of T', ϵ and $R \geq 1$; and N_R denotes the set $N_R = \{x \in \mathbb{R}^n; R \leq |x| \leq 2R\}$.

Proof. By multiplying ζ_R to (1.6.1), we obtain

$$\left\{\begin{aligned} &\partial_t(\zeta_R\phi_\infty) + \gamma\tilde{w}\cdot\nabla(\zeta_R\phi_\infty) + \gamma\mathrm{div}(\zeta_R w_\infty) = \zeta_R F^0_\infty + K_1(\nabla\zeta_R), \\ &\partial_t(\zeta_R w_\infty) - \nu\triangle(\zeta_R w_\infty) - \tilde{\nu}\nabla\mathrm{div}(\zeta_R w_\infty) + \gamma\nabla(\zeta_R\phi_\infty) = \zeta_R\tilde{F}_\infty + K_2(\nabla\zeta_R), \end{aligned}\right. \tag{1.6.3}$$

where

$$\begin{aligned} K_1(\nabla\zeta_R) &= \gamma(w_\infty\cdot\nabla\zeta_R+\tilde{w}\cdot\nabla\zeta_R\phi_\infty), \\ K_2(\nabla\zeta_R) &= -\nu([\zeta_R,\triangle]w_\infty)-\tilde{\nu}([\zeta_R,\nabla\mathrm{div}]w_\infty)+\gamma\nabla\zeta_R\phi_\infty. \end{aligned}$$

For a multi-index α satisfying $|\alpha|\le k$, we take the inner product of $\partial_x^\alpha(1.6.3)_1$ with $|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)$ to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\||x|^\ell\partial_x^\alpha(\zeta_R\phi_\infty)\|_{L^2}^2+\gamma(\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)) \\ &\quad=\sum_{j=1}^{2}I_{\alpha,\ell,R}^{(j)}+\mathscr{P}_{\alpha,\ell}^{(1)}[\zeta_R u_\infty]+Q_{1,\alpha,\ell}(\nabla\zeta_R), \end{aligned}\tag{1.6.4}$$

where

$$I_{\alpha,\ell,R}^{(1)} = -\gamma\left\{\frac{1}{2}(\mathrm{div}\tilde{w},|x|^{2\ell}|\partial_x^\alpha(\zeta_R\phi_\infty)|^2)+([\partial_x^\alpha,\tilde{w}]\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))\right\},$$

$$I_{\alpha,\ell,R}^{(2)} = (\partial_x^\alpha(\zeta_R F_\infty^0),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)),$$

$$\mathscr{P}_{\alpha,\ell}^{(1)}[\zeta_R u_\infty]=\frac{\gamma}{2}(\tilde{w}\cdot\nabla(|x|^{2\ell}),|\partial_x^\alpha(\zeta_R\phi_\infty)|^2),$$

$$Q_{1,\alpha,\ell}(\nabla\zeta_R)=(\partial_x^\alpha K_1(\nabla\zeta_R),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)).$$

Here we used

$$\begin{aligned} &(\partial_x^\alpha(\gamma\tilde{w}\cdot\nabla(\zeta_R\phi_\infty)),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)) \\ &= \gamma(\tilde{w}\cdot\nabla\partial_x^\alpha(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty))+\gamma([\partial_x^\alpha,\tilde{w}]\cdot\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)) \\ &= \frac{1}{2}\gamma(|x|^{2\ell}\tilde{w},\nabla|\partial_x^\alpha(\zeta_R\phi_\infty)|^2)+\gamma([\partial_x^\alpha,\tilde{w}]\cdot\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)) \\ &= -\frac{1}{2}\gamma(|x|^{2\ell}\mathrm{div}\tilde{w},|\partial_x^\alpha(\zeta_R\phi_\infty)|^2)-\frac{1}{2}\gamma(\tilde{w}\cdot\nabla(|x|^{2\ell}),|\partial_x^\alpha(\zeta_R\phi_\infty)|^2) \\ &\quad+\gamma([\partial_x^\alpha,\tilde{w}]\cdot\nabla(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)) \\ &= I_{\alpha,\ell,R}^{(1)}+\mathscr{P}_{\alpha,R}^{(1)}(\nabla(|x|^{2\ell})). \end{aligned}$$

This calculation can be justified by using the standard Friedrichs commutator argument.

We take the inner product of $\partial_x^\alpha(1.6.3)_2$ with $|x|^{2\ell}\partial_x^\alpha(\zeta_R w_\infty)$ and integrate by parts to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\||x|^\ell\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2+\nu\||x|^\ell\nabla\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2+\tilde{\nu}\||x|^\ell\mathrm{div}\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2 \\ &\quad-\gamma(\partial_x^\alpha(\zeta_R\phi_\infty),|x|^{2\ell}\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty)) \\ &\quad=I_{\alpha,\ell,R}^{(3)}+\mathscr{P}_{\alpha,\ell}^{(2)}[\zeta_R u_\infty]+Q_{2,\alpha,\ell}(\nabla\zeta_R), \end{aligned}\tag{1.6.5}$$

where

$$I^{(3)}_{\alpha,\ell,R} = \begin{cases} ((\zeta_R\tilde{F}_\infty), |x|^{2\ell}(\zeta_R w_\infty)) & (\alpha = 0), \\ -(\partial_x^{\alpha-1}(\zeta_R\tilde{F}_\infty), |x|^{2\ell}\partial_x^{\alpha+1}(\zeta_R w_\infty)) & (|\alpha| \geq 1), \end{cases}$$

$$\begin{aligned} \mathscr{P}^{(2)}_{\alpha,\ell}[\zeta_R u_\infty] &= (\nu\partial_x^\alpha\nabla(\zeta_R w_\infty) + \tilde{\nu}\partial_x^\alpha \mathrm{div}(\zeta_R w_\infty) + \gamma\partial_x^\alpha(\zeta_R\phi_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha(\zeta_R w_\infty)) \\ &\quad -(\partial_x^{\alpha-1}(\zeta_R\tilde{F}_\infty), \partial_x(|x|^{2\ell})\partial_x^\alpha(\zeta_R w_\infty)), \end{aligned}$$

$$Q_{2,\alpha,\ell}(\nabla\zeta_R) = (\partial_x^\alpha(K_2(\nabla\zeta_R)), |x|^{2\ell}\partial_x^\alpha(\zeta_R\phi_\infty)).$$

By adding (1.6.4) to (1.6.5), we see that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\||x|^\ell\partial_x^\alpha(\zeta_R\phi_\infty)\|_{L^2}^2 + \||x|^\ell\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2\right\} \\ &\quad +\nu\||x|^\ell\nabla\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2 + \tilde{\nu}\||x|^\ell \mathrm{div}\partial_x^\alpha(\zeta_R w_\infty)\|_{L^2}^2 \\ &= \sum_{j=1}^{3} I^{(j)}_{\alpha,\ell,R} + \mathscr{P}^{(1)}_{\alpha,\ell}[\zeta_R u_\infty] + \mathscr{P}^{(2)}_{\alpha,\ell}[\zeta_R u_\infty] + Q_{1,\alpha,\ell}(\nabla\zeta_R) + Q_{2,\alpha,\ell}(\nabla\zeta_R). \quad (1.6.6) \end{aligned}$$

By using Lemma 1.1.2 and Lemma 1.1.3, we obtain

$$\begin{aligned} |\sum_{|\alpha|\leq k}\sum_{j=1}^{3} I^{(j)}_{\alpha,\ell,R}| &\leq \epsilon|\zeta_R u_\infty|^2_{L^2_\ell} + \epsilon_1|\nabla(\zeta_R\phi_\infty)|^2_{H^{k-1}_\ell} + \epsilon_2|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell} \\ &\quad +C\|\nabla\tilde{w}\|_{H^m}\|\zeta_R\phi_\infty\|^2_{H^k_\ell} \\ &\quad +C\Big(\frac{1}{\epsilon} + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\Big)|\zeta_R F_\infty|^2_{H^k_\ell\times H^{k-1}_\ell}, \end{aligned}$$

$$\begin{aligned} |\sum_{|\alpha|\leq k}\sum_{j=1}^{2} \mathscr{P}^{(j)}_{\alpha,\ell}[\zeta_R u_\infty]| &\leq \epsilon|\zeta_R u_\infty|^2_{L^2_\ell} + \epsilon_1|\nabla(\zeta_R\phi_\infty)|^2_{H^{k-1}_\ell} + \epsilon_2|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell} \\ &\quad +C\ell^2\Big(1 + \frac{1}{\epsilon} + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\Big)|\zeta_R w_\infty|^2_{H^k_{\ell-1}} \\ &\quad +C\ell^2\Big(1 + \frac{1}{\epsilon} + \frac{1}{\epsilon_1}\Big)\|\tilde{w}\|^2_{H^m}|\zeta_R\phi_\infty|^2_{H^k_{\ell-1}} \\ &\quad +C\ell^2|\zeta_R F_\infty|^2_{H^k_\ell\times H^{k-1}_\ell}, \end{aligned}$$

$$\begin{aligned} |\sum_{|\alpha|\leq k}\sum_{j=1}^{2} Q_{j,\alpha,\ell}(\nabla\zeta_R)| &\leq \epsilon|\zeta_R u_\infty|^2_{L^2_\ell(N_R)} + \epsilon_1|\nabla(\zeta_R\phi_\infty)|^2_{H^{k-1}_\ell(N_R)} \\ &\quad +\epsilon_2|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell(N_R)} \\ &\quad +C\Big(1 + \frac{1}{\epsilon} + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\Big)\Big\{|w_\infty|^2_{H^{k+1}_{\ell-1}(N_R)} \\ &\quad +(1 + \|\tilde{w}\|^2_{H^m(N_R)})|\phi_\infty|^2_{H^k_{\ell-1}(N_R)}\Big\}^2. \end{aligned}$$

Taking $\epsilon_2 > 0$ suitably small, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\zeta_R u_\infty|^2_{H^k_\ell} &+ \frac{\nu}{2}|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell} + \frac{\tilde{\nu}}{2}|\mathrm{div}(\zeta_R w_\infty)|^2_{H^k_\ell} \\
\leq\ & \epsilon|\zeta_R u_\infty|^2_{L^2_\ell} + \epsilon_1|\nabla(\zeta_R \phi_\infty)|^2_{H^{k-1}_\ell} + C\Big(1 + \frac{1}{\epsilon} + \frac{1}{\epsilon_1}\Big)|\zeta_R F|^2_{H^k_\ell \times H^{k-1}_\ell} \\
&+ C\|\nabla\tilde{w}\|_{H^m}\|\zeta_R\phi_\infty\|^2_{H^k_\ell} \\
&+ C\ell^2\Big(1 + (\frac{1}{\epsilon} + \frac{1}{\epsilon_1})(1 + \|\tilde{w}\|^2_{H^m})\Big)|\zeta_R u_\infty|^2_{H^k_{\ell-1}} \\
&+ C\Big(1 + \frac{1}{\epsilon} + \frac{1}{\epsilon_1}\Big)(1 + \|\tilde{w}\|^2_{H^m})|u_\infty|^2_{H^k_{\ell-1}(N_R)\times H^{k+1}_{\ell-1}(N_R)}. \qquad (1.6.7)
\end{aligned}$$

We next estimate $\||x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty\|^2_{L^2}$ for α with $|\alpha| \leq k-1$. For a multi-index α satisfying $|\alpha| \leq k-1$, we take the inner product of $\partial_x^\alpha(1.6.3)_2$ with $|x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)$ to obtain

$$\begin{aligned}
&(\partial_t\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)) + \gamma|\nabla\partial_x^\alpha(\zeta_R\phi_\infty)|^2_{L^2_\ell} \\
&= \sum_{i=1}^{3} J^{(i)}_{\alpha,\ell,R} + \left(\partial_x^\alpha K_2(\nabla\zeta_R), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)\right), \qquad (1.6.8)
\end{aligned}$$

where

$$\begin{aligned}
J^{(1)}_{\alpha,\ell,R} &= (\nu(\partial_x^\alpha\triangle(\zeta_R w_\infty), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)), \\
J^{(2)}_{\alpha,\ell,R} &= (\tilde{\nu}(\partial_x^\alpha(\nabla\mathrm{div}(\zeta_R w_\infty)), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)), \\
J^{(3)}_{\alpha,\ell,R} &= (\partial_x^\alpha(\zeta_R\tilde{F}_\infty), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)).
\end{aligned}$$

As for the first term on the left-hand side, we have

$$\begin{aligned}
&(\partial_t\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty) \\
&= \frac{d}{dt}(\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty)) + (\partial_x^\alpha(\zeta_R w_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha\partial_t(\zeta_R\phi_\infty)) \\
&\quad + (\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha\partial_t(\zeta_R\phi_\infty)). \qquad (1.6.9)
\end{aligned}$$

By (1.6.3), we have

$$\partial_t(\zeta_R\phi_\infty) = -\gamma\tilde{w}\cdot\nabla(\zeta_R\phi_\infty) - \gamma\mathrm{div}(\zeta_R w_\infty) + \zeta_R F^0_\infty + K_1(\nabla\zeta_R).$$

Substituting this into (1.6.9), we obtain

$$\begin{aligned}
&(\partial_t\partial_x^\alpha(\zeta_R\phi_\infty), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R w_\infty)) \\
&= \frac{d}{dt}(\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty)) - \sum_{i=4}^{6} J^{(i)}_{\alpha,\ell,R} - \mathscr{P}^{(3)}_{\alpha,\ell}[\zeta_R u_\infty] \\
&\quad + (\partial_x^\alpha(\zeta_R w_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha K_1(\nabla\zeta_R)) + (\partial_x^\alpha\mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha K_1(\nabla\zeta_R)),
\end{aligned}$$

where

$$\begin{aligned} J^{(4)}_{\alpha,\ell,R} &= \gamma(\partial_x^\alpha \mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha(\tilde{w}\cdot\nabla(\zeta_R\phi_\infty))), \\ J^{(5)}_{\alpha,\ell,R} &= \gamma(\partial_x^\alpha \mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha(\mathrm{div}(\zeta_R w_\infty))), \\ J^{(6)}_{\alpha,\ell,R} &= -(\partial_x^\alpha \mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha(\zeta_R F^0_\infty)), \end{aligned}$$

and

$$\begin{aligned} \mathscr{P}^{(3)}_{\alpha,\ell}[\zeta_R u_\infty] &= \gamma(\partial_x^\alpha(\zeta_R w_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha(\tilde{w}\cdot\nabla(\zeta_R\phi_\infty))) \\ &\quad +\gamma(\partial_x^\alpha(\zeta_R w_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha \mathrm{div}(\zeta_R w_\infty)) \\ &\quad -(\partial_x^\alpha(\zeta_R w_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha(\zeta_R F^0_\infty)). \end{aligned}$$

This, together with (1.6.8), gives

$$\begin{aligned} &\frac{d}{dt}(\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty)) + \gamma|\nabla\partial_x^\alpha(\zeta_R\phi_\infty)|^2_{L^2_\ell} \\ &= \sum_{i=4}^{6} J^{(i)}_{\alpha,\ell,R} + \mathscr{P}^{(3)}_{\alpha,\ell}[\zeta_R u_\infty] + Q_{3,\alpha,\ell}(\nabla\zeta_R), \end{aligned} \tag{1.6.10}$$

where

$$\begin{aligned} Q_{3,\alpha,\ell} &= (\partial_x^\alpha K_2(\nabla\zeta_R), |x|^{2\ell}\nabla\partial_x^\alpha(\zeta_R\phi_\infty)) \\ &\quad -(\partial_x^\alpha(\zeta_R w_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha K_1(\nabla\zeta_R)) \\ &\quad -(\partial_x^\alpha \mathrm{div}(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha K_1(\nabla\zeta_R)). \end{aligned}$$

By Lemma 1.1.2 and Lemma 1.1.3, we obtain

$$\begin{aligned} \Big|\sum_{|\alpha|\le k-1}\sum_{i=1}^{6} J^{(i)}_{\alpha,\ell,R}\Big| &\le \frac{\gamma}{4}|\nabla\partial_x^\alpha(\zeta_R\phi_\infty)|^2_{L^2_\ell} + C\Big(\gamma+\frac{1}{\gamma}\Big)|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell} \\ &\quad +\gamma\|\tilde{w}\|^2_{H^m}\|\nabla(\zeta_R\phi_\infty)\|^2_{H^{k-1}_\ell} + \frac{C}{\gamma}|\zeta_R F_\infty|^2_{H^{k-1}_\ell}, \end{aligned}$$

$$\begin{aligned} \Big|\sum_{|\alpha|\le k-1}\mathscr{P}^{(3)}_{\alpha,\ell}[\zeta_R u_\infty]\Big| &\le \tilde{\epsilon}\ell|\zeta_R w_\infty|^2_{L^2_\ell} + \tilde{\epsilon}\ell|\nabla(\zeta_R w_\infty)|^2_{H^{k-1}_\ell} \\ &\quad +\frac{C\ell}{\tilde{\epsilon}}\|\tilde{w}\|^2_{H^m}\|\nabla(\zeta_R\phi_\infty)\|^2_{H^{k-1}_{\ell-1}} \\ &\quad +C\ell\Big(1+\frac{1}{\tilde{\epsilon}}\Big)|\zeta_R w_\infty|^2_{H^{k-1}_{\ell-1}} + C\ell|\zeta_R F^0_\infty|^2_{H^{k-1}_{\ell-1}}, \end{aligned}$$

$$\Big|\sum_{|\alpha|\le k-1} Q_{3,\alpha,\ell}(\nabla\zeta_R)\Big| \le \frac{\gamma}{4}|\nabla(\zeta_R\phi_\infty)|^2_{H^{k-1}_\ell(N_R)} + \tilde{\epsilon}\ell|\zeta_R w_\infty|^2_{L^2_\ell(N_R)}$$

$$+C|\nabla(\zeta_R w_\infty)|^2_{H^{k-1}_\ell(N_R)}$$

$$+C\Big(\frac{1}{\gamma}+\frac{1}{\tilde{\epsilon}}\Big)(1+\|\tilde{w}\|^2_{H^m(N_R)})|u_\infty|^2_{H^k_{\ell-1}(N_R)}$$

for any $\tilde{\epsilon} > 0$ with $C > 0$ independent of $\tilde{\epsilon}$.

Combining these estimates with (1.6.8) and (1.6.10), we see that

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le k-1}(\partial_x^\alpha(\zeta_R w_\infty), |x|^{2\ell}\partial_x^\alpha\nabla(\zeta_R\phi_\infty)) + \frac{\gamma}{2}|\nabla(\zeta_R\phi_\infty)|^2_{H^{k-1}_\ell} \\
&\le \tilde{\epsilon}\ell|\zeta_R w_\infty|^2_{L^2_\ell} \\
&\quad +C\Big\{|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell} + \Big(1+\frac{1}{\tilde{\epsilon}}\Big)\|\tilde{w}\|^2_{H^m}\|\nabla(\zeta_R\phi_\infty)\|^2_{H^{k-1}_\ell} + |\zeta_R F_\infty|^2_{H^{k-1}_\ell}\Big\} \\
&\quad +C\Big(1+\frac{1}{\tilde{\epsilon}}\Big)\Big\{\ell|\zeta_R w_\infty|^2_{H^{k-1}_{\ell-1}} + (1+\|\tilde{w}\|^2_{H^m(N_R)})|u_\infty|^2_{H^k_{\ell-1}(N_R)}\Big\} \qquad (1.6.11)
\end{aligned}$$

for any $\tilde{\epsilon} > 0$ with $C > 0$ independent of $\tilde{\epsilon}$.

Consider now $\kappa \times (1.6.7) + (1.6.11)$ with a constant $\kappa > 0$. Taking $\kappa > 0$ so large that $|\nabla(\zeta_R w_\infty)|^2_{H^k_\ell}$ on the right-hand side is absorbed into the left-hand side and setting $\epsilon_1 = \frac{\gamma}{4\kappa}$ and $\tilde{\epsilon} = \ell^{-1}\epsilon$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}E^k_\ell[\zeta_R u_\infty](t) + dD^k_\ell[\zeta_R u_\infty] \\
&\le \epsilon|\zeta_R u_\infty|^2_{L^2_\ell} + C\Big(\Big(1+\frac{\ell^2}{\epsilon}\Big)\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}\Big)\|\zeta_R\phi_\infty\|^2_{H^k_\ell} \\
&\quad +C\Big(1+\frac{1}{\epsilon}\Big)|\zeta_R F_\infty|^2_{H^k_\ell\times H^{k-1}_\ell} \\
&\quad +C\ell^2\Big(1+\frac{1}{\epsilon}\Big)(1+\|\tilde{w}\|^2_{H^m})|\zeta_R u_\infty|^2_{H^k_{\ell-1}} \\
&\quad +C\Big(1+\frac{1}{\epsilon}\Big)(1+\|\tilde{w}\|^2_{H^m(N_R)})|u_\infty|^2_{H^k_{\ell-1}(N_R)\times H^{k+1}_{\ell-1}(N_R)}
\end{aligned}$$

for any$\epsilon > 0$ with $C > 0$ independent of ϵ. This completes the proof. □

Remark 1.6.2. Similarly to the proof of Proposition 1.5.4, one can prove that if

$$\begin{aligned}
&\tilde{w} \in C([0,T']; H^m) \cap L^2(0,T'; H^{m+1}), \\
&u_{0\infty} \in H^k, \\
&F_\infty \in L^2(0,T'; H^k \times H^{k-1}),
\end{aligned}$$

then there exists a unique solution $u_\infty = {}^\top(\phi_\infty, w_\infty)$ of (1.6.1) in $C([0,T']; H^k) \cap L^2(0,T'; H^k \times H^{k+1})$. Furthermore, by setting $\ell = 0$ and $\zeta_R \equiv 1$ in the proof of Proposition 1.6.1, one can see that $E^k_0[u_\infty](t)$ is absolutely continuous in $t \in [0,T']$ and there holds the estimate

$$\frac{d}{dt}E^k_0[u_\infty] + dD^k_0[u_\infty] \le C\Big\{\epsilon\|u_\infty\|^2_2 + (\|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m})\|\nabla\phi_\infty\|^2_{H^{k-1}}$$

$$+\Big(1+\frac{1}{\epsilon}\Big)\|F_\infty\|^2_{H^k\times H^{k-1}}\Big\} \tag{1.6.12}$$

on $(0,T')$, where ϵ is any positive number; and C is a positive constant independent of T' and ϵ.

Remark 1.6.3. One can easily see that (1.6.2) holds with ζ_R and N_R replaced by $\zeta_R-\zeta_{R'}$ and $N_{R,R'}$ for $R'>R\geq 1$, where $N_{R,R'}$ denotes the set $N_{R,R'}=\{x\in\mathbb{R}^n; R\leq|x|\leq 2R'\}$.

Proposition 1.6.4. *Let m be a nonnegative integer satisfying $m\geq[\frac{n}{2}]+1$ and let ℓ be an integer satisfying $\ell\geq 1$. Assume that*

$$\begin{aligned}
&u_{0\infty}={}^\top(\phi_{0\infty},w_{0\infty})\in H^k_\ell,\\
&F_\infty={}^\top(F^0_\infty,\tilde{F}_\infty)\in L^2(0,T';H^k_\ell\times H^{k-1}_\ell)
\end{aligned}$$

for $k=m-1$ or $k=m$. Here T' is a given positive number. Assume also that $u_\infty={}^\top(\phi_\infty,w_\infty)$ is the solution of (1.6.1) with $\tilde{w}\in C([0,T'];H^m)\cap L^2(0,T';H^{m+1})$ and that $u_\infty={}^\top(\phi_\infty,w_\infty)$ satisfies

$$\phi_\infty\in C([0,T'];H^k),\ w_\infty\in C([0,T'];H^k)\cap L^2(0,T';H^{k+1}).$$

Then it holds that

$$\phi_\infty\in C([0,T'];H^k_\ell),\ \ w_\infty\in C([0,T'];H^k_\ell)\cap L^2(0,T';H^{k+1}_\ell).$$

Furthermore, there exist positive constants $\kappa\geq\kappa_0$ and $d>0$ such that $E^k_\ell[u_\infty](t)$ is absolutely continuous in $t\in[0,T']$ and there holds the estimate

$$\begin{aligned}
&\frac{d}{dt}E^k_\ell[u_\infty]+dD^k_\ell[u_\infty]\\
&\leq\ C\Big\{\epsilon|u_\infty|^2_{L^2_\ell}+\Big(\big(1+\frac{\ell^2}{\epsilon}\big)\|\tilde{w}\|^2_{H^m}+\|\nabla\tilde{w}\|_{H^m}\Big)\|\phi_\infty\|^2_{H^k_\ell}\\
&\qquad+\Big(1+\frac{1}{\epsilon}\Big)|F_\infty|^2_{H^k_\ell\times H^{k-1}_\ell}\\
&\qquad+\ell^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|^2_{H^m}\right)|u_\infty|^2_{H^k_{\ell-1}}\Big\}
\end{aligned} \tag{1.6.13}$$

on $(0,T')$, where ϵ is any positive number; C is a positive constant independent of T' and ϵ.

Proof. It suffices to prove that

$$\zeta_R u_\infty\to u_\infty\text{ in }C([0,T'];H^k_\ell)\cap L^2(0,T';H^k_\ell\times H^{k+1}_\ell)$$

as $R\to\infty$. We prove this by induction on ℓ.

We first observe that it holds that

$$\zeta_R u_\infty \to u_\infty \text{ in } C([0,T'];H^k)\cap L^2(0,T';H^k\times H^{k+1}) \tag{1.6.14}$$

as $R\to\infty$, since $u_\infty \in C([0,T'];H^k)\cap L^2(0,T';H^k\times H^{k+1})$. We also note that since $\operatorname{supp}(\zeta_R-\zeta_{R'})\subset N_{R,R'}=\{x\in\mathbb{R}^n; R\le |x|\le 2R'\}$ for $R'>R$, it holds that

$$\|\zeta_R u_\infty - \zeta_{R'}u_\infty\|_{H^k_\ell} \le C\|u_\infty\|_{H^k_\ell(N_{R,R'})}$$

for $R'>R\ge 1$.

Set

$$\begin{aligned}
\varphi_{\ell,R,R'}(t) &= |\zeta_R u_\infty(t)-\zeta_{R'}u_\infty(t)|^2_{H^k_\ell},\\
b(t) &= 1+\|\tilde{w}(t)\|^2_{H^m}+\|\nabla\tilde{w}(t)\|_{H^m} \in L^1(0,T'),\\
a_{\ell,R,R'}(t) &= |\zeta_R u_{0\infty}-\zeta_{R'}u_{0\infty}|^2_{H^k_\ell}+\int_0^t |\zeta_R F_\infty-\zeta_{R'}F_\infty|^2_{H^k_\ell\times H^{k-1}_\ell}\,d\tau\\
&\quad +\int_0^t (1+\|\tilde{w}(t)\|^2_{H^m})\|u_\infty\|^2_{H^k_{\ell-1}(N_{R,R'})\times H^{k+1}_{\ell-1}(N_{R,R'})}\,d\tau.
\end{aligned}$$

Let us prove Proposition 1.6.4 for $\ell=1$. By (1.6.2), we have

$$\begin{aligned}
&\varphi_{1,R,R'}(t)+\int_0^t D^k_1[\zeta_R u_\infty-\zeta_{R'}u_\infty]\,d\tau\\
&\quad\le C\Big\{a_{1,R,R'}(T')+\int_0^t b(\tau)\varphi_{1,R,R'}(\tau)\,d\tau\Big\}
\end{aligned} \tag{1.6.15}$$

for $t\in[0,T']$, where C is a constant depending on ϵ. By the Gronwall inequality, we obtain

$$\varphi_{1,R,R'}(t)\le C a_{1,R,R'}(T')e^{C\int_0^{T'} b(\tau)\,d\tau} \tag{1.6.16}$$

for $t\in[0,T']$. Since $a_{1,R,R'}(T')\to 0$ as $R,R'\to\infty$, we see that $\sup_{0\le t\le T'}\varphi_{1,R,R'}(t)\to 0$ as $R,R'\to\infty$. This, together with (1.6.15), yields that $\int_0^{T'} D^k_1[\zeta_R u_\infty-\zeta_{R'}u_\infty]\,d\tau\to 0$ as $R,R'\to\infty$. In view of (1.6.14), we thus conclude that $\{\zeta_R u_\infty\}$ is Cauchy in $C([0,T'];H^k_1)\cap L^2(0,T';H^k_1\times H^{k+1}_1)$ and

$$\zeta_R u_\infty\to u_\infty \text{ in } C([0,T'];H^k_1)\cap L^2(0,T';H^k_1\times H^{k+1}_1)$$

as $R\to\infty$. Letting $R\to\infty$ in (1.6.2) with $\ell=1$, we have the desired estimate in Proposition 1.6.4 with $\ell=1$. Proposition 1.6.4 thus holds for $\ell=1$.

We next suppose that Proposition 1.6.4 holds for $\ell=p$. We will prove that it also holds for $\ell=p+1$. By (1.6.2) and Remark 1.6.3, we have

$$\begin{aligned}
&\varphi_{p+1,R,R'}(t)+\int_0^t D^k_{p+1}[\zeta_R u_\infty-\zeta_{R'}u_\infty]\,d\tau\\
&\quad\le C\Big\{a_{p+1,R,R'}(T')+\int_0^t b(\tau)\varphi_{p+1,R,R'}(\tau)\,d\tau\Big\}
\end{aligned} \tag{1.6.17}$$

for $t \in [0, T']$, where C is a constant depending on ϵ and p. By the Gronwall inequality, we obtain

$$\varphi_{p+1,R,R'}(t) \le C a_{p+1,R,R'}(T') e^{C\int_0^{T'} b(\tau)\,d\tau} \tag{1.6.18}$$

for $t \in [0, T']$. By the induction assumption, we see that $a_{p+1,R,R'}(T') \to 0$ as $R, R' \to \infty$, and hence, by (1.6.18), $\sup_{0\le t\le T'} \varphi_{p+1,R,R'}(t) \to 0$ as $R, R' \to \infty$. This, together with (1.6.17), yields that $\int_0^{T'} D^k_{p+1}[\zeta_R u_\infty - \zeta_{R'} u_\infty]\,d\tau \to 0$ as $R, R' \to \infty$. It then follows that $\{\zeta_R u_\infty\}$ is Cauchy in $C([0,T']; H^k_{p+1}) \cap L^2(0,T'; H^k_{p+1} \times H^{k+1}_{p+1})$ and

$$\zeta_R u_\infty \to u_\infty \text{ in } C([0,T']; H^k_{p+1}) \cap L^2(0,T'; H^k_{p+1} \times H^{k+1}_{p+1})$$

as $R \to \infty$. It is not difficult to see that $\frac{d}{dt} E^k_\ell[u_\infty] = G^k_\ell$ on $(0, T')$ for some $G^k_\ell \in L^1(0, T')$, and, thus, $E^k_\ell[u_\infty](t)$ is absolutely continuous in $t \in [0, T']$. Letting $R \to \infty$ in (1.6.2) with $\ell = p+1$, we have the desired estimate in Proposition 1.6.4 with $\ell = p+1$. Proposition 1.6.4 thus holds for $\ell = p+1$. This completes the proof. □

We are now in a position to prove Proposition 1.5.8.

Proof of Proposition 1.5.8. Let $U = {}^\top(\Phi, W) \in C([0,T']; H^k_\ell) \cap L^2(0,T'; H^k_\ell \times H^{k+1}_\ell)$. Then, by Lemma 1.3.3, we see that

$$\|P_1 B[\tilde{u}]U\|_{H^k_\ell} \le C\|\tilde{w}\|_\infty \|\nabla\Phi\|_{L^2_\ell} \le C\delta \|U\|_{H^k_\ell}.$$

It then follows from Remark 1.6.3 and Proposition 1.6.4 that there exists a unique solution $U_\infty \in C([0,T']; H^k_\ell \times H^k_\ell) \cap L^2(0,T'; H^k_\ell \times H^{k+1}_\ell)$ of

$$\partial_t U_\infty + AU_\infty + B[\tilde{u}]U_\infty = F_\infty + P_1 B[\tilde{u}]U, \quad U_\infty|_{t=0} = u_{0\infty}, \tag{1.6.19}$$

and U_∞ satisfies

$$\begin{aligned} &\|U_\infty(t)\|_{H^k_\ell} + \int_0^t \|\nabla U_\infty\|^2_{H^{k-1}{}_\ell \times H^k_\ell}\,d\tau \\ &\le C_0\Big\{\|u_{0\infty}\|^2_{H^k_\ell} + \int_0^t \|F_\infty\|^2_{H^k_\ell \times H^{k-1}_\ell}\,d\tau \\ &\quad +\delta^2 \int_0^t \|U\|^2_{H^k_\ell}\,d\tau + \int_0^t b(\tau)\|U_\infty\|^2_{H^k_{\ell-1}}\,d\tau\Big\}. \end{aligned} \tag{1.6.20}$$

Here $b(\tau) = 1 + \|\tilde{w}\|^2_{H^m} + \|\nabla\tilde{w}\|_{H^m}$.

We set $U^{(0)}_\infty = 0$ and define $U^{(j)}_\infty$ $(j = 1, 2, \cdots)$ inductively by the solution of (1.6.19) with $U = U^{(j-1)}_\infty$. Applying the Gronwall inequality to (1.6.20) with $U_\infty = U^{(1)}_\infty$ and $U = 0$, we have

$$\|U^{(1)}_\infty(t)\| \le A_0$$

for $t \in [0, T']$, where

$$A_0 = C_0\Big\{\|u_{0\infty}\|^2_{H^k_\ell} + \int_0^{T'} \|F_\infty\|^2_{H^k_\ell \times H^{k-1}_\ell}\,d\tau\Big\} e^{C_0\|b\|_{L^1(0,T')}}.$$

Similarly, using (1.6.20) with $U_\infty = U_\infty^{(j)} - U_\infty^{(j-1)}$ and $U = U_\infty^{(j-1)} - U_\infty^{(j-2)}$ for $j = 2, 3, \cdots$, one can inductively see that

$$\|U_\infty^{(j)}(t) - U_\infty^{(j-1)}(t)\|_{H_\ell^k}^2 \le \frac{A_0(C_0K_0\delta^2 t)^{j-1}}{(j-1)!},$$

$$\int_0^t \|\nabla(U_\infty^{(j)} - U_\infty^{(j-1)})\|_{H_\ell^{k-1}\times H_\ell^k}^2 \, d\tau \le \frac{A_0}{K_0}\Big\{\frac{(C_0K_0\delta^2 t)^{j-1}}{(j-1)!} + \frac{\|b\|_{L^1(0,T')}}{\delta^2}\frac{(C_0K_0\delta^2 t)^j}{j!}\Big\}.$$

Here $K_0 = 1 + \|b\|_{L^1(0,T')}e^{C_0\|b\|_{L^1(0,T')}}$. It then follows that $U_\infty^{(j)}$ converges to a function U_∞ in $C([0,T']; H_\ell^k) \cap L^2(0,T'; H_\ell^k \times H_\ell^{k+1})$ as $j \to \infty$. One can easily see that U_∞ satisfies (1.6.19) with $U = U_\infty$, i.e., U_∞ is a solution of (1.3.2), and $U_\infty(t) \in H_{(\infty)}^k$ for all $t \in [0, T']$. By the uniqueness of solutions of (1.3.2) (see Proposition 1.5.4), we see that $U_\infty = u_\infty$.

Applying Remark 1.6.2 and Proposition 1.6.4 with F_∞ replaced by $F_\infty + P_1 B[\tilde{u}]u_\infty$, we have

$$\begin{aligned}
&\frac{d}{dt}E_j^k[u_\infty] + dD_j^k[u_\infty] \\
&\le C\Big\{\epsilon|u_\infty|_{L_j^2}^2 + \Big(\big(1+\frac{1}{\epsilon}\big)\|\tilde{w}\|_{H^m}^2 + \|\nabla\tilde{w}\|_{H^m}\Big)\|\phi_\infty\|_{H_j^k}^2 \\
&\quad + \Big(1+\frac{1}{\epsilon}\Big)|F_\infty|_{H_j^k\times H_j^{k-1}}^2 \\
&\quad + j^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|_{H^m}^2\right)|u_\infty|_{H_{j-1}^k}^2\Big\}
\end{aligned}$$

for $j = 0, 1, \cdots, \ell$. Using Lemma 1.3.4 (ii) and Lemma 1.3.6, we see that

$$\begin{aligned}
&\frac{d}{dt}E_j^k[u_\infty] + 2d_1|u_\infty|_{H_j^k\times H_j^{k+1}}^2 \\
&\le C\Big\{\epsilon|u_\infty|_{L_j^2}^2 + \Big(\big(1+\frac{1}{\epsilon}\big)\|\tilde{w}\|_{H^m}^2 + \|\nabla\tilde{w}\|_{H^m}\Big)\|\phi_\infty\|_{H_j^k}^2 \\
&\quad + \Big(1+\frac{1}{\epsilon}\Big)|F_\infty|_{H_j^k\times H_j^{k-1}}^2 + j^2\Big(1+\frac{1}{\epsilon}\Big)\left(1+\|\tilde{w}\|_{H^m}^2\right)|u_\infty|_{H_{j-1}^k}^2\Big\}
\end{aligned}$$

for $j = 0, 1, \cdots, \ell$, with some constant $d_1 > 0$. Taking $\epsilon > 0$ suitably small, we obtain

$$\begin{aligned}
&\frac{d}{dt}E_j^k[u_\infty] + d_1|u_\infty|_{H_j^k\times H_j^{k+1}}^2 \\
&\le C\Big\{\Big(\|\tilde{w}\|_{H^m}^2 + \|\nabla\tilde{w}\|_{H^m}\Big)\|\phi_\infty\|_{H_j^k}^2 + |F_\infty|_{H_j^k\times H_j^{k-1}}^2 \\
&\quad + j^2\left(1+\|\tilde{w}\|_{H^m}^2\right)|u_\infty|_{H_{j-1}^k}^2\Big\} \qquad (1.6.21)
\end{aligned}$$

for $j = 0, 1, \cdots, \ell$.

We now prove (1.5.12) by induction on ℓ. When $\ell = 0$, inequality (1.6.21) with $j = 0$ is nothing but (1.5.12) with $\ell = 0$. Assume that (1.5.12) holds for $\ell = j - 1$. Then by adding $\frac{d}{2Cj^2(1+\delta^2)}\times$(1.6.21) to (1.5.12) with $\ell = j - 1$, we obtain the desired inequality (1.5.12) for $\ell = j$. This completes the proof. □

1.7 Proof of Theorem 1.2.1

In this section we prove Theorem 1.2.1.

We first establish the estimates for nonlinear and inhomogeneous terms $F_1(u,g)$ and $F_\infty(u,g)$:

$$F_1(u,g) = P_1 \begin{pmatrix} -\gamma w \cdot \nabla \phi + f^0(u) \\ \tilde{f}(u,g) \end{pmatrix} =: \begin{pmatrix} F_1^0(u) \\ \tilde{F}_1(u,g) \end{pmatrix},$$

$$F_\infty(u,g) = P_\infty \begin{pmatrix} -\gamma w \cdot \nabla \phi_1 + f^0(u) \\ \tilde{f}(u,g) \end{pmatrix} =: \begin{pmatrix} F_\infty^0(u) \\ \tilde{F}_\infty(u,g) \end{pmatrix},$$

where $f^0(u)$ and $\tilde{f}(u,g)$ are the same ones defined in (0.0.19) and (0.0.20) with $u = u_1 + u_\infty$, $u = {}^\top(\phi, w)$, $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, \infty)$.

We first state the estimates for $F_1(u,g)$ and $F_\infty(u,g)$.

Proposition 1.7.1. *There hold the estimates*

(i) $$\|F_1^0(u)\|_{L^1_1} \le C(\|\phi\|_{L^2}\|\mathrm{div} w\|_{L^2_1} + \|w\|_{L^2}\|\nabla\phi\|_{L^2_1}),$$

(ii) $$\|\tilde{F}_1(u,g)\|_{L^1_1} \le C(\|\phi\|_{L^2}\|\partial_t w\|_{L^2_1} + \|w\|_{L^2}\|\nabla w\|_{L^2_1} + \|\phi\|_{L^2}\|\nabla\phi\|_{L^2_1} + \|\phi\|_{L^2}\|g\|_{L^2_1} + \|g\|_{L^1_1}),$$

(iii) $$\|F_\infty^0(u)\|_{H^m_1} \le C(\|\phi\|_{H^m}\|\mathrm{div} w\|_{H^m_1} + \|w\|_{H^m}\|\nabla\phi_1\|_{H^m_1}),$$

(iv) $$\|\tilde{F}_\infty(u,g))\|_{H^{m-1}_1} \le C\{\|w\|_{H^m}\|\nabla w\|_{H^{m-1}_1} + \|\phi\|_{H^m}\|\nabla\phi\|_{H^{m-1}_1} + \|\phi\|_{H^m}\|\partial_t w\|_{H^{m-1}_1} + (1 + \|\phi\|_{H^m})\|g\|_{H^{m-1}_1}\}$$

uniformly for $u = {}^\top(\phi, w) = u_1 + u_\infty$ *with* $u_k = {}^\top(\phi_k, w_k)$ $(k = 1, \infty)$ *satisfying* $\|\phi\|_{L^\infty} \le \frac{1}{2}$ *and* $\|u\|_{H^m} \le 1$.

Proposition 1.7.1 directly follows from Lemmas 1.1.1 and 1.1.3.

We next estimate $F_j(u^{(1)}, g) - F_j(u^{(2)}, g)$ $(j = 1, \infty)$.

Proposition 1.7.2. *There hold the estimates*

(i) $$\begin{aligned}&\|F_1^0(u^{(1)}) - F_1^0(u^{(2)})\|_{L^1_1} \\ &\quad \le C\{\|\phi^{(1)} - \phi^{(2)}\|_{L^2}\|\mathrm{div} w^{(1)}\|_{L^2_1} + \|\phi^{(2)}\|_{L^2}\|\mathrm{div}(w^{(1)} - w^{(2)})\|_{L^2_1} \\ &\qquad + \|w^{(1)} - w^{(2)}\|_{L^2}\|\nabla\phi^{(1)}\|_{L^2_1} + \|w^{(2)}\|_{L^2}\|\nabla(\phi^{(1)} - \phi^{(2)})\|_{L^2_1}\},\end{aligned}$$

(ii) $$\begin{aligned}&\|\tilde{F}_1(u^{(1)}, g) - \tilde{F}_1(u^{(2)}, g)\|_{L^1_1} \\ &\quad \le C\{\|w^{(1)} - w^{(2)}\|_{L^2}\|\nabla w^{(1)}\|_{L^2_1} + \|w^{(2)}\|_{L^2}\|\nabla(w^{(1)} - w^{(2)})\|_{L^2_1} \\ &\qquad + \|\phi^{(1)} - \phi^{(2)}\|_{L^2}(\|w^{(1)}\|_{L^\infty}\|\nabla w^{(1)}\|_{L^2_1} + \|\partial_t w^{(1)}\|_{L^2_1} + \|g\|_{L^2_1}) \\ &\qquad + \|\phi^{(2)}\|_{L^2}\|\partial_t(w^{(1)} - w^{(2)})\|_{L^2_1} \\ &\qquad + (\|\nabla\phi^{(1)}\|_{L^2_1} + \|\nabla\phi^{(2)}\|_{L^2_1})\|\phi^{(1)} - \phi^{(2)}\|_{L^2} + \|\phi^{(1)}\|_{L^2}\|\nabla(\phi^{(1)} - \phi^{(2)})\|_{L^2_1}\},\end{aligned}$$

(iii) $$\begin{aligned}&\|F^0_\infty(u^{(1)})-F^0_\infty(u^{(2)}))\|_{H^{m-1}_1}\\ &\le C\{\|\mathrm{div}w^{(1)}\|_{H^m_1}\|\phi^{(1)}-\phi^{(2)}\|_{H^{m-1}}+\|\phi^{(2)}\|_{H^m}\|\mathrm{div}(w^{(1)}-w^{(2)})\|_{H^{m-1}_1}\\ &\quad+\|\nabla\phi^{(1)}_1\|_{H^m_1}\|w^{(1)}-w^{(2)}\|_{H^{m-1}}+\|w^{(2)}\|_{H^m}\|\nabla(\phi^{(1)}_1-\phi^{(2)}_1)\|_{H^{m-1}_1}\},\end{aligned}$$

(iv) $$\begin{aligned}&\|\tilde F_\infty(u^{(1)},g)-\tilde F_\infty(u^{(2)},g))\|_{H^{m-2}_1}\\ &\le C\{\|\phi^{(1)}-\phi^{(2)}\|_{H^{m-1}}(\|w^{(1)}\|_{H^m}\|\nabla w^{(1)}\|_{H^{m-1}_1}+\|\partial_t w^{(1)}\|_{H^{m-1}_1}+\|g\|_{H^{m-1}_1})\\ &\quad+\|w^{(1)}-w^{(2)}\|_{H^{m-1}}\|\nabla w^{(1)}\|_{H^{m-1}_1}+\|w^{(2)}\|_{H^m}\|\nabla(w^{(1)}-w^{(2)})\|_{H^{m-2}_1}\\ &\quad+\|\phi^{(2)}\|_{H^m}\|\partial_t(w^{(1)}-w^{(2)})\|_{H^{m-2}_1}\\ &\quad+(\|\nabla\phi^{(1)}\|_{H^{m-1}_1}+\|\nabla\phi^{(2)}\|_{H^{m-1}_1})\|\phi^{(1)}-\phi^{(2)}\|_{H^{m-1}}\\ &\quad+\|\phi^{(1)}\|_{H^m}\|\nabla(\phi^{(1)}-\phi^{(2)})\|_{H^{m-2}_1}\}\end{aligned}$$

uniformly for $u^{(j)}={}^\top(\phi^{(j)},w^{(j)})=u^{(j)}_1+u^{(j)}_\infty$ *with* $u^{(j)}_k={}^\top(\phi^{(j)}_k,w^{(j)}_k)$ $(k=1,\infty)$ *satisfying* $\|\phi^{(j)}\|_{L^\infty}\le\frac{1}{2}$ *and* $\|u^{(j)}\|_{H^m}\le 1$ $(j=1,2)$.

Proposition 1.7.2 directly follows from Lemmas 1.1.1–1.1.3.

To prove Theorem 1.2.1, we next show the existence of a solution $\{u_1,u_\infty\}$ of (0.0.21)-(0.0.22) on $[0,T]$ satisfying $u_j(0)=u_j(T)$ $(j=1,\infty)$ by an iteration argument.

For $\ell=0$, we define $u^{(0)}_1={}^\top(\phi^{(0)}_1,w^{(0)}_1)$ and $u^{(0)}_\infty={}^\top(\phi^{(0)}_\infty,w^{(0)}_\infty)$ by

$$\begin{cases}u^{(0)}_1(t)=S_1(t)\mathscr{S}_1(T)[(I-S_1(T))^{-1}\mathbb{G}_1]+\mathscr{S}_1(t)[\mathbb{G}_1],\\ u^{(0)}_\infty(t)=S_{\infty,0}(t)(I-S_{\infty,0}(T))^{-1}\mathscr{S}_{\infty,0}(T)[\mathbb{G}_\infty]+\mathscr{S}_{\infty,0}(t)[\mathbb{G}_\infty],\end{cases}\tag{1.7.1}$$

where $t\in[0,T]$, $\mathbb{G}={}^\top(0,\frac{1}{\gamma}g(x,t))$, $\mathbb{G}_1=P_1\mathbb{G}$ and $\mathbb{G}_\infty=P_\infty\mathbb{G}$. Note that $u^{(0)}_j(0)=u^{(0)}_j(T)$ $(j=1,\infty)$.

For $\ell\ge1$, we define $u^{(\ell)}_1={}^\top(\phi^{(\ell)}_1,w^{(\ell)}_1)$ and $u^{(\ell)}_\infty={}^\top(\phi^{(\ell)}_\infty,w^{(\ell)}_\infty)$, inductively, by

$$\begin{cases}u^{(\ell)}_1(t)&=S_1(t)\mathscr{S}_1(T)[(I-S_1(T))^{-1}F_1(u^{(\ell-1)},g)]+\mathscr{S}_1(t)[F_1(u^{(\ell-1)},g)],\\ u^{(\ell)}_\infty(t)&=S_{\infty,u^{(\ell-1)}}(t)(I-S_{\infty,u^{(\ell-1)}}(T))^{-1}\mathscr{S}_{\infty,u^{(\ell-1)}}(T)[F_\infty(u^{(\ell-1)},g)]\\ &\quad+\mathscr{S}_{\infty,u^{(\ell-1)}}(t)[F_\infty(u^{(\ell-1)},g)],\end{cases}\tag{1.7.2}$$

where $t\in[0,T]$ and $u^{(\ell-1)}=u^{(\ell-1)}_1+u^{(\ell-1)}_\infty$. Note that $u^{(\ell)}_j(0)=u^{(\ell)}_j(T)$ for $j=1,\infty$ and $\ell\ge1$.

Proposition 1.7.3. *There exists a constant* $\delta_1>0$ *such that if* $[g]_m\le\delta_1$*, then there holds the estimates*

(i) $$\|\{u^{(\ell)}_1,u^{(\ell)}_\infty\}\|_{\mathscr{X}^m(0,T)}\le C_1[g]_m$$

for all $\ell \geq 0$, *and*

(ii)
$$\begin{aligned} &\|\{u_1^{(\ell+1)} - u_1^{(\ell)}, u_\infty^{(\ell+1)} - u_\infty^{(\ell)}\}\|_{\mathscr{X}^{m-1}(0,T)} \\ &\leq \ C_1[g]_m \|\{u_1^{(\ell)} - u_1^{(\ell-1)}, u_\infty^{(\ell)} - u_\infty^{(\ell-1)}\}\|_{\mathscr{X}^{m-1}(0,T)} \end{aligned}$$

for $\ell \geq 1$. *Here* C_1 *is a constant independent of* g *and* ℓ.

Proof. The estimate (i) follows from Propositions 1.4.4, 1.5.9, 1.7.1 and Lemma 1.3.3 (ii).

