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Holomorphic maps into the complex Grassmannian manifold induced by orthogonal product of a holomorphic line bundle

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Contents

Chapter 1

Introduction

Let (M, g) and (M', g) be Riemannian manifolds. A smooth map $f : M \longrightarrow M'$ is called *harmonic* if its tension field vanishes. Let $S^{n-1} \subset \mathbb{R}^n$ is the standard sphere in $\mathbb{R}^n$ and $x_1, \cdots, x_n$ the standard coordinate functions on $\mathbb{R}^n$, which are regarded as functions on S^{n-1} by restriction. In [28] Takahashi has shown that an isometric immersion $f : M \longrightarrow S^{n-1}$ is harmonic if and only if f satisfies $\Delta(x_A \circ f) = \lambda \cdot x_A \circ f$ for some constant λ and $A = 1, \cdots, n$, where Δ is the Laplace operator of (M, g). We focus on this result.

Pulling the tangent bundle $T\mathbb{R}^n$ on $\mathbb{R}^n$ back by ι, we obtain the following short exact sequence of vector bundles over S^{n-1}:

$$0 \longrightarrow TS^{n-1} \longrightarrow \iota^* T\mathbb{R}^n \longrightarrow N \longrightarrow 0, \tag{1.1}$$

where the bundle $N \to S^{n-1}$ is the normal bundle, which is isomorphic to orthogonal complement bundle of $TS^{n-1} \to S^{n-1}$. Since $N \to S^{n-1}$ is trivial, so is the pull-back bundle $f^*N \to M$. An arbitrary function on M can be regarded as a section of $f^*N \to M$. Thus for each $A = 1, \cdots, n$ $x_A \circ f$ is a section of $f^*N \to M$. In Takahashi's theorem, the properties that each $x_A \circ f$ is an eigenfunction of the Laplacian of (M, g) can be expressed in terms of a section of $f^*N \to M$.

In this regard, Nagatomo [19] generalized the above result to the following: Let $Gr_p(W)$ be the oriented Grassmannian manifold of p-planes in a vector space W. We fix an inner product $(\cdot, \cdot)_W$ on W. Then we have the following short exact sequence of vector bundles over $Gr_p(W)$.

$$0 \longrightarrow S \longrightarrow Gr_p(W) \times W \longrightarrow Q \longrightarrow 0,$$

where $S \to Gr_p(W)$ is the tautological bundle and $Q \to Gr_p(W)$ the universal quotient bundle over $Gr_p(W)$. (When $p = n - 1$, this short exact sequence of vector bundles coincides with (1.1). For a detail, see the next chapter.)

Theorem 1 ([19]). *Let $f : (M, g) \longrightarrow Gr_p(W)$ be a smooth map of a Riemannian manifold (M, g) into the oriented Grassmannian manifold of real p-planes in vector space W. We fix an inner product $(\cdot, \cdot)_W$ on W. Then, the followings are equivalent.*

(i) *The map $f : M \longrightarrow Gr_p(W)$ is harmonic.*

(ii) *W has the zero property for Laplacian. (which is explained below)*

Under these conditions, we have for an arbitrary vector $t \in W$,

$$\Delta t = -At, \qquad |df|^2 = -\text{trace } A, \tag{1.2}$$

where the vector space W is regarded as a space of sections of the pull-back bundle $f^*Q \to M$ and A is a certain operator which is determined by the second fundamental forms of the above short exact sequence of vector bundles.

In this theorem we say that W has the zero property for the Laplacian if for an arbitrary section $t \in W$ the zero locus of t is included that of Δt.

Moreover, Nagatomo [19] proved another theorem which is an extension of a result of do Carmo and Wallach [7] concerning classification of minimal immersions of spheres into another spheres.

In this thesis, we develop study of holomorphic maps of Kähler manifolds into the complex Grassmannian manifolds using these theorems as a main tool.

A typical example of the complex Grassmannian manifold is the complex projective space $\mathbb{C}P^n$. This case has been studied for a long time. For example, Calabi [4] has shown that full holomorphic isometric immersions $f : M \longrightarrow \mathbb{C}P^n$ of a Kähler manifold (M, h_M) into $\mathbb{C}P^n$ is unique up to equivalence of holomorphic automorphisms. Takeuchi [29] constructed all full holomorphic isometric immersions of homogeneous Kähler manifold into the complex projective spaces.

Inspiring by these results, we consider holomorphic maps of Kähler manifolds into the complex Grassmannian manifold of higher rank. There exist some results [9] and [17] for Kähler submanifold in 2-plane complex Grassmannian manifold using matrix computation. However there exist few papers about complex submanifolds of complex Grassmannian manifold of rank ≥ 2.

Main focus of this thesis is study of holomorphic maps from the viewpoint of holomorphic vector bundle theory.

In Chapter 2, we introduce the geometry of complex Grassmannian manifold and relation between holomorphic maps and holomorphic vector bundles. In Chapter 3 we define projectively flat maps and show the rigidity of holomorphic isometric projectively flat immersion of Hermitian symmetric space of compact type. In Chapter 4 we define strongly projectively flat maps, which are inspired by a key of a proof of a theorem in Chapter 3 and we show the rigidity of holomorphic equivariant strongly projectively flat map of compact simply connected homogeneous Kähler manifold.

Chapter 2

Preliminaries

2.1 The complex Grassmannian manifold

Let $(\mathbb{C}^n, (\cdot,\cdot)_n)$ be an n-dimensional complex vector space with Hermitian inner product. Let p be the integer which satisfy the inequality $0 < p < n$. We denote by $Gr_p(\mathbb{C}^n)$ the set of all complex p-dimensional subspaces in $\mathbb{C}^n$. This is called the *complex Grassmannian manifold* of p-planes. We have a homogeneous metric h_{Gr} which is induced by $(\cdot,\cdot)_n$. We denote by $\underline{\mathbb{C}^n} := Gr \times \mathbb{C}^n \to Gr$ the trivial bundle over $Gr_p(\mathbb{C}^n)$. We denote a subbundle of $\underline{\mathbb{C}^n}$ by

$$S := \{(x, v) \in \underline{\mathbb{C}^n} | v \in x\}. \tag{2.1}$$

This is called the *tautological vector bundle* over $Gr_p(\mathbb{C}^n)$. Since $S \to Gr$ is a subbundle of $\underline{\mathbb{C}^n} \to Gr$, we can take the quotient bundle $Q := \underline{\mathbb{C}^n}/S$. These bundle satisfy the following short exact sequence of vector bundles:

$$0 \longrightarrow S \longrightarrow \underline{\mathbb{C}^n} \longrightarrow Q \longrightarrow 0. \tag{2.2}$$

The bundle $Q \to Gr$ is called *universal quotient bundle*.

The trivial bundle $\underline{\mathbb{C}^n} \to Gr$ has a Hermitian fibre metric induced by the Hermitian inner product $(\cdot,\cdot)_n$, which is denoted by the same notation. Then we obtain an orthogonal complement bundle $S^\perp \to Gr$ of $S \to Gr$ in $\underline{\mathbb{C}^n} \to Gr$. As C^∞-complex vector bundle $S^\perp \to Gr$ is naturally isomorphic to $Q \to Gr$.

Since $S \to Gr$ and $S^\perp \to Gr$ are subbundles of $\underline{\mathbb{C}^n} \to Gr$, they have a Hermitian fibre metric h_S and $h_{S^\perp}$ induced from a Hermitian fibre metric of $\underline{\mathbb{C}^n} \to Gr$. The universal quotient bundle $Q \to Gr$ also have a Hermitian fibre metric h_Q by using the natural isomorphism between $Q \to Gr$ and $S^\perp \to Gr$. We denote by $i_S : S \longrightarrow \underline{\mathbb{C}^n}$ the inclusion of $S \to Gr$ into $\underline{\mathbb{C}^n} \to Gr$ and by $\pi_S : \underline{\mathbb{C}^n} \longrightarrow S$ the orthogonal projection of $\underline{\mathbb{C}^n} \to Gr$ onto $S \to Gr$. Similarly we have the inclusion and the orthogonal projection with respect to $S^\perp \to Gr$ and $\underline{\mathbb{C}^n} \to Gr$. The projection π_Q is nothing but the third map in (2.2). Their relations are expressed as the following:

$$0 \rightleftarrows S \underset{\pi_S}{\overset{i_S}{\rightleftarrows}} \mathbb{C}^n \underset{i_Q}{\overset{\pi_Q}{\rightleftarrows}} Q \rightleftarrows 0. \tag{2.3}$$

These vector bundles are all homogeneous: let $\tilde{G}$ be the special unitary group $SU(n)$ and $\tilde{K}_0 = S(U(p) \times U(q))$ a subgroup of $\tilde{G}$. Then the complex Grassmannian manifold of p-planes $Gr_p(\mathbb{C}^n)$ is expressed as the homogeneous manifold $\tilde{G}/\tilde{K}_0$. We assume that $\tilde{G}$ and $\tilde{K}_0$ acts on $\mathbb{C}^n$ and $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q$ is natural decomposition with respect to the action of $\tilde{K}_0$. Then we obtain

$$S = \tilde{G} \times_{\tilde{K}_0} \mathbb{C}^p, \qquad S^{\perp} = \tilde{G} \times_{\tilde{K}_0} \mathbb{C}^q, \qquad Q = \tilde{G} \times_{\tilde{K}_0} (\mathbb{C}^n/\mathbb{C}^p). \tag{2.4}$$

We denote by $\pi_p : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ and $\pi_q : \mathbb{C}^n \longrightarrow \mathbb{C}^q$ the orthogonal projections onto $\mathbb{C}^p$ and $\mathbb{C}^q$ respectively. Then maps i_S, i_Q, π_S, π_Q are expressed as the followings:

$$i_S([g,u]) = ([g], gu), \tag{2.5}$$
$$i_Q([g,v]) = ([g], gv), \tag{2.6}$$
$$\pi_S([g],w) = [g, \pi_p(g^{-1}w)], \tag{2.7}$$
$$\pi_Q([g],w) = [g, \pi_q(g^{-1}w)], \tag{2.8}$$

for $g \in G$, $u \in \mathbb{C}^p$, $v \in \mathbb{C}^q \cong \mathbb{C}^n/\mathbb{C}^p$ and $w \in \mathbb{C}^n$.

The Hermitian connection on $\underline{\mathbb{C}^n} \to Gr$ is the canonical exterionr derivative d since this is a trivial bundle. When $s \in \Gamma(S)$ be a section of $S \to Gr$, $i_S(s)$ is regarded as a $\mathbb{C}^n$-valued function. Then $di_S(s)$ is a $\mathbb{C}^n$-valued 1-form. The 1-form $di_S(s)$ is decomposed to the S-component and Q-component by

$$di_S(s) = \pi_S(di_S(s)) + \pi_Q(di_S(s)). \tag{2.9}$$

We set

$$\nabla^S s := \pi_S(di_S(s)), \qquad Hs := \pi_Q(di_S(s)). \tag{2.10}$$

The symbol ∇^S is a connection of $S \to Gr$. This is nothing but the canonical connection of $S \to Gr$.

The symbol H is a 1-form with values in $\mathrm{Hom}(S,Q) \cong S^* \otimes Q$. This is called *the second fundamental form* of $S \to Gr$ into $\underline{\mathbb{C}^n} \to Gr$.

Similarly let $t \in \Gamma(Q)$ be a section of $Q \to Gr$. Then we obtain a connection ∇^Q and the second fundamental form K of $Q \to Gr$ in $\underline{\mathbb{C}^n} \to Gr$:

$$di_Q(t) = Kt + \nabla^Q t. \tag{2.11}$$

Proposition 1 ([10]). *For each $x \in Gr_p(\mathbb{C}^n)$ the second fundamental form H and K satisfy*

$$h_Q(H_X s, t) + h_S(s, K_{\overline{X}} t) = 0, \tag{2.12}$$

for $s \in S_x$, $t \in Q_x$, X a $(1,0)$-tangent vector at x and $\overline{X}$ is the complex conjugate of X.

Proof. we extend s and t to local holomorphic sections of $S \to Gr$ and $Q \to Gr$ respectively. Then we obtain

$$\begin{aligned} h_Q(H_X s, t) = (\nabla^S_X s + H_X s, t)_n &= d(s,t)_n - (s, K_{\overline{X}} t + \nabla^Q_{\overline{X}} t)_n \\ &= -h_S(s, K_{\overline{X}} t). \end{aligned}$$

□

Proposition 2 ([10]). *The second fundamental form H is a $(1,0)$-form and K is a $(0,1)$-form.*

Proof. Let s be a local holomorphic section of $S \to Gr$. Since $i_S : S \longrightarrow \underline{\mathbb{C}^n}$ preserves holomorphic structure, $i_S(s)$ is a local holomorphic section of $\underline{\mathbb{C}^n} \to M$. It follows that if $\overline{X}$ is a $(0,1)$-tangent vector on $Gr_p(\mathbb{C}^n)$, then the covariant derivative $d_{\overline{X}} i_S(s)$ vanishes. Therefore we obtain

$$H_{\overline{X}} s = 0.$$

Since any element of $S \to Gr$ can be locally extended to a holomorphic section, the second fundamental form H is a $(1,0)$-form.

For the second fundamental form K, it follows from (2.12) that it is a $(0,1)$-form. □

By using the orthogonal projection π_S and π_Q, for any $w \in \mathbb{C}^n$ we obtain a smooth section

$$s := \pi_S(w) \in \Gamma(S), \qquad t := \pi_Q(w) \in \Gamma(Q)$$

of $S \to Gr$ and $Q \to Gr$ respectively.

