# Trivializing number of positive knots and diagrams of almost alternating positive links

井上, 和彦

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# Trivializing number of positive knots and diagrams of almost alternating positive links

Kazuhiko Inoue

Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

Abstract. The thesis consists of two chapters. In Chapter 1, we give a formula for the trivializing numbers of all minimal diagrams of positive 2-bridge knots and study the relationship between the trivializing number and the unknotting number for such knots. In particular, we show that for a certain class of positive 2-bridge knots, the trivializing number equals twice the unknotting number. We also give a related result for a certain class of positive pretzel knots. In Chapter 2, we show that a link which has a positive and almost alternating diagram is alternating. We also show that the Montesinos links with positive standard diagrams admit diagrams which turn into a positive and alternating diagram by one crossing change.

# Acknowledgements

The author would like to express his sincere gratitude to his supervisor Professor Osamu Saeki for the enormous support for his study, helpful suggestions and warm encouragement. Without his guidance and persistent help, this thesis would not have materialized. The author can never thank him enough. The author would also like to thank Kazuto Takao and Takayuki Okuda for useful comments and suggestions.

# Preface

The diagrams of knots and links are fundamental tools and important objects of study in Knot Theory. In this thesis, we focus on positive links and their diagrams.

In Chapter 1, we determine the trivializing numbers of all the standard diagrams of positive 2-bridge knots. The trivializing number of a diagram is defined as follows. First, we consider the associated projection, which has several double points without over/under information. We call such double points *pre-crossings*. On the other hand, the double points of a diagram have over/under information , and such double points are called *crossings*. Hanaki [**5**] introduced a new object, called a *pseudo-diagram*, which is a plane curve each of whose double points is either a pre-crossing or a crossing. Such a pseudo-diagram is said to be *trivial* if every diagram obtained by assigning arbitrary over/under information to the pre-crossings is a diagram of a trivial knot. Then, the *trivializing number* of a projection is the minimum number of pre-crossings to which over/under information should be assigned to get a trivial pseudo-diagram. The *trivializing number* of a diagram is that of the associated projection. One of the motivations for studying the trivializing number is the existence of DNA-knots, which can be observed by electron microscopes: the images do not have high resolution and often the over/under information of some double points is missing. Moreover, we give some results about the relationship between the trivializing number and the unknotting number. In fact, it is known that twice the unknotting number is smaller than or equal to the trivializing number, and Hanaki [**5**] conjectures that the equality holds for positive knots. We prove this conjecture affirmatively for some positive 2-bridge knots and pretzel knots.

In Chapter 2, we consider positive and almost alternating links. In particular, we consider their diagrams, and show that a link which has a positive and almost alternating diagram is alternating. Furthermore, we introduce a notion of an "almost PA-link". A diagram which is positive and alternating is called a *PA-diagram* and a link which admits such a diagram is called a *PA-link*. A diagram which turns into a PA-diagram by one crossing change is called an *almost PA-diagram*. A link is an *almost PA-link* if it admits an almost PA-diagram but does not admit a PA-diagram. Then, we show that Montesinos links with positive standard diagrams are either PA-links or almost PA-links.

Throughout the thesis, we work in the PL category. All knots and links are oriented, and all diagrams are considered on $S^2$. Note that a knots consists of one component, while a link consists of one or more components.

# Contents

CHAPTER 1

# Trivializing number of positive knots

## 1. Introduction

The trivializing number is a numerical invariant of knots, similar to the unknotting number, which measures certain complexity of a knot. In general, it is known that the trivializing number is greater than or equal to twice the unknotting number in general. Furthermore, Hanaki [**5**] has conjectured that the trivializing number of a positive knot always coincides with twice the unknotting number. In fact, Hanaki showed that for every positive knot $K$ up to 10 crossings, the equality $\mathrm{tr}(K) = 2u(K)$ holds, where $\mathrm{tr}(K)$ is the trivializing number of $K$ and $u(K)$ is the unknotting number of $K$. Our result in this chapter will give a partial positive answer to the conjecture.

Chapter 1 is organized as follows. In Section 2, we define the trivializing number of a diagram and the trivializing number of a knot. In Section 3, we shortly review the definitions and elementary properties of positive knots, 2-bridge knots, and their diagrams. In Section 4, we determine the standard diagrams of positive 2-bridge knots. In Section 5, we determine the trivializing numbers of standard (and hence, minimal) diagrams of positive 2-bridge knots. In Section 6, we show that for some positive 2-bridge knots $K$, the equality $\mathrm{tr}(K) = 2u(K)$ holds. In Section 7, we consider positive pretzel knots, and show that the equality above also holds for some of them.

## 2. Preliminaries

A *projection* of a knot $K$ in $\mathbb{R}^3$ is a regular projection image of $K$ in $\mathbb{R}^2 \cup \{\infty\} = S^2$. A *diagram* of $K$ is a projection endowed with over/under information for its double points. A *crossing* is a double point with over/under information, and a *pre-crossing* is a double point without over/under information. A *pseudo-diagram* of $K$ is a projection of $K$ whose double points are either crossings or pre-crossings. See Figure 1 for some examples.

A pseudo-diagram is said to be *trivial* if we always get a diagram of a trivial knot after giving arbitrary over/under information to all the pre-crossings. An example is given in Figure 2. It is known that we can change every projection into a trivial pseudo-diagram by giving appropriate over/under information to some of the pre-crossings.

The trivializing number has been defined by Hanaki [**5**] as follows.

**Definition 1.2.1.** The *trivializing number* of a projection $P$, denoted by $\mathrm{tr}(P)$, is the minimal number of pre-crossings of $P$ which should be transformed into crossings for getting a trivial pseudo-diagram.

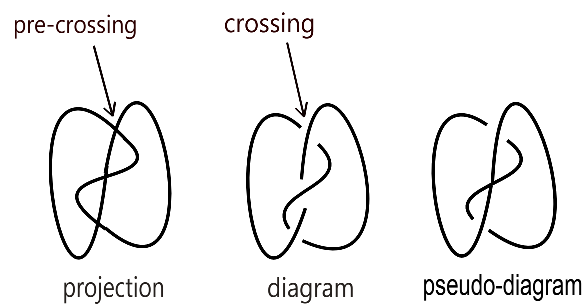



FIGURE 1. Projection, diagram, and pseudo-diagram

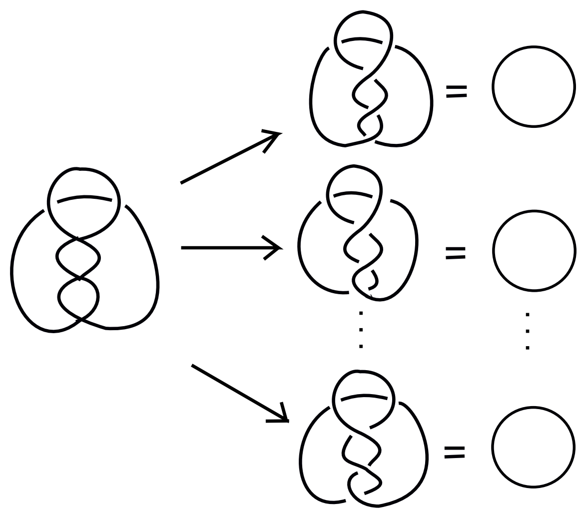


FIGURE 2. Example of a trivial pseudo-diagram

**Definition 1.2.2.** The *trivializing number* of a diagram $D$, denoted by $\mathrm{tr}(D)$, is by definition the trivializing number of the associated projection which is obtained from $D$ by ignoring the over/under information.

For example, for the diagram $D$ as shown in Figure 3, we get a trivial pseudo-diagram by transforming two pre-crossings $c_1$ and $c_2$ of the associated projection $P$ into crossings. Furthermore, it can be easily checked that we cannot get a trivial pseudo-diagram by transforming only one pre-crossing of $P$ into a crossing. Therefore, we have $\mathrm{tr}(D) = \mathrm{tr}(P) = 2$.

**Definition 1.2.3.** The *trivializing number* of a knot $K$, denoted by $\mathrm{tr}(K)$, is the minimum of $\mathrm{tr}(D)$, where the minimum is taken over all diagrams $D$ of $K$.

## 3. Positive 2-bridge knots

In general, the trivializing number of a knot is not always realized by its minimal diagram (a diagram that has the minimum number of crossings); in fact, we have counter examples (see [**5**]). Moreover, even for a given diagram, determining its

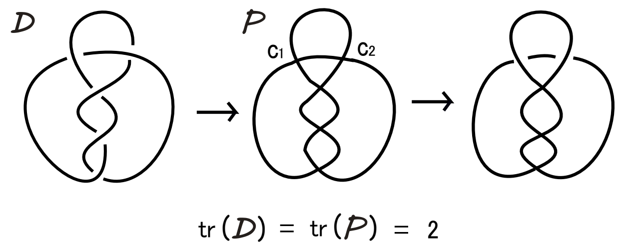



FIGURE 3. An operation for getting a trivial pseudo-diagram

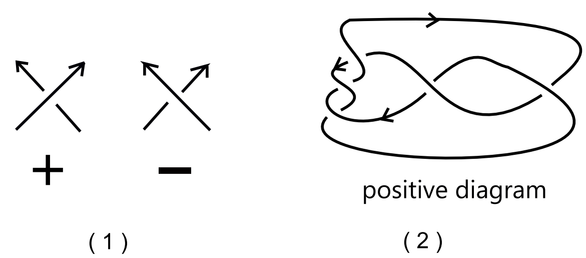



FIGURE 4. Sign of a crossing and an example of a positive diagram

trivializing number is not so easy in general. In Section 5, we give the trivializing numbers of all minimal diagrams of positive 2-bridge knots.

Let $D$ be an oriented diagram of a link. To each of its crossings, we associate sign $+$ or $-$ as shown in Figure 4 (1). If all the crossings in $D$ have the same sign $+$ (resp. $-$), then we say that $D$ is a *positive diagram* (resp. *negative diagram*).

When $D$ is a positive diagram, the mirror image of $D$, which is obtained by changing the over/under information of all crossings of $D$ and is denoted by $D^*$, is a negative diagram. Since $D$ and $D^*$ correspond to the same projection, we have $\mathrm{tr}(D) = \mathrm{tr}(D^*)$. A *positive link* is a link which has a positive diagram.

For a finite sequence $a_1, a_2, \ldots, a_m$ of integers, let us consider the link diagram $D(a_1, a_2, \ldots, a_m)$ as shown in Figure 5. In the figure, a rectangle in the upper row (resp. lower row), depicted by double lines (resp. simple lines), with integer $a$ represents $a$ left hand (resp. right hand) horizontal half-twists if $a \geq 0$, and $|a|$ right hand (resp. left hand) horizontal half-twists if $a < 0$. See Figure 6 for some explicit examples. We call a rectangle in the upper row (resp. lower row) an upper rectangle (resp. a lower rectangle) for short. A knot (or link) which is represented by such a diagram is called a 2-*bridge knot* (resp. 2-*bridge link*).

Note that in the case of links, the signs of crosings in $D$ may change if the orientation of a component is reversed, while in the case of knots, they do not change even if the orientation is reversed. Therefore, we choose an orientation and fix it for the orientation of $D$.

If $a_i > 0$ for all $i$ with $1 \leq i \leq m$ or if $a_i < 0$ for all $i$, then the diagram $D(a_1, a_2, \ldots, a_m)$ is reduced and alternating, and hence is a minimal diagram (see [**10**]). We call such a diagram a *standard diagram* of the 2-bridge knot or link.

It is known that every 2-bridge link has a unique standard diagram (see, for example, [**10**]). Therefore, a positive (resp. negative) 2-bridge link is a positive (resp. negative) alternating link. A positive alternating link may not have a diagram

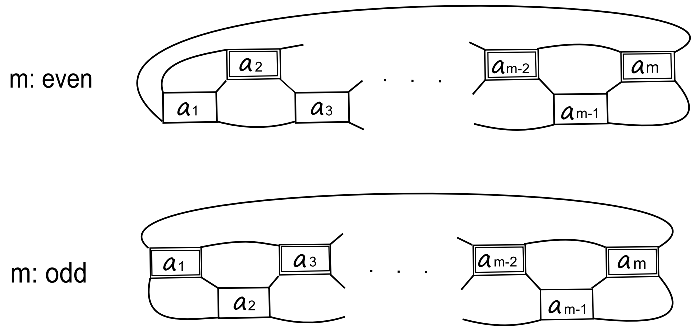



Figure 5. 2-bridge link diagrams

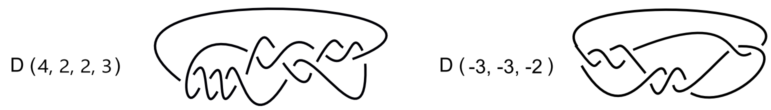



Figure 6. Examples of 2-bridge knot diagrams

which is both positive and alternating in general. However, Nakamura has shown the following.

**Theorem 1.3.1** (Nakamura [**11**]). *A reduced alternating diagram of a positive alternating link is positive.*

By the theorem above, the standard diagram of a positive (or negative) 2-bridge link is necessarily positive (resp. negative).

In order to study the trivializing number of the standard diagram $D$ of a positive or negative 2-bridge knot, by taking the mirror image, we may assume $a_i > 0$ for all $i$. Note that a positive knot may turn into a negative one by this operation.

## 4. Standard diagrams of positive 2-bridge knots

In this section, we determine the standard diagrams of positive 2-bridge knots.