Let us consider the estimate of the difference between $u^{(\ell+1)}$ and $u^{(\ell)}$. For $\ell \geq 0$, we set $\bar{\phi}_j^{(\ell)} = \phi_j^{(\ell+1)} - \phi_j^{(\ell)}$ and $\bar{w}_j^{(\ell)} = w_j^{(\ell+1)} - w_j^{(\ell)}$ for $j = 1, \infty$. Then by using (1.7.1) and (1.7.2), we see that $\bar{\phi}_j^{(\ell)}$ and $\bar{w}_j^{(\ell)}$ $(\ell \geq 1)$ satisfy

$$\left\{ \begin{array}{l} \partial_t \bar{\phi}_1^{(\ell)} + \gamma \mathrm{div} \bar{w}_1^{(\ell)} = F_{11}(\bar{u}^{(\ell-1)}), \\ \partial_t \bar{w}_1^{(\ell)} - \nu \triangle \bar{w}_1^{(\ell)} - \tilde{\nu} \nabla \mathrm{div} \bar{w}_1^{(\ell)} + \gamma \nabla \bar{\phi}_1^{(\ell)} = F_{12}(\bar{u}^{(\ell-1)}, g), \end{array} \right. \tag{1.7.3}$$

$$\left\{ \begin{array}{l} \partial_t \bar{\phi}_\infty^{(\ell)} + \gamma P_\infty (w^{(\ell)} \cdot \nabla \bar{\phi}_\infty^{(\ell)}) + \gamma \mathrm{div} \bar{w}_\infty^{(\ell)} = F_{\infty 1}(\bar{u}^{(\ell-1)}), \\ \partial_t \bar{w}_\infty^{(\ell)} - \nu \triangle \bar{w}_\infty^{(\ell)} - \tilde{\nu} \nabla \mathrm{div} \bar{w}_\infty^{(\ell)} + \gamma \nabla \bar{\phi}_\infty^{(\ell)} = F_{\infty 2}(\bar{u}^{(\ell-1)}, g), \end{array} \right. \tag{1.7.4}$$

where

$$\begin{aligned} &F_{11}(\bar{u}^{(\ell-1)}) = F_1^0(u^{(\ell)}) - F_1^0(u^{(\ell-1)}), \\ &F_{12}(\bar{u}^{(\ell-1)}, g) = \tilde{F}_1(u^{(\ell)}, g) - \tilde{F}_1(u^{(\ell-1)}, g), \\ &F_{\infty 1}(\bar{u}^{(\ell-1)}) = F_\infty^0(u^{(\ell)}) - F_\infty^0(u^{(\ell-1)}) - \gamma P_\infty((w^{(\ell)} - w^{(\ell-1)}) \cdot \nabla \phi_\infty^{(\ell)}), \\ &F_{\infty 2}(\bar{u}^{(\ell-1)}, g) = \tilde{F}_\infty(u^{(\ell)}, g) - \tilde{F}_\infty(u^{(\ell-1)}, g). \end{aligned}$$

The desired inequality (ii) can be obtained by applying Propositions 1.4.4, 1.5.9, 1.7.2, 1.7.3 (i) and Lemma 1.3.3 (ii). This completes the proof. □

We introduce a notation. We denote by $B_{\mathscr{X}^k(a,b)}(r)$ the closed unit ball of $\mathscr{X}^k(a,b)$ centered at 0 with radius r, i.e.,

$$B_{\mathscr{X}^k(a,b)}(r) = \left\{ \{u_1, u_\infty\} \in \mathscr{X}^k(a,b); \|\{u_1, u_\infty\}\|_{\mathscr{X}^k(a,b)} \leq r \right\}.$$

Proposition 1.7.4. *There exists a constant* $\delta_2 > 0$ *such that if* $[g]_m \leq \delta_2$, *then the system* (0.0.21)-(0.0.22) *has a unique solution* $\{u_1, u_\infty\}$ *on* $[0, T]$ *in* $B_{\mathscr{X}^m(0,T)}(C_1[g]_m)$ *satisfying* $u_j(0) = u_j(T)$ $(j = 1, \infty)$. *The uniqueness of solutions of* (0.0.21)-(0.0.22) *on* $[0, T]$ *satisfying* $u_j(0) = u_j(T)$ $(j = 1, \infty)$ *holds in* $B_{\mathscr{X}^m(0,T)}(C_1\delta_2)$.

Proof. Let $\delta_2 = \min\{\delta_1, \frac{1}{2C_1}\}$ with δ_1 given in Propositions 1.7.3. By Propositions 1.7.3, we see that if $[g]_m \leq \delta_2$, then $u_j^{(\ell)} = {}^\top(\phi_j^{(\ell)}, w_j^{(\ell)})$ $(j = 1, \infty)$ converges to $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, \infty)$ in the sense

$$\{u_1^{(\ell)}, u_\infty^{(\ell)}\} \ \to \ \{u_1, u_\infty\} \text{ in } \ \mathscr{X}^{m-1}(0, T),$$

$$u_\infty^{(\ell)} = {}^\top(\phi_\infty^{(\ell)}, w_\infty^{(\ell)}) \ \to \ u_\infty = {}^\top(\phi_\infty, w_\infty) \ *\text{-weakly in } L^\infty(0,T; H^m_{(\infty),1}),$$
$$w_\infty^{(\ell)} \ \to \ w_\infty \text{ weakly in } L^2(0,T; H^{m+1}_{(\infty),1}) \cap H^1(0,T; H^{m-1}_{(\infty),1}).$$

It is not difficult to see that $\{u_1, u_\infty\}$ is a solution of (0.0.21)-(0.0.22) satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

It remains to prove $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0,T]; H^m_1)$, which implies $\{u_1, u_\infty\} \in B_{\mathscr{X}^m(0,T)}(C_1[g]_m)$ with $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

As for w_∞, since $L^2(0,T; H^{m+1}) \cap H^1(0,T; H^{m-1}) \subset C([0,T]; H^m)$, we find that $w_\infty \in C([0,T]; H^m)$.

As for ϕ_∞, note that $\phi_\infty \in C([0,T]; H^1)$ and ϕ_∞ is the solution of

$$\begin{cases} \partial_t \phi_\infty + \gamma(w \cdot \nabla \phi_\infty) = g^0_\infty, \\ \phi_\infty|_{t=0} = \phi_{0\infty}, \end{cases} \tag{1.7.5}$$

where

$$g^0_\infty = -\gamma \mathrm{div}\, w_\infty + F^0_\infty(u) \in L^2(0,T; H^m), \ \phi_{0\infty} \in H^m.$$

On the other hand, by Lemma 1.5.1, we see that there exists a solution of (1.7.5) which belongs to $C([0,T]; H^m)$ and that the uniqueness of solutions of (1.7.5) holds in $C([0,T]; H^1)$. Therefore, we find that

$$\phi_\infty \in C([0,T]; H^m).$$

To prove that $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0,T]; H^m_1)$, we note that u_∞ is written as

$$u_\infty(t) = S_{\infty,u}(t)(I - S_{\infty,u}(T))^{-1} \mathscr{S}_{\infty,u}(T)[F_\infty(u,g)] + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)]$$

with $u = u_1 + u_\infty$. By Proposition 1.7.1 and Lemma 1.3.3 (ii), we see that $F_\infty(u,g) \in L^2(0,T; H^m_{(\infty),1} \times H^{m-1}_{(\infty),1})$. It then follows from Proposition 1.5.6 that if δ_2 is small such that $C_1\delta_2 \le \delta$, then $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0,T]; H^m_1)$. This completes the proof. $\square$

To complete the construction of a time periodic solution of (0.0.1), we use the following proposition on the unique existence of solutions to the initial value problem.

Proposition 1.7.5. *Let $s \in \mathbb{R}$ and let $U_0 = U_{01} + U_{0\infty}$ with $U_{01} \in \mathscr{H}^1_{(1),1}$ and $U_{0\infty} \in H^m_{(\infty),1}$. Then there exist constants $\delta_3 > 0$ and $C_2 > 0$ such that if*

$$M(U_{01}, U_{0\infty}, g) := \|U_{01}\|_{\mathscr{H}^1_{(1),1}} + \|U_{0\infty}\|_{H^m_{(\infty),1}} + [g]_m \le \delta_3,$$

there exists a solution $\{U_1, U_\infty\}$ of the initial value problem for (0.0.21)-(0.0.22) on $[s, s+T]$ in $B_{\mathscr{X}^m(s,s+T)}(C_2 M(U_{01}, U_{0\infty}, g))$ satisfying the initial condition $U_j|_{t=s} = U_{0j}$ $(j = 1, \infty)$. The uniqueness for this initial value problem holds in $B_{\mathscr{X}^m(s,s+T)}(C_2\delta_3)$.

Proof. Let $\tilde{u} = {}^{\top}(\tilde{\phi}, \tilde{w})$ be a given function in $C([s, s+T]; H^m) \cap L^2(s, s+T; H^{m+1})$. We define $S_{\infty,\tilde{u}}(t,s)u_{0\infty}$ and $\mathscr{S}_{\infty,\tilde{u}}(t,s)F_\infty$ by the solution operators for

$$\partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty, \quad u_\infty|_{t=s} = u_{0\infty}.$$

with $F_\infty = 0$ and $u_{0\infty} = 0$, respectively. As in the proof of Proposition 1.5.6, one can see that if $\tilde{w}$ satisfies

$$\|\tilde{w}\|_{C([s,s+T];H^m)\cap L^2(s,s+T;H^{m+1})} \leq \delta, \tag{1.7.6}$$

then it holds that $S_{\infty,\tilde{u}}(t,s)$ and $\mathscr{S}_{\infty,\tilde{u}}(t,s)$ satisfy the estimates

$$\|S_{\infty,\tilde{u}}(t,s)U_{0\infty}\|_{H^k_{(\infty),1}} \leq Ce^{-a(t-s)}\|U_{0\infty}\|_{H^k_{(\infty),1}}, \tag{1.7.7}$$

$$\|\mathscr{S}_{\infty,\tilde{u}}(t,s)[F_\infty]\|_{H^k_{(\infty),1}} \leq C\left\{\int_s^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^k_{(\infty),1}\times H^{k-1}_{(\infty),1}}\,d\tau\right\}^{\frac{1}{2}} \tag{1.7.8}$$

for $t \in [s, s+T]$, $F_{0\infty} \in H^k_{(\infty),1}$, $F_\infty \in L^2(s, s+T; H^k_{(\infty),1} \times H^{k-1}_{(\infty),1})$ and $k = m-1$ or m with $C = C(\delta, T) > 0$ uniformly for $s \in \mathbb{R}$ and $\tilde{u} = {}^{\top}(\tilde{\phi}, \tilde{w})$ satisfying (1.7.6).

To prove Proposition 1.7.5, it now suffices to show the unique existence of the solution $\{U_1, U_\infty\} \in B_{\mathscr{X}^m(s,s+T)}(C_2M(U_{01}, U_{0\infty}, g))$ of

$$\begin{cases} U_1(t) &= S_1(t-s)U_{01} + \mathscr{S}_1(t-s)[F_1(U,g)], \\ U_\infty(t) &= S_{\infty,U}(t,s)u_{0\infty} + \mathscr{S}_{\infty,U}(t,s)[F_\infty(U,g)], \end{cases} \tag{1.7.9}$$

with $U = U_1 + U_\infty$ for a constant $C_2 > 0$, provided that $M(U_{01}, U_{0\infty}, g)$ is sufficiently small. We solve this problem by an iteration argument as in the proof of Proposition 1.7.4.

For $\ell = 0$, we define $U_j^{(0)} = {}^{\top}(\Phi_j^{(0)}, W_j^{(0)})$ $(j = 1, \infty)$ by

$$\begin{cases} U_1^{(0)}(t) = S_1(t-s)U_{01} + \mathscr{S}_1(t-s)[\mathbb{G}_1], \\ U_\infty^{(0)}(t) = S_{\infty,0}(t,s)U_{0\infty} + \mathscr{S}_{\infty,0}(t,s)[\mathbb{G}_\infty], \end{cases}$$

where $t \in [s, s+T]$, $\mathbb{G} = {}^{\top}(0, \frac{1}{\gamma}g(x,t))$, $\mathbb{G}_1 = P_1\mathbb{G}$ and $\mathbb{G}_\infty = P_\infty\mathbb{G}$.

For $\ell \geq 1$, we define $U_j^{(\ell)} = {}^{\top}(\Phi_j^{(\ell)}, W_j^{(\ell)})$ $(j = 1, \infty)$, inductively, by

$$\begin{cases} U_1^{(\ell)}(t) &= S_1(t-s)U_{01} + \mathscr{S}_1(t-s)[F_1(U^{(\ell-1)}, g)], \\ U_\infty^{(\ell)}(t) &= S_{\infty,U^{(\ell-1)}}(t,s)U_{0\infty} + \mathscr{S}_{\infty,U^{(\ell-1)}}(t,s)[F_\infty(U^{(\ell-1)}, g)], \end{cases}$$

where $t \in [s, s+T]$ and $U^{(\ell-1)} = U_1^{(\ell-1)} + U_\infty^{(\ell-1)}$.

As in the proof of Proposition 1.7.3, by using Proposition 1.4.1, (1.7.7), (1.7.8), Propositions 1.7.1, 1.7.2 and Lemma 1.3.3 (ii), we can inductively show that if $M(U_{01}, U_{0\infty}, g)$ is sufficiently small, then there hold the estimates

$$\|\{U_1^{(\ell)}, U_\infty^{(\ell)}\}\|_{\mathscr{X}^m(s,s+T)} \leq C_2M(U_{01}, U_{0\infty}, g)$$

for all $\ell \geq 0$, and

$$\begin{aligned}
&\|\{U_1^{(\ell+1)} - U_1^{(\ell)}, U_\infty^{(\ell+1)} - U_\infty^{(\ell)}\}\|_{\mathscr{X}^{m-1}(s,s+T)} \\
&\leq \ C_2 M(U_{01}, U_{0\infty}, g)\|\{U_1^{(\ell)} - U_1^{(\ell-1)}, U_\infty^{(\ell)} - U_\infty^{(\ell-1)}\}\|_{\mathscr{X}^{m-1}(s,s+T)}
\end{aligned}$$

for all $\ell \geq 1$. Hence, in a similar manner to the proof of Proposition 1.7.4, we see that there exists a solution $\{U_1, U_\infty\} \in B_{\mathscr{X}^m(s,s+T)}(C_2 M(U_{01}, U_{0\infty}, g))$ of (0.0.21)-(0.0.22) satisfying $U_j|_{t=s} = U_{0j}$ $(j = 0, \infty)$, provided that $M(U_{01}, U_{0\infty}, g) \leq \delta_3$ for a small constant $\delta_3 > 0$. In view of the iteration argument, we can see that the uniqueness holds in $B_{\mathscr{X}^m(s,s+T)}(C_2\delta_3)$. This completes the proof. □

We are now in a position to prove Theorem 1.2.1.

Proof of Theorem 1.2.1. It suffices to prove the unique existence of a time periodic solution of (0.0.15). By Proposition 1.7.4, we see that if $[g]_m \leq \delta_2$, then (0.0.21)-(0.0.22) has a unique solution $\{u_1^{(0)}, u_\infty^{(0)}\} \in B_{\mathscr{X}^m(0,T)}(C_1[g]_m)$ satisfying $u_j^{(0)}(0) = u_j^{(0)}(T)$ $(j = 1, \infty)$. In particular, it holds that

$$\sup_{t\in[0,T]} \left\{ \|u_1^{(0)}(t)\|_{\mathscr{H}^1_{(1),1}} + \|u_\infty^{(0)}(t)\|_{H^m_{(\infty),1}} \right\} \leq C_1[g]_m. \tag{1.7.10}$$

Therefore, if g satisfies $(C_1 + 1)[g]_m \leq \delta_3$, then, by Proposition 1.7.5, we see that there exists a unique solution $\{u_1^{(1)}, u_\infty^{(1)}\} \in B_{\mathscr{X}^m(T,2T)}(C_2(C_1 + 1)[g]_m)$ of (0.0.21)-(0.0.22) satisfying $u_j^{(1)}|_{t=T} = u_j^{(0)}(T) = u_j^{(0)}(0)$ $(j = 1, \infty)$.

We introduce $\tilde{u}_j^{(1)}$ $(j = 1, \infty)$ and $\tilde{u}^{(1)}$ by

$$\tilde{u}_j^{(1)}(t) = u_j^{(1)}(t + T), \quad \tilde{u}^{(1)}(t) = \tilde{u}_1^{(1)}(t) + \tilde{u}_\infty^{(1)}(t) \ \text{ for } \ t \in [0, T].$$

Then we find that

$$\tilde{u}_1^{(1)}(0) = u_1^{(1)}(T) = u_1^{(0)}(T) = u_1^{(0)}(0),$$

$$\begin{aligned}
\partial_t \tilde{u}_1^{(1)}(t) + A\tilde{u}_1^{(1)}(t) = \partial_t u_1^{(1)}(t + T) + Au_1^{(1)}(t + T) &= F_1(u^{(1)}(t + T), g(t + T)) \\
&= F_1(\tilde{u}^{(1)}(t), g(t)).
\end{aligned}$$

Similarly, we see that

$$\tilde{u}_\infty^{(1)}(0) = u_\infty^{(0)}(0),$$
$$\partial_t \tilde{u}_\infty^{(1)}(t) + A\tilde{u}_\infty^{(1)}(t) + P_\infty(B[\tilde{u}^{(1)}(t)]\tilde{u}^{(1)}(t)) = F_\infty(\tilde{u}^{(1)}(t), g(t)).$$

Therefore, if $[g]_m \leq \delta_4 := \min\{\delta_2, \frac{C_2\delta_3}{C_1}, \frac{1}{(C_1+1)}\delta_3\}$, then, by the uniqueness of the solution, we find that $\{\tilde{u}_1^{(1)}(t), \tilde{u}_\infty^{(1)}(t)\} = \{u_1^{(0)}(t), u_\infty^{(0)}(t)\}$ for $t \in [0, T]$. Consequently, we have $\{u_1^{(1)}(t), u_\infty^{(1)}(t)\} = \{u_1^{(0)}(t - T), u_\infty^{(0)}(t - T)\}$ for $t \in [T, 2T]$.

We define $\{u_1(t), u_\infty(t)\}$ $(t \in [0, 2T])$ by $\{u_1(t), u_\infty(t)\} = \{u_1^{(k)}(t), u_\infty^{(k)}(t)\}$ for $t \in [kT, (k + 1)T]$, $k = 0, 1$. It then follows that $\{u_1(t + T), u_\infty(t + T)\} = \{u_1(t), u_\infty(t)\}$

for $t \in [0,T]$. Furthermore, we see from Proposition 1.7.5 and (1.7.10) that there exists a unique solution $\{v_1, v_\infty\} \in B_{\mathscr{X}^m(\frac{T}{2},\frac{3T}{2})}(C_2(C_1+1)[g]_m)$ of (0.0.21)-(0.0.22) on $[\frac{T}{2}, \frac{3T}{2}]$ satisfying $v_j|_{t=\frac{T}{2}} = u_j^{(0)}(\frac{T}{2})$ $(j = 1, \infty)$. By the uniqueness, it follows that $\{v_1, v_\infty\} = \{u_1, u_\infty\}$ on $[\frac{T}{2}, \frac{3T}{2}]$, which implies that $\{u_1, u_\infty\}$ is a solution of (0.0.21)-(0.0.22) on $[0, 2T]$ in $\mathscr{X}^m(0, 2T)$. Repeating this argument on intervals $[kT, (k+1)T]$ for $k = \pm 1, \pm 2 \cdots$, we obtain a solution $\{u_1, u_\infty\}$ of (0.0.21)-(0.0.22) in $\mathscr{X}^m_{per}(\mathbb{R})$ satisfying $\|\{u_1, u_\infty\}\|_{\mathscr{X}^m(0,T)} \leq C_1[g]_m$ that gives a time periodic solution $u = {}^\top(\phi, w)$ of (0.0.15) by setting $u = u_1 + u_\infty$, where $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, \infty)$, $\phi = \phi_1 + \phi_\infty$ and $w = w_1 + w_\infty$.

In view of the iteration argument in Propositions 1.7.3 and 1.7.4, one can see that the uniqueness of time periodic solutions for (0.0.15) holds in $\{u = {}^\top(\phi, w); \{P_1 u, P_\infty u\} \in \mathscr{X}^m_{per}(\mathbb{R}), \|\{P_1 u, P_\infty u\}\|_{\mathscr{X}^m(0,T)} \leq C_1 \delta_4\}$ if $[g]_m \leq \delta_4$. This completes the proof. $\square$

Chapter 2

On the existence and stability of time periodic solution to the compressible Navier-Stokes equation on the whole space

The existence of a time periodic solution of (0.0.1) on the whole space is proved for sufficiently small time periodic external force when the space dimension is greater than or equal to 3. The proof is based on the spectral properties of the time-T-map associated with the linearized problem around the motionless state with constant density in some weighted L^∞ and Sobolev spaces. The time periodic solution is shown to be asymptotically stable under sufficiently small initial perturbations and the L^∞ norm of the perturbation decays as time goes to infinity.

2.1 Preliminaries

In this chapter we use the following notation.

For a given Banach space X, the norm on X is denoted by $\|\cdot\|_X$.

Let $1 \leqq p \leqq \infty$. We denote by L^p the usual L^p space over $\mathbb{R}^n$. The inner product of L^2 is denoted by $(\cdot,\cdot)$. For a nonnegative integer k, we denote by H^k the usual L^2-Sobolev space of order k. (As usual, $H^0 = L^2$.)

We simply denote by L^p the set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in L^p$ $(j = 1, \cdots, n)$, i.e., $(L^p)^n$ and the norm $\|\cdot\|_{(L^p)^n}$ on it is denoted by $\|\cdot\|_{L^p}$ if no confusion will occur. Similarly, for a function space X, the set of all vector fields $w = {}^\top(w_1, \cdots, w_n)$ on $\mathbb{R}^n$ with $w_j \in X$ $(j = 1, \cdots, n)$, i.e., X^n, is simply denoted by X; and the norm $\|\cdot\|_{X^n}$ on it is denoted by $\|\cdot\|_X$ if no confusion will occur. (For example, $(H^k)^n$ is simply denoted by H^k and the norm $\|\cdot\|_{(H^k)^n}$ is denoted by $\|\cdot\|_{H^k}$.)

Let $u = {}^\top(\phi, w)$ with $\phi \in H^k$ and $w = {}^\top(w_1, \cdots, w_n) \in H^m$. We denote the norm of

u on $H^k \times H^m$ by $\|u\|_{H^k\times H^m}$:

$$\|u\|_{H^k\times H^m} = \left(\|\phi\|_{H^k}^2 + \|w\|_{H^m}^2\right)^{\frac{1}{2}}.$$

When $m = k$, the space $H^k \times (H^k)^n$ is simply denoted by H^k and the norm $\|u\|_{H^k\times(H^k)^n}$ by $\|u\|_{H^k}$ if no confusion will occur :

$$H^k := H^k \times (H^k)^n, \quad \|u\|_{H^k} := \|u\|_{H^k\times(H^k)^n} \quad (u = {}^\top(\phi, w)).$$

Similarly, for $u = {}^\top(\phi, w) \in X \times Y$ with $w = {}^\top(w_1, \cdots, w_n)$, we denote its norm by $\|u\|_{X\times Y}$:

$$\|u\|_{X\times Y} = \left(\|\phi\|_X^2 + \|w\|_Y^2\right)^{\frac{1}{2}} \quad (u = {}^\top(\phi, w)).$$

If $Y = X^n$, we simply denote $X \times X^n$ by X, and, its norm $\|u\|_{X\times X^n}$ by $\|u\|_X$:

$$X := X \times X^n, \quad \|u\|_X := \|u\|_{X\times X^n} \quad (u = {}^\top(\phi, w)).$$

We will work on function spaces with spatial weight. For a nonnegative integer ℓ and $1 \le p \le \infty$, we denote by L^p_ℓ the weighted L^p space defined by

$$L^p_\ell = \{u \in L^p; \|u\|_{L^p_\ell} := \|(1 + |x|)^\ell u\|_{L^p} < \infty\}.$$

We denote the Fourier transform of f by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \quad (\xi \in \mathbb{R}^n).$$

The inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}d\xi \quad (x \in \mathbb{R}^n).$$

Let k be a nonnegative integer and let r_1 and r_∞ be positive constants satisfying $r_1 < r_\infty$. We denote by $H^k_{(\infty)}$ the set of all $u \in H^k$ satisfying supp $\hat{u} \subset \{|\xi| \ge r_1\}$, and by $L^2_{(1)}$ the set of all $u \in L^2$ satisfying supp $\hat{u} \subset \{|\xi| \le r_\infty\}$. Note that $H^k \cap L^2_{(1)} = L^2_{(1)}$ for any nonnegative integer k. (Cf., Lemma 1.3.3 (ii).)

Let k and ℓ be nonnegative integers. We define the spaces H^k_ℓ and $H^k_{(\infty),\ell}$ by

$$H^k_\ell = \{u \in H^k; \|u\|_{H^k_\ell} < +\infty\},$$

where

$$\|u\|_{H^k_\ell} = \left(\sum_{j=0}^{\ell} |u|^2_{H^k_j}\right)^{\frac{1}{2}},$$

$$|u|_{H^k_\ell} = \left(\sum_{|\alpha|\le k} \| \, |x|^\ell \partial_x^\alpha u \, \|^2_{L^2}\right)^{\frac{1}{2}},$$

and

$$H^k_{(\infty),\ell} = \{u \in H^k_{(\infty)}; \|u\|_{H^k_\ell} < +\infty\}.$$

Let ℓ be a nonnegative integer. We denote $L^2_{(1),\ell}$ by

$$L^2_{(1),\ell} = \{f \in L^2_\ell; f \in L^2_{(1)}\}.$$

For $-\infty \le a < b \le \infty$, we denote by $C^k([a,b];X)$ the set of all C^k functions on $[a,b]$ with values in X. We denote the Bochner space on (a,b) by $L^p(a,b;X)$ and the L^2-Bochner-Sobolev space of order k by $H^k(a,b;X)$.

We define the space $\mathscr{X}_{(1)}$ by

$$\mathscr{X}_{(1)} = \{\phi \in L^\infty_{n-1}, \nabla\phi \in L^2_1; \mathrm{supp}\,\hat{\phi} \subset \{|\xi| \le r_\infty\}, \|\phi\|_{\mathscr{X}_{(1)}} < +\infty\},$$

where

$$\begin{aligned} &\|\phi\|_{\mathscr{X}_{(1)}} := \|\phi\|_{\mathscr{X}_{(1),L^\infty}} + \|\phi\|_{\mathscr{X}_{(1),L^2}}, \\ &\|\phi\|_{\mathscr{X}_{(1),L^\infty}} := \|(1+|x|)^{n-1}\phi\|_{L^\infty}, \ \|\phi\|_{\mathscr{X}_{(1),L^2}} := \|(1+|x|)\nabla\phi\|_{L^2}. \end{aligned}$$

The space $\mathscr{Y}_{(1)}$ is defined by

$$\mathscr{Y}_{(1)} = \{w \in L^\infty_{n-2}, \nabla w \in H^1; \mathrm{supp}\,\hat{w} \subset \{|\xi| \le r_\infty\}, \|w\|_{\mathscr{Y}_{(1)}} < +\infty\},$$

where

$$\begin{aligned} \|w\|_{\mathscr{Y}_{(1)}} &:= \|w\|_{\mathscr{Y}_{(1),L^\infty}} + \|w\|_{\mathscr{Y}_{(1),L^2}}, \\ \|w\|_{\mathscr{Y}_{(1),L^\infty}} &:= \sum_{j=0}^{1} \|(1+|x|)^{n-2+j}\nabla^j w\|_{L^\infty}, \\ \|w\|_{\mathscr{Y}_{(1),L^2}} &:= \sum_{j=1}^{2} \|(1+|x|)^{j-1}\nabla^j w\|_{L^2}. \end{aligned}$$

The space $\mathscr{Z}_{(1)}(a,b)$ is defined by

$$\mathscr{Z}_{(1)}(a,b) = C^1([a,b];\mathscr{X}_{(1)}) \times \Big[C([a,b];\mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})\Big].$$

Let ℓ be a nonnegative integer and let s be a nonnegative integer satisfying $s \ge \left[\frac{n}{2}\right]+1$. For $k = s-1, s$, the space $\mathscr{Z}^k_{(\infty),\ell}(a,b)$ is defined by

$$\begin{aligned} \mathscr{Z}^k_{(\infty),\ell}(a,b) = &\left[C([a,b];H^k_{(\infty),\ell}) \cap C^1([a,b];L^2_1)\right] \\ &\times \left[L^2(a,b;;H^{k+1}_{(\infty),\ell}) \cap C([a,b];H^k_{(\infty),\ell}) \cap H^1(a,b;H^{k-1}_{(\infty),\ell})\right]. \end{aligned}$$

Let s be a nonnegative integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$ and let $k = s-1, s$. The space $X^k(a,b)$ is defined by

$$\begin{aligned}
&X^k(a,b) \\
&= \bigl\{\{u_1, u_\infty\}; u_1 \in \mathscr{Z}_{(1)}(a,b), u_\infty \in \mathscr{Z}^k_{(\infty),n-1}(a,b), \\
&\qquad \partial_t \phi_\infty \in C([a,b]; L^2_1), u_j = {}^\top(\phi_j, w_j)\,(j=1,\infty)\bigr\},
\end{aligned}$$

equipped with the norm

$$\begin{aligned}
\|\{u_1, u_\infty\}\|_{X^k(a,b)} =& \|u_1\|_{\mathscr{Z}_{(1)}(a,b)} + \|u_\infty\|_{\mathscr{Z}^k_{(\infty),n-1}(a,b)} \\
&+ \|\partial_t \phi_\infty\|_{C([a,b];L^2_1)} + \|\partial_t u_1\|_{C([a,b];L^2)} + \|\partial_t \nabla u_1\|_{C([a,b];L^2_1)}.
\end{aligned}$$

We also introduce function spaces of T-periodic functions in t. We denote by $C_{per}(\mathbb{R}; X)$ the set of all T-periodic continuous functions with values in X equipped with the norm $\|\cdot\|_{C([0,T];X)}$; and we denote by $L^2_{per}(\mathbb{R}; X)$ the set of all T-periodic locally square integrable functions with values in X equipped with the norm $\|\cdot\|_{L^2(0,T;X)}$. Similarly, $H^1_{per}(\mathbb{R}; X)$ and $X^k_{per}(\mathbb{R})$, and so on, are defined.

For operators L_1 and L_2, $[L_1, L_2]$ denotes the commutator of L_1 and L_2:

$$[L_1, L_2]f = L_1(L_2 f) - L_2(L_1 f).$$

2.2 Main results of Chapter 2

In this section, we state our main results on the existence and stability of a time-periodic solution for system (0.0.1).

Recall that the following operators are introduced which decompose a function into its low and high frequency parts in Chapter 1. The operators P_1 and P_∞ on L^2 are defined by

$$P_j f = \mathcal{F}^{-1}(\hat{\chi}_j \mathcal{F}[f]) \quad (f \in L^2, j = 1, \infty),$$

where

$$\begin{aligned}
&\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^n) \quad (j=1,\infty), \quad 0 \leq \hat{\chi}_j \leq 1 \quad (j = 1,\infty), \\
&\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \leq r_1), \\ 0 & (|\xi| \geq r_\infty), \end{cases} \\
&\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi), \\
&0 < r_1 < r_\infty.
\end{aligned}$$

We fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu + \tilde{\nu}}$ in such a way that the estimate (2.4.6) in Lemma 2.4.3 below holds for $|\xi| \leq r_\infty$.

Our result on the existence of a time periodic solution is stated as follows.

Theorem 2.2.1. *Let $n \geq 3$ and let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies* (0.0.2) *and $g(x,t) \in C_{per}(\mathbb{R}; L^1 \cap L_n^\infty) \cap L^2_{per}(\mathbb{R}; H^{s-1}_{n-1})$. Set*

$$[g]_s = \|g\|_{C([0,T];L^1\cap L_n^\infty)} + \|g\|_{L^2(0,T;H^{s-1}_{n-1})}.$$

Then there exist constants $\delta > 0$ and $C > 0$ such that if $[g]_s \leq \delta$, then the system (0.0.15) *has a time-periodic solution $u = u_1 + u_\infty$ satisfying $\{u_1, u_\infty\} \in X^s_{per}(\mathbb{R})$ with $\|\{u_1, u_\infty\}\|_{X^s(0,T)} \leq C[g]_s$. Furthermore, the uniqueness of time periodic solutions of* (0.0.15) *holds in the class $\{u = {}^\top(\phi, w); \{P_1 u, P_\infty u\} \in X^s_{per}(\mathbb{R}),\ \|\{u_1, u_\infty\}\|_{X^s(0,T)} \leq C\delta\}$.*

We next consider the stability of the time-periodic solution obtained in Theorem 2.2.1.

Let ${}^\top(\rho_{per}, v_{per})$ be the periodic solution given in Theorem 2.2.1. We denote the perturbation by $u = {}^\top(\phi, w)$, where $\phi = \rho - \rho_{per}, w = v - v_{per}$. Substituting $\rho = \phi + \rho_{per}$ and $v = w + v_{per}$ into (0.0.1), we see that the perturbation $u = {}^\top(\phi, w)$ is governed by

$$\begin{cases} \partial_t \phi + v_{per} \cdot \nabla \phi + \phi \mathrm{div} v_{per} + \rho_{per} \mathrm{div} w + w \cdot \nabla \rho_{per} = f^0, \\ \partial_t w + v_{per} \cdot \nabla w + w \cdot \nabla v_{per} - \frac{\mu}{\rho_{per}} \Delta w - \frac{\mu+\mu'}{\rho_{per}} \nabla \mathrm{div} w \\ \quad + \frac{\phi}{\rho_{per}^2} (\mu \Delta v_{per} + (\mu+\mu') \nabla \mathrm{div} v_{per}) + \nabla(\frac{p'(\rho_{per})}{\rho_{per}} \phi) = \tilde{f}, \end{cases} \tag{2.2.1}$$

where

$$\begin{aligned} f^0 &= -\mathrm{div}(\phi w), \\ \tilde{f} &= -w \cdot \nabla w - \frac{\phi}{\rho_{per}(\rho_{per}+\phi)} (\mu \Delta w + (\mu+\mu') \nabla \mathrm{div} w) \\ &\quad + \frac{\phi}{\rho_{per}(\rho_{per}+\phi)} (\frac{\phi}{\rho_{per}} \mu \Delta v_{per} + \frac{\phi}{\rho_{per}} (\mu+\mu') \nabla \mathrm{div} v_{per}) \\ &\quad + \frac{\phi}{\rho_{per}^2} \nabla (p^{(2)}(\rho_{per}, \phi) \phi) + \frac{\phi^2}{\rho_{per}^2(\rho_{per}+\phi)} \nabla (p(\rho_{per}+\phi)) + \frac{1}{\rho_{per}} \nabla (p^{(3)}(\rho_{per}, \phi) \phi^2), \\ p^{(2)}(\rho_{per}, \phi) &= \int_0^1 p'(\rho_{per} + \theta\phi) d\theta, \\ p^{(3)}(\rho_{per}, \phi) &= \int_0^1 (1-\theta) p''(\rho_{per} + \theta\phi) d\theta. \end{aligned}$$

We consider the initial value problem for (2.2.1) under the initial condition

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{2.2.2}$$

Our result on the stability of the time-periodic solution is stated as follows.

Theorem 2.2.2. *Let $n \geq 3$ and let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$. Assume that $g(x,t)$ satisfies* (1.2) *and $g(x,t) \in C_{per}(\mathbb{R}; L^1 \cap L_n^\infty) \cap L^2_{per}(\mathbb{R}; H^s_{n-1})$. Let (ρ_{per}, v_{per}) be the time-periodic solution obtained in Theorem* 2.2.1*, and let $u_0 \in H^s$. Then there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that if*

$$[g]_{s+1} \leq \epsilon_1, \quad \|u_0\|_{H^s} \leq \epsilon_2,$$

there exists a unique global solution $u = {}^\top(\phi, w)$ *of* (2.2.1)-(2.2.2) *satisfying*

$$
\begin{aligned}
&u \in C([0,\infty); H^s),\\
&\|u(t)\|_{H^s}^2 + \int_0^t \|\nabla u(\tau)\|_{H^{s-1}\times H^s}^2 d\tau \le C\|u_0\|_{H^s}^2 \quad (t \in [0,\infty)),\\
&\|u(t)\|_{L^\infty} \to 0 \quad (t \to \infty).
\end{aligned}
$$

It is not difficult to see that Theorem 2.2.2 can be proved by the energy method ([16], [26]), since the Hardy inequality works well to deal with the linear terms including (ρ_{per}, v_{per}) due to the estimate for (ρ_{per}, v_{per}) in Theorem 2.2.2; and so the proof is omitted here.

2.3 Reformulation of the problem

In this section, we reformulate problem (0.0.15). As in Chapter 1, to solve the time periodic problem for (0.0.15), we decompose u into a low frequency part u_1 and a high frequency part u_∞, and then, we rewrite the problem into a system of equations for u_1 and u_∞.

As in Chapter 1, we set

$$u_1 = P_1 u, \quad u_\infty = P_\infty u.$$

Applying the operators P_1 and P_∞ to (0.0.15), we obtain,

$$\partial_t u_1 + A u_1 = F_1(u_1 + u_\infty, g), \tag{2.3.1}$$

$$\partial_t u_\infty + A u_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) = F_\infty(u_1 + u_\infty, g). \tag{2.3.2}$$

Here

$$
\begin{aligned}
F_1(u_1 + u_\infty, g) &= P_1[-B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g)],\\
F_\infty(u_1 + u_\infty, g) &= P_\infty[-B[u_1 + u_\infty]u_1 + G(u_1 + u_\infty, g)].
\end{aligned}
$$

Suppose that (2.3.1) and (2.3.2) are satisfied by some functions u_1 and u_∞. Then by adding (2.3.1) to (2.3.2), we obtain

$$
\begin{aligned}
\partial_t(u_1 + u_\infty) + A(u_1 + u_\infty) &= -P_\infty(B[u_1 + u_\infty]u_\infty) + (F_1 + F_\infty)(u_1 + u_\infty, g)\\
&= -B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g).
\end{aligned}
$$

Set $u = u_1 + u_\infty$, then we have

$$\partial_t u + Au + B[u]u = G(u, g).$$

Consequently, if we show the existence of a pair of functions $\{u_1, u_\infty\}$ satisfying (2.3.1)-(2.3.2), then we obtain a solution u of (0.0.15).

In this chapter, we consider the low frequency part u_1 in a weighted L^∞-space. To do so, the velocity formulation is not suitable, and, instead, we use the momentum formulation for the low frequency part.

Before introducing the momentum formulation, we prepare some inequalities for the low frequency part. The following inequality is concerned with the estimates of the weighted L^p norm for the low frequency part.

Lemma 2.3.1. *Let χ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then for a nonnegative integer ℓ and $1 \le p \le \infty$, there holds*

$$\||x|^\ell(\chi * f)\|_{L^p} \le C\{\||x|^\ell \chi\|_{L^1}\|f\|_{L^p} + \|\chi\|_{L^1}\||x|^\ell f\|_{L^p}\} \qquad (f \in L^p_\ell).$$

Here "$*$" denotes the convolution and C is a positive constant depending only on ℓ.

Proof. Let χ be a function which belongs to the Schwartz space on $\mathbb{R}^n$. Then

$$\begin{aligned} ||x|^\ell(\chi * f)| &\le |x|^\ell \int_{\mathbb{R}^n} |\chi(x-y)f(y)|dy \\ &\le C\int_{\mathbb{R}^n} |x-y|^\ell |\chi(x-y)||f(y)|dy + C\int_{\mathbb{R}^n} |\chi(x-y)||y|^\ell |f(y)|dy. \end{aligned}$$

Therefore, the Young inequality gives

$$\||x|^\ell(\chi * f)\|_{L^p} \le C\{\||x|^\ell \chi\|_{L^1}\|f\|_{L^p} + \|\chi\|_{L^1}\||x|^\ell f\|_{L^p}\} \qquad (f \in L^p_\ell).$$

This completes the proof. □

Applying Lemma 2.3.1, we have the following inequality for the weighted L^p norm of the low frequency part.

Lemma 2.3.2. *Let k and ℓ be nonnegative integers and let $1 \le p \le \infty$. Then there holds the estimate*

$$\||x|^\ell \nabla^k f_1\|_{L^p} \le C\||x|^\ell f_1\|_{L^p} \quad (f_1 \in L^2_{(1)} \cap L^p_\ell).$$

Proof. We define a cut-off function $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$ with $\hat{\chi}_0$ satisfying

$$\hat{\chi}_0 \in C^\infty(\mathbb{R}^n), \quad 0 \le \hat{\chi}_0 \le 1, \quad \hat{\chi}_0 = 1 \text{ on } \{|\xi| \le r_\infty\}, \quad \operatorname{supp}\hat{\chi}_0 \subset \{|\xi| \le 2r_\infty\}. \quad (2.3.3)$$

Since $f_1 \in L^2_{(1)}$, we see that $\nabla^k f_1 = (\nabla^k \chi_0) * f_1$ $(k \ge 0)$. Therefore, by Lemma 2.3.1, we obtain the desired estimate. This completes the proof. □

Since $n \ge 3$, applying the Hardy inequality and Lemma 2.3.2, we have the following inequality for the weighted L^2 norm of the low frequency part.

Lemma 2.3.3. *Let $\phi \in \mathscr{X}_{(1)}$ and $w_1 \in \mathscr{Y}_{(1)}$. Then, it holds that*

$$\|P_1(\phi w_1)\|_{\mathscr{Y}_{(1),L^2}} \leq C\|\phi\|_{L^\infty_{n-1}}\|\nabla w_1\|_{L^2}.$$

Here $C > 0$ is a constant depending only on n.

Proof. By Lemma 2.3.2, we see that

$$\|P_1(\phi w_1)\|_{\mathscr{Y}_{(1),L^2}} \leq C\|\phi w_1\|_{L^2_1}. \tag{2.3.4}$$

Since $n \geq 3$, by the Hardy inequality, we find that

$$\|\phi w_1\|_{L^2_1} \leq C\|\phi\|_{L^\infty_{n-1}}\|\nabla w_1\|_{L^2}. \tag{2.3.5}$$

By (2.3.4) and (2.3.5), we obtain the desired estimate. This completes the proof. □

Let us now reformulate the system (2.3.1)-(2.3.2) by using the momentum. We set m_1 and $u_{1,m}$ by

$$m_1 = w_1 + P_1(\phi w), \quad u_{1,m} = {}^\top(\phi_1, m_1), \tag{2.3.6}$$

where $\phi = \phi_1 + \phi_\infty$ and $w = w_1 + w_\infty$. Then, we see that $\{u_{1,m}, u_\infty\}$ defined by (2.3.6) satisfies the following system of equations.

Lemma 2.3.4. *Assume that $\{u_1, u_\infty\}$ satisfies the system (2.3.1)-(2.3.2). Then $\{u_{1,m}, u_\infty\}$ satisfies the following system:*

$$\begin{aligned} \partial_t u_{1,m} + Au_{1,m} &= F_{1,m}(u_1 + u_\infty, g), \\ \partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g). \end{aligned} \tag{2.3.7}$$

Here

$$\begin{aligned} F_{1,m}(u_1 + u_\infty, g) &= {}^\top(0, \tilde{F}_{1,m}(u_1 + u_\infty, g)), \\ \tilde{F}_{1,m}(u_1 + u_\infty, g) &= -P_1\{\mu\Delta(\phi w) + \tilde{\mu}\nabla\mathrm{div}\,(\phi w) + \frac{\rho_*}{\gamma}\nabla(p^{(1)}(\phi)\phi^2) \\ &\quad +\gamma\mathrm{div}\,((1+\phi)w \otimes w) - \frac{1}{\gamma}((1+\phi)g)\}. \end{aligned} \tag{2.3.8}$$

Proof. If $\{u_1, u_\infty\}$ satisfies the system (2.3.1)-(2.3.2), then $u = u_1 + u_\infty$ satisfies (0.0.15). Hence, we see that

$$\begin{aligned} (1+\phi)w \cdot \nabla w &= \mathrm{div}\,((1+\phi)w \otimes w) - w\mathrm{div}\,((1+\phi)w) \\ &= \mathrm{div}\,((1+\phi)w \otimes w) + \frac{w}{\gamma}\partial_t\phi. \end{aligned} \tag{2.3.9}$$

Therefore, substituting (2.3.9) into (2.3.1), we obtain the equation (2.3.7). This completes the proof. □

Conversely, one can see that the momentum formulation (2.3.2), (2.3.6) and (2.3.7) gives the solution $\{u_1, u_\infty\}$ of (2.3.1)-(2.3.2) if $\phi = \phi_1 + \phi_\infty$ is sufficiently small. In fact, we have the following Lemma.

Lemma 2.3.5. (i) *Let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$ and let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. Then there exists a positive constant δ_0 such that if $\phi = \phi_1 + \phi_\infty$ satisfies $\sup_{t\in[a,b]} \|\phi\|_{L^\infty_{n-1}} \leq \delta_0$, then there uniquely exists $w_1 \in C([a,b]; \mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})$ that satisfies*

$$w_1 = m_1 - P_1(\phi(w_1 + w_\infty)), \tag{2.3.10}$$

where $\phi = \phi_1 + \phi_\infty$. Furthermore, there hold the estimates

$$\begin{aligned} \|w_1\|_{C([a,b];\mathscr{Y}_{(1)})} &\leq C(\|m_1\|_{C([a,b];\mathscr{Y}_{(1)})} + \|w_\infty\|_{C([a,b];L^2)}), &(2.3.11)\\ \int_b^a \|\partial_t w_1(\tau)\|^2_{\mathscr{Y}_{(1)}} d\tau &\leq C((\|\partial_t \nabla\phi_1\|^2_{C([a,b];L^2_1)} + \|\partial_t\phi_\infty\|^2_{C([a,b];L^2_1)})\|w_1\|^2_{C([a,b];L^\infty_{n-2})} \\ &\quad +\|\partial_t\phi\|^2_{C([a,b];L^2)}\|w_1\|^2_{C([a,b];\mathscr{Y}_{(1),L^\infty})}) \\ &\quad + \int_b^a C\Big(\|\partial_t m_1(\tau)\|^2_{\mathscr{Y}_{(1)}} + \|\partial_t\phi\|^2_{(C[a,b];L^2)}\|w_\infty(\tau)\|^2_{H^s_{n-1}} \\ &\quad +\|\partial_t w_\infty(\tau)\|^2_{L^2}\Big)d\tau. &(2.3.12)\end{aligned}$$

(ii) *Let s be an integer satisfying $s \geq \left[\frac{n}{2}\right] + 1$ and let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. Assume that $\phi = \phi_1+\phi_\infty$ satisfies $\sup_{t\in[a,b]} \|\phi\|_{L^\infty_{n-1}} \leq \delta_0$ and $\{u_{1,m}, u_\infty\}$ satisfies*

$$\begin{aligned} \partial_t u_{1,m} + Au_{1,m} &= F_{1,m}(u_1 + u_\infty, g), \\ w_1 &= m_1 - P_1(\phi w), \\ \partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g). \end{aligned}$$

Here $w = w_1 + w_\infty$ with w_1 defined by (2.3.10). *Then $\{u_1, u_\infty\}$ with $u_1 = {}^\top(\phi_1, w_1)$ satisfies* (2.3.1)-(2.3.2).

Proof. (i) Let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy $\{u_{1,m}, u_\infty\} \in X^s(a,b)$. For $F_1 \in \mathscr{Y}_{(1)}$, we set $\mathscr{P}[\phi]F_1 := P_1(\phi F_1)$. By Lemma 2.3.2 and Lemma 2.3.3, we see that $\mathscr{P}[\phi]F_1 \in \mathscr{Y}_{(1)}$ and

$$\|\mathscr{P}[\phi]F_1\|_{\mathscr{Y}_{(1)}} \leq C\delta_0\{\|F_1\|_{L^\infty} + \|\nabla F_1\|_{L^2}\}.$$

Hence, if $\delta_0 \leq \frac{C}{2}$, then $(I+\mathscr{P}[\phi])$ is boundary invertible on $\mathscr{Y}_{(1)}$ and $(I+\mathscr{P}[\phi])^{-1}$ satisfies

$$\|(I + \mathscr{P}[\phi])^{-1}F_1\|_{\mathscr{Y}_{(1)}} \leq C\|F_1\|_{\mathscr{Y}_{(1)}}. \tag{2.3.13}$$

By Lemma 1.1.1 and Lemma 2.3.2, we see that $m_1 - P_1(\phi w_\infty) \in \mathscr{Y}_{(1)}$ and

$$\|m_1 - P_1(\phi w_\infty)\|_{\mathscr{Y}_{(1)}} \le C(\|m_1\|_{\mathscr{Y}_{(1)}} + \|w_\infty\|_{L^2}). \tag{2.3.14}$$

We define w_1 by

$$w_1 := (I + \mathscr{P}[\phi])^{-1}[m_1 - P_1(\phi w_\infty)].$$

Then, by (2.3.13) and (2.3.14), $w_1 \in \mathscr{Y}_{(1)}$ satisfies (2.3.10) and

$$\|w_1\|_{\mathscr{Y}_{(1)}} \le C(\|m_1\|_{\mathscr{Y}_{(1)}} + \|w_\infty\|_{L^2}). \tag{2.3.15}$$

It directly follows from (2.3.15) that $w_1 \in C([a,b]; \mathscr{Y}_{(1)})$ and w_1 satisfies (2.3.11).

We next show that $\partial_t w_1 \in L^2(a,b;\mathscr{Y}_{(1)})$ and $\partial_t w_1$ satisfies (2.3.12). We set $K_1 := m_1 - P_1(\phi w_\infty)$. By Lemma 1.1.1 and Lemma 2.3.2, we see that $-\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1 \in \mathscr{Y}_{(1)}$ and

$$\begin{aligned}\|-\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1\|_{\mathscr{Y}_{(1)}} \le\ & C\Big\{\|\partial_t m_1\|_{\mathscr{Y}_{(1)}} + \|\partial_t\phi\|_{L^2}\|w_1\|_{\mathscr{Y}_{(1),L^\infty}} \\ & +(\|\partial_t\nabla\phi_1\|_{L^2_1} + \|\partial_t\phi_\infty\|_{L^2_1})\|w_1\|_{L^\infty_{n-2}} \\ & +\|\partial_t\phi\|_{L^2}\|w_\infty\|_{H^s_{n-1}} + \|\partial_t w_\infty\|_{L^2}\Big\}.\end{aligned}$$

Therefore,

$$(I + \mathscr{P}[\phi])\partial_t w_1 = -\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1$$

and hence, $\partial_t w_1 = (I + \mathscr{P}[\phi])^{-1}[-\mathscr{P}[\partial_t\phi]w_1 + \partial_t K_1] \in L^2(a,b;\mathscr{Y}_{(1)})$ and $\partial_t w_1$ satisfies (2.3.12).

