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ハビバ, ウミュ

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A higher-order asymptotic theory for motion of a counter-rotating vortex pair of finite thickness

九州大学
KYUSHU UNIVERSITY

Ummu Habibah

Graduate School of Mathematics
Kyushu University
Fukuoka, Japan

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I would like to dedicate this thesis to my loving family . . .

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Abstract

A counter-rotating vortex pair is relevant to the wake vortices behind the wings of an airplane and controlling it has been demanded since jet planes started their commercial flight. We establish the traveling speed of a counter-rotating vortex pair moving in an incompressible fluid. The solution of the Navier-Stokes equation is constructed by use of the matched asymptotic expansions in a small parameter $\varepsilon = 1/\sqrt{Re}$ where $Re = \Gamma/\nu$ is the Reynolds number with ν being the kinematic viscosity of the fluid and $\pm\Gamma$ being the circulation of the vortices. The parameter ε is a measure for ratio of core radius σ to the half distance d between the vortices. The radius of vortex core is assumed to be much smaller than the distance between two centroids $(\varepsilon \ll 1)$. The Biot-Savart law is valid in the outer region, and its inner limit provides the boundary condition on the inner solution. Adapting Dyson's technique to two dimensions facilitates the evaluation the inner limit for an arbitrary vorticity distribution.

In Chapter 3, the asymptotic expansions are performed for the solution of the Navier-Stokes equation for a counter-rotating vortex pair up to 7th order. The 0th-order solution represents a circular vortex. The 1st-order solution produces the translation speed which is coincident with of the pair of point vortices. The 2nd, 3rd and 4th orders incorporate pure shear and higher order shear induced by the companion vortex where the core of a vortex pair deforms to an ellipse, and the perturbation vorticity at 2nd, 3rd and 4th orders represent quadrupole, hexapole and octapole structures respectively. At the 5th order, a correction due to the effect of finite thickness of the vortices makes the first appearance to the traveling speed for both of the viscous and the inviscid fluids. Moreover, a formula is established in surprisingly simple form to $O(\varepsilon^5)$ for the traveling speed of counter-rotating vortex pair for a general vorticity distribution. This general formula is supported by the analytical proof. The general formula is powerful since we can calculate fifth order traveling speed of a counter-rotating vortex pair only from second-order solution which is numerically calculated by shooting method. The next correction to the general formula for a counter-rotating vortex pair is given at 7th order. Furthermore the lateral motion of the centroids of vorticity are given through the conservation law of the hydrodynamics impulse. The end product is the general formula for the change in the lateral position of the center of the vortex pair that

occurs at $O(\varepsilon^0)$. The example of the Oseen vortex $O(\varepsilon^0)$ shows that the viscosity acts to increase the vortex-vortex distance with time.

Chapter 4 is concerned with illustration of a vortex pair in an inviscid fluid. As an example we consider the Rankine vortex, a circular vortex of uniform vorticity, at leading order. By using the matched asymptotic expansions up to 5th order in an inviscid fluid, the solution of the motion of an anti-parallel vortex pair is obtained and recovers the result of Yang and Kubota (1992). Thus the resulting solution serves to test our general formula of the traveling speed of a counter-rotating vortex that we obtain in Chapter 3.

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Chapter 1

Introduction

Motion and stability of an anti-parallel vortex pair is a long-standing problem which has boosted the field of vortex dynamics since the late 1960s when jet planes started their commercial flight. Specifically the need arises for determining the airplane's landing interval-time since the wake vortices behind wings of an airplane endanger any following airplane close behind. The viscosity diffuses vorticity and a thin core of concentrated vorticity is invalided after some time.

Finite-thickness effect of vortex tubes is a common problem in the dynamics of vortices, and has been intensively studied so far. An asymptotic theory for the two dimensional motion, at high Reynolds numbers, of a viscous vortex embedded in an external flow of an incompressible fluid was initiated by Ting and Tung (1965). The method of matched asymptotic expansions was formulated; the solution of a decaying single vortex, with the influences of the wall and surrounding vortices being incorporated as the external flow, is sought in a power series in small parameter which is a measure for the ratio ε of the core radius to the typical length-scale in spacial variation of the external flow. The inner solution is obtained by solving the Navier-Stokes equation perturbatively. The outer solution in the outer region with negligible vorticity is found from the Biot-Savart law. A well-known particular solution of the Navier-Stokes equations is the Oseen vortex, an axisymmetric diffusing vortex, with the axial vorticity ζ and the azimuthal velocity v given by

$$\zeta(r) = \frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}, \; v(r) = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right), \tag{1.1}$$

as functions of the time t and the distance r from the symmetric axis. Here Γ is the circulation of the vortex and ν is the kinematic viscosity. The Oseen vortex starts, at $t = 0$, with the axial vorticity concentrated along the symmetric axis $r = 0$ and serves as an example of the leading-order solution. If specialized to an anti-parallel vortex pair, the first-order

solution gives the same traveling speed as the pair of point vortices of opposite signs. This solution gets some support from the experiment by Leweke and Williamson (1997) in which they investigated the three-dimensional instability of a counter-rotating vortex pair to short waves. The experiments involve detailed visualizations and measurement to reveal the spatial structure of the instability for a vortex pair, which is generated underwater by two rotating plates. Another experiment was conducted by Williamson *et al* (2014) with extended three-dimensional instability of a counter-rotating vortex pair to long waves that evolve for an isolated a vortex pair, for a vortex pair interacting with a wall, including two-dimensional interactions of a vortex with each other or the companion vortex. These interactions are compatible with the solution of a vortex motion in this dissertation.

At the second-order, the influence of the companion vortex on the vortex under consideration takes the form an external pure shear flow (Moore and Saffman 1971), in addition to the influence to the traveling speed at the first order. Gaifullin and Zubtsov (2004) pursued the higher-order asymptotics for the special care of the Oseen vortex at the leading order and showed that the correction of finite-thickness effect to the traveling speed makes its appearance at $O(\varepsilon^5)$, along with change of the distance between vortices at $O(\varepsilon^6)$. Here $\varepsilon = 1/\sqrt{\nu/\Gamma}$ is a small parameter, with $\pm\Gamma$ being the circulation of the vortices. But they did not explain how to derive this change in the lateral position of counter-rotating vortex pair. In this dissertation we will give a clear explanation for deriving lateral position of a vortex pair at a higher order, exploiting the conservation law of the hydrodynamics impulse. As the end product, we construct the general formula for both the traveling speed and the change in the vortex-vortex distance. We then give an illustration of using this formula for the motion of a viscous vortex pair evolving from the delta-function cores. This exhibits a marked contrast with motion of a vortex ring, for which the small parameter is the ratio of the core radius to the ring radius. The correction of curvature origin appears at $O(\varepsilon)$ (Widnall *et al* 1970) and the influence of deformation of the core appears at $O(\varepsilon^3)$ (Fukumoto and Moffatt 2000). In a practical flow, the cores of a vortex pair are largely deformed into ellipse with shedding tails behind. A model of a vortex pair consisting of elliptic cores was shown to the fit well with numerical simulation contrived by Delbende and Rossi (2009). There is an attempt to include higher-order singularities, vortex dipoles, with taking account of desingularization by viscosity, to represent finite cores (Llwellyn Smith and Nagem 2013). First, they established a family of equations of motion for inviscid vortex dipoles and gave the discussions on limitations. Second, they investigated viscous vortex dipoles to obtain velocity propagation. In this dissertation we will show at $O(\varepsilon^2)$ and $O(\varepsilon^3)$, the cores of a vortex pair deform into elliptic and triangular shape by the shear induced by the companion vortex. For low-Reynolds-number motion, the numerical solution was obtained over a large time

by Dagan (1989). They used the pseudo-spectral method to efficiently handle the numerical boundary conditions and to validate the results that had been obtained by an asymptotic analysis. The large-time solution is obtainable for the vorticity from the Stokes equations and the decaying stage of a viscous vortex pair was described by sophisticated analyses (Cantwell and Rott 1988, Dommelen and Shankar 1995). Cantwell and Rott (1988) investigated the asymptotic behavior of the flow through expansions the solution in inverse powers of the time up to $O(\varepsilon^{-2})$. At $O(\varepsilon^{-1})$, for large time expansion their result showed that the solution turned out to be independent of the drift.

Vortex patches, or vortices of uniform vorticity, embedded in an inviscid incompressible fluid allow detailed analyses both theoretically and numerically. The numerical solution for a steadily translating vortex pair of equal shape and equal strength but of opposite signs was constructed over a wide range of core sizes with a limiting case of almost touching cores. Recently by a novel method, the steady state was extended to bifurcated solutions with asymmetric core shapes by Luzzatto-Fegiz (2014). Chaplygin-Lamb's dipole is well-known as a particular solution that describes steady motion of a vortex pair with non-uniform vorticity (Meleshko and van Heijst 1994). Luzzatto-Fegiz (2014) succeeded in extending his numerical method to deal with steady motion of vortices of non-uniform vorticity, by calculating a discretized version of Chaplygin-Lamb's dipole. In the virtually same spirit as the matched asymptotic expansions, the motion of vortex patches with, though thin, finite cores, in an inviscid incompressible fluid was calculated perturbatively in powers of a small parameter, the ratio of core radius to the vortex-vortex distance, to a high order. By extending Moore's idea (unpublished), Dhanak (1992) performed these asymptotic expansions for co-rotating vortices in the regular polygonal configuration. The angular velocity of rotation as the whole and the stability result compare well with the elaborate numerical result by Dritscel (1985), even for fat cores. For counter-rotating, equal and uniform vorticity of a pair of vortices, Saffman and Tanveer (1982) numerically produced a one-parameter family of steady flows as one progressively reduces the gap between the centroids of the vortices. When the gap is zero (such that vortices touch), the overall shape looks like a "rugby-ball", and at this smaller distance the corner reveal 90 degree. Yang and Kubota (1994) applied the technique similar to that of Dhanak (1992) to the asymptotic solution for a counter-rotating vortex pair with symmetric cores and derived a correction of the finite cores to the traveling speed of the vortex pair.

Despite the long history of the subject, theoretical results as regards the motion of a counter-rotating vortex pair is limited to uniform cores in an inviscid fluid and the special initial condition of the delta-function core in a viscous fluid. The available theories are not sufficient to predict the realistic motion of a vortex pair, at a high Reynolds number, governed

by the Navier-Stokes equations. The program envisioned by Ting and Tung (1965) is yet to be completed. In this dissertation, we establish a general formula of the traveling speed of a counter-rotating vortex pair in an inviscid as well as a viscous fluids, for an arbitrary initial condition up to $O(\varepsilon^5)$. The end-product exhibits a surprisingly simple structure; the strength of quadrupole of $O(\varepsilon^2)$ suffices to calculate the $O(\varepsilon^5)$ correction to the traveling speed. The formula is analytically prooved in Appendix A. Furthermore, we continue to pursue the solution of a counter-rotating vortex pair up to seventh order (ε^7) and derive a general formula. This formula needs verification. For $O(\varepsilon^7)$ Gaifullin and Zubtsov (2004) also gave an asymptotic value of the correction to the traveling speed for a viscous vortex pair starting from the Oseen vortex, unfortunately they did not describe how to derive the traveling speed of a vortex to this stage, and solution. For future work, the general formula of the translation of vortex pair of $O(\varepsilon^7)$ needs to be verified. Hopefully, the general formula of $O(\varepsilon^7)$ for a counter-rotating of a vortex pair has similar behavior as the general formula at $O(\varepsilon^5)$ where the higher order formula can be simplified to be expressed in terms of a lower-order multipole strength.

The fundamental equations as the base of our theory are explained in Chapter 2. The asymptotic solution of a viscous vortex pair is explained in Chapter 3 where the formulation of the matched asymptotic expansions is concisely described in sections 3.1 and 3.2. In section 3.3, we revisit the results of Nakagawa (2004) for the general asymptotic solution to (ε^5). With the example of the Oseen vortex at ε^0 we observe an accidental coincidence in the numerical values in the correction terms of Nakagawa's formula. We verify in Appendix A that this is indeed the case for a general distribution of vorticity at $O(\varepsilon^0)$, resulting in a highly tidy and powerful formula for the correction term at ε^5. The lateral motion of vortex pair caused by the viscous diffusion of vorticity change the distance between the vorticity centroids. In Section 3.4, the time evolution of the lateral position of the center of the vortex is shown in its general formula, exploiting the conservation law of the hydrodynamics impulse. Our numerical solution for the change of the lateral position of the center of the vortex coincides with Nakagawa (2004) and Gaifullin and Zubtsov (2004). Furthermore, the general formula for the travelin speed is tested using an inviscid fluid. An example of an inviscid case of a vortex pair is given, using the Rankine vortex at the leading order (Chapter 4). The procedure is a two dimensional version of the axisymmetric problem treated by Fukumoto (2002) for a higher order for the steady translation of the flow field around a thin axisymmetric vortex ring in an ideal fluid. Our general formula recovers the example of the motion of a vortex pair in a viscous fluid but also in an inviscid fluid. In this example, we reproduce the solution for the translation speed of a counter-rotating vortex pair with

symmetric cores by Yang and Kubota (1994). Finally we summarize the whole work of the motion of a vortex pair of finite thickness in Chapter 5.

Chapter 2

Fundamental equations

2.1 Equation motion

In general, for three-dimensional incompressible flow, velocity $\mathbf{u}$ of the flow is governed by Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \triangle \mathbf{u}, \tag{2.1}$$

and continuity equation

$$\nabla \cdot \mathbf{u} = 0, \tag{2.2}$$

where ρ is the fluid density (a constant due to incompressibility), p is pressure, and ν is kinematic viscosity (assumed to be constant). Vorticity $\boldsymbol{\omega}$ is defined as the curl of velocity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{2.3}$$

Upon taking the curl of the velocity equation (2.1), one obtains Navier-Stokes equation in terms of vorticity

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = -\boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \triangle \boldsymbol{\omega}, \tag{2.4}$$

where the fact $\nabla \times \nabla p = 0$ is already imposed. Also, continuity equation for vorticity is given by

$$\nabla \cdot \boldsymbol{\omega} = 0, \tag{2.5}$$

Velocity $\mathbf{u}(\mathbf{x},t)$ and vorticity $\boldsymbol{\omega}(\mathbf{x},t)$ are both functions of position vector $\mathbf{x}$ and time t. For two-dimensional incompressible flow, $\boldsymbol{\omega}$ is always perpendicular to the plane of dynamics, hence the stretching term $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ is always zero. The continuity equation (2.5) is automatically satisfied. The Navier-Stokes equation (2.4) is reduced to a scalar equation

$$\frac{\partial\zeta}{\partial t} = -\mathbf{u}\cdot\nabla\zeta + \nu\triangle\zeta, \tag{2.6}$$

where ζ is the only non-zero component of $\boldsymbol{\omega}$ in the 2D flow. We will deal exclusively with 2D incompressible flow in this work. If one is interested in inviscid flow, i.e. $\nu = 0$, the diffusion term $\nu\Delta\zeta$ is zero, then (2.6) reduces to Euler equation

$$\frac{\partial\zeta}{\partial t} = -\mathbf{u}\cdot\nabla\zeta, \tag{2.7}$$

and if pure diffusion problem is concerned, i.e. the convection term $\mathbf{u}\cdot\nabla\zeta$ is zero, then (2.6) becomes the heat conduction equation

$$\frac{\partial\zeta}{\partial t} = \nu\triangle\zeta, \tag{2.8}$$

where particular well-known solution is Oseen vortex.