**Proposition 1.4.1.** *Let $D = D(a_1, a_2, \ldots, a_m)$ be a standard diagram of a 2-bridge link such that $a_i > 0$ for all $i$ with $1 \leq i \leq m$. Then $D$ is a positive knot diagram or a negative knot diagram if and only if one of the following holds.*

(1) *When $m$ is even, say $m = 2n$, we have either*
- (a1) $a_{2i}$ *is even,* $1 \leq \forall i \leq n-1$,
- (a2) $a_{2n}$ *is odd, and*
- (a3) $\sum_{i=1}^{n} a_{2i-1}$ *is even,*

*or*
- (b1) $a_1$ *is odd,*
- (b2) $a_{2i-1}$ *is even,* $2 \leq \forall i \leq n$, *and*
- (b3) $\sum_{i=1}^{n} a_{2i}$ *is even.*

(2) *When $m$ is odd, say $m = 2n+1$, we have either*
- (c1) $a_{2i-1}$ *is even,* $2 \leq \forall i \leq n$,
- (c2) $a_1$ *and* $a_{2n+1}$ *are odd, and*
- (c3) $\sum_{i=1}^{n} a_{2i}$ *is odd,*

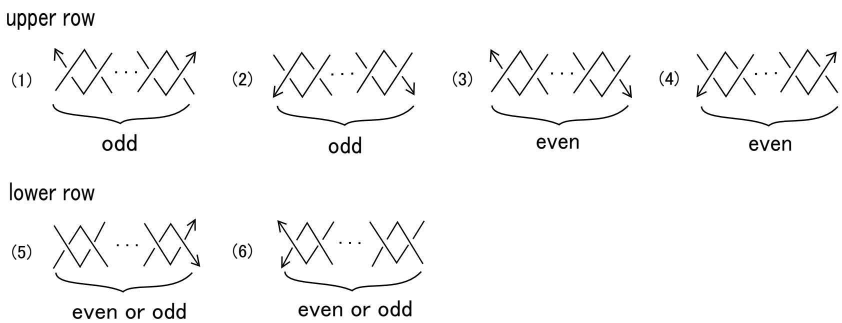



FIGURE 7. Orientations of two arcs with positive crossings

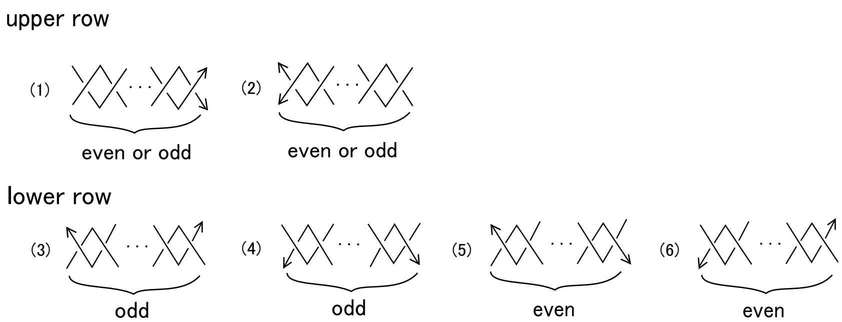



FIGURE 8. Orientations of two arcs with negative crossings

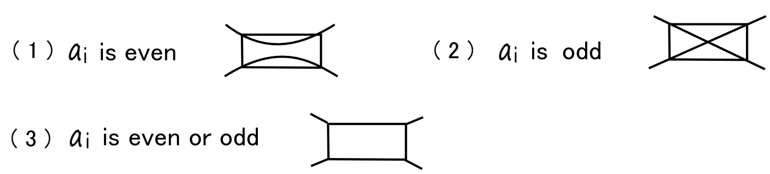



FIGURE 9. Symbolic conventions for depicting $(a_i)$

*or*
(d1) $a_{2i}$ *is even,* $1 \leq \forall i \leq n$*, and*
(d2) $\sum_{i=1}^{n+1} a_{2i-1}$ *is odd.*

Let us consider a rectangle with integer $a_i > 0$, as appearing in Figure 5, which corresponds to $a_i$ left hand (resp. right hand) half-twists if it is in the upper (resp. lower) row. In the following, such a rectangle will sometimes be denoted by $(a_i)$. If its crossings all have the same sign $+$, then the orientation of the two arcs are of a form as in Figure 7. If the crossings all have sign $-$, then they are of a form as in Figure 8. Furthermore, we adopt the symbolic convention as depicted in Figure 9.

These four forms are as shown in Figure 10 (a), (b), (c) and (d), up to orientation.

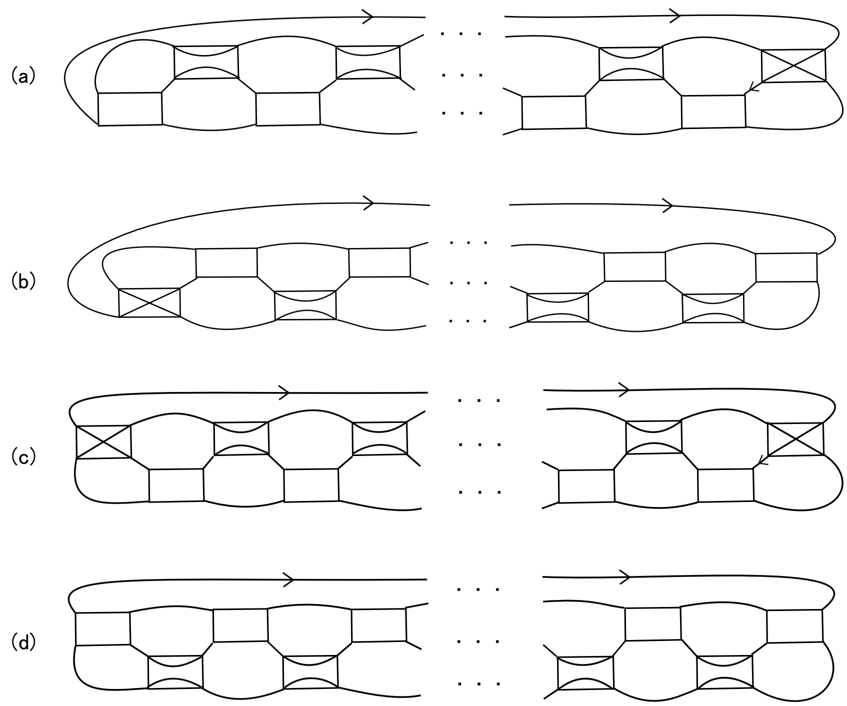



FIGURE 10. Four forms for $D$

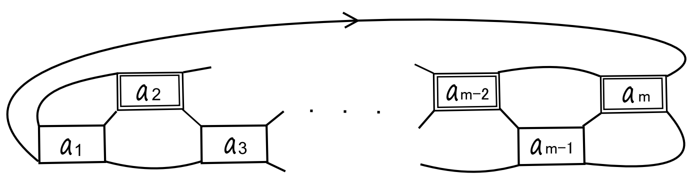



FIGURE 11. Orientation of diagram $D = D(a_1, a_2, \ldots, a_m)$ with $m = 2n$

PROOF OF PROPOSITION 1.4.1. (1) Suppose that $D$ is a positive or negative knot diagram. We may assume that the orientation of the diagram is as depicted in Figure 11, since it is a diagram of a knot.

When the crossings in $(a_i)$ with $1 \leq i \leq 2n$ all have the same sign $+$, the orientations of the two arcs of $(a_{2n})$ are as shown in Figure 12 (1) or (2). Then the orientation of the diagram is as shown in Figure 12 (4). Since the orientations of the arcs of $(a_{2n-1})$ must be as shown in Figure 12 (3), the orientations of the arcs of $(a_{2n})$ must be as in Figure 12 (1). In particular, $a_{2n}$ is necessarily odd. Due to the orientation of $(a_{2n-1})$, we see that the orientations of the other $(a_{2i-1})$, $1 \leq i \leq n$, are of the form as in Figure 12 (3). Furthermore, by chasing the oriented arcs, we can determine the orientations of all arcs in the remaining rectangles. Hence, the oriented diagram is as depicted in Figure 13.

Then, we see that $a_{2i}$, $1 \leq i \leq n-1$, are all even. Since this is a knot diagram, the oriented strand passing through $x$ and then $y$ in Figure 13 needs to pass through

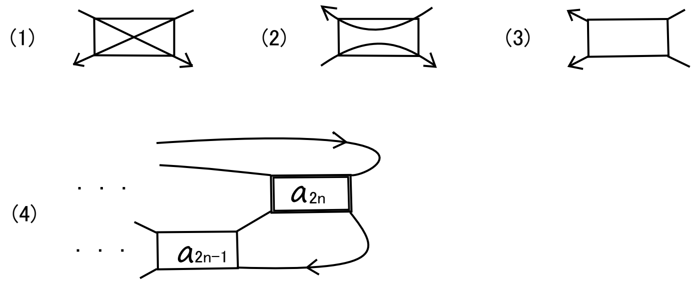



FIGURE 12. Orientations of arcs in rectangles

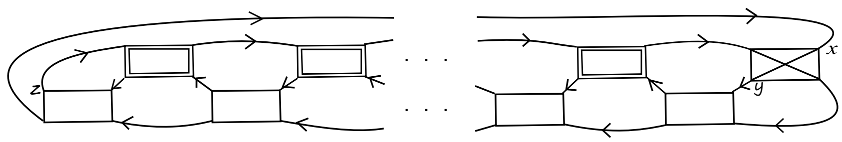



FIGURE 13. Orientations of arcs in $D$

$z$. Therefore, $\sum_{i=1}^{n} a_{2i-1}$ must necessarily be even. This shows that the conditions (a1), (a2) and (a3) are satisfied.

On the other hand, when the signs of crossings in $(a_i)$ are all $-$, by rotating the diagram as shown in Figure 14 (1) on the plane by 180 degrees, we get the diagram as in Figure 14 (2). Then, by isotoping the bottom arc on $S^2 = \mathbb{R} \cup \{\infty\}$, we get the diagram as in Figure 14 (3). Then, by reversing the orientaitions of all arcs, we get the diagram as in Figure 14 (4). Finally, by applying the crossing change to all the crossings, we get the positive diagram as shown in Figure 14 (5), which is of the form treated in the proof above. Consequently, we see that conditions (b1), (b2) and (b3) are satisfied.

(2) Let us now consider the case with $m = 2n + 1$. We may assume that the orientation of the diagram is as shown in Figure 15.

Let us first consider the case where all crossings in $(a_i)$ with $1 \leq i \leq 2n+1$, have the same sign $+$. Since the orientations of the two arcs of $(a_{2n+1})$ are as shown in Figure 16 (1) or (2), the orientation of the diagram must be as in Figure 16 (4). Furthermore, since the orientations of the arcs of $(a_{2n})$ are as shown in Figure 16 (3), the orientations of the arcs of $(a_{2n+1})$ must be as shown in Figure 16 (1). In particular, $a_{2n+1}$ is necessarily odd. Due to the orientation of $(a_{2n})$, we see that the orientations of the other $(a_{2i})$, $1 \leq i \leq n-1$, are of the form as in Figure 16 (3).

Hence, the oriented diagram must be as depicted in Figure 17, since this is a diagram of a knot. This shows that conditions (c1), (c2) and (c3) are satisfied.

If the signs of the crossings in $(a_i)$ are all negative, then the orientation of $(a_{2n+1})$ is as shown in Figure 18 (1). Therefore, the orientations of the other $(a_{2i+1})$, $1 \leq i \leq n-1$, must be of the same form as shown in Figure 18 (2). Thus the diagram must be of the form as shown in Figure 18 (3), and we can easily see that conditions (d1) and (d2) are satisfied.

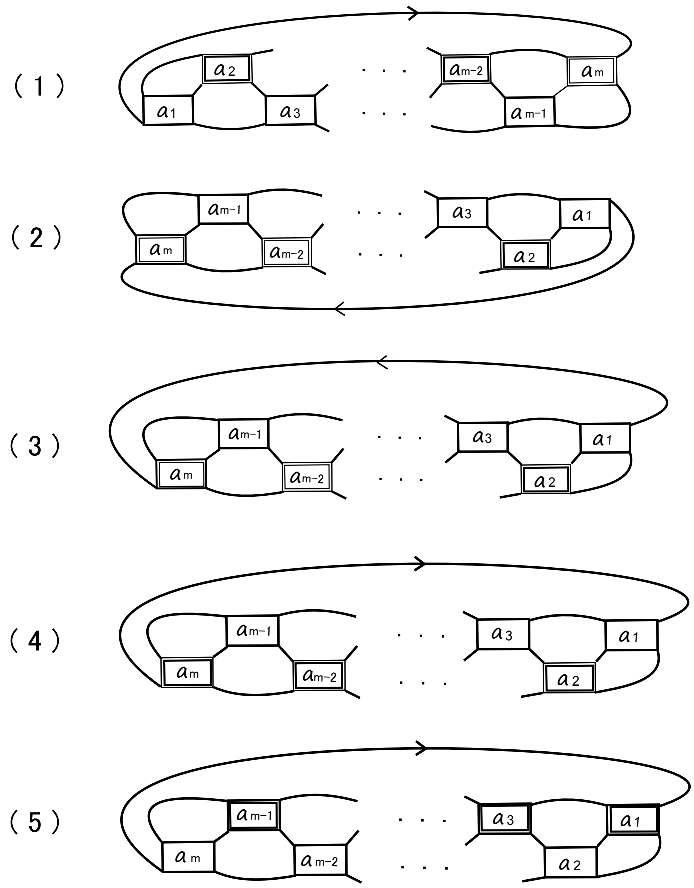



FIGURE 14. The transformation of $D$ with all signs $-$

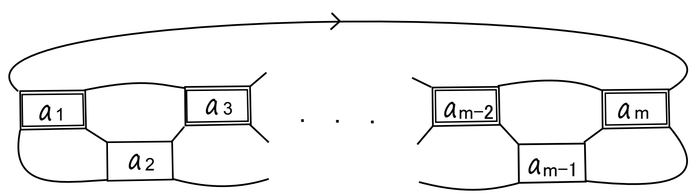



FIGURE 15. Orientation of diagram $D = D(a_1, a_2, \ldots, a_m)$ with $m = 2n + 1$

Finally, we see easily that if $D$ is of one of the four forms (a), (b), (c) and (d), then $D$ is a positive knot diagram or a negative knot diagram. This completes the proof. □

In the following, we refer to the diagram as in Proposition 1.4.1 (1) (a1)–(a3) as *type A*. Simillarly, we refer to the diagram as in (1) (b1)–(b3) as *type A'*, that in (2) (c1)–(c3) as *type B*, and that in (2) (d1)–(d2) as *type B'*.

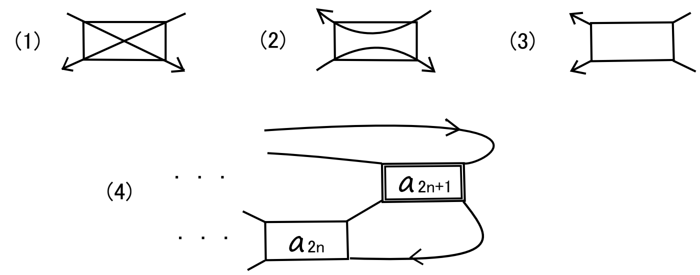



FIGURE 16. Orientations of arcs in rectangles

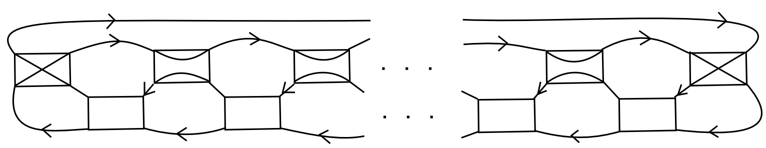


FIGURE 17. Orientations of arcs in $D = D(a_1, a_2, \ldots, a_m)$ with $m = 2n + 1$

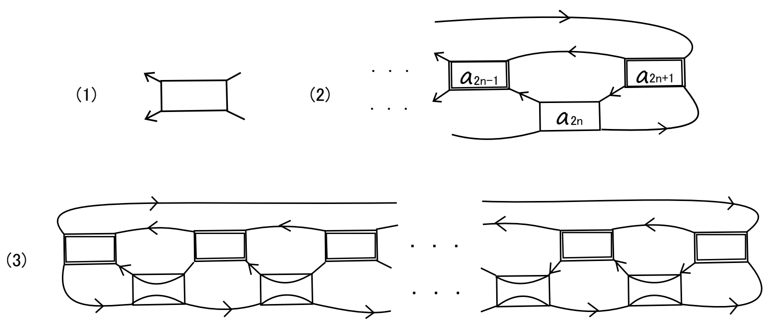



FIGURE 18. Orientations of arcs in $D = D(a_1, a_2, \ldots, a_m)$ with $m = 2n + 1$

## 5. Main theorem

For determining the trivializing number of a diagram, we can use chord diagrams. Let $P$ be a projection of a knot with $n$ pre-crossings. We may regard $P$ as the image of an immersion $\varphi : S^1 \to S^2$ with normal crossings, and each double point corresponds to a pre-crossing of $P$. A *chord diagram* of $P$, denoted by $CD_P$, is a circle with $n$ chords, depicted by dotted line segments or solid line segments, where the pair of points in the pre-image by $\varphi$ of each pre-crossing are connected by a chord (see [**5**]). Furthermore, a chord diagram of a diagram $D$ is by definition the chord diagram of the associated projection which is obtained from $D$. We give an example of the chord diagram of a knot projection in Figure 19.