(ii) We see from (i) that there uniquely exists $w_1 \in C([a,b];\mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})$ satisfying (2.3.10). Then substituting (2.3.10) into (2.3.7), we see that

$$\partial_t\phi_1 + \gamma w_1 = -\gamma P_1(\mathrm{div}\,(\phi w)). \tag{2.3.16}$$

On the other hand, by $(2.3.2)_1$, we have

$$\partial_t\phi_\infty + \gamma w_\infty = -\gamma P_\infty(\mathrm{div}\,(\phi w)). \tag{2.3.17}$$

Hence, by adding (2.3.16) to (2.3.17), we see that

$$\partial_t\phi + \gamma\mathrm{div}\,((1+\phi)w) = 0, \tag{2.3.18}$$

where $\phi = \phi_1 + \phi_\infty$ and $w = w_1 + w_\infty$. Substituting (2.3.10) into (2.3.7), and by using a similar computation as (2.3.9) based on (2.3.18), we see that $u_1 = {}^\top(\phi_1, w_1)$ satisfies (2.3.1). This completes the proof. □

By Lemma 2.3.5, if we show the existence of a pair of functions $\{u_{1,m}, u_\infty\} \in X^s(a,b)$ satisfying (2.3.2), (2.3.7) and (2.3.10), then we obtain a solution $\{u_1, u_\infty\} \in X^s(a,b)$ satisfying (2.3.1)-(2.3.2). Therefore, we will consider (2.3.2), (2.3.7) and (2.3.10) instead of (2.3.1)-(2.3.2).

We look for a time periodic solution u for the system (2.3.2), (2.3.7) and (2.3.10). To solve the time periodic problem for (2.3.2), (2.3.7) and (2.3.10), we introduce solution operators for the following linear problems:

$$\left\{\begin{array}{l}\partial_t u_{1,m} + Au_{1,m} = F_{1,m},\\ u_{1,m}|_{t=0} = u_{01,m},\end{array}\right. \tag{2.3.19}$$

and

$$\left\{\begin{array}{l}\partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty,\\ u_\infty|_{t=0} = u_{0\infty},\end{array}\right. \tag{2.3.20}$$

where $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), u_{01,m}, u_{0\infty}, F_{1,m}$ and F_∞ are given functions.

To formulate the time periodic problem, we denote by $S_1(t)$ the solution operator for (2.3.19) with $F_{1,m} = 0$, and by $\mathscr{S}_1(t)$ the solution operator for (2.3.19) with $u_{01,m} = 0$. We also denote by $S_{\infty,\tilde{u}}(t)$ the solution operator for (2.3.20) with $F_\infty = 0$ and by $\mathscr{S}_{\infty,\tilde{u}}(t)$ the solution operator for (2.3.20) with $u_{0\infty} = 0$. (The precise definition of these operators will be given later.)

As in Chapter 1, we will look for $\{u_{1,m}, u_\infty\}$ satisfying

$$\left\{\begin{array}{l}u_{1,m}(t) = S_1(t)u_{01,m} + \mathscr{S}_1(t)[F_{1,m}(u,g)],\\ u_\infty(t) = S_{\infty,u}(t)u_{0\infty} + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)],\end{array}\right. \tag{2.3.21}$$

where

$$\left\{\begin{array}{l}u_{01,m} = (I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m}(u,g)],\\ u_{0\infty} = (I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_\infty(u,g)],\end{array}\right. \tag{2.3.22}$$

$u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

Let us explain the relation between (2.3.21)-(2.3.22) and the time periodic problem (2.3.2), (2.3.7) and (2.3.10) for the reader's convenience.

If $\{u_{1,m}, u_\infty\}$ satisfies (2.3.2), (2.3.7) and (2.3.10), then $u_{1,m}(t)$ and $u_\infty(t)$ satisfy (2.3.21). Suppose that $\{u_{1,m}, u_\infty\}$ is a T-time periodic solution of (2.3.21). Then, since $u_{1,m}(T) = u_{1,m}(0)$ and $u_\infty(T) = u_\infty(0)$, we see that

$$\left\{\begin{array}{l}(I - S_1(T))u_{1,m}(0) = \mathscr{S}_1(T)[F_{1,m}(u,g)],\\ (I - S_{\infty,u}(T)u_\infty(0))\mathscr{S}_{\infty,u}(T)[F_\infty(u,g)],\end{array}\right.$$

where $u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

Therefore if $(I - S_1(T))$ and $(I - S_{\infty,u}(T))$ are invertible in a suitable sense, then one obtains (2.3.21)-(2.3.22). So, to obtain a T-time periodic solution of (2.3.2), (2.3.7) and

(2.3.10), we look for a pair of functions $\{u_{1,m}, u_\infty\}$ satisfying (2.3.21)-(2.3.22). We will investigate the solution operators $S_1(t)$ and $\mathscr{S}_1(t)$ in section 5; and we state some properties of $S_{\infty,u}(t)$ and $\mathscr{S}_{\infty,u}(t)$ in section 6.

In the remaining of this section we introduce some lemmas which will be used in the proof of Theorem 2.2.1.

For the analysis of the low frequency part, we will use the following well-known inequalities.

Lemma 2.3.6. *Let α and β be positive numbers satisfying $n < \alpha + \beta$. Then there holds the following estimate.*

$$\int_{\mathbb{R}^n} (1+|x-y|)^{-\alpha}(1+|y|^2)^{-\frac{\beta}{2}} dy \le C \begin{cases} (1+|x|)^{n-(\alpha+\beta)} & (\max\{\alpha,\beta\} < n), \\ (1+|x|)^{-\min\{\alpha,\beta\}} \log|x| & (\max\{\alpha,\beta\} = n), \\ (1+|x|)^{-\min\{\alpha,\beta\}} & (\max\{\alpha,\beta\} > n) \end{cases}$$

for $x \in \mathbb{R}^n$.

The following lemma is related to the estimates for the integral kernels which will appear in the analysis of the low frequency part.

Lemma 2.3.7. *Let ℓ be a nonnegative integer and let $E(x) = \mathscr{F}^{-1}\hat{\Phi}_\ell$ $(x \in \mathbb{R}^n)$, where $\hat{\Phi}_\ell \in C^\infty(\mathbb{R}^n - \{0\})$ is a function satisfying*

$$\partial_\xi^\alpha \hat{\Phi}_\ell \in L^1 \quad (|\alpha| \le n-3+\ell),$$
$$|\partial_\xi^\beta \hat{\Phi}_\ell| \le C|\xi|^{-2-|\beta|+\ell} \quad (\xi \ne 0,\ |\beta| \ge 0).$$

Then the following estimate holds for $x \ne 0$.

$$|E(x)| \le C|x|^{-(n-2+\ell)}.$$

Lemma 2.3.7 easily follows from a direct application of [31, Theorem 2.3]; and we omit the proof.

We will also use the following lemma for the analysis of the low frequency part.

Lemma 2.3.8. (i) *Let $E(x)$ $(x \in \mathbb{R}^n)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C}{(1+|x|)^{|\alpha|+n-2}} \quad (|\alpha| = 0,1,2). \tag{2.3.23}$$

Assume that f is a scalar function satisfying $\|f\|_{L_n^\infty \cap L^1} < \infty$. Then there holds the following estimate for $|\alpha| = 0,1$.

$$|[\partial_x^\alpha E * f](x)| \le \frac{C}{(1+|x|)^{|\alpha|+n-2}} \|f\|_{L_n^\infty \cap L^1}.$$

(ii) *Let* $E(x)$ $(x \in \mathbb{R}^n)$ *be a scalar function satisfying* (2.3.23). *Assume that* f *is a scalar function of the form:* $f = \partial_{x_j} f_1$ *for some* $1 \leq j \leq n$ *satisfying* $\|\partial_{x_j} f_1\|_{L^\infty_n} + \|f_1\|_{L^\infty_{n-1}} < \infty$. *Then there holds the following estimate for* $|\alpha| = 0, 1$.

$$|[\partial_x^\alpha E * f](x)| \leq \frac{C}{(1+|x|)^{|\alpha|+n-2}} (\|\partial_{x_j} f_1\|_{L^\infty_n} + \|f_1\|_{L^\infty_{n-1}}).$$

(iii) *Let* $E(x)$ $(x \in \mathbb{R}^n)$ *be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \leq \frac{C}{(1+|x|)^{|\alpha|+n-1}} \quad (|\alpha| = 0, 1).$$

Assume that f *is a scalar function satisfying* $\|f\|_{L^\infty_n} < \infty$. *Then there holds the following estimate for* $|\alpha| = 0, 1$.

$$|[\partial_x^\alpha E * f](x)| \leq \frac{C \log |x|}{(1+|x|)^{|\alpha|+n-1}} \|f\|_{L^\infty_n}.$$

Lemma 2.3.8 (i) and (ii) is given in [32, Lemma 2.5] for $n = 3$ and the case $n \geq 4$ can be proved similarly; the assertion (iii) can be proved by a direct computation based on Lemma 2.3.6; and so the details are omitted here.

The following inequalities will be used to estimate the low frequency part of nonlinear terms.

Lemma 2.3.9. (i) *Let* ℓ *be a nonnegative integer satisfying* $\ell \geq n-1$ *and* $E(x)$ *be a scaler function satisfying*

$$|E(x)| \leq \frac{C}{(1+|x|)^\ell} \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then for $f \in L^2_{n-1}$, *it holds that*

$$\|E * f\|_{L^\infty_{n-1}} \leq C\{\|(1+|y|)^{-\ell}\|_{L^2} \|f\|_{L^2_{n-1}} + \|f\|_{L^2_{n-1}}\}.$$

(ii) *Let* $E(x)$ *be a scaler function satisfying*

$$|E(x)| \leq \frac{C}{(1+|x|)^{n-2}} \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then for $f \in L^1_{n-1}$, *it holds that*

$$\|E * f\|_{L^\infty_{n-1}} \leq C\|f\|_{L^1_{n-1}}.$$

Lemma 2.3.9 easily follows from direct computations; and we omit the proof.

The following lemma is related to the weighted L^∞ estimate for the low frequency part.

Lemma 2.3.10.

$$\|F_1\|_{\mathscr{Y}_{(1),L^\infty}} \le C\|F_1\|_{L^2_{(1),n-1}}.$$

for $F_1 \in L^2_{(1),n-1}$.

Proof. We see that $\tilde{F}_1 = \chi_0 * F_1$, where $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$, $\hat{\chi}_0$ is the cut-off function defined by (2.3.3). Since $\hat{\chi}_0$ belongs to the Schwartz space on $\mathbb{R}^n$, we find that

$$|\partial_x^\alpha \chi_0(x)| \le C(1+|x|)^{-(n+|\alpha|)} \quad \text{for} \quad |\alpha| \ge 0. \tag{2.3.24}$$

Therefore, applying Lemma 2.3.2 and Lemma 2.3.9, we obtain the desired estimate. This completes the proof. □

As for the high frequency part, we have the following inequality.

Lemma 2.3.11. *Let* $\ell \in \mathbb{N}$. *Then there exists a positive constant* C *depending only on* ℓ *such that*

$$\|P_\infty f\|_{L^2_\ell} \le C\|\nabla f\|_{L^2_\ell}.$$

Lemma 2.3.11 follows from the inequalities

$$\||x|^k \nabla f_\infty\|_{L^2}^2 \ge \frac{r_1^2}{2}\||x|^k f_\infty\|_{L^2}^2 - C\||x|^{k-1} f_\infty\|_{L^2}^2 \quad (k = 1, \cdots, \ell)$$

for $f_\infty \in H^1_{(\infty),\ell}$ which are proved in Lemma 1.3.6 by using the Plancherel theorem.

To estimate nonlinear and inhomogeneous terms, we need to estimate $w_1^{(1)} - w_1^{(2)}$ in terms of $\phi_1^{(1)} - \phi_1^{(2)}$, $\phi_\infty^{(1)} - \phi_\infty^{(2)}$, $m_1^{(1)} - m_1^{(2)}$ and $w_\infty^{(1)} - w_\infty^{(2)}$.

Let s be an integer satisfying $s \ge [\frac{n}{2}] + 1$. Let $u_{1,m}^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ and $u_\infty^{(k)} = {}^\top(\phi_\infty^{(k)}, w_\infty^{(k)})$ satisfy $\{u_{1,m}^{(k)}, u_\infty^{(k)}\} \in X^s(a,b)$. Assume that $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ satisfies $\sup_{t\in[a,b]} \|\phi^{(k)}\|_{L^\infty_{n-1}} \le \delta_0$, where δ_0 is the one used in Lemma 2.3.5 for $(k = 1, 2)$. Then by Lemma 2.3.5 (i), there uniquely exist $w_1^{(k)} \in C([a,b]; \mathscr{Y}_{(1)}) \cap H^1(a,b; \mathscr{Y}_{(1)})$ satisfying

$$w_1^{(k)} = m_1^{(k)} + P_1(\phi^{(k)} w^{(k)}),$$

where $w^{(k)} = w_1^{(k)} + w_\infty^{(k)}$ for $k = 1, 2$. Then $w_1^{(1)} - w_1^{(2)}$ satisfies

$$\begin{aligned} & w_1^{(1)} - w_1^{(2)} \\ &= m_1^{(1)} - m_1^{(2)} - P_1(\phi^{(1)}(w^{(1)} - w^{(2)})) - P_1(w^{(2)}(\phi^{(1)} - \phi^{(2)})). \end{aligned} \tag{2.3.25}$$

We obtain the following estimate for $w_1^{(1)} - w_1^{(2)}$.

Lemma 2.3.12. *It holds that*

$$
\begin{aligned}
&\|w_1^{(1)} - w_1^{(2)}\|_{C([a,b];\mathscr{Y}_{(1)})\cap H^1(a,b;\mathscr{Y}_{(1)})} \\
&\quad \le \; C\left(1+\sum_{k=1}^{2}\|\{u_{1,m}^{(k)}, u_\infty^{(k)}\}\|_{X^s(a,b)}\right)\|\{u_{1,m}^{(1)}-u_{1,m}^{(2)}, u_\infty^{(1)}-u_\infty^{(2)}\}\|_{X^{s-1}(a,b)}.
\end{aligned}
$$

Lemma 2.3.12 directly follows from Lemma 1.1.1, Lemma 1.1.2, Lemma 2.3.2, Lemma 2.3.10 and (2.3.25); and we omit the proof.

2.4 Properties of $S_1(t)$ and $\mathscr{S}_1(t)$

In this section we investigate $S_1(t)$ and $\mathscr{S}_1(t)$ and establish estimates for a solution u_1 of

$$\partial_t u_1 + Au_1 = F_1 \tag{2.4.1}$$

satisfying $u_1(0) = u_1(T)$ where $F_1 = {}^\top(0, \tilde{F}_1)$.

We denote by A_1 the restriction of A on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$.

Proposition 2.4.1. (i) *A_1 is a bounded linear operator on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and $S_1(t) = e^{-tA_1}$ is a uniformly continuous semigroup on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. Furthermore, $S_1(t)$ satisfies*

$$S_1(t)u_1 \in C^1([0,T']; \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}), \quad \partial_t S_1(\cdot)u_1 \in C([0,T']; L^2)$$

for each $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and all $T' > 0$,

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1\,(= -AS_1(t)u_1),\ S_1(0)u_1 = u_1 \ \text{ for } \ u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty})} \le C\|u_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}},$$

$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2})} \le C\|u_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}}$$

for $u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, $k = 0, 1$,

$$\|\partial_t S_1(t)u_1\|_{C([0,T'];L^2)} \le C\|u_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}},$$

and

$$\|\partial_t \nabla S_1(t)u_1\|_{C([0,T'];L^2_1)} \le C\|u_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}$$

for $u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, where $T' > 0$ is any given positive number and C is a positive constant depending on T'.

(ii) *Let the operator $\mathscr{S}_1(t)$ be defined by*

$$\mathscr{S}_1(t)F_1 = \int_0^t S_1(t-\tau)F_1(\tau)\,d\tau$$

for $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$. *Then*

$$\mathscr{S}_1(\cdot)F_1 \in C^1([0,T];\mathscr{X}_{(1)}) \times [C([0,T];\mathscr{Y}_{(1)}) \times H^1(0,T;\mathscr{Y}_{(1)})]$$

for each $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$ *and*

$$\partial_t \mathscr{S}_1(t)F_1 + A_1\mathscr{S}_1(t)F_1 = F_1(t), \;\; \mathscr{S}_1(0)F_1 = 0,$$

$$\|\mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p})} \le C\|F_1\|_{C([0,T];\mathscr{X}_{(1),L^p})\times L^2(0,T;\mathscr{Y}_{(1),L^p})},$$

$$\|\partial_t\mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1),L^p})\times L^2(0,T;\mathscr{Y}_{(1),L^p})} \le C\|F_1\|_{C([0,T];\mathscr{X}_{(1),L^p})\times L^2(0,T;\mathscr{Y}_{(1),L^p})},$$

for $p = 2,\infty$, *where* C *is a positive constant depending on* T. *If, in addition,* $F_1 \in C([0,T];L^2_1)$, *then* $\partial_t\mathscr{S}_1(\cdot)F_1 \in C([0,T];L^2)$, $\partial_t\nabla\mathscr{S}_1(\cdot)F_1 \in C([0,T];L^2_1)$,

$$\|\partial_t\mathscr{S}_1(\cdot)F_1\|_{C([0,T];L^2)} \le C\|F_1\|_{C([0,T];L^2)},$$

and

$$\|\partial_t\nabla\mathscr{S}_1(\cdot)F_1\|_{C([0,T];L^2_1)} \le C\|F_1\|_{C([0,T];L^2_1)},$$

where C *is a positive constant depending on* T.

(iii) *It holds that*

$$S_1(t)\mathscr{S}_1(t')F_1 = \mathscr{S}_1(t')[S_1(t)F_1]$$

for any $t \ge 0$, $t' \in [0,T]$ *and* $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$.

Proof of Proposition 2.4.1. Let

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma^\top\xi \\ i\gamma\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top\xi \end{pmatrix} \quad (\xi \in \mathbb{R}^n).$$

Then, $\mathcal{F}(Au_1) = \hat{A}_\xi\hat{u}_1$. Hence, if $\operatorname{supp}\hat{u}_1 \subset \{\xi;|\xi| \le r_\infty\}$, then $\operatorname{supp}\hat{A}_\xi\hat{u}_1 \subset \{\xi;|\xi| \le r_\infty\}$. Furthermore, we see from Lemma 2.3.2 that

$$\|Au_1\|_{\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p}} \le C\|u_1\|_{\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p}}$$

for $p = 2,\infty$. Therefore, A_1 is a bounded linear operator on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. It then follows that $-A_1$ generates a uniformly continuous semigroup $S_1(t) = e^{-tA_1}$ that is given by

$$S_1(t)u_1 = \mathcal{F}^{-1}(e^{-t\hat{A}_\xi}\mathcal{F}u_1) \quad (u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}).$$

Furthermore, $S_1(t)$ satisfies $S_1(\cdot)u_1 \in C^1([0,T'];\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})$ for each $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, and

$$\partial_t S_1(t)u_1 = -A_1S_1(t)u_1\,(= -AS_1(t)u_1),\;\; S_1(0)u_1 = u_1 \;\text{ for }\; u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}.$$

It easily follows from the definition of $S_1(t)$ that

$$\|S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p})} \le C\|u_1\|_{\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p}} \;\; (p = 2,\infty) \;\text{ for }\; u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

and hence, by the relation $\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1$ and Lemma 2.3.2,

$$\|\partial_t S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p})} \leq C\|u_1\|_{\mathscr{X}_{(1),L^p}\times\mathscr{Y}_{(1),L^p}} \ (p=2,\infty) \text{ for } u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

where $T' > 0$ is any given positive number and C is a positive constant depending on T'. In addition, we see from the relation $\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1$ that $\partial_t S_1(\cdot)u_1 \in C([0,T'];L^2)$, $\partial_t \nabla S_1(\cdot)u_1 \in C([0,T'];L^2_1)$,

$$\|\partial_t S_1(\cdot)u_1\|_{C([0,T'];L^2)} \leq C\|u_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}$$

and

$$\|\partial_t \nabla S_1(\cdot)u_1\|_{C([0,T'];L^2_1)} \leq C\|u_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}.$$

The assertion (ii) follows from Lemma 2.3.2, the assertion (i) and the relation $\partial_t \mathscr{S}_1(t)[F_1] = -A_1\mathscr{S}_1(t)[F_1] + F_1(t)$. The assertion (iii) easily follows from the definitions of $S_1(t)$ and $\mathscr{S}_1(t)$. This completes the proof. □

We next investigate invertibility of $I - S_1(T)$.

Proposition 2.4.2. *If F_1 satisfies the conditions given in either* (i)-(iii)*, then there uniquely exists $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ that satisfies $(I - S_1(T))u = F_1$ and u satisfies the estimates in* (i)-(iii)*, respectively.*

(i) $F_1 \in L^2_{(1),1} \cap L^\infty \cap L^1$;

$$\|u\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \leq C\{\|F_1\|_{L^\infty_n} + \|F_1\|_{L^1}\}, \tag{2.4.2}$$

$$\|u\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \leq C(\|F_1\|_{L^1} + \|F_1\|_{L^2_1}). \tag{2.4.3}$$

(ii) $F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_n \cap L^2_{(1),1}$ *with* $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_{n-1}$ *for some* α *satisfying* $|\alpha| = 1$;

$$\|u\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \leq C\{\|F_1\|_{L^\infty_n} + \|F_1^{(1)}\|_{L^\infty_{n-1}}\},$$

$$\|u\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \leq C(\|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}).$$

(iii) $F_1 = \partial_x^\alpha F_1^{(1)} \in L^2_{(1)}$ *with* $F_1^{(1)} \in L^2_{(1),1} \cap L^\infty_n$ *for some* α *satisfying* $|\alpha| \geq 1$;

$$\|u\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \leq C\|F_1^{(1)}\|_{L^\infty_n}, \tag{2.4.4}$$

$$\|u\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \leq C\|F_1^{(1)}\|_{L^2_1}. \tag{2.4.5}$$

To prove Proposition 2.4.2, we prepare some lemmas. Recall that we have the following lemmas related to the linearized semigroup in Chapter 1.

Lemma 2.4.3. ([26]) (i) *The set of all eigenvalues of* $-\hat{A}_\xi$ *consists of* $\lambda_j(\xi)\,(j=1,\pm)$, *where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2, \\ \lambda_\pm(\xi) = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu+\tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}. \end{cases}$$

If $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$, *then*

$$\mathrm{Re}\,\lambda_\pm = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2, \quad \mathrm{Im}\,\lambda_\pm = \pm\gamma|\xi|\sqrt{1 - \frac{(\nu+\tilde{\nu})^2}{4\gamma^2}|\xi|^2}.$$

(ii) *For* $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$, $e^{-t\hat{A}_\xi}$ *has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1,\pm} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ *are eigenprojections for* $\lambda_j(\xi)\,(j=1,\pm)$, *and* $\Pi_j(\xi)\,(j=1,\pm)$ *satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_n - \frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix},$$
$$\Pi_\pm(\xi) = \pm\frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} -\lambda_\mp & -i\gamma^\top\xi \\ -i\gamma\xi & \lambda_\pm\frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix}.$$

Furthermore, if $0 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$, *then there exists a constant* $C > 0$ *such that the estimates*

$$\|\Pi_j(\xi)\| \le C\,(j=1,\pm) \tag{2.4.6}$$

hold for $|\xi| \le r_\infty$.

Hereafter we fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (2.4.6) in Lemma 2.4.3 holds for $|\xi| \le r_\infty$.

Lemma 2.4.4. *Let* α *be a multi-index. Then the following estimates hold true uniformly for* ξ *with* $|\xi| \le r_\infty$ *and* $t \in [0,T]$.

(i) $|\partial_\xi^\alpha\lambda_1| \le C|\xi|^{2-|\alpha|}$, $|\partial_\xi^\alpha\lambda_\pm| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 0)$.

(ii) $|(\partial_\xi^\alpha\Pi_1)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|$, $|(\partial_\xi^\alpha\Pi_\pm)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{F}_1|$ $(|\alpha| \ge 0)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(iii) $|\partial_\xi^\alpha(e^{\lambda_1 t})| \le C|\xi|^{2-|\alpha|}$ $(|\alpha| \ge 1)$.

(iv) $|\partial_\xi^\alpha(e^{\lambda_\pm t})| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 1)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}_1| \le C(|\xi|^{1-|\alpha|}|\hat{F}_1^0| + |\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|)$ $(|\alpha| \ge 1)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(vi) $|\partial_\xi^\alpha(I - e^{\lambda_1 t})^{-1}| \le C|\xi|^{-2-|\alpha|}$ $(|\alpha| \ge 0)$.

(vii) $|\partial_\xi^\alpha (I - e^{\lambda_\pm t})^{-1}| \le C|\xi|^{-1-|\alpha|}$ $(|\alpha| \ge 0)$.

Lemma 2.4.5. *Set*

$$E_{1,j}(x) := \mathcal{F}^{-1}(\hat{\chi}_0 (I - e^{\lambda_j T})^{-1}\Pi_j) \quad (j = 1, \pm) \quad (x \in \mathbb{R}^n),$$

where χ_0 is the cut-off function defined by (2.3.3). *Let α be a multi-index satisfying $|\alpha| \ge 0$. Then the following estimates hold true uniformly for $x \in \mathbb{R}^n$.*

(i) $|\partial_x^\alpha E_{1,1}(x)| \le C(1 + |x|)^{-(n-2+|\alpha|)}$.

(ii) $|\partial_x^\alpha E_{1,\pm}(x)| \le C(1 + |x|)^{-(n-1+|\alpha|)}$.

Proof. It follows from Lemma 2.4.4 that

$$\sum_j |\partial_x^\alpha E_{1,j}(x)| \le C \int_{|\xi| \le 2r_\infty} |\xi|^{-2} d\xi \quad (x \in \mathbb{R}^n).$$

Since $\int_{|\xi| \le r_\infty} |\xi|^{-2} d\xi < \infty$ for $n \ge 3$, we see that

$$\sum_j |\partial_x^\alpha E_{1,j}(x)| \le C \quad (x \in \mathbb{R}^n), \tag{2.4.7}$$

where $C > 0$ is a constant depending on α, T and n. By Lemma 2.4.4, we have

$$\begin{aligned} |\partial_\xi^\beta ((i\xi)^\alpha \hat{\chi}_0 (I - e^{\lambda_1 T})^{-1}\Pi_1)| &\le C|\xi|^{-2+|\alpha|-|\beta|} \text{ for } |\beta| \ge 0, \\ |\partial_\xi^\beta ((i\xi)^\alpha \hat{\chi}_0 (I - e^{\lambda_\pm T})^{-1}\Pi_\pm)| &\le C|\xi|^{-1+|\alpha|-|\beta|} \text{ for } |\beta| \ge 0. \end{aligned}$$

It then follows from Lemma 2.3.7 that

$$|\partial_x^\alpha E_{1,1}(x)| \le C|x|^{-(n-2+|\alpha|)} \text{ and } |\partial_x^\alpha E_{1,\pm}(x)| \le C|x|^{-(n-1+|\alpha|)}. \tag{2.4.8}$$

From (2.4.7) and (2.4.8), we obtain the desired estimates. This completes the proof. □

Let us now prove Proposition 2.4.2.

Proof of Proposition 2.4.2. We define a function u by

$$u = \mathcal{F}^{-1}\{(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1\}.$$

(i) By using Lemma 2.4.4, one can easily obtain (2.4.3). As for (2.4.2), note that

$$u = \mathcal{F}^{-1}((I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1) = \sum_j E_{1,j} * F_1,$$

where $E_{1,j}$ are the ones defined in Lemma 2.4.5. Then by Lemma 2.4.5, we see that $\sum_j E_{1,j}$ satisfies

$$|\partial_x^\alpha \sum_j E_{1,j}(x)| \le C(1+|x|)^{-(n-2+|\alpha|)} \quad (|\alpha| \ge 0).$$

Therefore, applying Lemma 2.3.8 (i), we obtain (2.4.2).

The assertion (ii) follows similarly from Lemma 2.3.8 (ii), Lemma 2.4.4 and Lemma 2.4.5.

(iii) By using Lemma 2.4.4, one can easily obtain (2.4.5). As for (2.4.4), if there exists a function $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_n$ satisfying $F_1 = \partial_x^\alpha F_1^{(1)}$ for some α satisfying $|\alpha| \ge 1$, then

$$u = \left(\sum_j \partial_x^\alpha E_{1,j}\right) * F_1^{(1)}.$$

Lemma 2.4.5 yields

$$|\sum_j \partial_x^{\alpha+\beta} E_{1,j}(x)| \le C(1+|x|)^{-(n-1+|\beta|)}$$

for $x \in \mathbb{R}^n$, $|\alpha| \ge 1$ and $|\beta| \ge 0$. It then follows from Lemma 2.3.8 (iii) that

$$\|u\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \le C\|F_1^{(1)}\|_{L^\infty_n}.$$

This completes the proof. □

In view of Proposition 2.4.2 (i), $I - S_1(T)$ has bounded inverse $(I - S_1(T))^{-1}$: $L^2_{(1),1} \cap L^\infty \cap L^1 \to \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and it holds that

$$\begin{aligned}
&\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L^\infty_n} + \|F_1\|_{L^1}\}, \\
&\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^2} \times \mathscr{Y}_{(1),L^2}} \le C(\|F_1\|_{L^1} + \|F_1\|_{L^2_1}).
\end{aligned}$$

If $F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_n \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_{n-1}$ for some α satisfying $|\alpha| = 1$, then $(I - S_1(T))^{-1}F_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and

$$\begin{aligned}
&\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L^\infty_n} + \|F_1^{(1)}\|_{L^\infty_{n-1}}\}, \\
&\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^2} \times \mathscr{Y}_{(1),L^2}} \le C(\|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}).
\end{aligned}$$

Furthermore, if $F_1 = \partial_x^\alpha F_1^{(1)} \in L^2_{(1)}$ with $F_1^{(1)} \in L^2_{(1),1} \cap L^\infty_n$ for some α satisfying $|\alpha| \ge 1$, then $(I - S_1(T))^{-1}F_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and

$$\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \le C\|F_1^{(1)}\|_{L^\infty_n},$$

$$\|(I-S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C\|F_1^{(1)}\|_{L^2_1}.$$

We next estimate $S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1$ and $\mathscr{S}_1(t)F_1$. Let $E_1(t,\sigma)$ and $E_2(t,\tau)$ be defined by

$$E_1(t,\sigma) = \mathcal{F}^{-1}\{\hat{\chi}_0 e^{-t\hat{A}_\xi}(I-e^{-T\hat{A}_\xi})^{-1}e^{-(T-\sigma)\hat{A}_\xi}\},$$
$$E_2(t,\tau) = \mathcal{F}^{-1}\{\hat{\chi}_0 e^{-(t-\tau)\hat{A}_\xi}\}$$

for $\sigma\in[0,T]$, $0\le\tau\le t\le T$, where $\hat{\chi}_0$ is the cut-off function defined by (4.9). Then $\mathscr{S}_1(t)F_1$ and $S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1$ are given by

$$S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1 = \int_0^T E_1(t,\sigma)*F_1(\sigma)d\sigma, \tag{2.4.9}$$

$$\mathscr{S}_1(t)F_1 = \int_0^t S_1(t-\tau)F_1(\tau)d\tau = \int_0^t E_2(t,\tau)*F_1(\tau)d\tau. \tag{2.4.10}$$

We have the following estimates for $E_1(t,\sigma)*F_1$ and $E_2(t,\tau)*F_1$.

Lemma 2.4.6. *If F_1 satisfies the conditions given in either* (i)-(iii)*, then $E_1(t,\sigma)*F_1\in\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}$, $E_2(t,\tau)*F_1\in\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}$ $(t,\sigma,\tau\in[0,T], j=1,2)$ and $E_1(t,\sigma)*F_1, E_2(t,\tau)*F_1$ satisfy the estimates in* (i)-(iii)*, respectively.*

(i) $F_1\in L^2_{(1),1}\cap L^\infty\cap L^1$;

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L^\infty_n}+\|F_1\|_{L^1}\}$$

and

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C(\|F_1\|_{L^1}+\|F_1\|_{L^2_1})$$

uniformly for $\sigma\in[0,T]$ and $0\le\tau\le t\le T$.

(ii) $F_1=\partial_x^\alpha F_1^{(1)}\in L^\infty_n\cap L^2_{(1),1}$ *with* $F_1^{(1)}\in L^2_{(1)}\cap L^\infty_{n-1}$ *for some α satisfying $|\alpha|=1$;*

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \le C\{\|F_1\|_{L^\infty_n}+\|F_1^{(1)}\|_{L^\infty_{n-1}}\}$$

and

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C(\|F_1^{(1)}\|_{L^2}+\|F_1\|_{L^2_1})$$

uniformly for $\sigma\in[0,T]$ and $0\le\tau\le t\le T$.

(iii) $F_1=\partial_x^\alpha F_1^{(1)}\in L^2_{(1)}$ *with* $F_1^{(1)}\in L^2_{(1),1}\cap L^\infty_n$ *for some α satisfying $|\alpha|\ge 1$;*

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^\infty}\times\mathscr{Y}_{(1),L^\infty}} \le C\|F_1^{(1)}\|_{L^\infty_n}$$

and

$$\|E_1(t,\sigma)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} + \|E_2(t,\tau)*F_1\|_{\mathscr{X}_{(1),L^2}\times\mathscr{Y}_{(1),L^2}} \le C\|F_1^{(1)}\|_{L^2_1}$$

uniformly for $\sigma\in[0,T]$ and $0\le\tau\le t\le T$.

Proof of Lemma 2.4.6. By Lemmas 2.4.3 and 2.4.4, we see that

$$|\partial_\xi^\beta(\hat{\chi}_0(i\xi)^\alpha e^{-t\hat{A}_\xi}(I-e^{-T\hat{A}_\xi})^{-1}e^{-(T-\sigma)\hat{A}_\xi})| \le C|\xi|^{-2+|\alpha|-|\beta|},$$
$$|\partial_\xi^\beta(\hat{\chi}_0(i\xi)^\alpha e^{-(t-\tau)\hat{A}_\xi})| \le C|\xi|^{|\alpha|-|\beta|}$$

for $\sigma \in [0,T]$, $0 \le \tau \le t \le T$ and $|\beta| \ge 0$. It then follows from Lemma 2.3.7 that

$$|\partial_x^\alpha E_1(x)| \le C(1+|x|)^{-(n-2+|\alpha|)}, \quad |\partial_x^\alpha E_2(x)| \le C(1+|x|)^{-(n+|\alpha|)} \tag{2.4.11}$$

for $|\alpha| \ge 0$. Therefore, in a similar manner to the proof of Proposition 2.4.2, we obtain the desired estimate by using Lemma 2.3.8 and Lemma 2.4.5. This completes the proof. □

We see from Proposition 2.4.1 (i), (ii) and Lemma 2.4.6 that the following estimates hold for $S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}$ and $\mathscr{S}_1(t)$.

Proposition 2.4.7. *Let Γ_1 and Γ_2 be defined by*

$$\Gamma_1[\tilde{F}_1](t) = S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}\begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}, \ \Gamma_2[\tilde{F}_1](t) = \mathscr{S}_1(t)\begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}. \tag{2.4.12}$$

If $\tilde{F}_1$ satisfies the conditions given in either (i)-(iii)*, then $\Gamma_j[\tilde{F}_1] \in C^1([0,T];\mathscr{X}_{(1)}) \times [C([0,T];\mathscr{Y}_{(1)}) \cap H^1(0,T;\mathscr{Y}_{(1)})]$ $(j=1,2)$ and $\Gamma_j[\tilde{F}_1]$ satisfy the estimates in* (i)-(iii) *for $j=1,2$, respectively.*

(i) $\tilde{F}_1 \in L^2(0,T;L^2_{(1),1} \cap L^\infty \cap L^1 \cap \mathscr{Y}_{(1)})$*;*

$$\|\Gamma_1[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \le C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)},$$
$$\|\partial_t\Gamma_1[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})} \le C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)},$$

and

$$\|\Gamma_2[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \le C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)},$$
$$\|\partial_t\Gamma_2[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^1\cap L^2_1)} + \|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}).$$

(ii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^\infty_n \cap L^2_{(1),1} \cap \mathscr{Y}_{(1)})$ *with* $F_1^{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_{n-1})$ *for some α satisfying $|\alpha|=1$;*

$$\|\Gamma_1[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)}),$$

$$\|\partial_t\Gamma_1[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)}),$$

and

$$\|\Gamma_2[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)}),$$

$$\|\partial_t \Gamma_2[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})} \leq C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1}\cap L^2)} + \|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}).$$

(iii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^2_{(1)} \cap \mathscr{Y}_{(1)})$ *with* $F_1^{(1)} \in L^2(0,T;L^2_{(1),1} \cap L^\infty_n)$ *for some* α *satisfying* $|\alpha| \geq 1$*;*

$$\|\Gamma_1[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)},$$

$$\|\partial_t\Gamma_1[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)}$$

and

$$\|\Gamma_2[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)},$$
$$\|\partial_t\Gamma_2[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)})\times L^2(0,T;\mathscr{Y}_{(1)})} \leq C(\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n\cap L^2_1)} + \|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}).$$

As for $\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^p})}$ $(p = 2,\infty)$, we have the following proposition.

Proposition 2.4.8. *If* $\tilde{F}_1$ *satisfies the conditions given in either* (i)-(iii)*, then* $\tilde{F}_1 \in L^2(0,T;\mathscr{Y}_{(1)})$ *and* $\tilde{F}_1$ *satisfies the estimates in* (i)-(iii)*, respectively.*

(i) $\tilde{F}_1 \in L^2(0,T;L^2_{(1),1} \cap L^\infty \cap L^1)$*;*

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^\infty})} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n\cap L^1)},$$
$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^2})} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^1\cap L^2_1)}.$$

(ii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^\infty_n \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_{n-1})$ *for some* α *satisfying* $|\alpha| = 1$ *;*

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^\infty})} \leq C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1})}),$$
$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^2})} \leq C(\|F_1^{(1)}\|_{L^2(0,T;L^2)} + \|\tilde{F}_1\|_{L^2(0,T;L^2_1)}).$$

(iii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^2_{(1)})$ *with* $F_1^{(1)} \in L^2(0,T;L^2_{(1),1}\cap L^\infty_n)$ *for some* α *satisfying* $|\alpha| \geq 1$;

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^\infty})} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n)},$$

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1),L^2})} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^2_1)}.$$

Proof of Proposition 2.4.8. We see that $\tilde{F}_1 = \chi_0 * \tilde{F}_1$, where $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$, $\hat{\chi}_0$ is the cut-off function defined by (2.3.3) satisfying (2.3.24). Therefore, in a similar manner to the proof of Proposition 2.4.2, we obtain the desired estimates. This completes the proof. □

We will also need another type of estimates for Γ_1 and Γ_2. We set

$$\Gamma_0[\tilde{F}_1] := (I - S_1(T))^{-1} \begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}.$$

Proposition 2.4.9. (i) *Let α be a multi-index satisfying $|\alpha| \geq 0$. Suppose that $\tilde{F}_1 \in L^1_{n-1} \cap L^2_{(1)}$. Then $\Gamma_0[\partial_x^\alpha \tilde{F}_1] \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and it holds that*

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \leq C \|\tilde{F}_1\|_{L^1_{n-1}}.$$

If $\tilde{F}_1 \in L^2(0,T; L^1_{n-1} \cap L^2_{(1)})$, then, for $j = 1, 2$, $\Gamma_j[\partial_x^\alpha \tilde{F}_1] \in \mathscr{Z}_{(1)}(0,T)$ and it holds that

$$\|\Gamma_j[\partial_x^\alpha \tilde{F}_1](t)\|_{\mathscr{Z}_{(1)}(0,T)} \leq C \|\tilde{F}_1\|_{L^2(0,T;L^1_{n-1})}.$$

(ii) *Let α be a multi-index satisfying $|\alpha| \geq 1$. Suppose that $\tilde{F}_1 \in L^2_{(1),n-1}$. Then $\Gamma_0[\partial_x^\alpha \tilde{F}_1] \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and it holds that*

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \leq C \|\tilde{F}_1\|_{L^2_{n-1}}.$$

If $\tilde{F}_1 \in L^2(0,T; L^2_{(1),n-1})$, then, for $j = 1, 2$, $\Gamma_j[\partial_x^\alpha \tilde{F}_1] \in \mathscr{Z}_{(1)}(0,T)$ and it holds that

$$\|\Gamma_j[\partial_x^\alpha \tilde{F}_1](t)\|_{\mathscr{Z}_{(1)}(0,T)} \leq C \|\tilde{F}_1\|_{L^2(0,T;L^2_{n-1})}.$$

Proof of Proposition 2.4.9. (i) We have already obtained the estimate for $\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1),L^2} \times \mathscr{Y}_{(1),L^2}}$ in (2.4.3). We see from Lemma 2.4.5 and Lemma 2.3.9 (ii) that

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{L^\infty_{n-1}} \leq C \|\tilde{F}_1\|_{L^1_{n-1}}.$$

Therefore, by Lemma 2.3.2, we find that

$$\|\Gamma_0[\partial_x^\alpha \tilde{F}_1]\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}} \leq C \|\tilde{F}_1\|_{L^1_{n-1}}.$$

Similarly, the estimates of Γ_j $(j = 1, 2)$ follow from (2.3.24), Lemma 2.3.9 (ii), Proposition 2.4.1, (2.4.9), (2.4.10) and (2.4.11).

The assertion (ii) can be proved similarly from (2.3.24), Lemma 2.3.9 (i), Proposition 2.4.1, (2.4.9), (2.4.10) and (2.4.11). This completes the proof. □

We are now in a position to give estimates for a solution of (2.4.1) satisfying $u_1(0) = u_1(T)$.

For $F_1 = {}^\top(0, \tilde{F}_1)$ we set

$$\Gamma[\tilde{F}_1] = S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}F_1 + \mathscr{S}_1(t)F_1.$$

Then $\Gamma[\tilde{F}_1]$ is written as

$$\Gamma[\tilde{F}_1](t) = \Gamma_1[\tilde{F}_1] + \Gamma_2[\tilde{F}_1], \tag{2.4.13}$$

where Γ_1 and Γ_2 were defined by (2.4.12).

Proposition 2.4.10. *If $\tilde{F}_1$ satisfies the conditions given in either* (i)-(v)*, then $\Gamma[\tilde{F}_1]$ is a solution of* (2.4.1) *with $F_1 = {}^\top(0, \tilde{F}_1)$ in $\mathscr{Z}_{(1)}(0,T)$ satisfying $\Gamma[\tilde{F}_1](0) = \Gamma[\tilde{F}_1](T)$ and $\Gamma[\tilde{F}_1]$ satisfies the estimate in* (i)-(v)*, respectively.*

(i) $\tilde{F}_1 \in L^2(0,T; L^2_{(1),1} \cap L^\infty \cap L^1)$*;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n \cap L^1 \cap L^2_1)}. \tag{2.4.14}$$

(ii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^\infty_n \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1)} \cap L^\infty_{n-1})$ *for some α satisfying $|\alpha| = 1$;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_n \cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_{n-1} \cap L^2)}). \tag{2.4.15}$$

(iii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^2_{(1)})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1),1} \cap L^\infty_n)$ *for some α satisfying $|\alpha| \geq 1$;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^\infty_n \cap L^2_1)}. \tag{2.4.16}$$

(iv) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^1_{n-1} \cap L^2_{(1)})$ *for some α satisfying $|\alpha| \geq 0$;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^1_{n-1})}. \tag{2.4.17}$$

(v) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^2_{(1),n-1})$ *for some α satisfying $|\alpha| \geq 1$;*

$$\|\Gamma[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|F_1^{(1)}\|_{L^2(0,T;L^2_{n-1})}. \tag{2.4.18}$$

Proof. We find from Proposition 2.4.1 (iii), Proposition 2.4.2 and Proposition 2.4.9 that $\Gamma[\tilde{F}_1]$ is a solution of (2.4.1) with $F_1 = {}^\top(0, \tilde{F}_1)$ satisfying $\Gamma[\tilde{F}_1](0) = \Gamma[\tilde{F}_1](T)$. The estimates of $\Gamma[\tilde{F}_1]$ in (i)-(iii) follow from Proposition 2.4.7 and Proposition 2.4.8. We obtain the estimates of $\Gamma[\tilde{F}_1]$ in (iv) and (v) by Proposition 2.4.9. This completes the proof. □

2.5 Properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$

In this section we state some properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ in weighted Sobolev spaces which were obtained in Chapter 1.

Let us consider the initial value problem (2.3.20). Concerning the solvability of (2.3.20), we have the following

Proposition 2.5.1. *Let $n \geq 3$ and let s be an integer satisfying $s \geq [\frac{n}{2}]+1$. Set $k = s-1$ or s. Assume that*

$$\begin{aligned} &\nabla\tilde{w} \in C([0,T']; H^{s-1}) \cap L^2(0,T'; H^s), \\ &u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)}, \\ &F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T'; H^k_{(\infty)} \times H^{k-1}_{(\infty)}). \end{aligned}$$

Here T' is a given positive number. Then there exists a unique solution $u_\infty = {}^\top(\phi_\infty, w_\infty)$ of (1.3.2) *satisfying*

$$\begin{aligned} &\phi_\infty \in C([0,T']; H^k_{(\infty)}), \\ &w_\infty \in C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)}) \cap H^1(0,T'; H^{k-1}_{(\infty)}). \end{aligned}$$

Proposition 2.5.1 can be verified in a similar manner to the proof of Proposition 1.5.4.

Remark 2.5.2. Concerning the condition for $\tilde{w}$, it is assumed in Proposition 1.5.4 that $\tilde{w} \in C([0,T']; H^s) \cap L^2(0,T'; H^{s+1})$. However, by taking a look at the proof of Proposition 1.5.4, it can be replaced by the condition that $\nabla\tilde{w} \in C([0,T']; H^{s-1}) \cap L^2(0,T'; H^s)$.

In view of Proposition 2.5.1, $S_{\infty,\tilde{u}}(t)$ $(t \geq 0)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ $(t \in [0,T])$ are defined as follows.

We fix an integer s satisfying $s \geq [\frac{n}{2}]+1$ and a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfying

$$\tilde{\phi} \in C_{per}(\mathbb{R}; H^s), \ \nabla\tilde{w} \in C_{per}(\mathbb{R}; H^{s-1}) \cap L^2_{per}(\mathbb{R}; H^s). \tag{2.5.1}$$

Let $k = s-1$ or s. The operator $S_{\infty,\tilde{u}}(t) : H^k_{(\infty)} \longrightarrow H^k_{(\infty)}$ $(t \geq 0)$ is defined by

$$u_\infty(t) = S_{\infty,\tilde{u}}(t)u_{0\infty} \ \text{ for } \ u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},$$

where $u_\infty(t)$ is the solution of (1.3.2) with $F_\infty = 0$; and the operator $\mathscr{S}_{\infty,\tilde{u}}(t) : L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}) \longrightarrow H^k_{(\infty)}$ $(t \in [0,T])$ is defined by

$$u_\infty(t) = \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \ \text{ for } \ F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}),$$

where $u_\infty(t)$ is the solution of (1.3.2) with $u_{0\infty} = 0$.

The operators $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ have the following properties.

Proposition 2.5.3. *Let $n \geq 3$ and let s be a nonnegative integer satisfying $s \geq [\frac{n}{2}] + 1$. Let $k = s - 1$ or s and let ℓ be a nonnegative integer. Assume that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies* (2.5.1)*. Then there exists a constant $\delta > 0$ such that the following assertions hold true if $\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta$.*

(i) *It holds that $S_{\infty,\tilde{u}}(\cdot)u_{0\infty} \in C([0,\infty); H^k_{(\infty),\ell})$ for each $u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),\ell}$ and there exist constants $a > 0$ and $C > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|S_{\infty,\tilde{u}}(t)u_{0\infty}\|_{H^k_{(\infty),\ell}} \leq Ce^{-at}\|u_{0\infty}\|_{H^k_{(\infty),\ell}}$$

for all $t \geq 0$ and $u_{0\infty} \in H^k_{(\infty),\ell}$.

(ii) *It holds that $\mathscr{S}_{\infty,\tilde{u}}(\cdot)F_\infty \in C([0,T]; H^k_{(\infty),\ell})$ for each $F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ satisfies the estimate*

$$\|\mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty]\|_{H^k_{(\infty),\ell}} \leq C\left\{\int_0^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^k_{(\infty),\ell}\times H^{k-1}_{(\infty),\ell}}\, d\tau\right\}^{\frac{1}{2}}$$

for $t \in [0,T]$ and $F_\infty \in L^2(0,T; H^k_{(\infty),\ell} \times H^{k-1}_{(\infty),\ell})$ with a positive constant C depending on T.

(iii) *It holds that $r_{H^k_{(\infty),\ell}}(S_{\infty,\tilde{u}}(T)) < 1$. Here $r_{H^k_{(\infty),\ell}}(S_{\infty,\tilde{u}}(T))$ denotes the spectral radius of $S_{\infty,\tilde{u}}(T)$ on $H^k_{(\infty),\ell}$.*

(iv) *$I - S_{\infty,\tilde{u}}(T)$ has a bounded inverse $(I - S_{\infty,\tilde{u}}(T))^{-1}$ on $H^k_{(\infty),\ell}$ and $(I - S_{\infty,\tilde{u}}(T))^{-1}$ satisfies*

$$\|(I - S_{\infty,\tilde{u}}(T))^{-1}u\|_{H^k_{(\infty),\ell}} \leq C\|u\|_{H^k_{(\infty),\ell}} \quad \text{for} \quad u \in H^k_{(\infty),\ell}.$$

Proposition 2.5.3 can be verified in a similar manner to the proof of Proposition 1.5.6

Remark 2.5.4. In Proposition 1.5.6, it is assumed that

$$\|\tilde{w}\|_{C([0,T];H^s)\cap L^2(0,T;H^{s+1})} \leq \delta.$$

However, by taking a look at the proof of Proposition 1.5.6 and Proposition 1.6.1, it can be replaced by the condition

$$\|\nabla\tilde{w}\|_{C([0,T];H^{s-1})\cap L^2(0,T;H^s)} \leq \delta.$$

Applying Proposition 2.5.3, we easily obtain the following estimate for a solution u_∞ of (1.3.2) satisfying $u_\infty(0) = u_\infty(T)$.