2.2 Circulation

Circulation Γ is defined to be the integral of velocity around a closed fluid loop and so is given by

$$\Gamma \equiv \oint_l \mathbf{u}\cdot d\mathbf{l} = \int_A \boldsymbol{\omega}\cdot\mathbf{n}dA, \tag{2.9}$$

where the second expression uses Stokes' theorem and A is any surface bounded by the loop $\mathbf{l}$; $\mathbf{n}$ is a unit normal vector. The circulation around the path is equal to the integral of the normal component of vorticity over any surface bounded by that path. The circulation is not a field like vorticity and velocity; rather, we think of the circulation around a particular material line of finite length, and so its value generally depends on the path chosen. If δA is an infinitesimal surface element whose normal points in the direction of the unit vector $\mathbf{n}$, then

$$\mathbf{n}\cdot(\nabla\times\mathrm{u}) = \frac{1}{\delta A}\oint_{\delta l} \boldsymbol{u}\cdot d\mathbf{l}, \tag{2.10}$$

where the line integral is around the infinitesimal area. Thus at a point the component of vorticity in the direction of **n** is proportional to the circulation around the surrounding infinitesimal fluid element, divided by the elemental area bounded by the path of the integral. A heuristic test for the presence of vorticity is to imagine a small paddle wheel in the flow; the paddle wheel acts as a 'circulation-meter' , and rotates if the vorticity is non-zero. Vorticity might seem to be similar to angular momentum, in that it is a measure of spin. However, unlike angular momentum,the value of vorticity at a point does not depend on the particular choice of an axis of rotation; indeed, the definition of vorticity makes no reference at all to an axis of rotation or to a coordinate system. Rather, vorticity is a measure of the local spin of a fluid element.

2.3 Vortex stucture

2.3.1 Lamb-Oseen vortex

We consider the evolution of an initial single point vortex in a viscous unbounded fluid. Due to axisymmetry, there is no preferable direction of movement of the vortex. In another word, the vortex does not induce velocity onto itself, the position of the vortex center remains fixed for all time. Therefore, this is a pure diffusion problem, and vorticity is governed by

$$\frac{\partial \zeta}{\partial t} = \nu \Delta \zeta, \tag{2.11}$$

where $\zeta = \boldsymbol{\omega}_z$ is the axial component of vorticity. This equation coincides with the heat transfer equation for a problem of heat spreading from a linear source in a uniform medium. This solution is well known

$$\zeta = \frac{c}{4\pi\nu t} e^{-r^2/4\nu t}, \tag{2.12}$$

where constant c follows from initial condition, which is given through the Stokes theorem (2.9)

$$\Gamma = \Gamma(r,t)|_{t=0} = 2\pi \int_0^r \zeta r dr|_{t=0}, \tag{2.13}$$

using the solution 2.12 we obtain

$$\Gamma = c\frac{2\pi}{4\pi\nu t} \int_0^r e^{-r^2/4\nu t} r dr|_{t=0} = c. \tag{2.14}$$

Thus

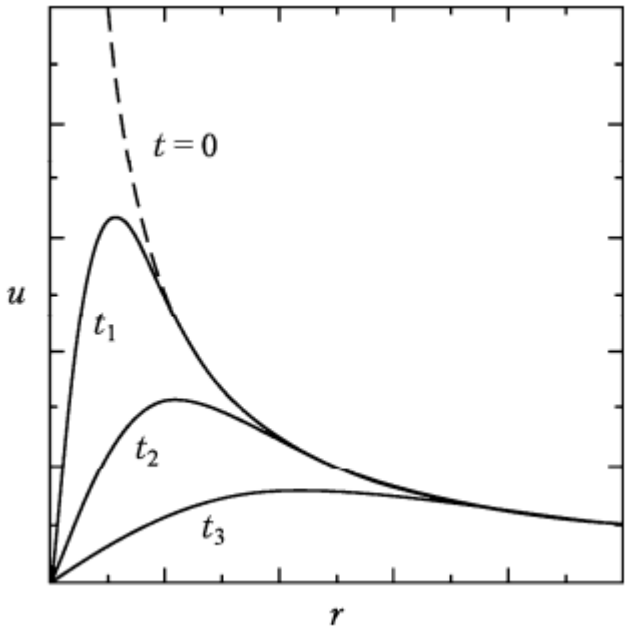

Fig. 2.1 Radial distributions of tangential velocity at different time instants t for the Lamb-Oseen vortex

$$\zeta(r,t) = \frac{\Gamma}{4\pi\nu t} e^{-r^2/4\nu t}, \tag{2.15}$$

and the tangential velocity u is as follows:

$$u = \frac{1}{r}\int_0^r \zeta r dr = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/4\nu t}\right). \tag{2.16}$$

The distribution of velocity u and vorticity ζ and are plotted in Figs. 2.1 and 2.2 respectively for different instants of time. At $t = 0$ we have the velocity distribution induced by the infinitely thin vortex filament: $u = \Gamma/2\pi r$. For $t > 0$, a local maximum appears on these profiles and it drifts with time to infinity, with simultaneous decrease in the amplitude of the maximum. For $r << \sqrt{4\nu t}$, by Taylor series of (2.16), the velocity $u = \Gamma r/8\pi\nu t$, i.e., the liquid in the vortex core rotates as a solid body with the angular velocity $\Gamma/8\pi\nu t$. Thus the vorticity diffuses into the entire space filled with the fluid. This flow example was named the Lamb-Oseen vortex(Lamb, 1932).

All the above analysis proves that taking into account the viscosity in the bulk of a fluid leads to vorticity diffusion, but in no way it is responsible for its generation.

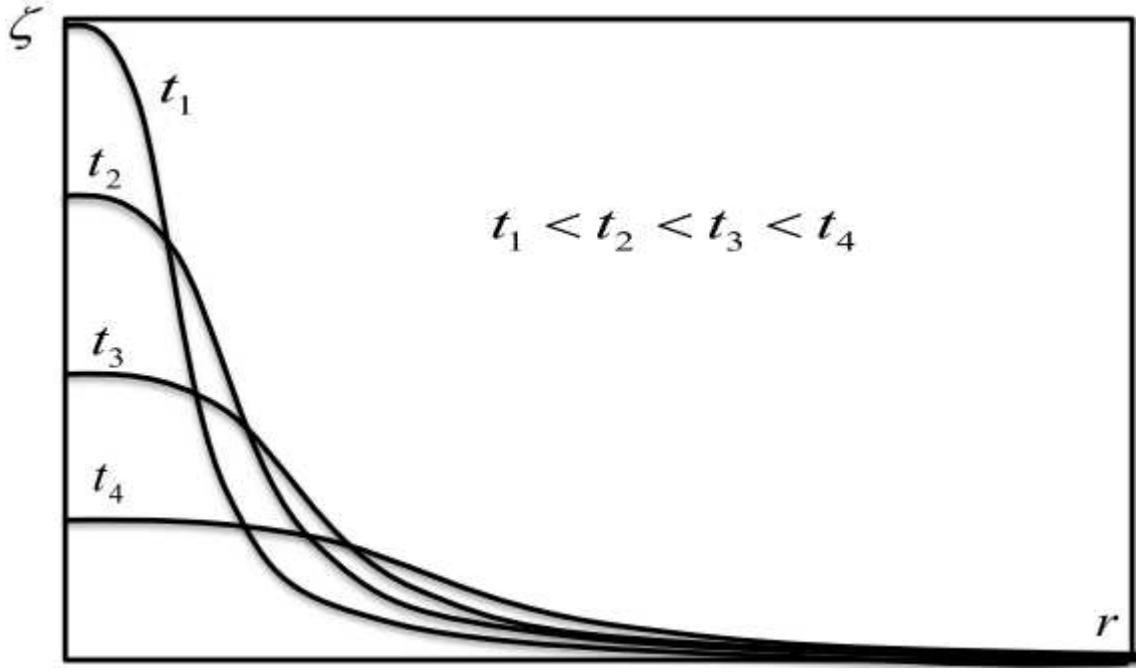

Fig. 2.2 Vorticity ζ at different time instants t for the Lamb-Oseen vortex

2.3.2 Rankine vortex

The model of a cylinder vortex with a finite round core of radius a, where a is a constant vorticity ζ inside it is more realistic than the model of an infinitely thin vortex filament. Outside the core, the flow is assumed to be irrotational. As in the case of a vortex sheet, this vortex can be approximated by the continuous distribution of rectilinear vortex filaments in the core. Then, according to Stokes theorem (2.9), the contribution of the core cross-section element dA to circulation $d\Gamma$, equals

$$d\Gamma = \zeta dA. \tag{2.17}$$

The circulation around any circuit once and enclosing the whole vortex core is

$$\Gamma = \zeta \pi a^2 = const. \tag{2.18}$$

From Stokes theorem (2.9), for a circle with radius $r > 0$, we have $2\pi r u = \Gamma$, and further taking into account (2.18) we find the expression for velocity in the region of irrotational (potential) flow

$$u = \frac{a^2 \zeta}{2r} = \frac{\Gamma}{2\pi r}, \; r > a. \tag{2.19}$$

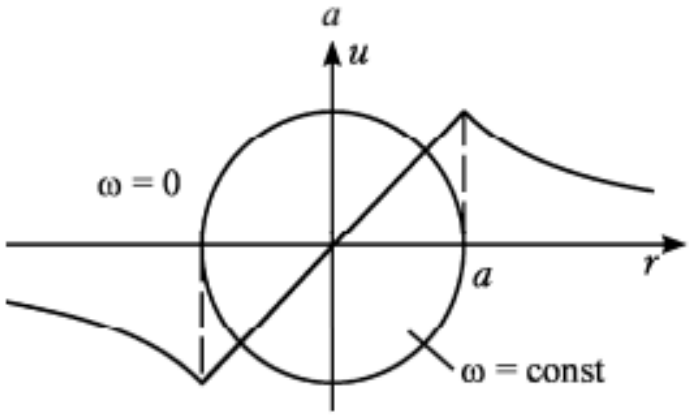

Fig. 2.3 Vorticity ω and tangential velocity v_θ profiles

As in the case of a cylindrical sheet, this distribution coincides with the velocity field induced by an infinitely thin vortex filament with intensity Γ at distance $r > a$. Inside the core, in the same way we obtain

$$2\pi r u = \pi r^2 \zeta, \tag{2.20}$$

or

$$u = \frac{\Gamma r}{2\pi a^2}, \; r < a. \tag{2.21}$$

Linearity of the profile indicates the solid-body rotation of fluid in the vortex core with angular velocity Ω equal to

$$\Omega = \frac{\Gamma}{2\pi a^2}. \tag{2.22}$$

The resultant velocity distribution is shown in Fig. 2.3. Apparently, there is a break in the velocity profile at the boundary of the core $r = a$, caused by a vorticity jump. Nevertheless, this model called the Rankine vortex is the most popular. It reflects the main features of concentrated vortices.

2.4 Shooting method

Another popular method for solving boundary value problems is called the shooting method. In it one utilizes an initial value method in the following way. One guesses $y'(a) = \gamma_1$ and

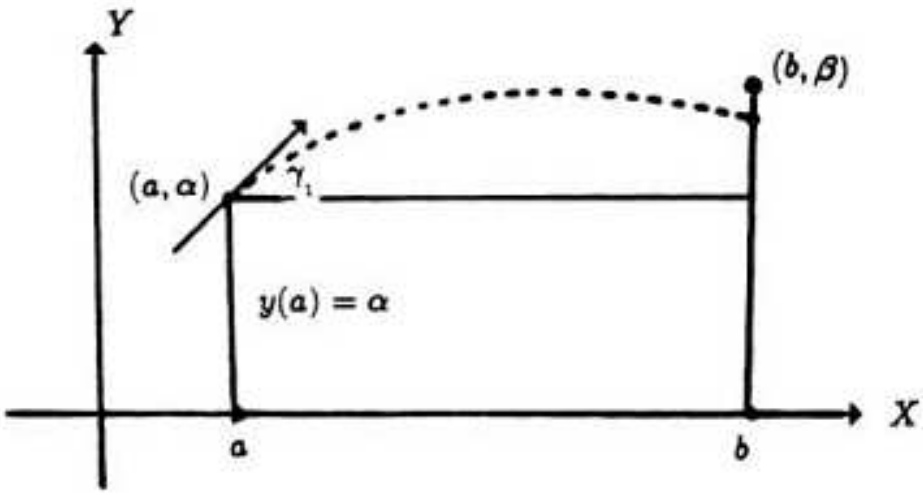

Fig. 2.4 A shooting method

solves the initial value problem

$$y = f(x, y, y'), \tag{2.23}$$

$$y(a) = \alpha,\ y'(a) = \gamma_1. \tag{2.24}$$

As shown in Fig. (2.4), one computes until $x = b$ and compares the numerical result with the desired result $y = \beta$. The figure also suggests that, physically, one has shot a projectile from the point (a, α) and its path at $x = b$ is below the desired height β. So, one next adjusts the initial angle γ_1 to, say, γ_2, in which $\gamma_2 > \gamma_1$. The numerical calculations are repeated with γ_1 replaced by γ_2 in (2.24). If the numerical solution this time at $x = b$ is, say, greater than β at $x = b$, then one repeats the process with a new angle γ_3 in the range $\gamma_1 < \gamma_3 < \gamma_2$. Hopefully, proceeding with such a process of refinement leads to numerical results which converge to $y = \beta$ at $x = b$.

2.5 Heaviside and Dirac delta functions

The Heaviside step function appears in many places in fluid mechanics. Simply put, it is a function whose value is zero for $x < 1$ and one for $x > 1$ as in Fig. 2.5. Explicitly,

$$H(x-1) \equiv \begin{cases} 0 & \text{for } x < 1, \\ 1 & \text{for } x > 1. \end{cases} \tag{2.25}$$

The derivative of Heaviside step function (2.25) is called Dirac delta function which is given by

$$\delta(x-1) = \frac{dH(x-1)}{dx} \equiv \begin{cases} 0 & \text{for } x \neq 1, \\ \infty & \text{for } x = 1. \end{cases} \tag{2.26}$$

This function has the properties

$$\int_{-\infty}^{\infty} \delta(x-1)dx = 1, \tag{2.27}$$

and

$$\int_{-\infty}^{\infty} \delta(x-1)f(x)dx = f(1), \tag{2.28}$$

for every continuous function $f(x)$. We can characterize the delta function by its sifting property:

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0). \tag{2.29}$$

Dirac has used a simple argument, based on the integration by parts formula, to get the sifting property of the derivative δ' of the delta function:

$$\int_{-\infty}^{\infty} \delta'(x)f(x)dx = -f'(0). \tag{2.30}$$

The theory of distributions as linear functionals, instead of defining the integral of a distribution and so proving that it satisfies some kind of integration by parts formula, just uses the formula deduced by Dirac for the delta function:

$$\int_{-\infty}^{\infty} \delta'(x)f(x)dx = -\int_{-\infty}^{\infty} \delta(x)f'(x)dx, \tag{2.31}$$

as “distributions derivative definition”.

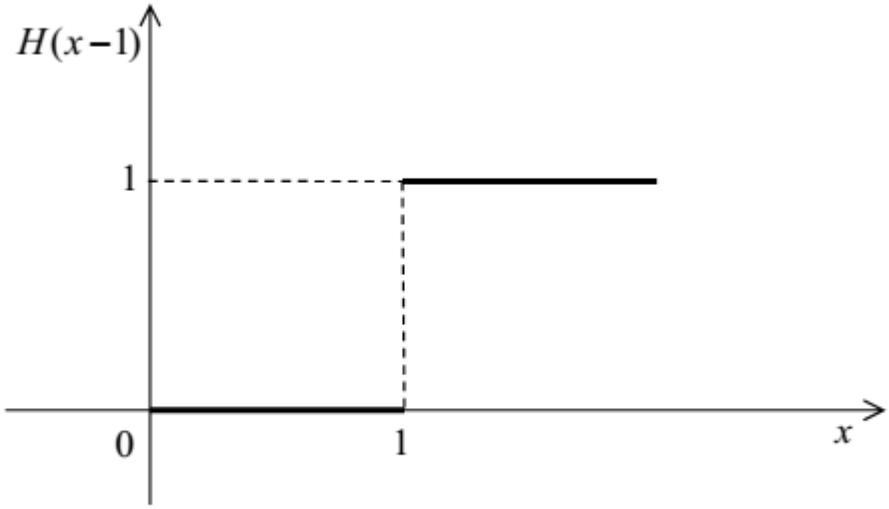

Fig. 2.5 Heaviside step function

Chapter 3

Finite thickness effect on speed of a vortex pair

3.1 Formulation of matched asymptotic expansions

We consider a counter-rotating vortex pair rotates with circulations $\pm\Gamma$ moving in an inviscid fluid or a viscous fluid with the kinematic viscosity ν. The core radius σ of the two vortices is assumed to be much smaller than the distance $2d$ between the centroids of the two vortices. We introduce the Cartesian coordinates (x,y), fixed in space, with the x-axis parallel to the direction of the line connecting the centroids. At the same time, we introduce local polar coordinates (r,θ), centered at the centroid (X,Y) of one of the vortices, moving with it. The angle is measured from the direction parallel to the x-axis, and therefore the laboratory and the moving frames are viewed with each other through $x = X + r\cos\theta$ and $y = Y + r\sin\theta$ (figure 3.1).