Proposition 3 ([19])**.** *For any $w \in \mathbb{C}^n$, we set $s := \pi_S(w)$ and $t := \pi_Q(w)$. Then for any $(1,0)$-tangent vector X we have*

$$\nabla^S_{\overline{X}} s = -K_{\overline{X}} t, \tag{2.13}$$

$$\nabla^Q_{X} t = -H_X s. \tag{2.14}$$

Proof. Since $i_S(s([g])) + i_Q(t([g])) = ([g], w)$, we obtain

$$0 = di_S(s) + di_Q(t) = \nabla^S s + Hs + Kt + \nabla^Q t.$$

Therefore we obtain the above equations. □

Sections in Proposition 3 are sometimes called *the sections corresponding to w*.

Let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\tilde{G}$ and $\tilde{\mathfrak{k}}$ the Lie algebra of $\tilde{K}_0$, which is a subalgebra of $\tilde{\mathfrak{g}}$. We have a standard decomposition of $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{m}}. \tag{2.15}$$

Then the real tangent bundle TGr of $Gr_p(\mathbb{C}^n)$ can be expressed as

$$TGr = \tilde{G} \times_{\tilde{K}_0} \tilde{\mathfrak{m}}. \tag{2.16}$$

Since the complex Grassmannian manifold is a Hermitian symmetric space, there exists a complex structure on $\tilde{\mathfrak{m}}$. Thus the complexification $\tilde{\mathfrak{m}}_{\mathbb{C}}$ of $\tilde{\mathfrak{m}}$ is decomposed to the eigenspaces of the complex structure:

$$\tilde{\mathfrak{m}}_{\mathbb{C}} = \tilde{\mathfrak{m}}_{(1,0)} \oplus \tilde{\mathfrak{m}}_{(0,1)}, \tag{2.17}$$

where $\tilde{\mathfrak{m}}_{(1,0)}$ (resp. $\tilde{\mathfrak{m}}_{(0,1)}$) is the eigenspace with the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). The holomorphic tangent bundle $T_{(1,0)}Gr$ of $Gr_p(\mathbb{C}^n)$ is expressed as

$$T_{(1,0)}Gr = \tilde{G} \times_{\tilde{K}} \tilde{\mathfrak{m}}_{(1,0)}. \tag{2.18}$$

Let $\pi_{\tilde{\mathfrak{m}}}$ be the orthogonal projection of $\tilde{\mathfrak{g}}$ onto $\tilde{\mathfrak{m}}$. For $w \in \mathbb{C}^n$, we compute the covariant derivative and the second fundamental form of the sections $s \in \Gamma(S)$ and $t \in \Gamma(Q)$ by using (2.10), (2.11), and (2.13) - (2.15). Sections s and t is expressed as the following:

$$s([g]) = [g, \pi_p(g^{-1}w)], \tag{2.19}$$
$$t([g]) = [g, \pi_q(g^{-1}w)], \tag{2.20}$$

for $g \in G$. Then we compute the covariant derivative $\nabla^S s$ as follows:

$$\begin{aligned} \nabla^S s &= \pi_S d i_S(s) = \pi_S d([g], g\pi_p(g^{-1}w)) \\ &= \pi_S([g], dg\pi_p(g^{-1}w) - g\pi_p(g^{-1}dgg^{-1}w)) \\ &= [g, \pi_p(g^{-1}dg\pi_p(g^{-1}w))] - [g, \pi_p(g^{-1}dgg^{-1}w)], \end{aligned} \tag{2.21}$$

where we use the equation $d(g^{-1}) = g^{-1}dgg^{-1}$. Since $g^{-1}dg$ is the Maurer-Cartan form, the first term of the last side of (2.21) vanishes.

On the other hand, Hs is computed as follows:

$$Hs = [g, \pi_q(g^{-1}dg\pi_p(g^{-1}w))]. \tag{2.22}$$

The Maurer-Cartan form $g^{-1}dg$ is a 1-form with values in $\tilde{\mathfrak{g}}$. The Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{su}(n)$ is the set of all trace-free skew-Hermitian matrices. $\tilde{\mathfrak{m}} \subset \tilde{\mathfrak{g}}$ is expressed as the all elements vanishing entries of $\mathfrak{u}(p) \oplus \mathfrak{u}(q)$ in $\tilde{\mathfrak{g}}$. Thus (2.22) is rewriten as

$$Hs = [g, \pi_{\tilde{\mathfrak{m}}}(g^{-1}dg)\pi_p(g^{-1}w)]. \tag{2.23}$$

Similarly for a section t of $Q \to Gr$ we obtain the following computation:

$$Kt = [g, \pi_{\tilde{\mathfrak{m}}}(g^{-1}dg)\pi_q(g^{-1}w)], \tag{2.24}$$
$$\nabla^Q t = -[g, \pi_q(g^{-1}dgg^{-1}w)]. \tag{2.25}$$

The holomorphic tangent bundle $T_{(1,0)}Gr \to Gr$ is naturally isomorphic to $S^* \otimes Q \to Gr$, where $S^* \to Gr$ is the dual bundle of $S \to Gr$. Since the 1-form H is an homomorphism of $T_{(1,0)}Gr \to Gr$ to $S^* \otimes Q \to Gr$, this is also considered as an endomorphism of $T_{(1,0)}Gr \to Gr$ by using the natural identification.

Proposition 4 ([19]). *The second fundamental form H is the identity automorphism of $T_{(1,0)}Gr$.*

Proof. We use Killing vector fields. □

Corollary 1. *The second fundamental forms H and K are parallel.*

From this proposition, any $(1,0)$-vector X at $x \in Gr_p(\mathbb{C}^n)$ is identified with the homomorphism H_X of S_x into Q_x, where S_x and Q_x is fibres of $S \to Gr$ and $Q \to Gr$ at x respectively.

The natural identification induce a Hermitian metric h_{Gr} of $Gr_p(\mathbb{C}^n)$ from the metric $h_{S^*} \otimes h_Q$ of $S^* \otimes Q \to Gr$. We call this metric *the Hermitian metric of Fubini study type.*

Let $e_1, \cdots, e_n$ be the canonical unitary basis of $\mathbb{C}^n$ and s_A, t_A the corresponding sections of $S \to Gr$, $Q \to Gr$ for $A = 1, \cdots, n$ respectively.

Proposition 5 ([19])**.** *For any* $(1,0)$*-tangent vectors* X *and* Y *at* $x \in Gr_p(\mathbb{C}^n)$*, we have*

$$h_{Gr}(X,Y) = \sum_{A=1}^{n} h_S(K_{\overline{Y}}t_A, K_{\overline{X}}t_A) = \sum_{A=1}^{n} h_Q(H_X s_A, H_Y s_A). \tag{2.26}$$

This proposition implies the following equation:

$$h_{Gr}(X,Y) = -\mathrm{trace}_Q H_X K_{\overline{Y}} = -\overline{\mathrm{trace}_S K_{\overline{Y}} H_X}, \qquad X,Y \in T_{(1,0)_x}Gr, \tag{2.27}$$

where $x \in Gr_p(\mathbb{C}^n)$ and the symbol trace_Q and trace_S means to take the trace of the linear homomorphism of $Q \to Gr$ and $S \to Gr$ respectively.

Proposition 6 ([19],[15])**.** *The curvatures* R^S *and* R^Q *of* ∇^S *and* ∇^Q *are expressed as*

$$R^S(X,\overline{Y}) = K_{\overline{Y}}H_X, \tag{2.28}$$
$$R^Q(X,\overline{Y}) = -H_X K_{\overline{Y}}, \tag{2.29}$$

for $(1,0)$*-tangent vectors* X*,* Y*.*

Proof. If necessary we extend a $(1,0)$-tangent vector (resp.$(0,1)$-tangent vector) to a local holomorphic section (resp. local anti-holomorphic sction). By definition of a curvature we have

$$R^S(X,\overline{Y})s = \nabla^S_X \nabla^S_{\overline{Y}} s - \nabla^S_{\overline{Y}} \nabla^S_X s - \nabla^S_{[X,\overline{Y}]} s, \qquad X,Y \in T_{(1,0)_x}Gr, \tag{2.30}$$

where $s \in \Gamma(S)$ is a local section around $x \in Gr_p(\mathbb{C}^n)$ corresponding to a certain vector $w \in \mathbb{C}^n$. (We can take such a w since $\pi_S : \underline{\mathbb{C}^n} \longrightarrow S$ is surjective.) From general theory of complex manifolds the third term in right hand side vanishes. It follows from (2.13), (2.14) and Corollary 1 that we obtain

$$\begin{aligned} R^S(X,\overline{Y}) =& \nabla^S_X(\nabla^S_{\overline{Y}} s) - \nabla^S_{\overline{Y}}(\nabla^S_X s) = -\nabla^S_X(\nabla^S_{\overline{Y}} s)\nabla^S_{\overline{Y}}(K_X t) \\ =& -(\nabla_X K)_{\overline{Y}} t - K_{\nabla_X \overline{Y}} t - K_{\overline{Y}}(\nabla^Q_X t) \\ =& K_{\overline{Y}} H_X s. \end{aligned}$$

Similarly we also obtain (2.29). □

Remark 1. It follows from (2.27), (2.28) and (2.29) that we obtain

$$h_{Gr} = \mathrm{trace} R^Q = -\overline{\mathrm{trace} R^S}. \tag{2.31}$$

Proposition 7 (cf.[15])**.** *The curvature* R^{Gr} *of* ∇^{Gr} *is expressed as the following:*

$$R^{Gr}(X,\overline{Y})Z = -H_Z K_{\overline{Y}} H_X - H_X K_{\overline{Y}} H_Z, \tag{2.32}$$

for $(1,0)$*-tangent vectors* X, Y, Z *at* $x \in Gr_p(\mathbb{C}^n)$*.*

Proof. Since the metric h_{Gr} of $Gr_p(\mathbb{C}^n)$ is equivalent to the product metric $h_{S^*} \otimes h_Q$ of $S^* \otimes Q \to Gr$, the Hermitian connection is equivalent to the product connection $h_{S^*} \otimes h_Q$, and the corresponding curvature R^{Gr} is equivalent to $R^{S^*} \otimes \mathrm{Id}_Q + \mathrm{Id}_{S^*} \otimes Q$, where Id is the identity automorphism. Therefore for $(1,0)$-tangent vectors X, Y, Z at $x \in Gr_p(\mathbb{C}^n)$, we obtain

$$R^{Gr}(X,\overline{Y})Z = (R^{S^*}(X,\overline{Y}) \otimes \mathrm{Id}_Q)H_Z + (\mathrm{Id}_{S^*} \otimes R^Q(X,\overline{Y}))H_Z. \tag{2.33}$$

We compute each terms of right hand side respectively.

Since H_Z is an element in $S_x^* \otimes Q_x$, this is expressed as $\sum f_i \otimes t_i$, where $f_i \in S_x^*$ and $t_i \in Q_x$ for finite integers i. For an arbitrary element $s \in S_x$ it follows from (2.28) that we compute

$$\begin{aligned}
(R^{S^*}(X,\overline{Y}) \otimes \mathrm{Id}_Q)(H_Z)(s) =&(R^{S^*}(X,\overline{Y}) \otimes \mathrm{Id}_Q)(\textstyle\sum f_i \otimes t_i)(s)\\
=&(R^{S^*}(X,\overline{Y})f_i) \otimes t_i(s)\\
=&(R^{S^*}(X,\overline{Y})f_i)(s)t_i\\
=&-f_i(R^S(X,\overline{Y})s)t_i\\
=&-(f_i \otimes t_i)(R^S(X,\overline{Y})s)\\
=&-H_Z K_{\overline{Y}} H_X s,
\end{aligned} \tag{2.34}$$

where we omit the symbol $\sum$ to compute clearly.