Proposition 2.5.5. *Let $n \geq 3$ and let s be a nonnegative integer satisfying $s \geq [\frac{n}{2}] + 1$. Assume that*

$$F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),n-1} \times H^{k-1}_{(\infty),n-1})$$

with $k = s-1$ or s. Assume also that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (2.5.1). *Then there exists a positive constant δ such that the following assertion holds true if*

$$\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta.$$

The function

$$u_\infty(t) := S_{\infty,\tilde{u}}(t)(I - S_{\infty,\tilde{u}}(T))^{-1} \mathscr{S}_{\infty,\tilde{u}}(T)[F_\infty] + \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \tag{2.5.2}$$

is a solution of (1.3.2) *in $\mathscr{Z}^k_{(\infty),n-1}(0,T)$ satisfying $u_\infty(0) = u_\infty(T)$ and the estimate*

$$\|u_\infty\|_{\mathscr{Z}^k_{(\infty),n-1}(0,T)} \leq C \|F_\infty\|_{L^2(0,T;H^k_{(\infty),n-1} \times H^{k-1}_{(\infty),n-1})}.$$

2.6 Proof of Theorem 2.2.1

In this section we give a proof of Theorem 2.2.1.

We first establish the estimates for the nonlinear and inhomogeneous terms $F_{1,m}(u,g)$ and $F_\infty(u,g)$:

$$F_{1,m}(u,g) = \begin{pmatrix} 0 \\ \tilde{F}_{1,m}(u,g) \end{pmatrix},$$

$$F_\infty(u,g) = P_\infty \begin{pmatrix} -\gamma w \cdot \nabla \phi_1 + F^0(u) \\ \tilde{F}(u,g) \end{pmatrix} =: \begin{pmatrix} F^0_\infty(u) \\ \tilde{F}_\infty(u,g) \end{pmatrix},$$

where $\tilde{F}_{1,m}(u,g)$, $F^0(u)$ and $\tilde{F}(u,g)$ were defined in (2.3.8), (0.0.19) and (0.0.20), respectively, $u = {}^\top(\phi, w)$ is the function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

We first state the estimates for $F_{1,m}(u,g)$ and $F_\infty(u,g)$.

For the estimates of the low frequency part, we recall that

$$\Gamma[\tilde{F}_1](t) := S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1} \begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix} + \mathscr{S}_1(t) \begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}.$$

We first show the estimate of $\|\Gamma[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}_{(1)}(0,T)}$.

Proposition 2.6.1. *Let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy*

$$\sup_{0\le t\le T} \|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u_\infty(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T} \|\phi(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 is the one in Lemma 2.3.5 (i) and $\phi = \phi_1 + \phi_\infty$. Then it holds that

$$\|\Gamma[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{1,m}, u_\infty\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{1,m}, u_\infty\}\|_{X^s(0,T)}\Big)[g]_s$$

uniformly for $u_{1,m}$ and u_∞.

Proof. For $u^{(j)} = {}^\top(\phi^{(j)}, w^{(j)})$ $(j = 1, \infty)$, we set

$$\begin{aligned}
G_1(u^{(1)}, u^{(2)}) &= -P_1(\gamma \mathrm{div}\, w^{(1)} \otimes w^{(2)}),\\
G_2(u^{(1)}, u^{(2)}) &= -P_1(\mu\Delta(\phi^{(1)} w^{(2)}) + \tilde{\mu}\nabla \mathrm{div}\,(\phi^{(1)} w^{(2)})),\\
G_3(\phi, u^{(1)}, u^{(2)}) &= -P_1\left(\frac{\rho_*}{\gamma}\nabla(P^{(1)}(\phi)\phi^{(1)}\phi^{(2)}) + \gamma \mathrm{div}\,(\phi w^{(1)} \otimes w^{(2)})\right),\\
H_k(u^{(1)}, u^{(2)}) &= G_k(u^{(1)}, u^{(2)}) + G_k(u^{(2)}, u^{(1)}), \quad (k = 1, 2),\\
H_3(\phi, u^{(1)}, u^{(2)}) &= G_3(\phi, u^{(1)}, u^{(2)}) + G_3(\phi, u^{(2)}, u^{(1)}).
\end{aligned}$$

Then, $\Gamma[\tilde{F}_{1,m}(u,g)]$ is written as

$$\begin{aligned}
\Gamma[\tilde{F}_{1,m}(u,g)] &= \sum_{k=1}^{2} (\Gamma[G_k(u_1, u_1)] + \Gamma[H_k(u_1, u_\infty)] + \Gamma[G_k(u_\infty, u_\infty)])\\
&\quad + \Gamma[G_3(\phi, u_1, u_1)] + \Gamma[H_3(\phi, u_1, u_\infty)] + \Gamma[G_3(\phi, u_\infty, u_\infty)]\\
&\quad + \Gamma\left[\frac{1}{\gamma}(1+\phi_1)g\right] + \Gamma\left[\frac{1}{\gamma}\phi_\infty g\right].
\end{aligned}$$

Applying (2.4.15) to $\Gamma[G_1(u_1, u_1)]$, we have

$$\|\Gamma[G_1(u_1, u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1, u_\infty\}\|^2_{X^s(0,T)}.$$

As for $\Gamma[G_2(u_1, u_1)]$ and $\Gamma[G_3(\phi, u_1, u_1)]$, we apply (2.4.16) with $F_1^{(1)} = \phi_1 w_1$ $(|\alpha| = 2)$, $F_1^{(1)} = P^{(1)}(\phi)\phi_1^2$ $(|\alpha| = 1)$, and $F_1^{(1)} = \phi w_1 \otimes w_1$ $(|\alpha| = 1)$ to obtain

$$\begin{aligned}
&\|\Gamma[G_2(u_1, u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1, u_\infty\}\|^2_{X^s(0,T)},\\
&\|\Gamma[G_3(\phi, u_1, u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1, u_\infty\}\|^2_{X^s(0,T)}.
\end{aligned}$$

By (2.4.17), we have

$$\|\sum_{k=1}^{2} \Gamma[G_k(u_\infty, u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1, u_\infty\}\|^2_{X^s(0,T)},$$

$$\|\Gamma[G_3(\phi,u_\infty,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.$$

By (2.4.18), we also have

$$\|\sum_{k=1}^{2}\Gamma[G_k(u_1,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)},$$
$$\|\Gamma[G_3(\phi,u_1,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.$$

Concerning $\Gamma[(1+\phi_1)g]$ and $\Gamma[\phi_\infty g]$, we see from (2.4.14) and (2.4.17) that

$$\|\Gamma[(1+\phi_1)g]\|_{\mathscr{Z}_{(1)}(0,T)} + \|\Gamma[\phi_\infty g]\|_{\mathscr{Z}_{(1)}(0,T)} \le C(1+\|\{u_1,u_\infty\}\|_{X^s(0,T)})[g]_s.$$

Therefore, we find that

$$\|\Gamma[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)} + C\Big(1+\|\{u_1,u_\infty\}\|_{X^s(0,T)}\Big)[g]_s.$$

Applying Lemma 2.3.5 (i), we obtain the desired estimate. This completes the proof. □

We next show the estimates for the nonlinear and inhomogeneous terms of the high frequency part.

Proposition 2.6.2. *Let $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfy*

$$\sup_{0\le t\le T}\|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T}\|u_\infty(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T}\|\phi(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 is the one in Lemma 2.3.5 (i) and $\phi = \phi_1+\phi_\infty$. Then it holds that

$$\begin{aligned}&\|F_\infty(u,g)\|_{L^2(0,T;H^s_{n-1}\times H^{s-1}_{n-1})}\\ &\quad\le C\|\{u_{1,m},u_\infty\}\|^2_{X^s(0,T)} + C\Big(1+\|\{u_{1,m},u_\infty\}\|_{X^s(0,T)}\Big)[g]_s\end{aligned}$$

uniformly for $u_{1,m}$ and u_∞.

Proof. We here estimate only $P_\infty(w\cdot\nabla w)$, since the computation is not straightforward due to the slow decay of w_1 as $|x|\to\infty$. By Lemma 2.3.11, we see that

$$\begin{aligned}\|P_\infty(w\cdot\nabla w)\|_{L^2_{n-1}} &\le \|\nabla(w\cdot\nabla w)\|_{L^2_{n-1}}\\ &\le C\|\nabla w\cdot\nabla w\|_{L^2_{n-1}} + \|w\cdot\nabla^2 w\|_{L^2_{n-1}}\\ &\le C(\|(1+|x|)^{n-1}\nabla w\|_{L^\infty}\|\nabla w\|_{L^2}\\ &\qquad +\|(1+|x|)^{n-2}w\|_{L^\infty}\|(1+|x|)\nabla^2 w\|_{L^2}). \qquad (2.6.1)\end{aligned}$$

For $1\le|\alpha|\le s-1$, by Lemma 1.1.1, Lemma 1.1.3 Lemma 1.3.4 and Lemma 2.3.2, we see that

$$\|P_\infty\partial_x^\alpha(w\cdot\nabla w)\|_{L^2_{n-1}}$$

$$
\begin{aligned}
&\leq \|w\cdot\partial_x^\alpha\nabla w\|_{L^2_{n-1}}+\|[\partial_x^\alpha,w]\cdot\nabla w\|_{L^2_{n-1}}\\
&\leq C\Big\{\sum_{j=0}^{1}(\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty}+\|w_\infty\|_{H^s_{n-1}})\Big\}\\
&\qquad\times\Big\{\sum_{j=1}^{2}(\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}})\Big\}. \qquad (2.6.2)
\end{aligned}
$$

It follows from (2.6.1) and (2.6.2) that

$$
\begin{aligned}
&\|P_\infty(w\cdot\nabla w)\|_{H^{s-1}_{n-1}}\\
&\leq C\Big\{\sum_{j=0}^{1}\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty}+\|w_\infty\|_{H^s_{n-1}}\Big\}\\
&\qquad\times\Big\{\sum_{j=1}^{2}\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}})\Big\}.
\end{aligned}
$$

Similarly to (2.6.2), the remaining terms can be estimated by a straightforward application of Lemma 1.1.1, Lemma 1.1.3 Lemma 1.3.4 and Lemma 2.3.2. We thus arrive at

$$
\begin{aligned}
&\|F^0_\infty(u)\|_{H^s_{n-1}}\\
&\leq C\{(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty}+\|\nabla\phi_1\|_{L^2}+\|\phi_\infty\|_{H^s_{n-1}})\\
&\qquad\times(\|(1+|x|)^{n-1}\nabla w_1\|_{L^\infty}+\|\nabla w_1\|_{L^2}+\|w_\infty\|_{H^{s+1}_{n-1}})\\
&\quad+(\|(1+|x|)^{n-2}w_1\|_{L^\infty}+\|\nabla w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}})\\
&\qquad\times(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty}+\|(1+|x|)\nabla\phi_1\|_{L^2})\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\tilde F_\infty(u,g))\|_{H^{s-1}_{n-1}}\\
&\leq C\Big\{\Big(\sum_{j=0}^{1}\|(1+|x|)^{n-2+j}\nabla^j w_1\|_{L^\infty}+\|w_\infty\|_{H^s_{n-1}}\Big)\\
&\qquad\times\Big(\sum_{j=1}^{2}\|(1+|x|)^{j-1}\nabla^j w_1\|_{L^2}+\|w_\infty\|_{H^s_{n-1}}\Big)\\
&\qquad+(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty}+\|\phi_\infty\|_{H^s_{n-1}})(\|\nabla\phi\|_{H^{s-1}}+\|\partial_t w\|_{H^{s-1}}+\|g\|_{H^{s-1}})\Big\}.
\end{aligned}
$$

Integrating these inequalities on $(0,T)$ and applying Lemma 2.3.5 (i), we obtain the desired estimate. This completes the proof. □

We next estimate $F_{1,m}(u^{(1)},g)-F_{1,m}(u^{(2)},g)$.

Proposition 2.6.3. *Let* $u_{1,m}^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ *and* $u_\infty^{(k)} = {}^\top(\phi_\infty^{(k)}, w_\infty^{(k)})$ *satisfy*

$$\sup_{0\le t\le T} \|u_{1,m}^{(k)}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u_\infty^{(k)}(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T} \|\phi^{(k)}(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 2.3.5 (i) *and* $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ $(k = 1, 2)$*. Then it holds that*

$$\begin{aligned}
&\|\Gamma[\tilde{F}_{1,m}(u^{(1)}, g) - \tilde{F}_{1,m}(u^{(2)}, g)]\|_{\mathscr{Z}_{(1)}(0,T)} \\
&\le \ C\sum_{k=1}^{2} \|\{u_{1,m}^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \\
&\qquad +C[g]_s\|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}
\end{aligned}$$

uniformly for $u_{1,m}^{(k)}$ *and* $u_\infty^{(k)}$*.*

Proposition 2.6.3 can be proved in a similar manner to the proof of Proposition 2.6.1; and we omit the proof.

We next estimate $F_\infty(u^{(1)}, g) - F_\infty(u^{(2)}, g)$.

Proposition 2.6.4. *Let* $u_{1,m}^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ *and* $u_\infty^{(k)} = {}^\top(\phi_\infty^{(k)}, w_\infty^{(k)})$ *satisfy*

$$\sup_{0\le t\le T} \|u_{1,m}^{(k)}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u_\infty^{(k)}(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T} \|\phi^{(k)}(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 2.3.5 (i) *and* $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ $(k = 1, 2)$*. Then it holds that*

$$\begin{aligned}
&\|F_\infty(u^{(1)}, g) - F_\infty(u^{(2)}, g)]\|_{L^2(0,T;H^{s-1}_{n-1}\times H^{s-2}_{n-1})} \\
&\le \ C\sum_{k=1}^{2} \|\{u_{1,m}^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \\
&\qquad +C[g]_s\|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)}
\end{aligned}$$

uniformly for $u_{1,m}^{(k)}$ *and* $u_\infty^{(k)}$*.*

Proposition 2.6.4 directly follows from Lemmas 1.1.1–1.1.3, Lemma 1.3.4, Lemma 2.3.2 and Lemma 2.3.11 in a similar manner to the proof of Proposition 2.6.2.

We next show the following estimate which will be used in the proof of Proposition 2.6.6.

Proposition 2.6.5. (i) *Let* $u_{1,m} = {}^{\top}(\phi_1, m_1)$ *and* $u_\infty = {}^{\top}(\phi_\infty, w_\infty)$ *satisfy*

$$\sup_{0\le t\le T} \|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u_\infty(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T} \|\phi(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 2.3.5 (i) *and* $\phi = \phi_1 + \phi_\infty$*. Then it holds that*

$$\|F_{1,m}(u,g)\|_{C([0,T];L^2_1)} \le C\|\{u_{1,m}, u_\infty\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{1,m}, u_\infty\}\|_{X^s(0,T)}\Big)[g]_s$$

uniformly for $u_{1,m}$ *and* u_∞*.*

(ii) *Let* $u^{(k)}_{1,m} = {}^{\top}(\phi^{(k)}_1, m^{(k)}_1)$ *and* $u^{(k)}_\infty = {}^{\top}(\phi^{(k)}_\infty, w^{(k)}_\infty)$ *satisfy*

$$\sup_{0\le t\le T} \|u^{(k)}_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u^{(k)}_\infty(t)\|_{H^s_{n-1}} + \sup_{0\le t\le T} \|\phi^{(k)}(t)\|_{L^\infty} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 2.3.5 (i) *and* $\phi^{(k)} = \phi^{(k)}_1 + \phi^{(k)}_\infty$ $(k = 1, 2)$*. Then it holds that*

$$\begin{aligned}
&\|F_{1,m}(u^{(1)}, g) - F_{1,m}(u^{(2)}, g)\|_{L^2_1} \\
\le\ & C\sum_{k=1}^{2} \|\{u^{(k)}_{1,m}, u^{(k)}_\infty\}\|_{X^s(0,T)} \|\{u^{(1)}_{1,m} - u^{(2)}_{1,m}, u^{(1)}_\infty - u^{(2)}_\infty\}\|_{X^{s-1}(0,T)} \\
& + C[g]_s \|\{u^{(1)}_{1,m} - u^{(2)}_{1,m}, u^{(1)}_\infty - u^{(2)}_\infty\}\|_{X^{s-1}(0,T)}
\end{aligned}$$

uniformly for $u^{(k)}_{1,m}$ *and* $u^{(k)}_\infty$*.*

Proof. As for (i), since $n \ge 3$, we see from the Hardy inequality that

$$\|\phi g\|_{L^2_1} \le C\|\frac{\phi}{|x|}\|_{L^2} \|(1+|x|)^{n-1} g\|_{L^\infty} \le C\|\nabla\phi\|_{L^2} \|(1+|x|)^{n-1} g\|_{L^\infty}.$$

Similarly, we can estimate the remaining terms by using Lemma 1.1.1, Lemma 2.3.2 and the Hardy inequality to obtain

$$\begin{aligned}
&\|F_{1,m}(u,g)\|_{L^2_1} \\
\le\ & C\Big\{(\|(1+|x|)^{n-1}\phi\|_{L^\infty} + \|(1+|x|)w_1\|_{L^\infty} + \|w_\infty\|_{H^s_1})(\|\nabla w_1\|_{L^2} + \|\nabla w_\infty\|_{L^2}) \\
& + \|\nabla\phi\|_{L^2}(\|(1+|x|)^{n-1}\phi_1\|_{L^\infty} + \|\phi_\infty\|_{H^s_{n-1}} + \|(1+|x|)^{n-1} g\|_{L^\infty}) + \|g\|_{L^2_1}\Big\}.
\end{aligned}$$

Applying Lemma 2.3.5 (i), we obtain the desired estimate (i).

The desired estimate in (ii) can be similarly obtained by applying Lemma 1.1.1, Lemma 1.1.2, Lemma 2.3.2 and the Hardy inequality. This completes the proof. □

To prove Theorem 2.2.1, we next show the existence of a solution $\{u_{1,m}, u_\infty\}$ of (2.3.2), (2.3.7) and (2.3.10) on $[0,T]$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$ by an iteration argument.

For $N = 0$, we define $u_{1,m}^{(0)} = {}^\top(\phi_1^{(0)}, m_1^{(0)})$ and $u_\infty^{(0)} = {}^\top(\phi_\infty^{(0)}, w_\infty^{(0)})$ by

$$\begin{cases} u_{1,m}^{(0)}(t) &= S_1(t)\mathscr{S}_1(T)[(I - S_1(T))^{-1}\mathbb{G}_1] + \mathscr{S}_1(t)[\mathbb{G}_1], \\ w_1^{(0)} &= m_1^{(1)} - P_1(\phi^{(0)}w^{(0)}), \\ u_\infty^{(0)}(t) &= S_{\infty,0}(t)(I - S_{\infty,0}(T))^{-1}\mathscr{S}_{\infty,0}(T)[\mathbb{G}_\infty] + \mathscr{S}_{\infty,0}(t)[\mathbb{G}_\infty], \end{cases} \tag{2.6.3}$$

where $t \in [0, T]$, $\mathbb{G} = {}^\top(0, \frac{1}{\gamma}g(x,t))$, $\mathbb{G}_1 = P_1\mathbb{G}$, $\mathbb{G}_\infty = P_\infty\mathbb{G}$, $\phi^{(0)} = \phi_1^{(0)} + \phi_\infty^{(0)}$ and $w^{(0)} = w_1^{(0)} + w_\infty^{(0)}$. Note that $u_{1,m}^{(0)}(0) = u_{1,m}^{(0)}(T)$ and $u_\infty^{(0)}(0) = u_\infty^{(0)}(T)$.

For $N \geq 1$, we define $u_{1,m}^{(N)} = {}^\top(\phi_1^{(N)}, m_1^{(N)})$ and $u_\infty^{(N)} = {}^\top(\phi_\infty^{(N)}, w_\infty^{(N)})$, inductively, by

$$\begin{cases} u_{1,m}^{(N)}(t) &= S_1(t)\mathscr{S}_1(T)[(I - S_1(T))^{-1}F_{1,m}(u^{(N-1)}, g)] + \mathscr{S}_1(t)[F_{1,m}(u^{(N-1)}, g)], \\ w_1^{(N)} &= m_1^{(N)} - P_1(\phi^{(N)}w^{(N)}), \\ u_\infty^{(N)}(t) &= S_{\infty,u^{(N-1)}}(t)(I - S_{\infty,u^{(N-1)}}(T))^{-1}\mathscr{S}_{\infty,u^{(N-1)}}(T)[F_\infty(u^{(N-1)}, g)] \\ &\quad + \mathscr{S}_{\infty,u^{(N-1)}}(t)[F_\infty(u^{(N-1)}, g)], \end{cases} \tag{2.6.4}$$

where $t \in [0, T]$, $u^{(N-1)} = u_1^{(N-1)} + u_\infty^{(N-1)}$, $u_1^{(N-1)} = {}^\top(\phi_1^{(N-1)}, w_1^{(N-1)})$, $\phi^{(N)} = \phi_1^{(N)} + \phi_\infty^{(N)}$ and $w^{(N)} = w_1^{(N)} + w_\infty^{(N)}$. Note that $u_{1,m}^{(N)}(0) = u_{1,m}^{(N)}(T)$ and $u_\infty^{(0)}(0) = u_\infty^{(0)}(T)$.

Proposition 2.6.6. *There exists a constant $\delta_1 > 0$ such that if $[g]_s \leq \delta_1$, then there holds the estimates*

(i)
$$\|\{u_{1,m}^{(N)}, u_\infty^{(N)}\}\|_{X^s(0,T)} \leq C_1[g]_s$$

for all $N \geq 0$, and

(ii)
$$\begin{aligned} &\|\{u_{1,m}^{(N+1)} - u_{1,m}^{(N)}, u_\infty^{(N+1)} - u_\infty^{(N)}\}\|_{X^{s-1}(0,T)} \\ &\quad\leq\ C_1[g]_s\|\{u_{1,m}^{(N)} - u_{1,m}^{(N-1)}, u_\infty^{(N)} - u_\infty^{(N-1)}\}\|_{X^{s-1}(0,T)} \end{aligned}$$

for $N \geq 1$. Here C_1 is a constant independent of g and N.

Proof. If $[g]_s \leq \delta_1$ for sufficiently small δ_1, the estimate (i) easily follows from Propositions 2.4.1, 2.5.5, 2.6.1, 2.6.2, and 2.6.5.

Let us consider the estimate of the difference $\{u_{1,m}^{(N+1)} - u_{1,m}^{(N)}, u_\infty^{(N+1)} - u_\infty^{(N)}\}$. For $N \geq 0$, we set $\bar{\phi}_j^{(N)} = \phi_j^{(N+1)} - \phi_j^{(N)}$ for $j = 1, \infty$, $\bar{m}_1^{(N)} = m_1^{(N+1)} - m_1^{(N)}$, and $\bar{w}_\infty^{(N)} = w_\infty^{(N+1)} - w_\infty^{(N)}$. Then by using (2.6.3) and (2.6.4), we see that $\bar{\phi}_j^{(N)}$, $\bar{m}_1^{(N)}$ and $\bar{w}_\infty^{(N)}$ $(N \geq 1)$ satisfy

$$\begin{cases} \partial_t\bar{\phi}_1^{(N)} + \gamma\mathrm{div}\bar{w}_1^{(N)} = 0, \\ \partial_t\bar{m}_1^{(N)} - \nu\Delta\bar{m}_1^{(N)} - \tilde{\nu}\nabla\mathrm{div}\bar{m}_1^{(N)} + \gamma\nabla\bar{\phi}_1^{(N)} = F_{1,m,2}(\bar{u}^{(N-1)}, g), \\ \bar{w}_1^{(N)} = \bar{m}_1^{(N)} - P_1(\phi^{(N+1)}\bar{w}_1^{(N)}) - P_1(w^{(N)}\bar{\phi}^{(N)}), \end{cases} \tag{2.6.5}$$

$$\begin{cases} \partial_t \bar{\phi}_\infty^{(N)} + \gamma P_\infty(w^{(N)} \cdot \nabla \bar{\phi}_\infty^{(N)}) + \gamma \mathrm{div} \bar{w}_\infty^{(N)} = F_{\infty 1}(\bar{u}^{(N-1)}), \\ \partial_t \bar{w}_\infty^{(N)} - \nu \Delta \bar{w}_\infty^{(N)} - \tilde{\nu} \nabla \mathrm{div} \bar{w}_\infty^{(N)} + \gamma \nabla \bar{\phi}_\infty^{(N)} = F_{\infty 2}(\bar{u}^{(N-1)}, g), \end{cases} \tag{2.6.6}$$

where

$$\begin{aligned} &F_{1,m,2}(\bar{u}^{(N-1)}, g) = \tilde{F}_{1,m}(u^{(N)}, g) - \tilde{F}_{1,m}(u^{(N-1)}, g), \\ &F_{\infty 1}(\bar{u}^{(N-1)}) = F_\infty^0(u^{(N)}) - F_\infty^0(u^{(N-1)}) - \gamma P_\infty((w^{(N)} - w^{(N-1)}) \cdot \nabla \phi_\infty^{(N)}), \\ &F_{\infty 2}(\bar{u}^{(N-1)}, g) = \tilde{F}_\infty(u^{(N)}, g) - \tilde{F}_\infty(u^{(N-1)}, g). \end{aligned}$$

The desired inequality (ii) can be obtained by applying Lemma 2.3.12, Propositions 2.4.1, 2.5.5, 2.6.3, 2.6.4, 2.6.5, and 2.6.6 (i). This completes the proof. $\square$

Before going further, we introduce new notation. We denote by $B_{X^k(a,b)}(r)$ the closed unit ball in $X^k(a,b)$ centered at 0 with radius r, i.e.,

$$B_{X^k(a,b)}(r) = \left\{ \{u_{1,m}, u_\infty\} \in X^k(a,b); \|\{u_{1,m}, u_\infty\}\|_{X^k(a,b)} \le r \right\}.$$

Proposition 2.6.7. *There exists a constant $\delta_2 > 0$ such that if $[g]_s \le \delta_2$, then the system* (2.3.2), (2.3.7) *and* (2.3.10) *has a unique solution $\{u_{1,m}, u_\infty\}$ on $[0,T]$ in $B_{X^s(0,T)}(C_1[g]_s)$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$. The uniqueness of solutions of* (2.3.2), (2.3.7) *and* (2.3.10) *on $[0,T]$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$ holds in $B_{X^s(0,T)}(C_1\delta_2)$.*

Proof. Let $\delta_2 = \min\{\delta_1, \frac{1}{2C_1}\}$ with δ_1 given in Proposition 2.6.6. By Propositions 2.6.6, we see that if $[g]_s \le \delta_2$, then $u_{1,m}^{(N)} = {}^\top(\phi_1^{(N)}, m_1^{(N)})$ and $u_\infty^{(N)} = {}^\top(\phi_\infty^{(N)}, w_\infty^{(N)})$ converge to some $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$, respectively, in the sense

$$\{u_{1,m}^{(N)}, u_\infty^{(N)}\} \ \to \ \{u_{1,m}, u_\infty\} \text{ in } X^{s-1}(0,T),$$

$$u_\infty^{(N)} = {}^\top(\phi_\infty^{(N)}, w_\infty^{(N)}) \ \to \ u_\infty = {}^\top(\phi_\infty, w_\infty) \ *\text{-weakly in } L^\infty(0,T; H^s_{(\infty),n-1}),$$

$$w_\infty^{(N)} \ \to \ w_\infty \text{ weakly in } L^2(0,T; H^{s+1}_{(\infty),n-1}) \cap H^1(0,T; H^{s-1}_{(\infty),n-1}).$$

It is not difficult to see that $\{u_{1,m}, u_\infty\}$ is a solution of (2.3.2), (2.3.7) and (2.3.10) satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$.

It remains to prove $u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0,T]; H^s_{n-1})$, which implies $\{u_{1,m}, u_\infty\} \in B_{X^s(0,T)}(C_1[g]_s)$ with $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$. But this can be shown in the same way as in the proof of Proposition 1.7.4. This completes the proof. $\square$

By Lemma 2.3.5 and Proposition 2.6.7, we can show the existence of the solution of the system (2.3.1)-(2.3.2) satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ in terms of the velocity field w_1.

Corollary 2.6.8. *There exists a constant $\delta_3 > 0$ such that if $[g]_s \leq \delta_3$, then the system* (2.3.1)-(2.3.2) *has a unique solution $\{u_1, u_\infty\}$ on $[0,T]$ in $B_{X^s(0,T)}(C_2[g]_s)$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ where $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, \infty)$ and C_2 is a constant independent of g. The uniqueness of solutions of* (2.3.1)-(2.3.2) *on $[0,T]$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ holds in $B_{X^s(0,T)}(C_2\delta_3)$.*

Proof. Let $[g]_s \leq \delta_2$. By Proposition 2.6.7, we see that the system (2.3.2), (2.3.7) and (2.3.10) has a unique solution $\{u_{1,m}, u_\infty\}$ on $[0,T]$ in $B_{X^s(0,T)}(C_1[g]_s)$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$. The uniqueness of the solution holds in $B_{X^s(0,T)}(C_1\delta_2)$. Therefore, by Lemma 2.3.5, the system (2.3.1)-(2.3.2) has a solution $\{u_1, u_\infty\}$ in $X^s(0,T)$ on $[0,T]$ satisfying

$$\|\{u_1, u_\infty\}\|_{X^s(0,T)} \leq C_2[g]_s$$

and $u_j(0) = u_j(T)$ $(j = 1, \infty)$.

We show the uniqueness of the solution. Let $\{u_1^{(k)}, u_\infty^{(k)}\}$ $(k = 1, 2)$ be solutions of the system (2.3.1)-(2.3.2) in $X^s(0,T)$ on $[0,T]$ satisfying

$$\|\{u_1^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \leq C_2[g]_s$$

and $u_j^{(k)}(0) = u_j^{(k)}(T)$ $(j = 1, \infty)$. We set $u_{1,m}^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)})$ where $m_1^{(k)} = w_1^{(k)} - P_1(\phi^{(k)}w^{(k)})$, $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ and $w^{(k)} = w_1^{(k)} + w_\infty^{(k)}$ $(k = 1, 2)$. Then by Lemmas (1.1.1), (2.3.2), (2.3.3) and (2.3.4), $\{u_{1,m}^{(k)}, u_\infty^{(k)}\}$ are solutions of the system (2.3.2), (2.3.7) and (2.3.10) on $[0,T]$ in $B_{X^s(0,T)}(CC_2[g]_s)$ satisfying $u_{1,m}^{(k)}(0) = u_{1,m}^{(k)}(T)$ and $u_\infty^{(k)}(0) = u_\infty^{(k)}(T)$ $(k = 1, 2)$. If $\delta_3 = \min\{\frac{C_1}{CC_2}\delta_2, \delta_2\}$ and $[g]_s \leq \delta_3$, then $\{u_{1,m}^{(k)}, u_\infty^{(k)}\} \in B_{X^s(0,T)}(C_1\delta_2)$ $(k = 1, 2)$. Therefore, by the uniqueness of the solution of (2.3.2), (2.3.7) and (2.3.10), we see that $u_{1,m}^{(1)} = u_{1,m}^{(2)}$ and $u_\infty^{(1)} = u_\infty^{(2)}$. It follows from Lemma 1.1.1 and Lemma 2.3.2 that $m_1^{(k)} - P_1(\phi^{(k)}w_\infty^{(k)}) \in \mathscr{Y}^{(1)}$ $(k = 1, 2)$, hence,

$$\begin{aligned} w_1^{(1)} &= (I - \mathscr{P}[\phi^{(1)}])^{-1}[m^{(1)} - P_1(\phi^{(1)}w_\infty^{(1)})] \\ &= (I - \mathscr{P}[\phi^{(2)}])^{-1}[m^{(2)} - P_1(\phi^{(2)}w_\infty^{(2)})] \\ &= w_1^{(2)}, \end{aligned}$$

where $\mathscr{P}$ is the one in the proof of Lemma 2.3.5 (i). Therefore, we see that $u_1^{(1)} = u_1^{(2)}$ and $u_\infty^{(1)} = u_\infty^{(2)}$. This completes the proof. □

We can now construct a time periodic solution of (2.3.1)-(2.3.2) by the same argument as that in Chapter 1. As in Chapter 1, based on the estimates in sections 6-8, one can show the following proposition on the unique existence of solutions of the initial value problem.

Proposition 2.6.9. *Let $h \in \mathbb{R}$ and let $U_0 = U_{01} + U_{0\infty}$ with $U_{01} \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and $U_{0\infty} \in H^s_{(\infty),n-1}$. Then there exist constants $\delta_4 > 0$ and $C_3 > 0$ such that if*

$$M(U_{01}, U_{0\infty}, g) := \|U_{01}\|_{\mathscr{X}^{(1)} \times \mathscr{Y}^{(1)}} + \|U_{0\infty}\|_{H^s_{(\infty),n-1}} + [g]_s \leq \delta_4,$$

there exists a solution $\{U_1, U_\infty\}$ *of the initial value problem for* (2.3.1)-(2.3.2) *on* $[h, h+T]$ *in* $B_{X^s(h,h+T)}(C_3 M(U_{01}, U_{0\infty}, g))$ *satisfying the initial condition* $U_j|_{t=h} = U_{0j}$ $(j = 0, \infty)$. *The uniqueness for this initial value problem holds in* $B_{X^s(h,h+T)}(C_3\delta_4)$.

By using Corollary 2.6.8 and Proposition 2.6.9, one can extend $\{u_1, u_\infty\}$ periodically on $\mathbb{R}$ as a time periodic solution of (2.3.1)-(2.3.2). Since the argument for extension is the same as that given in Chapter, we omit the details here. Consequently, we obtain Theorem 2.2.1. This completes the proof.

Chapter 3

Existence and stability of time periodic solution to the compressible Navier-Stokes-Korteweg system on $\mathbb{R}^3$

(0.0.3)-(0.0.5) is considered on $\mathbb{R}^3$. The existence of a time periodic solution is proved for a sufficiently small time-periodic external force by using the time-T-map related to the linearized problem around the motionless state with constant density and absolute temperature. The spectral properties of the time-T-map is investigated by a potential theoretic method and an energy method in some weighted spaces. The stability of the time periodic solution is proved for sufficiently small initial perturbations. It is also shown that the L^∞ norm of the perturbation decays as time goes to infinity.

3.1 Preliminaries

In this section we use the following notations.

For a given Banach space X, the norm on X is denoted by $\|\cdot\|_X$. We denote by L^p the usual L^p space over $\mathbb{R}^3$. The inner product of L^2 is denoted by $(\cdot,\cdot)$. The symbol H^k stands for the usual L^2-Sobolev space of order k. (As usual, $H^0 = L^2$.)

We also denote by L^p the set of all vector fields $w = {}^\top(w_1, w_2, w_3)$ on $\mathbb{R}^3$ with $w_j \in L^p$ $(j = 1, 2, 3)$, i.e., $(L^p)^3$, and the norm $\|\cdot\|_{(L^p)^3}$ is denoted by $\|\cdot\|_{L^p}$, if no confusion will occur. Similarly, for a function space X, we denote by X the set of all vector fields $w = {}^\top(w_1, w_2, w_3)$ on $\mathbb{R}^3$ with $w_j \in X$ $(j = 1, 2, 3)$, i.e., X^3; and the norm $\|\cdot\|_{X^3}$ on it is denoted by $\|\cdot\|_X$. (For example, $(H^k)^3$ is simply denoted by H^k and the norm $\|\cdot\|_{(H^k)^3}$ is denoted by $\|\cdot\|_{H^k}$.)

For $u = {}^\top(\phi, w, \vartheta)$ with $\phi \in H^k$, $w = {}^\top(w_1, w_2, w_3) \in H^m$ and $\vartheta \in H^j$, we denote the norm of u on $H^k \times H^m \times H^j$ by $\|u\|_{H^k\times H^m\times H^j}$:

$$\|u\|_{H^k\times H^m\times H^j} = \left(\|\phi\|_{H^k}^2 + \|w\|_{H^m}^2 + \|\vartheta\|_{H^j}^2\right)^{\frac{1}{2}}.$$

When $m = k = j$, the space $H^k \times (H^k)^3 \times H^k$ is simply denoted by H^k, and, also, the norm $\|u\|_{H^k\times(H^k)^3\times H^k}$ by $\|u\|_{H^k}$ if no confusion will occur:

$$H^k := H^k \times (H^k)^3 \times H^k, \quad \|u\|_{H^k} := \|u\|_{H^k\times(H^k)^3\times H^k} \quad (u = {}^\top(\phi, w, \vartheta)).$$

Similarly, for $u = {}^\top(\phi, w, \vartheta) \in X \times Y \times Z$ with $w = {}^\top(w_1, w_2, w_3)$, we denote its norm by $\|u\|_{X\times Y\times Z}$:

$$\|u\|_{X\times Y\times Z} = \left(\|\phi\|_X^2 + \|w\|_Y^2 + \|\vartheta\|_Z^2\right)^{\frac{1}{2}} \quad (u = {}^\top(\phi, w, \vartheta)).$$

If $X = Z$ and $Y = X^3$, we simply denote $X \times X^3 \times X$ by X, and, its norm $\|u\|_{X\times X^3\times X}$ by $\|u\|_X$:

$$X := X \times X^3 \times X, \quad \|u\|_X := \|u\|_{X\times X^3\times X} \quad (u = {}^\top(\phi, w, \vartheta)).$$

Similar expressions are used for norms of $u = {}^\top(w, \vartheta) \in Y \times Z$ with $w = {}^\top(w_1, w_2, w_3)$.

The Fourier transform of f is denoted by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}dx \quad (\xi \in \mathbb{R}^3).$$

The inverse Fourier transform of f is denoted by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](x) = (2\pi)^{-3}\int_{\mathbb{R}^3} f(\xi)e^{i\xi\cdot x}d\xi \quad (x \in \mathbb{R}^3).$$

For $-\infty \le a < b \le \infty$, the symbols $C^k([a, b]; X)$, $L^p(a, b; X)$ and $H^k(a, b; X)$ stand for the set of all C^k functions on $[a, b]$, the Bochner space on (a, b) and the L^2-Bochner-Sobolev space of order k on (a, b) with values in X, respectively.

We next introduce function spaces with spatial weights. For a nonnegative integer ℓ and $1 \le p \le \infty$, the symbol L^p_ℓ stands for the weighted L^p space defined by

$$L^p_\ell = \{u \in L^p; \|u\|_{L^p_\ell} := \|(1 + |x|)^\ell u\|_{L^p} < \infty\}.$$

Let k and ℓ be nonnegative integers. The spaces H^k_ℓ is defined by

$$H^k_\ell = \{u \in H^k; \|u\|_{H^k_\ell} < +\infty\},$$

where

$$\|u\|_{H^k_\ell} = \left(\sum_{j=0}^{\ell} |u|^2_{H^k_j}\right)^{\frac{1}{2}},$$

$$|u|_{H^k_\ell} = \left(\sum_{|\alpha|\le k} \| \, |x|^\ell \partial_x^\alpha u \, \|^2_{L^2}\right)^{\frac{1}{2}}.$$

We next introduce the weighted $L^\infty \cap L^2$ space. We define $\mathscr{X}$ by

$$\mathscr{X} = \{w \in L^\infty_1, \nabla w \in H^1; \|w\|_{\mathscr{X}} < +\infty\},$$

where

$$\|w\|_{\mathscr{X}} := \sum_{j=0}^{1} \|(1+|x|)^{1+j}\nabla^j w\|_{L^\infty} + \sum_{j=1}^{2} \|(1+|x|)^{j-1}\nabla^j w\|_{L^2}.$$

For a nonnegative integer s satisfying $s \geq 2$ we also define $\mathscr{X}^s$ by

$$\mathscr{X}^s = \{w \in \mathscr{X}; \nabla w \in H^s\}$$

and the norm is defined by

$$\|w\|_{\mathscr{X}^s} = \|w\|_{\mathscr{X}} + \|\nabla w\|_{H^s}.$$

Let ℓ be a nonnegative integer and let s be a nonnegative integer satisfying $s \geq 2$. We define the weighted L^2-Sobolev space $\mathscr{Y}^s_\ell(a,b)$ by

$$\begin{aligned} \mathscr{Y}^s_\ell(a,b) = &\left[C([a,b]; H^{s+1}_\ell) \cap L^2(a,b;; H^{s+2}_\ell)\right] \\ &\times \left[C([a,b]; H^s_\ell) \cap L^2(a,b; H^{s+1}_\ell)\right]. \end{aligned}$$

Recall that the following operators are introduced which decompose a function into its low and high frequency parts in Chapter 1. The operators P_1 and P_∞ on L^2 are defined by

$$P_j f = \mathcal{F}^{-1}\hat{\chi}_j \mathcal{F}[f] \quad (f \in L^2, j = 1, \infty),$$

where $\hat{\chi}_j$ $(j = 1, \infty)$ are the cut-off functions defined by

$$\begin{aligned} &\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^3) \quad (j = 1, \infty), \quad 0 \leq \hat{\chi}_j \leq 1 \quad (j = 1, \infty), \\ &\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \leq r_1), \\ 0 & (|\xi| \geq r_\infty), \end{cases} \\ &\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi) \end{aligned}$$

for constants r_1 and r_∞ satisfying $0 < r_1 < r_\infty$. Clearly, it holds that

$$u = P_1 u + P_\infty u.$$

We fix the constants r_1 and r_∞ in the definitions of P_1 and P_∞ in such a way that the estimate (3.3.1) in Lemma 3.3.2 below holds for $|\xi| \leq r_\infty$.

Let s be a nonnegative integer satisfying $s \geq 2$. We define the space $\mathscr{Z}^s(a,b)$ by

$$\mathscr{Z}^s(a,b) = \left\{u;\ P_1 u \in C(a,b;\mathscr{X}),\ P_\infty u \in \mathscr{Y}^s_2(a,b)\right\},$$

and the norm is defined by

$$\|u\|_{\mathscr{Z}^s(a,b)} = \|P_1 u\|_{C(a,b;\mathscr{X})} + \|P_\infty u\|_{\mathscr{Y}^s_2(a,b)}.$$

We also introduce function spaces of time periodic functions in t with period T. The symbol $C_{per}(\mathbb{R};X)$ stands for the set of all time periodic continuous functions with values in X with period T; and the norm is defined by$\|\cdot\|_{C([0,T];X)}$. We denote by $L^2_{per}(\mathbb{R};X)$ the set of all time periodic locally square integrable functions with values in X with period T; and the norm is defined by $\|\cdot\|_{L^2(0,T;X)}$. Similarly, $H^1_{per}(\mathbb{R};X)$, $X^k_{per}(\mathbb{R})$, and so on, are defined.

For a bounded linear operator L on a Banach space X, the spectral radius of L is denoted by $r_X(L)$.

$B_Z(r)$ stands for the closed ball of a norm space Z centered at 0 with radius r, i.e.,

$$B_Z(r) = \{u \in Z; \|u\|_Z \le r\}.$$

The commutator of L_1 and L_2 is denoted by $[L_1, L_2]$:

$$[L_1, L_2]f = L_1(L_2 f) - L_2(L_1 f).$$

3.2 Main results of Chapter 3

In this section, we state our results on the existence and stability of a time-periodic solution for system (0.0.3)-(0.0.5). Our result on the existence of a time periodic solution is stated as follows.

Theorem 3.2.1. *Let s be an integer satisfying $s \ge 2$. Assume that $g(x,t)$ satisfies* (0.0.7) *and $g \in C_{per}(\mathbb{R}; L^1 \cap L^\infty_3) \cap L^2_{per}(\mathbb{R}; H^{s-1}_2)$. Set*

$$[g]_s = \|g\|_{C([0,T];L^1\cap L^\infty_3)} + \|g\|_{L^2(0,T;H^{s-1}_2)}.$$

Then there exist constants $\delta > 0$ and $C > 0$ such that if $[g]_s \le \delta$, then the system (0.0.3)-(0.0.5) *has a time periodic solution ${}^\top(\rho_{per} - \rho_*, M_{per}, E_{per} - E_*)$ with period T satisfying ${}^\top(\rho_{per} - \rho_*, M_{per}, E_{per} - E_*) \in B_{\mathscr{L}^s_{per}(\mathbb{R})}(C[g]_s)$. Furthermore, the uniqueness of time periodic solutions of* (0.0.3)-(0.0.5) *holds in the class $\{{}^\top(\rho, M, E); {}^\top(\rho - \rho_*, M, E - E_*) \in B_{\mathscr{L}^s_{per}(\mathbb{R})}(C\delta).\}$*

Our next issue to study the stability of the time periodic solution obtained in Theorem 3.2.1. Let ${}^\top(\rho_{per}, M_{per}, E_{per})$ be the time-periodic solution obtained in Theorem 3.2.1, let the perturbation be denoted by $\tilde{u} = {}^\top(\tilde{\rho}, \tilde{M}, \tilde{E})$, where $\tilde{\rho} = \rho - \rho_{per}$, $\tilde{M} = M - M_{per}$, $\tilde{E} = E - E_{per}$ and let the initial perturbation be denoted by

$$\tilde{u}_0 = \tilde{u}|_{t=0} = {}^\top(\rho(0) - \rho_{per}(0), M(0) - M_{per}(0), E(0) - E_{per}(0)).$$

We have the following stability result of the time periodic solution.

Theorem 3.2.2. *Let s be an integer satisfying $s \geq 2$. Assume that $g(x,t)$ satisfies* (0.0.7) *and $g \in C_{per}(\mathbb{R}; L^1 \cap L^\infty_3) \cap L^2_{per}(\mathbb{R}; H^s_2)$. Let ${}^\top(\rho_{per}, M_{per}, E_{per})$ be the time-periodic solution obtained in Theorem* 3.2.1 *and let $\tilde{u}_0 \in H^{s+1} \times H^s$. Then there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that if*

$$[g]_{s+1} \leq \epsilon_1, \quad \|\tilde{u}_0\|_{H^{s+1}\times H^s} \leq \epsilon_2,$$

then $\tilde{u}(t)$ exists globally in time and $\tilde{u}(t)$ satisfies

$$\begin{aligned}
&\tilde{u} \in C([0,\infty); H^{s+1} \times H^s),\\
&\|\tilde{u}(t)\|^2_{H^{s+1}\times H^s} + \int_0^t \|\nabla \tilde{u}(\tau)\|^2_{H^{s+1}\times H^s} d\tau \leq C\|\tilde{u}_0\|^2_{H^{s+1}\times H^s} \quad (t \in [0,\infty)),\\
&\|\tilde{u}(t)\|_{L^\infty} \to 0 \quad (t \to \infty).
\end{aligned}$$

Theorem 3.2.2 is proved as follows. We write (0.0.3)-(0.0.5) into (0.0.8)-(0.0.10). Let ${}^\top(\rho_{per}, M_{per}, E_{per})$ be the periodic solution given in Theorem 3.2.1. We set v_{per}, θ_{per} and U_{per} by

$$v_{per} = \frac{M_{per}}{\rho_{per}}, \ \theta_{per} = \frac{1}{C_v}\Big(E_{per} - \frac{|M_{per}|^2}{2\rho_{per}^2}\Big), \ U_{per} = {}^\top(\rho_{per}, v_{per}, \theta_{per}).$$

It directly follows from Lemma 1.1.1 and Lemma 1.1.3 that U_{per} satisfies the estimate

$$\|{}^\top(v_{per}, \theta_{per} - \theta_*)\|_{C([0,T];\mathscr{X}^s)} \leq C[g]_{s+1}. \tag{3.2.1}$$

Let the perturbation be denoted by $U = {}^\top(\phi, w, \vartheta)$, where $\phi = \rho - \rho_{per}, w = v - v_{per}, \vartheta = \theta - \theta_{per}$. Then the perturbation $U = {}^\top(\phi, w, \vartheta)$ is governed by

$$\begin{cases}
\partial_t \phi + v_{per} \cdot \nabla \phi + \phi \mathrm{div} v_{per} + \rho_{per} \mathrm{div} w + w \cdot \nabla \rho_{per} = f^1,\\
\partial_t w - \frac{1}{\rho_{per}}\{\mu \Delta w + (\mu + \mu')\nabla \mathrm{div}\, w\} + B_1(U, U_{per})\nabla\phi - \kappa \nabla \Delta \phi + B_2(U, U_{per})\nabla \vartheta = f^2,\\
\partial_t \vartheta - \tilde{\alpha} B_3(U_{per})\Delta\vartheta + B_4(U, U_{per})\mathrm{div}\, w = f^3,
\end{cases} \tag{3.2.2}$$

where

$$f^1 = -\mathrm{div}(\phi w),$$

$$\begin{aligned}
f^2 = &-(v_{per}\cdot\nabla)w - (w\cdot\nabla)(v_{per} + w) - (B_1(U, U_{per}) - B_1(U_{per}))\nabla\rho_{per}\\
&- (B_2(U, U_{per}) - B_2(U_{per}))\nabla\theta_{per} - \frac{\phi}{\rho_{per}(\rho_{per} + \phi)}\Big\{\mu\Delta(v_{per} + w) + (\mu + \mu')\nabla\mathrm{div}\,(v_{per} + w)\Big\},
\end{aligned}$$

$$\begin{aligned}
f^3 = &-(v_{per}\cdot\nabla)\vartheta - (w\cdot\nabla)(\theta_{per} + \vartheta) + \tilde{\alpha}(B_3(U, U_{per}) - B_3(U_{per}))\Delta(\theta_{per} + \vartheta)\\
&+ (B_3(U, U_{per}) - B_3(U_{per}))(\Psi(v_{per}) + \tilde{\Phi}(\rho_{per}, v_{per}))\\
&+ B_3(U, U_{per})\Big\{\Psi(v) - \Psi(v_{per}) + \tilde{\Phi}(\rho, v) - \tilde{\Phi}(\rho_{per}, v_{per})\Big\} - (B_4(U, U_{per}) - B_4(U_{per}))\mathrm{div}\, v_{per},
\end{aligned}$$

$$B_1(U,U_{per}) = \frac{P_\rho(\rho_{per}+\phi,\theta_{per}+\theta)}{\rho_{per}+\phi}, \quad B_2(U,U_{per}) = \frac{P_\theta(\rho_{per}+\phi,\theta_{per}+\theta)}{\rho_{per}+\phi},$$
$$B_3(U,U_{per}) = \frac{1}{C_v(\rho_{per}+\phi)}, \quad B_4(U,U_{per}) = \frac{(\theta_{per}+\theta)P_\theta(\rho_{per}+\phi,\theta_{per}+\theta)}{C_v(\rho_{per}+\phi)}$$

with

$$B_1(U_{per}) = \frac{P_\rho(\rho_{per},\theta_{per})}{\rho_{per}}, \quad B_2(U_{per}) = \frac{P_\theta(\rho_{per},\theta_{per})}{\rho_{per}},$$
$$B_3(U_{per}) = \frac{1}{C_v\rho_{per}}, \quad B_4(U_{per}) = \frac{\theta_{per}P_\theta(\rho_{per},\theta_{per})}{C_v\rho_{per}}.$$

We consider the initial value problem for (3.2.2) under the initial condition

$$U|_{t=0} = U_0 = {}^\top(\phi_0, w_0, \vartheta_0).$$

One can show that if $[g]_{s+1}$ and $\|U_0\|_{H^{s+1}\times H^s}$ are sufficiently small, then $U(t)$ exists globally in time and $U(t)$ satisfies

$$\begin{aligned}
&U \in C([0,\infty); H^{s+1}\times H^s),\\
&\|U(t)\|^2_{H^{s+1}\times H^s} + \int_0^t \|\nabla U(\tau)\|^2_{H^{s+1}\times H^s} d\tau \le C\|U_0\|^2_{H^{s+1}\times H^s} \quad (t\in[0,\infty)),\\
&\|U(t)\|_{L^\infty} \to 0 \quad (t\to\infty).
\end{aligned}$$

These can be proved by similar methods as those in [4, 16], since the Hardy inequality works well to deal with the linear terms including ${}^\top(\rho_{per}, v_{per}, \theta_{per})$ due to the estimates for ${}^\top(\rho_{per}, v_{per}, \theta_{per})$ in Theorem 3.2.1 and (3.2.1). We thus omit the details.