The governing equations of the problem are the Navier-Stokes equations for the velocity field $\boldsymbol{u}(\boldsymbol{x},t) = (u(x,y,t), v(x,y,t))$. The z component $\zeta = \zeta(x,y,t)$ of the vorticity is defined by $\zeta = \partial v/\partial x - \partial u/\partial y$. By taking the curl or the Navier-Stokes equation, we are left with the vorticity equation for ζ

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu\triangle\zeta, \tag{3.1}$$

where Δ is the two-dimensional Laplace operator. We introduce the streamfunction ψ by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. The streamfunction ψ has a link with the vorticity via

$$\zeta = -\triangle\psi. \tag{3.2}$$

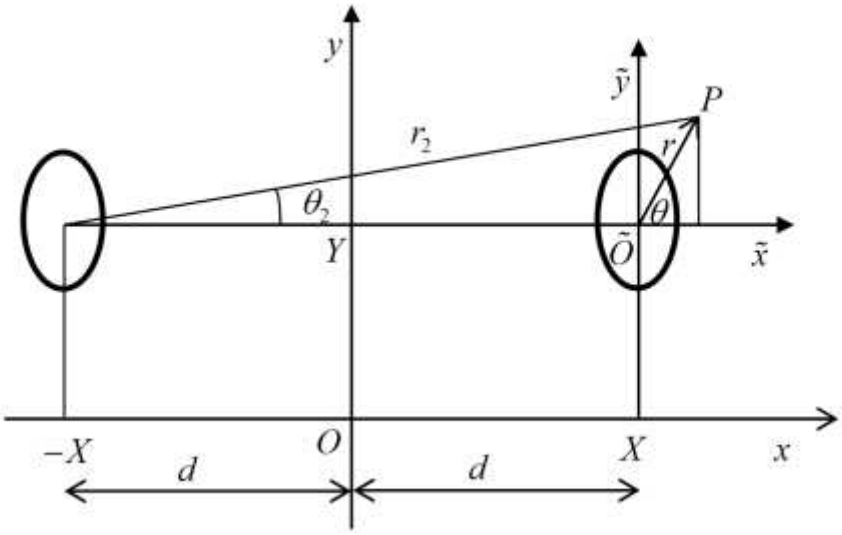

Fig. 3.1 Two dimensional viscous vortex pair in the coordinates systems. The Cartesian coordinates system fixed in space is denoted by (x,y), and (r,θ) is the polar coordinates system centered on (X,Y) in moving frame.

We suppose that the circulation of the Reynolds number Re is very large $Re = \Gamma/\nu \gg 1$. The solution of the vorticity equation (3.1) and the subsidiary relation (3.2) is constructed by use of the matched asymptotic expansions in a small parameter $\varepsilon = \sqrt{\nu/\Gamma}(\ll 1)$. The parameter ε is regarded as the ratio of the core radius σ to the distance between the centroids of the vortices $2d$. We introduce dimensionless inner variables r^*, by normalizing r by the core radius $\sigma = \varepsilon d$, and the dimensionless distance r_2^*, by normalizing r by d. By making an appropriate normalization for other flow variables, we have the following dimensionless variables with superscript $*$.

$$r = \varepsilon d r^*,\ r_2 = d r_2^*,\ t = \frac{d^2}{\Gamma} t^*,\ \psi = \Gamma \psi^*,\ \zeta = \frac{\Gamma}{\varepsilon^2 d^2}\zeta^*,$$
$$\mathbf{u} = \frac{\Gamma}{\varepsilon d}\mathbf{u}^*,\ (\dot{X},\dot{Y}) = \frac{\Gamma}{d}(\dot{X}^*,\dot{Y}^*), \tag{3.3}$$

with variables with asterisk representing dimensionless variables. The symbols $\dot{X}$ and $\dot{Y}$ are the x- and y-components of the velocity of the movement of the vortex core as a whole. We work out the solution in local co-moving polar coordinates (r,θ), in which the coupled

system of the vorticity equation (3.1) and the relation (3.2) can be written as

$$\frac{\partial \zeta}{\partial t} + \frac{1}{\varepsilon^2}[\zeta, \tilde{\psi}] = \hat{\nu}\triangle\zeta, \tag{3.4}$$

$$\zeta = -\triangle\tilde{\psi}, \tag{3.5}$$

where $\tilde{\psi}$ is the streamfunction for the flow relative to the moving frame

$$\tilde{\psi} \equiv \psi - \varepsilon r\left(\dot{X}\sin\theta - \dot{Y}\cos\theta\right). \tag{3.6}$$

in order to deal with the both inviscid and viscous cases, we introduce

$$\hat{\nu} = \begin{cases} 0, & \nu = 0, \\ 1, & \nu \neq 0. \end{cases} \tag{3.7}$$

and $[\cdot,\cdot]$ is the Jacobian

$$[\zeta, \tilde{\psi}] \equiv \frac{1}{r}\left(\frac{\partial\zeta}{\partial r}\frac{\partial\tilde{\psi}}{\partial\theta} - \frac{\partial\zeta}{\partial\theta}\frac{\partial\tilde{\psi}}{\partial r}\right). \tag{3.8}$$

The solution of the above equations is sought in a power series in ε as

$$\zeta = \zeta^{(0)} + \varepsilon\zeta^{(1)} + \varepsilon^2\zeta^{(2)} + \varepsilon^3\zeta^{(3)} + \varepsilon^4\zeta^{(4)} + \varepsilon^5\zeta^{(5)} + \varepsilon^6\zeta^{(6)} + \varepsilon^7\zeta^{(7)} + \ldots, \tag{3.9}$$

$$\psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \varepsilon^3\psi^{(3)} + \varepsilon^4\psi^{(4)} + \varepsilon^5\psi^{(5)} + \varepsilon^6\psi^{(6)} + \varepsilon^7\psi^{(7)} + \ldots, \tag{3.10}$$

$$\dot{X} = \dot{X}^{(0)} + \varepsilon\dot{X}^{(1)} + \varepsilon^2\dot{X}^{(2)} + \varepsilon^3\dot{X}^{(3)} + \varepsilon^4\dot{X}^{(4)} + \varepsilon^5\dot{X}^{(5)} + \varepsilon^6\dot{X}^{(6)} + \varepsilon^7\dot{X}^{(7)} + \ldots, \tag{3.11}$$

$$\dot{Y} = \dot{Y}^{(0)} + \varepsilon\dot{Y}^{(1)} + \varepsilon^2\dot{Y}^{(2)} + \varepsilon^3\dot{Y}^{(3)} + \varepsilon^4\dot{Y}^{(4)} + \varepsilon^5\dot{Y}^{(5)} + \varepsilon^6\dot{Y}^{(6)} + \varepsilon^7\dot{Y}^{(7)} + \ldots, \tag{3.12}$$

where $\zeta^{(i)}$ and $\psi^{(i)}$ for $i = 0,1,2,3,\ldots$ are functions of r, θ and, in the viscous case, of t. The streamfunction $\tilde{\psi}$ for the flow relative to the moving frame is expanded in the same way as (3.10), and, in view of (3.6), has relation with the expansion of ψ via

$$\tilde{\psi}^{(j)} = \psi^{(j)} - r\left(\dot{X}^{(j-1)}\sin\theta - \dot{Y}^{(j-1)}\cos\theta\right), \tag{3.13}$$

for $j = 1,2,3,\ldots$

3.2 Outer solution: Dyson's technique

In order to take account of the influence of finite size of the vortex cores on the motion of the vortex pair, we perform an extension, to higher orders in the small parameter ε, extension of the method of matched asymptotic expansions contrived by Ting and Tung (1965).

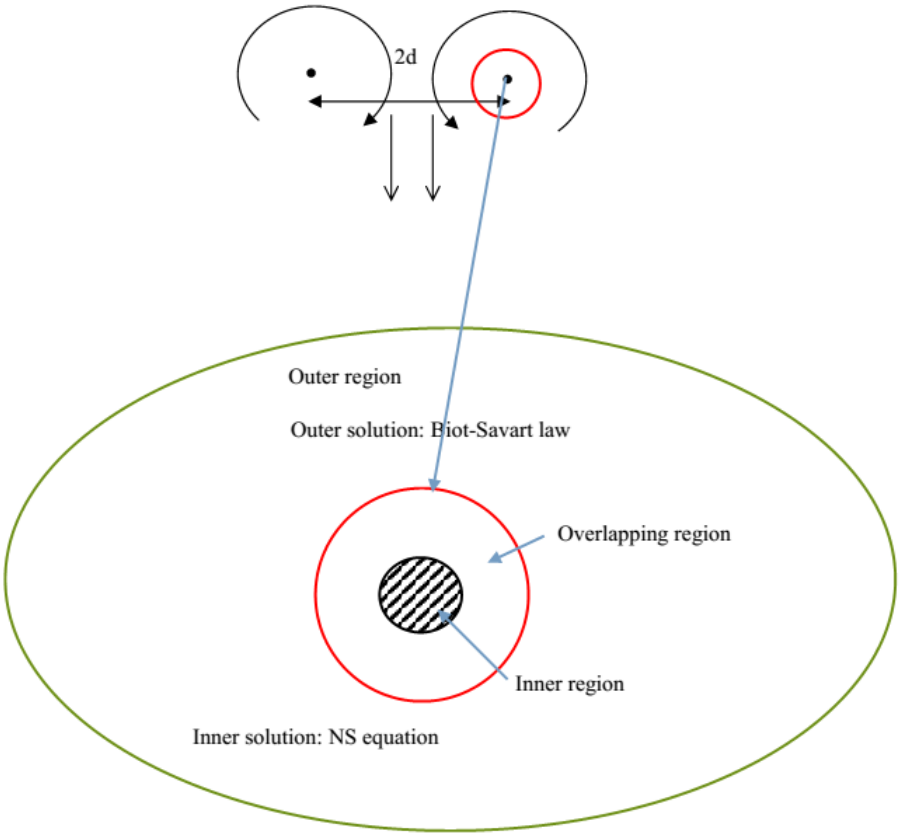

Fig. 3.2 The matched asymptotic expansions

We pay attention to one of the two vortices, specifically the right vortex in figure 3.1. The domain is separated into two, the inner and the outer regions. The inner region signifies the vortex core and the surrounding region with the distance from the core center of order the core radius. The region exterior to it is the outer region. The characteristic length scale of the outer region is the distance between the centroids of the two vortices. We seek the solution in a power series in ε of the Navier-Stokes or the Euler equations, which is exposed at some length in the next section (section 3.3). The Biot-Savart law provides the solution of the Navier-Stokes and the Euler equations in the outer region where the vorticity is negligibly small, subject to the condition that the vorticity distribution is precisely given. For this, an input from the knowledge of the detail of the inner solution is indispensable. This input, and vice versa are unimplemented by matching the outer solution to the inner solution in the overlapping (common) region. The matched asymptotic expansion can be seen as Figure 3.2.

Given the vorticity distribution $\zeta(x',y')$, the Biot-Savart law is represented for the stream-function $\psi(x,y)$ as

$$\psi(x,y) = -\frac{1}{2\pi}\int \zeta(x',y')\log\sqrt{(x-x')^2+(y-y')^2}\mathrm{d}x'\mathrm{d}y', \tag{3.14}$$

The integration domain is virtually limited to the two vortex cores on which ζ is non-negligible. Hence the outer solution comprises contributions from the left and the right vortices: $\psi = \psi_{\text{left}} + \psi_{\text{right}}$. We are requested to evaluate the near field, valid $\sigma \ll r \ll d$, around the right vortex, in order to gain the matching condition on the inner solution.

We evaluate the near field of ψ_{right}, the self-induced flow by the right vortex. To this end, we adapt Dyson's technique (Dyson 1893), being originally developed for an axisymmetric flow, to the planer flow. As a first step, we rewrite (3.14), using the shift operator, into

$$\psi_{\text{right}}(x,y) = -\frac{1}{2\pi}\int \zeta(x',y')e^{-x'\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y}}\log r\mathrm{d}x'\mathrm{d}y', \tag{3.15}$$

where $r=(x^2+y^2)^{1/2}$. We then expand the exponential function in a Taylor series in the exponent to give

$$\begin{aligned}\psi_{\text{right}} = -\frac{1}{2\pi}\int \zeta(x',y')\Bigg\{&1-\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}\right)+\frac{1}{2!}\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}\right)^2\\ &-\frac{1}{3!}\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}\right)^3+\frac{1}{4!}\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}\right)^4+\cdots\Bigg\}\log r\mathrm{d}x'\mathrm{d}y'.\end{aligned} \tag{3.16}$$

We insert the general form of the vorticity distribution $\zeta(x',y') = \zeta_0 + \sum_{j=1}^{\infty}\left\{\zeta_{j1}\cos j\theta + \zeta_{j2}\sin j\theta\right\}$ into (3.16), the detailed information of which is supplied by the inner solution, and perform integration in θ first, leaving

$$\begin{aligned}\psi_{\text{right}} = &-\frac{\Gamma}{2\pi}\log r+\frac{1}{2}\left(\int_0^\infty \zeta_{11}r^2\mathrm{d}r\right)\frac{\cos\theta}{r}+\frac{1}{2}\left(\int_0^\infty \zeta_{12}r^2\mathrm{d}r\right)\frac{\sin\theta}{r}\\ &+\frac{1}{4}\left(\int_0^\infty \zeta_{21}r^3\mathrm{d}r\right)\frac{\cos 2\theta}{r^2}+\frac{1}{4}\left(\int_0^\infty \zeta_{22}r^3\mathrm{d}r\right)\frac{\sin 2\theta}{r^2}\\ &+\frac{1}{6}\left(\int_0^\infty \zeta_{31}r^4\mathrm{d}r\right)\frac{\cos 3\theta}{r^3}+\frac{1}{6}\left(\int_0^\infty \zeta_{32}r^4\mathrm{d}r\right)\frac{\sin 3\theta}{r^3}+\cdots,\end{aligned} \tag{3.17}$$

where $\Gamma = \int \zeta_0(x',y')\mathrm{d}x'\mathrm{d}y'$, and repeated use of

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log r = 0 \tag{3.18}$$

has facilitated the calculation. The contribution ψ_{left} from the left vortex takes the same form except that the sign of vorticity is opposite and that r and θ are replaced by r_2 and θ_2 as defined by figure 3.1.

$$\begin{aligned}\psi_{\text{left}} = \frac{\Gamma}{2\pi}\log r_2 &- \frac{1}{2}\left(\int_0^\infty \zeta_{11} r^2 \mathrm{d}r\right)\frac{\cos\theta_2}{r_2} - \frac{1}{2}\left(\int_0^\infty \zeta_{12} r^2 \mathrm{d}r\right)\frac{\sin\theta_2}{r_2} \\ &-\frac{1}{4}\left(\int_0^\infty \zeta_{21} r^3 \mathrm{d}r\right)\frac{\cos 2\theta_2}{r_2^2} - \frac{1}{4}\left(\int_0^\infty \zeta_{22} r^3 \mathrm{d}r\right)\frac{\sin 2\theta_2}{r_2^2} \\ &-\frac{1}{6}\left(\int_0^\infty \zeta_{31} r^4 \mathrm{d}r\right)\frac{\cos 3\theta_2}{r_2^3} - \frac{1}{6}\left(\int_0^\infty \zeta_{32} r^4 \mathrm{d}r\right)\frac{\sin 3\theta_2}{r_2^3} + \cdots, \end{aligned} \tag{3.19}$$

where $\log r_2$ is derived in Appendix B. We thus obtain a multi-pole expansion form of the outer solution, which is written in terms of dimensionless variables (3.3) as

$$\begin{aligned}\psi^*(x^*,y^*) &= \psi^*_{\text{right}}(x^*,y^*) + \psi^*_{\text{left}}(x^*,y^*) \\ &= -\frac{1}{2\pi}\log(\varepsilon d) r^* + \frac{1}{2\pi}\log d r_2^* \\ &\quad + \frac{1}{2}\left(\int_0^\infty \zeta^*_{11} r^{*2} \mathrm{d}r^*\right)\frac{\cos\theta}{r^*} - \frac{\varepsilon}{2}\left(\int_0^\infty \zeta^*_{11} r^{*2} \mathrm{d}r^*\right)\frac{\cos\theta_2}{r_2^*} \\ &\quad + \frac{1}{2}\left(\int_0^\infty \zeta^*_{12} r^{*2} \mathrm{d}r^*\right)\frac{\sin\theta}{r^*} - \frac{\varepsilon}{2}\left(\int_0^\infty \zeta^*_{12} r^{*2} \mathrm{d}r^*\right)\frac{\sin\theta_2}{r_2^*} \\ &\quad + \frac{1}{4}\left(\int_0^\infty \zeta^*_{21} r^{*3} \mathrm{d}r^*\right)\frac{\cos 2\theta}{r^{*2}} - \frac{\varepsilon^2}{4}\left(\int_0^\infty \zeta^*_{21} r^{*3} \mathrm{d}r^*\right)\frac{\cos 2\theta_2}{r_2^{*2}} \\ &\quad + \frac{1}{4}\left(\int_0^\infty \zeta^*_{22} r^{*3} \mathrm{d}r^*\right)\frac{\sin 2\theta}{r^{*2}} - \frac{\varepsilon^2}{4}\left(\int_0^\infty \zeta^*_{22} r^{*3} \mathrm{d}r^*\right)\frac{\sin 2\theta_2}{r_2^{*2}} \\ &\quad + \cdots. \end{aligned} \tag{3.20}$$

This representation serves as a basis for deriving the inner-limit of the outer solution. This limit then provides us with the matching condition on the inner solution in the overlapping region $\sigma \ll r \ll d$.