A chord diagram is said to be *parallel* if the chords have no intersection. For example, the chord diagram in Figure 20 is parallel.

The following is known.

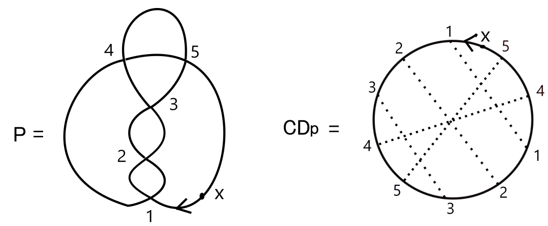



FIGURE 19. A chord diagram

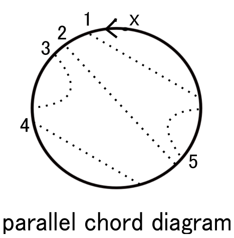



FIGURE 20. A parallel chord diagram

**Theorem 1.5.1** (Hanaki [**5**]). *Let $P$ be a knot projection. Then, tr$(P)$ is equal to the minimum number of chords which must be deleted from $CD_P$ for getting a parallel chord diagram. Moreover, tr$(P)$ is always even.*

A *sub-chord diagram* is a partial chord diagram correponding to a subset of a projection. For a diagram $D = D(a_1, a_2, \ldots, a_m)$, we consider the sub-chord diagram correponding to each $(a_i)$ by regarding $(a_i)$ as a subset of a diagram.

**Lemma 1.5.2.** *Let $D = D(a_1.a_2, \ldots, a_m)$ be the standard diagram of a 2-bridge knot $K$. Let $SC_{a_i}$ be the sub-chord diagram corresponding to the diagram $(a_i)$ in $D$.*

(1) *If the two arcs of $(a_i)$ enter from the same side as shown in Figure 21* (1) *or* (2)*, then every pair of distinct chords in $SC_{a_i}$ intersect each other as shown in Figure 21* (7)*.*
(2) *If the two arcs of $(a_i)$ enter from opposite sides as shown in Figure 21* (3)*,* (4) *when $a_i$ is odd and* (5)*,* (6) *when $a_i$ is even, then there are no intersections in $SC_{a_i}$ as shown in Figure 21* (8)*. In other words, $SC_{a_i}$ is parallel.*

PROOF. Let $c_1, c_2, \ldots, c_k$ be the crossings in the diagram $(a_i)$, situated from left to right in this order.

If an arc enters from the left-hand side (resp. from the right-hand side), then it passes through the crossings $c_1, c_2, \ldots, c_k$ (resp. $c_k, c_{k-1}, \ldots, c_1$) in this order. Since the other arc enters from the left-hand side (resp. from the right-hand side) again, it passes through the crossings in the same order. Therefore, the sub-chord diagram corresponding to $(a_i)$ must be as shown in Figure 22 (1).

Similarly, we see easily that when the arcs enter from opposite sides of $(a_i)$, the sub-chord diagram corresponding to $(a_i)$ must be as shown in Figure 22 (2). □

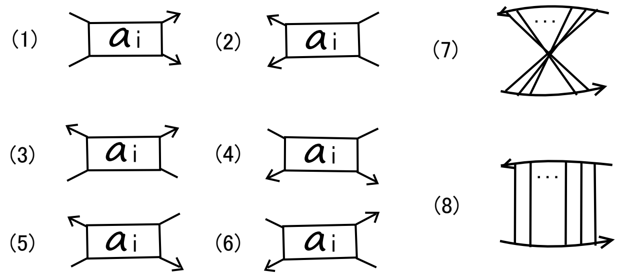



FIGURE 21. Orientations of two arcs in $(a_i)$ and the sub-chord diagram corresponding to $(a_i)$

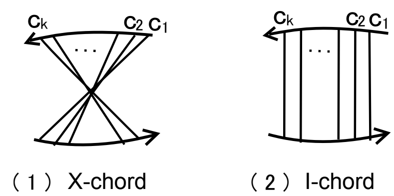



FIGURE 22. Sub-chord diagram corresponding to $(a_i)$

By the above lemma, we may regard the sub-chord diagram corresponding to $(a_i)$ as a "bunch" of chords. In other words, we can gather all chords in the sub-chord diagram corresponding to $(a_i)$ into one chord group denoted by $\overline{a}_i$. Furthermore, in the case of Lemma 1.5.2 (1), we call the chord group an *X-chord*, which we depict by a dotted line, while in the case of Lemma 1.5.2 (2) we call the chord group a *I-chord*, which we depict by a solid line (see Figure 22).

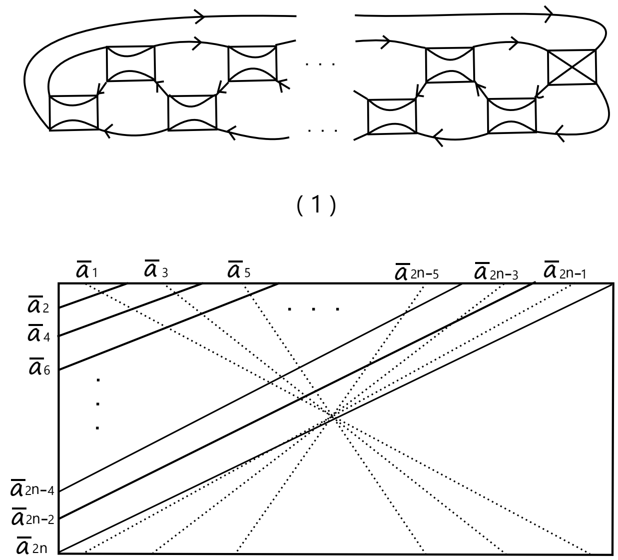
Let us now consider the chord diagram $CD_P$ corresponding to the positive (or negative) standard diagram $D$ of a 2-bridge knot $K$. We will use Theorem 1.5.1 to determine the trivializing number $tr(D)$ of the diagram $D$.

(1) The case of type A; When $D$ is of type A of Proposition 1.4.1, by considering the orientations of arcs in each rectangle, we see that $\overline{a}_{2i}$ for all $i$ with $1 \leq i \leq n$, is an I-chord, and $\overline{a}_{2i-1}$ for all $i$ with$1 \leq i \leq n$, is an X-chord. If every $a_{2i-1}$, $1 \leq i \leq n$, is even, then we obtain the diagram as shown in Figure 23 (1), and the associated chord diagram as in Figure 23 (2). (For convenience, we often represent a chord diagram not by a circle, but by a quadrangle.)

In general, the order of the rectangles that we pass through depends on whether $a_{2i-1}$ are odd or even. Let us rename the lower rectangles which consist of an odd number of half-twists (the number of such rectangles is even), as $(b_1), (b_2), \ldots, (b_{2r})$ from left to right. Moreover, we also rename all the upper rectangles as follows:

- the upper rectangles on the left hand side of $(b_1)$; $(c_0^1), (c_0^2), \ldots, (c_0^{q_0})$,
- the upper rectangles between $(b_j)$ and $(b_{j+1})$; $(c_j^1), (c_j^2), \ldots, (c_j^{q_j})$,
- the upper rectangles on the right hand side of $(b_{2r})$; $(c_{2r}^1), (c_{2r}^2), \ldots, (c_{2r}^{q_{2r}})$,

always from left to right.




(2)

FIGURE 23. Diagram $D$ of type A with every $a_{2i-1}$ even, and the associated chord diagram

Then we can obtain the sub-chord diagram corresponding to the rectangles between $(b_j)$ and $(b_{j+1})$ as shown in Figure 24, where $\overline{c_j^k}$ $(1 \leq k \leq q_j)$ consists of some parallel chords which correspond to the crossings in the rectangle $(c_j^k)$.

Furthermore, any two I-chords in $\overline{c_j^1}, \overline{c_j^2}, \ldots, \overline{c_j^{q_j}}$ do not cross each other, so we can bundle them again into one I-chord. Now we represent them by a solid line. In other words, we consider that $c_j = c_j^1 + c_j^2 + \cdots + c_j^{q_j}$. Since the chord groups which correspond to the lower rectangles between $(b_j)$ and $(b_{j+1})$ are all X-chords, in the situation of Theorem 1.5.1, the number of chords which we can leave is at most one. Furthermore, among the X-chords in Figure 24(1),we have only $\overline{b}_{j+1}$ that does not intersect $\overline{c}_j$: in Figure 24 (2), we have only $\overline{b}_j$. Note that all the chords which are out of this sub-chord diagram cross either all the X-chords between $\overline{b}_j$ and $\overline{b}_{j+1}$ or no X-chord between $\overline{b}_j$ and $\overline{b}_{j+1}$. So we have only to consider the chord diagram in which all the X-chords between $\overline{b}_{j+1}$ and $\overline{b}_j$ are delated as shown in Figure 25.

(2)The case of type A'; In the case of a diagram of type A', we can proceed in a similar fashion. When every $a_{2i}$ is even, the diagram is as shown in Figure 26 (1), and the associated chord diagram is as shown in Figure 26 (2). Otherwise, the sub-chord diagram between $(b_j)$ and $(b_{j+1})$ is also as shown in Figure 24 (1) or (2), and we get the chord diagram as shown in Figure 27.

(3)The case of type B or B'; On the other hand, in the case of a diagrams of type B or type B', when $j$ is even, the sub-chord diagram is as shown in Figure 24 (1), and when $j$ is odd, the sub-chord diagram is as shown in Figure 24 (2). Thus, we get the chord diagrams as shown in Figure 28 and 29.

Using the above notation, we can state the main theorem of this section as follows.

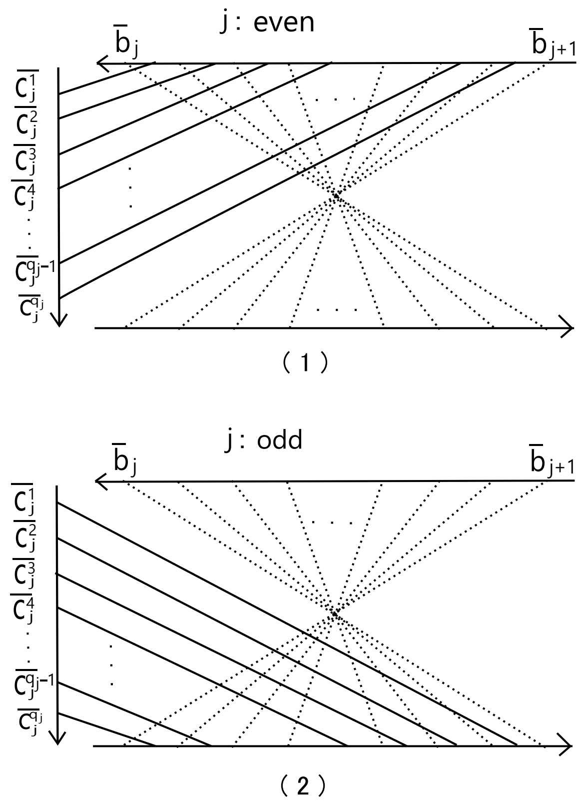



FIGURE 24. Sub-chord diagram corresponding to the part between $(b_j)$ and $(b_{j+1})$

**Theorem 1.5.3.** *Let $D = D(a_1, a_2, \ldots, a_m)$ be a positive diagram or a negative diagram of a 2-bridge knot such that $a_i > 0$ for all $i$ with $1 \leq i \leq m$. Then the trivializing number* $\mathrm{tr}(D)$ *of $D$ is given by the following formulas.*

(1) *When $D$ is of type* A $(m = 2n)$.
  (a) If every $a_{2i-1}$ is even, then $\mathrm{tr}(D) = \sum_{i=1}^{n} a_{2i-1}$.
  (b) Otherwise,

$$\mathrm{tr}(D) = \min\{A_0, A_1, \ldots, A_r\},$$

  where

$$A_r = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{r} c_{2j-1},$$

$$A_0 = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{r} c_{2j} - 1.$$

  and for $p = 1, 2, \ldots, r-1$, we set

$$A_p = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{p} c_{2j-1} + \sum_{j=p+1}^{r} c_{2j} - 1.$$

(2) When $D$ is of type A' $(m = 2n)$.

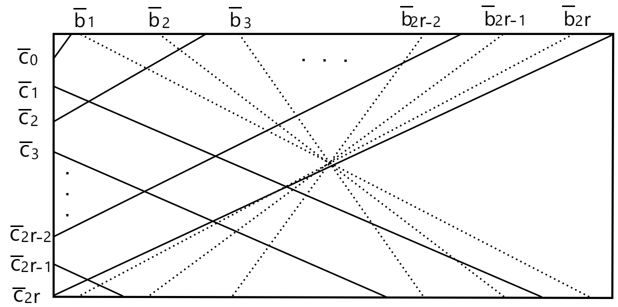



FIGURE 25. The chord diagram corresponding to a diagram of type A

(a) If every $a_{2i}$ is even, then $\mathrm{tr}(D) = \sum_{i=1}^{n} a_{2i}$.
(b) Otherwise,

$$\mathrm{tr}(D) = \min\{A'_0, A'_1, \dots, A'_r\},$$

where

$$A'_r = \sum_{i=1}^{n} a_{2i} + \sum_{j=1}^{r} c_{2j-1},$$

and for $p = 0, 1, \dots, r-1$, we set

$$A'_p = \sum_{i=1}^{n} a_{2i} + \sum_{j=0}^{p} c_{2j} + \sum_{j=p+2}^{r} c_{2j-1} - 1.$$

(3) When $D$ is of type B ($m = 2n+1$).

$$\mathrm{tr}(D) = \min\{B_0, B_1, \dots, B_{r+1}\},$$

where

$$B_{r+1} = \sum_{i=1}^{n} a_{2i} + \sum_{j=0}^{r} c_{2j+1},$$

and for $p = 0, 1, \dots, r$, we set

$$B_p = \sum_{i=1}^{n} a_{2i} + \sum_{j=0}^{p} c_{2j} + \sum_{j=p+1}^{r} c_{2j+1}.$$

(4) When $D$ is of type B' ($m = 2n+1$).

$$\mathrm{tr}(D) = \min\{B'_0, B'_1, \dots, B'_r\},$$

where

$$B'_r = \sum_{i=0}^{n} a_{2i+1} + \sum_{j=1}^{r} c_{2j} - 1$$

and for $p = 0, 1, \dots, r-1$, we set

$$B'_p = \sum_{i=0}^{n} a_{2i+1} + \sum_{j=0}^{p} c_{2j+1} + \sum_{j=p+2}^{r} c_{2j} - 1.$$