To prove Theorem 3.2.1, we rewrite (0.0.3)-(0.0.5) as follows. Let

$$\gamma_1 = \sqrt{P_\rho(\rho_*,\theta_*)}, \quad \gamma_2 = \gamma_1\sqrt{\frac{C_vP(\rho_*,\theta_*)}{P_\theta(\rho_*,\theta_*)}}, \quad \gamma_3 = \frac{P_\theta(\rho_*,\theta_*)\gamma_2}{\gamma_1 C_v}.$$

We define ϕ, m and ε by $\phi = \rho - \rho_*$, $m = \frac{M}{\gamma_1}$ and $\varepsilon = (\rho_*+\phi)\frac{E-E_*}{\gamma_2}$, respectively. Then (0.0.3)-(0.0.5) is rewritten as

$$\partial_t u + Au = F(u,g), \tag{3.2.3}$$

where

$$u = {}^\top(\phi, m, \varepsilon), \quad A = \begin{pmatrix} 0 & \gamma_1\mathrm{div} & 0 \\ \gamma_1\nabla - \kappa_0\nabla\Delta & -\nu\Delta - \tilde{\nu}\nabla\mathrm{div} & \zeta\nabla \\ 0 & \zeta\mathrm{div} & -\alpha_0\Delta \end{pmatrix}, \tag{3.2.4}$$

$$\nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu+\mu'}{\rho_*}, \quad \zeta = \frac{\gamma_1 P(\rho_*,\theta_*)}{\gamma_2\rho_*}, \quad \kappa_0 = \frac{\kappa\rho_*}{\gamma_1}, \quad \alpha_0 = \frac{\tilde{\alpha}}{C_v\rho_*}$$

and

$$
\begin{aligned}
F(u,g) &= \begin{pmatrix} 0 \\ F^2(u,g) \\ F^3(u) \end{pmatrix}, && (3.2.5)\\
F^2(u,g) &= -\Big\{\frac{\rho_*}{\gamma_1}\mathrm{div}\,(m\otimes m) + \gamma_1\mathrm{div}\,(P^{(1)}(\phi)\phi m\otimes m) \\
&\quad +\rho_*\nu\Delta(P^{(1)}(\phi)\phi m) + \rho_*\tilde{\nu}\nabla\mathrm{div}\,(P^{(1)}(\phi)\phi m) + \gamma_3\nabla(P^{(1)}(\phi)\phi\varepsilon) \\
&\quad +\frac{1}{\gamma_1}\nabla(P^{(2)}(\phi)\phi^2) - \frac{1}{\gamma_1}\nabla\Big(P_\theta(\rho_*,\theta_*)\frac{\gamma_1^2|m|^2}{2C_v(\rho_*+\phi)^2}\Big) \\
&\quad +\frac{1}{C_v^2\gamma_1}\nabla\Big\{P^{(3)}(\theta)\Big(\Big(\frac{\gamma_1^2|m|^2}{2(\rho_*+\phi)^2}\Big)^2 - \frac{\gamma_1^2\gamma_2\varepsilon|m|^2}{(\rho_*+\phi)^3} + \frac{\gamma_2^2\varepsilon^2}{(\rho_*+\phi)^2}\Big)\Big\} \\
&\quad +\frac{1}{C_v\gamma_1}\nabla\Big\{P^{(4)}(\theta)\Big(\frac{\gamma_2\phi\varepsilon}{\rho_*+\phi} - \frac{\gamma_1^2|m|^2\phi}{2(\rho_*+\phi)^2}\Big)\Big\} \\
&\quad -\frac{1}{\gamma_1}\mathrm{div}\,\Phi(\phi) - \frac{1}{\gamma_1}(\rho_*+\phi)g\Big\}, && (3.2.6)\\
\theta &= \frac{1}{C_v}\Big(E_* + \frac{\gamma_2\varepsilon}{\rho_*+\phi} - \gamma_1^2\frac{|m|^2}{2(\rho_*+\phi)^2}\Big),\\
P^{(1)}(\phi) &= \int_0^1 f'(\rho_*+\tau\phi)d\tau, \quad f(\tau)=\frac{1}{\tau} \quad (\tau\in\mathbb{R}),\\
P^{(2)}(\phi,\theta) &= \int_0^1 (1-\tau)P_{\rho\rho}\Big(\rho_*+\tau\phi,\theta\Big)d\tau,\\
P^{(3)}(\theta) &= \int_0^1 (1-\tau)P_{\theta\theta}\Big(\rho_*,\theta_*+\tau(\theta-\theta_*)\Big)d\tau,\\
P^{(4)}(\theta) &= \int_0^1 P_{\rho\theta}\Big(\rho_*,\theta_*+\tau(\theta-\theta_*)\Big)d\tau,\\
\Phi(\phi) &= \kappa\Big\{\phi\Delta\phi I + (\nabla\phi)\cdot(\nabla\phi)I - \frac{|\nabla\phi|^2}{2}I - \nabla\phi\otimes\nabla\phi\Big\},\\
F^3(u) &= -\Big\{\frac{\gamma_1}{\rho_*}\mathrm{div}\,(\varepsilon m) + \gamma_1\mathrm{div}\,(P^{(1)}(\phi)\phi\varepsilon) - \alpha_0\rho_*\Delta(P^{(1)}(\phi)\phi\varepsilon), \\
&\quad +\frac{\alpha_0}{C_v\gamma_2}\Delta\Big(\frac{\gamma_1^2|m|^2}{2(\rho_*+\phi)^2}\Big) + \frac{\gamma_1}{\gamma_2}\mathrm{div}\,(P^{(1)}(\phi)\phi mP(\rho_*+\phi,\theta)) \\
&\quad +\frac{\gamma_1}{\rho_*\gamma_2}\mathrm{div}\,(mP^{(5)}(\phi,\theta)\phi) \\
&\quad +\frac{\gamma_1}{C_v\rho_*\gamma_2}\mathrm{div}\,\Big(mP^{(6)}(\theta)\Big(\frac{\gamma_2\varepsilon}{(\rho_*+\phi)} - \frac{\gamma_1^2|m|^2}{2(\rho_*+\phi)^2}\Big)\Big) \\
&\quad -\frac{\gamma_1}{\gamma_2}\mathrm{div}\,\Big(\Big(\mathcal{S}(\frac{\gamma_1 m}{\rho_*+\phi}) + \mathcal{K}(\rho_*+\phi)\Big)\frac{m}{\rho_*+\phi}\Big) - \frac{1}{\gamma_2}mg\Big\}, && (3.2.7)\\
P^{(5)}(\phi,\theta) &= \int_0^1 P_\rho(\rho_*+\tau\phi,\theta)d\tau,
\end{aligned}
$$

$$P^{(6)}(\theta) \quad = \quad \int_0^1 P_\theta(\rho_*, \theta_* + \tau(\theta - \theta_*))d\tau.$$

Let us introduce a semigroup $S(t) = e^{-tA}$ generated by A;

$$S(t) = e^{-tA} = \mathcal{F}^{-1} e^{-t\hat{A}_\xi} \mathcal{F},$$

where

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma_1{}^\top\xi & 0 \\ i\gamma_1\xi + i\kappa_0|\xi|^2\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top\xi & i\zeta\xi \\ 0 & i\zeta^\top\xi & \alpha_0|\xi|^2 \end{pmatrix} \quad (\xi \in \mathbb{R}^3).$$

Then $S(t)$ has the following properties.

Proposition 3.2.3. *Let s be a nonnegative integer satisfying $s \geq 2$. Then $S(t) = e^{-tA}$ is a contraction semigroup on $H^s \times H^{s-1} \times H^{s-1}$. In addition, for each $u \in H^s \times H^{s-1} \times H^{s-1}$ and all $T' > 0$, $S(t)$ satisfies*

$$S(\cdot)u \in C([0, T']; H^s \times H^{s-1} \times H^{s-1}), \ S(0)u = u$$

and there hold the estimates

$$\|S(t)u\|_{H^s \times H^{s-1} \times H^{s-1}} \leq \|u\|_{H^s \times H^{s-1} \times H^{s-1}} \tag{3.2.8}$$

for $u \in H^s \times H^{s-1} \times H^{s-1}$ and $t \geq 0$.

Proof. Let $F = {}^\top(F^1, F^2, F^3) \in H^s \times H^{s-1} \times H^{s-1}$. We consider the following resolvent problem

$$\lambda u + Au = F, \tag{3.2.9}$$

where $\lambda \in \mathbb{C}$ is a parameter. Here we regard A as an operator on $H^s \times H^{s-1} \times H^{s-1}$ with domain $D(A) = H^{s+2} \times H^{s+1} \times H^{s+1}$. Taking the Fourier transform of (3.2.9), we obtain

$$\lambda \hat{u} + \hat{A}_\xi \hat{u} = \hat{F}. \tag{3.2.10}$$

Then, by a similar manner to the proof of Proposition 3.4.4 below, one can see that

$$\begin{aligned} &\mathrm{Re}\lambda \sum_{|\alpha|=0}^{s-1} \left(|(i\xi)^\alpha \hat{u}|^2 + \frac{\kappa_0}{\gamma_1} |(i\xi)^\alpha (i\xi)\hat{\phi}|^2 \right) \\ &\leq \sum_{|\alpha|=0}^{s-1} \left\{ |(i\xi)^\alpha \hat{F}|^2 + \frac{\kappa_0}{\gamma_1} |(i\xi)^\alpha (i\xi) \hat{F}^1|^2 \right\}, \end{aligned} \tag{3.2.11}$$

$$\mathrm{Re}\lambda \left\{ \sum_{|\alpha|=0}^{s} \left(\kappa_1 |(i\xi)^\alpha \hat{u}|^2 + \frac{\kappa_1\kappa_0}{\gamma_1} |(i\xi)^\alpha (i\xi)\hat{\phi}|^2 + (i\xi)^\alpha \hat{m} \cdot \overline{(i\xi)^\alpha (i\xi)\hat{\phi}} \right) \right\}$$

$$+ \quad d_1\Big(\sum_{|\alpha|=0}^{s} |(i\xi)^\alpha (i\xi)\ {}^\top(\hat{m},\hat{h})|^2 + \sum_{|\alpha|=0}^{s} |(i\xi)^\alpha (i\xi)(i\xi)\hat{\phi}|^2 \Big)$$
$$\leq C\Big\{ \sum_{|\alpha|=0}^{s} |(i\xi)^\alpha \hat{F}^1|^2 + \sum_{|\alpha|=0}^{s-1} (|(i\xi)^\alpha \hat{F}^2|^2 + |(i\xi)^\alpha \hat{F}^3|^2) \Big\}, \qquad (3.2.12)$$

for $\xi \in \mathbb{R}^3$, where κ_1 and d_1 are the same ones in Proposition 3.4.4. Therefore, if $\mathrm{Re}\lambda > 0$, then $(\lambda + \hat{A}_\xi)^{-1}$ exists for each $\xi \in \mathbb{R}^3$ and $\hat{u}$ is given by $\hat{u} = (\lambda + \hat{A}_\xi)^{-1}\hat{F}$. We define the norm $|||\cdot|||_s$ on $H^s \times H^{s-1} \times H^{s-1}$ by

$$|||u|||_s = \Big(\sum_{|\alpha|=0}^{s-1} \Big\{ \|\partial_x^\alpha u\|_{L^2}^2 + \frac{\kappa_0}{\gamma_1} \|\partial_x^\alpha \nabla\phi\|_{L^2}^2 \Big\} \Big)^{\frac{1}{2}}$$

for $u = {}^\top(\phi, m, \varepsilon)$. It follows from (3.2.11) and (3.2.12) that

$$\mathrm{Re}\lambda |||u|||_s \leq |||F|||_s$$

and if $\mathrm{Re}\lambda > 0$, then it holds that

$$\|u\|_{H^{s+2}\times H^{s+1}\times H^{s+1}} \leq C|||F|||_s.$$

Hence

$$\{\lambda;\ \mathrm{Re}\lambda > 0\} \subset \rho(-A),$$

where $\rho(-A)$ denotes the resolvent set of $-A$ and it holds that

$$|||(\lambda + A)^{-1}F|||_s \leq \frac{1}{\mathrm{Re}\lambda} |||F|||_s.$$

This implies that $S(t) = e^{-tA}$ is a contraction semigroup on $H^s \times H^{s-1} \times H^{s-1}$, and we obtain (3.2.8). This completes the proof. □

We set an operator Γ using the time-T-map by

$$\Gamma[F] = S(t)(I - S(T))^{-1}\mathscr{S}(T)F + \mathscr{S}(t)F \quad (t \in [0,T]), \qquad (3.2.13)$$

where

$$\mathscr{S}(t)F := \int_0^t S(t-\tau)F(\tau)d\tau.$$

To solve the time periodic problem for (3.2.3), as in Chapter 1, we look for a fixed point u of $\Gamma[F(u,g)]$, i.e.,

$$u = \Gamma[F(u,g)] \quad (t \in [0,T]), \qquad (3.2.14)$$

where $u = {}^\top(\phi, m, \varepsilon)$ and $F(u,g)$ is given by (3.2.5)-(3.2.7). From (3.2.13) and (3.2.14), it holds that $u(T) = u(0)$. Therefore, we will investigate properties of the map Γ. Observe that

$$P_j\Gamma[F(u,g)] = \Gamma[P_jF(u,g)] \quad (j = 1, \infty)$$

and

$$\text{supp } \widehat{P_1 F}(u,g) \subset \{|\xi| \le r_\infty\},$$
$$\text{supp } \widehat{P_\infty F}(u,g) \subset \{|\xi| \ge r_1\}.$$

So we will investigate the restriction of Γ to the space of functions whose Fourier transforms have support in $\{|\xi| \le r_\infty\}$ and will then establish estimates for ΓP_1. Likewise, the restriction of Γ to the high frequency part will be investigated to establish estimates for ΓP_∞ in section 5.

3.3 Estimates of Γ for the low frequency part

In this section we estimate Γ for the low frequency part. We begin with function spaces for the low frequency part.

The symbol $L^2_{(1)}$ stands for the set of all $u \in L^2$ satisfying supp $\hat{f} \subset \{|\xi| \le r_\infty\}$. For any nonnegative integer k, we see that $H^k \cap L^2_{(1)} = L^2_{(1)}$. (Cf., Lemma 1.3.3 (ii).)

We define the spaces $\mathscr{X}_{(1)}$ by

$$\mathscr{X}_{(1)} = \mathscr{X} \cap \{f \in \mathcal{S}'(\mathbb{R}^3); \text{supp } \hat{f} \subset \{|\xi| \le r_\infty\},$$

where $\mathcal{S}'(\mathbb{R}^3)$ denotes the set of all of distributions on $\mathcal{S}(\mathbb{R}^3)$, $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space on $\mathbb{R}^3$.

We set operators $S_1(t)$ and $\mathscr{S}_1(t)$ by

$$S_1(t) = S(t)|_{\mathscr{X}_{(1)}}, \quad \mathscr{S}_1(t)F_1 = \int_0^t S_1(t-\tau)F_1(\tau)\,d\tau.$$

Then we have the following

Proposition 3.3.1. (i) *$S_1(t)$ is a uniformly continuous semigroup on $\mathscr{X}_{(1)}$. In addition, for each $u_1 \in \mathscr{X}_{(1)}$ and all $T' > 0$, $S_1(t)$ satisfies*

$$S_1(t)u_1 \in C^1([0,T']; \mathscr{X}_{(1)}),$$

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 \,(= -AS_1(t)u_1), \; S_1(0)u_1 = u_1,$$

and there hold the estimates

$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1)})} \le C\|u_1\|_{\mathscr{X}_{(1)}}$$

for $u_1 \in \mathscr{X}_{(1)}$, $k = 0, 1$, where $T' > 0$ is any given positive number and C is a positive constant depending on T'.

(ii)

$$\mathscr{S}_1(t) : L^2(0,T;\mathscr{X}_{(1)}) \to C([0,T];\mathscr{X}_{(1)}) \cap H^1(0,T;\mathscr{X}_{(1)})$$

is a bounded linear operator for $t \in [0,T]$ satisfying

$$\partial_t \mathscr{S}_1(t)F_1 + A_1 \mathscr{S}_1(t)F_1 = F_1(t),\ \mathscr{S}_1(0)F_1 = 0,$$

$$\|\mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1)})} \le C\|F_1\|_{L^2(0,T;\mathscr{X}_{(1)})},$$

$$\|\partial_t \mathscr{S}_1(\cdot)F_1\|_{L^2(0,T;\mathscr{X}_{(1)})} \le C\|F_1\|_{L^2(0,T;\mathscr{X}_{(1)})}.$$

for $F_1 \in L^2(0,T;\mathscr{X}_{(1)})$, where C is a positive constant depending on T.

(iii) *It holds that*

$$S_1(t)\mathscr{S}_1(t')F_1 = \mathscr{S}_1(t')[S_1(t)F_1]$$

for any $t \ge 0$, $t' \in [0,T]$ and $F_1 \in L^2(0,T;\mathscr{X}_{(1)})$.

Proposition 3.3.1 can be proved in a similar manner to the proof of Proposition 2.4.1; and we omit the proof.

To investigate the invertibility of $I - S_1(T)$, we prepare some lemmas. The following lemma plays an important role to investigate the spatial decay properties of the time-T-map.

Lemma 3.3.2. (i) *Let*

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma_1{}^\top\xi & 0 \\ i\gamma_1\xi + i\kappa_0|\xi|^2\xi & \nu|\xi|^2 I_n + \tilde{\nu}\xi^\top\xi & i\zeta\xi \\ 0 & i\zeta^\top\xi & \alpha_0|\xi|^2 \end{pmatrix} \quad (\xi \in \mathbb{R}^3).$$

Then there exists $\delta_0 > 0$ such that if $0 < r_\infty \le \delta_0$, the set of all eigenvalues of $-\hat{A}_\xi$ consists of $\lambda_j(\xi)\,(j = 1, \cdots 4)$, where

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2 + O(|\xi|^3), \\ \lambda_2(\xi) = -\frac{\alpha_0\gamma_1^2}{\gamma_1^2+\zeta^2}|\xi|^2 + O(|\xi|^3), \\ \lambda_3(\xi) = i\sqrt{\gamma_1^2+\zeta^2}|\xi| - \frac{\nu+\tilde{\nu}}{2}|\xi|^2 - \frac{\alpha_0\zeta^2}{2(\gamma_1^2+\zeta^2)}|\xi|^2 + O(|\xi|^3), \\ \lambda_4(\xi) = \bar{\lambda_3} \text{ (complex conjugate)}. \end{cases}$$

(ii) *For $|\xi| \le \delta_0$, $e^{-t\hat{A}_\xi}$ has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1}^{4} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ is eigenprojections for $\lambda_j(\xi)\,(j = 1, \cdots, 4)$, and $\Pi_j(\xi)\,(j = 1, \cdots, 4)$ satisfy

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_3 - \frac{\xi^\top\xi}{|\xi|^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(|\xi|),$$

$$\Pi_2(\xi) = \begin{pmatrix} 1-\frac{\gamma_1^2}{\gamma_1^2+\zeta^2} & 0 & -\frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} \\ 0 & 0 & 0 \\ -\frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} & 0 & 1-\frac{\zeta^2}{\gamma_1^2+\zeta^2} \end{pmatrix} + O(|\xi|),$$

$$\Pi_3(\xi) = \frac{1}{2}\begin{pmatrix} \frac{\gamma_1^2}{\gamma_1^2+\zeta^2} & -\frac{i\gamma_1{}^\top\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} \\ -\frac{i\gamma_1\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\xi^\top\xi}{|\xi|^2} & -\frac{i\zeta\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} \\ \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} & -\frac{i\zeta^\top\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\zeta^2}{\gamma_1^2+\zeta^2} \end{pmatrix} + O(|\xi|),$$

$$\Pi_4(\xi) = \frac{1}{2}\begin{pmatrix} \frac{\gamma_1^2}{\gamma_1^2+\zeta^2} & \frac{i\gamma_1{}^\top\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} \\ \frac{i\gamma_1\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\xi^\top\xi}{|\xi|^2} & \frac{i\zeta\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} \\ \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} & \frac{i\zeta^\top\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\zeta^2}{\gamma_1^2+\zeta^2} \end{pmatrix} + O(|\xi|).$$

Furthermore, there exist a constant $C > 0$ such that the estimates

$$\|\Pi_j(\xi)\| \le C\,(j = 1, \cdots, 4) \tag{3.3.1}$$

hold for $|\xi| \le r_\infty$.

Lemma 3.3.2 is proved by the analytic perturbation theory ([21]). We set

$$\xi = |\xi|\omega, \quad \omega = \frac{\xi}{|\xi|}, \quad -\hat{A}_\xi = r\tilde{A}_\xi, \quad \tilde{A}_\xi = L_1 + rL_2 + r^2L_3,$$

where $r = |\xi|$,

$$L_1 = -i\begin{pmatrix} 0 & \gamma_1{}^\top\omega & 0 \\ \gamma_1\omega & 0 & \zeta\omega \\ 0 & \zeta^\top\omega & 0 \end{pmatrix}, \quad L_2 = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu I_3 + \omega^\top\omega & 0 \\ 0 & 0 & \alpha_0 \end{pmatrix}$$

and

$$L_3 = -\begin{pmatrix} 0 & 0 & 0 \\ i\kappa_0\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying the reduction process ([21, Section II-2-3]), we can prove Lemma 3.3.2. See also [26, Lemma 3.1].

Hereafter we fix $0 < r_1 < r_\infty \le \delta_0$ so that (3.3.1) in Lemma 3.3.2 holds for $|\xi| \le r_\infty$.

Lemma 3.3.3. *Let α be a multi-index. Then the following estimates hold true uniformly for ξ with $|\xi| \le r_\infty$ and $t \in [0, T]$.*

(i) $|\partial_\xi^\alpha \lambda_j| \le C|\xi|^{2-|\alpha|} \quad ((|\alpha| \ge 0,\ j = 1, 2),\ |\partial_\xi^\alpha \lambda_j| \le C|\xi|^{1-|\alpha|} \quad ((|\alpha| \ge 0,\ j = 3, 4).$

(ii) $|(\partial_\xi^\alpha \Pi_j)\hat{F}| \le C|\xi|^{-|\alpha|}|\hat{F}|\ (|\alpha| \ge 0).$

(iii) $|\partial_\xi^\alpha(e^{\lambda_j t})| \le C|\xi|^{2-|\alpha|}$ $(|\alpha| \ge 1,\ j = 1, 2)$.

(iv) $|\partial_\xi^\alpha(e^{\lambda_j t})| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 1,\ j = 3, 4)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}| \le C|\xi|^{-|\alpha|}|\hat{F}|$ $(|\alpha| \ge 1)$.

(vi) $|\partial_\xi^\alpha(I - e^{\lambda_j t})^{-1}| \le C|\xi|^{-2-|\alpha|}$ $(|\alpha| \ge 0,\ j = 1, 2)$.

(vii) $|\partial_\xi^\alpha(I - e^{\lambda_j t})^{-1}| \le C|\xi|^{-1-|\alpha|}$ $(|\alpha| \ge 0,\ j = 3, 4)$.

Lemma 3.3.3 can be verified by direct computations based on Lemma 3.3.2.

Lemma 3.3.4. *Set*

$$E_{1,j}(x) := \mathcal{F}^{-1}(\hat{\chi}_0(I - e^{\lambda_j T})^{-1}\Pi_j) \quad (j = 1, \cdots, 4), \quad (x \in \mathbb{R}^3)$$

where $\hat{\chi}_0$ *is the one defined by* (2.3.3)*. Then the following estimates hold true uniformly for* $x \in \mathbb{R}^3$.

(i) $|\partial_x^\alpha E_{1,j}(x)| \le C(1 + |x|)^{-(1+|\alpha|)}$ $\quad (j = 1, 2)$,

(ii) $|\partial_x^\alpha E_{1,j}(x)| \le C(1 + |x|)^{-(2+|\alpha|)}$ $\quad (j = 3, 4)$.

In a similar manner to the proof of Lemma 2.4.5, Lemma 3.3.4 is proved by Lemma 2.3.7 and Lemma 3.3.3.

We are now in a position to investigate the invertibility of $I - S(T)$ for the low frequency part. We consider the following equation

$$(I - S_1(T))u_1 = F_1 \tag{3.3.2}$$

for a given F_1. By using Lemma 2.3.8, Lemma 3.3.3, and Lemma 3.3.4, one can show the following proposition in a similar manner to the proof of Proposition 2.4.2.

Proposition 3.3.5. (i) *Assume that* $F_1 \in L^2_{(1),1} \cap L^\infty_3 \cap L^1$*. Then there uniquely exists the solution* $u_1 \in \mathscr{X}_{(1)}$ *for* (3.3.2) *which satisfies*

$$\|u_1\|_{\mathscr{X}_{(1)}} \le C(\|F_1\|_{L^\infty_3} + \|F_1\|_{L^1} + \|F_1\|_{L^2_1}).$$

(ii) *Let* $F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_3 \cap L^2_{(1),1}$ *with* $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$ *for some* α *satisfying* $|\alpha| = 1$. *Then* (3.3.2) *has a unique solution* $u_1 \in \mathscr{X}_{(1)}$ *satisfying*

$$\|u_1\|_{\mathscr{X}_{(1)}} \le C(\|F_1\|_{L^\infty_3} + \|F_1\|_{L^2_1} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2}).$$

(iii) *Suppose that* $F_1 = \partial_x^\alpha F_1^{(1)} \in L^2_{(1)}$ *with* $F_1^{(1)} \in L^2_{(1),1} \cap L^\infty_3$ *for some* α *satisfying* $|\alpha| \ge 1$*. Then there uniquely exists the solution* $u_1 \in \mathscr{X}_{(1)}$ *for* (3.3.2) *satisfying*

$$\|u_1\|_{\mathscr{X}_{(1)}} \le C(\|F_1^{(1)}\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^2_1}).$$

Proposition 3.3.5 (i) implies that $I - S_1(T)$ has a bounded inverse $(I - S_1(T))^{-1}$: $L^2_{(1),1} \cap L^\infty_3 \cap L^1 \to \mathscr{X}_{(1)}$ and it holds that

$$\|(I - S_1(T))^{-1} F_1\|_{\mathscr{X}_{(1)}} \le C(\|F_1\|_{L^\infty_3} + \|F_1\|_{L^1} + \|F_1\|_{L^2_1}).$$

In the case $F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_3 \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| = 1$, we have $(I - S_1(T))^{-1} F_1 \in \mathscr{X}_{(1)}$ and

$$\|(I - S_1(T))^{-1} F_1\|_{\mathscr{X}_{(1)}} \le C(\|F_1\|_{L^\infty_3} + \|F_1\|_{L^2_1} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2}).$$

We also see that for $F_1 = \partial_x^\alpha F_1^{(1)} \in L^2_{(1)}$ with $F_1^{(1)} \in L^2_{(1),1} \cap L^\infty_3$ with some α satisfying $|\alpha| \ge 1$, there hold $(I - S_1(T))^{-1} F_1 \in \mathscr{X}_{(1)}$ and

$$\|(I - S_1(T))^{-1} F_1\|_{\mathscr{X}_{(1)}} \le C(\|F_1^{(1)}\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^2_1}).$$

By the above argument, $\Gamma[P_1 F]$ makes sense and satisfies the following estimates. We set

$$\Gamma_1[P_1 F](t) = S(t)\mathscr{S}(T)(I - S(T))^{-1}(P_1 F),\ \Gamma_2[P_1 F](t) = \mathscr{S}(t)(P_1 F), \qquad (3.3.3)$$

for given F. By Proposition 3.3.1 (i), (ii) and Proposition 3.3.5, we have the following estimates for $\Gamma_j[P_1 F]$ $(j = 1, 2)$.

Proposition 3.3.6. (i) *Assume that $F \in L^2(0, T; L^2_1 \cap L^\infty_3 \cap L^1)$. Then $\Gamma_j[P_1 F] \in C([0, T]; \mathscr{X}_{(1)})$ $(j = 1, 2)$ and $\Gamma_j[P_1 F]$ satisfy the estimates*

$$\|\Gamma_j[P_1 F]\|_{C([0,T];\mathscr{X})} \le C\|F\|_{L^2(0,T;L^\infty_3 \cap L^1 \cap L^2_1)} \quad (j = 1, 2).$$

(ii) *For each $F \in L^2(0, T; L^\infty_3 \cap L^2_1)$ satisfying $F = \partial_x^\alpha F^{(1)}$ with $F^{(1)} \in L^2(0, T; L^2 \cap L^\infty_2)$ for some α satisfying $|\alpha| = 1$, $\Gamma_j[P_1 F] \in C([0, T]; \mathscr{X}_{(1)})$ $(j = 1, 2)$ and $\Gamma_j[P_1 F]$ satisfy the estimates*

$$\|\Gamma_j[P_1 F]\|_{C([0,T];\mathscr{X})} \le C(\|F\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}) \quad (j = 1, 2).$$

(iii) *$\Gamma_j[P_1 F] \in C([0, T]; \mathscr{X}_{(1)})$ $(j = 1, 2)$ for $F = \partial_x^\alpha F^{(1)} \in L^2(0, T; L^2)$ with $F^{(1)} \in L^2(0, T; L^2_1 \cap L^\infty_3)$ for some α satisfying $|\alpha| \ge 1$ and $\Gamma_j[P_1 F]$ satisfy the estimates*

$$\|\Gamma_j[P_1 F]\|_{C([0,T];\mathscr{X})} \le C\|F^{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} \quad (j = 1, 2).$$

We have the following another type of estimates for $\Gamma_j[P_1 F]$ $(j = 1, 2)$.

Proposition 3.3.7. (i) *In the case* $F = \partial_x^\alpha F^{(1)} \in L^2(0,T;L^2)$ *with* $F^{(1)} \in L^2(0,T;L^1_2)$ *for some* α *satisfying* $|\alpha| \geq 0$, $\Gamma_j[P_1F] \in C([0,T];\mathscr{X}_{(1)})$ $(j = 1,2)$ *and* $\Gamma_j[P_1F]$ *satisfy the estimates*

$$\|\Gamma_j[P_1F]\|_{C([0,T];\mathscr{X})} \leq C\|F^{(1)}\|_{L^2(0,T;L^1_2)}.$$

(ii) *Let* $F \in L^2(0,T;L^2)$ *and let* $F = \partial_x^\alpha F^{(1)}$ *with* $F^{(1)} \in L^2(0,T;L^2_2)$ *for some* α *satisfying* $|\alpha| \geq 1$. *Then* $\Gamma_j[P_1F] \in C([0,T];\mathscr{X}_{(1)})$ $(j = 1,2)$ *and* $\Gamma_j[P_1F]$ *satisfy the estimates*

$$\|\Gamma_j[P_1F]\|_{C([0,T];\mathscr{X})} \leq C\|F^{(1)}\|_{L^2(0,T;L^2_2)}.$$

Proposition 3.3.7 can be easily verified by Lemma 2.3.7, Lemma 2.3.9, Lemma 3.3.3 and Lemma 3.3.4.

We recall that

$$\Gamma[F] = S(t)\mathscr{S}(T)(I - S(T))^{-1}F + \mathscr{S}(t)F$$

for $F = {}^\top(0, F_2, F_3)$. The following estimates for $\Gamma[P_1F]$ directly follow from Proposition 3.3.6 and Proposition 3.3.7.

Proposition 3.3.8. (i) *Assume that* $F \in L^2(0,T;L^2_1 \cap L^\infty_3 \cap L^1)$. *Then* $\Gamma[P_1F]$ *satisfies the estimate*

$$\|\Gamma[P_1F_1]\|_{C([0,T];\mathscr{X})} \leq C\|F\|_{L^2(0,T;L^\infty_3 \cap L^1 \cap L^2_1)}. \tag{3.3.4}$$

(ii) *For each* $F \in L^2(0,T;L^\infty_3 \cap L^2_1)$ *satisfying* $F = \partial_x^\alpha F^{(1)}$ *with* $F^{(1)} \in L^2(0,T;L^2 \cap L^\infty_2)$ *for some* α *satisfying* $|\alpha| = 1$, *it holds that*

$$\|\Gamma[P_1F]\|_{C([0,T];\mathscr{X})} \leq C(\|F\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}). \tag{3.3.5}$$

(iii) *Let* $F = \partial_x^\alpha F^{(1)} \in L^2(0,T;L^2)$ *with* $F^{(1)} \in L^2(0,T;L^2_1 \cap L^\infty_3)$ *for some* α *satisfying* $|\alpha| \geq 1$. *Then* $\Gamma[P_1F]$ *satisfies the estimate*

$$\|\Gamma[P_1F]\|_{C([0,T];\mathscr{X})} \leq C\|F^{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)}. \tag{3.3.6}$$

(iv) *In the case* $F = \partial_x^\alpha F^{(1)} \in L^2(0,T;L^2)$ *with* $F^{(1)} \in L^2(0,T;L^1_2)$ *for some* α *satisfying* $|\alpha| \geq 0$, $\Gamma[P_1F]$ *satisfies the estimate*

$$\|\Gamma[P_1F]\|_{C([0,T];\mathscr{X})} \leq C\|F^{(1)}\|_{L^2(0,T;L^1_2)}. \tag{3.3.7}$$

(v) *Let* $F \in L^2(0,T;L^2)$ *and let* $F = \partial_x^\alpha F^{(1)}$ *with* $F^{(1)} \in L^2(0,T;L^2_2)$ *for some* α *satisfying* $|\alpha| \geq 1$. *Then it holds that*

$$\|\Gamma[P_1F]\|_{C([0,T];\mathscr{X})} \leq C\|F^{(1)}\|_{L^2(0,T;L^2_2)}. \tag{3.3.8}$$

3.4 Estimates of Γ for the high frequency part

In this section we establish estimates Γ for the high frequency part. We begin with to introduce function spaces for the high frequency part.

Let k and ℓ be nonnegative integers. The symbol $H^k_{(\infty)}$ stands for the set of all $u \in H^k$ satisfying supp $\hat{u} \subset \{|\xi| \geq r_1\}$ and the space $H^k_{(\infty),\ell}$ is defined by

$$H^k_{(\infty),\ell} = \{u \in H^k_{(\infty)}; \|u\|_{H^k_\ell} < +\infty\}.$$

Let s be a nonnegative integer satisfying $s \geq 2$. By Proposition 3.2.3, for $u_\infty \in H^{s+1}_{(\infty)} \times H^s_{(\infty)}$ and $F_\infty \in L^2(0,T; H^s_{(\infty)} \times H^{s-1}_{(\infty)})$, the operators

$$S_\infty(t) : H^{s+1}_{(\infty)} \times H^s_{(\infty)} \longrightarrow H^{s+1}_{(\infty)} \times H^s_{(\infty)} \quad (t \geq 0)$$

and

$$\mathscr{S}_\infty(t) : L^2(0,T; H^s_{(\infty)} \times H^{s-1}_{(\infty)}) \longrightarrow H^{s+1}_{(\infty)} \times H^s_{(\infty)} \quad (t \in [0,T])$$

are defined by $S_\infty(t)u_\infty = S(t)u_\infty$ and

$$\mathscr{S}_\infty(t)F_\infty = \int_0^t S_\infty(t-\tau)F_\infty(\tau)\,d\tau.$$

The operators $S_\infty(t)$ and $\mathscr{S}_\infty(t)$ have the following properties in weighted L^2-Sobolev spaces.

Proposition 3.4.1. (i) *It holds that $S_\infty(\cdot)u_{0\infty} \in C([0,\infty); H^{s+1}_{(\infty),2} \times H^s_{(\infty),2})$ for each $u_{0\infty} = {}^\top(\phi_{0\infty}, m_{0\infty}, \varepsilon_{0\infty}) \in H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}$ and there exist constants $a > 0$ and $C > 0$ such that $S_\infty(t)$ satisfies the estimate*

$$\|S_\infty(t)u_{0\infty}\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \leq Ce^{-at}\|u_{0\infty}\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}}$$

for all $t \geq 0$ and $u_{0\infty} \in H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}$. Furthermore, $r_{H^s_{(\infty),2}}(S_\infty(T)) < 1$; and $I - S_\infty(T)$ has a bounded inverse $(I - S_\infty(T))^{-1}$ on $H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}$ and $(I - S_\infty(T))^{-1}$ satisfies

$$\|(I - S_\infty(T))^{-1}u\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \leq C\|u\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \quad \text{for} \quad u \in H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}.$$

(ii) *It holds that $\mathscr{S}_\infty(\cdot)F_\infty \in C([0,T]; H^{s+1}_{(\infty),2} \times H^s_{(\infty),2})$ for each $F_\infty = {}^\top(F^1_\infty, F^2_\infty, F^3_\infty) \in L^2(0,T; H^s_{(\infty),2} \times H^{s-1}_{(\infty),2})$ and $\mathscr{S}_\infty(t)$ satisfies the estimate*

$$\|\mathscr{S}_\infty(t)[F_\infty]\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \leq C\left\{\int_0^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^s_{(\infty),2} \times H^{s-1}_{(\infty),2}} d\tau\right\}^{\frac{1}{2}}$$

for $t \in [0,T]$ and $F_\infty \in L^2(0,T; H^s_{(\infty),2} \times H^{s-1}_{(\infty),2})$ with a positive constant C depending on T.

Proposition 3.4.1 can be proved by the weighted energy method. In fact, Proposition 3.4.1 is an immediate consequence of the following proposition.

Proposition 3.4.2. *Let s be a nonnegative integer satisfying $s \geq 2$. Assume that*

$$u_{0\infty} = {}^\top(\phi_{0\infty}, m_{0\infty}, \varepsilon_{0\infty}) \in H^{s+1}_{(\infty),2} \times H^{s}_{(\infty),2},$$
$$F_\infty = {}^\top(F^1_\infty, F^2_\infty, F^3_\infty) \in L^2(0,T'; H^s_{(\infty),2} \times H^{s-1}_{(\infty),2})$$

for all $T' > 0$. Assume also that $u_\infty = {}^\top(\phi_\infty, m_\infty, \varepsilon_\infty)$ satisfies

$$\begin{cases} \partial_t u_\infty + A u_\infty = F_\infty, \\ u_\infty|_{t=0} = u_{0\infty}. \end{cases} \tag{3.4.1}$$

and

$$\phi_\infty \in C([0,T']; H^{s+1}_{(\infty)}) \cap L^2(0,T'; H^{s+2}_{(\infty)}),\ \ {}^\top(m_\infty, \varepsilon_\infty) \in C([0,T']; H^s_{(\infty)}) \cap L^2(0,T'; H^{s+1}_{(\infty)})$$

Then u_∞ satisfies

$$\phi_\infty \in C([0,T']; H^{s+1}_{(\infty),2}) \cap L^2(0,T'; H^{s+2}_{(\infty),2}),\ \ {}^\top(m_\infty, \varepsilon_\infty) \in C([0,T']; H^s_{(\infty),2}) \cap L^2(0,T'; H^{s+1}_{(\infty),2})$$

for all $T' > 0$ and there exists an energy functional $\mathcal{E}^s[u_\infty]$ such that there holds the estimate

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}^s[u_\infty](t) + d(\|\phi_\infty(t)\|^2_{H^{s+2}_2} + \|m_\infty(t)\|^2_{H^{s+1}_2} + \|\varepsilon_\infty(t)\|^2_{H^{s+1}_2}) \\ &\qquad \leq C\|F_\infty(t)\|^2_{H^s_2 \times H^{s-1}_2} \end{aligned} \tag{3.4.2}$$

on $(0,T')$ for all $T' > 0$. Here d is a positive constant; C is a positive constant depending on T but not on T'; $\mathcal{E}^s[u_\infty]$ is equivalent to $\|u_\infty\|^2_{H^{s+1}_2 \times H^s_2}$, i.e,

$$C^{-1}\|u_\infty\|^2_{H^{s+1}_2 \times H^s_2} \leq \mathcal{E}^s[u_\infty] \leq C\|u_\infty\|^2_{H^{s+1}_2 \times H^s_2};$$

and $\mathcal{E}^s[u_\infty](t)$ is absolutely continuous in $t \in [0,T']$ for all $T' > 0$.

We prove Proposition 3.4.2 by a weighted energy method. We introduce some notations. We define the energy functional $E^s_j[u_\infty]$ by

$$E^s_j[u_\infty] = (\kappa_1 |u_\infty|^2_{H^s_j} + \frac{\kappa_0\kappa_1}{\gamma_1}|\nabla\phi_\infty|^2_{H^s_j}) + \sum_{|\alpha|\leq s}(\partial^\alpha_x m_\infty, |x|^{2j}\nabla\partial^\alpha_x\phi_\infty).$$

Here κ_1 is a positive constant to be determined later.

Note that there exists a constant $\kappa_2 > 0$ such that if $\kappa_1 \geq \kappa_2$, then $E^s_j[u_\infty]$ is equivalent to $|u_\infty|^2_{H^{s+1}_j \times H^s_j}$, i.e.,

$$C^{-1}|u_\infty|^2_{H^{s+1}_j \times H^s_j} \leq E^s_j[u_\infty] \leq C|u_\infty|^2_{H^{s+1}_j \times H^s_j}$$

for some constant $C > 0$.

We also define $D_j^s[u_\infty]$ by

$$D_j^s[u_\infty] = |\nabla u_\infty|_{H_j^s}^2 + |\nabla^2 \phi_\infty|_{H_j^s}^2.$$

We begin with the estimate for $E_0^s[u_\infty]$ and $D_0^s[u_\infty]$.

Proposition 3.4.3. *Let s be a nonnegative integer satisfying $s \geq 2$. Assume that*

$$u_{0\infty} \in H^{s+1} \times H^s, \quad F_\infty \in L^2(0, T'; H^s \times H^{s-1}).$$

Here T' is a given positive number. Assume also that $u_\infty = {}^\top(\phi_\infty, m_\infty, \varepsilon_\infty)$ is the solution of (3.4.1). Then $u_\infty = {}^\top(\phi_\infty, m_\infty, \varepsilon_\infty)$ satisfies that $u_\infty \in C([0, T']; H^{s+1} \times H^s) \cap L^2(0, T'; H^{s+2} \times H^s)$ and that $E_0^s[u_\infty](t)$ is absolutely continuous in $t \in [0, T']$ and there exist positive constants $\kappa_3 \geq \kappa_2$ and $d_0 > 0$ such that the estimate

$$\frac{d}{dt} E_0^s[u_\infty] + d_0 D_0^s[u_\infty] \leq C\Big\{\epsilon \|u_\infty\|_2^2 + \Big(1 + \frac{1}{\epsilon}\Big) \|F_\infty\|_{H^s \times H^{s-1}}^2\Big\}$$

holds on $(0, T')$ for any $\kappa_1 \geq \kappa_3$, where κ_1 is the constant in the definition of $E_0^s[u_\infty]$; κ_2 is the constant in p.24*; ϵ is any positive number; and C is a positive constant independent of T' and ϵ.*

Proposition 3.4.3 can be proved by the energy method as in [2, 13]. (In fact, the estimate in Proposition 3.4.3 can be obtained by setting $\ell = 0$ in the proof of Proposition 3.4.4 bellow.)

We next derive the estimate for $E_\ell^s[u_\infty]$ and $D_\ell^s[u_\infty]$ for $\ell = 1, 2$. We show the following

Proposition 3.4.4. *Let s be a nonnegative integer satisfying $s \geq 2$ and let $\ell = 1, 2$. Assume that*

$$u_{0\infty} = {}^\top(\phi_{0\infty}, m_{0\infty}, \varepsilon_{0\infty}) \in H_\ell^{s+1} \times H_\ell^s,$$
$$F_\infty = {}^\top(F_\infty^1, F_\infty^2, F_\infty^3) \in L^2(0, T'; H_\ell^s \times H_\ell^{s-1}).$$

Here T' is a given positive number. Assume also that $u_\infty = {}^\top(\phi_\infty, m_\infty, \varepsilon_\infty)$ is the solution of (3.4.1) and that

$$\phi_\infty \in C([0, T']; H^{s+1}) \cap L^2(0, T'; H^{s+2}),\ {}^\top(m_\infty, \varepsilon_\infty) \in C([0, T']; H^s) \cap L^2(0, T'; H^{s+1}).$$

Then $u_\infty = {}^\top(\phi_\infty, m_\infty, \varepsilon_\infty)$ satisfies

$$\phi_\infty \in C([0, T']; H_\ell^{s+1}) \cap L^2(0, T'; H_\ell^{s+2}),\ {}^\top(m_\infty, \varepsilon_\infty) \in C([0, T']; H_\ell^s) \cap L^2(0, T'; H_\ell^{s+1}).$$

Furthermore, $E_\ell^s[u_\infty](t)$ are absolutely continuous in $t \in [0, T']$ and there exist positive constants $\kappa_4 \geq \kappa_2$ and $d_1 > 0$ such that the estimate

$$\frac{d}{dt} E_\ell^s[u_\infty] + d_1 D_\ell^s[u_\infty] \leq C\Big\{\epsilon |u_\infty|_{L_\ell^2}^2 + \Big(1 + \frac{1}{\epsilon}\Big) |F_\infty|_{H_\ell^s \times H_\ell^{s-1}}^2$$

$$+\ell^2\Big(1+\frac{1}{\epsilon}\Big)(|u_\infty|^2_{H^s_{\ell-1}}+|\nabla\phi_\infty|^2_{H^s_{\ell-1}})\Big\} \tag{3.4.3}$$

holds on $(0,T')$ *for any* $\kappa_1\geq\kappa_4$. *Here* κ_1 *is the constant in the definition of* $E^s_\ell[u_\infty]$; κ_2 *is the constant in* p.24; ϵ *is any positive number;* C *is a positive constant independent of* T', ϵ.