Similarly we also compute

$$\begin{aligned}
(\mathrm{Id}_{S^*} \otimes R^Q(X,\overline{Y}))(H_Z)(s) =&f_i \otimes (R^Q(X,\overline{Y})t_i)(s)\\
=&-f_i(s) H_X K_{\overline{Y}} t_i\\
=&-H_X K_{\overline{Y}}(f_i(s)t_i)\\
=&-H_X K_{\overline{Y}}(f_i \otimes t_i)s\\
=&-H_X K_{\overline{Y}} H_Z s.
\end{aligned} \tag{2.35}$$

□

Remark 2. Let us compute the holomorphic sectional curvature Hol of $Gr_{n-1}(\mathbb{C}^n)$, the complex projective space with the metric of Fubini Study type. In this case, the rank of the universal quotient bundle $Q \to Gr$ is one. Thus it follows from (2.31) that we have

$$-H_X K_{\overline{Y}} = R^Q(X,\overline{Y}) = h_{Gr}(X,Y)\mathrm{Id}_{Q_x}, \qquad X, Y \in T_{(1,0)_x}Gr. \tag{2.36}$$

Therefore we obtain

$$R^{Gr}(X,\overline{Y})Z = -H_Z K_{\overline{Y}} H_X - H_X K_{\overline{Y}} H_Z = h_{Gr}(Z,Y)X + h_{Gr}(X,Y)Z. \tag{2.37}$$

for $(1,0)$-tangent vectors X, Y, Z. This implies that the complex Grassmannian $Gr_{n-1}(\mathbb{C}^n)$ with the metric of Fubini-Study type is the complex space form. Moreover for unit $(1,0)$-tangent vector X we can compute that

$$\mathrm{Hol}(X) = h_{Gr}(R^{Gr}(X,\overline{X})X,X) = h_{S^* \otimes Q}(-2H_X K_{\overline{X}} H_X, H_X) = 2. \tag{2.38}$$

For general complex Grassmannian manifold we obtain the following proposition:

Proposition 8. *Let $Gr_p(\mathbb{C}^n)$ be the complex Grassmanifold of complex p-planes in $\mathbb{C}^n$ with the metric of Fubini-Study type. We set $q := n - p$. Then the holomorphic sectional curvature* Hol *of $Gr_p(\mathbb{C}^n)$ satisfies the following inequality:*

$$\frac{2}{\min\{p,q\}} \leqq \mathrm{Hol} \leqq 2. \tag{2.39}$$

Proof. For a unit $(1,0)$-tangent vector X at $x \in Gr_p(\mathbb{C}^n)$, the holomorphic sectional curvature $H(X)$ is expressed as

$$H(X) = 2\mathrm{trace} H_X K_{\overline{X}} H_X K_{\overline{X}} \tag{2.40}$$

since the Hermitian metric h_{Gr} satisfies (2.27). It follows from Proposition 1 that $H_X K_{\overline{X}}$ is a Hermitian operator on Q_x. Thus $H_X K_{\overline{X}}$ has non-positive eigenvalues $\lambda_1, \cdots, \lambda_q$. The Hermitian operator $H_X K_{\overline{X}}$ is the composition of H_X and $K_{\overline{X}}$:

$$Q_x \xrightarrow{K_{\overline{X}}} S_x \xrightarrow{H_X} Q_x. \tag{2.41}$$

It follows that the number of nonzero eigenvalues is less than or equal to $\min\{p,q\}$.

Here we assume that $q \leqq p$. Then we obtain

$$H(X) = 2(\lambda_1^2 + \cdots + \lambda_q^2). \tag{2.42}$$

Since X is a unit vector, we have

$$1 = h_{Gr}(X,X) = -\mathrm{trace} H_X K_{\overline{X}} = -(\lambda_1 + \cdots + \lambda_q). \tag{2.43}$$

It follows from the method of Lagrange multiplier that we obtain (2.39). □

2.2 Holomorphic vector bundles and holomorphic maps into the complex Grassmannian

Let M be a m-dimensional compact Kähler manifold, $V \to M$ a holomorphic Hermitian vector bundle of rank q and $V \to M$ and W a finite-dimensional subspace of the space of sections.

Definition 1 ([19])**.** The vector bundle $V \to M$ is called *globally generated* by W if the bundle homomorphism

$$ev : M \times W \longrightarrow V \ : \ (x,t) \longmapsto t(x) \tag{2.44}$$

is surjective. Here the bundle homomorphism ev is called an *evaluation map*.

When $V \to M$ is globally generated by W, we define a map

$$f_0 : M \longrightarrow Gr(W) \ : \ x \longmapsto \mathrm{Ker}\ ev_x, \tag{2.45}$$

where $ev_x := ev(x,*) : W \longrightarrow V_x$ and Ker $ev_x = \{t \in W | t(x) = 0\}$. The map f_0 is called an *induced map* by $(V \to M, W)$.

Let W be the space of holomorphic sections. Since M is compact, W is a finite dimensional complex vector space, denoted by $N :=$ dim W. We set a Hermitian inner product $(\cdot,\cdot)_W$ of W by

$$(t_1, t_2)_W = \int_M h_V(t_1(x), t_2(x))dx, \qquad t_1, t_2 \in W, \tag{2.46}$$

where h_V is the Hermitian fibre metric of $V \to M$ and dx is the volume form of M. In this case, the induced map $f_0 : M \longrightarrow Gr_{N-q}(W)$ by $(V \to M, W)$ is called the *standard map* induced by $V \to M$.

Proposition 9. *The standard map $f_0 : M \longrightarrow Gr_{N-q}(W)$ is a holomorphic map.*

Proof. Since the bundle homomorphism $ev : \underline{W} \longrightarrow V$ is surjective, we have a short exact sequence

$$0 \longrightarrow \text{Ker } ev \longrightarrow \underline{W} \longrightarrow V \longrightarrow 0, \tag{2.47}$$

where Ker $ev \to M$ is the holomorphic vector bundle of fibre Ker ev_x at $x \in M$. Let $(s_1, \cdots, s_{N-q})$ be a local holomorphic frame of Ker $ev \to M$. By using the bundle injection Ker $ev \longrightarrow \underline{W}$, $s_1, \cdots, s_{N-q}$ are regarded as holomorphic maps into W which is linearly independent at each point $x \in M$. Therefore we obtain a local holomorphic map $x \longmapsto (s_1, \cdots, s_{N-q})$ of M into $Gr_{N-q}(\mathbb{C}^n)$, which is nothing but f_0. $\square$

On the other hand, let $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ be a holomorphic map. Then we have a pull-back bundle $f^*Q \to M$ of $Q \to Gr$ by f. Since the complex vector space $\mathbb{C}^n$ is regarded as the space of holomoprhic sections of $Q \to Gr$, we obtain a linear map $\iota : \mathbb{C}^n \longrightarrow H^0(f^*Q)$, where $H^0(f^*Q)$ is the space of holomorphic sections. Since $\underline{\mathbb{C}^n} \longrightarrow Q$ is surjective, the evaluation map

$$ev : M \times \mathbb{C}^n \longrightarrow f^*Q \ : \ (x,t) \longmapsto \iota(t)(x) \tag{2.48}$$

is also surjective. Thus we obtain an induced map by $(f^*Q \to M, \mathbb{C}^n)$. This is nothing but f.

Definition 2 ([19])**.** A holomorphic map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is called *full* if the linear map $\iota : \mathbb{C}^n \longrightarrow H^0(f^*Q)$ is injective.

Let us consider the case that a holomorphic map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is NOT full.

We denote by $ev : M \times \mathbb{C}^n \longrightarrow f^*Q$ an evaluation map inducing f. Then the linear map $\iota : \mathbb{C}^n \longrightarrow H^0(f^*Q)$ has non-trivial kernel Ker ι. Let W' be the orthogonal complement of Ker ι in $\mathbb{C}^n$. An evaluation map $ev' : M \times W' \longrightarrow f^*Q$ is surjective. We obtain an *full* induced map

$$f' : M \longrightarrow Gr(W') \ : \ x \longmapsto \text{Ker } ev'_x. \tag{2.49}$$

Then the holomorphic map f is expressed as the following:

$$f : M \xrightarrow{f'} Gr(W') \xrightarrow{\iota} Gr_p(\mathbb{C}^n) \ : \ x \longmapsto f'(x) \longmapsto f'(x) \oplus \text{Ker } \iota. \tag{2.50}$$

It follows that a not-full map into $Gr_p(\mathbb{C}^n)$ is realized by a full map into the proper subspace in $Gr_p(\mathbb{C}^n)$.

This notion is the same as fullness of maps into the complex projective space.

Let M be a compact Kähler manifold, $V \to M$ a holomorphic Hermitian vector bundle and W the space of holomorphic sections of $V \to M$ with $\mathcal{L}_2$-Hermitian inner product $(\cdot,\cdot)_W$.

Definition 3. A map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is called *induced by* $(V \to M, (\mathbb{C}^n, (\cdot,\cdot)_n)$ if the pull-back bundle $f^*Q \to M$ is holomorphic isomorphic to $V \to M$ with metrics and connections.

In this definition, $(\cdot,\cdot)_n$ is the standard Hermitian inner product of $\mathbb{C}^n$ and $Gr_p(\mathbb{C}^n)$ is considered as the Kähler manifold with the Kähler metric induced by $(\cdot,\cdot)_n$.

Let $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ be a full holomorphic map induced by $V \to M$. Since $\mathbb{C}^n$ is regarded as the space of holomorphic sections of $Q \to Gr$, there exists a complex linear map $\iota : \mathbb{C}^n \longrightarrow W$. By using ι, $\mathbb{C}^n$ is regarded as the subspace of W. We notice that in general ι does not preserve Hermitian inner products. Thus now there exist two Hermitian inner products $(\cdot,\cdot)_n$ and $(\cdot,\cdot)_W$ in $\mathbb{C}^n$. Let $\underline{T} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the positive Hermitian endomorphism of $\mathbb{C}^n$ which satisfies the following equality:

$$(\underline{T}w_1, \underline{T}w_2)_n = (w_1, w_2)_W, \qquad w_1, w_2 \in \mathbb{C}^n. \tag{2.51}$$

This induces an isometry

$$\underline{T}^{-1} : (Gr_p(\mathbb{C}^n), (\cdot,\cdot)_n) \longrightarrow (Gr_p(\mathbb{C}^n), (\cdot,\cdot)_W) \ : \ x \longmapsto \underline{T}^{-1}x, \tag{2.52}$$

which is denoted by the same notation of the inverse map of $\underline{T}$.

We denote by $ev_{\mathbb{C}} : \underline{\mathbb{C}^n} \longrightarrow V$ and $ev : W \longrightarrow V$ evaluation maps. Then $ev_{\mathbb{C}}$ is considered as the restriction of ev to $\underline{\mathbb{C}}$. Since $f : M \longrightarrow (Gr_p(\mathbb{C}^n), (\cdot,\cdot)_n)$ is considered as the induced map by $(V \to M, \mathbb{C}^n)$ by the evaluation map $ev_{\mathbb{C}}$, we obtain

$$f(x) = \text{Ker } ev_{\mathbb{C}_x} = \text{Ker } ev_x \cap \mathbb{C}^n, \qquad x \in M. \tag{2.53}$$

It follows from (2.52) that f is congruent to the following holomorphic map:

$$f : M \longrightarrow (Gr_p(\mathbb{C}^n), (\cdot,\cdot)_W) : x \longmapsto \underline{T}^{-1}(f_0(x) \cap \mathbb{C}^n), \tag{2.54}$$

where $f_0 : M \longrightarrow Gr_p(W)$ is the standard map induced by $V \to M$. When we denote by $\pi : W \to \mathbb{C}^n$ the orthogonal projection onto $\mathbb{C}^n$, we denote by $T := \underline{T} \circ \pi$ a semi-positive Hermitian endomorphism of W. Then $\mathbb{C}^n$ is expressed as the orthogonal complement of Ker T in W.

This means that holomorphic maps induced by $V \to M$ is expressed as a deformation of the standard map induced by $V \to M$ by a semi-positive Hermitian endomorphism.

At the end of this section, we define a equivalence of holomorphic maps into the complex Grassmannian manifold.

Definition 4 ([19])**.** (i) Let f_1 and f_2 be holomorphic sections of a compact complex manifold M into the complex Grassmannian. Then f_1 and f_2 are called *image equivalent* if there exists an isometry $\psi : Gr_p(\mathbb{C}^n) \longrightarrow Gr_p(\mathbb{C}^n)$ such that $f_1 = f_2 \circ \psi$, where the right hand side of the equality is the composition of f_2 and ψ.

(ii) Let $V \to M$ be a holomorphic vector bundle, $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ a holomorphic map, $\phi : V \longrightarrow f^*Q$ a holomorphic bundle isomorphism. Then pairs (f_1, ψ_1) and (f_2, ψ_2) are called *gauge equivalent* if there exists an isometry $\phi : Gr_p(\mathbb{C}^n) \longrightarrow Gr_p(\mathbb{C}^n)$ such that $f_1 = f_2 \circ \phi$ and $\psi_1 = \psi_2 \circ \tilde{\phi}$, where $\tilde{\phi} : Q \longrightarrow Q$ is the bundle isomorphism induced by ϕ.

By definition gauge equivalence induces image equivalence.

2.3 Homogeneous case

In this section we assume that the compact Kähler manifold $M = G/K_0$ is homogeneous, where G is a compact Lie group and K_0 a closed subgroup.

Let V_0 be an K_0-representation space and $V := G \times_{K_0} V_0 \to M$ the holomorphic homogeneous vector bundle. A point in $V \to M$ is expressed as the following:

$$[g, v] \in V, \qquad g \in G,\ v \in V_0. \tag{2.55}$$

The Lie group G acts on V:

$$g \cdot [g_0, v] := [gg_0, v], \qquad g, g_0 \in G,\ v \in V_0. \tag{2.56}$$

For a section $t \in \Gamma(V)$ of $V \to M$ and $g \in G$, we have a new section $g \cdot t \in \Gamma(V)$:

$$(g \cdot t)(x) := g(t(g^{-1}x)), \qquad x \in M, \tag{2.57}$$

where $g^{-1}x$ is the natural action of G to M.

We denote by W the space of holomorphic sections of $V \to M$. Since for $g \in G$ and $t \in W$ $g \cdot t$ is also holomorphic sections of $V \to M$, W is a G-representation space.

We assume that the evaluation map $ev : M \times W \longrightarrow V$ is surjective.