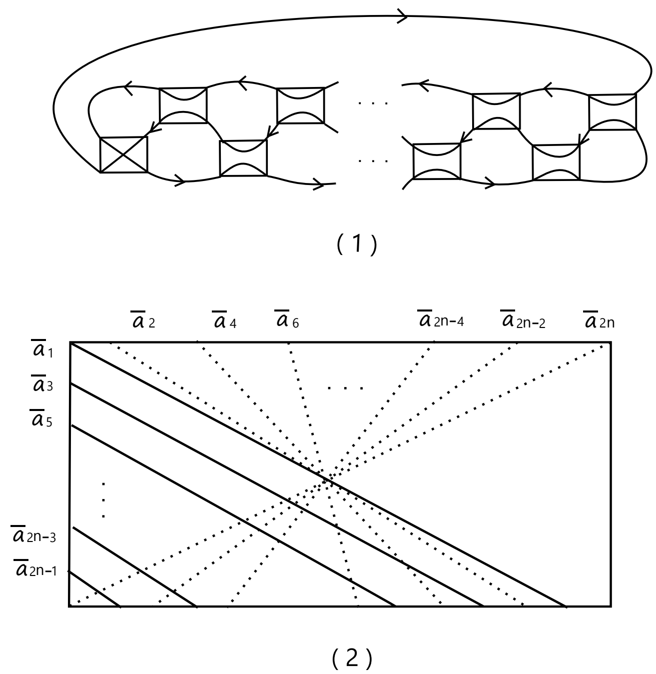



FIGURE 26. Diagram $D$ of type A' with every $a_{2i}$ even, and the associated chord diagram

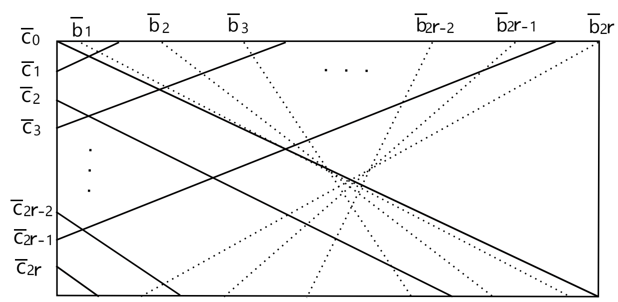



FIGURE 27. The chord diagram corresponding to a diagram of type A'

PROOF. 1) In the case where $D$ is of type A.

If every $a_{2i-1}$ is even, then we have the chord diagram as shown in Figure 23 (2). Since any two X-chords in this chord diagram cross each other, we can leave at most one X-chord when we attempt to gain a parallel chord diagram. Moreover, every two chords in any X-chord also cross each other. This means that the number of the chords corresponding to the crossings in lower rectangles which we can leave is at most one.

In addition, the I-chords corresponding to the crossings in upper rectangles are all parallel and any I-chord crosses at least one X-chord. Hence, the minimal number of the chords which we must delate in order to get a parallel chord diagram

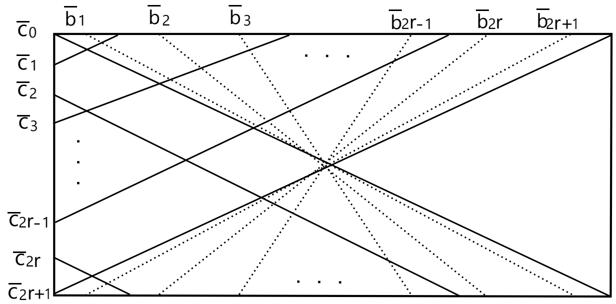



FIGURE 28. The chord diagram corresponding to a diagram of type B

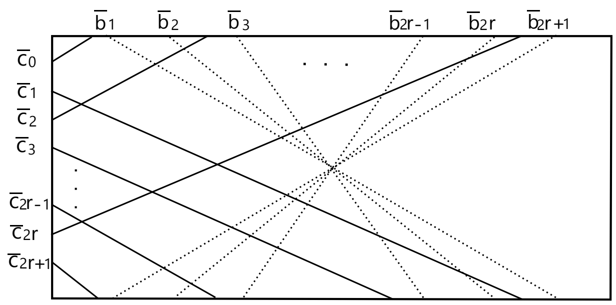



FIGURE 29. The chord diagram corresponding to a diagram of type B'

is the number of all the chords which correspond to the crossings in lower rectangles. Therefore, we have the following.

$\mathrm{tr(D)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{a_{2i-1}}$

Otherwise, the chord diagram is as shown in Figure 25, and we can see the I-chord represented by $\overline{c}_{2r}$ crosses all I-chords represented by $\overline{c}_{2j-1}$ $(1 \leq j \leq r)$ and all X-chords represented by $\overline{b}_k$ $(1 \leq k \leq 2r)$. (This means that $\overline{c}_{2r}$ crosses all chords corresponding to the crossings in lower rectangles.) So if we leave $\overline{c}_{2r}$, then we must delete all these chords which cross $\overline{c}_{2r}$. That is to say, the number of chords we must delete is

$$A_r = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{r} c_{2j-1}$$

(see Figure 30 (1)).

When we delete $\overline{c}_{2r}$, we can leave all chords in I-chord $\overline{c}_{2r-1}$ and only one chord in X-chord $\overline{b}_{2r-1}$. So the number of chords we need to delete is

$$A_{r-1} = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{r-1} c_{2j-1} + c_{2r} - 1$$

(see Figure 30 (2)).

Next we attempt to delete the I-chords which correspond to $\overline{c}_{2j}$, $(1 \leq j \leq r)$, step by step in the way as following: $\overline{c}_{2r} \to \overline{c}_{2r}, \overline{c}_{2r-2} \to \overline{c}_{2r}, \overline{c}_{2r-2}, \overline{c}_{2r-4} \to \cdots \to \overline{c}_{2r}, \overline{c}_{2r-2}, \cdots, \overline{c}_2$.

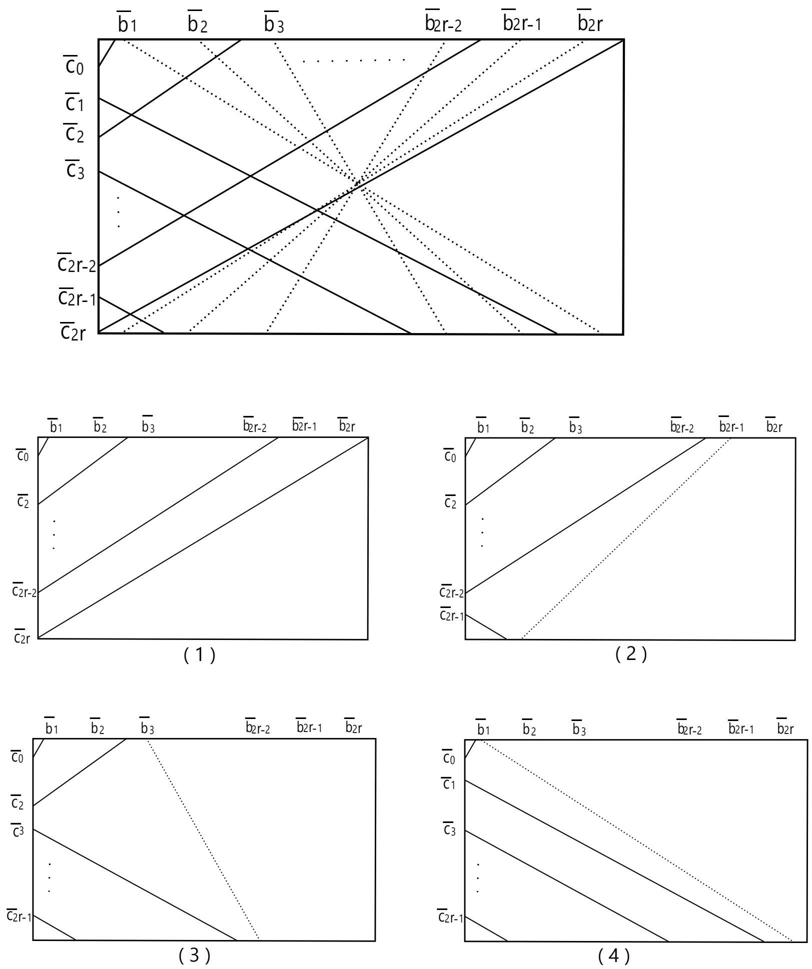



FIGURE 30. The operation of deleting some I-chords

By these operations we can also get a trivial chord diagram even if we leave the I-chords which correspond to $\overline{c}_{2j-1}$, $(1 \leq j \leq r)$, step by step in the way as following: $\overline{c}_{2r-1} \to \overline{c}_{2r-1}, \overline{c}_{2r-3} \to \overline{c}_{2r-1}, \overline{c}_{2r-3}, \overline{c}_{2r-5} \to \cdots \to \overline{c}_{2r-1}, \overline{c}_{2r-3} \cdots \overline{c}_1$.

There is an one-to-one correlation between these two operations. Consequently, the minimum of these numbers is the trivializing number of the diagram, and the following holds.

(1) If every $a_{2i-1}$ is even, then $\mathrm{tr}(D) = \sum_{i=1}^{n} a_{2i-1}$.
(2) Otherwise,

$$\mathrm{tr}(D) = \min\{A_0, A_1, \ldots, A_r\},$$

where

$$A_r = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{r} c_{2j-1},$$

$$A_0 = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{r} c_{2j} - 1.$$

and for $p = 1, 2, \ldots, r-1$, we set

$$A_p = \sum_{i=1}^{n} a_{2i-1} + \sum_{j=1}^{p} c_{2j-1} + \sum_{j=p+1}^{r} c_{2j} - 1.$$

2) In the case where $D$ is of type A'.

If every $a_{2i}$ is even, then we can consider in a similar fashion to type A and can easily see $\mathrm{tr}(D) = \sum_{i=1}^{n} a_{2i}$.

Otherwise, from the chord diagram as shown in Figure 27, we know the I-chord $\overline{c}_{2r}$ in Figure 25 is replaced by $\overline{c}_0$ in Figure 27. In this case, if we delete the I-chords represented by $\overline{c}_{2j}$, $(0 \leq j \leq s)$, step by step in the way $\overline{c}_0 \to \overline{c}_0, \overline{c}_2 \to \overline{c}_0, \overline{c}_2, \overline{c}_4 \cdots$, then we can leave $\overline{c}_1 \to \overline{c}_1, \overline{c}_3 \to \overline{c}_1, \overline{c}_3, \overline{c}_5 \cdots$, by way of compensation. Thus, the following holds.

(1) If every $a_{2i}$ is even, then $\mathrm{tr}(D) = \sum_{i=1}^{n} a_{2i}$.

(2) Otherwise,

$$\mathrm{tr}(D) = \min\{A'_0, A'_1, \ldots, A'_r\},$$

where

$$A'_r = \sum_{i=1}^{n} a_{2i} + \sum_{j=1}^{r} c_{2j-1},$$

and for $p = 0, 1, \ldots, r-1$, we set

$$A'_p = \sum_{i=1}^{n} a_{2i} + \sum_{j=0}^{p} c_{2j} + \sum_{j=p+2}^{r} c_{2j-1} - 1.$$

3) In the case where $D$ is of type B.

In this case, the chord diagram is as shown in Figure 28, and we see that every X-chord which corresponds to $(b_j)$, $(1 \leq j \leq 2r+1)$, necessarily crosses two I-chords which correspond to $(c_0)$ and $(c_{2r+1})$. So we can leave none of these X-chords unless we delete both $\overline{c}_0$ and $\overline{c}_{2r+1}$. In addition, we consider the relation of I-chords which correspond to $(c_j)$, $(1 \leq j \leq 2r+1)$. If we leave every $\overline{c}_{2j}$, $(0 \leq j \leq r)$, then we must delete every $\overline{c}_{2j+1}$, $(0 \leq j \leq r)$. Hence, the number of chords which we need to delete is the following:

$$B_{r+1} = \sum_{i=1}^{n} a_{2i} + \sum_{j=0}^{r} c_{2j+1}.$$

Furthermore, there exists a relation among the I-chords in this chord diagram. That is, if we delete $\overline{c}_0$ then we can leave $\overline{c}_1$, if we delete $\overline{c}_0, \overline{c}_2$ then we can leave $\overline{c}_1, \overline{c}_3$, and so on. Because of this, the following holds.

$$\mathrm{tr}(D) = \min\{B_0, B_1, \ldots, B_{r+1}\},$$

where

$$B_{r+1} = \sum_{i=1}^{n} a_{2i} + \sum_{j=0}^{r} c_{2j+1},$$

and for $p = 0, 1, \ldots, r$, we set

$$B_p = \sum_{i=1}^{n} a_{2i} + \sum_{j=0}^{p} c_{2j} + \sum_{j=p+1}^{r} c_{2j+1}.$$

4) In the case where $D$ is of type B'.

In this case, the chord diagram is as shown in Figure 29. In this chord diagram, $\overline{c}_0$ and $\overline{c}_{2r+1}$ dose not cross each other. Moreover, they dose not cross any other I-chord or X-chord. Therefore, we can leave both $\overline{c}_0$ and $\overline{c}_{2r+1}$. However, for X-chords $\overline{b}_j$, $(1 \leq j \leq 2r+1)$, any two of them cross each other, so we can leave at most one X-chord among $\{\overline{b}_j\}$. Thus, if we leave all I-chords corresponding to $(c_{2k+1})$, $(1 \leq k \leq r)$, we must delete all I-chords corresponding to $(c_{2k})$, $(1 \leq k \leq r)$. Hence, the number of all chords which we must delete is

$$B'_r = \sum_{i=0}^{n} a_{2i+1} + \sum_{j=0}^{r} c_{2j} - 1.$$

Besides that, if we orderly delete some I-chords step by step such as $\overline{c}_{2r} \to \overline{c}_{2r}, \overline{c}_{2r-2} \to \overline{c}_{2r}, \overline{c}_{2r-2}, \overline{c}_{2r-4} \cdots$ , we can leave other I-chords such as $\overline{c}_{2r-1} \to \overline{c}_{2r-1}, \overline{c}_{2r-3} \to \overline{c}_{2r-1}, \overline{c}_{2r-3}, \overline{c}_{2r-5} \cdots$ by way of compensation. Finally, the following holds.

$$\mathrm{tr}(D) = \min\{B'_0, B'_1, \ldots, B'_r\},$$

where

$$B'_r = \sum_{i=0}^{n} a_{2i+1} + \sum_{j=1}^{r} c_{2j} - 1$$

and for $p = 0, 1, \ldots, r-1$, we set

$$B'_p = \sum_{i=0}^{n} a_{2i+1} + \sum_{j=0}^{p} c_{2j+1} + \sum_{j=p+2}^{r} c_{2j} - 1.$$

We have just completed the proof of Theorem 1.5.3. □

## 6. Trivializing number and unknotting number

In this section, we study the relation between the trivializing number and the unknotting number. The definitions of the unknotting number of a diagram and the unknotting number of a knot are the following:

**Definition 1.6.1.** The *unknotting number* of a diagram $D$, denoted by $u(D)$, is the minimal number of crossings of $D$ whose over/under information should be changed for getting a diagram of a trivial knot.