Proof. For a multi-index α satisfying $|\alpha|\leq s$, we take the inner product of $\partial_x^\alpha(3.4.1)_1$ with $|x|^{2\ell}\partial_x^\alpha\phi_\infty$ to obtain

$$\frac{1}{2}\frac{d}{dt}\||x|^\ell\partial_x^\alpha\phi_\infty\|^2_{L^2}+\gamma_1(\partial_x^\alpha\mathrm{div}m_\infty,|x|^{2\ell}\partial_x^\alpha\phi_\infty)=I^{(1)}_{\alpha,\ell}, \tag{3.4.4}$$

where

$$I^{(1)}_{\alpha,\ell} = (\partial_x^\alpha F^1_\infty,|x|^{2\ell}\partial_x^\alpha\phi_\infty).$$

We take the inner product of $\partial_x^\alpha(3.4.1)_2$ with $|x|^{2\ell}\partial_x^\alpha m_\infty$ and integrate by parts to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\||x|^\ell\partial_x^\alpha m_\infty\|^2_{L^2}-\kappa_0(\partial_x^\alpha\nabla\Delta\phi_\infty,|x|^2\partial_x^\alpha m_\infty)\\ &\quad+\nu\||x|^\ell\nabla\partial_x^\alpha m_\infty\|^2_{L^2}+\tilde\nu\||x|^\ell\mathrm{div}\partial_x^\alpha m_\infty\|^2_{L^2}\\ &\quad-\gamma_1(\partial_x^\alpha\phi_\infty,|x|^{2\ell}\partial_x^\alpha\mathrm{div}m_\infty)-\zeta(\partial_x^\alpha\varepsilon_\infty,|x|^{2\ell}\partial_x^\alpha\mathrm{div}m_\infty)\\ &\quad=I^{(2)}_{\alpha,\ell}+\mathscr{P}^{(1)}_{\alpha,\ell}[u_\infty],\end{aligned} \tag{3.4.5}$$

where

$$I^{(2)}_{\alpha,\ell}=\begin{cases}(F^2_\infty,|x|^{2\ell}m_\infty) & (\alpha=0),\\ -(\partial_x^{\alpha-1}F^2_\infty,|x|^{2\ell}\partial_x^{\alpha+1}m_\infty) & (|\alpha|\geq 1),\end{cases}$$

$$\begin{aligned}&\mathscr{P}^{(1)}_{\alpha,\ell}[u_\infty]\\ &= (-\nu\partial_x^\alpha\nabla m_\infty-\tilde\nu\partial_x^\alpha\mathrm{div}m_\infty+\gamma_1\partial_x^\alpha\phi_\infty+\zeta\partial_x^\alpha\varepsilon_\infty,\nabla(|x|^{2\ell})\partial_x^\alpha m_\infty)\\ &\quad-(\partial_x^{\alpha-1}F^2_\infty,\partial_x(|x|^{2\ell})\partial_x^\alpha m_\infty).\end{aligned}$$

As for the second term on the left-hand side, we have

$$\begin{aligned}&(\partial_x^\alpha\nabla\Delta\phi_\infty,|x|^{2\ell}\partial_x^\alpha m_\infty)\\ &= -(\partial_x^\alpha\Delta\phi_\infty,|x|^{2\ell}\partial_x^\alpha\mathrm{div}\,m_\infty)-(\partial_x^\alpha\Delta\phi_\infty,\nabla(|x|^{2\ell})\partial_x^\alpha m_\infty).\end{aligned} \tag{3.4.6}$$

By (3.4.1), we have

$$\gamma_1\mathrm{div}m_\infty=-\partial_t\phi_\infty+F^1_\infty.$$

Substituting this into (3.4.6), we obtain

$$\begin{aligned}&(\partial_x^\alpha\nabla\Delta\phi_\infty,|x|^{2\ell}\partial_x^\alpha m_\infty)\\ &= \frac{1}{\gamma_1}(\partial_x^\alpha\Delta\phi_\infty,|x|^{2\ell}\partial_x^\alpha\partial_t\phi_\infty)-\frac{1}{\gamma_1}(\partial_x^\alpha\Delta\phi_\infty,|x|^{2\ell}\partial_x^\alpha F^1_\infty)\end{aligned}$$

$$
\begin{aligned}
&-(\partial_x^\alpha \Delta\phi_\infty, \nabla(|x|^{2\ell})\partial_x^\alpha m_\infty) \\
= \quad & -\frac{1}{2\gamma_1}\frac{d}{dt}\|\partial_x^\alpha \nabla\phi_\infty\|_{L^2}^2 - \frac{1}{\gamma_1}(\partial_x^\alpha \Delta\phi_\infty, |x|^{2\ell}\partial_x^\alpha F_\infty^1) \\
&-(\partial_x^\alpha \Delta\phi_\infty, \nabla(|x|^{2\ell})\partial_x^\alpha m_\infty).
\end{aligned}
$$

This, together with (3.4.5), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\||x|^\ell \partial_x^\alpha m_\infty\|_{L^2}^2 + \frac{\kappa_0}{2\gamma_1}\frac{d}{dt}\||x|^\ell \partial_x^\alpha \nabla\phi_\infty\|_{L^2}^2 \\
&\quad +\nu\||x|^\ell \nabla\partial_x^\alpha m_\infty\|_{L^2}^2 + \tilde{\nu}\||x|^\ell \mathrm{div}\partial_x^\alpha m_\infty\|_{L^2}^2 \\
&\quad -\gamma_1(\partial_x^\alpha \phi_\infty, |x|^{2\ell}\partial_x^\alpha \mathrm{div} m_\infty) - \zeta(\partial_x^\alpha \varepsilon_\infty, |x|^{2\ell}\partial_x^\alpha \mathrm{div} m_\infty) \\
&\quad = \sum_{j=2}^{3} I_{\alpha,\ell}^{(j)} + \mathscr{P}_{\alpha,\ell}^{(2)}[u_\infty],
\end{aligned} \tag{3.4.7}
$$

where

$$
I_{\alpha,\ell}^{(3)} = -\frac{\kappa_0}{\gamma_1}(\partial_x^\alpha \Delta\phi_\infty, |x|^{2\ell}\partial_x^\alpha F_\infty^1),
$$

$$
\begin{aligned}
\mathscr{P}_{\alpha,\ell}^{(2)}[u_\infty] \quad = \quad & (-\nu\partial_x^\alpha \nabla m_\infty - \tilde{\nu}\partial_x^\alpha \mathrm{div} m_\infty + \gamma_1\partial_x^\alpha \phi_\infty \\
& -\kappa_0\partial_x^\alpha \Delta\phi_\infty + \zeta\partial_x^\alpha \varepsilon_\infty), \nabla(|x|^{2\ell})\partial_x^\alpha m_\infty) \\
& -(\partial_x^{\alpha-1}F_\infty^2, \partial_x(|x|^{2\ell})\partial_x^\alpha m_\infty).
\end{aligned}
$$

We take the inner product of $\partial_x^\alpha(3.4.1)_3$ with $|x|^{2\ell}\partial_x^\alpha \varepsilon_\infty$ and integrate by parts to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\||x|^\ell \partial_x^\alpha \varepsilon_\infty\|_{L^2}^2 + \alpha_0\||x|^\ell \nabla\partial_x^\alpha \varepsilon_\infty\|_{L^2}^2 + \zeta(\partial_x^\alpha \varepsilon_\infty, |x|^{2\ell}\partial_x^\alpha \mathrm{div} m_\infty) \\
&\quad = I_{\alpha,\ell}^{(4)} + \mathscr{P}_{\alpha,\ell}^{(2)}[u_\infty],
\end{aligned} \tag{3.4.8}
$$

where

$$
I_{\alpha,\ell}^{(4)} = \begin{cases} (F_\infty^3, |x|^{2\ell}\varepsilon_\infty) & (\alpha = 0), \\ -(\partial_x^{\alpha-1}F_\infty^3, |x|^{2\ell}\partial_x^{\alpha+1}\varepsilon_\infty) & (|\alpha| \geq 1), \end{cases}
$$

$$
\mathscr{P}_{\alpha,\ell}^{(2)}[u_\infty] \quad = \quad -\alpha_0(\partial_x^\alpha \nabla\varepsilon_\infty, \nabla(|x|^{2\ell})\partial_x^\alpha \varepsilon_\infty) - (\partial_x^{\alpha-1}F_\infty^3, \partial_x(|x|^{2\ell})\partial_x^\alpha \varepsilon_\infty).
$$

By adding (3.4.4) and (3.4.7) to (3.4.8), we see that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\||x|^\ell \partial_x^\alpha \phi_\infty\|_{L^2}^2 + \frac{\kappa_0}{\gamma_1}\||x|^\ell \partial_x^\alpha \nabla\phi_\infty\|_{L^2}^2 \\
&\quad +\||x|^\ell \partial_x^\alpha m_\infty\|_{L^2}^2 + \||x|^\ell \partial_x^\alpha \varepsilon_\infty\|_{L^2}^2\Big\} \\
&\qquad +\nu\||x|^\ell \nabla\partial_x^\alpha m_\infty\|_{L^2}^2 + \tilde{\nu}\||x|^\ell \mathrm{div}\partial_x^\alpha m_\infty\|_{L^2}^2 + \alpha_0\||x|^\ell \nabla\partial_x^\alpha \varepsilon_\infty\|_{L^2}^2 \\
&\quad = \sum_{j=1}^{4} I_{\alpha,\ell}^{(j)} + \mathscr{P}_{\alpha,\ell}^{(2)}[u_\infty] + \mathscr{P}_{\alpha,\ell}^{(3)}[u_\infty].
\end{aligned} \tag{3.4.9}
$$

By using Lemma 1.1.2 and the Hölder inequality, we obtain

$$
\begin{aligned}
|\sum_{|\alpha|\le s}\sum_{j=1}^{4} I_{\alpha,\ell}^{(j)}| \ \le\ & \epsilon|u_\infty|_{L^2_\ell}^2+\epsilon_1(|\nabla\phi_\infty|_{H_\ell^{s-1}}^2+|\Delta\phi_\infty|_{H_\ell^s}^2)\\
&+\epsilon_2|\nabla m_\infty|_{H_\ell^s}^2+\epsilon_3|\nabla\varepsilon_\infty|_{H_\ell^s}^2\\
&+C\Big(\frac{1}{\epsilon}+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}+\frac{1}{\epsilon_3}\Big)|F_\infty|_{H_\ell^s\times H_\ell^{s-1}}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{|\alpha|\le s}\sum_{j=2}^{3}\mathscr{P}_{\alpha,\ell}^{(j)}[u_\infty]| \ \le\ & \epsilon|u_\infty|_{L^2_\ell}^2+\epsilon_1(|\nabla\phi_\infty|_{H_\ell^{s-1}}^2+|\Delta\phi_\infty|_{H_\ell^s}^2)\\
&+\epsilon_2|\nabla m_\infty|_{H_\ell^s}^2+\epsilon_3|\nabla\varepsilon_\infty|_{H_\ell^s}^2\\
&+C\ell^2\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}+\frac{1}{\epsilon_3}\Big)|m_\infty|_{H_{\ell-1}^s}^2\\
&+C\ell^2|F_\infty|_{H_\ell^s\times H_\ell^{s-1}}^2.
\end{aligned}
$$

Taking $\epsilon_2>0$ and $\epsilon_3>0$ suitably small, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_\infty|_{H_\ell^s}^2+\frac{\kappa_0}{2\gamma_1}\frac{d}{dt}|\nabla\phi_\infty|_{H_\ell^s}^2+\frac{\nu}{2}|\nabla m_\infty|_{H_\ell^s}^2+\frac{\tilde{\nu}}{2}|\mathrm{div} m_\infty|_{H_\ell^s}^2+\frac{\alpha_0}{2}|\nabla\varepsilon_\infty|_{H_\ell^s}^2\\
&\le\ \epsilon|u_\infty|_{L_1^2}^2+\epsilon_1(|\nabla\phi_\infty|_{H_\ell^{s-1}}^2+|\nabla^2\phi_\infty|_{H_\ell^s}^2)+C\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}\Big)|F_\infty|_{H_\ell^s\times H_\ell^{s-1}}^2\\
&\quad+C\Big(1+\frac{1}{\epsilon}+\frac{1}{\epsilon_1}\Big)\|u_\infty\|_{H_{\ell-1}^s}^2. \qquad (3.4.10)
\end{aligned}
$$

We next estimate $\||x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty\|_{L^2}^2+\||x|^{2\ell}\Delta\partial_x^\alpha\phi_\infty\|_{L^2}^2$ for α with $|\alpha|\le s$. For a multi-index α satisfying $|\alpha|\le s$, we take the inner product of $\partial_x^\alpha(3.4.1)_2$ with $|x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty$ to obtain

$$
(\partial_t\partial_x^\alpha m_\infty,|x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty)+\gamma_1|\nabla\partial_x^\alpha\phi_\infty|_{L^2_\ell}^2+\kappa_0|\partial_x^\alpha\Delta\phi_\infty|_{L^2_\ell}^2=\sum_{i=1}^{4}J_{\alpha,\ell}^{(i)}+\mathscr{P}_{\alpha,\ell}^{(4)}[u_\infty], \qquad (3.4.11)
$$

where

$$
\begin{aligned}
J_{\alpha,\ell}^{(1)}&=-\nu(\partial_x^\alpha\nabla m_\infty,|x|^{2\ell}\nabla^2\partial_x^\alpha\phi_\infty),\\
J_{\alpha,\ell}^{(2)}&=-\tilde{\nu}(\partial_x^\alpha\mathrm{div} m_\infty,|x|^{2\ell}\Delta\partial_x^\alpha\phi_\infty),\\
J_{\alpha,\ell}^{(3)}&=\begin{cases}-(\partial_x^{\alpha-1}F_\infty^2,|x|^{2\ell}\nabla\partial_x^{\alpha+1}\phi_\infty) & (\alpha\ge 1),\\ (F_\infty^2,|x|^{2\ell}\nabla\phi_\infty) & (|\alpha|=0),\end{cases}\\
J_{\alpha,\ell}^{(4)}&=-\zeta(\partial_x^\alpha\nabla\varepsilon_\infty,|x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty),
\end{aligned}
$$

$$
\mathscr{P}_{\alpha,\ell}^{(4)}[u_\infty] \ =\ -\kappa_0(\partial_x^\alpha\Delta\phi_\infty,\nabla(|x|^{2\ell})\nabla\partial_x^\alpha\phi_\infty)-\nu(\partial_x^\alpha\nabla m_\infty,\nabla(|x|^{2\ell})\nabla\partial_x^\alpha\phi_\infty)
$$

$$-\tilde{\nu}(\partial_x^\alpha \mathrm{div}\, m_\infty, \nabla(|x|^{2\ell})\nabla\partial_x^\alpha\phi_\infty) - (\partial_x^{\alpha-1}F_\infty^2, \partial_x(|x|^{2\ell})\nabla\partial_x^\alpha\phi_\infty).$$

As for the first term on the left-hand side, we have

$$\begin{aligned}
&(\partial_t\partial_x^\alpha m_\infty, |x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty) \\
&= \frac{d}{dt}(\partial_x^\alpha m_\infty, |x|^{2\ell}\partial_x^\alpha\nabla\phi_\infty) + (\partial_x^\alpha m_\infty, \nabla(|x|^{2\ell})\partial_x^\alpha\partial_t\phi_\infty) \\
&\quad +(\partial_x^\alpha \mathrm{div} m_\infty, |x|^{2\ell}\partial_x^\alpha\partial_t\phi_\infty).
\end{aligned} \tag{3.4.12}$$

By (3.4.1), we have

$$\partial_t\phi_\infty = -\gamma_1 \mathrm{div} m_\infty + F_\infty^1.$$

Substituting this into (3.4.12), we obtain

$$(\partial_t\partial_x^\alpha m_\infty, |x|^{2\ell}\nabla\partial_x^\alpha\phi_\infty) = \frac{d}{dt}(\partial_x^\alpha m_\infty, |x|^{2\ell}\partial_x^\alpha\nabla\phi_\infty) - \sum_{i=4}^{5} J_{\alpha,\ell}^{(i)} - \mathscr{P}_{\alpha,\ell}^{(5)}[u_\infty],$$

where

$$\begin{aligned}
J_{\alpha,\ell}^{(5)} &= \gamma_1(\partial_x^\alpha \mathrm{div} m_\infty, |x|^{2\ell}\partial_x^\alpha \mathrm{div} m_\infty), \\
J_{\alpha,\ell}^{(6)} &= -(\partial_x^\alpha \mathrm{div} m_\infty, |x|^{2\ell}\partial_x^\alpha F_\infty^1),
\end{aligned}$$

and

$$\mathscr{P}_{\alpha,\ell}^{(5)}[u_\infty] = \gamma_1(\partial_x^\alpha m_\infty, \nabla(|x|^{2\ell})\partial_x^\alpha \mathrm{div} m_\infty) - (\partial_x^\alpha m_\infty, \nabla(|x|^{2\ell})\partial_x^\alpha F_\infty^1).$$

This, together with (3.4.11), gives

$$\begin{aligned}
&\frac{d}{dt}(\partial_x^\alpha m_\infty, |x|^{2\ell}\partial_x^\alpha\nabla\phi_\infty) + \gamma_1|\nabla\partial_x^\alpha\phi_\infty|_{L^2_\ell}^2 + \kappa_0|\partial_x^\alpha\Delta\phi_\infty|_{L^2_\ell}^2 \\
&\quad = \sum_{i=1}^{6} J_{\alpha,\ell}^{(i)} + \sum_{i=4}^{5} \mathscr{P}_{\alpha,\ell}^{(i)}[u_\infty].
\end{aligned} \tag{3.4.13}$$

By Lemma 1.1.2 and the Hölder inequality, we obtain

$$\begin{aligned}
|\sum_{|\alpha|\le s}\sum_{i=1}^{6} J_{\alpha,\ell}^{(i)}| \;\le\; & \frac{\kappa_0}{6}|\nabla^2\phi_\infty|_{H^s_\ell} + \frac{\gamma_1}{4}|\nabla\phi_\infty|_{H^s_\ell}^2 \\
& +C\Big(\gamma_1 + \frac{1}{\kappa_0}\Big)|\nabla m_\infty|_{H^s_\ell}^2 + \frac{C}{\gamma_1}|\nabla\varepsilon_\infty|_{H^s_\ell}^2 \\
& +C\Big(\frac{1}{\kappa_0} + \frac{1}{\gamma_1}\Big)|F_\infty|_{H^s_\ell\times H^{s-1}_\ell}^2,
\end{aligned}$$

$$|\sum_{|\alpha|\le s}\mathscr{P}_{\alpha,\ell}^{(4)}[u_\infty]| \;\le\; \frac{\gamma_1}{4}|\nabla\phi_\infty|_{H^s_\ell}^2 + \frac{\kappa_0}{6}|\nabla^2\phi_\infty|_{H^s_\ell}^2 + \tilde{\epsilon}\ell|\nabla m_\infty|_{H^s_\ell}^2$$

$$+C\Big(\frac{\ell}{\tilde{\epsilon}}+\frac{\ell^2}{\gamma_1}+\frac{\ell^2}{\kappa_0}\Big)(|\nabla\phi_\infty|^2_{H^s_{\ell-1}}+|F^2_\infty|^2_{H^{s-1}_\ell}),$$

$$\begin{aligned}|\sum_{|\alpha|\le s}\mathscr{P}^{(5)}_{\alpha,\ell}[u_\infty]| &\le \tilde{\epsilon}\ell|m_\infty|^2_{L^2_\ell}+\tilde{\epsilon}\ell|\nabla m_\infty|^2_{H^s_\ell}\\ &\quad+\frac{C\ell}{\tilde{\epsilon}}|m_\infty|^2_{H^s_{\ell-1}}+C\ell|F^1_\infty|^2_{H^s_\ell}\end{aligned}$$

for any $\tilde{\epsilon}>0$ with $C>0$ independent of $\tilde{\epsilon}$.

By integration by parts, we see that

$$|\nabla^2\partial^\alpha_x\phi_\infty|_{L^2_\ell}=|\Delta\partial^\alpha_x\phi_\infty|_{L^2_\ell}+\mathscr{P}^{(6)}_{\alpha,\ell}[u_\infty],$$

where

$$\begin{aligned}\mathscr{P}^{(6)}_{\alpha,\ell}[u_\infty] &= \sum_{i,j=1}^{3}(\partial_{x_i}\partial^\alpha_x\phi_\infty,\partial_{x_i}(|x|^{2\ell})\partial_{x_j}\partial_{x_j}\partial^\alpha_x\phi_\infty)\\ &\quad-\sum_{i,j=1}^{3}(\partial_{x_i}\partial^\alpha_x\phi_\infty,\partial_{x_j}(|x|^{2\ell})\partial_{x_i}\partial_{x_j}\partial^\alpha_x\phi_\infty)\\ &\le\frac{\kappa_0}{6}|\nabla^2\phi_\infty|^2_{H^s_\ell}+\frac{C}{\kappa_0}\ell^2|\nabla\phi_\infty|^2_{H^s_{\ell-1}}.\end{aligned}$$

Combining these estimates with (3.4.11) and (3.4.13), we see that

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\le s}(\partial^\alpha_x m_\infty,|x|^{2\ell}\partial^\alpha_x\nabla\phi_\infty)+\frac{\gamma_1}{2}|\nabla\phi_\infty|^2_{H^s_\ell}+\frac{\kappa_0}{2}|\nabla^2\phi_\infty|^2_{H^s_\ell}\\ &\le \tilde{\epsilon}\ell|m_\infty|^2_{L^2_\ell}+C\Big\{|\nabla m_\infty|^2_{H^s_\ell}+|\nabla\varepsilon_\infty|^2_{H^s_\ell}+|F_\infty|^2_{H^s_\ell\times H^{s-1}_\ell}\Big\}\\ &\quad+C\Big(\ell^2+\frac{\ell}{\tilde{\epsilon}}\Big)\Big\{|m_\infty|^2_{H^s_{\ell-1}}+|\nabla\phi_\infty|^2_{H^s_{\ell-1}}\Big\}\end{aligned}\qquad(3.4.14)$$

for any $\tilde{\epsilon}>0$ with $C>0$ independent of $\tilde{\epsilon}$.

Consider now $\kappa_1\times(3.4.10)+(3.4.14)$ with a constant $\kappa_1>0$. We take $\kappa_4\ge\kappa_2$ so large that if κ_1 satisfies $\kappa_1\ge\kappa_4$, then $|\nabla m_\infty|^2_{H^s_\ell}+|\nabla\varepsilon_\infty|^2_{H^s_\ell}$ on the right-hand side is absorbed into the left-hand side. Setting $\epsilon_1=\min\Big\{\frac{\gamma_1}{4\kappa_1},\frac{\kappa_0}{4\kappa_1}\Big\}$ and $\tilde{\epsilon}=\ell^{-1}\epsilon$, we arrive at

$$\begin{aligned}&\frac{d}{dt}E^s_\ell[u_\infty](t)+dD^s_\ell[u_\infty]\\ &\le \epsilon|u_\infty|^2_{L^2_\ell}+C\Big(1+\frac{1}{\epsilon}\Big)|F_\infty|^2_{H^s_\ell\times H^{s-1}_\ell}\\ &\quad+C\ell^2\Big(1+\frac{1}{\epsilon}\Big)(|u_\infty|^2_{H^s_{\ell-1}}+|\nabla\phi_\infty|^2_{H^s_{\ell-1}})\end{aligned}$$

for any$\epsilon>0$ with $C>0$ independent of ϵ. The computation above is formal, but it can be justified by using the cut-off argument as in Chapter 1. This completes the proof. □

By Proposition 3.4.4 and Lemma 1.3.4, and Lemma 2.3.11, we obtain Proposition 3.4.2 in a similar argument to that in Proposition 1.5.8.

Proposition 3.4.1 yields the following estimate for Γ.

Proposition 3.4.5. *Let s be a nonnegative integer satisfying $s \geq 2$. Then for*

$$F = {}^\top(0, F^2, F^3) \in L^2(0, T; H_2^{s-1})$$

$\Gamma[P_\infty F]$ satisfies the estimate

$$\|\Gamma[P_\infty F]\|_{\mathscr{Y}_2^s(0,T)} \leq C\|P_\infty F\|_{L^2(0,T;H_2^{s-1})}.$$

3.5 Proof of Theorem 3.2.1

In this section we give a proof of Theorem 3.2.1.

We first establish the estimates for the nonlinear and inhomogeneous terms $P_1 F(u, g)$ and $P_\infty F(u, g)$. where $F^2(u, g)$, $F^3(u)$ are the same ones defined in (3.2.6), (3.2.7), respectively.

For the estimates of the low frequency part, we recall that

$$\Gamma[P_1 F](t) := S(t)\mathscr{S}(T)(I - S(T))^{-1}(P_1 F) + \mathscr{S}(t)(P_1 F).$$

We first show the estimate of $\|\Gamma[P_1 F(u, g)]\|_{C([0,T];\mathscr{X})}$.

Proposition 3.5.1. *Suppose that $u = {}^\top(\phi, m, \varepsilon) \in \mathscr{Z}^s(0, T)$ satisfies*

$$\sup_{0\leq t\leq T} \|P_1 u(t)\|_{\mathscr{X}} + \sup_{0\leq t\leq T} \|P_\infty u(t)\|_{H_2^{s+1}\times H_2^s} + \sup_{0\leq t\leq T} \|\phi(t)\|_{L^\infty} \leq \frac{1}{2}.$$

Then there holds

$$\|\Gamma[P_1 F(u, g)]\|_{C([0,T];\mathscr{X})} \leq C\|u\|^2_{\mathscr{Z}^s(0,T)} + C\Big(1 + \|u\|_{\mathscr{Z}^s(0,T)}\Big)[g]_s$$

uniformly for u.

Proof. For $u^{(j)} = {}^\top(\phi^{(j)}, m^{(j)}, \varepsilon^{(j)})$ $(j = 1, 2)$ we set

$$\begin{aligned} \mathbb{G}_1(u^{(1)}, u^{(2)}) &= -\begin{pmatrix} 0 \\ G_{1,1}(u^{(1)}, u^{(2)}) \\ G_{1,2}(u^{(1)}, u^{(2)}) \end{pmatrix}, \\ \mathbb{G}_2(\phi, m, \varepsilon, u^{(1)}, u^{(2)}) &= -\begin{pmatrix} 0 \\ G_{2,1}(\phi, m, \varepsilon, u^{(1)}, u^{(2)}) \\ G_{2,2}(\phi, m, \varepsilon, u^{(1)}, u^{(2)}) \end{pmatrix}, \end{aligned}$$

$$
\begin{aligned}
\mathbb{G}_3(u^{(1)},u^{(2)}) &= -\begin{pmatrix} 0 \\ G_{3,1}(u^{(1)},u^{(2)}) \\ 0 \end{pmatrix}, \\
\mathbb{G}_4(\phi,m,\varepsilon,u^{(1)},u^{(2)}) &= -\begin{pmatrix} 0 \\ G_{4,1}(\phi,m,\varepsilon,u^{(1)},u^{(2)}) \\ G_{4,2}(\phi,m,\varepsilon,u^{(1)},u^{(2)}) \end{pmatrix}, \\
\mathbb{G}_5(\phi,m,g) &= \begin{pmatrix} 0 \\ \frac{1}{\gamma_1}(1+\phi)g \\ \frac{1}{\gamma_2}mg \end{pmatrix}, \\
\mathbb{H}_k(u^{(1)},u^{(2)}) &= \mathbb{G}_k(u^{(1)},u^{(2)}) + \mathbb{G}_k(u^{(2)},u^{(1)}), \quad (k=1,2), \\
\mathbb{H}_k(\phi,m,\varepsilon,u^{(1)},u^{(2)}) &= \mathbb{G}_k(\phi,m,\varepsilon,u^{(1)},u^{(2)}) + \mathbb{G}_k(\phi,m,\varepsilon,u^{(2)},u^{(1)}) \quad (k=3,4),
\end{aligned}
$$

where

$$
\begin{aligned}
G_{1,1}(u^{(1)},u^{(2)}) &= \frac{\gamma_1}{\rho_*}\mathrm{div}\,(m^{(1)}\otimes m^{(2)}), \\
G_{1,2}(u^{(1)},u^{(2)}) &= \frac{\gamma_1}{\rho_*}\mathrm{div}\,(\varepsilon^{(1)}\otimes m^{(2)}), \\
G_{2,1}(\phi,m,\varepsilon,u^{(1)},u^{(2)}) &= \rho_*\nu\Delta(P^{(1)}(\phi)\phi^{(1)}m^{(2)}) + \rho_*\tilde{\nu}\nabla\mathrm{div}\,(P^{(1)}(\phi)\phi^{(1)}m^{(2)}) \\
&\quad +\gamma_3\nabla(P^{(1)}(\phi)\phi^{(1)}\varepsilon^{(1)}) + \frac{1}{\gamma_1}\nabla(P^{(2)}(\phi)\phi^{(1)}\phi^{(2)}) \\
&\quad +\frac{1}{\gamma_1 C_v^2}\nabla\Big(P^{(3)}(\theta)\frac{\gamma_2^2\varepsilon^{(1)}\varepsilon^{(2)}}{(\rho_*+\phi)^2}\Big) + \frac{\gamma_2}{\gamma_1 C_v}\nabla\Big(P^{(4)}(\theta)\frac{\phi^{(1)}\varepsilon^{(2)}}{(\rho_*+\phi)}\Big) \\
&\quad -\frac{P_\theta(\rho_*,\theta_*)}{\gamma_1}\nabla\Big(\frac{\gamma_1^2|m^{(1)}||m^{(2)}|}{2C_v(\rho_*+\phi)^2}\Big), \\
G_{2,2}(\phi,m,\varepsilon,u^{(1)},u^{(2)}) &= \gamma_1\mathrm{div}\,(P^{(1)}(\phi)\phi^{(1)}\varepsilon^{(2)}) - \alpha_0\rho_*\Delta(P^{(1)}(\phi)\phi^{(1)}\varepsilon^{(2)}) \\
&\quad +\frac{\alpha_0}{C_v\gamma_2}\Delta\Big(\frac{\gamma_1^2|m^{(1)}||m^{(2)}|}{2(\rho_*+\phi)^2}\Big) + \frac{\gamma_1}{\gamma_2}\mathrm{div}\,(P^{(1)}(\phi)\phi^{(1)}m^{(2)}P(\rho_*+\phi,\theta)) \\
&\quad +\frac{\gamma_1}{\rho_*\gamma_2}\mathrm{div}\,(m^{(1)}P^{(5)}(\phi,\theta)\phi^{(2)}) + \frac{\gamma_1}{C_v\rho_*\gamma_2}\mathrm{div}\,\Big(m^{(1)}P^{(6)}(\theta)\frac{\gamma_2\varepsilon^{(2)}}{\rho_*+\phi}\Big) \\
&\quad -\frac{\gamma_1}{\gamma_2}\mathrm{div}\,\Big(\mathcal{S}\Big(\frac{\gamma_1 m^{(1)}}{\rho_*+\phi}\Big)\frac{m^{(2)}}{\rho_*+\phi}\Big), \\
G_{3,1}(u^{(1)},u^{(2)}) &= -\frac{1}{\gamma_1}\mathrm{div}\,\Phi(\phi^{(1)},\phi^{(2)}), \\
G_{4,1}(\phi,m,\varepsilon,u^{(1)},u^{(2)}) &= \gamma_1\mathrm{div}\,(P^{(1)}(\phi)\phi m^{(1)}\otimes m^{(2)}) \\
&\quad +\frac{1}{C_v^2\gamma_1}\nabla\Big\{P^{(3)}(\theta)\Big(\frac{\gamma_1^4|m^{(1)}||m^{(2)}||m|^2}{4(\rho_*+\phi)^4} - \frac{\gamma_1^2\gamma_2\varepsilon|m^{(1)}||m^{(2)}|}{(\rho_*+\phi)^3}\Big)\Big\} \\
&\quad -\frac{1}{C_v\gamma_1}\nabla\Big(P^{(4)}(\theta)\frac{\gamma_1^2\phi|m^{(1)}||m^{(2)}|}{2(\rho_*+\phi)^2}\Big), \\
G_{4,2}(\phi,m,\varepsilon,u^{(1)},u^{(2)}) &= -\frac{\gamma_1}{\gamma_2}\mathrm{div}\,\Big(\mathcal{K}(\phi^{(1)},\phi^{(2)})\frac{m}{\rho_*+\phi}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\gamma_1}{C_v\rho_*\gamma_2}\operatorname{div}\Big(m^{(1)}P^{(6)}(\theta)\Big(\frac{\gamma_1^2|m^{(2)}||m|}{2(\rho_*+\phi)^2}\Big)\Big),\\
\theta &= \frac{1}{C_v}\Big(E_* + \frac{\gamma_2\varepsilon}{\rho_*+\phi} - \gamma_1^2\frac{m^2}{2(\rho_*+\phi)^2}\Big),\\
\Phi(\phi^{(1)},\phi^{(2)}) &= \kappa\Big\{\phi^{(1)}\Delta\phi^{(2)}I + (\nabla\phi^{(1)})\cdot(\nabla\phi^{(1)})I\\
&\quad -\frac{|\nabla\phi^{(1)}||\nabla\phi^{(2)}|}{2}I - \nabla\phi^{(1)}\otimes\nabla\phi^{(2)}\Big\},\\
\mathcal{K}(\phi^{(1)},\phi^{(2)}) &= \frac{\kappa}{2}(\Delta\{(\rho_*+\phi^{(1)})(\rho_*+\phi^{(2)})\} - |\nabla\phi^{(1)}||\nabla\phi^{(2)}|)I - \kappa\nabla\phi^{(1)}\otimes\nabla\phi^{(2)}.
\end{aligned}
$$

Then, $\Gamma[P_1F(u,g)]$ is written as

$$
\begin{aligned}
\Gamma[P_1F(u,g)] &= \sum_{k\in\{1,3\}}\Big\{\Gamma[P_1\mathbb{G}_k(P_1u,P_1u)] + \Gamma[P_1\mathbb{H}_k(P_1u,P_\infty u)] + \Gamma[P_1\mathbb{G}_k(P_\infty u,P_\infty u)]\Big\}\\
&\quad + \sum_{k\in\{2,4\}}\Big\{\Gamma[P_1\mathbb{G}_k(\phi,m,\varepsilon,P_1u,P_1u)] + \Gamma[P_1\mathbb{H}_k(\phi,m,\epsilon,P_1u,P_\infty u)]\\
&\quad + \Gamma[\mathbb{G}_k(\phi,m,\epsilon,P_\infty u,P_\infty u)]\Big\} + \Gamma\left[P_1\mathbb{G}_5(\phi,m,g)\right].
\end{aligned}
$$

Applying (3.3.5) to $\Gamma[P_1\mathbb{G}_1(P_1u,P_1u)]$, we have

$$
\|\Gamma[P_1\mathbb{G}_1(P_1u,P_1u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)}.
$$

As for $\Gamma[P_1\mathbb{G}_2(\phi,m,\varepsilon,P_1u,P_1u)]$, we apply (3.3.5) with $F^{(1)} = P^{(1)}(\phi)P_1\phi P_1\varepsilon$, $F^{(1)} = P^{(2)}(\phi)(P_1\phi)^2$, $F^{(1)} = \frac{\gamma_1^2|P_1m|^2}{2C_v(\rho_*+\phi)^2}$, $= P^{(1)}(\phi)P_1\phi P_1mP(\rho_*+\phi,\theta)$, $F^{(1)} = P^{(3)}(\theta)\frac{(P_1\varepsilon)^2}{\rho_*+\phi}$, $F^{(1)} = P^{(4)}(\theta)\frac{P_1\phi P_1\varepsilon}{\rho_*+\phi}$, $F^{(1)} = P_1mP^{(5)}(\phi,\theta)P_1\phi$, $F^{(1)} = P_1mP^{(6)}(\theta)P_1\varepsilon$, $F^{(1)} = \mathcal{S}\Big(\frac{\gamma_1P_1m}{\rho_*+\phi}\Big)\frac{P_1m}{\rho_*+\phi}$ and we also apply (3.3.6) with $F^{(1)} = \nabla(P^{(1)}(\phi)P_1\phi P_1m)$ $(|\alpha|=1)$, $F^{(1)} = \operatorname{div}(P^{(1)}(\phi)P_1\phi P_1m)$ $(|\alpha|=1)$, $F^{(1)} = \nabla(P^{(1)}(\phi)P_1\phi P_1\varepsilon)$ $(|\alpha|=1)$, $F^{(1)} = \nabla\Big(\frac{\gamma_1^2|P_1m|^2}{2C_v(\rho_*+\phi)^2}\Big)$ $(|\alpha|=1)$ to obtain

$$
\|\Gamma[P_1\mathbb{G}_2(\phi,m,\varepsilon,P_1u,P_1u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)}.
$$

As for $\Gamma[\mathbb{G}_3(P_1u,P_1u)]$, using (3.3.6) with $F^{(1)} = \Phi(P_1\phi,P_1\phi)$ $(|\alpha|=1)$, we have

$$
\|\Gamma[\mathbb{G}_3(P_1u,P_1u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)}.
$$

As for $\Gamma[P_1\mathbb{G}_4(\phi,m,\varepsilon,P_1u,P_1u)]$, we apply (3.3.6) with $F^{(1)} = P^{(1)}(\phi)\phi P_1m\otimes P_1m$ $(|\alpha|=1)$, $F^{(1)} = P^{(3)}(\theta)\frac{\gamma_1^4|P_1m|^2|m|^2}{4C_v^2(\rho_*+\phi)^4}$ $(|\alpha|=1)$, $F^{(1)} = \frac{\varepsilon|P_1m|^2}{(\rho_*+\phi)^3}$ $(|\alpha|=1)$, $F^{(1)} = P^{(4)}(\theta)\frac{\phi|P_1m|^2}{(\rho_*+\phi)^2}$ $(|\alpha|=1)$, $F^{(1)} = \mathcal{K}(P_1\phi,P_1\phi)\frac{m}{\rho_*+\phi}$ $(|\alpha|=1)$, $F^{(1)} = P_1mP^{(6)}(\theta)\Big(\frac{\gamma_1^2|P_1m||m|}{2(\rho_*+\phi)^2}\Big)$ $(|\alpha|=1)$ to obtain

$$
\|\Gamma[P_1\mathbb{G}_2(\phi,m,\varepsilon,P_1u,P_1u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)}.
$$

By (3.3.7), we have

$$\begin{aligned}
&\| \sum_{k\in\{1,3\}} \Gamma[\mathbb{G}_k(P_\infty u, P_\infty u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)},\\
&\| \sum_{k\in\{2,4\}} \Gamma[\mathbb{G}_k(\phi, m, \varepsilon, P_\infty u, P_\infty u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)}.
\end{aligned}$$

By (3.3.8), we also have

$$\begin{aligned}
&\| \sum_{k\in\{1,3\}} \Gamma[\mathbb{H}_k(P_1 u, P_\infty u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)},\\
&\| \sum_{k\in\{2,4\}} \Gamma[\mathbb{H}_k(\phi, m, \varepsilon, P_1 u, P_\infty u)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)}.
\end{aligned}$$

Concerning $\Gamma[P_1\mathbb{G}_5(\phi, m, g)]$, we see from (3.3.4) and (3.3.7) that

$$\|\Gamma[P_1\mathbb{G}_5(\phi, m, g)]\|_{C([0,T];\mathscr{X})} \le C(1 + \|u\|_{\mathscr{Z}^s(0,T)})[g]_s.$$

Therefore, we find that

$$\|\Gamma[P_1 F(u, g)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)} + C\Big(1 + \|u\|_{\mathscr{Z}^s(0,T)}\Big)[g]_s.$$

This completes the proof. □

We next show the estimates for the nonlinear and inhomogeneous terms of the high frequency part.

Proposition 3.5.2. *Assume that* $u = {}^\top(\phi, m, \varepsilon) \in \mathscr{Z}^s(0,T)$ *satisfies*

$$\sup_{0\le t\le T} \|P_1 u(t)\|_{\mathscr{X}} + \sup_{0\le t\le T} \|P_\infty u(t)\|_{H^{s+1}_2 \times H^s_2} + \sup_{0\le t\le T} \|\phi(t)\|_{L^\infty} \le \frac{1}{2}.$$

Then there holds

$$\begin{aligned}
&\|P_\infty F(u, g)\|_{L^2(0,T;H^s_2\times H^{s-1}_2)}\\
&\quad \le \ C\|u\|^2_{\mathscr{Z}^s(0,T)} + C\Big(1 + \|u\|_{\mathscr{Z}^s(0,T)}\Big)[g]_s
\end{aligned}$$

uniformly for u.

Proposition 3.5.2 directly follows from Lemma 1.1.1, Lemma 1.1.3, Lemma 1.3.4 (i) and Lemma 2.3.2 (C.f., the proof of Proposition 2.6.2).

For the estimates $P_j F(u^{(1)}, g) - P_j F(u^{(2)}, g)$ $(j = 1, \infty)$, we have

Proposition 3.5.3. *Suppose that* $u^{(k)} = {}^\top(\phi^{(k)}, m^{(k)}, \varepsilon^{(k)}) \in \mathscr{Z}^s(0,T)$ $(k=1,2)$ *satisfy*

$$\sup_{0\le t\le T}\|P_1 u^{(k)}(t)\|_{\mathscr{X}} + \sup_{0\le t\le T}\|P_\infty u^{(k)}(t)\|_{H^{s+1}_2\times H^s_2} + \sup_{0\le t\le T}\|\phi^{(k)}(t)\|_{L^\infty} \le \frac{1}{2}.$$

Then there hold

$$\begin{aligned}
&\|\Gamma[P_1F(u^{(1)},g) - P_1F(u^{(2)},g)]\|_{C([0,T];\mathscr{X})} \\
&\le \ C\sum_{k=1}^{2}\|u^{(k)}\|_{X^s(0,T)}\|u^{(1)}-u^{(2)}\|_{\mathscr{Z}^s(0,T)} \\
&\quad +C[g]_s\|u^{(1)}-u^{(2)}\|_{\mathscr{Z}^s(0,T)}
\end{aligned}$$

and

$$\begin{aligned}
&\|P_\infty F(u^{(1)},g) - P_\infty F(u^{(2)},g)]\|_{L^2(0,T;H^s_2\times H^{s-1}_2)} \\
&\le \ C\sum_{k=1}^{2}\|u^{(k)}\|_{\mathscr{Z}^s(0,T)}\|u^{(1)}-u^{(2)}\|_{\mathscr{Z}^s(0,T)} \\
&\quad +C[g]_s\|u^{(1)}-u^{(2)}\|_{\mathscr{Z}^s(0,T)}
\end{aligned}$$

uniformly for $u^{(k)}$.

Proposition 3.5.3 directly follows from Lemma 1.1.1, Lemma 1.1.3, Lemma 1.3.4 (i). Lemma 2.3.2 and Proposition 3.3.8.

Applying Lemma 1.1.1, propositions 3.4.5 and 3.5.1-3.5.3, we obtain the following estimate for Γ in $\mathscr{Z}^s(0,T)$.

Corollary 3.5.4. (i) *Assume that* $u = {}^\top(\phi, m, \varepsilon) \in B_{\mathscr{Z}^s(0,T)}(c_0)$, *where* $c_0 = \min\{\frac{1}{2}, \frac{1}{2C_0}\}$, C_0 *is the same one in Lemma* 1.1.1. *Then there holds*

$$\|\Gamma[F(u,g)]\|_{\mathscr{Z}^s(0,T)} \le C_1\|u\|^2_{\mathscr{Z}^s(0,T)} + C_1\Big(1+\|u\|_{\mathscr{Z}^s(0,T)}\Big)[g]_s$$

uniformly for u. *Here* C_1 *is a constant independent of* g.

(ii) *Suppose that* $u^{(k)} = {}^\top(\phi^{(k)}, m^{(k)}, \varepsilon^{(k)}) \in B_{\mathscr{Z}^s(0,T)}(c_0)$ $(k=1,2)$. *Then there holds*

$$\begin{aligned}
&\|\Gamma[F(u^{(1)},g) - F(u^{(2)},g)]\|_{\mathscr{Z}^s(0,T)} \\
&\le \ C_1\sum_{k=1}^{2}\|u^{(k)}\|_{\mathscr{Z}^s(0,T)}\|u^{(1)}-u^{(2)}\|_{\mathscr{Z}^s(0,T)} \\
&\quad +C_1[g]_s\|u^{(1)}-u^{(2)}\|_{\mathscr{Z}^s(0,T)}
\end{aligned}$$

uniformly for $u^{(k)}$.

By Corollary 3.5.4, one can show the following proposition for the existence of a solution u of (3.2.3) on $[0,T]$ satisfying $u(0) = u(T)$.

Proposition 3.5.5. *There exists a constant $\delta_1 > 0$ such that if $[g]_s \le \delta_1$, then the system* (3.2.3) *has a unique solution u on $[0,T]$ in $B_{\mathscr{Z}^s(0,T)}(\frac{3}{2}C_1[g]_s)$ satisfying $u(0) = u(T)$. The uniqueness of solutions of* (3.2.3) *on $[0,T]$ satisfying $u(0) = u(T)$ holds in $B_{\mathscr{Z}^s(0,T)}(\frac{3}{2}C_1\delta_1)$.*

Proof. If g satisfies

$$[g]_s \le \min\Big\{\frac{2c_0}{3C_1},\ \frac{1}{2C_1(3C_1+2)}\Big\},$$

then by Corollary 3.5.4, one can see that Γ is a map on $B_{\mathscr{Z}^s(0,T)}(\frac{3}{2}C_1[g]_s)$ and Γ satisfies the estimate

$$\|\Gamma[F(u^{(1)},g) - F(u^{(2)},g)]\|_{\mathscr{Z}^s(0,T)} \le \frac{1}{2}\|u^{(1)} - u^{(2)}\|_{\mathscr{Z}^s(0,T)}.$$

Therefore, by the contraction mapping principle, we obtain Proposition 3.5.5. This completes the proof. □

We are now in a position to construct a time periodic solution of (0.0.3)-(0.0.5). By using Proposition 3.5.5, we are able to extend u periodically on $\mathbb{R}$ as a time periodic solution of (0.0.3)-(0.0.5) by the same way as that given in Chapter 1. Consequently, we obtain Theorem 3.2.1. This completes the proof.

Chapter 4

Time periodic problem for the compressible Navier-Stokes equation on $\mathbb{R}^2$ with antisymmetry

We show the existence of a time periodic solution of (0.0.1) on $\mathbb{R}^{\nvDash}$ for sufficiently small time periodic external force. We prove the result by using the time-T-map associated with the linearized problem around the motionless state with constant density. In some weighted L^∞ and Sobolev spaces we investigate the spectral properties of the time-T-map by a potential theoretic method and an energy method.

4.1 Preliminaries

In this chapter we use the following notations Furthermore, we introduce a lemma which will be useful in the proof of the main results.

We define the norm on X by $\|\cdot\|_X$ for a given Banach space X.

Let $1 \leqq p \leqq \infty$. L^p stands for the usual L^p space on $\mathbb{R}^2$. We define the inner product of L^2 by $(\cdot,\cdot)$. Let k be a nonnegative integer. H^k denotes the usual L^2 Sobolev space of order k. (As usual, we define that $H^0 = L^2$.)

For simplicity, L^p stands for the set of all vector fields $w = {}^\top(w_1, w_2)$ on $\mathbb{R}^2$ with $w_j \in L^p$ $(j = 1, 2)$, and we define by $\|\cdot\|_{L^p}$ the norm $\|\cdot\|_{(L^p)^2}$ if no confusion will occur. Similarly, we denote by a function space X the set of all vector fields $w = {}^\top(w_1, w_2)$ on $\mathbb{R}^2$ with $w_j \in X$ $(j = 1, 2)$; and we define the norm $\|\cdot\|_{X^2}$ on it by $\|\cdot\|_X$ if no confusion will occur.

We take $u = {}^\top(\phi, w)$ with $\phi \in H^k$ and $w = {}^\top(w_1, w_2) \in H^m$. Then the norm of u on $H^k \times H^m$ is denoted by $\|u\|_{H^k \times H^m}$, that is, we define

$$\|u\|_{H^k\times H^m} := \left(\|\phi\|_{H^k}^2 + \|w\|_{H^m}^2\right)^{\frac{1}{2}}.$$

When $m = k$, we simply denote $H^k \times (H^k)^2$ by H^k. We also simply denote the norm

$\|u\|_{H^k\times(H^k)^2}$ by $\|u\|_{H^k}$, i.e., we define that

$$H^k := H^k \times (H^k)^2, \quad \|u\|_{H^k} := \|u\|_{H^k\times(H^k)^2} \quad (u = {}^\top(\phi, w)).$$

Similarly, for $u = {}^\top(\phi, w) \in X \times Y$ with $w = {}^\top(w_1, w_2)$, the norm $\|u\|_{X\times Y}$ stands for:

$$\|u\|_{X\times Y} = \left(\|\phi\|_X^2 + \|w\|_Y^2\right)^{\frac{1}{2}} \quad (u = {}^\top(\phi, w)).$$

If $Y = X^2$, the symbol X stands for $X\times X^2$ for simplicity, and we define its norm $\|u\|_{X\times X^2}$ by $\|u\|_X$;

$$X := X \times X^2, \quad \|u\|_X := \|u\|_{X\times X^2} \quad (u = {}^\top(\phi, w)).$$

A function space with spatial weight is defined as follows. For a nonnegative integer ℓ and $1 \le p \le \infty$, the symbol L^p_ℓ denotes the weighted L^p space which is defined by

$$L^p_\ell = \{u \in L^p; \|u\|_{L^p_\ell} := \|(1 + |x|)^\ell u\|_{L^p} < \infty\}.$$

The notations $\hat{f}$ and $\mathcal{F}[f]$ denotes the Fourier transform of f:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^2} f(x)e^{-ix\cdot\xi}dx \quad (\xi \in \mathbb{R}^2),$$

In addition, we denote the inverse Fourier transform of f by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](x) = (2\pi)^{-2}\int_{\mathbb{R}^2} f(\xi)e^{i\xi\cdot x}d\xi \quad (x \in \mathbb{R}^2).$$

Let k be a nonnegative integer and let r_1 and r_∞ be positive constants satisfying $r_1 < r_\infty$. The symbol $H^k_{(\infty)}$ stands for the set of all $u \in H^k$ satisfying supp $\hat{u} \subset \{|\xi| \ge r_1\}$, and the symbol $L^2_{(1)}$ stands for the set of all $u \in L^2$ satisfying supp $\hat{u} \subset \{|\xi| \le r_\infty\}$. It follows from Lemma 1.3.3 (ii) that $H^k \cap L^2_{(1)} = L^2_{(1)}$ for any nonnegative integer k.

Let k and ℓ be nonnegative integers. The weighted L^2 Sobolev space H^k_ℓ is defined by

$$H^k_\ell = \{u \in H^k; \|u\|_{H^k_\ell} < +\infty\},$$

where

$$\begin{aligned}
\|u\|_{H^k_\ell} &= \left(\sum_{j=0}^{\ell} |u|^2_{H^k_j}\right)^{\frac{1}{2}}, \\
|u|_{H^k_\ell} &= \left(\sum_{|\alpha|\le k} \|\, |x|^\ell \partial_x^\alpha u \,\|^2_{L^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Moreover, $H^k_{(\infty),\ell}$ denotes the weighted L^2 Sobolev space for the high frequency part defined by

$$H^k_{(\infty),\ell} = \{u \in H^k_{(\infty)}; \|u\|_{H^k_\ell} < +\infty\}.$$

Let ℓ be a nonnegative integer. The symbol $L^2_{(1),\ell}$ stands for the weighted L^2 space for the low frequency part defined by

$$L^2_{(1),\ell} = \{f \in L^2_\ell; f \in L^2_{(1)}\}.$$

For $-\infty \le a < b \le \infty$, the symbol $C^k([a,b];X)$ denotes the set of all C^k functions on $[a,b]$ with values in X. Similarly, $L^p(a,b;X)$ and $H^k(a,b;X)$ denote the L^p-Bochner space on (a,b) and the L^2-Bochner-Sobolev space of order k respectively.