Proposition 10 ([19])**.** *V_0 is considered as a subspace of W.*

Proof. We denote by e the unit element in G and K_0-representation space V_0 is identified with $V_{[e]}$. We set $\pi_0 := ev_{[e]} : W \longrightarrow V_0$ a surjective linear map. For $k \in K_0$ and $t \in W$ we have

$$\pi_0(k \cdot t) = ev([e], k \cdot t) = (k \cdot t)([e]) = k(t(k^{-1}[e])) = k(t([e])) = k\pi_0(t). \tag{2.58}$$

Thus π is K_0-equivariant. Consequently Ker π_0 and (Ker $\pi_0)^\perp$ are K_0-representation space and (Ker $\pi_0)^\perp$ is identified with V_0 as K_0-representation space. □

By definition of π_0 and identification between V_0 and $V_{[e]}$, we compute as follows:

$$t([e]) = [e, \pi_0(t)], \qquad t \in W. \tag{2.59}$$

For $g \in G$ and $t \in W$ we can compute that

$$\begin{aligned} ev([g], t) =& t([g]) = (gg^{-1})(t(g \cdot [e])) \\ =& g((g^{-1} \cdot t)([e])) = g \cdot [e, \pi_0(g^{-1} \cdot t)] = [g, \pi_0(g^{-1} \cdot t)]. \end{aligned} \tag{2.60}$$

Therefore we can define an adjoint map $ev^* : V \longrightarrow \underline{W}$ of ev by

$$ev^*([g, v]) = ([g], gv). \tag{2.61}$$

We set $U_0 := \text{Ker } ev_{[e]}$. Then it follows from (2.60) that we obtain

$$\begin{aligned} \text{Ker } ev_{[g]} =& \{t \in W \mid \pi_0(g^{-1}t) = 0\} \\ =& \{t \in W \mid g^{-1}t \in U_0\} = gU_0. \end{aligned} \tag{2.62}$$

This does not depend on the choice of a representative in $[g]$.

Consequently the standard map $f_0 : M \longrightarrow Gr(W)$ induced by $V \to M$ is the G-equivariant map expressed as the following:

$$f_0([g]) = gU_0. \tag{2.63}$$

Let $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ be a holomorphic map induced by $V \to M$. Then there exists a semi-positive Hermitian endomorphism $T : W \longrightarrow W$ such that f is congruent to the following map by (2.54):

$$f : M \longrightarrow Gr(\text{Ker } T) : [g] \longmapsto T^{-1}(gU_0 \cap (\text{Ker } T)^{\perp}). \tag{2.64}$$

The semi-positive Hermitian endomorphism T has some properties, which has been shown by Nagatomo in [19]. To introduce them, we prepare some notations.

Let $H(W)$ be the space of all Hermitian endomorphism of W. The Lie group G acts on W by the following way:

$$(g \cdot A)(w) := g(A(g^{-1}w)), \qquad g \in G,\ A \in H(W),\ w \in W. \tag{2.65}$$

We define an Hermitian inner product $(\cdot, \cdot)_H$ on $H(W)$ by

$$(A, B)_H := \text{trace} A\overline{B}, \qquad A, B \in H(W). \tag{2.66}$$

For $u, v \in W$, we define a symmetric or Hermitain endomorphism $H(u, v)$ by

$$H(u, v)w := \frac{1}{2}\{(w, v)_W u + (w, u)_W v\}, \qquad w \in W. \tag{2.67}$$

For subspaces $U, V \subset W$, we define subspaces of $H(W)$ by the following:

$$H(U, V) := \text{Span}_{\mathbb{R}}\{H(u, v) | u \in U, v \in V\}, \tag{2.68}$$

$$GH(U, V) := \text{Span}_{\mathbb{R}}\{g \cdot A | g \in G, A \in H(U, V)\}. \tag{2.69}$$

In [19] Nagatomo has shown the following theorem:

Theorem 2 ([19], Theorem 5.20)**.** *Let $M := G/K_0$ be e a compact reductive Riemannian homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Fix a homogeneous vector bundle $V = G \times_{K_0} V_0 \to G/K_0$ of rank q.*

Let $f : M \longrightarrow Gr_p(\mathbb{K}^n)$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, be a full harmonic map satisfying following two conditions:

(i) *The pull-back bundle $f^*Q \to M$ with the pull-back metric and connection is gauge equivalent to $V \to M$ with the invariant metric and the canonical connection. (Hence, $q = n - p$.)*

(ii) *The mean curvature operator $A \in \Gamma(\mathrm{End}V)$ of a map f is expressed as $-\mu \mathrm{Id}_V$ for some positive real number μ.*

Then there exists an eigenspace $W \subset \Gamma(V)$ of the Laplacian of an eigenvalue μ equipped with L_2-scalar product $(\cdot,\cdot)_W$ and a semi-positive symmetric or Hermitian endomorphism $T \in \mathrm{End}(W)$. Regard W as $\mathfrak{g}$-representation (ρ, W). The pair (W,T) satisfies the following conditions.

(I) *The vector space $\mathbb{K}^n$ is a subspace of W with the inclusion $\iota : \mathbb{K}^n \longrightarrow W$ and $V \to M$ is globally generated by $\mathbb{K}^n$.*

(II) *As a subspace, $\mathbb{K}^n = (\mathrm{Ker}T)^\perp$, and the restriction of T is positive endomorphism of $\mathbb{K}^n$.*

(III) *The endomorphism T satisfies*

$$(T^2 - \mathrm{Id}_W, GH(V_0,V_0))_H = 0, \quad (T^2, GH(\rho(\mathfrak{m})V_0, V_0))_H = 0, \tag{2.70}$$

where V_0 is regarded as a subspace of W.

(IV) *The endomorphism T gives an embedding of $Gr_p(\mathbb{K}^n)$ into $Gr_{p'}(W)$, where $p' = n + \dim \mathrm{Ker}T$ and also gives a bundle isomorphism $\phi : V \longrightarrow f^*Q$.*

Then, $f : M \longrightarrow Gr_p(\mathbb{K}^n)$ can be expressed as

$$f(x) = (\iota^* T \iota)^{-1} \left(f_0(x) \cap (\mathrm{Ker}T)^\perp \right), \tag{2.71}$$

where ι^* denotes the adjoint operator of ι under the induced scalar product on $\mathbb{K}^n$ from $(\cdot,\cdot)_W$ on W and f_0 the standard map induced by W.

The pairs (f_1,ϕ_1) and (f_2,ϕ_2) are gauge equivalent if and only if

$$\iota_1^* T_1 \iota_1 = \iota_2^* T_2 \iota_2, \tag{2.72}$$

where (T_i, ι_i) correspond to f_i under the expression in (2.71) respectively.

Conversely, suppose that a vector space $\mathbb{K}^n$, an eigenspace $W \subset \Gamma(V)$ with eigenvalue μ and a semi-positive symmetric or Hermitian endomorphism $T \in \mathrm{End}(W)$ satisfying condition (I), (II) and (III) are given. Then there exists a unique embedding of $Gr_p(\mathbb{K}^n)$ into $Gr_{p'}(W)$ and the map $f : M \to Gr_p(\mathbb{K}^n)$ defined in (2.71) is a full harmonic map into $Gr_p(\mathbb{K}^n)$ satisfying condition (i) and (ii) with bundle isomorphism $V \cong f^*Q$.

When we assume that M is a Hermitian symmetric space, $V \to M$ holomorphic and f a holomorphic, we obtain the following corollary.

Corollary 2. *Let $M := G/K_0$ be a Hermitian symmetric space of compact type such that (G,K_0) is a Hermitian symmetric pair. Fix a holomorphic homogeneous Hermitian bundle $V := G \times_{K_0} V_0 \to M$ with Hermitian connection and we denote by W the space* *of holomorphic sections of $V \to M$ with L_2-inner product $(\cdot,\cdot)_W$. Let $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ be a full holomorphic isometric immersion with the condition that the pull-back bundle $f^*Q \to M$ is gauge equivalent to $V \to M$. Regard W as $\mathfrak{g}$-representation (ρ, W). Then there exists a semi-positive Hermitian endomorphism $T \in H(W)$ of W such that T satisfies the following conditions.*

(I) *$\mathbb{C}^n$ is the subspace of W with the inclusion $\iota : \mathbb{C}^n \longrightarrow W$ and $V \to M$ is globally generated by K.*

(II) *As a subspace, $\mathbb{C}^n = (\mathrm{Ker}T)^\perp$, and the restriction of T to $\mathbb{C}^n$ is positive Hermitian endomorphism of $\mathbb{C}^n$.*

(III) *The endomorphism T satisfy*

$$(T^2 - \mathrm{Id}_W, GH(V_0, V_0))_H = 0, \quad (T^2, GH(\rho(\mathfrak{m})V_0, V_0))_H = 0, \tag{2.73}$$

where V_0 is regarded as a subspace of W.

(IV) *The endomorphism T gives an embedding of $Gr_p(\mathbb{C}^n)$ into $Gr_{p'}(W)$, where $p' = n + \dim \mathrm{Ker}T$ and also gives a bundle isomorphism $\phi : V \longrightarrow f^*Q$.*

Then, $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ can be expressed as

$$f(x) = (\iota^* T \iota)^{-1} \left(f_0(x) \cap (\mathrm{Ker}T)^\perp \right), \tag{2.74}$$

where ι^* denotes the adjoint operator of ι under the induced scalar product on $\mathbb{C}^n$ from $(\cdot,\cdot)_W$ on W and f_0 the standard map induced by W.

The pairs (f_1, ϕ_1) and (f_2, ϕ_2) are gauge equivalent if and only if

$$\iota_1^* T_1 \iota_1 = \iota_2^* T_2 \iota_2, \tag{2.75}$$

where (T_i, ι_i) correspond to f_i under the expression in (2.75) respectively.

Conversely, suppose that a vector space $\mathbb{C}^n$, the space W of holomorphic sections of $V \to M$ and a semi-positive Hermitian endomorphism $T \in \mathrm{End}(W)$ satisfying condition (I), (II) and (III) are given. Then there exists a unique embedding of $Gr_p(\mathbb{C}^n)$ into $Gr_{p'}(W)$ and the map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ defined in (2.74) is a full harmonic map into $Gr_p(\mathbb{C}^n)$ satisfying condition (i) and (ii) with bundle isomorphism $V \cong f^*Q$.

Chapter 3

Projectively flat immersions of Hermitian symmetric space of compact type

3.1 Definition of projectively flatness

First of all, we define a projectively flat map.

Definition 5. Let M be a complex manifold. A holomorphic map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is called *projectively flat* if the pull-back bundle $f^*Q \to M$ of the universal quotient bundle over $Gr_p(\mathbb{C}^n)$ is projectively flat with respect to the pull-back connection.

In the present paper, we use the following definition of the projectively flatness of vector bundles.

Definition 6. Let M be a complex manifold, $V \to M$ a holomorphic Hermitian vector bundle. The Hermitian connection ∇^V of $V \to M$ is called *projectively flat* if the curvature R^V of ∇^V satisfy the following equality:

$$R^V = \alpha \mathrm{Id}_V, \tag{3.1}$$

where α is a 2-form on M and Id_V a identity isomorphism on $V \to M$.

Remark 3. In the case that $p = n-1$, where $Gr_{n-1}(\mathbb{C}^n)$ is the complex projective space, an arbitrary holomorphic map $f : M \longrightarrow Gr_{n-1}(\mathbb{C}^n)$ is projectively flat since the rank of $Q \to M$ is one. Thus projectively flatness is a kind of extension of holomorphic maps into the complex projective space.

In this capter, our goal is to prove the following theorem.

Theorem 3. *Let M be a Hermitian symmetric space of compact type and $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ a full holomorphic isometric projectively flat immersion. Then there exists a holomorphic line bundle $L \to M$ such that f is congruent to the standard map of $\tilde{L} \to M$, where $\tilde{L} \to M$ is the orthogonal direct sum of copies of $L \to M$.*

3.2 Normal decompsition

The argument in this section is introduced by Nagatomo in [19].

Let G be a compact Lie group and K_0 a closed subgroup of G. Let W be a G-representation space and $V_0 \subset W$ a K_0-representation subspace of W.

Definition 7 ([19], Definition 7.1)**.** Let $\mathfrak{g}$ and $\mathfrak{k}$ Lie algebras of G and K_0 respectively. We assume that there exists a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We denote by $N_1 = V_0^{\perp} \subset W$ the orthogonal complement of V_0. We define $B_1 : \mathfrak{m} \otimes V_0 \longrightarrow N_1$ as

$$B_1(\xi \otimes t) = \pi_1(\xi t), \qquad \xi \in \mathfrak{m},\ t \in V_0, \tag{3.2}$$

where $\pi_1 : W \longrightarrow N_1$ is the orthogonal projection into N_1. Inductively we denote by N_k the orthogonal complement of $L_0 \oplus \bigoplus_{i=1}^{k-1} \mathrm{Im}B_i$ and by π_k orthogonal projection into N_k. $B_k : S^k\mathfrak{m} \otimes V_0 \to N_k$ is defined as

$$B_k(\xi_1 \cdots \xi_k \otimes t) := \pi_k \left(\frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)} t \right). \tag{3.3}$$

Since B_k is K_0-equivariant, $\mathrm{Im}B_k$ is a K_0-representation space. If there exists a positive integer k such that

$$W = V_0 \oplus \mathrm{Im}B_1 \oplus \cdots \oplus \mathrm{Im}B_k, \tag{3.4}$$

then (W, V_0) is said to have a *normal decomposition*.