**Definition 1.6.2.** The *unknotting number* of a knot $K$, denoted by $u(K)$, is the minimum of u($D$), where the minimum is taken over all diagrams $D$ of $K$.

There is some relationship between the unknotting number and the signature. The signature $\sigma(D)$ of a diagram $D$ is defined by using the Seifert matrix M of $D$. Let A be the matrix such that $A = (M + M^T)$, where M is the Seifert matrix of $D$ and $\mathrm{M}^{\mathrm{T}}$ is the transposed matrix of M, and let B be a diagonal matrix obtained from A. Then $\sigma(D) :=$ (the number of positive elements in the diagonal elements of B) $-$ (the number of positive elements in the diagonal elements of B). (About

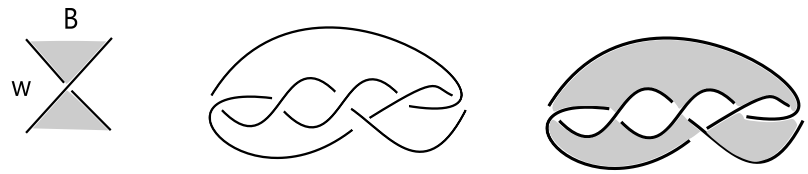



FIGURE 31. An example of the checkerboard coloring and local writhes

the signature, there is a detailed explanation in [**10**]). The signature is an invariant of knots, and in general the following holds.

**Theorem 1.6.3** (Murasugi [**10**]). *Let $K$ be a knot and $D$ be a diagram of $K$. Then the following holds.*

$$\frac{1}{2}|\sigma(D)| = \frac{1}{2}|\sigma(K)| \leq u(K) \leq u(D),$$

*where $\sigma(K)$ is the signature of $K$.*

In addition, for a alternating diagram, it is known ([**14**]) that

$$\sigma(D) = -w(D)/2 + (W - B)/2,$$

where $w(D)$ is the sum of local writhes of all crossings, $B$ is the number of domains colored with a grayish color when we give checkerboard coloring as shown in Figure 31, and $W$ is the number of domains which are not colored. For example, in the case as shown in Figure 31, the number of $+$ crossings is 2, and that of $-$ crossings is 4, then $\sigma(D) = 2 + (-4) = -2$, and $W = 5$, $B = 3$. Therefore, we can get $\sigma(D) = \sigma(K) = -(-2)/2 + (5-3)/2 = 2$, and $|\sigma(D)|/2 = 1 \leq u(K) \leq u(D)$. In actually, we can obtain a diagram of a trivial knot with one crossing change, hence $u(D) = u(K) = 1$.

About the relation between the trivializing number and the unknotting number, it is known that $2u(D) \leq \mathrm{tr}(D)$ and $2u(K) \leq \mathrm{tr}(K)$ hold in general. However, particularly for positive knots, there exists a conjecture that $2u(K) = \mathrm{tr}(K)$ ([**5**]). And as the partial positive answer of this, we have the next corollary to Theorem 1.5.3 and Theorem 1.6.3.

**Corollary 1.6.4.** *Let $K$ be a positive 2-bridge knot and $D$ be the standard diagram of $K$ such that $D = D(a_1, a_2, \ldots, a_{2n})$ $a_i > 0$ for all $i$ with $1 \leq i \leq 2n$. If $a_{2i-1}$ is even for all $i$ with $1 \leq i \leq n$, or $a_{2i}$ is even for all $i$ with $1 \leq i \leq n$, then $2u(K) = \mathrm{tr}(K)$.*

PROOF. First we prove the case where any $a_{2i-1}$ is an even number. In this case, $D$ is a minimal diagram of $K$, so by Theorem 1.3.1, $D$ is a positive-alternating diagram. Besides that, by the Proposition 1.4.1, $a_{2n}$ must be an odd number and other $a_{2i}$ $(1 \leq i \leq n-1)$ are necessarily all even numbers. Moreover, the sign of any crossing is $+$, thereby $w(D) = \sum_{i=1}^{2n} a_i$. The checkerboard coloring is like as shown in Figure 32, and we know $W = \sum_{i=1}^{n} a_{2i} + 1$, $B = \sum_{i=1}^{n} a_{2i-1} + 1$.

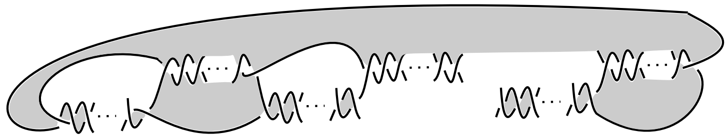


FIGURE 32. The checkerboard coloring of a 2-bridge knot diagram in which any $a_{2i-1}$ is an even number

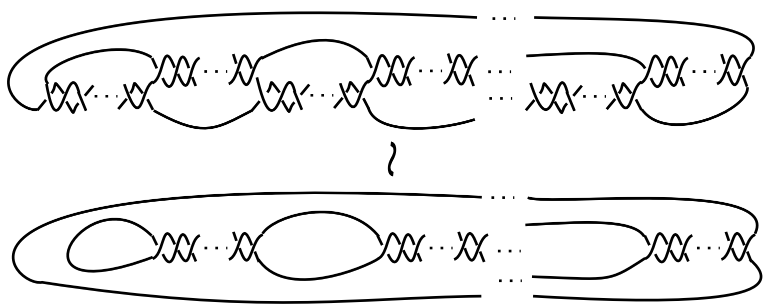


FIGURE 33. By some crossing changes, we can obtain a trivial knot diagram.

Therefore, next equality holds.

$$\sigma(D) = -\frac{1}{2}(w(D)) + \frac{1}{2}(W - B)$$
$$= \frac{1}{2}(-\sum_{i=1}^{2n} a_i + \sum_{i=1}^{n} a_{2i} - \sum_{i=1}^{n} a_{2i-1})$$
$$= -\sum_{i=1}^{n} a_{2i-1}$$

Furthermore, by Theorem 1.6.3, we can see $(|\sigma(D)|)/2 = (\sum_{i=1}^n a_{2i-1})/2 \leq u(K) = u(D)$. In actually, as shown in Figure 33, we can obtain a trivial diagram by some crossing changes of the crossings which correspondent to lower tangles, and the number of these crossing changes is $(\sum_{i=1}^n a_{2i-1})/2$. Hence, $u(D) = u(K) = (\sum_{i=1}^n a_{2i-1})/2$. Finally, we can get the inequality $2u(K) \leq \mathrm{tr}(K) \leq \mathrm{tr}(D)$ and the equality $2u(K) = \mathrm{tr}(D)$. Thus, $2u(K) = \mathrm{tr}(K)$ holds.

In the case that every $a_{2i}$ is an even number, we can also gain this equality in a similar fashion. □

This result is for the special case of positive 2-bridge knots. So whether $2u(D) = \mathrm{tr}(D)$ holds for any minimal diagram of positive 2-bridge knots or not, and whether $2u(K) = \mathrm{tr}(K)$ holds or not, these questions are our theme of the future.

## 7. Positive pretzel knots

In this section, we consider positive pretzel knots, and for some type of them, we can get the following theorem.

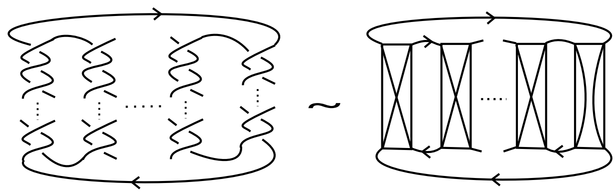


FIGURE 34. The standard diagram $D$ of $K$

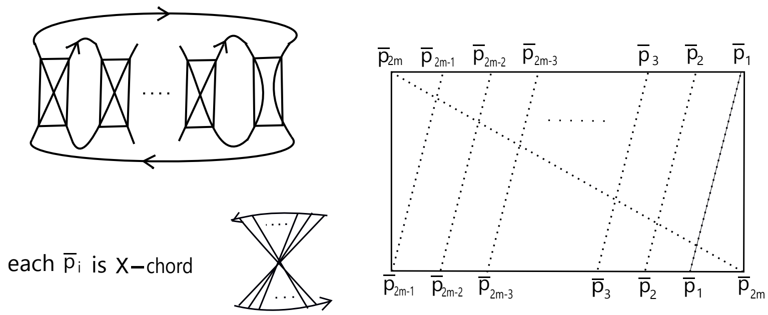



FIGURE 35. The chord diagram of $K$

**Theorem 1.7.1.** *Let $K$ be a pretzel knot $P(p_1, p_2, \ldots, p_{2n})$ $p_i > 0$ for all $i$ with $1 \leq i \leq 2n$, $p_{2n}$ is even and other $p_i$s are all odd $(1 \leq i \leq 2n-1)$, then the following holds.*

$tr(K) = 2u(K) = \sum_{i=1}^{2n} p_i - 2n + 1$

PROOF. The standard diagram $D$ of $K$ is as shown in Figure 34, and we know this diagram is positive and alternating. Moreover, the sub-chord diagram which corresponds to each $\overline{p}_i$ is an X-chord. So the chord diagram of $D$ is as shown in Figure 35.

Then we can easily obtain the trivializing number of $D$. Namely, $\mathrm{tr}(D) = \sum_{i=1}^{2n} p_i - 2n + 1$. Furthermore, by the checkerboard coloring as shown in Figure 36, the signature of $K$ is the following:

$$\sigma(K) = \sigma(D) = -\frac{1}{2}w(D) + \frac{1}{2}(W - B) = -(\sum_{i=1}^{2n} p_i - 2n + 1)$$

By the inequality $|\sigma(K)| \leq 2u(K) \leq \mathrm{tr}(K) \leq \mathrm{tr}(D)$, we can conclude that $\mathrm{tr}(K) = 2u(K)$. This completes the proof. $\square$

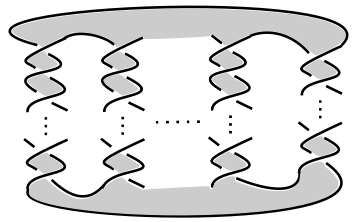


FIGURE 36. An example of checkerboard coloring

CHAPTER 2

# Diagrams of positive and almost alternating links

## 1. Introduction

In Chapter 1, we have seen that every positive and alternating link has a positive and alternating (we say positive-alternating for short) diagram (Theorem 1.3.1). A positive-alternating diagrarm is also called a *PA-diagram*, for short. A link is a *PA-link* if it admits a PA-diagram. Then, we see that a positive and alternating link is a PA-link. Note that when we say a link is positive, we always choose and fix a suitable orientation for the link and its diagrams.

In Chapter 2, we consider positive and almost alternating diagrams and show that show that if a link has a positive and almost alternating diagram, then it is necessarily alternating (Theorem 2.3.2).

A diagram is called an *almost PA-diagram* if it turns into a PA-diagram after one crossing change. It is known that every positive and almost alternating link with eleven or less crossings has an almost PA-diagram (see [**3**]). Furthermore, Jong and Kishimoto [**6**] have shown that every positive knot with genus one or two admits a PA-diagram or an almost PA-diagram. On the other hand, Abe, Jong and Kishimoto[**1**] have shown that every Montesinos link is either alternating or almost alternating. As an analogue of these results, we will show that a Montesinos link whose standard diagram is positive admits either a PA-diagram or an almost PA-diagram (Theorem 2.4.2).

Chapter 2 is organized as follows. In Section 2, we briefly review almost alternating links and positive alternating links. In Section 3, we prove Theorem 2.3.2. In Section 4, we characterize Montesinos links whose standard diagrams are positive, and prove Theorem 2.4.2.

## 2. Preliminaries

A diagram is *almost positive* if one crossing change makes it into a positive diagram, and a link is *almost positive* if it has an almost positive diagram and no positive diagram. In a similar way, a diagram is *almost alternating* if one crossing change makes it into an alternating diagram, and a link is *almost alternating* if it has an almost alternating diagram and no alternating diagram. Our concern at the moment is a *positive and almost alternating link*, that is to say, a link which has a positive diagram and an almost alternating diagram and has no alternating diagram.

A *flype* is an isotopy move applied on a sub tangle of the form $[\pm 1] + t$, and it fixes the endpoints of the sub tangle (see Figure 1). A flype preserves the alternating structure of a diagram ([**7**]). Moreover, we distinguish a positive tangle from a positive diagram as the following: A tangle is positive if it is as shown in Figure 2 (1) and negative as shown in (2).

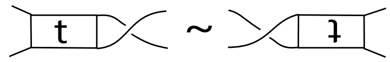



FIGURE 1. A flype

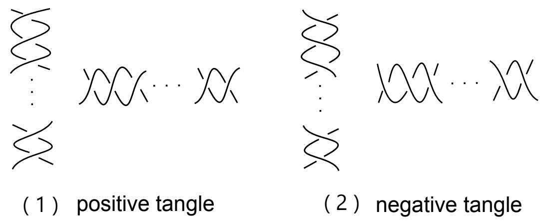



FIGURE 2. Examples of positive(negative) tangle


FIGURE 3. Almost alternating diagrams of a trefoil knot

Next, we introduce some results about almost alternating links. Every alternating link involving a trivial link has an almost alternating diagram. Moreover, any alternating link has infinite almost alternating diagrams(see [**2**]). For example, we can make infinite almost alternating diagrams from a trefoil knot as shown in Figure 3 . Every diagram turns into an alternating diagram if we change the over/under information of the crossing point $d$.

## 3. Positive and almost alternating diagrams

When a link is positive or almost positive, alternating or almost alternating, we can think these four types of links by combination, positive and alternating link, positive and almost alternating link, almost positive and alternating link and almost positive and almost alternating link. However, Stoimenow showed there are no almost positive and alternating links.

**Theorem 2.3.1** (Stoimenow, [**13**]). *Let $L$ be a link. If $L$ is almost positive then $L$ is not alternating.*

Besides that, we already know that a positive and alternating link is a PA-link. Therefore, we can have Table 1.

TABLE 1. Property of the link

| property of the link | altenating | almost alternating |
|---|---|---|
| positive | PA-link (Nakamura) | ? |
| almost positive | There exist no links (Stoimenow) | ? |

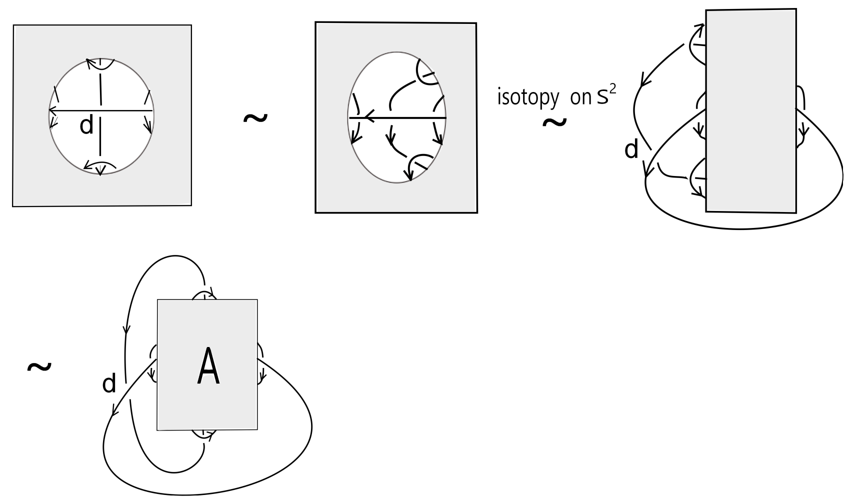



FIGURE 4. Positive and almost alternating diagram

Then, naturally we have the following question: "Does a positive and almost alternating link have a positive and almost alternating diagram? " In this section, we give the negative answer to this question.