The time periodic problem is considered in function spaces with the following antisymmetry. Γ_j $(j=1,2,3)$ are defined by

$$(\Gamma_1 u)(x) = {}^\top(\phi(-x), -w_1(-x), w_2(-x)), \quad (\Gamma_2 u)(x) = {}^\top(\phi(-x), w_1(-x), -w_2(-x)),$$
$$(\Gamma_3 u)(x_1,x_2) = {}^\top(\phi(x_2,x_1), w_2(x_2,x_1), w_1(x_2,x_1))$$

for $u(x) = {}^\top(\phi(x), w_1(x), w_2(x)),\ x \in \mathbb{R}^2$. For a function space X on $\mathbb{R}^2$, $X_\diamond$ denotes the set of all $f \in X$ satisfying

$$f(-x_1,x_2) = f(x_1,x_2), \quad f(x_1,-x_2) = f(x_1,x_2),$$
$$f(x_2,x_1) = f(x_1,x_2).$$

The subscript $\cdot_\#$ denotes function spaces satisfying the antisymmetric condition. Exactly, $X_\#$ denotes the set of all $f = {}^\top(f_1,f_2) \in X$ satisfying

$$\begin{cases} f_1(-x_1,x_2) = -f_1(x_1,x_2), \quad f_1(x_1,-x_2) = f_1(x_1,x_2), \\ f_2(-x_1,x_2) = f_2(x_1,x_2), \quad f_2(x_1,-x_2) = -f_2(x_1,x_2), \\ f_1(x_2,x_1) = f_2(x_1,x_2), \quad f_2(x_2,x_1) = f_1(x_1,x_2). \end{cases}$$

The space X_{sym} denotes the set of all $u = {}^\top(\phi, w_1, w_2) \in X$ satisfying $\Gamma_j u = u$ $(j=1,2,3)$.

The space $\mathscr{X}_{(1)}$ is defined by

$$\mathscr{X}_{(1)} = \{\phi \in L^\infty_1 \cap L^2; \mathrm{supp}\ \hat{\phi} \subset \{|\xi| \le r_\infty\}, \|\phi\|_{\mathscr{X}_{(1)}} < +\infty\},$$

where the norm is defined by

$$\|\phi\|_{\mathscr{X}_{(1)}} := \|\phi\|_{\mathscr{X}_{(1),L^\infty}} + \|\phi\|_{\mathscr{X}_{(1),L^2}},$$
$$\|\phi\|_{\mathscr{X}_{(1),L^\infty}} := \sum_{k=0}^{1} \|\nabla^k \phi\|_{L^\infty_{k+1}}, \ \|\phi\|_{\mathscr{X}_{(1),L^2}} := \sum_{k=0}^{1} \|\nabla^k \phi\|_{L^2_k}.$$

On the other hand, $\mathscr{Y}_{(1)}$ is defined by

$$\mathscr{Y}_{(1)} = \{w \in L^\infty_1, \nabla w \in H^1; \mathrm{supp}\ \hat{w} \subset \{|\xi| \le r_\infty\}, \|w\|_{\mathscr{Y}_{(1)}} < +\infty\},$$

where

$$\|w\|_{\mathscr{Y}_{(1)}} := \|w\|_{\mathscr{X}_{(1),L^\infty}} + \|w\|_{\mathscr{Y}_{(1),L^2}},$$

$$\|w\|_{\mathscr{Y}_{(1),L^2}} := \sum_{j=1}^{2} \|(1+|x|)^{j-1}\nabla^j w\|_{L^2}.$$

We define a weighted space for the low frequency part $\mathscr{Z}_{(1)}(a,b)$ by

$$\mathscr{Z}_{(1)}(a,b) = C^1([a,b];\mathscr{X}_{(1)}) \times \Big[C([a,b];\mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})\Big].$$

Let s be a nonnegative integer satisfying $s \geq 3$. We denote by the space $\mathscr{Z}^k_{(\infty),1}(a,b)$ $(k = s-1, s)$ the weighted space for the high frequency part defined by

$$\begin{aligned}\mathscr{Z}^k_{(\infty),1}(a,b) = &\big[C([a,b];H^k_{(\infty),2}) \cap C^1([a,b];L^2_2)\big] \\ &\times \big[L^2(a,b;;H^{k+1}_{(\infty),2}) \cap C([a,b];H^k_{(\infty),2}) \cap H^1(a,b;H^{k-1}_{(\infty),2})\big].\end{aligned}$$

Let s be a nonnegative integer satisfying $s \geq 3$ and let $k = s-1, s$. We define a space $X^k(a,b)$ by

$$\begin{aligned}&X^k(a,b) \\ = &\big\{\{u_1,u_\infty\}; u_1 \in \mathscr{Z}_{(1)}(a,b), u_\infty \in \mathscr{Z}^k_{(\infty),2}(a,b), \\ &\qquad \partial_t\phi_\infty \in C([a,b];L^2_1), u_j = {}^\top(\phi_j, w_j)\,(j=1,\infty)\big\},\end{aligned}$$

and we define the norm by

$$\begin{aligned}\|\{u_1,u_\infty\}\|_{X^k(a,b)} :=&\|u_1\|_{\mathscr{Z}_{(1)}(a,b)} + \|u_\infty\|_{\mathscr{Z}^k_{(\infty),2}(a,b)} \\ &+ \|\partial_t\phi_\infty\|_{C([a,b];L^2_1)} + \|\partial_t u_1\|_{C([a,b];L^2)} + \|\partial_t\nabla u_1\|_{C([a,b];L^2_1)}.\end{aligned}$$

Function spaces of time periodic functions with period T are introduced as follows. $C_{per}(\mathbb{R};X)$ stands for the set of all time periodic continuous functions with values in X and period T whose the norm is defined by $\|\cdot\|_{C([0,T];X)}$; Similarly, $L^2_{per}(\mathbb{R};X)$ denotes the set of all time periodic locally square integrable functions with values in X and period T whose the norm is defined by $\|\cdot\|_{L^2(0,T;X)}$. Similarly, $H^1_{per}(\mathbb{R};X)$ and $X^k_{per}(\mathbb{R})$, and so on, are defined.

For operators L_1 and L_2, we denote by $[L_1,L_2]$ the commutator of L_1 and L_2 i.e.,

$$[L_1,L_2]f := L_1(L_2 f) - L_2(L_1 f).$$

We next state a lemma which will be used in the proof of the main result. The following Hardy inequality is known for a function satisfying the oddness conditions in (0.0.30) on $\mathbb{R}^2$.

Lemma 4.1.1. *Let $u \in H^1$ and we assume that u satisfies*

$$u(-x_1,x_2) = -u(x_1,x_2) \text{ or } u(x_1,-x_2) = -u(x_1,x_2) \tag{4.1.1}$$

for $x = {}^\top(x_1,x_2)$. Then there holds the inequality

$$\Big\|\frac{u}{|x|}\Big\| \leq C\|\nabla u\|_{L^2}.$$

See, e.g., [9] for the proof of Lemma 4.1.1.

4.2 Main result of Chapter 4

In this section, we state our main result on the existence of a time periodic solution for (0.0.1).

To state our result, Recall that the following operators are introduced which decompose a function into its low and high frequency parts respectively in Chapter 1. The operators P_1 and P_∞ on L^2 are defined by

$$P_j f = \mathcal{F}^{-1}(\hat{\chi}_j \mathcal{F}[f]) \quad (f \in L^2, j = 1, \infty),$$

where

$$\begin{aligned}
&\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^2) \quad (j = 1, \infty), \quad 0 \le \hat{\chi}_j \le 1 \quad (j = 1, \infty), \\
&\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \le r_1), \\ 0 & (|\xi| \ge r_\infty), \end{cases} \\
&\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi), \\
&0 < r_1 < r_\infty.
\end{aligned}$$

r_1 and r_∞ are positive constants satisfying $0 < r_1 < r_\infty < \frac{2\gamma}{\nu + \tilde{\nu}}$ in such a way that the estimate (4.4.7) in Lemma 4.4.3 below holds for $|\xi| \le r_\infty$.

We are in a position to state our result on the existence of a time periodic solution.

Theorem 4.2.1. *Let s be an integer satisfying $s \ge 3$. Assume that $g(x,t)$ satisfies* (0.0.2), (0.0.30) *and $g \in C_{per}(\mathbb{R}; L^1_1 \cap L^\infty_3) \cap L^2_{per}(\mathbb{R}; H^{s-1}_2)$. We define the norm of g by*

$$[g]_s = \|g\|_{C([0,T]; L^1_1 \cap L^\infty_3)} + \|g\|_{L^2(0,T; H^{s-1}_2)}.$$

Then there exist constants $\delta > 0$ and $C > 0$ such that if $[g]_s \le \delta$, then the system (0.0.15) *has a time periodic solution $u = u_1 + u_\infty$ satisfying $\{u_1, u_\infty\} \in X^s_{sym,per}(\mathbb{R})$ with $\|\{u_1, u_\infty\}\|_{X^s(0,T)} \le C[g]_s$. Furthermore, the uniqueness of time periodic solutions of* (0.0.15) *holds in the class $\{u = {}^\top(\phi, w); u = u_1 + u_\infty, \{u_1, u_\infty\} \in X^s_{sym,per}(\mathbb{R}), \|\{u_1, u_\infty\}\|_{X^s(0,T)} \le C\delta\}$.*

4.3 Reformulation of the problem

In this section, we reformulate (0.0.15). we begin with to decompose u into a low frequency part u_1 and a high frequency part u_∞, and then, we rewrite (0.0.15) into equations for u_1 and u_∞ as in Chapter 1.

Similarly to Chapter 1, we define

$$u_1 = P_1 u, \quad u_\infty = P_\infty u.$$

Applying the operators P_1 and P_∞ to (0.0.15), we see

$$\partial_t u_1 + Au_1 = F_1(u_1 + u_\infty, g), \tag{4.3.1}$$

$$\partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) = F_\infty(u_1 + u_\infty, g). \tag{4.3.2}$$

Here

$$\begin{aligned} F_1(u_1 + u_\infty, g) &= P_1[-B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g)], \\ F_\infty(u_1 + u_\infty, g) &= P_\infty[-B[u_1 + u_\infty]u_1 + G(u_1 + u_\infty, g)]. \end{aligned}$$

On the other hand, if some functions u_1 and u_∞ satisfy (4.3.1) and (4.3.2), then adding (4.3.1) to (4.3.2), we derive that

$$\begin{aligned} \partial_t(u_1 + u_\infty) + A(u_1 + u_\infty) &= -P_\infty(B[u_1 + u_\infty]u_\infty) + (F_1 + F_\infty)(u_1 + u_\infty, g) \\ &= -B[u_1 + u_\infty](u_1 + u_\infty) + G(u_1 + u_\infty, g). \end{aligned}$$

Defining $u = u_1 + u_\infty$, we get

$$\partial_t u + Au + B[u]u = G(u, g).$$

Therefore, to obtain a solution u of (0.0.15), we look for a solution $\{u_1, u_\infty\}$ satisfying (4.3.1)-(4.3.2).

Concerning antisymmetry of (0.0.15) and (4.3.1)-(4.3.2), We state the following lemmas. Recall that Γ_j $(j = 1, 2, 3)$ is defined by

$$(\Gamma_1 u)(x) = {}^\top(\phi(-x), -w_1(-x), w_2(-x)), \quad (\Gamma_2 u)(x) = {}^\top(\phi(-x), w_1(-x), -w_2(-x)),$$

$$(\Gamma_3 u)(x_1, x_2) = {}^\top(\phi(x_2, x_1), w_2(x_2, x_1), w_1(x_2, x_1))$$

for $u(x) = {}^\top(\phi(x), w_1(x), w_2(x))$, $x \in \mathbb{R}^2$.

Lemma 4.3.1. *We define* $\boldsymbol{g}(x, t) = {}^\top(0, g(x, t))$ *and let* g *satisfy* $(\Gamma_j \boldsymbol{g})(x, t) = \boldsymbol{g}(x, t)$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1, 2, 3)$.

(i) $\Gamma_j u$ $(j = 1, 2, 3)$ *is a solution of* (0.0.15) *if* $u = {}^\top(\phi, w)$ *is a solution of* (0.0.15).

(ii) $\{\Gamma_j u_1, \Gamma_j u_\infty\}$ $(j = 1, 2, 3)$ *is a solution of* (4.3.1)-(4.3.2) *if* $\{u_1, u_\infty\}$ *is a solution of* (4.3.1)-(4.3.2).

Lemma 4.3.2. *Let g satisfy $(\Gamma_j \boldsymbol{g})(x,t) = \boldsymbol{g}(x,t)$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1,2,3)$.*

(i) *There holds*

$$[\Gamma_j(\partial_t u + Au + B[u]u - G(u,g))](x,t) = [\partial_t u + Au + B[u]u - G(u,g)](x,t)$$

for $x \in \mathbb{R}^2, t \in \mathbb{R},\ j = 1,2,3$ if $(\Gamma_j u)(x,t) = u(x,t)$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1,2,3)$.

(ii) *There hold*

$$[\Gamma_j(\partial_t u_1 + Au_1 - F_1(u_1 + u_\infty, g))](x,t) = [\partial_t u_1 + Au_1 - F_1(u_1 + u_\infty, g)](x,t)$$

and

$$\begin{aligned} &[\Gamma_j(\partial_t u_\infty + Au_\infty + P_\infty(B[u_1+u_\infty]u_\infty) - F_\infty(u_1+u_\infty, g))](x,t) \\ &= \ [\partial_t u_\infty + Au_\infty + P_\infty(B[u_1+u_\infty]u_\infty) - F_\infty(u_1+u_\infty, g)](x,t) \end{aligned}$$

for $x \in \mathbb{R}^n, t \in \mathbb{R},\ j = 1,2,3$ if $\{\Gamma_j u_1(x,t), \Gamma_j u_\infty(x,t)\} = \{u_1(x,t), u_\infty(x,t)\}$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1,2,3)$.

Direct computations verify Lemma 4.3.1 (i) and Lemma 4.3.2 (i). As for Lemma 4.3.1 (ii) and Lemma 4.3.2 (ii), since it holds that $\mathcal{F}\Gamma_j = -\Gamma_j\mathcal{F}$ $(j = 1,2)$, $\mathcal{F}\Gamma_3 = \Gamma_3\mathcal{F}$, $\chi_j(-\xi_1,\xi_2) = \chi_j(\xi_1,-\xi_2) = \chi_j(\xi_2,\xi_1) = \chi_j(\xi_1,\xi_2)$ $(j = 1,\infty)$, we find that $\Gamma_k P_j = P_j \Gamma_k$ $(k = 1,2,3,\ j = 1,\infty)$. Hence Lemma 4.3.1 (ii) and Lemma 4.3.2 (ii) follow from the above relation by a direct computation.

Therefore, we consider (4.3.1)-(4.3.2) in space of functions satisfying $\{\Gamma_j u_1, \Gamma_j u_\infty\} = \{u_1, u_\infty\}$ $(j = 1,2,3)$ by Lemma 4.3.1 and Lemma 4.3.2.

To prove the existence of time periodic solution on $\mathbb{R}^2$, we use the momentum formulation for the low frequency part due to the slow decay of the low frequency part u_1 in a weighted L^∞ space. Applying the momentum formulation was used in Chapter 2 for the low frequency part.

To state the momentum formulation, the following inequality holds for the weighted L^2 norm of the low frequency part.

Lemma 4.3.3. *Let $\phi \in \mathscr{X}_{(1)}$ and $w_1 \in \mathscr{Y}_{(1)}$. Then, it holds that*

$$\|P_1(\phi w_1)\|_{\mathscr{Y}_{(1),L^2}} \le C(\|\phi\|_{L^\infty_1} + \|\nabla\phi\|_{L^2_1})(\|w\|_{L^\infty_1} + \|\nabla w_1\|_{L^2})$$

uniformly for ϕ and w.

Lemma 4.3.3 follows directly from Lemma 2.3.2.

We are in a position to reformulate the system (4.3.1)-(4.3.2) by using the momentum for the low frequency part as in Chapter 2.

We introduce m_1 and $u_{1,m}$ by

$$m_1 = w_1 + P_1(\phi w), \quad u_{1,m} = {}^\top(\phi_1, m_1), \tag{4.3.3}$$

where $\phi = \phi_1 + \phi_\infty$ and $w = w_1 + w_\infty$. Here we write the vector w with values in $\mathbb{R}^2$ as $w = {}^\top(w^1, w^2)$. We directly obtain the following Lemma from Lemma 2.3.4.

Lemma 4.3.4. *Assume that* $\{u_1, u_\infty\}$ *satisfies the system* (4.3.1)-(4.3.2). *Then* $\{u_{1,m}, u_\infty\}$ *satisfies the following system:*

$$\begin{aligned} \partial_t u_{1,m} + Au_{1,m} &= F_{1,m}(u_1 + u_\infty, g), \\ \partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g). \end{aligned} \tag{4.3.4}$$

Here

$$\begin{aligned} F_{1,m}(u_1 + u_\infty, g) &= {}^\top(0, \tilde{F}_{1,m}(u_1 + u_\infty, g)), \\ \tilde{F}_{1,m}(u_1 + u_\infty, g) &= -P_1\{\mu\Delta(\phi w) + \tilde{\mu}\nabla \mathrm{div}\,(\phi w) + \frac{\rho_*}{\gamma}\nabla(p^{(1)}(\phi)\phi^2) \\ &\quad + \gamma \mathrm{div}\,(\phi w \otimes w) - \frac{1}{\gamma}((1+\phi)g) \\ &\quad + \gamma\partial_{x_2}\begin{pmatrix} w^1w^2 \\ (w^2)^2 - (w^1)^2 \end{pmatrix} + \gamma\partial_{x_1}\begin{pmatrix} 0 \\ w^2w^1 \end{pmatrix} + \gamma\nabla(w^1)^2\}. \end{aligned} \tag{4.3.5}$$

Remark 4.3.5. Here we rewrite the convection term $\mathrm{div}\,(w \otimes w)$ by using the relation

$$\mathrm{div}\,(w \otimes w) = \partial_{x_2}\begin{pmatrix} w^1w^2 \\ (w^2)^2 - (w^1)^2 \end{pmatrix} + \partial_{x_1}\begin{pmatrix} 0 \\ w^2w^1 \end{pmatrix} + \nabla(w^1)^2$$

to estimate with the antisymmetry. See Proposition 7.1.

Similarly to Lemma 4.3.2, the following lemma follows from direct computations which implies that the antisymmetry of (4.3.4) holds.

Lemma 4.3.6. (i) $\Gamma_j u_{1,m}$ $(j = 1,2,3)$ *is a solution of* (4.3.4) *if* $u_{1,m} = {}^\top(\phi_1, m_1)$ *is a solution of* (4.3.4).

(ii) *Let* g *satisfy* $(\Gamma_j \boldsymbol{g})(x,t) = \boldsymbol{g}(x,t)$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1,2,3)$. *Then there hold*

$$[\Gamma_j(\partial_t u_{1,m} + Au_{1,m} - F_{1,m}(u_{1,m} + u_\infty, g))](x,t) = [\partial_t u_{1,m} + Au_{1,m} - F_{1,m}(u_{1,m} + u_\infty, g)](x,t)$$

for $x \in \mathbb{R}^n, t \in \mathbb{R}$, $j = 1,2,3$ *if* $\{\Gamma_j u_{1,m}(x,t), \Gamma_j u_\infty(x,t)\} = \{u_{1,m}(x,t), u_\infty(x,t)\}$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1,2,3)$.

If $\phi = \phi_1 + \phi_\infty$ is sufficiently small, we obtain the solution $\{u_1, u_\infty\}$ of (4.3.1)-(4.3.2) from the solution of (4.3.2), (4.3.3) and (4.3.4), i.e., we have the following.

Lemma 4.3.7. (i) *Let s be an integer satisfying $s \geq 3$ and We choose $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfying $\{u_{1,m}, u_\infty\} \in X^s_{sym}(a,b)$. Then there exists a positive constant δ_0 such that there uniquely exists $w_1 \in C([a,b]; \mathscr{Y}_{(1),\#}) \cap H^1(a,b; \mathscr{Y}_{(1),\#})$ and w_1 satisfies the following inequality if $\phi = \phi_1 + \phi_\infty$ satisfies $\sup_{t\in[a,b]}(\|\phi\|_{L^\infty_2} + \|\nabla\phi\|_{L^2}) \leq \delta_0$.*

$$w_1 = m_1 - P_1(\phi(w_1 + w_\infty)), \tag{4.3.6}$$

where $\phi = \phi_1 + \phi_\infty$. Furthermore, we have the estimates

$$\begin{aligned}
\|w_1\|_{C([a,b];\mathscr{Y}_{(1)})} &\leq C(\|m_1\|_{C([a,b];\mathscr{Y}_{(1)})} + \|w_\infty\|_{C([a,b];L^2_1)}), &(4.3.7)\\
\int_b^a \|\partial_t w_1(\tau)\|^2_{\mathscr{Y}_{(1)}} d\tau &\leq C((\|\partial_t \nabla\phi_1\|^2_{C([a,b];L^2_1)} + \|\partial_t\phi_\infty\|^2_{C([a,b];L^2_1)})\|w_1\|^2_{C([a,b];L^\infty_1)}\\
&\quad +\|\partial_t\phi\|^2_{C([a,b];L^2)}\|w_1\|^2_{C([a,b];\mathscr{X}_{(1),L^\infty})})\\
&\quad + \int_b^a C\Big(\|\partial_t m_1(\tau)\|^2_{\mathscr{Y}_{(1)}} + \|\partial_t\phi\|^2_{C([a,b];L^2)}\|w_\infty(\tau)\|^2_{H^s_2}\\
&\quad +\|\partial_t w_\infty(\tau)\|^2_{L^2_1}\Big)d\tau. &(4.3.8)
\end{aligned}$$

(ii) *Let s be an integer satisfying $s \geq 3$ and We choose $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ satisfying $\{u_{1,m}, u_\infty\} \in X^s_{sym}(a,b)$. We suppose that $\phi = \phi_1 + \phi_\infty$ satisfies $\sup_{t\in[a,b]}(\|\phi\|_{L^\infty_1} + \|\nabla\phi\|_{L^2}) \leq \delta_0$ and $\{u_{1,m}, u_\infty\}$ satisfies*

$$\begin{aligned}
\partial_t u_{1,m} + Au_{1,m} &= F_{1,m}(u_1 + u_\infty, g),\\
w_1 &= m_1 - P_1(\phi w),\\
\partial_t u_\infty + Au_\infty + P_\infty(B[u_1 + u_\infty]u_\infty) &= F_\infty(u_1 + u_\infty, g).
\end{aligned}$$

Here $w = w_1 + w_\infty$ and w_1 defined by (4.3.6). *Then $\{u_1, u_\infty\}$ satisfies* (4.3.1)-(4.3.2) *with $u_1 = {}^\top(\phi_1, w_1)$.*

Lemma 4.3.7 can be proved by the same way as the proof of Lemma 2.3.5 using Lemma 1.1.1 and Lemma 2.3.2 and we omit the details.

Therefore, we consider (4.3.2), (4.3.4) and (4.3.6) because if we show the existence of a solution $\{u_{1,m}, u_\infty\} \in X^s_{sym}(a,b)$ satisfying (4.3.2), (4.3.4) and (4.3.6), then by Lemma 4.3.7, we obtain a solution $\{u_1, u_\infty\} \in X^s_{sym}(a,b)$ satisfying (4.3.1)-(4.3.2).

As in Chapter 2, we formulate (4.3.2), (4.3.4) and (4.3.6) by using time-T-mapping to solve the time periodic problem. We consider the following linear problems for the low frequency part and the high frequency part respectively:

$$\begin{cases} \partial_t u_{1,m} + Au_{1,m} = F_{1,m},\\ u_{1,m}|_{t=0} = u_{01,m}, \end{cases} \tag{4.3.9}$$

and

$$\begin{cases} \partial_t u_\infty + Au_\infty + P_\infty(B[\tilde{u}]u_\infty) = F_\infty,\\ u_\infty|_{t=0} = u_{0\infty}, \end{cases} \tag{4.3.10}$$

where $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), u_{01,m}, u_{0\infty}, F_{1,m}$ and F_∞ are given functions.

The solution operators are introduced as follows. (The precise definition of these operators will be given later.) $S_1(t)$ stands for the solution operator for (4.3.9) with $F_{1,m} = 0$, and $\mathscr{S}_1(t)$ stands for the solution operator for (4.3.9) with $u_{01,m} = 0$. On the other hand, $S_{\infty,\tilde{u}}(t)$ stands for the solution operator for (4.3.10) with $F_\infty = 0$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ stands for the solution operator for (4.3.10) with $u_{0\infty} = 0$.

As in Chapter 2, we will look for $\{u_{1,m}, u_\infty\}$ satisfying

$$\begin{cases} u_{1,m}(t) = S_1(t)u_{01,m} + \mathscr{S}_1(t)[F_{1,m}(u,g)], \\ u_\infty(t) = S_{\infty,u}(t)u_{0\infty} + \mathscr{S}_{\infty,u}(t)[F_\infty(u,g)], \end{cases} \tag{4.3.11}$$

where

$$\begin{cases} u_{01,m} = (I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{1,m}(u,g)], \\ u_{0\infty} = (I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_\infty(u,g)], \end{cases} \tag{4.3.12}$$

$u = {}^\top(\phi, w)$ is a function given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w).$$

From (4.3.11) and (4.3.12), it holds that $u_{1,m}(T) = u_{1,m}(0)$, $u_\infty(T) = u_\infty(0)$. Hence we look for a pair of functions $\{u_{1,m}, u_\infty\}$ satisfying (4.3.11)-(4.3.12). The solution operators $S_1(t)$ and $\mathscr{S}_1(t)$ are investigated and we state the estimate of a solution for the low frequency part in section 5; Some properties of $S_{\infty,u}(t)$ and $\mathscr{S}_{\infty,u}(t)$ will be stated and we estimate a solution for the high frequency part in section 6.

In the remaining of this section some lemmas are stated which will be used in the proof of Theorem 4.2.1. The following lemma prays important roles to estimate a convolution with antisymmetry for the low frequency part.

Lemma 4.3.8. *Let $E(x)$ $(x \in \mathbb{R}^2)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C}{(1+|x|)^{|\alpha|+1}} \quad (|\alpha| \ge 0) \tag{4.3.13}$$

and let f be a scalar function satisfying $f \in L^\infty_2$. We assume that f satisfies

$$f(-x_1, x_2) = -f(x_1, x_2) \text{ or } f(x_1, -x_2) = -f(x_1, x_2) \text{ or } f(x_2, x_1) = -f(x_1, x_2) \tag{4.3.14}$$

Then there holds the following estimate.

$$|E * f(x)| \le \frac{C\|f\|_{L^\infty_2}}{(1+|x|)}. \tag{4.3.15}$$

Proof. We first assume $|x| \ge 1$. We set $R = \frac{|x|}{2}$. Then we see that

$$E * f(x) = \int_{\mathbb{R}^2} E(x-y)f(y)dy$$

$$\begin{aligned} &= \int_{|x-y|\geq R,\,|y|\geq R} E(x-y)f(y)dy \\ &\qquad + \int_{|x-y|\leq R} E(x-y)f(y)dy + \int_{|y|\leq R} E(x-y)f(y)dy \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where,

$$I_1 = \int_{|x-y|\geq R,\,|y|\geq R} E(x-y)f(y)dy,\ I_2 = \int_{|x-y|\leq R} E(x-y)f(y)dy,\ I_3 = \int_{|y|\leq R} E(x-y)f(y)dy.$$

Concerning the estimate for I_1, since $|y| \leq |x| + |x-y| \leq 3|x-y|$ if $|x-y| \geq R$ and $|y| \geq R$, it follows from (4.3.13) that

$$|I_1| \leq C\|f\|_{L^\infty_2} \int_{|y|\geq R} \frac{1}{(1+|y|)^3} dy \leq \frac{C\|f\|_{L^\infty_2}}{1+|x|}.$$

We next derive the estimate of I_2. Because it holds that $|y| \geq |x| - |x-y| \geq R$ if $|x-y| \leq R$, we obtain from (4.3.13) that

$$|I_2| \leq \frac{C\|f\|_{L^\infty_2}}{R^2} \int_{|x-y|\geq R} \frac{1}{(1+|x-y|)} dy \leq \frac{C\|f\|_{L^\infty_2}}{1+|x|}.$$

As for the estimate of I_3, we consider the case such that f satisfies $f_1(-x_1, x_2) = -f_1(x_1, x_2)$. We define $\tilde{y} = {}^\top(-y_1, y_2)$ for $y = {}^\top(y_1, y_2)$ on $\mathbb{R}^2$ satisfying $y_1 \geq 0$. Note that $f(\tilde{y}) = -f(y)$ due to (4.3.14). This implies that

$$\begin{aligned} I_3 &= \int_{|y|\leq R,\, y_1\geq 0} E(x-y)f(y)dy + \int_{|y|\leq R,\, y_1\geq 0} E(x-\tilde{y})f(\tilde{y})dy \\ &= \int_{|y|\leq R,\, y_1\geq 0} \{E(x-y) - E(x-\tilde{y})\} f(y)dy. \end{aligned}$$

In addition, we see from (4.3.13) that

$$|E(x-y) - E(x-\tilde{y})| \leq \frac{C|y|}{1+|x-y|^2} \leq \frac{C|y|}{1+R^2} \tag{4.3.16}$$

for $|y| \leq R$. Hence we arrive at

$$|I_3| \leq \frac{C\|f\|_{L^\infty_2}}{(1+R)^2} \int_{|y|\geq R} \frac{1}{1+|y|} dy \leq \frac{C\|f\|_{L^\infty_2}}{1+|x|}.$$

Similarly, we obtain (4.3.15) in the case such that f satisfies $f(x_1, -x_2) = -f(x_1, x_2)$. If f satisfies $f(x_2, x_1) = -f(x_1, x_2)$, by setting $\tilde{y} = {}^\top(y_2, y_1)$ for $y = {}^\top(y_1, y_2)$ on $\mathbb{R}^2$, $|I_3|$ is written as

$$|I_3| = \left| \int_{|y|\leq R,\, y_2\geq y_1} E(x-y)f(y)dy + \int_{|y|\leq R,\, y_2\geq y_1} E(x-\tilde{y})f(\tilde{y})dy \right|$$

$$= \left| \int_{|y|\le R,\, y_2\ge y_1} \{E(x-y) - E(x-\tilde{y})\} f(y)dy \right|.$$

This together with (4.3.16) yields the required estimate (4.3.15). By using the estimates for I_j ($j = 1, 2, 3$), we get the required estimate (4.3.15) for $|x| \ge 1$.

As for the case $|x| \le 1$, the required estimate (4.3.15) can be verified by direct computations, hence we omit the details. This completes the proof. □

By Lemma 4.3.8, we have the following assertion which is useful for the estimate of a convolution with external force.

Lemma 4.3.9. *Let $f \in \mathscr{S}'(\mathbb{R}^n)$ satisfying that supp $\hat{f} \subset \{|\xi| \le r_\infty\}$ and $\nabla f \in L^\infty_2$, where $\mathscr{S}'(\mathbb{R}^n)$ denotes a set of all tempered distributions on $\mathbb{R}^n$. We assume that for $j = 1$ or 2, $\partial_{x_j} f$ satisfies*

$$\partial_{x_j} f(-x_1, x_2) = -\partial_{x_j} f(x_1, x_2) \text{ or } \partial_{x_j} f(x_1, -x_2) = -\partial_{x_j} f(x_1, x_2) \qquad (4.3.17)$$

Then $f \in L^\infty_1$ and it holds that

$$\|f\|_{L^\infty_1} \le C \|\nabla f\|_{L^\infty_2}.$$

Proof. We assume that $\partial_{x_1} f$ satisfies (4.3.17). We see that

$$f = \mathcal{F}^{-1}\Big\{ \Big(\frac{\chi_0}{i\xi_1} \Big) (i\xi_1) \hat{f} \Big\} = \mathcal{F}^{-1} \Big(\frac{\chi_0}{i\xi_1} \Big) * \partial_{x_1} f,$$

where χ_0 is a cut-off function defined by $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$ and $\hat{\chi}_0$ is the one defined by (2.3.3). Note that

$$\Big| \mathcal{F}^{-1} \Big(\frac{\chi_0}{i\xi_1} \Big) \Big| \le C$$

where $C > 0$ is a positive constant. This together with Lemma 2.3.7 implies that

$$\Big| \partial_x^\alpha \mathcal{F}^{-1} \Big(\frac{\chi_0}{i\xi_1} \Big) \Big| \le \frac{C}{(1 + |x|)^{1+|\alpha|}}$$

for $|\alpha| \ge 0$. Therefore, we derive from Lemma 4.3.8 that

$$\|f\|_{L^\infty_1} \le C \|\nabla f\|_{L^\infty_2}.$$

If $\partial_{x_2} f$ satisfies (4.3.17), we obtain the required estimate similarly to the proof for $\partial_{x_1} f$. This completes the proof. □

We use the following another type estimates for a convolution with antisymmetry.

Lemma 4.3.10. (i) *Let $E(x)$ $(x \in \mathbb{R}^2)$ be a scalar function satisfying* (4.3.13)*. Assume that f is a scalar function satisfying $\|f\|_{L^\infty_3} + \|f\|_{L^1_1} < \infty$. We also assume that f satisfies* (4.3.14)*. Then there holds the following estimate.*

$$|E * f(x)| \leq \frac{C}{(1+|x|)^2}(\|f\|_{L^\infty_3} + \|f\|_{L^1_1}).$$

(ii) *Let $E(x)$ $(x \in \mathbb{R}^2)$ be a scalar function satisfying* (4.3.13) *and let f be a scalar function which is written as $f = \partial_{x_j} f_1$ for $j = 1$ or 2 and satisfy $\|\partial_{x_j} f_1\|_{L^\infty_3} + \|f_1\|_{L^\infty_2} < \infty$. We assume that f_1 satisfies* (4.3.14)*. Then the following estimate is true.*

$$|E * f(x)| \leq \frac{C}{(1+|x|)^2}(\|\partial_{x_j} f_1\|_{L^\infty_3} + \|f_1\|_{L^\infty_2}).$$

(iii) *Let $E(x)$ $(x \in \mathbb{R}^2)$ be a scalar function satisfying* (4.3.13) *and let f be a scalar function of the form: $f = \partial_{x_j} f_1$ for $j = 1$ or 2 and it holds that $\|\partial_{x_j} f_1\|_{L^\infty_3} + \|f_1\|_{L^\infty_2} < \infty$. Then we have the following estimate.*

$$|\partial^\alpha_x E * f(x)| \leq \frac{C}{(1+|x|)^{1+|\alpha|}}(\|\partial_{x_j} f_1\|_{L^\infty_3} + \|f_1\|_{L^\infty_2}).$$

Lemma 4.3.10 yields in a similar manner to the proof of Lemma 4.3.8 and we omit the proofs.

The following L^2 estimates holds for the external force in the low frequency part.

Lemma 4.3.11. (i) *Let $E(\xi)$ $(\xi \in \mathbb{R}^2)$ be a scalar function satisfying* $\operatorname{supp} E \subset \{|\xi| \leq r_\infty\}$ *and*

$$|E(\xi)| \leq \frac{C}{|\xi|^2} \text{ for } |\xi| \leq r_\infty, \ |\xi| \neq 0.$$

Let f belong to $L^2_{(1),1} \cap L^1_1$ and we assume that the following case (1) *or* (2) *hold ;*

$$(1) \quad f(-x_1, x_2) = -f(x_1, x_2), \quad f(x_1, -x_2) = f(x_1, x_2),$$

$$(2) \quad f(-x_1, x_2) = f(x_1, x_2), \quad f(x_1, -x_2) = -f(x_1, x_2).$$

Then we have the estimate

$$\|\mathcal{F}^{-1}(E\hat{f})\|_{\mathscr{Y}_{(1),L^2}} \leq C\|f\|_{L^2_1 \cap L^1_1}.$$

(ii) *We suppose that $E(\xi)$ $(\xi \in \mathbb{R}^2)$ is a scalar function satisfying* $\operatorname{supp} E \subset \{|\xi| \leq r_\infty\}$ *and*

$$|E(\xi)| \leq \frac{C}{|\xi|} \text{ for } |\xi| \leq r_\infty, \ |\xi| \neq 0.$$

and f belongs to $L^2_{(1),1} \cap L^1_1$ which satisfies the following case (1) or (2);

$$(1) \quad f(-x_1, x_2) = -f(x_1, x_2), \quad f(x_1, -x_2) = f(x_1, x_2),$$

$$(2) \quad f(-x_1, x_2) = f(x_1, x_2), \quad f(x_1, -x_2) = -f(x_1, x_2).$$

Then there hols the estimate

$$\|\mathcal{F}^{-1}(E\hat{f})\|_{\mathscr{X}_{(1),L^2}} \leq C\|f\|_{L^2_1 \cap L^1_1}.$$

Proof. (i) We assume that f satisfies (1) without loss of generality. Since $\hat{f}(\xi_1, -\xi_2) = -\hat{f}(\xi_1, \xi_2)$, we see that

$$\begin{aligned}
\|\nabla\{\mathcal{F}^{-1}(E\hat{f})\}\|_{L^2} &\leq C\Big\|\frac{1}{|\xi|}\hat{f}\Big\|_{L^2} \\
&\leq C\Big\|\xi_2\frac{1}{|\xi|}\Big\|_{L^2(|\xi|\leq r_\infty)}\Big\|\int_0^1 \partial_{\xi_2}\hat{f}(\xi_1, \tau\xi_2)d\tau\Big\|_{L^2(|\xi|\leq r_\infty)} \\
&\leq C\|xf\|_{L^1}.
\end{aligned}$$

Similarly, we obtain the estimate

$$\|\nabla^2\{\mathcal{F}^{-1}(E\hat{f})\}\|_{L^2_1} \leq C\|f\|_{L^1_1 \cap L^2_1}.$$

The assertion (ii) can be proved by the same way as that for (i). This completes the proof. □

We find the following estimate for the nonlinear term on the low frequency part in weighted L^2 spaces.

Lemma 4.3.12. (i) *Let $w_1 \in \mathscr{Y}_{(1),\#}$. Then, it holds that*

$$\|(w_1)^2\|_{L^2} + \|w_1\partial_{x_j}w_1\|_{L^2_1} \leq C\|w_1\|^2_{\mathscr{Y}_{(1)}} \quad (j = 1, 2).$$

(ii) *Let $\phi \in \mathscr{X}_{(1)}$ and $w_1 \in \mathscr{Y}_{(1),\#}$. Then, there holds the estimate*

$$\|\phi w_1\|_{L^2} + \|\partial_{x_j}(\phi w_1)\|_{L^2_1} \leq C\|\phi\|_{\mathscr{X}_{(1)}}\|w_1\|_{\mathscr{Y}_{(1)}} \quad (j = 1, 2).$$

Proof. Concerning the assertion (i), applying Lemma 4.1.1, we see that

$$\|(w_1)^2\|_{L^2} \leq C\|w_1\|_{L^\infty_1}\Big\|\frac{w_1}{|x|}\Big\|_{L^2} \leq C\|w_1\|_{L^\infty_1}\|\nabla w_1\|_{L^2}.$$

Similarly we derive that

$$\|w_1\partial_{x_j}w_1\|_{L^2_1} \leq C\|w_1\|^2_{\mathscr{Y}_{(1)}}.$$

The assertion (ii) yields similarly to the proof of the estimate for (i). This completes the proof. □

4.4 Estimates for solution on the low frequency part

In this section we estimate a solution u_1 satisfying $u_1(0) = u_1(T)$ and

$$\partial_t u_1 + Au_1 = F_1, \tag{4.4.1}$$

where $F_1 = {}^\top(0, \tilde{F}_1)$.

We define A_1 by the restriction of A on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. The symbol S_1 and $\mathscr{S}_1(t)$ are defined by $S_1(t) = e^{-tA_1}$ and

$$\mathscr{S}_1(t)F_1 = \int_0^t S_1(t-\tau)F_1(\tau)\, d\tau.$$

Recall that Γ_j $(j = 1, 2, 3)$ are defined by

$$(\Gamma_1 u)(x) = {}^\top(\phi(-x), -w_1(-x), w_2(-x)), \quad (\Gamma_2 u)(x) = {}^\top(\phi(-x), w_1(-x), -w_2(-x)),$$
$$(\Gamma_3 u)(x_1, x_2) = {}^\top(\phi(x_2, x_1), w_2(x_2, x_1), w_1(x_2, x_1))$$

for $u(x) = {}^\top(\phi(x), w_1(x), w_2(x)), \quad x \in \mathbb{R}^2$. We have the following.

Proposition 4.4.1. (i) *A_1 is a bounded linear operator on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. Moreover, $S_1(t)$ is a uniformly continuous semigroup on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. and $S_1(t)$ satisfies the following estimates for all $T' > 0$;*

$$S_1(t)u_1 \in C^1([0, T']; \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}), \quad \partial_t S_1(\cdot)u_1 \in C([0, T']; L^2),$$

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 \,(= -AS_1(t)u_1), \;\; S_1(0)u_1 = u_1 \;\; \text{for} \;\; u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$
$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})} \le C\|u_1\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}},$$

for $u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, $k = 0, 1$

$$\|\partial_t S_1(t)u_1\|_{C([0,T'];L^2)} \le C\|u_1\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}},$$

and

$$\|\partial_t \nabla S_1(t)u_1\|_{C([0,T'];L^2_1)} \le C\|u_1\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}},$$

for $u_1 \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, where C is a positive constant depending on T'.

(ii) *It holds for each $F_1 \in C([0, T]; \mathscr{X}_{(1)}) \times L^2(0, T; \mathscr{Y}_{(1)})$ that*

$$\mathscr{S}_1(\cdot)F_1 \in C^1([0, T]; \mathscr{X}_{(1)}) \times [C([0, T]; \mathscr{Y}_{(1)}) \times H^1(0, T; \mathscr{Y}_{(1)})],$$

and

$$\partial_t \mathscr{S}_1(t)F_1 + A_1 \mathscr{S}_1(t)F_1 = F_1(t), \;\; \mathscr{S}_1(0)F_1 = 0,$$
$$\|\mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})} \le C\|F_1\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})},$$
$$\|\partial_t \mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C\|F_1\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})},$$

where C is a positive constant depending on T. In addition, $\partial_t \mathscr{S}_1(\cdot)F_1 \in C([0,T];L^2)$, $\partial_t \nabla \mathscr{S}_1(\cdot)F_1 \in C([0,T];L^2_1)$ for $F_1 \in C([0,T];L^2_1)$ and we have

$$\|\partial_t \mathscr{S}_1(\cdot)F_1\|_{C([0,T];L^2)} \le C\|F_1\|_{C([0,T];L^2)},$$

and

$$\|\partial_t \nabla \mathscr{S}_1(\cdot)F_1\|_{C([0,T];L^2_1)} \le C\|\nabla F_1\|_{C([0,T];L^2_1)},$$

where C is a positive constant depending on T.

(iii) *There holds the following relation between S_1 and $\mathscr{S}_1$.*

$$S_1(t)\mathscr{S}_1(t')F_1 = \mathscr{S}_1(t')[S_1(t)F_1]$$

for any $t \ge 0$, $t' \in [0,T]$ and $F_1 \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$.

(iv) *$\Gamma_j S_1(t) = S_1(t)\Gamma_j$ and $\Gamma_j \mathscr{S}_1(t) = \mathscr{S}_1(t)\Gamma_j$ for $j = 1,2,3$. Therefore the assertions* (i)–(iii) *above hold with function spaces $\mathscr{X}_{(1)}$ and $\mathscr{Y}_{(1)}$ replaced by $(\mathscr{X}_{(1)})_\diamond$ and $(\mathscr{Y}_{(1)})_\#$, respectively.*

The assertion (i)–(iii) follows by the same way as that in Proposition 2.4.1. The assertion (iv) is verified by the fact $\Gamma A_1 = A_1\Gamma$, which derive that $\Gamma S_1(t) = S_1(t)\Gamma$ and we omit the details.

We next investigate invertibility of $I - S_1(T)$.

Proposition 4.4.2. *There uniquely exists $u \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ that satisfies $(I - S_1(T))u = F_1$ and u satisfies the estimate in each* (i)-(iv) *for F_1 satisfying the conditions given in either* (i)-(iv)*, respectively.*

(i) *$F_1 \in L^2_{(1)} \cap L^\infty_{3,sym} \cap L^1_1$;*

$$\|u\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\{\|F_1\|_{L^\infty_3} + \|F_1\|_{L^1_1}\}. \tag{4.4.2}$$

(ii) *$F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| = 1$ and $F_1^{(1)}$ satisfies the following condition*

$$F_1^{(1)}(-x_1,x_2) = -F_1^{(1)}(x_1,x_2) \text{ or } F_1^{(1)}(x_1,-x_2) = -F_1^{(1)}(x_1,x_2)$$
$$\text{or } F_1^{(1)}(x_2,x_1) = -F_1^{(1)}(x_1,x_2); \tag{4.4.3}$$

$$\|u\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\{\|F_1\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}\}. \tag{4.4.4}$$

(iii) *$F_1 = {}^\top(0, \nabla F_1^{(1)}) \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$;*

$$\|u\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\{\|F_1\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}\}. \tag{4.4.5}$$

(iv) $F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| \geq 2$;

$$\|u\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \leq C\{\|F_1\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}\}. \tag{4.4.6}$$

To prove Proposition 4.4.2, we use the following lemmas. Similarly to Lemmas 2.4.3 and 2.4.4, we have the following lemmas related to the linearized semigroup in two space-dimensional case.

Lemma 4.4.3. ([26]) (i) *The set of all eigenvalues of* $-\hat{A}_\xi$ *consists of* $\lambda_j(\xi)\,(j = 1, \pm)$, *where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2, \\ \lambda_\pm(\xi) = -\frac{1}{2}(\nu + \tilde{\nu})|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu + \tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}. \end{cases}$$

If $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$, *then*

$$\mathrm{Re}\,\lambda_\pm = -\frac{1}{2}(\nu + \tilde{\nu})|\xi|^2, \quad \mathrm{Im}\,\lambda_\pm = \pm\gamma|\xi|\sqrt{1 - \frac{(\nu + \tilde{\nu})^2}{4\gamma^2}|\xi|^2}.$$

(ii) *For* $|\xi| < \frac{2\gamma}{\nu+\tilde{\nu}}$, $e^{-t\hat{A}_\xi}$ *has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1,\pm} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ *are eigenprojections for* $\lambda_j(\xi)\,(j = 1, \pm)$, *and* $\Pi_j(\xi)\,(j = 1, \pm)$ *satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_2 - \frac{\xi^\top \xi}{|\xi|^2} \end{pmatrix},$$

$$\Pi_\pm(\xi) = \pm\frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} -\lambda_\mp & -i\gamma^\top\xi \\ -i\gamma\xi & \lambda_\pm\frac{\xi^\top\xi}{|\xi|^2} \end{pmatrix}.$$

Furthermore, if $0 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$, *then there exists a constant* $C > 0$ *such that the estimates*

$$\|\Pi_j(\xi)\| \leq C\,(j = 1, \pm), \tag{4.4.7}$$

hold for $|\xi| \leq r_\infty$.

Hereafter we fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ so that (4.4.7) in Lemma 4.4.3 holds for $|\xi| \leq r_\infty$.

Lemma 4.4.4. *Let* α *be a multi-index. Then the following estimates hold true uniformly for* ξ *with* $|\xi| \leq r_\infty$ *and* $t \in [0, T]$.

(i) $|\partial_\xi^\alpha \lambda_1| \leq C|\xi|^{2-|\alpha|}$, $|\partial_\xi^\alpha \lambda_\pm| \leq C|\xi|^{1-|\alpha|}$ $(|\alpha| \geq 0)$.

(ii) $|(\partial_\xi^\alpha \Pi_1)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|$, $|(\partial_\xi^\alpha \Pi_\pm)\hat{F}_1| \le C|\xi|^{-|\alpha|}|\hat{F}_1|$ $(|\alpha| \ge 0)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(iii) $|\partial_\xi^\alpha(e^{\lambda_1 t})| \le C|\xi|^{2-|\alpha|}$ $(|\alpha| \ge 1)$.

(iv) $|\partial_\xi^\alpha(e^{\lambda_\pm t})| \le C|\xi|^{1-|\alpha|}$ $(|\alpha| \ge 1)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}_1| \le C(|\xi|^{1-|\alpha|}|\hat{F}_1^0| + |\xi|^{-|\alpha|}|\hat{\tilde{F}}_1|)$ $(|\alpha| \ge 1)$, *where* $F_1 = {}^\top(F_1^0, \tilde{F}_1)$.

(vi) $|\partial_\xi^\alpha(I - e^{\lambda_1 t})^{-1}| \le C|\xi|^{-2-|\alpha|}$ $(|\alpha| \ge 0)$.

(vii) $|\partial_\xi^\alpha(I - e^{\lambda_\pm t})^{-1}| \le C|\xi|^{-1-|\alpha|}$ $(|\alpha| \ge 0)$.

We define

$$E_{1,j}(x) := \mathcal{F}^{-1}(\hat{\chi}_0(I - e^{\lambda_j T})^{-1}\Pi_j) \quad (j = 1, \pm) \quad (x \in \mathbb{R}^2), \tag{4.4.8}$$

where χ_0 is the cut-off function defined by (2.3.3). We have the following estimates for $E_{1,j}$.

Lemma 4.4.5. *There hold*

$$|\partial_x^\alpha E_{1,1}(x)| \le C(1 + |x|)^{-(1+|\alpha|)}$$

for $|\alpha| \ge 1, x \in \mathbb{R}^2$ *and*

$$|\partial_x^\alpha E_{1,\pm}(x)| \le C(1 + |x|)^{-(1+|\alpha|)}$$

for $|\alpha| \ge 0, x \in \mathbb{R}^2$.

By using Lemma 2.3.7 and Lemma 4.4.4, Lemma 4.4.5 can be proved in a similar manner to the proof of Lemma 2.4.5 and we omit the details.

We derive the following property for Π_1 from direct computations.

Lemma 4.4.6. *It holds that*

$$\Pi_1(\xi)\widehat{\nabla F}(\xi) = 0 \quad (\xi \ne 0, \, |\xi| \le r_\infty),$$

where F *is a scalar function in* H^1.

We are now in a position to prove Proposition 2.4.2.