Proposition 11 ([19], Lemma 7.5)**.** *If (G, K_0) is a symmetri pair, then for $\xi_1, \cdots, \xi_k \in \mathfrak{m}$ and $v \in V_0$ we have*

$$B_k(\xi_1 \cdots \xi_k \otimes v) = \pi_k(\xi_1(\cdots(\xi_k \otimes v)\ldots)). \tag{3.5}$$

Proof. we compute that

$$\xi_1\xi_2\xi_3 \cdots \xi_k v - \xi_2\xi_1\xi_3 \cdots \xi_k = [\xi_1, \xi_2]\xi_3 \cdots \xi_k. \tag{3.6}$$

By assumption for an arbitrary $\xi, \eta \in \mathfrak{m}$ we have $[\xi, \eta] \in \mathfrak{k}$. Therefore the right hand side in (3.6) is an element in $V_0 \oplus \mathrm{Im}B_1 \oplus \cdots \oplus \mathrm{Im}B_{k-2}$. It follows that $\pi_k(\xi_1\xi_2\xi_3 \cdots \xi_k v) = \pi_k(\xi_2\xi_1\xi_3 \cdots \xi_k v)$. Consequently we obtain the assigned equality. □

Proposition 12 ([19], Proposition 7.9)**.** *Let W be an irreducible G-representation space and $V_0 \subset W$ a K_0-representation subspace of W. Since W is irreducible, (W, V_0) has a normal decomposition. We assume that if $i \neq j$, then $\mathrm{Im}B_i$ and $\mathrm{Im}B_j$ has no common irreducible K_0-representation subspaces.*

Let $\tilde{W}$ be a direct sum of N-copies of W and $\tilde{V}_0$ be a direct sum of N-copies of V_0 which is regarded as a subspace of $\tilde{W}$ in a natural way. Then an arbitrary class one representations of (G, K_0) in $H(\tilde{W})$ is a submodule of $GH(\tilde{V}_0, \tilde{V}_0)$.

Proof. See [19]. □

3.3 A proof of Theorem 3

Let $M = G/K_0$ be a Hermitian symmetric space of compact type such that (G, K_0) be a Hermitian symmetric pair and $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ be a holomorphic isometric projectively flat immersion. By definition of the projectively flatness there exists a complex $(1,1)$-differential form α on M such that

$$R^{f^*Q}(X, \overline{Y}) = \alpha(X, \overline{Y})\mathrm{Id}_{Q_{f(x)}}. \tag{3.7}$$

Since f is isometric, we have $h_M(X, Y) = h_{Gr}(df(X), df(Y))$. It follows from (2.31) that we obtain

$$\begin{aligned} q \cdot \alpha(X, \overline{Y}) &= \mathrm{trace} R^{f^*Q}(X, \overline{Y}) = \mathrm{trace} R^Q(df(X), df(\overline{Y})) \\ &= h_{Gr}(df(X), df(Y)) = h_M(X, Y). \end{aligned} \tag{3.8}$$

Consequently

$$R^{f^*Q}(X, \overline{Y}) = -\frac{1}{q}\sqrt{-1}\omega_M(X, \overline{Y})\mathrm{Id}_{Q_{f(x)}}, \tag{3.9}$$

where ω_M is the Kähler form on M.

Since ω_M is parallel with respect to the Kähler connection, it follows from Holonomy theorem that there exists a holomorphic Hermitian line bundle $L \to M$ with the curvature tensor

$$R^L(X, \overline{Y}) = -\frac{1}{q}\sqrt{-1}\omega_M(X, \overline{Y})\mathrm{Id}_{Q_{f(x)}} \tag{3.10}$$

such that $f^*Q \to M$ is holomorphic isomorphic to $\tilde{L} \to M$ with metrics and curvatures, where $\tilde{L} \to M$ is orthogonal direct sum of copies of $L \to M$.

Since $M = G/K_0$ is a Hermitian symmetric space of compact type, holomorphic line bundle is homogeneous. Thus there exists a 1-dimensional K_0-representaiton space L_0 such that

$$L = G \times_{K_0} L_0. \tag{3.11}$$

When we denote by $\tilde{L}_0$ the direct sum of L_0, $\tilde{L} \to M$ is regarded as $\tilde{L} = G \times_{K_0} \tilde{L}_0$.

We denote by W and $\tilde{W}$ the spaces of holomorphic sections of $L \to M$ and $\tilde{L} \to M$ respectively, which are G-representation spaces. It follows from Borel-Weil theory that W is irreducible since L_0 is irreducible as a K_0-representation space and $\tilde{W}$ is regarded as a orthogonal direct sum of W as a G-representation space. We set $N = \dim W$.

Lemma 1 ([19], Proposition 7.7). *W has a normal decomposition $W = L_0 \oplus \mathrm{Im}B_1 \oplus \cdots \oplus \mathrm{Im}B_k$.*

Proof. Since the dimension of W is finite, there exists a positive integer k such that $B_{\hat{k}} = 0$ for an arbitrary integer $\hat{k} > k$. Therefore we have a $\mathfrak{g}$-module $L_0 \oplus \mathrm{Im}B_1 \oplus \cdots \oplus \mathrm{Im}B_{\hat{k}}$. Since W is an irreducible G-representation space, this is also an irreducible $\mathfrak{g}$-module, which implies the result. □

Since M is a Hermitian symmetric space of compact type, M is decomposed to direct product of irreducible Hermitian symmetric spaces by a theorem of de Rham. We denote by $M \cong M_1 \times \cdots \times M_l$ the de Rham decomposition of M, where M_i is an irreducible Hermitian symmetric space of compact type for $i = 1, \cdots, l$. G and K_0 are also decomposed to $G \cong G_1 \times \cdots \times G_l$ and $K_0 \cong K_1 \times \cdots \times K_l$ respectively such that (G_i, K_i) is an irreducible Hermitian symmetric pair and $M_i = G_i/K_i$.

Lemma 2 (cf. Theorem 7.18 in [19])**.** *If $i \neq j$, then* $\mathrm{Im}B_i$ *and* $\mathrm{Im}B_j$ *has no common irreducible K_0-representation subspace.*

Proof. Let $\mathfrak{g}_i$ and $\mathfrak{k}_i$ be Lie algebras of G_i and K_i respectively. Let $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{m}_i$ be the standard decomposition of Lie algebras. Since (G_i, K_i) is an Hermitian symmetric pair, then $\mathfrak{k}_i$ can be decomposed to $kk_i = \mathfrak{u}_i \oplus \mathfrak{k}'_i$, where $\mathfrak{u}_i \cong \mathfrak{u}(1)$ is the center of $\mathfrak{k}_i$. Let $\mathfrak{m}_i^{\mathbb{C}}$ be the complexification of $\mathfrak{m}_i$ and $\mathfrak{m}_{i,(0,1)}$ (resp. $\mathfrak{m}_{i,(1,0)}$) the eigenspace of the complex structure on $\mathfrak{m}_i$ corresponding to the eigenvalue $-\sqrt{-1}$ (resp. $\sqrt{-1}$). Since $\mathfrak{m}_{i,(0,1)}$ is an irreducible $\mathfrak{k}_i$-module, then it can be expressed as $\mathfrak{m}_{i,(0,1)} = \mathbb{C}_{d_i} \otimes T_i$, where $\mathbb{C}_{d_i}$ is a 1-dimensional $\mathfrak{u}_i$-module with weight $d_i \neq 0$ and T_i is an irreducible $\mathfrak{k}'_i$-module. Thus the tangent space of type $(0,1)$ at $o \in M$ is expressed as

$$\bigoplus_{i=1}^{l} \mathfrak{m}_{i,(0,1)} = \bigoplus_{i=1}^{l} \left(\mathbb{C}_{d_i} \otimes T_i\right). \tag{3.12}$$

Then j-th symmetric tenser space of $\bigoplus_{i=1}^{l} \mathfrak{m}_{i,(0,1)}$ can be expressed as

$$S^j\left(\bigoplus_{i=1}^{l} \mathfrak{m}_{i,(0,1)}\right) = \bigoplus_{\substack{r_1+\cdots+r_l=j \\ r_i \geq 0}} \left(\bigotimes_{i=1}^{l} \mathbb{C}_{r_i d_i}\right) \otimes T_{r_1,\cdots,r_l}, \tag{3.13}$$

where $\mathbb{C}_{r_i d_i}$ is a 1-dimensional $\mathfrak{u}_i$-module with weight $r_i d_i$ and $T_{r_1,\cdots,r_l}$ is a $\mathfrak{k}'_1 \oplus \cdots \oplus \mathfrak{k}'_l$-module.

Since L_0 is a 1-dimensional $(K_1 \times \cdots \times K_l)$-representation space, L_0 is expressed as

$$L_0 = \left(\bigotimes_{i=1}^{l} \mathbb{C}_{k_i}\right) \otimes V,$$

where $\mathbb{C}_{k_i}$ is a 1-dimensional $\mathfrak{u}_i$-module with weight $k_i \neq 0$ and V is a 1-dimensional $(\mathfrak{k}'_1 \oplus \cdots \oplus \mathfrak{k}_l)$-module.

We denote by $ev : \underline{W} \longrightarrow L$ an evaluation map of $L \to M$. Then $B_1 : (\mathfrak{m}_1^{\mathbb{C}} \oplus \cdots \oplus \mathfrak{m}_l^{\mathbb{C}}) \otimes L_0 \to N_1 \subset W$ is nothing but the second fundamental form of vector bundle of $L \to M$ in $\underline{W} \to M$. Thus

$$B_1((\mathfrak{m}_{1,(1,0)} \oplus \cdots \oplus \mathfrak{m}_{l,(1,0)}) \otimes L_0) = \{0\}. \tag{3.14}$$

It follows from Propositon 11 that we can ignore $\mathfrak{m}_{i,(1,0)}$ for $i = 1, \cdots, l$. Since

$$B_j : S^j\left(\bigoplus_{i=1}^{l} \mathfrak{m}_{i,(0,1)}\right) \otimes L_0 \longrightarrow N_j \tag{3.15}$$

is K_0-equivariant, $\mathrm{Im}B_j$ can be expressed as

$$\mathrm{Im}B_j = \bigoplus_{\substack{r_1+\cdots+r_l=j \\ r_i \geq 0}} \left(\bigotimes_{i=1}^{l} \mathbb{C}_{r_i d_i + k_i}\right) \otimes U_{r_1,\cdots,r_l}, \tag{3.16}$$

where $U_{r_1,\cdots,r_l}$ is a $\mathfrak{k}'_1 \oplus \cdots \oplus \mathfrak{k}'_l$-module. It follows that $\mathrm{Im}B_i$ and $\mathrm{Im}B_j$ has no common irreducible $(K_1 \times \cdots \times K_l)$-submodule if $i \neq j$. $\square$

It follows from Lemma 2 and Proposition 12 that an arbitrary class one representations of (G, K_0) in $H(\tilde{W})$ is a submodule of $GH(\tilde{L_0}, \tilde{L_0})$.

On the other hand, the following result has proved in [19].

Theorem 4 ([19], Theorem 7.19)**.** *Let G/K_0 be a compact simply connectet homogeneous Kähler manifold and $L \to G/K_0$ a holomorphic homogeneous line bundle. We denote by W the space of holomorphic sections of $L \to G/K_0$. If $L \to G/K_0$ is a positive line bundle, then the set of Hermitian endomorphisms $H(W)$ on W consists of class one representations of (G, K_0).*

Thus we otain $H(\tilde{W}) = GH(\tilde{L_0}, \tilde{L_0})$. It follows from equation (2.73) in Corollary 2 that the semi-positive Hermitian endomorphism $T \in H(\tilde{W})$ obtained by $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is the identity endomorphism. This implies that $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is the standard map induced by $\tilde{L} \to M$.

Since $\tilde{L}$ is orthogonal direct sum of q-copies of $L \to M$, the space $\tilde{W}$ of holomorphic sections of $\tilde{L} \to M$ is decomposed to the orthogonal direct sum of q-copies of W. We denote by $ev : M \times W \longrightarrow L$ an evaluation map of $L \to M$ and by $\tilde{ev} : M \times \tilde{W} \longrightarrow L$ an evaluation map of $\tilde{L} \to M$. The evaluation map $\tilde{ev}$ is expressed as the following:

$$\tilde{ev} : M \times (W \oplus \cdots \oplus W) \longrightarrow L \oplus \cdots \oplus L : (x, t_1 + \cdots + t_q) \mapsto t_1(x) + \cdots + t_q(x),$$

where $W \oplus \cdots \oplus W = \tilde{W}$ and $L \oplus \cdots \oplus L = \tilde{L}$. Thus $\mathrm{Ker}\tilde{ev}_x = \mathrm{Ker}ev_x \oplus \cdots \oplus \mathrm{Ker}ev_x$ for any $x \in M$. Let $f_0 : M \longrightarrow Gr_{N-1}(W)$ be the standard map induced by $L \to M$, where $N = \mathrm{dim}W$. Since $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is the standard map induced by $\tilde{L} \to M$, f is expressed as

$$f(x) = \mathrm{Ker}\tilde{ev}_x = \mathrm{Ker}ev_x \oplus \cdots \oplus \mathrm{Ker}ev_x = f_0(x) \oplus \cdots \oplus f_0(x). \tag{3.17}$$

This is the composed map of the following two maps:

$$f_1 : M \longrightarrow Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) : x \longmapsto (f_0(x), \cdots, f_0(x)), \tag{3.18}$$

$$\begin{aligned} f_2 : Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) \longrightarrow Gr_{q(N-1)}(\bigoplus^{q} W) :& \\ (U_1, \cdots, U_q) \longmapsto U_1 \oplus \cdots \oplus U_q.& \end{aligned} \tag{3.19}$$

Since f and f_2 is isometric maps, f_1 is also isometric. Therefore f_0 is isometric up to homothety.