**Theorem 2.3.2.** *Let $L$ be an oriented link. If $L$ has a positive and almost alternating diagram then $L$ is alternating.*

PROOF OF THEOREM 2.3.2. By the assumption above, $L$ has an almost alternating diagram, so we can see $L$ is alternating or almost alternating. Our claim is that every positive and almost alternating diagram of $L$ is equivalent to an alternating diagram.

In general, positive and almost alternating diagrams are as shown in Figure 4, where the diagram in every shaded portion is positive-alternating. Although we depict four crossings around the crossing d, the over/under information of which must be changed for getting a alternating diagram, these crossings are not necessarily different four crossings, in other words, they may be duplicate.

The shaded portion in the rightmost figure is equivalent to a disk and we denote this region by $A$. Note that there does not happen the case as shown in Figure 5, because this diagram is not positive.

Assume disk $A$ separates into disk $A_1$ and disk $A_2$. Since each diagram in $A_1$ and $A_2$ is alternating, then the diagram $D$ is equivalent to an alternating diagram $D$' (see Figure 6). Hence, $L$ is alternating.

Next we prove that disk $A$ actually separates into disk $A_1$ and disk $A_2$. We name five crossing points outside of $A$, $\alpha$, $\alpha'$, $\beta$, $\beta'$, d as shown in Figure 7. Moreover, $\overline{\alpha}$ denotes the strand which passes through $\alpha$ and enters into $A$, similarly $\overline{\beta}$ denotes the strand which passes through $\beta$ and enters into $A$.

When the strand which passed under the strand $\overline{\alpha}$ at $\alpha$ crosses next strand, there can be three cases as shown in Figure 8 $(1), (2), (3)$.

Since the diagram $D$ is positive and alternating in region $A$, in any case of $(1), (2), (3)$, the next strand passes under this strand from the right-hand side to the left-hand side as shown in Figure 9(1). We name these crossing points $p_1, p_2, \cdots$.

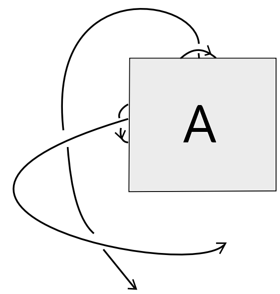



FIGURE 5. This diagram is not positive

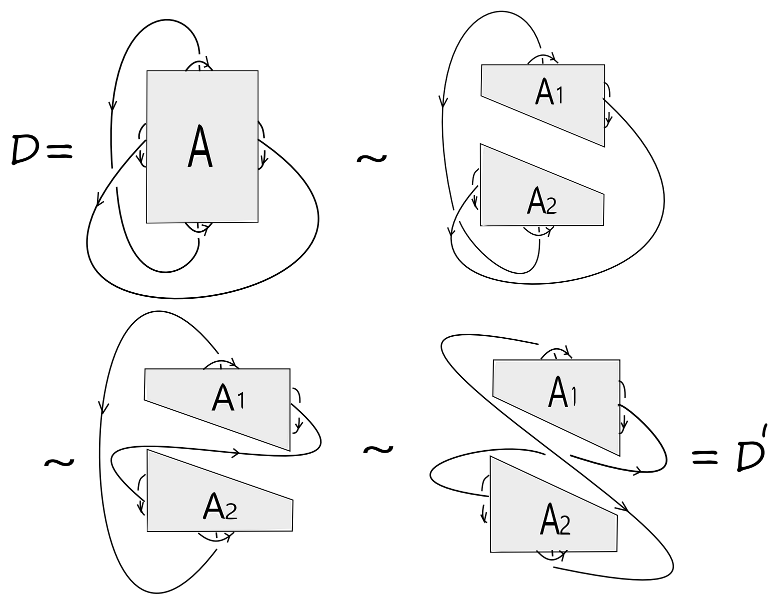



FIGURE 6. $D$ is equivalent to an alternating diagram $D$'

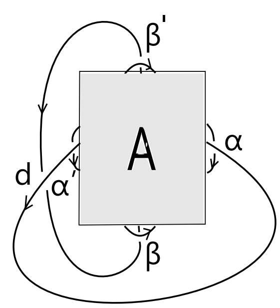



FIGURE 7. Five crossings $\alpha$, $\alpha'$, $\beta$, $\beta'$ and d

Furthermore, $\overline{p}_0$ denotes the arc from $\alpha$ to $p_1$, and $\overline{p}_1$ denotes the arc from $p_1$ to $p_2$, similarly $\overline{p}_2$, $\overline{p}_3$, $\cdots$ and so on. Besides that, we also consider the strand which

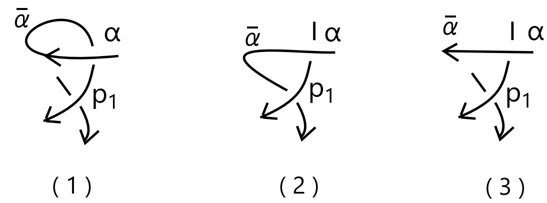



FIGURE 8. There can be three cases when the strand which passed under the strand $\overline{\alpha}$ at $\alpha$ crosses next strand

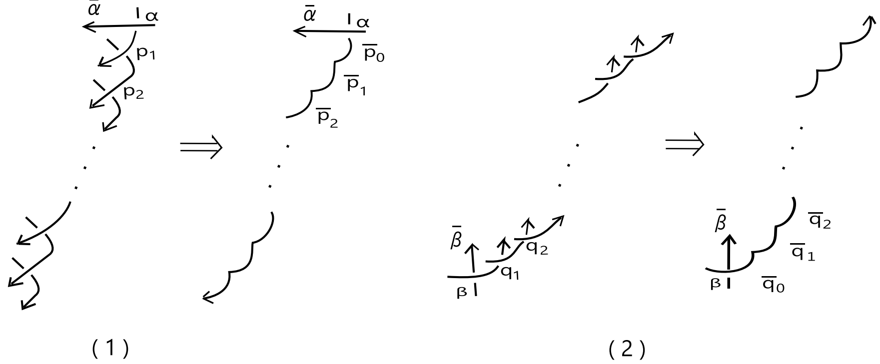



FIGURE 9. The strand which passes under $\overline{\alpha}$ and the strand which passes over $\overline{\beta}$

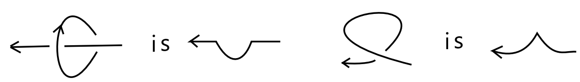



FIGURE 10. In the case where a strand crosses a loop or a strand crosses by itself

passes over $\overline{\beta}$ as shown in Figure 9(2). Then we name the crossing points, $q_1, q_2, \cdots$ and the arcs, $\overline{q}_0$, $\overline{q}_1$, $\cdots$ and so on.

In the case where a strand crosses a loop or a strand crosses by itself, we regard as shown in Figure 10.

Finally, there are two sequences of arcs in $A$ and they are both oriented as shown in Figure 11 (1). If $\overline{p}_m$ and $\overline{q}_n$ cross each other then $\overline{p}_m$ passes over $\overline{q}_n$ from the left hand side to the right hand side as shown in Figure 11(2).

We denote this crossing point by $c$, then there is a polygon with vertices $\alpha$, $d$, $\beta$, $q_1$, $q_2$, $\ldots$, $q_n$, $c$, $p_m$, $p_{m-1}$, $\ldots$, $p_2$, $p_1$. And two arcs $\overline{p}_m$, $\overline{q}_n$ enter this polygon as shown in Figure 12. This is the contradiction to The Jordan curve theorem ([**5**]).

**Theorem 2.3.3.** (*Jordan curve theorem*)

*Let $C$ be the image of the unit circle, that is $C = \{(x, y); x^2 + y^2 = 1\}$ under an injective continuous mapping $\gamma$ into $\mathbb{R}^2$. Then $\mathbb{R}^2 \backslash C$ is disconnected and consists of two component.*

Furthermore, if $\overline{p}_m$ or $\overline{q}_n$ crosses some arc in $\{\overline{p}_i\}$ or $\{\overline{q}_j\}$ then next it crosses the same arc and enter this polygon again. Because each $\overline{p}_i$ is an arc from under

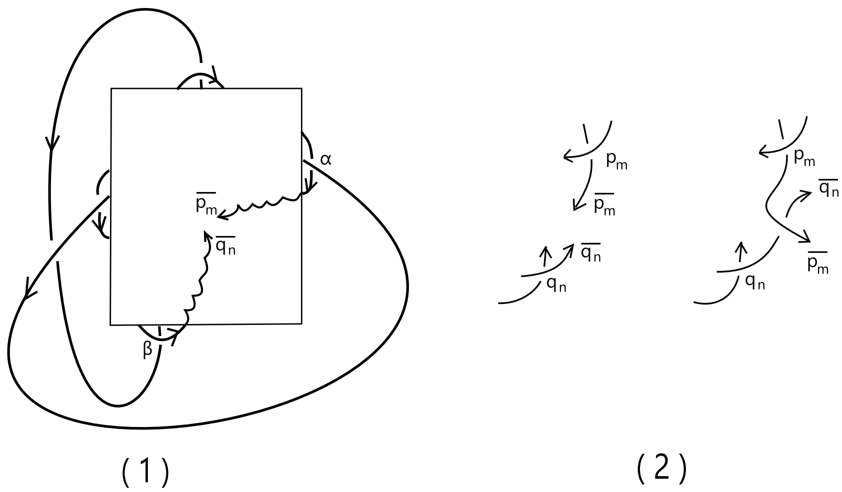



FIGURE 11. The relation between $\overline{p}_m$ and $\overline{q}_n$

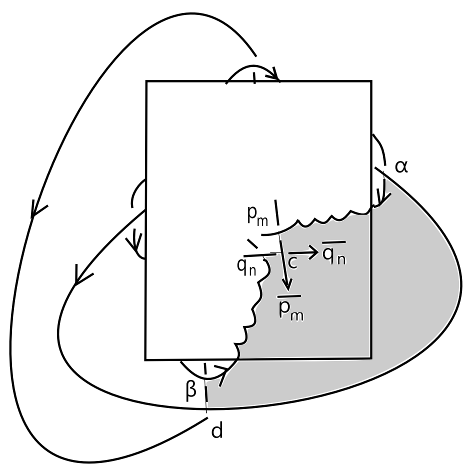



FIGURE 12. This is the contradiction to The Jordan curve theorem

crossing to over crossing and each $\overline{q}_j$ is an arc from over crossing to under crossing. After all, we can see that $\overline{p}_m$ and $\overline{q}_n$ never cross each other in $A$.

For this reason, $A$ must separate into $A_1$ and $A_2$, hence $D$ is equivalent to an alternating diagram $D'$. This completes the proof of Theorem 2.3.2. □

From the theorem above, we know that a positive and almost alternating link has no positive and almost alternating diagram. Therefore, we think another question: "How is the diagram of a positive and almost alternating link?" For answering this question, we introduce a notion "almost PA".

A diagram is *almost positive-alternating* if one crossing change makes it into a PA-diagram. We call such a diagram an *almost PA-diagram.* Furthermore, we call a link an *almost PA-link* if it has an almost PA-diagram and has no PA-diagram. For almost PA-links, the following are known.

**Theorem 2.3.4** (Cromwell [**3**]). *Every positive and almost alternating knot is almost positive-alternating with up to eleven crossings.*

**Theorem 2.3.5** (Jong–Kishimoto [**6**]). *Positive knots up to genus two are positive-alternating or almost positive-alternating.*

## 4. Montesinos links with positive standard diagrams

In this section, we would like to study an oriented Montesinos link $L$, which has the standard diagram $D$ as shown in Figure 13 (1), where $R_i$ represents the standard diagram of a rational tangle. In general, $R_i$ is represented $R_i = T(a_{i1}, a_{i2}, \ldots, a_{in_i})$, where each $a_{ij}$ is an integral which signifies the number of half-twists. Moreover, the standard diagram means a diagram in which $a_{ij} > 0$ for all $j$ with $1 \leq j \leq n_i$, or $a_{ij} < 0$ for all $j$ with $1 \leq j \leq n_i$. About rational tangles, see [**10**]. We show tha standard diagram of a rational tangle with $a_{ij} > 0 forall a_{ij}$ in Figure13 (2) or (3). By these $a_{i1}, a_{i2}, \ldots, a_{in_i}$, we can get the following continued fraction.

$$a_{in_i} + \cfrac{1}{a_{i,n_i-1} + \cfrac{1}{\ddots \cfrac{1}{a_{i1}}}} = \frac{\alpha_i}{\beta_i}$$

Note that $\alpha_i/\beta_i \in \mathbb{Q} \cup \infty$ is either an irreducible fraction or an infinity. We sometimes represent $R_1$ by this rational number (or $\infty$). If $\alpha_i = 0$ then $R_i$ is the standard diagram of a 0-tangle, and if $\beta_i = 0$ then $R_i$ is the standard diagram of an $\infty$-tangle (see Figure 14).

For Montesinos links, Abe and Kishimoto showed the following.

**Theorem 2.4.1** (Abe–Jong–Kishimoto [**1**]). *Non-alternating Montesinos links are almost alternating.*

By the theorem above, we know that a Montesinos link $L$ is alternating or almost alternating. Then we would like to consider the case where the standard diagram $D$ of $L$ is positive. If $L$ is alternating then $L$ is positive and alternating, that is to say, $L$ is clearly a PA-link. Moreover, we show that if $L$ is almost alternating then $L$ is an almost PA-link as the following.

**Theorem 2.4.2.** *Let $L$ be an oriented Montesinos link and be denoted by $C(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_m/\beta_m)$ where $\alpha_i/\beta_i \in \mathbb{Q} \cup \infty$, and $D$ the standard diagram of $L$ such that $D(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_m/\beta_m)$. If $D$ is positive then $L$ has an almost PA-diagram.*

It is to be noted that in general if a link $L$ has a PA-diagram, then $L$ has also an almost PA-diagram. Because we can transform a PA-diagram $D$ of $L$ into an almost PA-diagram $D'$ (see Figure 15).

Before prooving the theorem above, we prove two propositions and one lemma needed later.