To extract the information on the field near the right vortex with distance from the its center $r \ll d$, we subsitute

$$\begin{aligned} r_2 &= \left(4d^2 + 4dy + r^2\right)^{1/2}, \\ \cos\theta_2 &= \frac{2d+y}{r_2}, \quad \sin\theta_2 = \frac{x}{r_2}, \end{aligned} \tag{3.21}$$

and expand the resulting expressions in powers of a small parameter r/d. For an inviscid flow, $\zeta_0(r)$ may be freely given, but otherwise the distribution of vorticity ζ_0, ζ_{11}, ζ_{12}, ζ_{211},

ζ_{22}, ζ_{31}, ζ_{32}, ... are as yet unknown. These are determined order by order by building the inner solution. In the following section, we proceed to manipulating the inner solution.

3.3 Inner solution

This section manipulates the asymptotic solution of the vorticity equation (3.1), concomitantly with the streamfunction from (3.2), in the inner region. The equations for determining the asymptotic solution can be obtained by collecting terms of the same order in small parameter ε in (3.1) and (3.2). The solution is constructed at each order so as to satisfy the matching condition of the corresponding order, to be derived from (3.20) supplemented by (3.21). We start from the zeroth order and go up to the seventh order in ε. Hereafter we work exclusively with the dimensionless variables, for which we drop off the superscript asterisk $*$.

3.3.1 Zeroth-order solution

Equations governing the leading-order vorticity $\zeta^{(0)}$ are supplied by the $O(\varepsilon^{-2})$ terms in (3.4) and the $O(\varepsilon^{0})$ terms in (3.5)

$$\left[\zeta^{(0)}, \psi^{(0)}\right] = 0, \quad \zeta^{(0)} = -\triangle\tilde{\psi}^{(0)}. \tag{3.22}$$

The Jacobian form of the left dictates a functional relation $\zeta^{(0)} = \mathscr{F}(\psi^{(0)})$, for some function $\mathscr{F}$. Suppose that the flow has a single stagnation point at $r = 0$, all the streamlines being closed around that point, then the solution of $\triangle\tilde{\psi^{(0)}} = -\mathscr{F}(\psi^{(0)})$ must be radial, namely, $\psi^{(0)} = \psi^{(0)}(r)$. The streamlines are all circles (Fraenkel 1970, Fukumoto and Moffatt 2000). The functional form of $\psi^{(0)}$ and $\zeta^{(0)}$ remains undetermined at this level of approximation. Rather, $\zeta^{(0)}$ will be governed by the axisymmetric part of $O(\varepsilon^{0})$ terms in (3.4), to be determined subsequently

The condition to be imposed on the vorticity distribution by (3.20) is that it decay sufficiently rapidly so that the moments of vorticity exist.

The functions $\tilde{\psi}^{(0)}$ and $\zeta^{(0)}$ are undetermined at this level of approximation. Therefore, from the second order equation we can find that particular solution of heat transfer equation

$$-\frac{\partial\zeta^{(0)}}{\partial t} + \hat{\nu}\Delta\zeta^{(0)} = 0. \tag{3.23}$$

When vorticity is concentrated at the origin with strength Γ at the initial instant $t = 0$, $\zeta^{(0)}(r,0) = \Gamma\delta(x)\delta(y)$, the solution is the well-known Oseen vortex

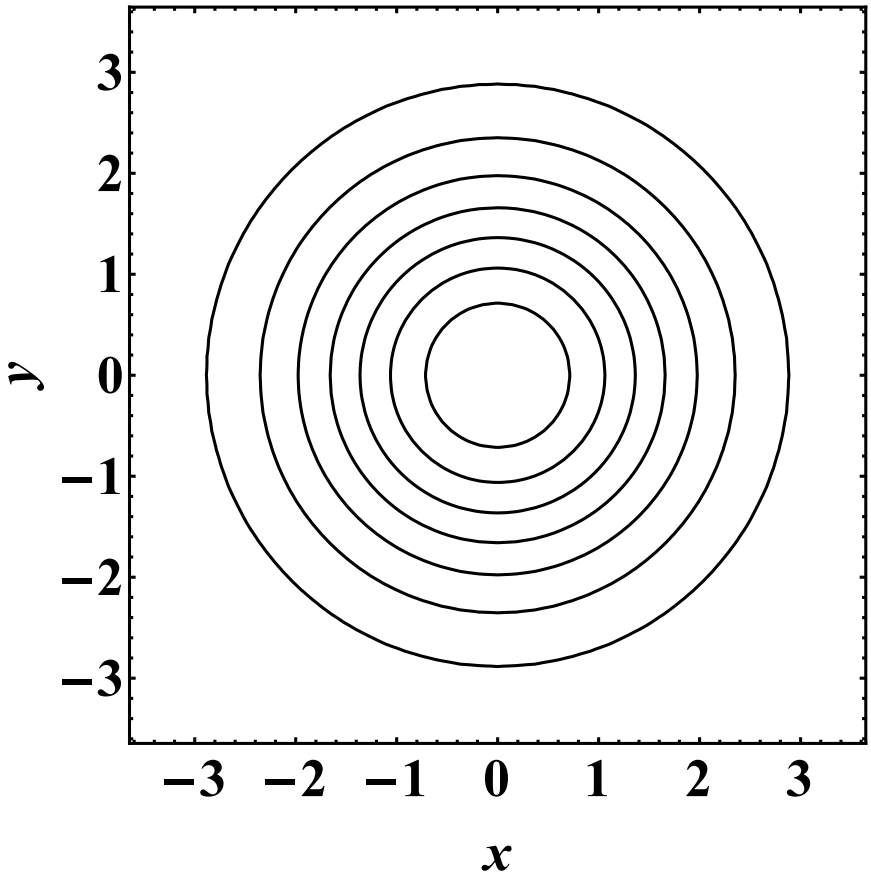

Fig. 3.3 Gaussian vorticity distribution

$$\zeta^{(0)} = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right), \tag{3.24}$$

$$v^{(0)} = \frac{\Gamma}{2\pi r}\left[1-\exp\left(-\frac{r^2}{4\nu t}\right)\right]. \tag{3.25}$$

The Gaussian vorticity distribution (3.24) at leading order $O(\varepsilon^0)$ is circles, Fig. 3.3. Remark that azimuthal velocity is $v^{(0)} = -\partial\tilde{\psi}^{(0)}/\partial r$ so that the streamfunction can be obtained simply by taking minus integration of the azimuthal velocity with respect to r

$$\tilde{\psi}^{(0)} = -\frac{\Gamma}{2\pi}\int_0^r \left[1-\exp\left(-\frac{r^2}{4\nu t}\right)\right] r'^{-1}dr'. \tag{3.26}$$

This solution automatically meets the requirement of the matching condition at leading order of (3.20)

$$\psi^{(0)} \sim -\frac{1}{2\pi}\log\varepsilon dr \quad \text{as } r\to\infty \tag{3.27}$$

3.3.2 First-order solution

Collecting $O(\varepsilon^1)$ terms in the vorticity equation (3.4) and $O(\varepsilon^1)$ terms in (3.5), we have

$$[\zeta^{(0)}, \tilde{\psi}^{(1)}] + [\zeta^{(1)}, \psi^{(0)}] = 0, \tag{3.28}$$

$$\zeta^{(1)} = -\triangle\tilde{\psi}^{(1)}, \tag{3.29}$$

where $\tilde{\psi}^{(1)}$ is the streamfunction, of $O(\varepsilon)$, for the flow relative to the moving frame as defined by (3.13) with $j = 1$:

$$\tilde{\psi}^{(1)} = \psi^{(1)} - r\left(\dot{X}^{(0)}\sin\theta - \dot{Y}^{(0)}\cos\theta\right). \tag{3.30}$$

The $O(\varepsilon)$ terms of (3.20) provide the matching condition on $\tilde{\psi}^{(1)}$.

Substituting from (3.21) and making an expansion in powers of ε, we see from (3.20) that the left vortex induces, around the right vortex, a dipole field of (ε) in proportion to $\cos\theta$. In accordance, $\cos\theta$-component is induced at $O(\varepsilon)$ in addition to the monopole component

$$\psi^{(1)} = \psi_0^{(1)} + \psi_{11}^{(1)}\cos\theta, \tag{3.31}$$

and $\zeta^{(1)}$ has the same dependence on θ. Correspondingly, we have only to consider the monopole and dipole field proportional to $\cos\theta$. Substituting (3.31) and (3.21) into (3.20) yields the matching condition at $O(\varepsilon)$ as

$$\psi^{(1)} \sim \left(\frac{r}{4\pi} + \frac{1}{2r}\int_0^\infty \zeta_{11}^{(1)} r^2 \mathrm{d}r\right)\cos\theta \quad \text{as } r \to \infty, \tag{3.32}$$

supplemented by

$$\int_0^\infty \zeta_0^{(1)} r\mathrm{d}r = 0. \tag{3.33}$$

By virtue of axisymmetry of $\zeta^{(0)}$ and $\psi^{(0)}$, (3.28) is integrated with respect to θ to yield

$$\zeta^{(1)} = a\tilde{\psi}^{(1)} + b_1(r,t), \tag{3.34}$$

where $b_1(r,t)$ is an arbitrary function of r and t, and

$$a = -\frac{1}{v^{(0)}}\frac{\partial\zeta^{(0)}}{\partial r}, \tag{3.35}$$

with $v^{(0)} = -\partial\psi^{(0)}/\partial r$ being the azimuthal velocity field. Combining (3.30) with (3.34), we obtain equation governing $\tilde{\psi}_{11}^{(1)} = \psi_{11}^{(1)} + r\dot{Y}^{(0)}$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + a\right)\tilde{\psi}_{11}^{(1)} = 0, \tag{3.36}$$

where $\tilde{\psi}_{11}^{(1)} = \psi^{(1)} + r\dot{Y}^{(0)}$ is the streamfunction in moving polar coordinates system that moves together with local coordinates and the vortex core. Note that:

$$\zeta^{(0)} = \frac{1}{r}\frac{\partial}{\partial r}\left(rv^{(0)}\right). \tag{3.37}$$

equation (3.35) can be written as

$$a = -\frac{1}{v^{(0)}}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rv^{(0)}\right)\right). \tag{3.38}$$

By substituting equation (3.38) into (3.36) it is clear that $v^{(0)}$ is the solution (Fukumoto and Moffat (2000)). The general solution is then obtained as

$$\tilde{\psi}_{11}^{(1)} = c_1 v^{(0)}(r) + c_2 v^{(0)}(r)\int_0^r \frac{1}{rv_0^2}dr, \tag{3.39}$$

for arbitrary constants c_1 and c_2. Since velocity $v^{(0)}$ is the homogeneous solution of the second-order differential equation (3.36), we can choose $c_1 = 0$. The second term of (3.39) is divergent when $r = 0$, while the solution should be non zero. For non zero r, by choosing the Rankine vortex (Crowdy, 2002) at leading order $\varepsilon^{(0)}$

$$v^{(0)} = \begin{cases} \frac{1}{2}\omega_0 r & \text{for } r < \sigma, \\ \frac{1}{2r}\omega_0\sigma^2 & \text{for } r > \sigma, \end{cases} \tag{3.40}$$

yields the second term of (3.39) is infinite, where σ is the core radius and ω_0 is the uniform vorticity distribution in the circular region by using definition of the circulation $\Gamma = \int \omega_0 dA = \pi\sigma^2\omega_0$. To make the solution finite we may choose $c_2 = 0$ hence equation (3.39) equals zero. Alternatively we give another point of view of choosing the coefficients of equation (3.39). Since a viscous vortex pair moves with local coordinates, small changing coordinates of the center of a vortex will not contribute to the local coordinates; or we can say that a vortex pair stays on their position. Hence without loss of generality of the solution, we may put the arbitrary constants c_1 and c_2 equal zero. Then equation (3.39) becomes $\tilde{\psi}_{11}^{(1)} = 0$. Applying this result into equation (3.29) we get $\zeta_{11}^{(1)} = 0$. By matching condition (3.32), the inner

solution for streamfunction is

$$\psi^{(1)} \sim \frac{r}{4\pi}\cos\theta \quad \text{as } r \to \infty. \tag{3.41}$$

Therefore the traveling speed of a viscous vortex pair is given by

$$\dot{Y}^{(0)} = -\frac{1}{4\pi},\ \dot{X}^{(0)} = 0. \tag{3.42}$$

The traveling speed of a vortex pair in a viscous or an inviscid fluid begins with that of a pair of point vortices as expected (Ting and Tung 1965). Influence of finiteness of the vortex cores makes its appearance at a higher order.

An arbitrary constant c_1 corresponds to the freedom of choosing of the origin of the local co-moving coordinates $(\tilde{x},\tilde{y})$ within the accuracy of $O(\varepsilon^2)$ (Fukumoto and Moffatt 2000). Since the solution has the fore-and-aft symmetry, the origin should be maintained at the core center in the y direction by selecting $c_1 = 0$. In the Appendix E, we shall show that the arbitrary function $b_1(r,t)$ is zero. Thus we may conveniently take $\zeta^{(1)} \equiv 0$ and $\tilde{\psi}^{(1)} \equiv 0$.