3.4 Application: with parallel second fundamental form

In this section, we show the following theorem.

Theorem 5. *Let M be a compact Kähler manifold and $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ be a holomorphic isometric immersion. Assume that f is projectively flat. Then the holomorphic sectional curvature of M is greater than or equal to $\frac{1}{q}$ if and only if f has parallel second fundamental form.*

This is a kind of extension of a theorem of A. Ros in [24], which claimed that a holomorphic map of a compact Kähler manifold into the complex projective space has holomorhic sectional curvature greater than or equal to $\frac{1}{2}$ if and only if it has parallel second fundamental form. When $q=1$, Theorem 5 claims that a holomorphic map of a compact Kähler manifold into the complex projective space has holomorhic sectional curvature greater than or equal to 1 if and only if it has parallel second fundamental form, which is distinct from $\frac{1}{2}$.

This is because we take a metric of Fubini-Study type with constant holomorphic sectional curvature 2 (see Remark 2) although holomorphic sectional curvature of complex projective space in a theorem of Ros is identically 1.

It follows from Theorem 3 that we show the following theorem, which is a rigidity theorem of holomorphic projectively flat immersions with parallel second fundamental form.

Theorem 6. *Let M be a compact Kähler manifold and $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ a holomorphic isometric immersion. If f is projectively flat and has parallel second fundamental form, then f is expressed as the composed map of* (3.18) *and* (3.19)*.*

Proof. First of all, we show the following lemma.

Lemma 3. *The compact Kähler manifold M is a Hermitian symmetric space of compact type.*

Proof. Since f has parallel second fundamental form, it follows from equation of Gauss that M is locally symmetric. Let $\tilde{M} \to M$ be the universal covering over M. Since $\tilde{M}$ is simply connected and locally symmetric, $\tilde{M}$ is a Hermitian symmetric space. It follows from Theorem 5 that $\tilde{M}$ has holomorphic sectional curvature greater than or equal to $1/q$. It follows that $\tilde{M}$ is of compact type. $\tilde{M}$ has non-negative Ricci tensor and so is M. It follows from [11] that M is simply connected. □

It follows from Theorem 3 that $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is expressed as the composed map of (3.18) and (3.19). Since f has parallel second fundamental form and f_2 is totally geodesic, $f_0 : M \longrightarrow Gr_{N-1}(W)$ has parallel second fundamental form. Therefore a holomorphic map $f_0 : M \to Gr_{N-1}(W)$ is an isometric immersion with parallel second fundamental form, which is classified in [22]. □

3.5 A proof of Theorem 5

We have a short exact sequence of holomorphic vector bundles:

$$0 \longrightarrow T_{1,0}M \longrightarrow T_{1,0}Gr|_M \longrightarrow N \longrightarrow 0, \tag{3.20}$$

where $T_{1,0}Gr_p(\mathbb{C}^n)|_M \to M$ is a holomorphic vector bundle induced by f from the holomorphic tangent bundle over $Gr_p(\mathbb{C}^n)$ and $N \to M$ is the quotient bundle. We obtain second fundamental forms σ and A of TM and N in $T_{1,0}Gr|_M$:

$$\nabla^{Gr}_X Y = \nabla^M_X Y + \sigma(X,Y), \qquad X \in T_{\mathbb{C}}M,\ Y \in \Gamma(T_{1,0}M), \tag{3.21}$$

$$\nabla^{Gr}_X \xi = -A_\xi X + \nabla^N_X \xi, \qquad U \in T_{\mathbb{C}}M,\ \xi \in \Gamma(N). \tag{3.22}$$

For each point $x \in M$, $\sigma : T_{1,0_x}M \times T_{1,0_x}M \longrightarrow N_x$ is a symmetric bilinear mapping. This is called the second fundamental form of f. The second fundamental form $A : N_x \times T_{0,1_x}M \longrightarrow T_{1,0_x}M$ is a bilinear mapping. This is called the shape operator of M. We follow a convention of submanifold theory to define the shape operator. Second fundamental forms σ and A satisfy the following formulas.

Formulas 1. *For any $X, Y, Z, W \in T_{1,0_x}M$, we have*

- $\sigma(\overline{X}, Y) = 0, \qquad A_\xi X = 0,$
- $h_{Gr}\left(\sigma(X,Y),\xi\right) = h_{Gr}\left(Y, A_\xi \overline{X}\right),$
- $h_{Gr}\left(R^M(X,\overline{Y})Z, W\right) = h_{Gr}\left(R^{Gr}(X,\overline{Y})Z, W\right) - h_{Gr}\left(\sigma(X,Z),\sigma(Y,W)\right),$
- $h_{Gr}\left(R^N(X,\overline{Y})\xi, \eta\right) = h_{Gr}\left(R^{Gr}(X,\overline{Y})\xi, \eta\right) + h_{Gr}\left(A_\xi \overline{Y}, A_\eta \overline{X}\right),$
- $\left(\nabla_Y \sigma\right)(X,Z) = \left(\nabla_X \sigma\right)(Y,Z),$
- $\left(\nabla_{\overline{Y}} \sigma\right)(U,Z) = -\left(R^{Gr}(X,\overline{Y})Z\right)^{\perp}.$

Note that the quotient bundle $N \to M$ is isomorphic to the orthogonal complement bundle $T^{\perp}_{1,0}M \to M$ as a C^∞ complex vector bundle. The third, fourth and fifth formulas are called the equation of Gauss, the equation of Ricci and the equation of Codazzi respectively. From the equation of Codazzi,

$$\nabla\sigma : T_{1,0_x}M \otimes T_{1,0_x}M \otimes T_{1,0_x}M \longrightarrow N_x \tag{3.23}$$

is a symmetric tensor for any $x \in M$.

Since f is holomorphic, isometric and projectively flat, we obtain

$$R^{f^*Q}(X,\overline{Y}) = \frac{1}{q} h_M(X,Y)\mathrm{Id}_{Q_{f(x)}}, \qquad \text{for } X, Y \in T_{1,0_x}M. \tag{$*$}$$

We denote by Hol the holomorphic sectional curvature of a Kähler manifold. By the equation of Gauss , if X is a unit (1,0)-vector on M, then

$$\begin{aligned} \mathrm{Hol}^M(X) = h_M(R^M(X,\overline{X})X,X) &= h_{Gr}(R^{Gr}(X,\overline{X})X,X) - \|\sigma(X,X)\|^2 \\ &= \mathrm{Hol}^{Gr}(X) - \|\sigma(X,X)\|^2. \end{aligned} \tag{3.24}$$

Lemma 4. *Let f be a holomorphic isometric immersion. Then f is projectively flat if and only if*

$$\mathrm{Hol}^{Gr}(X) = \frac{2}{q}, \qquad \text{for any unit } (1,0)\text{−vector } X. \tag{3.25}$$

Proof. Let X be a unit (1,0)-vector at $x \in M$. By the equation ($*$), we have

$$-H_X K_{\overline{X}} = R^{f^*Q}(X,\overline{X}) = \frac{1}{q}\mathrm{Id}_{Q_x}. \tag{3.26}$$

It follows from equations (2.37) and (3.26) that

$$\begin{aligned} \mathrm{Hol}^{Gr}(X) = h_{Gr}(R^{Gr}(X,\overline{X})X,X) &= -2h_{S^*\otimes Q}(H_X K_{\overline{X}} H_X, H_X) \\ &= \frac{2}{q} h_{S^*\otimes Q}(H_X,H_X) = \frac{2}{q}. \end{aligned} \tag{3.27}$$

□

Lemma 5. *For any* $(0,1)$*-vector* $\overline{Y}$ *on* M *we have*

$$\nabla^M_{\overline{Y}}\sigma = 0. \tag{3.28}$$

Proof. By the equation (2.32) and the equation $(*)$, it follows that

$$\begin{aligned} R^{Gr}(X,\overline{Y})Z &= -H_Z K_{\overline{Y}} H_X - H_X K_{\overline{Y}} H_Z \\ &= \frac{1}{q} h_{Gr}(Z,Y)X + \frac{1}{q} h_{Gr}(X,Y)Z. \end{aligned} \tag{3.29}$$

By the equation of Codazzi, we have

$$\nabla^M_{\overline{Y}}\sigma(X,Z) = -(R^{Gr}(X,\overline{Y})Z)^{\perp} = 0. \tag{3.30}$$

$\square$

In [24] A.Ros has proved the following Lemma.

Lemma 6 (A.Ros,[24])**.** *Let* T *be a* k*-covariant tensor on a compact Riemannian manifold* M*. Then*

$$\int_{UM} (\nabla T)(X, \cdots, X) dX = 0, \tag{3.31}$$

where UM *is the unit tangent bundle of* M *and* dX *is the canonical measure of* UM *induced by the Riemannian metric on* M*.*

For a proof, see [24].
We use the complexification of the above lemma.

Lemma 7. *Let* T *be a* (p,q)*-covariant tensor on an* m*-dimensional compact Kähler manifold* (M, h_M)*. We consider* M *as an* $2m$*-dimensional real manifold with the almost complex structure* J*. We denote by* g_M *the Riemannian metric induced by* h_M*. Then we have the canonical measure* dX *of* UM*. We obtain the following equality:*

$$\int_{UM} (\nabla T)(\overline{U}_X, U_X, \cdots, U_X, \overline{U}_X, \cdots, \overline{U}_X) dX = 0, \tag{3.32}$$

where $U_X = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$ *and* $\overline{U}_X = \frac{1}{\sqrt{2}}(X + \sqrt{-1}JX)$ *and* X *is a real tangent vector on* M*.*

Proof. We define real valued k-covariant tensors on Riemannian manifold (M, g_M) by

$$2K(X_1, \cdots, X_k) = T(U_1, \cdots, U_p, \overline{U}_{p+1}, \cdots, \overline{U}_k) + \overline{T(U_1, \cdots, U_p, \overline{U}_{p+1}, \cdots, \overline{U}_k)}, \tag{3.33}$$

$$\begin{aligned} 2L(X_1, \cdots, X_k) =& \sqrt{-1}\{T(U_1, \cdots, U_p, \overline{U}_{p+1}, \cdots, \overline{U}_k) \\ &- \overline{T(U_1, \cdots, U_p, \overline{U}_{p+1}, \cdots, \overline{U}_k)}\}, \end{aligned} \tag{3.34}$$

where $k = p + q$, $U_i = U_{X_i}$ for $i = 1, \cdots, k$. Then T, K and L satisfy the following equation:

$$T(U_1, \cdots, U_p, \overline{U}_{p+1}, \cdots, \overline{U}_k) = K(X_1, \cdots, X_k) - \sqrt{-1}L(X_1, \cdots, X_k). \tag{3.35}$$

We get the covariant derivative of both sides of this equation:

$$\begin{aligned}(\nabla_{\overline{U}_X}T)(U_X,\cdots,\overline{U}_X,\cdots) &= \frac{1}{\sqrt{2}}(\nabla_{X+\sqrt{-1}JX}K)(X,\cdots,X)\\ &\quad-\frac{\sqrt{-1}}{\sqrt{2}}(\nabla_{X+\sqrt{-1}JX}L)(X,\cdots,X).\end{aligned}\tag{3.36}$$

Since the covariant derivative is linear, then

$$(\nabla_{X+\sqrt{-1}JX}K)(X,\cdots,X)=(\nabla_X K)(X,\cdots,X)+\sqrt{-1}(\nabla_{JX}K)(X,\cdots,X).\tag{3.37}$$

Consequently it follows from Lemma 6 that we obtain

$$\begin{aligned}\int_{UM}(\nabla T)(\overline{U_X},U_X,\cdots,U_X,\overline{U_X},\cdots,\overline{U_X})dX &= \frac{\sqrt{-1}}{\sqrt{2}}\int_{UM}(\nabla_{JX}K)(X,\cdots,X)dX\\ &\quad+\frac{1}{\sqrt{2}}\int_{UM}(\nabla_{JX}L)(X,\cdots,X)dX.\end{aligned}$$

For the covariant tensor field K, we define a new covariant tensor fields $\tilde{K}$ by

$$\tilde{K}(X_1,\cdots,X_k)=K(JX_1,\cdots,JK_k),\qquad X_1,\cdots,X_k\in T_xM,\ x\in M.\tag{3.38}$$

Since the almost complex structure J is parallel and preserves the inner product and orientation of each tangent space of M, it follows that

$$\begin{aligned}\int_{UM}(\nabla_{JX}K)(X,\cdots,X)dX &= (-1)^k\int_{UM}(\nabla_{JX}K)(J(JX),\cdots,J(JX))dX\\ &= (-1)^k\int_{UM}(\nabla_{JX}\tilde{K})(JX,\cdots,JX)dX\\ &= (-1)^k\int_{UM}(\nabla_X\tilde{K})(X,\cdots,X)dX\\ &= 0.\end{aligned}\tag{3.39}$$

The last equality follows from Lemma 6. Similarly we have

$$\int_{UM}(\nabla_{JX}L)(X,\cdots,X)dX=0.\tag{3.40}$$

Therefore we obtain the equality in Lemma 7. □

From now on we show Theorem 5.