**Proposition 2.4.3.** *Let $L$ be an oriented link and $D$ be a diagram of $L$ such that $D = D_1 \sharp D_2 \sharp \cdots \sharp D_m$ where $D_t$ is an alternating diagram for any $t$ with $1 \leq t \leq m$. If $D$ is positive, then $L$ has a PA-diagram.*

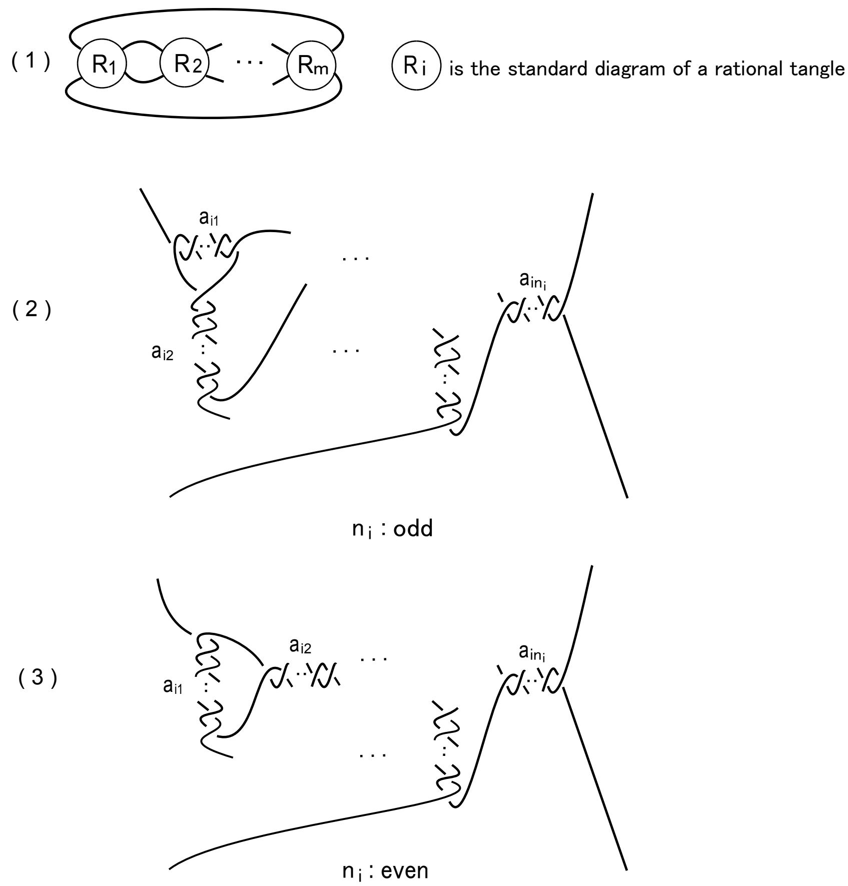



FIGURE 13. The standard diagram of a Montesinos link


FIGURE 14. The standard diagrams of a 0-tangle and an ∞-tangle

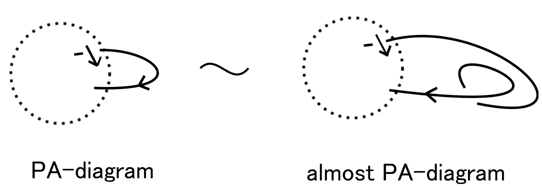



FIGURE 15. An almost PA-diagram of a PA-link

PROOF. Assume $D = D_1 \sharp D_2 \sharp \cdots \sharp D_m$ is positive and $A = D_s \sharp D_{s+1} \sharp \cdots \sharp D_m$ ($1 \leq s \leq m$) is alternating as shown in Figure 16. We consider the orientations of two arcs on the left hand side of $A$ and the over/under informations of the leftmost crossings of $A$ and the rightmost crossings of $D_{s-1}$. Then we can see the four

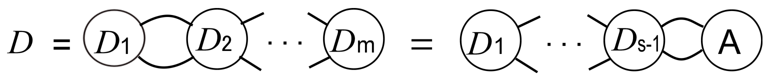


FIGURE 16. $D = D_1 \sharp D_2 \sharp \cdots \sharp D_m = D_1 \sharp D_2 \sharp \cdots \sharp D_{s-1} \sharp A$

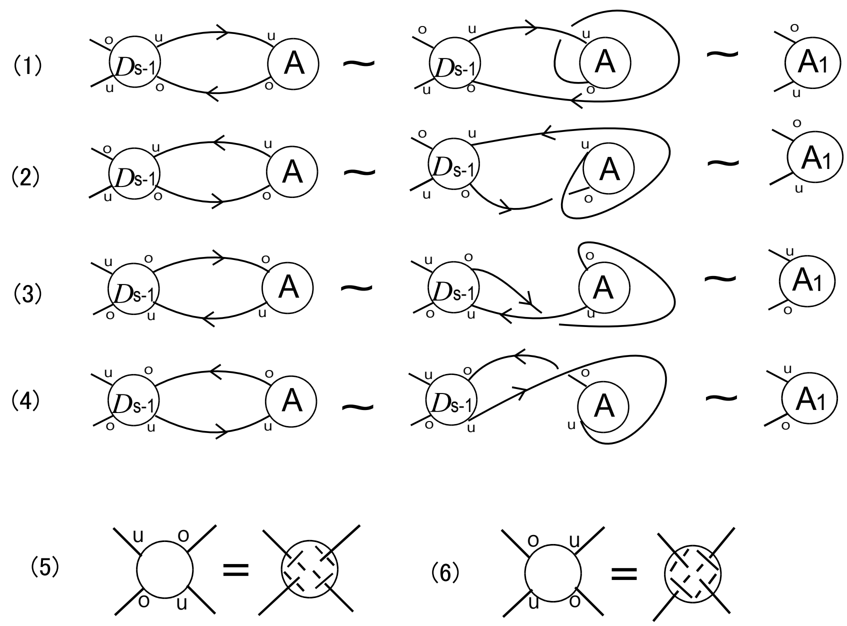


FIGURE 17. Four conditions of $D_{s-1} \sharp A$

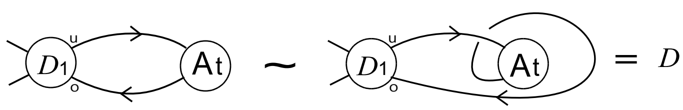


FIGURE 18. $D_1 \sharp A_t$ is equivalent to a PA-diagram

conditions as shown in Figure 17 (1), (2), (3) and (4), where the simbol $o$ (resp. $u$) means that an over-crossing (resp. under-crossing) appears first when we traverse the component from the end point ([**1**]). For the diagram of an alternating tagle, we can check, for example by using the checkerboard coloring, that $o$ and $u$ appear alternately when we make a round of the boundary of the disk. By repeating this transformation we can finally obtain a PA-diagram of $L$ as shown in Figure 17 (5) and (6). □

**Proposition 2.4.4.** *Let $L$ be an oriented Montesinos link, and $D$ be the standard diagram of $L$ denoted by $D(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_m/\beta_m)$, $\alpha_i/\beta_i \in \mathbb{Q}$. Suppose that $D$ is positive, $|\alpha_i/\beta_i| \geq 1$ and $\beta_i \neq 0$ for any $i$ with $1 \leq i \leq m$. Then $D$ is alternating.*

PROOF. First, we consider the case where the orientations of left-hand side arcs of $R_1$ are parallel. In this case, naturally the orientations of the right-hand side arcs of $R_m$ are also parallel as shown in Figure 19 (1). In addition, it is easy to see that

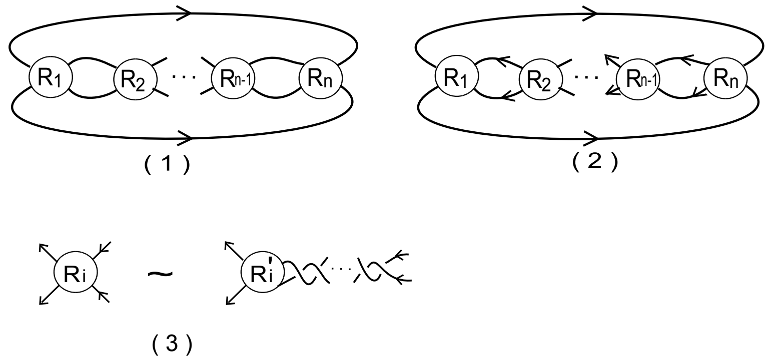



FIGURE 19. The case where the orientations of left-hand side arcs of $R_1$ are parallel

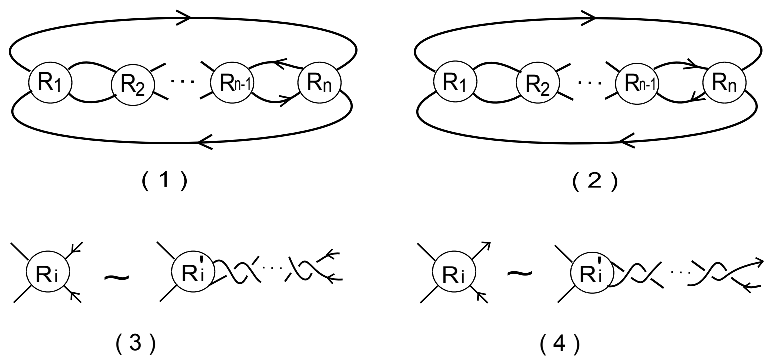



FIGURE 20. The case where the directions of left-hand side arcs of $R_1$ are oppsite

these orientations hold in the case of $R_{m-1}, R_{m-2}, \dots, R_2$ as shown in Figure 19 (2). That is to say, the orientations of the left-hand side arcs of any tangle $R_i$ are all the same as shown in Figure 19 (3). Since each tangle $R_i = R(a_{i1}, a_{i2}, \dots, a_{in})$ is positive and alternating, we know that $a_{ij} \leq 0$ for any $j$ for any $j$ with $1 \leq j \leq n$. Hence $\alpha_i/\beta_i < 0$ for any $i$ with $1 \leq i \leq m$. Then $D$ is necessarily alternating.

Next we consider the case where the orientations of the left-hand side arcs of $R_1$ are opposite. In this case, the orientations of the right-hand side arcs of $R_m$ are as shown in Figure 20 (1) or (2). So for any tangle $R_i$, the orientations of the right-hand side arcs are as shown in Figure 20 (3) or (4). In any case, we know that $a_{ij} > 0$ for any $j$ with $1 \leq j \leq n$, because any $R_i$ is positive and alternating. Therefore, $\alpha_i/\beta_i > 0$ for any $i$ with $1 \leq i \leq m$ and $D$ must be positive. This completes the proof of the proposition. □

In addition, when we meditate upon oriented rational tangles, we can classify them into three types as shown in Figure 21. What is more, we can have the next lemma.

**Lemma 2.4.5.** *Let $R$ be the standard diagram of an oriented rational tangle denoted by $(\alpha/\beta)$, where $\alpha/\beta \in \mathbb{Q}$, $\alpha \neq 0$, $\beta \neq 0$. If any crossing in $R$ has the same sign $+$, then the following holds.*

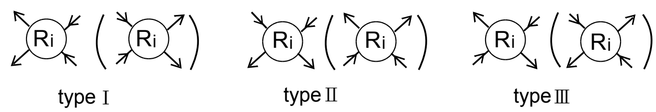



FIGURE 21. Three types of oriented tangles

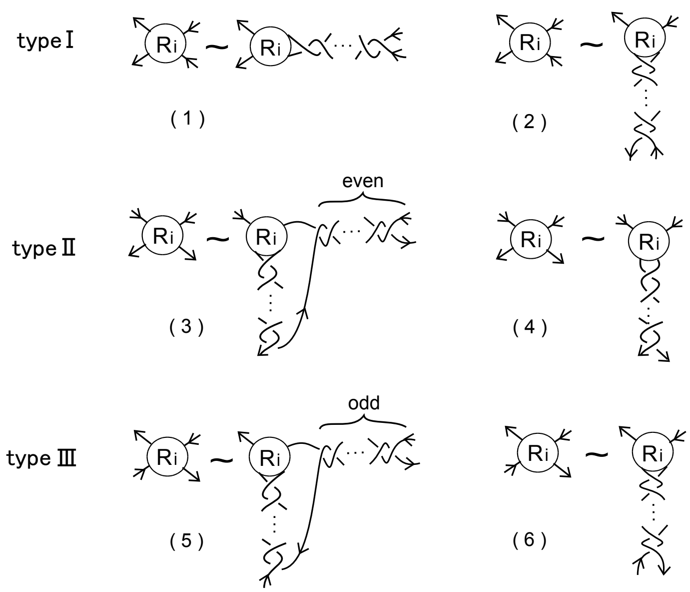



FIGURE 22. The orientations of rightmost arcs of tangles of type I, type II and type III

(1) *If $R$ is of type* I*, then* $\alpha/\beta < 0$.
(2) *If $R$ is of type* II*, then* $\alpha/\beta > 0$.
(3) *If $R$ is of type* III *and* $|\alpha/\beta| \geq 1$*, then* $\alpha/\beta > 0$.
(4) *If $R$ is of type* III *and* $|\alpha/\beta| < 1$*, then* $\alpha/\beta < 0$.

PROOF. In the case where $R$ is of type I, the oriented tangle $R$ is naturally as shown in Figure 22 (1) or (2), and in both cases $\alpha/\beta < 0$. If we reverse all orientations, we can prove in exactly the same way. In the case where $R$ is of type II, $R$ is as shown in Figure 22 (3) or (4), and it is easy to see in both cases $\alpha/\beta > 0$. Besides, when $R$ is of type III and $|\alpha/\beta| \geq 1$, $R$ is neccessarily as shown in Figure 22 (5), and $\alpha/\beta > 0$. On the contrary if $|\alpha/\beta| < 1$, $R$ must be as shown in Figure 22 (6), and $\alpha/\beta < 0$. We have thus proved the lemma. □

In fact, there are two types in type III as shown in Figure 22 (5) and (6). So next we rename type III as shown in Figure 22 (5) type $\text{III}_+$, and as shown in Figure 22 (6) type $\text{III}_-$. Now we are ready to prove Theorem 2.4.2

PROOF OF THEOREM 2.4.2. Let $L$ be a Montesinos link. Let $D$ be the standard diagram of $L$, $D = D(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_m/\beta_m)$, where each $\alpha_i/\beta_i$ represents a rational number (or $\infty$) corresponding to $R_i$, which is the standard diagram of a rational tangle.

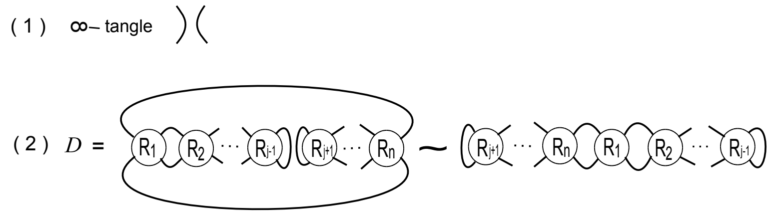



FIGURE 23. The case where some $\beta_j = 0$

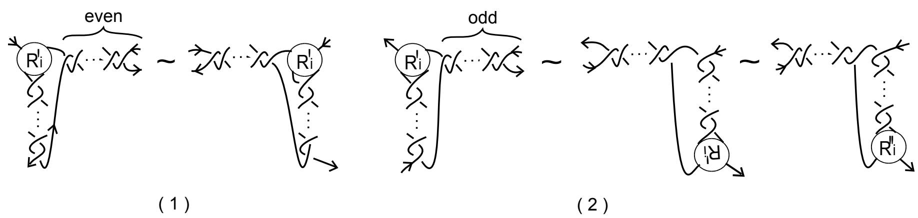



FIGURE 24. The transformation of oriented tangles

First, we consider the case where some $\beta_j = 0$. In this case, $R_i$ is the standard diagram of an $\infty$-tangle as shown in Figure 23 (1), and the diagram $D$ is like as shown in Figure 23 (2), where each $R_k(1 \leq k \leq j-1,\ j+1 \leq k \leq m)$ is an alternating tangle. This is similar to the figure in Proposition 2.4.3. Therefore, by this proposition, $L$ has a PA-diagram. Thus, $L$ has also an almost PA-diagram. For this reason, throughout the following argument, we assume that $\beta_i \neq 0$ for all $i$ with $1 \leq i \leq m$. On the other hand, in the case where $\alpha_i = 0$, the diagram is exactly equivalent to the diagram which is obtained by removing the standard diagram of the tangle $\alpha_i/\beta_i$. Hence, we may also assume that $\alpha_i \neq 0$ for all $i$ with $1 \leq i \leq m$.