3.3.3 Second-order solution

Collecting $O(\varepsilon^0)$ terms in the vorticity equation (3.4) and $O(\varepsilon^2)$ terms in (3.5), we have

$$[\zeta^{(2)},\psi^{(0)}] + [\zeta^{(1)},\tilde{\psi}^{(1)}] + [\zeta^{(0)},\tilde{\psi}^{(2)}] = -\frac{\partial\zeta^{(0)}}{\partial t} + \hat{\nu}\Delta\zeta^{(0)}, \tag{3.43}$$

$$\zeta^{(2)} = -\triangle\tilde{\psi}^{(2)}, \tag{3.44}$$

where $\tilde{\psi}^{(2)}$ is the streamfunction, of $O(\varepsilon^2)$, for the flow relative to the moving frame as defined by (3.13) with $j = 2$:

$$\tilde{\psi}^{(2)} = \psi^{(2)} - r\left(\dot{X}^{(1)}\sin\theta - \dot{Y}^{(1)}\cos\theta\right). \tag{3.45}$$

In the same way as at $O(\varepsilon)$, we see from (3.20) that the left vortex induces, around the right vortex, a quadrupole field of $O(\varepsilon^2)$ in proportion to $\cos 2\theta$. In accordance, we have only to consider the monopole and quadrupole field proportional to $\cos 2\theta$.

$$\psi^{(2)} = \psi_0^{(2)} + \psi_{21}^{(2)}\cos 2\theta. \tag{3.46}$$

with the same θ-dependence for $\zeta^{(2)}$. With this form, the matching condition to be imposed on $\psi^{(2)}$ is found from (3.20) to be

$$\psi^{(2)} \sim \left(-\frac{r^2}{16\pi} + \frac{q_2}{r^2}\right)\cos 2\theta \ \text{ as } r \to \infty, \tag{3.47}$$

where

$$q_2 = \frac{1}{4}\int_0^\infty \zeta_{21}^{(2)} r^3 \mathrm{d}r, \tag{3.48}$$

is the strength of the quadrupole of $O(\varepsilon^2)$. It immediately follows from this argument that the motion of the vortex pair as a whole is uninfluenced at $O(\varepsilon^2)$:

$$\dot{X}^{(1)} = 0, \ \dot{Y}^{(1)} = 0. \tag{3.49}$$

We are now prepared to tackle with (3.43). Recalling that $\zeta^{(1)} = \tilde{\psi}^{(1)} \equiv 0$ and that the leading-order fields $\zeta^{(0)}$ and $\psi^{(0)}$ are axisymmetric, we rewrite (3.43) into

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{\partial\zeta^{(0)}}{\partial r}\tilde{\psi}^{(2)} - \zeta^{(2)}\frac{\partial\psi^{(0)}}{\partial r}\right\} = -\frac{\partial\zeta^{(0)}}{\partial t} + \hat{\nu}\Delta\zeta^{(0)}. \tag{3.50}$$

Intregration of (3.50) with respect to θ over $[0, 2\pi)$ yields the heat conduction equation

$$\frac{\partial\zeta^{(0)}}{\partial t} - \hat{\nu}\Delta\zeta^{(0)} = 0. \tag{3.51}$$

For illustration, we consider an example where vorticity, with unit strength, is concentrated at the origin at the initial instant $t = 0$, namely, $\zeta^{(0)}(x, y, 0) = \delta(x)\delta(y)$, with use of the Dirac delta function. Then, in the subsequent evolution, the vorticity takes the Gaussian distribution (Ting and Tung 1965, Fukumoto and Moffatt 2000, Gaifullin and Zubtsov 2004),

$$\zeta^{(0)} = \frac{1}{4\pi\hat{\nu}t}\exp\left(-\frac{r^2}{4\hat{\nu}t}\right), \tag{3.52}$$

and the corresponding azimuthal velocity $v^{(0)}$ is

$$v^{(0)} = \frac{1}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\hat{\nu}t}\right)\right]. \tag{3.53}$$

This is called the Oseen vortex. Notice that, in the inviscid case ($\hat{\nu} = 0$), (3.23) tells that, in the inviscid case, $\zeta^{(0)}$ should be steady (Fukumoto and Moffatt 2000), but can otherwise be arbitrary.

The rest of (3.50) brings us

$$\zeta^{(2)} = a\tilde{\psi}^{(2)} + b_2(r,t), \tag{3.54}$$

where a is defined by (3.35), and $b_2(r,t)$ is independent of θ but otherwise an arbitrary function of r and t. As in the case of b_1, we can prove for the viscous case that $b_2(r,t) \equiv 0$ and, by enforcing $\int_0^\infty \zeta_0^{(2)} r\mathrm{d}r = 0$ similarly as (3.33), that $\psi_0^{(2)} \equiv 0$. The second-order field (3.46) comprises the quadrupole component only, and equation ruling $\psi_{21}^{(2)}$ is derived from (3.54), combined with (3.44), as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2} + a\right)\tilde{\psi}_{21}^{(2)} = 0. \tag{3.55}$$

The boundary condition (3.47) on $\psi_{21}^{(2)}$ reads from (3.47)

$$\psi_{21}^{(2)} \sim -\frac{r^2}{16\pi} + \frac{q_2}{r^2} \quad \text{as } r \to \infty. \tag{3.56}$$

It turns out that the value of the strength q_2 of the quadrupole plays the key role for the motion of vortex pair. This is determined by numerically integrating (3.55) subject to (3.56). We illustrate this procedure for the case of the Oseen vortex, being given by (3.52) and (3.53), at $O(\varepsilon^0)$.

To this aim, we use the shooting method (see Moffatt *et al* 1994). We rewrite (3.55), for a function $\phi_{21}^{(2)} = \phi_{21}^{(2)}(\xi)$ of $\xi = r/\sqrt{\hat{\nu}t}$ defined through

$$\psi_{21}^{(2)} = \frac{\hat{\nu}t}{4\pi}\phi_{21}^{(2)}(\xi) - \frac{r^2}{16\pi}, \tag{3.57}$$

into

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{4}{\xi^2}\right)\phi_{21}^{(2)}(\xi) = -\frac{\xi^2}{4}\left(e^{\frac{\xi^2}{4}} - 1\right)^{-1}\left(\phi_{21}^{(2)}(\xi) - \frac{\xi^2}{4}\right). \tag{3.58}$$

The local non-singular solution around $\xi = 0$ is manipulated with ease as

$$\phi_{21}^{(2)}(\xi) = \alpha_2\xi^2 + \frac{1}{12}\left(\frac{1}{4} - \alpha_2\right)\left(\xi^4 - \frac{5}{64}\xi^6 + \frac{17}{3840}\xi^8 - \cdots\right), \tag{3.59}$$

with arbitrary constant α_2 at our disposal. The constant α_2 should be determined in such a way that $\phi_{21}^{(2)}$ decays at large values of ξ as $\phi_{21}^{(2)} \propto \xi^{-2}$ as $\xi \to \infty$. Our numerical calculation produces $\alpha_2 \approx -0.38117483$, and $\xi^2\phi_{21}^{(2)}(\xi) \to E_2$ as $\xi \to \infty$ with $E_2 \approx -17.4725096948$.

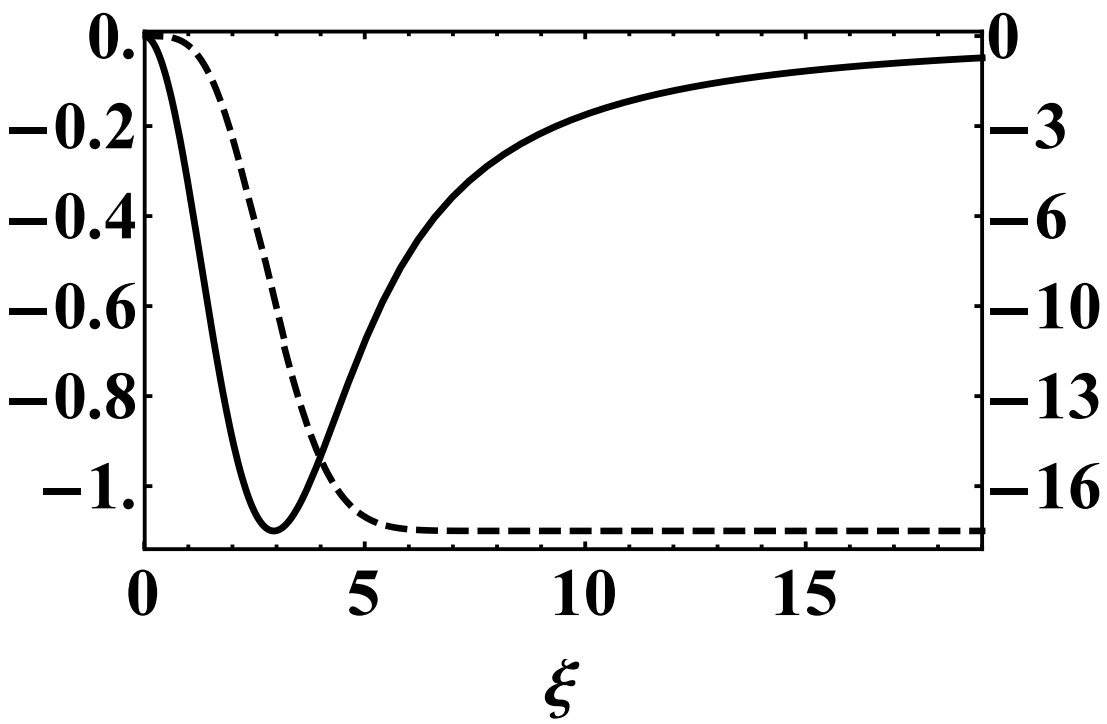

Fig. 3.4 The solution $\phi_{21}^{(2)}(\xi)$ (solid line) of (3.58) using shooting method. The dashed line is $\xi^2\phi_{21}^{(2)}(\xi)$.

These values coincides with those of Gaifullin and Zubtsov (2004) to 5 digits. The quadrupole strength q_2 has link with E_2 via

$$q_2 = \frac{(\hat{v}t)^2}{4\pi}E_2. \tag{3.60}$$

Fig. 3.4 displays the solution $\phi_{21}^{(2)}(\xi)$. The dashed line displays $\xi^2\phi_{21}^{(2)}(\xi)$.

The second-order solution represents elliptical deformation of the core by the pure shear induced by the companion vortex. To see this, we draw in Fig. 3.5 contours of the second-order vorticity field $\zeta^{(2)} = \zeta_{21}^{(2)}(r)\cos 2\theta$. Contours of the total vorticity $\zeta = \zeta^{(2)} + \varepsilon^2\zeta^{(2)}$ to $O(\varepsilon^2)$ and the corresponding contours of the streamfunction $\tilde{\psi} = \psi^{(0)} + \varepsilon^2\zeta_{21}^{(2)}(r)\cos 2\theta$ for the flow relative to the frame moving with the vortex pair are displayed in figures (3.6) and (3.7).

The elliptical deformation of the core at $O(\varepsilon^2)$ does not react back on the motion of the vortex pair as a whole at $O(\varepsilon^2)$. In order to pursue the influence of the core deformation on the motion, we proceed to higher orders.

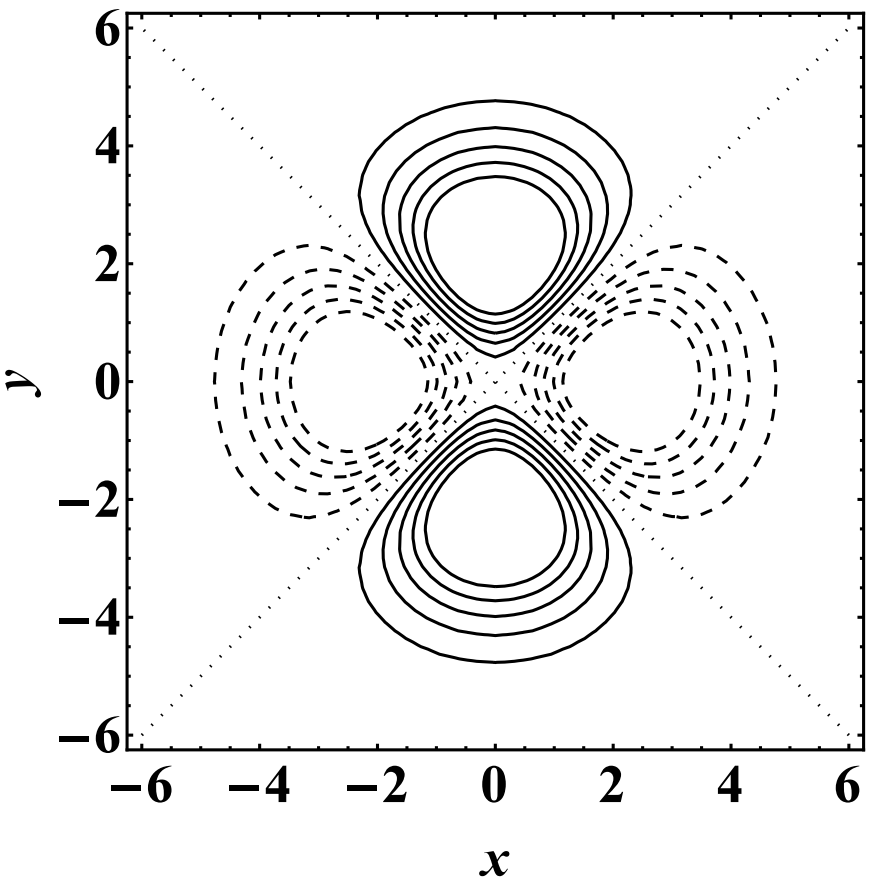

Fig. 3.5 The vorticity $\zeta_{21}^{(2)}(r)\cos 2\theta$ represents quadrupole at origin due to interference the outer flow.

3.3.4 Third-order solution

Collecting $O(\varepsilon^1)$ terms in the vorticity equation (3.4) and $O(\varepsilon^3)$ terms in (3.5), we have

$$[\zeta^{(3)}, \psi^{(0)}] + [\zeta^{(2)}, \tilde{\psi}^{(1)}] + [\zeta^{(1)}, \tilde{\psi}^{(2)}] + [\zeta^{(0)}, \tilde{\psi}^{(3)}] = -\frac{\partial \zeta^{(1)}}{\partial t} + \hat{\nu}\Delta\zeta^{(1)}, \tag{3.61}$$

$$\zeta^{(3)} = -\triangle\tilde{\psi}^{(3)}, \tag{3.62}$$

where $\tilde{\psi}^{(3)}$ is the streamfunction, of $O(\varepsilon^3)$, for the flow relative to the moving frame as defined by (3.13) with $j = 3$:

$$\tilde{\psi}^{(3)} = \psi^{(3)} - r\left(\dot{X}^{(2)}\sin\theta - \dot{Y}^{(2)}\cos\theta\right). \tag{3.63}$$

According to (3.20) supplemented by (3.21), the left vortex induces, around the right vortex, a hexapole field of (ε^3) proportional to $\cos 3\theta$. Remembering the result of section

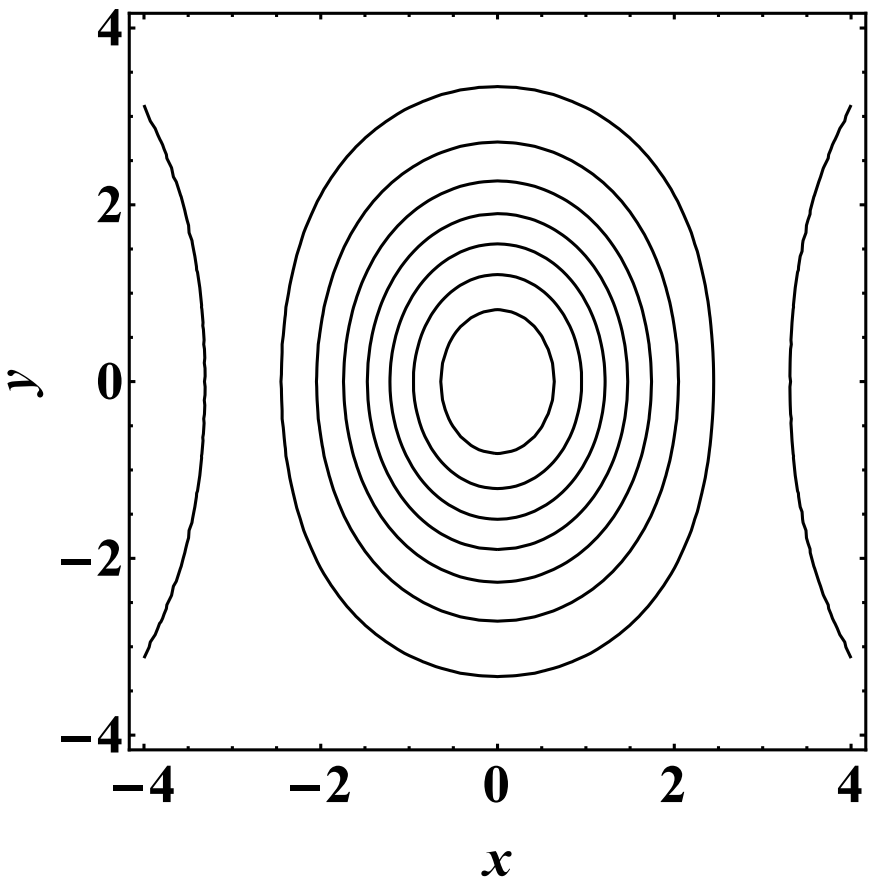

Fig. 3.6 The contour of vorticity field forms ellipse using $\zeta = \zeta^{(2)} + \varepsilon^2 \zeta^{(2)}$ with $\varepsilon = 0.3$.