We define a (2,2)-covariant tensor T on M by

$$T(X,Y,\overline{Z},\overline{W})=h_{Gr}\left(\sigma(X,Y),\sigma(Z,W)\right),\tag{3.41}$$

where U, V, Z, W are (1,0)-vectors on M. Using the equation of Ricci and the equation of Codazzi, we obtain

$$(\nabla^2T)(\overline{X},X,X,X,\overline{X},\overline{X})=h_M\left((\nabla^2\sigma)(\overline{X},X,X,X),\sigma(X,X)\right)+\|(\nabla\sigma)(X,X,X)\|^2.\tag{3.42}$$

Using the Ricci identity, we obtain

$$(\nabla^2\sigma)(X,\overline{X},X,X) - (\nabla^2\sigma)(\overline{X},X,X,X) = R^N(X,\overline{X})\left(\sigma(X,X)\right) - 2\sigma\left(R^M(X,\overline{X})X,X\right). \tag{3.43}$$

It follows from Lemma 5 that

$$(\nabla^2\sigma)(\overline{X},X,X,X) = -R^N(X,\overline{X})\left(\sigma(X,X)\right) + 2\sigma\left(R^M(X,\overline{X})X,X\right). \tag{3.44}$$

Therefore, we obtain

$$\begin{aligned}(\nabla^2 T)(\overline{X},X,X,X,\overline{X},\overline{X}) = &- h_{Gr}\left(R^N(X,\overline{X})(\sigma(X,X)),\sigma(X,X)\right)\\ &+ 2h_{Gr}\left(\sigma(R^M(X,\overline{X})X,X),\sigma(X,X)\right)\\ &+ \|(\nabla\sigma)(X,X,X)\|^2.\end{aligned} \tag{3.45}$$

From the equation of Ricci and (2.32), we have

$$\begin{aligned}h_{Gr}\left(R^N(X,\overline{X})(\sigma(X,X)),\sigma(X,X)\right) =& h_{Gr}\left(R^{Gr}(X,\overline{X})(\sigma(X,X)),\sigma(X,X)\right)\\ &+ \|A_{\sigma(X,X)}\overline{X}\|^2\\ =& h_{Gr}(-H_{\sigma(X,X)}K_{\overline{X}}H_X, H_{\sigma(X,X)})\\ &+ h_{Gr}(-H_X K_{\overline{X}} H_{\sigma(X,X)}, H_{\sigma(X,X)})\\ &+ \|A_{\sigma(X,X)}\overline{X}\|^2.\end{aligned} \tag{3.46}$$

In the following computation, we extend $(1,0)$-vectors to local holomorphic vector fields if necessary.

Lemma 8. *For any* $(1,0)$*-vectors* X,Y,Z *on* M*, we have*

$$-H_{\sigma(X,Z)}K_{\overline{Y}} = \left(\nabla_Z R^{f^*Q}\right)(X,\overline{Y}) \tag{3.47}$$

Proof. We have

$$\left(\nabla_Z R^{f^*Q}\right)(X,\overline{Y}) = -\nabla_Z(H_X K_{\overline{Y}}) + H_{\nabla_Z X}K_{\overline{Y}} = -(\nabla_Z H)(U)K_{\overline{Y}}. \tag{3.48}$$

Since we can easily show that $H_{\sigma(X,Z)} = (\nabla_X H)(Z)$, we obtain

$$-H_{\sigma(X,Z)}K_{\overline{Y}} = (\nabla_X H)(Z)K_{\overline{Y}} = \left(\nabla_Z R^{f^*Q}\right)(X,\overline{Y}). \tag{3.49}$$

$\square$

It follows from $(*)$ that

$$\begin{aligned}\left(\nabla_Z R^{f^*Q}\right)(X,\overline{Y}) =& \nabla_Z^{f^*Q}\left(R^{f^*Q}(X,\overline{Y})\right) - R^{f^*Q}(\nabla_Z^M X,\overline{Y})\\ =& \frac{1}{q}\nabla_Z^M\left(h_M(X,Y)\right)\mathrm{Id}_Q - \frac{1}{q}h_M(\nabla_Z^M X,Y)\mathrm{Id}_Q = 0,\end{aligned} \tag{3.50}$$

where X, Y, Z are $(1,0)$-vectors on M. Then it follows from Lemma 8, (3.26) and (3.46) that

$$\begin{aligned}h_{Gr}\left(R^N(X,\overline{X})(\sigma(X,X)),\sigma(X,X)\right) =& h_{Gr}(-H_X K_{\overline{X}} H_{\sigma(X,X)}, H_{\sigma(X,X)})\\ &+ \|A_{\sigma(X,X)}\overline{X}\|^2\\ =& \frac{1}{q}\|\sigma(X,X)\|^2 + \|A_{\sigma(X,X)}\overline{X}\|^2.\end{aligned} \tag{3.51}$$

Using the equation of Gauss and the equation (3.29), we have

$$\begin{aligned} h_{Gr}\left(\sigma(R^M(X,\overline{X})X,X),\sigma(X,X)\right) &= h_{Gr}\left(R^M(X,\overline{X})X, A_{\sigma(X,X)}\overline{X}\right) \\ &= h_{Gr}\left(R^{Gr}(X,\overline{X})X, A_{\sigma(X,X)}\overline{X}\right) - X\|A_{\sigma(X,X)}\overline{X}\|^2 \\ &= -2h_{Gr}(H_X K_{\overline{X}} H_X, H_{A_{\sigma(X,X)}\overline{X}}) - \|A_{\sigma(X,X)}\overline{X}\|^2 \\ &= \frac{2}{q}\|\sigma(X,X)\|^2 - \|A_{\sigma(X,X)}\overline{X}\|^2. \end{aligned} \tag{3.52}$$

Combining the equations (3.51) and (3.52) with (3.45), we obtain

$$\begin{aligned} (\nabla^2 T)(\overline{X},X,X,X,\overline{X},\overline{X}) = &- \left(\frac{1}{q}\|\sigma(X,X)\|^2 + \|A_{\sigma(X,X)}\overline{X}\|^2\right) \\ &+ 2\left(\frac{2}{q}\|\sigma(X,X)\|^2 - \|A_{\sigma(X,X)}\overline{X}\|^2\right) + \|(\nabla\sigma)(X,X,X)\|^2 \\ =&\frac{3}{q}\left(\|\sigma(U,U)\|^2 - q\|A_{\sigma(X,X)}\overline{X}\|^2\right) + \|(\nabla\sigma)(X,X,X)\|^2. \end{aligned} \tag{3.53}$$

By integrating both sides of (3.53) ($X = U_X$), Lemma 7 yields

$$\frac{3}{q}\int_{UM}\left(\|\sigma(U_X,U_X)\|^2 - q\|A_{\sigma(U_X,U_X)}\overline{U}_X\|^2\right)dX + \int_{UM}\|(\nabla\sigma)(U_X,U_X,U_X)\|^2 dX = 0. \tag{3.54}$$

From now on we assume that the holomorphic sectional curvature of M is greater than or equal to $\frac{1}{q}$. Let us compute the first term of the left hand side of the equation (3.54). We define $\xi \in N$ as $\sigma(X,X) = \|\sigma(X,X)\|\xi$. Then we have

$$A_{\sigma(X,X)}\overline{X} = \|\sigma(X,X)\| A_\xi \overline{X}. \tag{3.55}$$

We denote by τ the involutive anti-holomorphic transformation of the complexification $T_{\mathbb{C}}M$ of TM having TM as the fixed point set. We set $B := A_\xi \circ \tau$. Then B is an anti-linear transformation and satisfies the following equation:

$$h_{Gr}(BX,Y) = h_{Gr}(BY,X), \qquad X,Y \in T_{1,0_x}M,\ x \in M. \tag{3.56}$$

If we regard B as a real linear transformation on the real vector space with an inner product $\mathfrak{Re}(h_{Gr}(\cdot,\cdot))$, then B is a symmetric transformation. Let λ be the eigenvalue of B whose absolute value is maximum and e the corresponding unit eigenvector. By Cauchy-Schwarz inequality, we have

$$\lambda = h_{Gr}(Be,e) = h_{Gr}(A_\xi\overline{e},e) = h_{Gr}(\xi,\sigma(e,e)) \leq \|\sigma(e,e)\|. \tag{3.57}$$

It follows from (3.24), Lemma 4 and the hypothesis that

$$\|A_\xi\overline{X}\|^2 \leq \lambda^2 \leq \|\sigma(e,e)\|^2 \leq \frac{1}{q}. \tag{3.58}$$

It follows that

$$\begin{aligned}\|\sigma(X,X)\|^2 - q\|A_{\sigma(X,X)}\overline{X}\|^2 &= \|\sigma(U,U)\|^2(1-q\|A_\xi\overline{X}\|^2)\\ &\geq \|\sigma(X,X)\|^2(1-q\cdot\frac{1}{q}) = 0.\end{aligned} \tag{3.59}$$

Thus it follows from (3.54) that

$$\|(\nabla\sigma)(X,X,X)\|^2 = 0. \tag{3.60}$$

Since $\nabla\sigma$ is a symmetric tensor, $\nabla\sigma$ vanishes.

Conversely, we assume that M has parallel second fundamental form. From (3.24) and Lemmas 4 and 5, it is enough to prove that $\|\sigma(X,X)\|^2 \leq \frac{1}{q}$, where X is an arbitrary unit $(1,0)$-vector on M. Let T be a $(2,2)$-covariant tensor on M defined by the equation (3.41). Since the second fundamental form σ is parallel, T is also parallel and so $\nabla^2 T = 0$. It follows from (3.53) that

$$\|\sigma(X,X)\|^2 - q\|A_{\sigma(X,X)}\overline{X}\|^2 = 0. \tag{3.61}$$

The Cauchy-Schwarz inequality and (3.61) imply that

$$\begin{aligned}\|\sigma(X,X)\|^2 = h_{Gr}\left(\sigma(X,X),\sigma(X,X)\right) &= h_{Gr}\left(X, A_{\sigma(X,X)}\overline{X}\right)\\ \leq \|A_{\sigma(X,X)}\overline{X}\| &= \frac{1}{\sqrt{q}}\|\sigma(X,X)\|.\end{aligned} \tag{3.62}$$

Therefore, $\|\sigma(X,X)\|^2 \leq \frac{1}{q}$.

Chapter 4

Strongly projectively flat maps of compact homogeneous Kähler manifolds

4.1 Definition of Strongly projectively flatness

First of all, we define a strongly projectively flat map.

Definition 8. Let M be a compact complex manifold. A holomorphic map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is called *strongly projectively flat* if there exists a holomorphic Hermitian line bundle $L \to M$ such that the pull-back bundle $f^*Q \to M$ of the universal quotient bundle is holomorphic isometric to $\tilde{L} \to M$ preserving Hermitian metrics, where $\tilde{L} \to M$ is the orthogonal direct sum of copies of $L \to M$.

Remark 4. Since $L \to M$ is of rank 1, there exists a $(1,1)$-form α such that the curvature R^L of the Hermitian connection ∇^L on $L \to M$ is expressed as $R^L = \alpha \mathrm{Id}_L$. It follows that the curvature $R^{\tilde{L}}$ of the Hermitian connection $\nabla^{\tilde{L}}$ on $\tilde{L} \to M$ is also expressed as

$$R^L = \alpha \mathrm{Id}_L. \tag{4.1}$$

Therefore $\tilde{L} \to M$ is projectively flat and a holomorphic strongly projectively flat map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is projectively flat.

In general the inverse of the above assertion is not true. However If a holomorphic map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is an isometric immersion, then f is strongly projectively flat if and only if it is strongly projectively flat by the observation in Section 3.3.

A holomorphic map $f : M \longrightarrow Gr_{n-1}(\mathbb{C}^n)$ is strongly projectively flat since the rank of $Q \to M$ is one. Therefore strongly projectively flatness is also a kind of extension of holomorphic maps into the complex projective space.