In general, a PA-link has an almost PA-diagram, so we may assume that $D$ is non-alternating. If there are at least one standard diagram of a rational tangle of type I, then all the other standard diagrams of rational tangles must be type I because of the orientations of arcs. Therefore, by Lemma2.4.5, $\alpha_i/\beta_i < 0$ for all i with $i$, $1 \leq i \leq m$ and $D$ is naturally alternating. By the assumption, we see that in $D$, there are no standard diagrams of rational tangles of type I. Moreover, if all the standard diagrams of rational tangles are of type III$_-$, then $\alpha_i/\beta_i < 0$ for all $i$ with $1 \leq i \leq m$ and $D$ must be alternating. Similarlly, if all the standard diagrams of rational tangles are of type II or type III$_+$, we can easily see that $D$ is necessarily alternating. Thus, there must be at least one standard diagram of a rational tangle of type III$_-$ and at least one standard diagram of a rational tangle of type II or type III$_+$.

Since the standard diagrams of rational tangles, the rightmost arcs of which are horizontal half-twists, are of type II or type III$_+$, by Lemma2.4.5 for the rational number $\alpha_i/\beta_i$, which corresponds to the standard diagram of this tangle, $\alpha_i/\beta_i > 0$. By transforming the standard daiagrams of these tangles as shown in Figure 24 (1) and (2) (or the orientations of all the arcs are reversed), we can get rational tangles without rightmost horizontal half-twists (these diagrams are not necessarily standard diagrams).

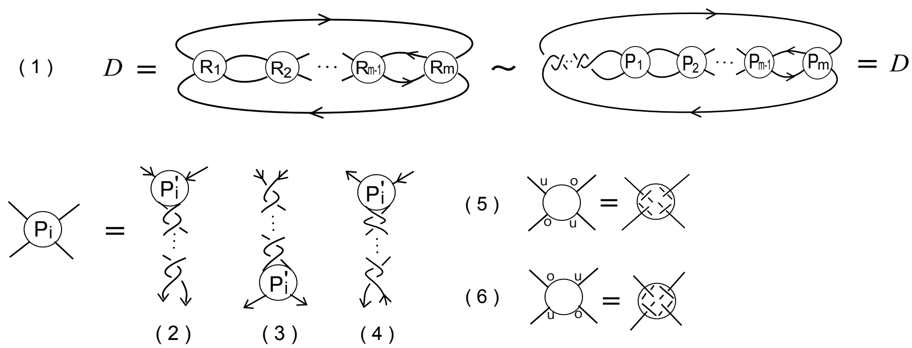



FIGURE 25. The transformation of $D$ and tangle diagram $P_i$

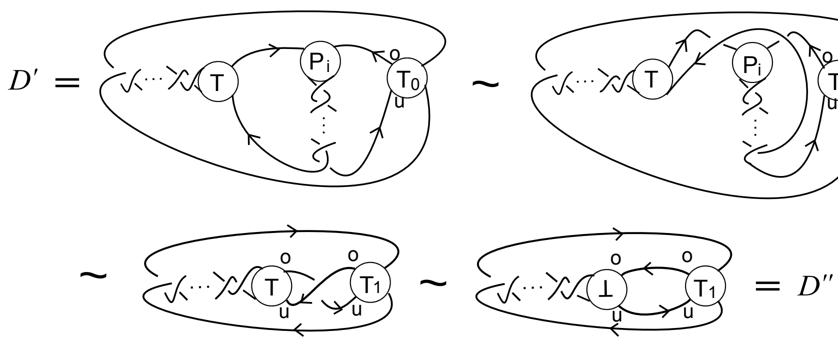



FIGURE 26. The transformation of diagram $D$

In the case of Figure 24 (2), tangle $R'_i$ rotates vertically by 180°, so we rename this tangle$R''_i$. Note that $R''_i$ is positive and alternating. Throughout these transformations, the orientations of arcs with horizontal half-twists, which move from the right-hand side of $R'_i$ to the left-hand side, are both reversed and the sign of each crossing is not changed, that is to say, the positivity is preserved in the case of (1) and (2). Moreover, the alternativity is also preserved in this transformation. By using these transformations, we can move all the horizontal half-twists in the right-hand side of diagrams of rational tangles to the left-hand side, and $D$ is equivalent to $D'$ (or all the orientations of arcs are reversed) as shown in Figure 25 (1). In the case of type II or type III$_+$, tangle diagram $P_i$, $1 \leq i \leq m$, is as shown in Figure25 (2) or (3), and in the case of type III$_-$, $P_i$ is as shown in Figure 25 (4) (or all the orientations of arcs are reversed). Furthermore, $P'_i$ is a PA-diagram.

Assume that $P_s$ is the rightmost tangle of type II or type III$_+$ in $D'$, then $P_j$ are of type III$_-$ for all $j$ with $s+1 \leq j \leq m$. Thus, when we denote the sum of these tangles by $T_0 = P_{s+1} + P_{s+2} + \cdots + P_m$, $T_0$ must be a PA-diagram and the depicted symbols of $T_0$ is as shown in Figure 25 (6). Moreover, we denote the sum of tangles, which are on the left-hand side of $P_s$. Thereby, we can transform $D'$ as shown in Figure 26, and have a diagram $D''$, which is equivalent to $D'$ where $T_1$ is a PA-diagram.

The orientations of arcs on the left-hand side of $T_1$ are as same as those of $T_0$, and it is obvious that if $P_{s-1}$ is of type III$_-$ then the tangle sum $P_{s-1} + T_1$ is a

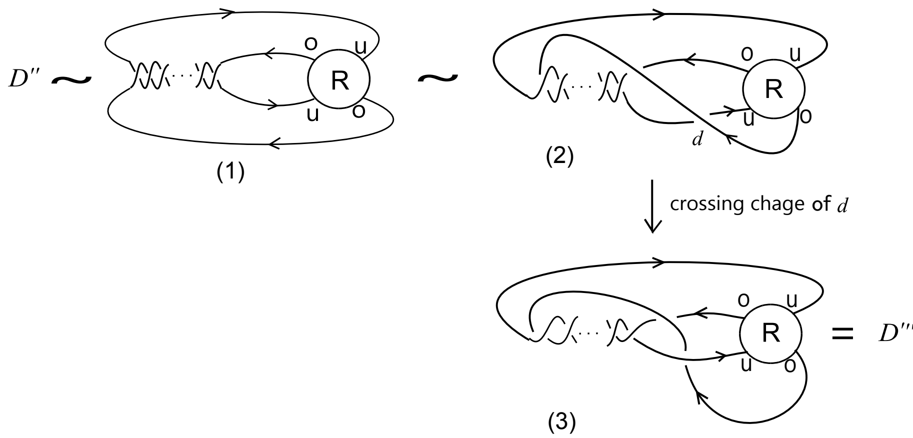



FIGURE 27. $D'$ is equivalent to an almost PA-diagram

PA-diagram. On the other hand, if $P_{s-1}$ is of type II or type III$_+$, we can obtain a PA-diagram $T_2$ in a similar fashion like Figure 26. By using these transformations, we can get the diagram as shown in Figure 27 (1) (or the orientations of all arcs are reversed). Since there are no standard diagram of rational tangle of type I, we can limit the orientations of the arcs on the left-hand side of $R$ to this (or the orientations of both arcs are reversed). Note that $R$ is a PA-diagram.

Next, by moving the arc which is on the foot of the diagram, we can get the diagram as shown in Figure 27 (2). Moreover, by crossing change of $d$, we can also get a PA-diagram $D'''$. Hence, we can see that the diagram as shown in Figure 27 (2) is an almost PA-diagram. This completes the proof. □

From the theorem above, we can see that there can be some almost PA-links with over Genus 2. For example, let us think about pretzel knot $K = P(2p+1, 2q+1, -2r)$, $p > 0$, $q > 0$, $r > 0$, whose standard diagram $D$ is as shown in Figure 28 (1) (or the orientations of all the arcs are reversed), and $D$ is positive. Besides that, $K$ is almost alternating (see [**8**]). For a positive knot $K$ and a positive diagram $D$ of $K$, Nakamura [**12**] showed that

$$2g(K) = c(D) - s(D) + 1,$$

where g(K) is the genus of $K$, $c(D)$ is the crossing number of $D$ and $s(D)$ is the number of Seifert circles obtained from $D$. For the diagram $D$, we can easily see that $c(D) = 2p+2q+2r+2$ and also see $s(D) = 2r+1$ from Figure 28 (2). Hence, $2g(K) = (2p+2q+2r+2) - (2r+1) + 1 = 2p+2q+2$ and $g(K) = p+q+1$. Therefore, we can think infinitely great genus for $K$.

Next, we transform $D$ into the diagram as shown in Figure 29 (2), (3) and (4), where $A$ is the diagram surrounded by a dotted line. In a similar fashion, we can also obtain the diagram as shown in Figure 29 (6), where $A'$ is the diagram surrounded by a dotted line, and it is obvious that $A$ and $A'$ are both PA-diagrams. Moreover, the form of the diagram as shown in Figure 29 (6) is as same as the form of the diagram as shown in Figure 27 (1), and hence, we can see that this diagram is an almost PA-diagram. Thus, we can make sure that there are infinite positive and almost alternating links with genus greater than 2, which have almost PA-diagrams.

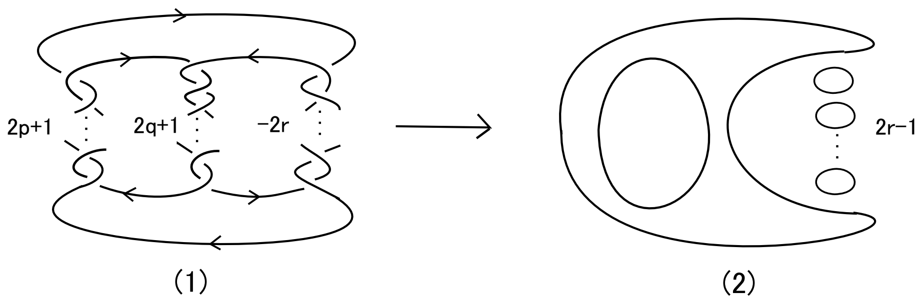



FIGURE 28. The standard diagram $D$ of pretzel knot $P(2p+1,\ 2q+1,\ -2r)$ and Seifert circles of $D$

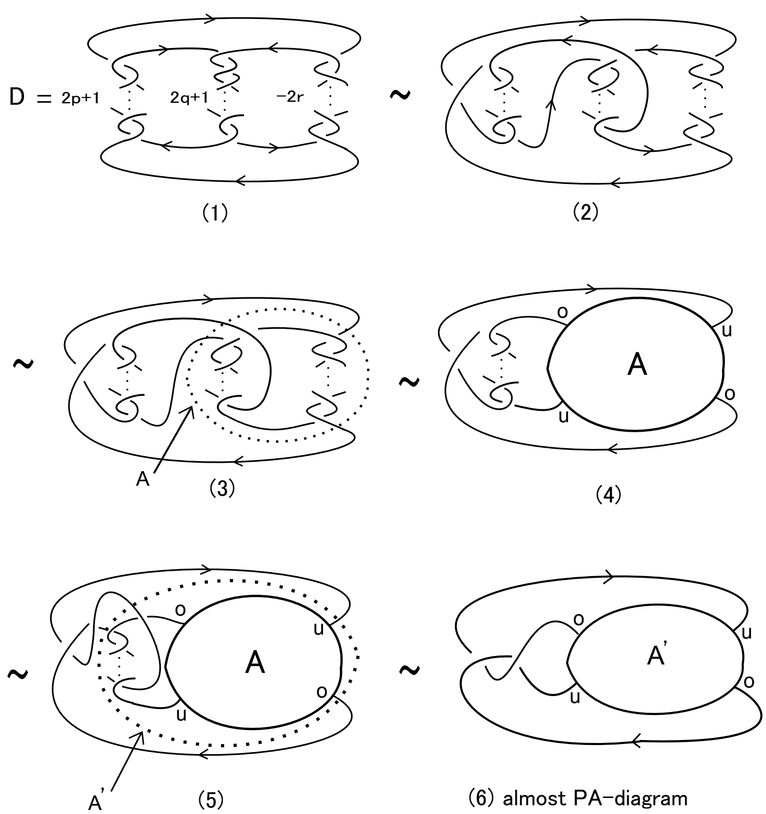



FIGURE 29. Pretzel knot $P(2p+1, 2q+2, -2r)$ is an almost PA-knot

If $L$ is an almost PA-link, by Theorem 2.3.1 $L$ is naturally positive and almost alternating or almost positive and almost alternating. For almost positive and almost alternating links, Stoimenow [**13**] showed that the mirror image of the knot which is represented by $10_{145}$ on The Rolfsen Knot Table is an almost positive and almost alternating link and has an almost PA-diagram.

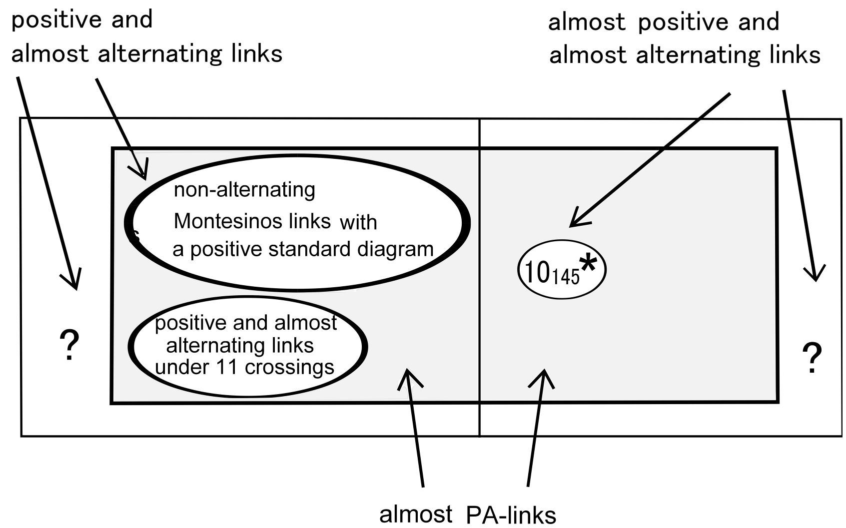



Figure 30. Venn diagram for almost PA-links

Finally, we can obtain a Venn diagram as shown in Figure 30, and have a newer question, "Does there exist a region on which we put the question mark? " This is the problem which now confronts us.

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