3.3.2 that $\tilde{\psi}^{(1)} \equiv 0$ and $\zeta^{(1)} \equiv 0$. we conclude that, at $O(\varepsilon^3)$,

$$\psi^{(3)} = \psi_0^{(3)} + \psi_{31}^{(3)} \cos 3\theta, \tag{3.64}$$

with the same θ-dependence for $\zeta^{(3)}$, and simultaneously that

$$\dot{X}^{(2)} = 0, \ \dot{Y}^{(2)} = 0. \tag{3.65}$$

In the sequel, we calculate $\psi_{31}^{(3)}$. The form of (3.61) implies, in the absence of $\tilde{\psi}^{(1)}$ and $\zeta^{(1)}$,

$$\zeta^{(3)} = a\tilde{\psi}^{(3)} + b_3(r,t). \tag{3.66}$$

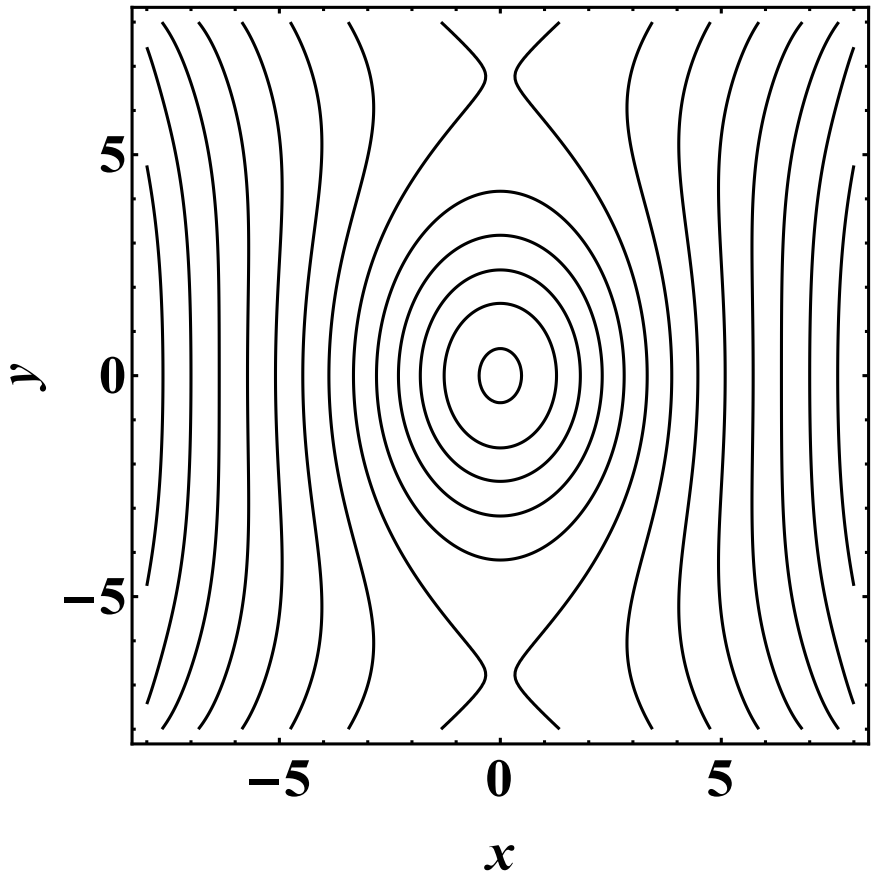

Fig. 3.7 The contour of streamfunction using $\tilde{\psi} = \psi^{(0)} + \varepsilon^2 \zeta_{21}^{(2)}(r)\cos 2\theta$ with $\varepsilon = 0.3$.

where $b_3(r,t)$ is an arbitrary function of r and t. Along the same line of reasoning for the $b_1(r,t)$ and $b_2(r,t)$, we may determine $b_3(r,t) \equiv 0$. Combination of (3.62) with (3.66) yields

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{9}{r^2} + a\right)\tilde{\psi}_{31}^{(3)} = 0. \tag{3.67}$$

The matching condition to be imposed on $\psi_{31}^{(3)}$ is found from (3.20) to be

$$\psi_{31}^{(3)} \sim \frac{r^3}{48\pi} + \frac{h_3}{r^3} \quad \text{as } r \to \infty, \tag{3.68}$$

with

$$h_3 = \frac{1}{6}\int_0^\infty \zeta_{31}^{(3)} r^4 \, \mathrm{d}r. \tag{3.69}$$

Again, we solve (3.67), subject to (3.68), using shooting method,

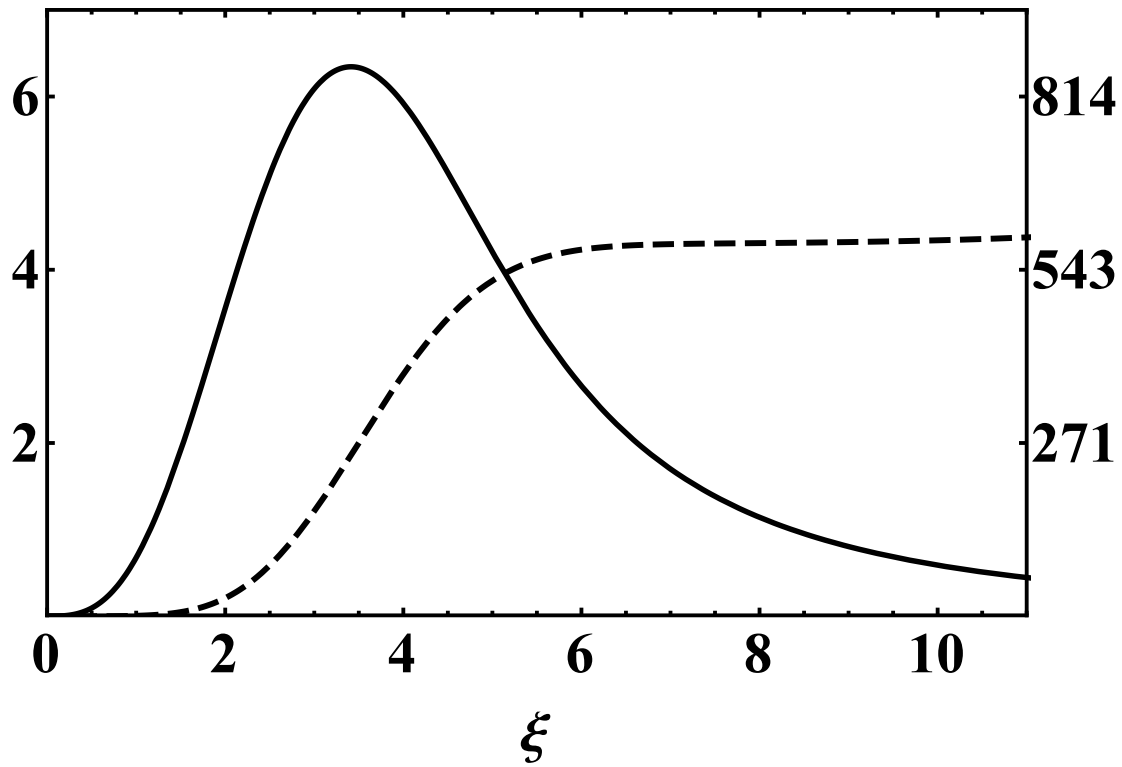

Fig. 3.8 The solution $\phi_{31}^{(3)}(\xi)$ (solid line). The dashed line draws $\xi^3\phi_{31}^{(3)}(\xi)$.

We rewrite (3.67) in term of the similarity variable $\xi = r/\sqrt{\hat{\nu}t}$ as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}-\frac{9}{\xi^2}\right)\phi_{31}^{(3)}(\xi) = -\frac{\xi^2}{4}\left(e^{\frac{\xi^2}{4}}-1\right)^{-1}\left(\phi_{31}^{(3)}(\xi)+\xi^2\right), \tag{3.70}$$

for

$$\psi_{31}^{(3)} = \frac{(\hat{\nu}t)^{3/2}}{48\pi}\phi_{31}^{(3)}(\xi)+\frac{r^3}{48\pi}. \tag{3.71}$$

A solution of (3.70) non-singular at $\xi =$ is obtained in a Taylor series in ξ as

$$\phi_{31}^{(3)} = \alpha_3\xi^3 - \frac{1}{16}(1+\alpha_3)\left(\xi^5-\frac{3}{40}\xi^7+174320\xi^9-\cdots\right), \tag{3.72}$$

with a disposable parameter α_3. This parameter is numerically determined in such a way that $\phi_{31}^{(3)}$ decreases to zero as $\phi_{31}^{(3)} \propto \xi^{-3}$ at large values of ξ, resulting in $\alpha_3 \approx 0.7797441$ and

$$h_3 \approx 583.147972\frac{(\hat{\nu}t)^3}{48\pi}. \tag{3.73}$$

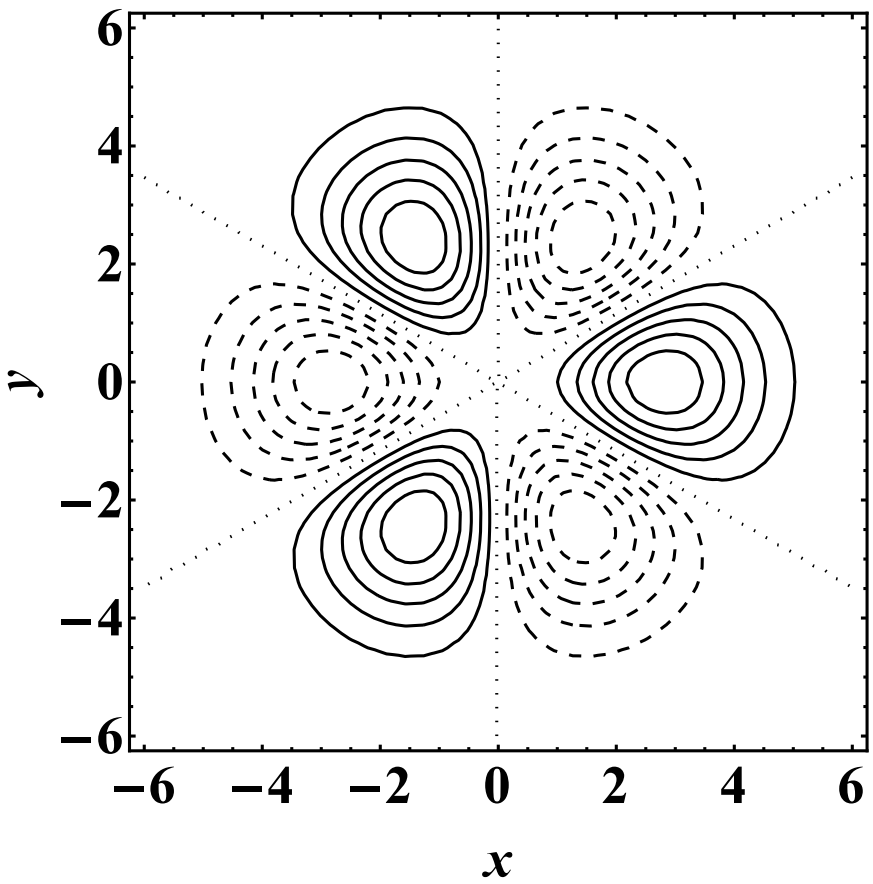

Fig. 3.9 The vorticity $\zeta_{31}^{(3)}(r)\cos 3\theta$ represents hexapole at origin due to interference the outer flow.

The solution $\phi_{31}^{(3)}(\xi)$ is drawn in Fig. 3.8. For clarity, the dashed line draws $\xi^3\phi_{31}^{(3)}(\xi)$. The contour plot of the perturbation vorticity $\zeta_{31}^{(3)}(r)\cos 3\theta$ of $O(\varepsilon^3)$ is shown in Fig. 3.9, and that of the vorticity $\zeta = \zeta^{(0)} + \varepsilon^2\zeta^{(2)} + \varepsilon^3\zeta^{(3)}$ to $O(\varepsilon^3)$ is shown, with a choice of $\varepsilon = 0.3$. We observe from Fig. 3.9 the hexapole structure of $\zeta^{(3)}$. The corresponding streamlines to $O(\varepsilon^3)$, viewed from the co-moving frame, are drawn in Fig. 3.11.

3.3.5 Fourth-order solution

Collecting $O(\varepsilon^2)$ terms in the vorticity equation (3.4) and $O(\varepsilon^4)$ terms in (3.5) with $\zeta^{(1)} \equiv 0$ and $\tilde{\psi}^{(1)} \equiv 0$ taken into consideration, $O(\varepsilon^2)$ terms in (3.4) and $O(\varepsilon^4)$ terms in (3.5) are

$$[\zeta^{(4)}, \psi^{(0)}] + [\zeta^{(2)}, \tilde{\psi}^{(2)}] + [\zeta^{(0)}, \tilde{\psi}^{(4)}] = -\frac{\partial \zeta^{(2)}}{\partial t} + \hat{\nu}\Delta\zeta^{(2)}, \tag{3.74}$$

$$\zeta^{(4)} = -\triangle\tilde{\psi}^{(4)}. \tag{3.75}$$

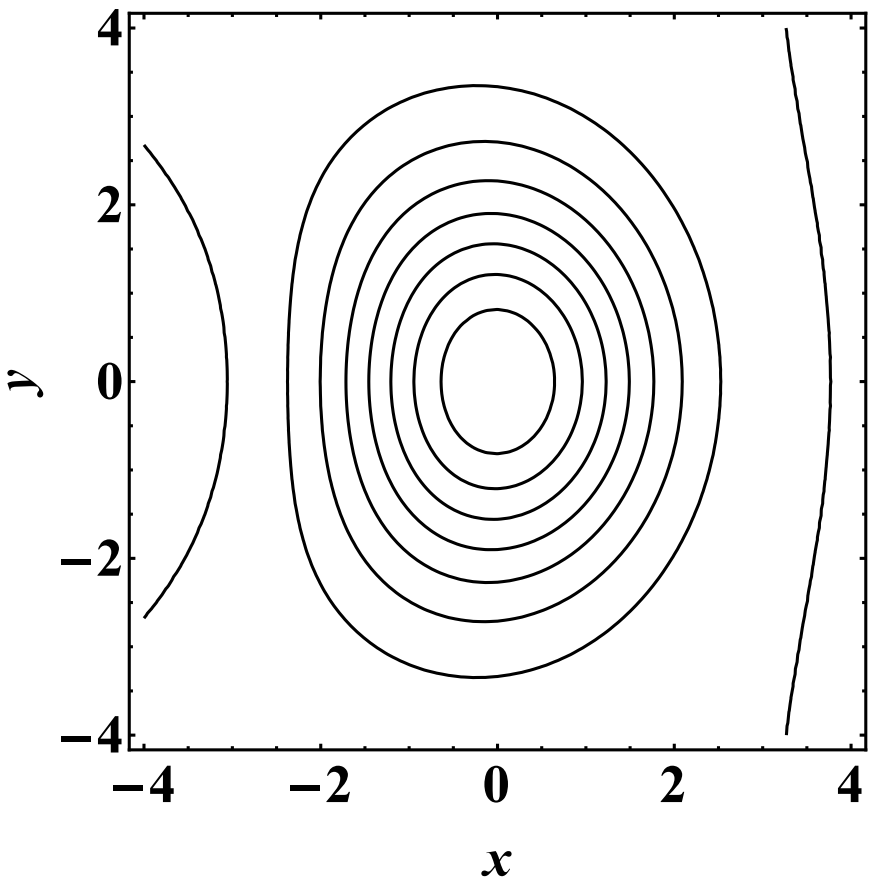

Fig. 3.10 The contour of vorticity field using $\zeta = \zeta^{(0)} + \varepsilon^2\zeta^{(2)} + \varepsilon^3\zeta^{(3)}$ to $O(\varepsilon^3)$ with $\varepsilon = 0.3$.

where $\tilde{\psi}^{(4)}$ is the streamfunction, of $O(\varepsilon^4)$, for the flow relative to the moving frame as defined by (3.13) with $j = 4$:

$$\tilde{\psi}^{(4)} = \psi^{(4)} - r\left(\dot{X}^{(3)}\sin\theta - \dot{Y}^{(3)}\cos\theta\right). \tag{3.76}$$

According to (3.20) supplemented by (3.21) and remembering the result of section 3.3.3 that $\tilde{\psi}^{(2)} \equiv 0$ and $\zeta^{(2)} \equiv 0$. These equations, together with the inner limit of the outer solution (3.20), inform us of the dependence of $\psi^{(4)}$ and $\zeta^{(4)}$ on θ to be

$$\psi^{(4)} = \psi_0^{(4)} + \psi_{22}^{(4)}\sin 2\theta + \psi_{41}^{(4)}\cos 4\theta, \tag{3.77}$$

with the same θ-dependence for $\zeta^{(4)}$.