4.2 Strongly projectively flat maps of compact homogeneous Kähler manifolds

Let M be a compact simply connected homogeneous Kähler manifold, G the identity component of the isometry group of M and K_0 an isotropy subgroup of G. Let $f : M \longrightarrow$

$Gr_p(\mathbb{C}^n)$ be a full holomorphic strongly projectively flat map. By definition of strongly projectively flatness there exists a Hermitian line bundle $L \to M$ such that $f^*Q \to M$ is isomorphic to $\tilde{L} \to M$ as a Hermitian vector bundle, where $\tilde{L} \to M$ is orthogonal direct sum of q-copies of $L \to M$. Since M is a compact simply connected homogeneous Kähler, $L \to M$ is homogeneous. We set L_0 the 1-dimensional K_0-representation space such that $L = G \times_{K_0} L_0$. Then we have

$$\tilde{L} = L \oplus \cdots \oplus L = G \times_{K_0} (L_0 \oplus \cdots \oplus L_0) = G \times_{K_0} \tilde{L}_0, \tag{4.2}$$

where $\tilde{L}_0$ is q-orthogonal direct sum of L_0. We denote by W and $\tilde{W}$ the spaces of holomorphic sections of $L \to M$ and $\tilde{L} \to M$ respectively and we set N the dimension of W. By definition of $\tilde{L} \to M$, $\tilde{W}$ is regarded as q-orthogonal direct sum of W. Let $\pi_j : \tilde{W} \longrightarrow W$ be the orthogonal projection onto the j-th component of $\tilde{W}$. It follows from Proposition 10 that L_0 is a subspace of W and $\tilde{L}_0$ is a subspace of $\tilde{W}$ as a K_0-representation space. When we restrict π_j to $\tilde{L}_0$, $\pi_j|_{\tilde{L}_0}$ is orthogonal projection of $\tilde{L}_0$ onto the j-th component of $\tilde{L}_0$. We denote by

$$ev : M \times W \longrightarrow L, \qquad \tilde{ev} : M \times \tilde{W} \longrightarrow \tilde{L} \tag{4.3}$$

the evaluation maps respectively and by $f_0 : M \longrightarrow Gr_{N-1}(W)$ and $\tilde{f}_0 : M \longrightarrow Gr_{q(N-1)}(\tilde{W})$ the standard maps induced by $L \to M$ and $\tilde{L} \to M$ respectively. Since $\tilde{W}$ is orthogonal direct sum of q-copies of W, we have

$$\begin{aligned} \tilde{ev}([g], t) &= \tilde{ev}([g], t_1 \oplus \cdots \oplus t_q) \\ &= ev([g], t_1) \oplus \cdots \oplus ev([g], t_q) \in L \oplus \cdots \oplus L, \end{aligned} \tag{4.4}$$

where $t = t_1 \oplus \cdots \oplus t_q$ is the orthogonal decomposition with respect to $\tilde{W} = W \oplus \cdots \oplus W$. We set $U_0 := \mathrm{Ker} ev_{[e]}$, where e is the unit element in G. Then it follows from (2.63) that $f_0([g]) = gU_0$. It follows from (4.4) that the map $\tilde{f}_0$ is expressed as

$$\begin{aligned} \tilde{f}_0 : &M \longrightarrow Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) \longrightarrow Gr_{q(N-1)}(\tilde{W}), \\ &[g] \longmapsto (gU_0, \cdots, gU_0) \longmapsto gU_0 \oplus \cdots \oplus gU_0 = g \cdot (U_0 \oplus \cdots \oplus U_0). \end{aligned} \tag{4.5}$$

Since $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is full, $\mathbb{C}^n$ is regarded as a subspace of $\tilde{W}$. It follows from Section 2.2 that there exists a semi-positive Hermitian endomorphism T of $\tilde{W}$ and a bundle isomorphism $\phi : \tilde{L} \longrightarrow f^*Q$ such that maps $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ and $\phi : \tilde{L} \longrightarrow f^*Q$ are expressed as follows:

$$f([g]) = T^{-1}\left(\tilde{f}_0([g]) \cap (\mathrm{Ker}T)^{\perp}\right), \tag{4.6}$$

$$\phi([g, v]) = ([g], Tgv), \tag{4.7}$$

for $g \in G$ and $v \in \tilde{L}_0$.

Our goal in this chapter is to show the following theorem.

Theorem 7. *Let M be a compact simply connected homogeneous Kähler manifold and G the unit isometry group of M. Let $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ be a full holomorphic strongly projectively flat map into the complex Grassmannian manifold. Then f is G-equivariant if and onlyl if f is the standard map.*

Here G-equivariance means that there exists a Lie group homomorphism $\rho : G \longrightarrow SU(n)$ which satisfies the following equation:

$$f(gx) = \rho(g)f(x), \qquad g \in G,\ x \in M. \tag{4.8}$$

In order to prove this theorem, it is sufficient to show that the semi-positive Hermitian endomorphism $T : \tilde{W} \longrightarrow \tilde{W}$ is the identity map of $\tilde{W}$.

From now on, we assume that $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is G-equivariant. Then there exists a Lie group homomorphism $\rho : G \longrightarrow SU(n)$ which satisfies the following equation:

$$f(g[\tilde{g}]) = \rho(g)f([\tilde{g}]), \qquad g, \tilde{g} \in G. \tag{4.9}$$

By definition $\mathbb{C}^n$ is G-representation space and a vector subspace of $\tilde{W}$.

Lemma 9. *$f^*Q \to M$ is homogeneous.*

Proof. The definition of the pull-back bundle $f^*Q \to M$ is that

$$f^*Q = \{([g], v) \in M \times Q | f([g]) = \pi(v)\}, \tag{4.10}$$

where $\pi : Q \to Gr_p(\mathbb{C}^n)$ is the natural projection. For any $([\tilde{g}], v) \in f^*Q$ and $g \in G$, we have an action of G to $f^*Q \to M$ by

$$g \cdot ([\tilde{g}], v) = (g[\tilde{g}], \rho(g)v). \tag{4.11}$$

Since G acts on M transitively, $f^*Q \to M$ is homogeneous. □

Since $f^*Q \to M$ is homogeneous, the space of holomorphic sections of $f^*Q \to M$ is G-representation space. Let t be a holomorphic section of $f^*Q \to M$. For $g \in G$ and $x \in M$, we have

$$(g \cdot t)(x) = g\left(t(g^{-1}x)\right). \tag{4.12}$$

For $t \in \mathbb{C}^n$, we obtain a holomorphic section of $f^*Q \to M$ which is expressed as

$$t(x) = (x, t(f(x))), \qquad \text{for } x \in M. \tag{4.13}$$

Thus for $g, \tilde{g} \in G$ we obtain

$$\begin{aligned}(g \cdot t)(x) &= g(t(g^{-1}x)) = g\left(g^{-1}x, t(f(g^{-1}x))\right), \\ &= \left(x, \rho(g)t(\rho(g^{-1})f(x))\right) = (x, (\rho(g)t)(f(x))), \\ &= (\rho(g)t)(x).\end{aligned} \tag{4.14}$$

Therefore $\mathbb{C}^n$ is a G-representation subspace of the space of holomorphic sections of $f^*Q \to M$.

Lemma 10. *The holomorphic isomorphism $\phi : \tilde{L} \longrightarrow f^*Q$ is G-equivariant.*

Proof. At first we show that $f^*Q \to M$ is isomorphic to $\tilde{L}$ as a homogeneous vector bundle. Since ϕ preserves Hermitian connections, bundles $\tilde{L} \to M$ and $f^*Q \to M$ have same holonomy groups and ϕ is holonomy equivariant. Since the action of K_0 to $f^*Q \to M$ and $L \to M$ at $[e]$, where e is the unit element in G, is expressed as a action of the holonomy group, ϕ is K-equivariant. Thus $f^*Q_{[e]}$ is isomorphic to $L_0 \oplus \cdots \oplus L_0$ as a K_0-representation space. Therefore $f^*Q \to M$ is isomorphic to $\tilde{L} \to M$ as a homogeneous vector bundle.

Finally we show that a holomorphic isomorphism $\phi : \tilde{L} \longrightarrow \tilde{L}$ is G-equivariant. We denote by $\tilde{L} \cong L_1 \oplus \cdots \oplus L_q$ and $L_j = G \times_{K_0} L_{(j)}$, where $L_j \to M$ is the j-th component and $L_{(j)}$ is isomorphic to L_0 as a K_0-representation space for $j = 1, \cdots, q$. Then we have

$$\tilde{L} = G \times_{K_0} (L_{(1)} \oplus \cdots \oplus L_{(q)}). \tag{4.15}$$

Let $\phi_j : L_j \longrightarrow \tilde{L}$ be the restriction of $\phi : \tilde{L} \longrightarrow \tilde{L}$ to $L_j \to M$. Then ϕ_j is expressed as the following:

$$\phi_j([g, v]) = [g, \varphi_1(g)(v) \oplus \cdots \oplus \varphi_q(g)(v)], \qquad \text{for } g \in G, v \in L_{(j)}, \tag{4.16}$$

where $\varphi_i(g) : L_{(j)} \longrightarrow L_{(i)}$ is a linear map for $i = 1, \cdots, q$. Since $L_{(i)}$ and $L_{(j)}$ are isomorphic 1-dimensional K_0-representation spaces, there exists a complex number $\alpha_i(g)$ such that $\varphi_i(g)(v) = \alpha_i(g)v$. Thus we have

$$\phi_j([g, v]) = [g, \alpha_1(g)v \oplus \cdots \oplus \alpha_q(g)v], \qquad \text{for } g \in G, v \in L_{(j)}. \tag{4.17}$$

Since ϕ_j is a bundle homomorphism, we obtain

$$\begin{aligned}
\phi_j([gk, v]) &= [gk, \alpha_1(gk)v \oplus \cdots \oplus \alpha_q(gk)v] = [g, \alpha_1(gk)kv \oplus \cdots \oplus \alpha_q(gk)kv],\\
\phi_j([g, kv]) &= [g, \alpha_1(g)kv \oplus \cdots \oplus \alpha_q(g)kv],
\end{aligned}$$

for $g \in G$, $k \in K_0$ and $v \in L_{(j)}$. It follows that $\alpha_i(gk) = \alpha_i(g)$ for $i = 1 \cdots, q$, $g \in G$ and $k \in K_0$. Therefore α_i is a complex valued function on G/K_0. Since ϕ_j is holomorphic, so is α_i for $i = 1, \cdots q$, which implies that α_i is a constant function for each i since G/K_0 is compact. We regard α_i as a complex number. Then we have

$$\phi_j([g, v]) = [g, \alpha_1 v \oplus \cdots \oplus \alpha_q v], \qquad \text{for } g \in G, v \in L_{(j)}. \tag{4.18}$$

This is G-equivariant for each $j = 1, \cdots, q$. Consequently $\phi : \tilde{L} \longrightarrow \tilde{L}$ is G-equivariant. $\square$

It follows from Lemma 9 and Lemma 10 that $\mathbb{C}^n$ is a G-representation subspace of $\mathbb{C}^n$.

Lemma 11. *The semi-positive Hermitian endomorphism $T : \tilde{W} \longrightarrow \tilde{W}$ is G-equivariant.*

Proof. Since $\phi : \tilde{L} \longrightarrow f^*Q$ is G-equivariant, it follows from (4.7) that we can compute that

$$\phi(g_1 \cdot [g_2, v]) = \phi([g_1 g_2, v]) = ([g_1 g_2], T g_1 g_2 v), \tag{4.19}$$

$$\phi(g_1 \cdot [g_2, v]) = g_1 \cdot \phi([g_2, v]) = g_1 \cdot ([g_2], T g_2 v) = ([g_1 g_2], g_1 T g_2 v). \tag{4.20}$$

Therefore we have

$$Tgv = gTv, \qquad \text{for } g \in G, v \in \tilde{L}_0. \tag{4.21}$$

We denote by GL_0 an subspace of W spanned by gv for any $g \in G$ and $v \in L_0$ and similarly we denote by $G\tilde{L}_0$. Then $G\tilde{L}_0$ is regarded as q-orthogonal direct sum of GL_0.

GL_0 is a G-representation subspace of W. Since W is irreducible and GL_0 is not empty, we obtain $W = GL_0$. Consequently we obtain $\tilde{W} = G\tilde{L}_0$. It follows that for any $w \in \tilde{W}$ there exists $\alpha_i \in \mathbb{C}$, $g_i \in G$ and $v_i \in \tilde{L}_0$ such that $w = \sum \alpha_i g_i v_i$, where the right hand side of this equation is a finite sum. For any $g \in G$, we have

$$Tgw = Tg \sum \alpha_i g_i v_i = \sum Tg\alpha_i g_i v_i = \sum gT\alpha_i g_i v_i = gT \sum \alpha_i g_i v_i = gTw. \tag{4.22}$$

Therefore T is G-equivariant. □

Since $T : \tilde{W} \longrightarrow \tilde{W}$ is G-equivariant, T is also K_0-eqivariant.

Lemma 12.

$$T(\tilde{L}_0) \subset \tilde{L}_0. \tag{4.23}$$

Proof. Since the orthogonal projection $\pi_j : \tilde{W} \longrightarrow W$ is K_0-equivariant for each $j = 1, \cdots, q$, $\pi_j \circ T : \tilde{W} \longrightarrow W$ is a K_0-equivariant endomorphism. Thus $\pi_j \circ T(\tilde{L}_0) \subset W$ is a K_0-representation subspace of W. It follows from Schur's lemma and Borel-Weil theory that $\pi_j \circ T(\tilde{L}_0) \subset L_0$. Concequently $T(\tilde{L}_0) \subset (\tilde{L}_0)$. □

We denote by the same notation $T : \tilde{L}_0 \longrightarrow \tilde{L}_0$ the restriction of $T : \tilde{W} \longrightarrow \tilde{W}$ to $\tilde{L}_0$.

Theorem 8. *The endomorphism $T : \tilde{W} \longrightarrow \tilde{W}$ is the identity map.*

Proof. Since the bundle isomorphism $\phi : \tilde{L} \longrightarrow f^*Q$ preserves fiber metrics and T is Hermitian, we have

$$(v_1, v_2)_{\tilde{L}_0} = ([e, v_1], [e, v_2])_{\tilde{L}} = ([e, Tv_1], [e, Tv_2])_{\tilde{L}} = (Tv_1, Tv_2)_{\tilde{L}_0} = (T^2 v_1, v_2)_{\tilde{L}_0},$$

for any $v_1, v_2 \in \tilde{L}_0$. Therefore $T^2 : \tilde{L}_0 \longrightarrow \tilde{L}_0$ is the identity map. Since W is G-irreducible and T is G-equivariant, $T^2 : \tilde{W} \longrightarrow \tilde{W}$ is the identity map and so is T because T is semi-positive Hermitian. □

Consequently, a holomorphic strongly projectively flat G-equivariant map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is the standard map induced by a q-orthogonal direct sum bundle of Hermitian line bundle $L \to M$, which is the end of the proof of Theorem 7.

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