Proof of Proposition 4.4.2. (i) We define a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}^1, \tilde{w}^2)$ by

$$\tilde{u} = \mathcal{F}^{-1}((i\xi_2)(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1).$$

$\tilde{u}$ can be rewrite as

$$\tilde{u} = \mathcal{F}^{-1}((i\xi_2)(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1) = \mathcal{E} * F_1,$$

where

$$\mathcal{E} = \mathcal{F}^{-1}\{(i\xi_2 \sum_j \hat{E}_{1,j})\}, \tag{4.4.9}$$

$E_{1,j}$ are the ones defined in (4.4.8). We obtain from Lemma 4.4.5 that

$$|\partial_x^\alpha \mathcal{E}(x)| \leq C(1+|x|)^{-(1+|\alpha|)} \tag{4.4.10}$$

for $|\alpha| \geq 0$, $x \in \mathbb{R}^2$. Hence, It follows from Lemma 2.3.2, Lemma 4.3.10 (i) and Lemma 4.4.4 that there uniquely exists $\tilde{u} \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ that satisfies $(I - S_1(T))\tilde{u} = \partial_{x_2} F_1$ and $\tilde{u}$ satisfies the estimates

$$\|\tilde{u}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \leq C\{\|F_1\|_{L_3^\infty} + \|F_1\|_{L_1^1}\} \tag{4.4.11}$$

and

$$\|\tilde{u}\|_{L_2^\infty} \leq C\{\|F_1\|_{L_3^\infty} + \|F_1\|_{L_1^1}\}. \tag{4.4.12}$$

Note that $\partial_{x_2} F_1$ satisfies $\Gamma_1(\partial_{x_2} F_1) = \partial_{x_2} F_1$. Therefore, by Proposition **??** (i) and (iii) we obtain that $\Gamma_1 \tilde{u} = \tilde{u}$, especially,

$$\tilde{w}^1(-x_1, x_2) = -\tilde{w}^1(x_1, x_2) \quad \text{for} \quad x \in \mathbb{R}^2. \tag{4.4.13}$$

We set $u = {}^\top(\phi, w^1, w^2)$ by

$$u = \mathcal{F}^{-1}((I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1).$$

Since $\tilde{u} = \partial_{x_2} u$, we see from Lemma 4.3.9, (4.4.12) and (4.4.13) that $w^1 \in L_1^\infty$, $\partial_{x_2} w^1 \in L_2^\infty$, $\tilde{w}^1 = \partial_{x_2} w^1$ and w^1 satisfies the estimate

$$\|w^1\|_{L_1^\infty} \leq C\|\partial_{x_2} w^1\|_{L_2^\infty} \leq C\{\|F_1\|_{L_3^\infty} + \|F_1\|_{L_1^1}\}. \tag{4.4.14}$$

Replacing $\tilde{u}$ to

$$\tilde{u} = \mathcal{F}^{-1}((i\xi_1)(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1),$$

in a similar manner to the estimate for w^1, we derive that $w^2 \in L_1^\infty$, $\partial_{x_1} w^2 \in L_2^\infty$,

$$\|w^2\|_{L_1^\infty} \leq C\|\partial_{x_1} w^2\|_{L_2^\infty} \leq C\{\|F_1\|_{L_3^\infty} + \|F_1\|_{L_1^1}\} \tag{4.4.15}$$

and

$$\|\partial_{x_1} u\|_{L_2^\infty} \leq C\{\|F_1\|_{L_3^\infty} + \|F_1\|_{L_1^1}\}. \tag{4.4.16}$$

Concerning the estimate for ϕ, We also obtain from Lemma 2.3.2, Lemma 4.3.10 (i) and Lemma 4.4.5 that

$$\|\phi\|_{\mathscr{X}_{(1),L^\infty}} \leq C\{\|F_1\|_{L_3^\infty} + \|F_1\|_{L_1^1}\}.$$

This together with Lemma 4.3.11, (4.4.12), (4.4.14), (4.4.15) and (4.4.16), we get that $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, $(I - S_1(T))u = F_1$ and u satisfies the estimate (4.4.2). By the assumption of F_1 and Proposition 4.4.1 (i) and (iii) we see that $\Gamma_j u = u$ $(j = 1, 2, 3)$, i.e., $u \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$.

(ii) We suppose that $F_1 = \partial_{x_2} F_1^{(1)}$ without loss of generality. We define $u = {}^\top(\phi, w^1, w^2)$ by

$$\begin{aligned} u &= \mathcal{F}^{-1}((I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1) \\ &= \mathcal{F}^{-1}((i\xi_2)(I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1^{(1)}) = \mathcal{E} * F_1^{(1)}, \end{aligned}$$

where $\mathcal{E}(x)$ is the same one used in (4.4.9). Therefore, by Lemma 4.3.8, Lemma 4.3.10 (ii), Lemma 4.4.4 and (4.4.10), we find the assertion (ii).

(iii) By Lemma 4.4.6, we derive that

$$u = \mathcal{F}^{-1}((I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_1) = \mathcal{F}^{-1}\{\sum_{j\in\{\pm\}} \hat{E}_{1,j}\hat{F}_1\}$$

for $F_1 = {}^\top(0, \nabla F_1^{(1)}) \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$. It then follows from Lemma 4.3.10 (iii), Proposition 4.4.1, Lemma 4.4.4 and Lemma 4.4.5 that $u \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$, $(I - S_1(T))u = F_1$ and u satisfies the estimate

$$\|u\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\{\|F_1\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}\}.$$

We arrive at the assertion (iv) from Lemma 4.3.10 (iii), Lemma 4.4.4 and Lemma 4.4.5 similarly to the assertion (iii). This completes the proof. □.

In view of Proposition 4.4.2, if F_1 satisfies the each condition (i)-(iv) bellow, the $I - S_1(T)$ has bounded inverse $(I - S_1(T))^{-1}$ in $(\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ satisfying the estimate in (i)-(iv) respectively;

(i) $F_1 \in L^2_{(1)} \cap L^\infty_{3,sym} \cap L^1_1$;

$$\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\{\|F_1\|_{L^\infty_3} + \|F_1\|_{L^1_1}\}.$$

(ii) $F_1 = \partial_x^\alpha F_1^{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| = 1$ and $F_1^{(1)}$ satisfies (4.4.3);

$$\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\{\|F_1\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}\}.$$

(iii) $F_1 = {}^\top(0, \nabla F_1^{(1)}) \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $F_1^{(1)} \in L^2_{(1)} \cap L^\infty_2$;

$$\|(I - S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \le C\{\|F_1\|_{L^\infty_3} + \|F_1^{(1)}\|_{L^\infty_2} + \|F_1^{(1)}\|_{L^2} + \|F_1\|_{L^2_1}\}.$$

(iv) $F_1=\partial_x^\alpha F_1^{(1)}\in L^\infty_{3,sym}\cap L^2_{(1),1}$ with $F_1^{(1)}\in L^2_{(1)}\cap L^\infty_2$ for some α satisfying $|\alpha|\geq 2$;

$$\|(I-S_1(T))^{-1}F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}\leq C\{\|F_1\|_{L^\infty_3}+\|F_1^{(1)}\|_{L^\infty_2}+\|F_1^{(1)}\|_{L^2}+\|F_1\|_{L^2_1}\}.$$

We can write $\mathscr{S}_1(t)F_1$ and $S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1$ as

$$S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}F_1=\int_0^T E_1(t,\sigma)*F_1(\sigma)d\sigma, \tag{4.4.17}$$

$$\mathscr{S}_1(t)F_1=\int_0^t S_1(t-\tau)F_1(\tau)d\tau=\int_0^t E_2(t,\tau)*F_1(\tau)d\tau, \tag{4.4.18}$$

where $E_1(t,\sigma)$ and $E_2(t,\tau)$ are defined by

$$E_1(t,\sigma)=\mathcal{F}^{-1}\{\hat{\chi}_0e^{-t\hat{A}_\xi}(I-e^{-T\hat{A}_\xi})^{-1}e^{-(T-\sigma)\hat{A}_\xi}\},$$
$$E_2(t,\tau)=\mathcal{F}^{-1}\{\hat{\chi}_0e^{-(t-\tau)\hat{A}_\xi}\}$$

for $\sigma\in[0,T]$, $0\leq\tau\leq t\leq T$, $\hat{\chi}_0$ is the cut-off function defined by (2.3.3). Then $E_1(t,\sigma)*F_1$ and $E_2(t,\tau)*F_1$ are estimated as follows.

Lemma 4.4.7. *$E_j(t,\sigma)*F_1\in(\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})_{sym}$ $(t,\sigma,\tau\in[0,T], j=1,2)$ if F_1 satisfies the conditions given in either* (i)-(iv) *and $E_1(t,\sigma)*F_1, E_2(t,\tau)*F_1$ satisfy the following estimate in each* (i)-(iv).

(i) $F_1\in L^2_{(1)}\cap L^\infty_{3,sym}\cap L^1_1$;

$$\sum_j\|E_j(t,\sigma)*F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}\leq C\{\|F_1\|_{L^\infty_3}+\|F_1\|_{L^1_1}\}$$

uniformly for $\sigma\in[0,T]$ and $0\leq\tau\leq t\leq T$.

(ii) $F_1=\partial_x^\alpha F_1^{(1)}\in L^\infty_{3,sym}\cap L^2_{(1),1}$ *with* $F_1^{(1)}\in L^2_{(1)}\cap L^\infty_2$ *for some α satisfying $|\alpha|=1$ and $F_1^{(1)}$ satisfies* (4.4.3);

$$\sum_j\|E_j(t,\sigma)*F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}\leq C\{\|F_1\|_{L^\infty_3}+\|F_1^{(1)}\|_{L^\infty_2}+\|F_1^{(1)}\|_{L^2}+\|F_1\|_{L^2_1}\}$$

uniformly for $\sigma\in[0,T]$ and $0\leq\tau\leq t\leq T$.

(iii) $F_1={}^\top(0,\nabla F_1^{(1)})\in L^\infty_{3,sym}\cap L^2_{(1),1}$ *with* $F_1^{(1)}\in L^2_{(1)}\cap L^\infty_2$;

$$\sum_j\|E_j(t,\sigma)*F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}\leq C\{\|F_1\|_{L^\infty_3}+\|F_1^{(1)}\|_{L^\infty_2}+\|F_1^{(1)}\|_{L^2}+\|F_1\|_{L^2_1}\}$$

uniformly for $\sigma\in[0,T]$ and $0\leq\tau\leq t\leq T$.

(iv) $F_1=\partial_x^\alpha F_1^{(1)}\in L^\infty_{3,sym}\cap L^2_{(1),1}$ *with* $F_1^{(1)}\in L^2_{(1)}\cap L^\infty_2$ *for some α satisfying $|\alpha|\geq 2$;*

$$\sum_j\|E_j(t,\sigma)*F_1\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}\leq C\{\|F_1\|_{L^\infty_3}+\|F_1^{(1)}\|_{L^\infty_2}+\|F_1^{(1)}\|_{L^2}+\|F_1\|_{L^2_1}\}$$

uniformly for $\sigma\in[0,T]$ and $0\leq\tau\leq t\leq T$.

Proof of Lemma 4.4.7. It follows from Lemmas 4.4.3 and 4.4.4 that

$$|\partial_\xi^\beta(\hat{\chi}_0(i\xi)^\alpha e^{-t\hat{A}_\xi}(I-e^{-T\hat{A}_\xi})^{-1}e^{-(T-\sigma)\hat{A}_\xi})| \le C|\xi|^{-2+|\alpha|-|\beta|},$$
$$|\partial_\xi^\beta(\hat{\chi}_0(i\xi)^\alpha e^{-(t-\tau)\hat{A}_\xi})| \le C|\xi|^{|\alpha|-|\beta|},$$

for $\sigma \in [0,T]$, $0 \le \tau \le t \le T$ and $|\alpha|$, $|\beta| \ge 0$. Hence by Lemma 2.3.7 we see that

$$|\partial_x^\alpha E_1(x)| \le C(1+|x|)^{-|\alpha|} \quad (|\alpha| \ge 1), \tag{4.4.19}$$
$$|\partial_x^\alpha E_2(x)| \le C(1+|x|)^{-(2+|\alpha|)} \quad (|\alpha| \ge 0). \tag{4.4.20}$$

This together with Lemma 4.3.8, Lemma 4.3.9 and Lemma 4.3.10 we obtain the desired estimate in a similar manner to the proof of Proposition 4.4.2. This completes the proof. □

The symbol Ψ_1 and Ψ_2 stand for

$$\Psi_1[\tilde{F}_1](t) = S_1(t)\mathscr{S}_1(T)(I-S_1(T))^{-1}\begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}, \quad \Psi_2[\tilde{F}_1](t) = \mathscr{S}_1(t)\begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}. \tag{4.4.21}$$

For Ψ_1 and Ψ_2 we derive the following estimates.

Proposition 4.4.8. (i) *For $\tilde{F}_1 \in L^2(0,T;L^2_{(1)} \cap L^\infty_{3,\#} \cap L^1_1)$ it holds that*

$$\Psi_j[\tilde{F}_1] \in C^1([0,T];\mathscr{X}_{(1),\diamond}) \times [C([0,T];\mathscr{Y}_{(1),\#}) \cap H^1(0,T;\mathscr{Y}_{(1),\#})]$$

for $j=1,2$ and $\Psi_j[\tilde{F}_1]$ satisfy the following.

$$\|\partial_t^k \Psi_j[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^1_1)}$$

for $k=0,1$ and $j=1,2$.

(ii) *If $\tilde{F}_1$ satisfies $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T;L^\infty_{3,\#} \cap L^2_{(1),1})$ with $F_1^{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_2)$ for some α satisfying $|\alpha|=1$ and $F_1^{(1)}$ satisfies* (4.4.3)*, then $\Psi_j[\tilde{F}_1] \in C^1([0,T];\mathscr{X}_{(1),\diamond}) \times [C([0,T];\mathscr{Y}_{(1),\#}) \cap H^1(0,T;\mathscr{Y}_{(1),\#})]$ $(j=1,2)$ and $\Psi_j[\tilde{F}_1]$ satisfy the following estimates.*

$$\|\partial_t^k \Psi_j[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)})$$

for $k=0,1$ and $j=1,2$.

(iii) *We have that $\Psi_j[\tilde{F}_1] \in C^1([0,T];\mathscr{X}_{(1),\diamond}) \times [C([0,T];\mathscr{Y}_{(1),\#}) \cap H^1(0,T;\mathscr{Y}_{(1),\#})]$ $(j=1,2)$ for $\tilde{F}_1 = \nabla F_1^{(1)} \in L^2(0,T;L^\infty_{3,\#} \cap L^2_{(1),1})$ with $F_1^{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_2)$ and $\Psi_j[\tilde{F}_1]$ satisfy the estimates*

$$\|\partial_t^k \Psi_j[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)})$$

for $k=0,1$ and $j=1,2$.

(iv) *Let* $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^\infty_{3,\#} \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1)} \cap L^\infty_2)$ *for some* α *satisfying* $|\alpha| \geq 2$*. Then* $\Psi_j[\tilde{F}_1] \in C^1([0,T]; \mathscr{X}_{(1),\diamond}) \times [C([0,T]; \mathscr{Y}_{(1),\#}) \cap H^1(0,T; \mathscr{Y}_{(1),\#})]$ $(j = 1,2)$ *and* $\Psi_j[\tilde{F}_1]$ *satisfy the estimates*

$$\|\partial_t^k \Psi_j[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \leq C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)})$$

for $k = 0,1$ *and* $j = 1,2$.

Proof. As for the assertion (i), it follows from Proposition 4.4.1 (i), (ii) and Lemma 4.4.7 that

$$\|\Psi_j[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^1_1)}$$

for $j = 1,2$,

$$\|\partial_t \Psi_1[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^1_1)},$$

and

$$\|\partial_t \Psi_2[\tilde{F}_1]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^1_1)} + \|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}).$$

Note that $\tilde{F}_1 = \chi_0 * \tilde{F}_1$, where $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$, $\hat{\chi}_0$ is the cut-off function defined by (2.3.3). Since $\hat{\chi}_0$ belongs to the Schwartz space on $\mathbb{R}^2$, we get that

$$|\partial_x^\alpha \chi_0(x)| \leq C(1+|x|)^{-(2+|\alpha|)} \quad \text{for} \quad |\alpha| \geq 0.$$

Therefore, we derive the following estimate for $\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1)})}$ in a similar manner to the proof of Proposition 4.4.2.

$$\|\tilde{F}_1\|_{L^2(0,T;\mathscr{Y}_{(1)})} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^1_1)}.$$

Consequently, we obtain the desired estimate in (i). Similarly, we can verify the assertion (ii)-(iv). This completes the proof. $\square$

By using Proposition 4.4.8, we give estimates for a solution of (4.4.1) satisfying $u_1(0) = u_1(T)$.

Proposition 4.4.9. *Set*

$$\Psi[\tilde{F}_1](t) = \Psi_1[\tilde{F}_1] + \Psi_2[\tilde{F}_1], \tag{4.4.22}$$

for $F_1 = {}^\top(0, \tilde{F}_1)$*, where* Ψ_1 *and* Ψ_2 *were defined by* (4.4.21)*. If* $\tilde{F}_1$ *satisfies the conditions given in either* (i)-(iv)*, then* $\Psi[\tilde{F}_1]$ *is a solution of* (4.4.1) *with* $F_1 = {}^\top(0, \tilde{F}_1)$ *in* $\mathscr{Z}_{(1),sym}(0,T)$ *satisfying* $\Psi[\tilde{F}_1](0) = \Psi[\tilde{F}_1](T)$ *and* $\Psi[\tilde{F}_1]$ *satisfies the estimate in each* (i)-(iv)*, respectively.*

(i) $\tilde{F}_1 \in L^2(0,T; L^2_{(1)} \cap L^\infty_{3,\#} \cap L^1_1)$;

$$\|\Psi[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^1_1)}. \tag{4.4.23}$$

(ii) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^\infty_{3,\#} \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1)} \cap L^\infty_2)$ *for some* α *satisfying* $|\alpha| = 1$ *and* $F_1^{(1)}$ *satisfies* (4.4.3)*;*

$$\|\Psi[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}). \tag{4.4.24}$$

(iii) $\tilde{F}_1 = \nabla F_1^{(1)} \in L^2(0,T; L^\infty_{3,\#} \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1)} \cap L^\infty_2)$*;*

$$\|\Psi[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}). \tag{4.4.25}$$

(iv) $\tilde{F}_1 = \partial_x^\alpha F_1^{(1)} \in L^2(0,T; L^\infty_{3,\#} \cap L^2_{(1),1})$ *with* $F_1^{(1)} \in L^2(0,T; L^2_{(1)} \cap L^\infty_2)$ *for some* α *satisfying* $|\alpha| \ge 2$*;*

$$\|\Psi[\tilde{F}_1]\|_{\mathscr{Z}_{(1)}(0,T)} \le C(\|\tilde{F}_1\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|F_1^{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}). \tag{4.4.26}$$

Proof. By Proposition 4.4.1 (iii) and Proposition 4.4.2 we see that $\Psi[\tilde{F}_1]$ is a solution of (4.4.1) with $F_1 = {}^\top(0, \tilde{F}_1)$ and satisfies $\Psi[\tilde{F}_1](0) = \Psi[\tilde{F}_1](T)$. The estimates and antisymmetry of $\Psi[\tilde{F}_1]$ in (i)-(iv) are verified by Proposition 4.4.8. This completes the proof. □

4.5 Estimates for solution on the high frequency part

In this section we estimate a solution for the high frequency part. We begin with some properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$.

As for the solvability of (4.3.10), we state the following proposition.

Proposition 4.5.1. *Let* s *be an integer satisfying* $s \ge 3$*. Set* $k = s - 1$ *or* s*. Assume that*

$$\begin{aligned}
&\nabla\tilde{w} \in C([0,T']; H^{s-1}) \cap L^2(0,T'; H^s),\\
&u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},\\
&F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T'; H^k_{(\infty)} \times H^{k-1}_{(\infty)}).
\end{aligned}$$

Here T' *is a given positive number. Then there exists a unique solution* $u_\infty = {}^\top(\phi_\infty, w_\infty)$ *of* (1.3.2) *satisfying*

$$\begin{aligned}
&\phi_\infty \in C([0,T']; H^k_{(\infty)}),\\
&w_\infty \in C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)}) \cap H^1(0,T'; H^{k-1}_{(\infty)}).
\end{aligned}$$

One can verify Proposition 4.5.1 in a similar manner to the proof of Proposition 1.5.4 and we omit the details.

Remark 4.5.2. Concerning the space dimension n, in Proposition 1.5.4 we assume that $n \geq 3$. But we can replace the space dimension to $n = 2$ by taking a look at the fact that [16, Theorem 4.1] holds for the space dimension $n = 2$ and the proof of Proposition 1.5.4. See also Remark 2.5.2 for the condition of $\tilde{w}$.

Therefore, it follows from Proposition 4.5.1 that we can define $S_{\infty,\tilde{u}}(t)$ $(t \geq 0)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ $(t \in [0,T])$ as follows.

Let an integer s satisfy $s \geq 3$ and a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfy

$$\tilde{\phi} \in C_{per}(\mathbb{R}; H^s), \ \nabla \tilde{w} \in C_{per}(\mathbb{R}; H^{s-1}) \cap L^2_{per}(\mathbb{R}; H^s). \tag{4.5.1}$$

Let $k = s-1$ or s. We define and operator $S_{\infty,\tilde{u}}(t) : H^k_{(\infty)} \longrightarrow H^k_{(\infty)}$ $(t \geq 0)$ by

$$u_\infty(t) = S_{\infty,\tilde{u}}(t) u_{0\infty} \ \text{ for } \ u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},$$

where $u_\infty(t)$ is the solution of (1.3.2) with $F_\infty = 0$. Moreover, we define an operator $\mathscr{S}_{\infty,\tilde{u}}(t) : L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}) \longrightarrow H^k_{(\infty)}$ $(t \in [0,T])$ by

$$u_\infty(t) = \mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty] \ \text{ for } \ F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}),$$

where $u_\infty(t)$ is the solution of (1.3.2) with $u_{0\infty} = 0$.

We have the following properties for $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ in the weighted L^2 Sobolev spaces.

Proposition 4.5.3. *Let s be a nonnegative integer satisfying $s \geq 3$ and let $k = s-1$ or s. We suppose that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (4.5.1). Then there exists a constant $\delta > 0$ such that if $\|\nabla \tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta$, then the following assertions hold true.*

(i) *For $u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),2}$, there holds $S_{\infty,\tilde{u}}(\cdot) u_{0\infty} \in C([0,\infty); H^k_{(\infty),2})$ and there exist constants $a > 0$ and $C > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the following estimate for all $t \geq 0$ and $u_{0\infty} \in H^k_{(\infty),2}$.*

$$\|S_{\infty,\tilde{u}}(t) u_{0\infty}\|_{H^k_{(\infty),2}} \leq C e^{-at} \|u_{0\infty}\|_{H^k_{(\infty),2}}.$$

(ii) *For $F_\infty = {}^\top(F^0_\infty, \tilde{F}_\infty) \in L^2(0,T; H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})$, there holds $\mathscr{S}_{\infty,\tilde{u}}(\cdot) F_\infty \in C([0,T]; H^k_{(\infty),2})$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ satisfies the following estimate for $t \in [0,T]$ and $F_\infty \in L^2(0,T; H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})$ with a positive constant C depending on T.*

$$\|\mathscr{S}_{\infty,\tilde{u}}(t)[F_\infty]\|_{H^k_{(\infty),2}} \leq C \left\{ \int_0^t e^{-a(t-\tau)} \|F_\infty\|^2_{H^k_{(\infty),2} \times H^{k-1}_{(\infty),2}} d\tau \right\}^{\frac{1}{2}}.$$

(iii) *We define* $r_{H^k_{(\infty),2}}(S_{\infty,\tilde u}(T))$ *by the spectral radius of* $S_{\infty,\tilde u}(T)$ *on* $H^k_{(\infty),2}$. *Then it holds that* $r_{H^k_{(\infty),2}}(S_{\infty,\tilde u}(T)) < 1$.

(iv) $I - S_{\infty,\tilde u}(T)$ *has a bounded inverse* $(I - S_{\infty,\tilde u}(T))^{-1}$ *on* $H^k_{(\infty),2}$ *satisfying*

$$\|(I - S_{\infty,\tilde u}(T))^{-1}u\|_{H^k_{(\infty),2}} \le C\|u\|_{H^k_{(\infty),2}} \quad \textit{for} \quad u \in H^k_{(\infty),2}.$$

(v) *Suppose that* $\Gamma_j\tilde u = \tilde u$ *for* $j = 1, 2, 3$. *Then it holds that* $\Gamma_j S_{\infty,\tilde u}(t) = S_{\infty,\tilde u}(t)\Gamma_j$ *and* $\Gamma_j \mathscr{S}_{\infty,\tilde u}(t) = \mathscr{S}_{\infty,\tilde u}(t)\Gamma_j$. *Accordingly, the assertions* (i)–(iv) *hold true in function spaces* $H^k_{\infty,2}$ *and* $H^k_{\infty,2} \times H^{k-1}_{\infty,2}$ *replaced by* $(H^k_{\infty,2})_{sym}$ *and* $(H^k_{\infty,2} \times H^{k-1}_{\infty,2})_{sym}$, *respectively if* $\Gamma_j\tilde u = \tilde u$ $(j = 1, 2, 3)$.

We can verify Proposition 4.5.3 in a similar manner to the proof of Proposition 1.5.6 and we omit the proof.

Remark 4.5.4. As for the space dimension n, in Proposition 1.5.6 it is assumed that $n \ge 3$. But it is replaced by $n = 2$ due to taking a look at the proof of Proposition 1.5.6. See also Remark 2.5.4 for the condition of $\tilde w$.

We are now in a position to give the following estimate for a solution u_∞ of (4.3.10) satisfying $u_\infty(0) = u_\infty(T)$.

Proposition 4.5.5. *Let* s *be a nonnegative integer satisfying* $s \ge 3$. *We suppose that*

$$F_\infty = {}^\top(F^0_\infty, \tilde F_\infty) \in L^2(0, T; (H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})_{sym}),$$

with $k = s - 1$ *or* s. *We also assume that* $\tilde u = {}^\top(\tilde\phi, \tilde w)$ *satisfies* (4.5.1). *Then there exists a positive constant* δ *such that if*

$$\|\nabla \tilde w\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \le \delta,$$

then the following assertion holds true.

The function

$$u_\infty(t) := S_{\infty,\tilde u}(t)(I - S_{\infty,\tilde u}(T))^{-1}\mathscr{S}_{\infty,\tilde u}(T)[F_\infty] + \mathscr{S}_{\infty,\tilde u}(t)[F_\infty] \qquad (4.5.2)$$

is a solution of (1.3.2) *in* $\mathscr{Z}^k_{(\infty),2,sym}(0, T)$ *satisfying* $u_\infty(0) = u_\infty(T)$ *and the estimate*

$$\|u_\infty\|_{\mathscr{Z}^k_{(\infty),2}(0,T)} \le C\|F_\infty\|_{L^2(0,T;H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})}.$$

Proposition 4.5.5 is directly derived by Proposition 4.5.3.

4.6 Proof of Theorem 4.2.1

In this section we prove Theorem 4.2.1.

The estimates for the nonlinear and inhomogeneous terms are established here. We set $F_{1,m}(u,g)$ and $F_\infty(u,g)$ by

$$F_{1,m}(u,g) = \begin{pmatrix} 0 \\ \tilde{F}_{1,m}(u,g) \end{pmatrix},$$

$$F_\infty(u,g) = P_\infty \begin{pmatrix} -\gamma w\cdot\nabla\phi_1 + F^0(u) \\ \tilde{F}(u,g) \end{pmatrix} =: \begin{pmatrix} F^0_\infty(u) \\ \tilde{F}_\infty(u,g) \end{pmatrix},$$

where $u = {}^\top(\phi, w)$ is given by $u_{1,m} = {}^\top(\phi_1, m_1)$ and $u_\infty = {}^\top(\phi_\infty, w_\infty)$ through the relation

$$\phi = \phi_1 + \phi_\infty, \quad w = w_1 + w_\infty, \quad w_1 = m_1 - P_1(\phi w),$$

$\tilde{F}_{1,m}(u,g)$, $F^0(u)$ and $\tilde{F}(u,g)$ were given in (4.3.5), (0.0.19) and (0.0.20), respectively,

As for the estimate $F_{1,m}(u,g)$, we use the notation Ψ introduced in section 5, i.e.,

$$\Psi[\tilde{F}_1](t) := S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}\begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix} + \mathscr{S}_1(t)\begin{pmatrix} 0 \\ \tilde{F}_1 \end{pmatrix}.$$

We have the following estimate for $\Psi[\tilde{F}_{1,m}(u,g)]$ in $\mathscr{Z}_{(1),sym}(0,T)$.

Proposition 4.6.1. *Let $u_{1,m} = {}^\top(\phi_1, m_1) \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ and $u_\infty = {}^\top(\phi_\infty, w_\infty) \in H^s_{2,sym}$ satisfying*

$$\sup_{0\le t\le T}\|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T}\|u_\infty(t)\|_{H^s_2}$$
$$+ \sup_{0\le t\le T}\|\phi(t)\|_{L^\infty} + \sup_{0\le t\le T}\|\nabla\phi(t)\|_{L^2_1} \le \min\{\delta_0, \frac{1}{2}\},$$

where δ_0 is the one in Lemma 4.3.7 (i) and $\phi = \phi_1 + \phi_\infty$. Then we obtain the following estimate

$$\|\Psi[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{1,m}, u_\infty\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{1,m}, u_\infty\}\|_{X^s(0,T)}\Big)[g]_s,$$

uniformly for $u_{1,m}$ and u_∞.

Proof. Let $u^{(j)} = {}^\top(\phi^{(j)}, w^{(j)})$ $(j = 1, \infty)$, $w^{(j)} = {}^\top(w^{(j),1}, w^{(j),2})$ and we define

$$G_1(u^{(1)}, u^{(2)}) = -P_1\Big\{\gamma\partial_{x_2}\begin{pmatrix} w^{(1),1}w^{(2),2} \\ w^{(1),2}w^{(2),2} - w^{(1),1}w^{(2),1} \end{pmatrix} + \gamma\partial_{x_1}\begin{pmatrix} 0 \\ w^{(1),1}w^{(2),2} \end{pmatrix}\Big\},$$

$$G_2(u^{(1)}, u^{(2)}) = -P_1(\gamma\nabla(w^{(1),1}w^{(2),1}))$$

$$
\begin{aligned}
G_3(u^{(1)}, u^{(2)}) &= -P_1(\mu\Delta(\phi^{(1)}w^{(2)}) + \tilde{\mu}\nabla\mathrm{div}\,(\phi^{(1)}w^{(2)})), \\
G_4(\phi, u^{(1)}, u^{(2)}) &= -P_1\left(\frac{\rho_*}{\gamma}\nabla(p^{(1)}(\phi)\phi^{(1)}\phi^{(2)})\right), \\
G_5(\phi, u^{(1)}, u^{(2)}) &= -P_1(\gamma\mathrm{div}\,(\phi w^{(1)} \otimes w^{(2)})), \\
H_k(u^{(1)}, u^{(2)}) &= G_k(u^{(1)}, u^{(2)}) + G_k(u^{(2)}, u^{(1)}), \quad (k = 1, 2, 3), \\
H_k(\phi, u^{(1)}, u^{(2)}) &= G_k(\phi, u^{(1)}, u^{(2)}) + G_k(\phi, u^{(2)}, u^{(1)}), \quad (k = 4, 5).
\end{aligned}
$$

and we then write $\Psi[\tilde{F}_{1,m}(u,g)]$ as

$$
\begin{aligned}
\Psi[\tilde{F}_{1,m}(u,g)] &= \sum_{k=1}^{3}(\Psi[G_k(u_1,u_1)] + \Psi[H_k(u_1,u_\infty)] + \Psi[G_k(u_\infty,u_\infty)]) \\
&\quad + \sum_{k=4}^{5}\Psi[G_k(\phi,u_1,u_1)] + \Psi[H_k(\phi,u_1,u_\infty)] + \Psi[G_k(\phi,u_\infty,u_\infty)] \\
&\quad + \Psi\left[\frac{1}{\gamma}(1+\phi_1)g\right] + \Psi\left[\frac{1}{\gamma}\phi_\infty g\right].
\end{aligned}
$$

Using Lemma 4.3.12 and (4.4.24) we have the following estimate for $\Psi[G_1(u_1,u_1)]$.

$$
\|\Psi[G_1(u_1,u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
$$

Concerning the estimates $\Psi[G_2(u_1,u_1)]$ and $\Psi[G_4(\phi,u_1,u_1)]$, applying Lemma 4.3.12 and (4.4.25) with $F_1^{(1)} = (w_1^1)^2$ and $F_1^{(1)} = p^{(1)}(\phi)\phi_1^2$ we obtain the estimates

$$
\begin{aligned}
&\|\Psi[G_2(u_1,u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}, \\
&\|\Psi[G_4(\phi,u_1,u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
\end{aligned}
$$

By using Lemma 4.3.12 and (4.4.26) we arrive at the following estimate for $\Psi[G_3(u_1,u_1)]$.

$$
\|\Psi[G_3(u_1,u_1)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
$$

It follows from Lemma 2.3.2, Lemma 4.3.12 and (4.4.24) that we get

$$
\begin{aligned}
&\|\Psi[G_1(u_1,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}, \\
&\|\Psi[G_1(u_\infty,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
\end{aligned}
$$

Similarly, by Lemma 2.3.2, Lemma 4.3.12 and (4.4.25) we obtain for $k = 2, 3$ that

$$
\begin{aligned}
&\|\Psi[G_k(u_1,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} + \|\Psi[G_4(\phi,u_1,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \\
&\quad + \|\Psi[G_k(u_\infty,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)}\|\Psi[G_4(\phi,u_\infty,u_\infty)]\|_{\mathscr{Z}_{(1)}(0,T)} \\
&\quad \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.
\end{aligned}
$$

$G_5(\phi,u,u)$ is estimated by same way as that in the estimate for $\Psi[G_1(u_1,u_1)]$ and we see that

$$\|\Psi[G_5(\phi,u,u)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)}.$$

As for the estimates for $\Psi[(1+\phi_1)g]$ and $\Psi[\phi_\infty g]$, it holds from (4.4.23) that

$$\|\Psi[(1+\phi_1)g]\|_{\mathscr{Z}_{(1)}(0,T)} + \|\Psi[\phi_\infty g]\|_{\mathscr{Z}_{(1)}(0,T)} \le C(1+\|\{u_1,u_\infty\}\|_{X^s(0,T)})[g]_s.$$

Therefore, we find that

$$\|\Psi[\tilde{F}_{1,m}(u,g)]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_1,u_\infty\}\|^2_{X^s(0,T)} + C\Big(1+\|\{u_1,u_\infty\}\|_{X^s(0,T)}\Big)[g]_s.$$

Consequently, we obtain the desired estimate by applying Lemma 4.3.7 (i). This completes the proof. $\square$

We state the estimates for the nonlinear and inhomogeneous terms of the high frequency part.

Proposition 4.6.2. *Let $u_{1,m} = {}^\top(\phi_1,m_1) \in (\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})_{sym}$ and $u_\infty = {}^\top(\phi_\infty,w_\infty) \in H^s_{2,sym}$ satisfying*

$$\sup_{0\le t\le T}\|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T}\|u_\infty(t)\|_{H^s_2} + \sup_{0\le t\le T}\|\phi(t)\|_{L^\infty} + \sup_{0\le t\le T}\|\nabla\phi(t)\|_{L^2_1} \le \min\{\delta_0,\frac{1}{2}\},$$

where δ_0 is the one in Lemma 4.3.7 (i) and $\phi=\phi_1+\phi_\infty$. Then we have the estimate

$$\begin{aligned}&\|F_\infty(u,g)\|_{L^2(0,T;H^s_2\times H^{s-1}_2)}\\ &\le\ C\|\{u_{1,m},u_\infty\}\|^2_{X^s(0,T)} + C\Big(1+\|\{u_{1,m},u_\infty\}\|_{X^s(0,T)}\Big)[g]_s,\end{aligned}$$

uniformly for $u_{1,m}$ and u_∞.

Proposition 4.6.2 follows in a similar manner to the proof of Proposition 2.6.2 and we omit the details.

By the same way as that in the proof of Proposition 4.6.1, we have the following estimate for $F_{1,m}(u^{(1)},g) - F_{1,m}(u^{(2)},g)$.

Proposition 4.6.3. *Let $u^{(k)}_{1,m} = {}^\top(\phi^{(k)}_1,m^{(k)}_1) \in (\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})_{sym}$ and $u^{(k)}_\infty = {}^\top(\phi^{(k)}_\infty,w^{(k)}_\infty) \in H^s_2$ satisfying*

$$\sup_{0\le t\le T}\|u^{(k)}_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T}\|u^{(k)}_\infty(t)\|_{H^s_2} + \sup_{0\le t\le T}\|\phi^{(k)}(t)\|_{L^\infty} + \sup_{0\le t\le T}\|\nabla\phi^{(k)}(t)\|_{L^2_1} \le \min\{\delta_0,\frac{1}{2}\},$$

where δ_0 *is the one in Lemma* 4.3.7 (i) *and* $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ $(k = 1, 2)$*. Then it holds that*

$$\begin{aligned}
&\|\Psi[\tilde{F}_{1,m}(u^{(1)}, g) - \tilde{F}_{1,m}(u^{(2)}, g)]\|_{\mathscr{Z}_{(1)}(0,T)} \\
&\le \; C \sum_{k=1}^{2} \|\{u_{1,m}^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \\
&\quad + C[g]_s \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)},
\end{aligned}$$

uniformly for $u_{1,m}^{(k)}$ *and* $u_\infty^{(k)}$*.*

We next estimate $F_\infty(u^{(1)}, g) - F_\infty(u^{(2)}, g)$.

Proposition 4.6.4. *Let* $u_{1,m}^{(k)} = {}^\top(\phi_1^{(k)}, m_1^{(k)}) \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ *and* $u_\infty^{(k)} = {}^\top(\phi_\infty^{(k)}, w_\infty^{(k)}) \in H_2^s$ *satisfying*

$$\begin{aligned}
&\sup_{0\le t\le T} \|u_{1,m}^{(k)}(t)\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u_\infty^{(k)}(t)\|_{H_2^s} \\
&\qquad + \sup_{0\le t\le T} \|\phi^{(k)}(t)\|_{L^\infty} + \sup_{0\le t\le T} \|\nabla\phi^{(k)}(t)\|_{L_1^2} \le \min\{\delta_0, \frac{1}{2}\},
\end{aligned}$$

where δ_0 *is the one in Lemma* 4.3.7 (i) *and* $\phi^{(k)} = \phi_1^{(k)} + \phi_\infty^{(k)}$ $(k = 1, 2)$*. Then it holds that*

$$\begin{aligned}
&\|F_\infty(u^{(1)}, g) - F_\infty(u^{(2)}, g)]\|_{L^2(0,T;H_2^{s-1} \times H_2^{s-2})} \\
&\le C \sum_{k=1}^{2} \|\{u_{1,m}^{(k)}, u_\infty^{(k)}\}\|_{X^s(0,T)} \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)} \\
&\quad + C[g]_s \|\{u_{1,m}^{(1)} - u_{1,m}^{(2)}, u_\infty^{(1)} - u_\infty^{(2)}\}\|_{X^{s-1}(0,T)},
\end{aligned}$$

uniformly for $u_{1,m}^{(k)}$ *and* $u_\infty^{(k)}$*.*

Proposition 4.6.4 easily follows from Lemmas 1.1.1–1.1.3, Lemma 1.3.4, Lemma 2.3.2, and Lemma 2.3.11 in a similar manner to the proof of Proposition 4.6.2.

The following estimate is concerned with to state Proposition 4.6.6.

Proposition 4.6.5. (i) *Let* $u_{1,m} = {}^\top(\phi_1, m_1) \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ *and* $u_\infty = {}^\top(\phi_\infty, w_\infty) \in H_{2,sym}^s$ *satisfying*

$$\begin{aligned}
&\sup_{0\le t\le T} \|u_{1,m}(t)\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} + \sup_{0\le t\le T} \|u_\infty(t)\|_{H_2^s} \\
&\qquad + \sup_{0\le t\le T} \|\phi(t)\|_{L^\infty} + \sup_{0\le t\le T} \|\nabla\phi(t)\|_{L_1^2} \le \min\{\delta_0, \frac{1}{2}\},
\end{aligned}$$

where δ_0 *is the one in Lemma* 4.3.7 (i) *and* $\phi = \phi_1 + \phi_\infty$*. Then it holds that*

$$\begin{aligned} &\|F_{1,m}(u,g)\|_{C([0,T];L^2)} + \|\nabla F_{1,m}(u,g)\|_{C([0,T];L^2_1)} \\ &\leq \; C\|\{u_{1,m}, u_\infty\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{1,m}, u_\infty\}\|_{X^s(0,T)}\Big)[g]_s, \end{aligned}$$

uniformly for $u_{1,m}$ *and* u_∞*.*

(ii) *Let* $u^{(k)}_{1,m} = {}^\top(\phi^{(k)}_1, m^{(k)}_1) \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ *and* $u^{(k)}_\infty = {}^\top(\phi^{(k)}_\infty, w^{(k)}_\infty) \in H^s_2$ *satisfying*

$$\begin{aligned} &\sup_{0\leq t\leq T} \|u_{1,m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\leq t\leq T} \|u_\infty(t)\|_{H^s_2} \\ &\quad + \sup_{0\leq t\leq T} \|\phi(t)\|_{L^\infty} + \sup_{0\leq t\leq T} \|\nabla\phi(t)\|_{L^2_1} \leq \min\{\delta_0, \frac{1}{2}\}, \end{aligned}$$

where δ_0 *is the one in Lemma* 4.3.7 (i) *and* $\phi^{(k)} = \phi^{(k)}_1 + \phi^{(k)}_\infty$ $(k = 1, 2)$*. Then it holds that*

$$\begin{aligned} &\|F_{1,m}(u^{(1)}, g) - F_{1,m}(u^{(2)}, g)\|_{L^2} + \|\nabla F_{1,m}(u^{(1)}, g) - F_{1,m}(u^{(2)}, g)\|_{L^2_1} \\ &\leq \; C\sum_{k=1}^{2} \|\{u^{(k)}_{1,m}, u^{(k)}_\infty\}\|_{X^s(0,T)}\|\{u^{(1)}_{1,m} - u^{(2)}_{1,m}, u^{(1)}_\infty - u^{(2)}_\infty\}\|_{X^{s-1}(0,T)} \\ &\quad +C[g]_s\|\{u^{(1)}_{1,m} - u^{(2)}_{1,m}, u^{(1)}_\infty - u^{(2)}_\infty\}\|_{X^{s-1}(0,T)}, \end{aligned}$$

uniformly for $u^{(k)}_{1,m}$ *and* $u^{(k)}_\infty$*.*

Proposition 4.6.5 follows from direct computations based on Lemma 4.3.12.

We obtain the existence of a solution $\{u_{1,m}, u_\infty\}$ of (4.3.2), (4.3.4) and (4.3.6) on $[0, T]$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$ by similar iteration argument to that in [28].

$u^{(0)}_{1,m} = {}^\top(\phi^{(0)}_1, m^{(0)}_1)$ and $u^{(0)}_\infty = {}^\top(\phi^{(0)}_\infty, w^{(0)}_\infty)$ are defined by

$$\begin{cases} u^{(0)}_{1,m}(t) &= S_1(t)\mathscr{S}_1(T)[(I - S_1(T))^{-1}\mathbb{G}_1] + \mathscr{S}_1(t)[\mathbb{G}_1], \\ w^{(0)}_1 &= m^{(1)}_1 - P_1(\phi^{(0)}w^{(0)}), \\ u^{(0)}_\infty(t) &= S_{\infty,0}(t)(I - S_{\infty,0}(T))^{-1}\mathscr{S}_{\infty,0}(T)[\mathbb{G}_\infty] + \mathscr{S}_{\infty,0}(t)[\mathbb{G}_\infty], \end{cases} \tag{4.6.1}$$

where $t \in [0, T]$, $\mathbb{G} = {}^\top(0, \frac{1}{\gamma}g(x, t))$, $\mathbb{G}_1 = P_1\mathbb{G}$, $\mathbb{G}_\infty = P_\infty\mathbb{G}$, $\phi^{(0)} = \phi^{(0)}_1 + \phi^{(0)}_\infty$ and $w^{(0)} = w^{(0)}_1 + w^{(0)}_\infty$. Note that $u^{(0)}_{1,m}(0) = u^{(0)}_{1,m}(T)$ and $u^{(0)}_\infty(0) = u^{(0)}_\infty(T)$.

$u^{(N)}_{1,m} = {}^\top(\phi^{(N)}_1, m^{(N)}_1)$ and $u^{(N)}_\infty = {}^\top(\phi^{(N)}_\infty, w^{(N)}_\infty)$ are defined, inductively for $N \geq 1$, by

$$\begin{cases} u^{(N)}_{1,m}(t) &= S_1(t)\mathscr{S}_1(T)[(I - S_1(T))^{-1}F_{1,m}(u^{(N-1)}, g)] + \mathscr{S}_1(t)[F_{1,m}(u^{(N-1)}, g)], \\ w^{(N)}_1 &= m^{(N)}_1 - P_1(\phi^{(N)}w^{(N)}), \\ u^{(N)}_\infty(t) &= S_{\infty,u^{(N-1)}}(t)(I - S_{\infty,u^{(N-1)}}(T))^{-1}\mathscr{S}_{\infty,u^{(N-1)}}(T)[F_\infty(u^{(N-1)}, g)] \\ &\quad +\mathscr{S}_{\infty,u^{(N-1)}}(t)[F_\infty(u^{(N-1)}, g)], \end{cases} \tag{4.6.2}$$

where $t \in [0,T]$, $u^{(N-1)} = u_1^{(N-1)} + u_\infty^{(N-1)}$, $u_1^{(N-1)} = {}^\top(\phi_1^{(N-1)}, w_1^{(N-1)})$, $\phi^{(N)} = \phi_1^{(N)} + \phi_\infty^{(N)}$ and $w^{(N)} = w_1^{(N)} + w_\infty^{(N)}$. Note that $u_{1,m}^{(N)}(0) = u_{1,m}^{(N)}(T)$ and $u_\infty^{(0)}(0) = u_\infty^{(0)}(T)$.

The symbol $B_{X^k_{sym}(a,b)}(r)$ stands for the closed unit ball in $X^k_{sym}(a,b)$ centered at 0 with radius r, i.e.,

$$B_{X^k_{sym}(a,b)}(r) = \left\{ \{u_{1,m}, u_\infty\} \in X^k_{sym}(a,b); \|\{u_{1,m}, u_\infty\}\|_{X^k(a,b)} \le r \right\}.$$

We have the following proposition from Propositions 4.4.1, 4.5.5, 4.6.1, 4.6.2, and 4.6.5 by the same argument as that in Chapter 2.

Proposition 4.6.6. *There exists a constant $\delta_1 > 0$ such that if $[g]_s \le \delta_1$, then it holds that*

$$\text{(i)} \qquad \|\{u_{1,m}^{(N)}, u_\infty^{(N)}\}\|_{X^s(0,T)} \le C_1[g]_s,$$

for all $N \ge 0$, and

$$\text{(ii)} \qquad \begin{aligned} &\|\{u_{1,m}^{(N+1)} - u_{1,m}^{(N)}, u_\infty^{(N+1)} - u_\infty^{(N)}\}\|_{X^{s-1}(0,T)} \\ \le\ & C_1[g]_s \|\{u_{1,m}^{(N)} - u_{1,m}^{(N-1)}, u_\infty^{(N)} - u_\infty^{(N-1)}\}\|_{X^{s-1}(0,T)}, \end{aligned}$$

for $N \ge 1$. Here C_1 is a constant independent of g and N.

Concerning the existence of a solution $\{u_{1,m}, u_\infty\}$ of (4.3.2), (4.3.4) and (4.3.6) on $[0,T]$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$, we state the following

Proposition 4.6.7. *There exists a constant $\delta_2 > 0$ such that if $[g]_s \le \delta_2$, then the system* (4.3.2), (4.3.4) *and* (4.3.6) *has a unique solution $\{u_{1,m}, u_\infty\}$ on $[0,T]$ in $B_{X^s_{sym}(0,T)}(C_1[g]_s)$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$. The uniqueness of solutions of* (4.3.2), (4.3.4) *and* (4.3.6) *on $[0,T]$ satisfying $u_{1,m}(0) = u_{1,m}(T)$ and $u_\infty(0) = u_\infty(T)$ holds in $B_{X^s_{sym}(0,T)}(C_1\delta_2)$.*

Corollary 4.6.8. *There exists a constant $\delta_3 > 0$ such that if $[g]_s \le \delta_3$, then the system* (4.3.1)-(4.3.2) *has a unique solution $\{u_1, u_\infty\}$ on $[0,T]$ in $B_{X^s_{sym}(0,T)}(C_2[g]_s)$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ where $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, \infty)$ and C_2 is a constant independent of g. The uniqueness of solutions of* (4.3.1)-(4.3.2) *on $[0,T]$ satisfying $u_j(0) = u_j(T)$ $(j = 1, \infty)$ holds in $B_{X^s_{sym}(0,T)}(C_2\delta_3)$.*

Proposition 4.6.7 and Corollary 4.6.8 follows from Lemma 4.3.7 (i) and Proposition 4.6.7 by the same way as that in Chapter 2 and we omit the proofs.

As for the unique existence of solutions of the initial value problem, (4.3.1)-(4.3.2), the following proposition can be proved from the estimates in sections 6-8, as in chapters 1 and 2.

Proposition 4.6.9. *Let $h \in \mathbb{R}$ and let $U_0 = U_{01} + U_{0\infty}$ with $U_{01} \in \mathscr{X}_{(1),sym} \times \mathscr{Y}_{(1),syn}$ and $U_{0\infty} \in H^s_{(\infty),2}$. Then there exist constants $\delta_4 > 0$ and $C_3 > 0$ such that if*

$$M(U_{01}, U_{0\infty}, g) := \|U_{01}\|_{(\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}} + \|U_{0\infty}\|_{H^s_{(\infty),2}} + [g]_s \leq \delta_4,$$

there exists a solution $\{U_1, U_\infty\}$ of the initial value problem for (4.3.1)-(4.3.2) *on $[h, h+T]$ in $B_{X^s_{sym}(h,h+T)}(C_3 M(U_{01}, U_{0\infty}, g))$ satisfying the initial condition $U_j|_{t=h} = U_{0j}$ $(j = 0, \infty)$. The uniqueness for this initial value problem holds in $B_{X^s_{sym}(h,h+T)}(C_3\delta_4)$.*

Therefore, we can extend $\{u_1, u_\infty\}$ periodically on $\mathbb{R}$ as a time periodic solution of (4.3.1)-(4.3.2) by using Corollary 4.6.8 and Proposition 4.6.9 in the same argument as that given in Chapter 1. Consequently, we obtain Theorem 4.2.1. This completes the proof.

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