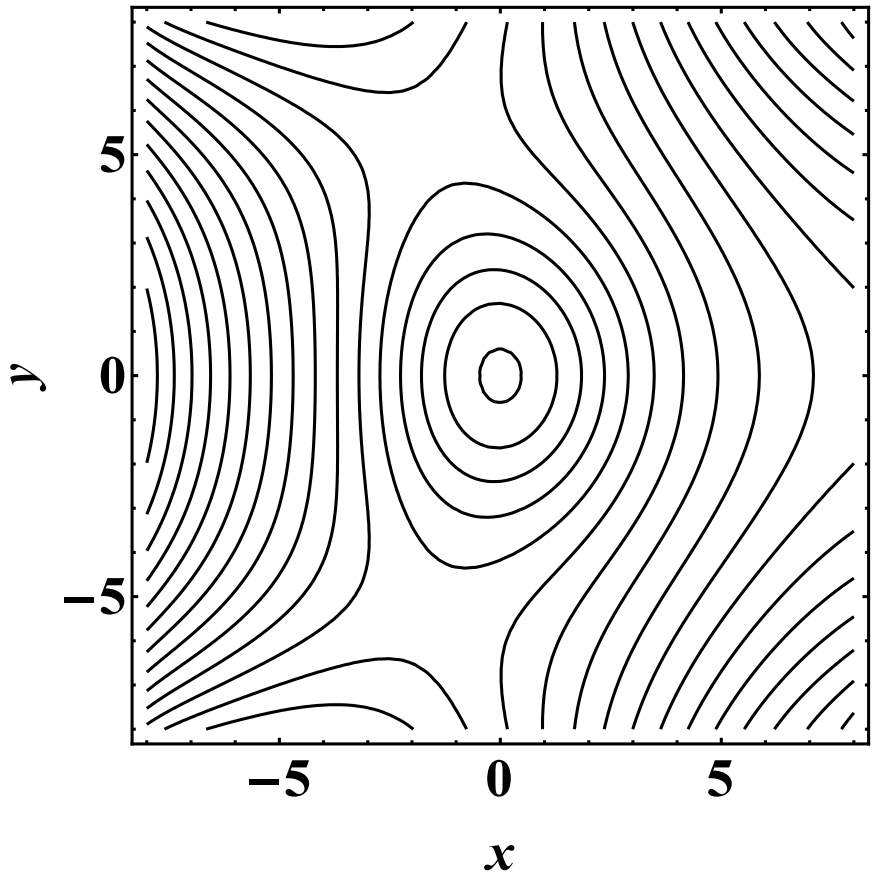

Fig. 3.11 The contour of streamfunction using $\psi = \psi^{(0)} + \varepsilon^2\psi^{(2)} + \varepsilon^3\psi^{(3)}$ to $O(\varepsilon^3)$ with $\varepsilon = 0.3$.

The immediate consequence of absence of components of $\cos\theta$ and $\sin\theta$ is

$$\dot{X}^{(3)} = 0,\ \dot{Y}^{(3)} = 0. \tag{3.78}$$

We will see that the correction to the traveling speed enters at $O(\varepsilon^5)$, the next order. Since $\zeta^{(1)}$ and $\tilde{\psi}^{(1)}$ are absent, $\psi^{(4)}$ and $\zeta^{(4)}$ do not contribute to the flow field of $O(\varepsilon^5)$ or therefore to traveling speed at $O(\varepsilon^5)$. We leave the flow field of $O(\varepsilon^4)$ untouched and will tackle with the solution of $O(\varepsilon^5)$, but we will calculate the coefficient of $\cos 4\theta$ that contribute to the solution of $O(\varepsilon^7)$

In the same as deriving second and third order solutions, the vorticity equation (3.74) and the streamfunction equation (3.75) are solved simultaneously where $\tilde{\psi}^{(4)}$ is the streamfunction, of $O(\varepsilon^4)$, for the flow relative to the moving frame as defined by (3.6):

$$\tilde{\psi}^{(4)} = \psi^{(4)} - r\left(\dot{X}^{(3)}\sin\theta - \dot{Y}^{(3)}\cos\theta\right). \tag{3.79}$$

The vorticity at $O(\varepsilon^4)$

$$\zeta^{(4)} = a\hat{\psi}^{(4)} + \frac{1}{4\nu_0}\left(\frac{\partial \zeta_{21}^{(2)}}{\partial r}\psi_{21}^{(2)} - \zeta_{21}^{(2)}\frac{\partial \psi_{21}^{(2)}}{\partial r}\right)\cos 4\theta$$

$$+\frac{r}{2\nu_0}\left\{-\frac{\partial \zeta_{21}^{(2)}}{\partial t} + \hat{\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\zeta_{21}^{(2)}\right\}\sin 2\theta + \frac{b_4(r,t)}{\nu_0}, \tag{3.80}$$

where $b_4(r,t)$ is an arbitrary function of r and t. Along the same line of reasoning for the $b_1(r,t)$, $b_2(r,t)$ and $b_3(r,t)$, we may determine $b_4(r,t) \equiv 0$. Combination of (3.75) with (3.80) yields

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{16}{r^2} + a\right)\psi_{41}^{(4)} = \frac{1}{4\nu_0}\left(\frac{\partial \zeta_{21}^{(2)}}{\partial r}\psi_{21}^{(2)} - \zeta_{21}^{(2)}\frac{\partial \psi_{21}^{(2)}}{\partial r}\right), \tag{3.81}$$

From the matching condition equation (3.20) to be imposed on $\psi_{41}^{(4)}$ as following

$$\psi_{41}^{(4)} \sim \frac{-r^4}{128\pi} + \frac{\Theta_4}{r^4} \quad \text{as } r \to \infty, \tag{3.82}$$

with $\Theta_4 = \int_0^\infty \zeta_{41}^{(4)} r^5 \, \mathrm{d}r/8$. Again, we solve (3.81), subject to (3.82), using shooting method.

We rewrite (3.81) in term of the similarity variable $\xi = r/\sqrt{\hat{\nu}t}$ as

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} - \left(\frac{16}{\xi^2} - a\right)\right\}\phi_{41}^{(4)}(\xi) = 4\xi\left(1 - e^{-\frac{\xi^2}{4}}\right)^{-1}\frac{\partial a}{\partial \xi}\left(\phi_{41}^{(4)}(\xi) - \frac{\xi^2}{4}\right)^2 + a\xi^2, \tag{3.83}$$

where

$$a = -\frac{1}{\nu^{(0)}}\frac{\partial \zeta^{(0)}}{\partial r}, \tag{3.84}$$

with $\nu^{(0)} = -\partial \psi^{(0)}/\partial r$ being the azimuthal velocity field, and

$$\psi_{41}^{(4)} = \frac{(\hat{\nu}t)^2}{128\pi}\phi_{41}^{(4)}(\xi) - \frac{r^4}{128\pi}. \tag{3.85}$$

A solution of (3.83) non-singular at $\xi =$ is obtained in a Taylor series in ξ as

$$\phi_{41}^{(4)} \sim \alpha_4 r^4 + \frac{1}{80}(3 + 8\alpha_2 - 16\alpha_2^2 - 4\alpha_4)r^6 + \frac{7}{7680}(-3 - 8\alpha_2 + 16\alpha_2^2 + 4\alpha_4)r^8$$

$$+\frac{1}{322560}(41 + 136\alpha_2 - 272\alpha_2^2 - 58\alpha_4)r^{10} + \cdots, \tag{3.86}$$

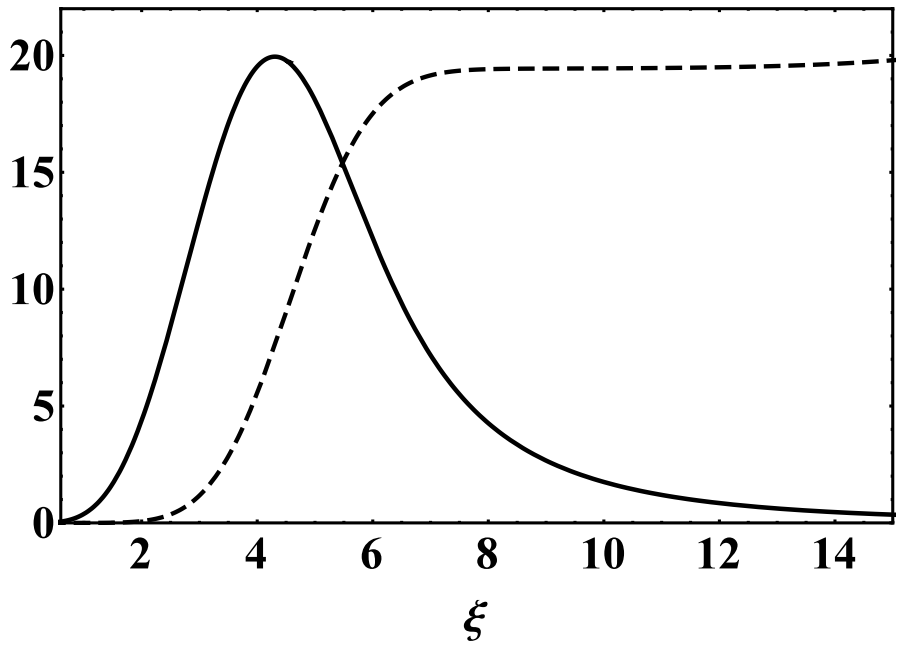

Fig. 3.12 The solution $\phi_{41}^{(4)}(\xi)$ (solid line). The dashed line draws $\xi^4\phi_{41}^{(4)}(\xi)$.

with a disposable parameter α_4. This parameter is numerically determined in such a way that $\phi_{41}^{(4)}$ decreases to zero as $\phi_{41}^{(4)} \propto \xi^{-4}$ at large values of ξ, resulting in $\alpha_4 \approx 0.43181$ and

$$\Theta_4 \approx 17489.374778\frac{(\hat{v}t)^4}{1024\pi}. \tag{3.87}$$

The solution $\phi_{41}^{(4)}(\xi)$ is drawn in Fig. 3.12. For clarity, the dashed line draws $\xi^4\phi_{41}^{(4)}(\xi)$. The contour plot of the perturbation vorticity $\zeta_{41}^{(4)}(r)\cos 4\theta$ of $O(\varepsilon^4)$ is shown in Fig. 3.13 where it is drawn by following equation

$$\zeta_{41}^{(4)} = \left(a\psi_{41}^{(4)} + \frac{1}{4\nu_0}\left(\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{21}^{(2)} - \zeta_{21}^{(2)}\frac{\partial\psi_{21}^{(2)}}{\partial r}\right)\right)\cos 4\theta, \tag{3.88}$$

because $\zeta_{21}^{(2)} = a\psi_{21}^{(2)}$, we may write equation (3.88) as following

$$\zeta_{41}^{(4)} = \left(a\psi_{41}^{(4)} + \frac{1}{4\nu_0}\frac{\partial a}{\partial r}\left(\psi_{21}^{(2)}\right)^2\right)\cos 4\theta. \tag{3.89}$$

The streamfunction contour $\psi_{41}^{(4)}(r)\cos 4\theta$ is plotted in Fig. 3.14. We only solve the fourth-order solution with respect to the coefficient of $\cos 4\theta$ where this solution will be used to catch seventh order solution of $O(\varepsilon^7)$.

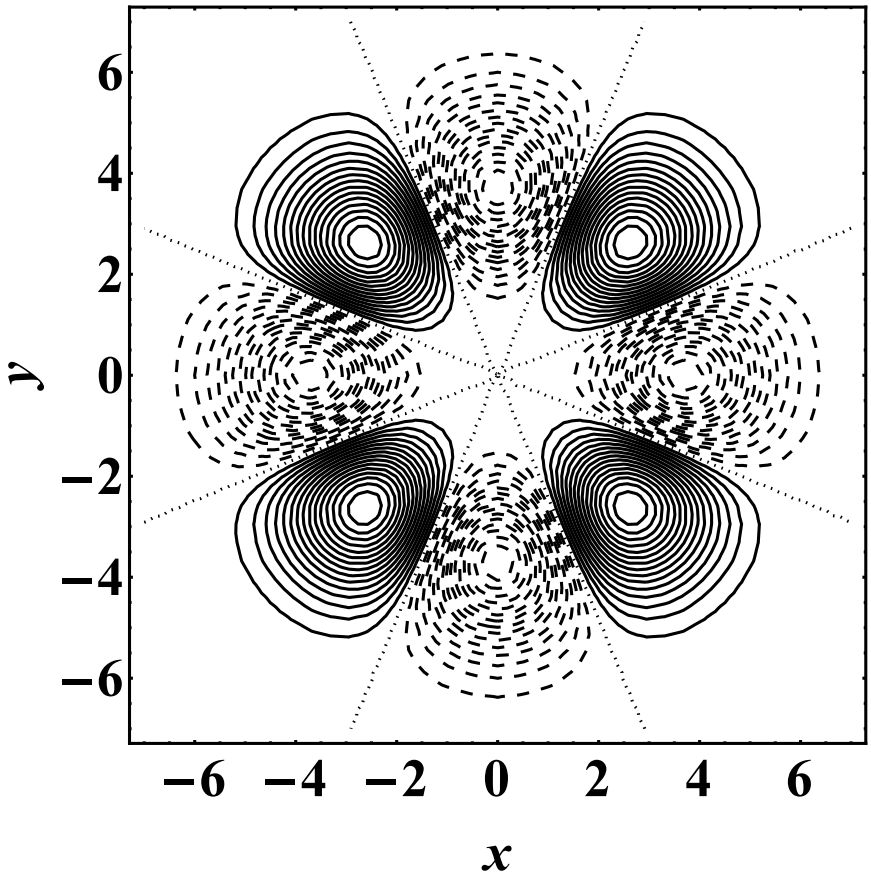

Fig. 3.13 The vorticity $\zeta_{41}^{(4)}(r)\cos 4\theta$ represents octapole at origin due to interference the outer flow.

3.3.6 Fifth-order solution

Collecting $O(\varepsilon^5)$ terms in the vorticity equation (3.4) and $O(\varepsilon^5)$ terms in (3.5), we have

$$[\zeta^{(5)},\psi^{(0)}]+[\zeta^{(3)},\tilde{\psi}^{(2)}]+[\zeta^{(2)},\tilde{\psi}^{(3)}]+[\zeta^{(0)},\tilde{\psi}^{(5)}]=-\frac{\partial\zeta^{(3)}}{\partial t}+\hat{\nu}\Delta\zeta^{(3)}, \tag{3.90}$$

$$\zeta^{(5)}=-\triangle\tilde{\psi}^{(5)}. \tag{3.91}$$

where $\tilde{\psi}^{(4)}$ is the streamfunction, of $O(\varepsilon^4)$, for the flow relative to the moving frame as defined by (3.13) with $j=5$:

$$\tilde{\psi}^{(5)}=\psi^{(5)}-r\left(\dot{X}^{(4)}\sin\theta-\dot{Y}^{(4)}\cos\theta\right). \tag{3.92}$$

Upon substituting from (3.46), (3.64) and their vorticity counterparts, namely, $\tilde{\psi}^{(2)}=\psi_{21}^{(2)}\cos 2\theta$, $\zeta^{(2)}=\zeta_{21}^{(2)}\cos 2\theta$, $\tilde{\psi}^{(3)}=\psi_0^{(3)}+\psi_{31}^{(3)}\cos 3\theta$ and $\zeta^{(3)}=\zeta_0^{(3)}+\zeta_{31}^{(3)}\cos 3\theta$,

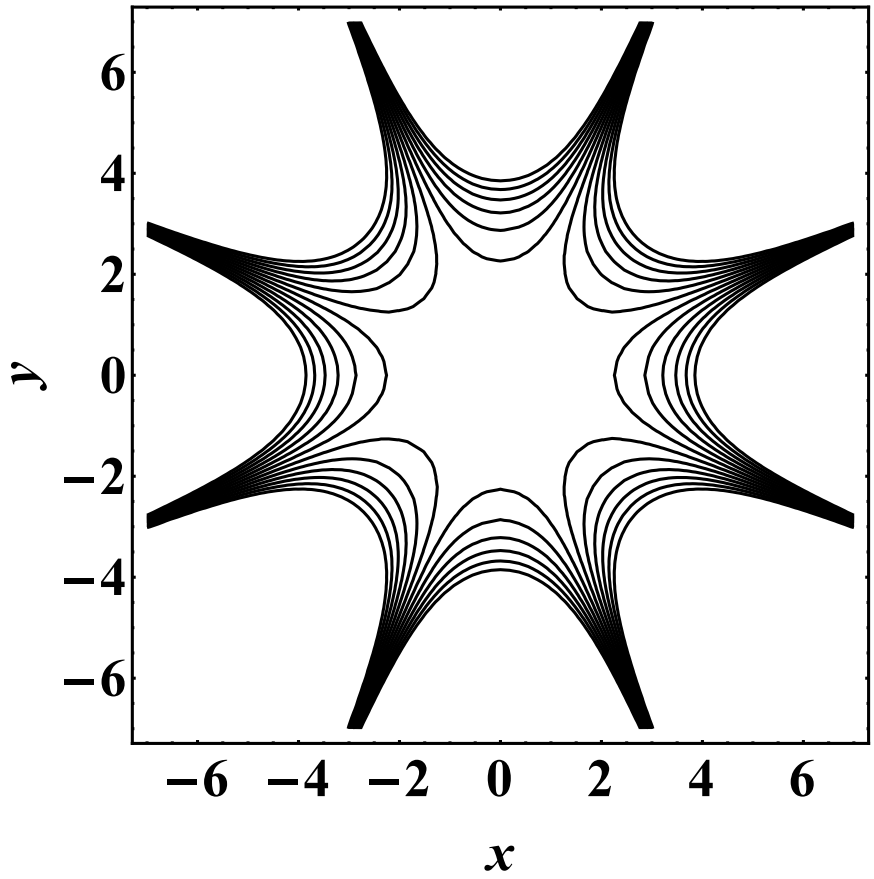

Fig. 3.14 The streamfunction contour $\psi_{41}^{(4)}(r)\cos 4\theta$.

(3.90) becomes

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{\partial\zeta^{(0)}}{\partial r}\tilde{\psi}^{(5)}-\zeta^{(5)}\frac{\partial\psi^{(0)}}{\partial r}\right.\\
&\quad-\left(\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{21}^{(2)}-\zeta_{21}^{(2)}\frac{\partial\psi_{31}^{(3)}}{\partial r}+\frac{3}{2}\zeta_{31}^{(3)}\frac{\partial\psi_{21}^{(2)}}{\partial r}-\frac{3}{2}\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{31}^{(3)}\right)\cos\theta\\
&\quad+2\left(\frac{\partial\zeta_{0}^{(3)}}{\partial r}\psi_{21}^{(2)}-\zeta_{21}^{(2)}\frac{\partial\psi_{0}^{(3)}}{\partial r}\right)\cos 2\theta\\
&\quad\left.-\frac{1}{5}\left(\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{31}^{(3)}-\zeta_{31}^{(3)}\frac{\partial\psi_{21}^{(2)}}{\partial r}+\frac{3}{2}\zeta_{31}^{(3)}\frac{\partial\psi_{21}^{(2)}}{\partial r}-\frac{3}{2}\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{31}^{(3)}\right)\cos 5\theta\right\}\\
&=\quad-\frac{\partial\zeta_{0}^{(3)}}{\partial t}+\hat{\nu}\Delta\zeta_{0}^{(3)}\\
&\quad+\left\{-\frac{\partial\zeta_{31}^{(3)}}{\partial t}+\hat{\nu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{9}{r^2}\right)\zeta_{31}^{(3)}\right\}\cos 3\theta. \qquad (3.93)
\end{aligned}
$$

An immediate consequence of the form of right-hand side is that, if we impose $\int \zeta_0^{(3)} r\mathrm{d}r = 0$ for the viscous case, $\zeta_0^{(3)} = \psi_0^{(3)} \equiv 0$. We may take this also for the inviscid case ($\hat{\nu} = 0$), without loss of generality. Upon integration with respect to θ over $[0, 2\pi)$ and subsitution from $\zeta_{21}^{(2)} = a\psi_{21}^{(2)}$ and $\zeta_{31}^{(3)} = a\psi_{31}^{(3)}$, (3.93) reduces, when divided by $v^{(0)} = -\partial\psi^{(0)}/\partial r$, to

$$\zeta^{(5)} = a\tilde{\psi}^{(5)} - \frac{1}{2v_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}\left(\cos\theta + \frac{1}{5}\cos 5\theta\right) + \frac{r}{3v_0}\left\{-\frac{\partial \zeta_{31}^{(3)}}{\partial t} + \hat{\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{9}{r^2}\right)\zeta_{31}^{(3)}\right\}\sin 3\theta + b_5(r,t), \tag{3.94}$$

where $b_5(r,t)$ is an arbitrary function of r and t. It is at this order that a non-trivial dipole component, being proportional to $\cos\theta$, makes its appearance for the first time, implying that the traveling speed $\dot{Y}$ of the vortical cores experiences modification due to the finite-thickness effect. Neither the vorticity equation (3.94) nor the inner limit of the Biot-Savart law (3.20) produces terms proportional to $\sin\theta$ at $O(\varepsilon^5)$. Hence we conclude no relative motion $\dot{X}^{(5)} = 0$ along the x-direction. In the sequel, we exclusively deal with the $\cos\theta$-component and deduce the correction term to the traveling velocity.

Combined with (3.91), $\cos\theta$-component of (3.93) yields, for $\tilde{\psi}_{11}^{(5)} = \psi_{11}^{(5)} + r\dot{Y}^{(4)}$,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + a\right)\tilde{\psi}_{11}^{(5)} = \frac{1}{2v^{(0)}}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}, \tag{3.95}$$

where use has been made of $\zeta_{21}^{(2)} = a\psi_{21}^{(2)}$ and $\zeta_{31}^{(3)} = a\psi_{31}^{(3)}$. The matching condition on (3.95) at large distances is manipulated from $\cos\theta$-component, of $O(\varepsilon^5)$, of the inner limit of (3.20) as

$$\psi_{11}^{(5)} \sim q_2\frac{r}{4} + \frac{1}{2r}\left(\int_0^\infty \zeta_{11}^{(5)} r^2 \mathrm{d}r\right) \quad \text{as } r \to \infty, \tag{3.96}$$

where the definition (3.48) of the quadrupole strength q_2 $O(\varepsilon^2)$ is to be remembered.

The general solution of (3.95), non-singular at $r = 0$, is

$$\tilde{\psi}_{11}^{(5)} = c_{11}^{(5)} v^{(0)} + \frac{v^{(0)}}{2}\int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)}(r')]^2}\left(\int_0^{r'} \frac{\partial a(r,t)}{\partial r''}\psi_{21}^{(2)}\psi_{31}^{(3)} r''\mathrm{d}r''\right), \tag{3.97}$$

where $c_{11}^{(5)}$ is an arbitrary constant. Noting generically that $v^{(0)} = 1/(2\pi r)$ as $r \to \infty$, the asymptotic behavior of (3.97) is manipulated as

$$\lim_{r\to\infty} \tilde{\psi}_{11}^{(5)} = \frac{\pi r}{2}\int_0^\infty \frac{\partial a(r,t)}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r. \tag{3.98}$$

The matching condition (3.96) for $\psi_{11}^{(5)} = \tilde{\psi}_{11}^{(5)} - r\dot{Y}^{(4)}$ reads

$$\dot{Y}^{(4)} = \frac{\pi}{2}\int_0^\infty \frac{\partial a(r,t)}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r - \frac{q_2}{4}. \tag{3.99}$$

Thus, along with (3.42), we reach a general formula of the translation velocity of a vortex pair

$$\dot{Y} = -\frac{1}{4\pi}\left\{1 - \pi\varepsilon^4\left(2\pi\int_0^\infty \frac{\partial a(r,t)}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r - q_2\right)\right\}. \tag{3.100}$$

This formula is applicable to both viscous and inviscid vortex pairs.

We apply this general formula to two example: the viscous vortex pair starting from a delta-function core $\zeta(x,y,0) = \delta(x)\delta(y)$ at the initial instant $t = 0$ and an inviscid vortex pair with the leading-order vorticity $\zeta^{(0)}$ given by the Rankine vortex. The latter example is relegated to Chapter 4. There we show that (3.100) recovers the result of Yang and Kubota (1994) which made use of a method developed specifically for the motion of vortex patches, finite vortices of uniform vorticity (Dhanak 1992)

For the former example, the leading-order flow field is the Oseen vortex (3.52) and (3.53). The strength q_2 of the quadrupole of $O(\varepsilon^2)$ is calculated as (3.60) or

$$q_2 = -1.3904181431\ldots \times (\hat{\nu}t)^2. \tag{3.101}$$

The self-induced part $\hat{\nu}t\phi_{21}^{(2)}/(4\pi)$ of the secon-order quadrupole, defined through (3.57), is numerically evaluated as shown by figure 3.4. The self-induced part $(\hat{\nu}t)^{2/3}\phi_{31}^{(3)}/(48\pi)$ of the third-order tri-pole field, defined through (3.71), is numerically calculated as shown by figure 3.8. These data produces numerically

$$2\pi\int_0^\infty \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r = 1.3904181425\ldots \times (\hat{\nu}t)^2. \tag{3.102}$$

With substitution of (3.101) and (3.102) into (3.100), the dimensional form of the travelin speed of the viscous vortex pair is obtained as

$$\dot{Y} \approx -\frac{\Gamma}{4\pi d}\left\{1 - 8.73625485\frac{(\nu t)^2}{d^4}\right\}. \tag{3.103}$$

The result agrees with and improve those of Gaifullin and Zubtsov (2004) and Nakagawa (2004). Gaifullin and Zubtsov (2004) gave 4 digits for the correction of $O(\varepsilon^5)$. The quadrupole field, of $O(\varepsilon^2)$, of the companion vortex induces a flow field (3.96) at $O(\varepsilon^5)$ of decelerating the traveling speed, and the nonlinear interaction between the $O(\varepsilon^2)$-quadrupole

and $O(\varepsilon^3)$ makes the same action. As indicated by Fig. 3.5, the $O(\varepsilon^2)$-quadrupole field may be regarded as a combination of four secondary vortices. Among them, the nearest one of the secondary vortices, being the dominat ones, is of opposite sign, induce the ownward motion flow as a whole.

Notice the coincidence, to a sufficient number of digits, of the numerical values in (3.101) and (3.102), though they are obtainedgained by totally diffierent procedures. This coincidence also occurs for the other example of vortex patches described in Chapter 4. We are thus lead to a belief that this coincidence is generally true

$$q_2 = -2\pi \int_0^\infty \frac{\partial a}{\partial r} \psi_{21}^{(2)} \psi_{31}^{(3)} r \mathrm{d}r. \tag{3.104}$$

whether the viscosity is present or not. In Appendix A, we prove that this is indeed the case, resulting in a surprisingly simple form for the correction term (3.99) $\dot{Y}^{(4)} = -q_2/2$. The evaluation of q_2 is by far simpler than that of the integral (3.102). The latter implements numericall integration of a function including $\phi_{21}^{(2)}/(4\pi)$ and $\phi_{31}^{(3)}$, both being obtained only numerically. In contrast, q_2 necessitates no global data of functions, but is evaluated from the asymptoptic behavior, at large argument, of the solution, in the process of numerically solving the second-order ordinary differential equation (3.58) by the shooting method. The eventual formula for the translation speed of a vortex pair inludes the strength of the second-order quadrupole field only and is expressed, in terms of the dimensional variables,

$$\dot{Y} = -\frac{\Gamma}{4\pi d}\left\{1 + \frac{2\pi}{\Gamma d^2} q\right\}; \quad q = \varepsilon^2 q_2, \tag{3.105}$$

where q is the strength of the quadrupole field of self-induced origin (3.18)

$$\psi_{\text{right}} = -\frac{\Gamma}{2\pi}\log r + q\frac{\cos 2\theta}{r^2} + \cdots, \tag{3.106}$$

arising at $O(\varepsilon^2)$. The representation of the quadrupole strength $q = \varepsilon^2 q_2$ in terms of the vorticity distribution (3.48) helps to recover dimensions. The dimensional form q_2 of the $O(\varepsilon^2)$-quadrupole strength related with the dimensionless one q_2^* through $q_2 = \Gamma(\varepsilon d)^2 q_2^*$.

We note that this formula supports for the attempt of Delbende and Rossi (2009). They made calculation of motion of a counter-rotating vortex pair, by relying on the observation that they are well fitted by a pair of elliptic vortices.

3.3.7 Sixth-order asymptotic solution

Collecting $O(\varepsilon^4)$ terms in the vorticity equation (3.4) and $O(\varepsilon^6)$ terms in (3.5), we have

$$[\zeta^{(6)},\tilde{\psi}^{(0)}]+[\zeta^{(5)},\tilde{\psi}^{(1)}]+[\zeta^{(4)},\tilde{\psi}^{(2)}]+[\zeta^{(3)},\tilde{\psi}^{(3)}]+[\zeta^{(2)},\tilde{\psi}^{(4)}]+[\zeta^{(1)},\tilde{\psi}^{(5)}]+[\zeta^{(0)},\tilde{\psi}^{(6)}]$$

$$=-\frac{\partial\zeta^{(4)}}{\partial t}+\hat{\nu}\Delta\zeta^{(4)}, \tag{3.107}$$

$$\zeta^{(6)}=-\triangle\tilde{\psi}^{(6)}. \tag{3.108}$$

where $\tilde{\psi}^{(6)}$ is the streamfunction, of $O(\varepsilon^6)$, for the flow relative to the moving frame as defined by (3.13) with $j=6$:

$$\tilde{\psi}^{(6)}=\psi^{(6)}-r\left(\dot{X}^{(5)}\sin\theta-\dot{Y}^{(5)}\cos\theta\right). \tag{3.109}$$

According to (3.20) supplemented by (3.21) and remembering the result of section 3.3.3, 3.3.3 and 3.3.6

$$\tilde{\psi}^{(2)}=\psi_{21}^{(2)}\cos 2\theta,\ \zeta^{(2)}=\zeta_{21}^{(2)}\cos 2\theta, \tag{3.110}$$

$$\tilde{\psi}^{(3)}=\psi_{31}^{(3)}\cos 3\theta,\ \zeta^{(3)}=\zeta_{31}^{(3)}\cos 3\theta, \tag{3.111}$$

$$\tilde{\psi}^{(4)}=\psi_{0}^{(4)}+\psi_{22}^{(4)}\sin 2\theta+\psi_{41}^{(4)}\cos 4\theta,\ \zeta^{(42)}=\zeta_{0}^{(4)}+\zeta_{22}^{(4)}\sin 2\theta+\zeta_{41}^{(4)}\cos 4\theta, \tag{3.112}$$

We gain differential equation at $O(\varepsilon^6)$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{\partial\zeta^{(0)}}{\partial r}\tilde{\psi}^{(6)}-\zeta^{(6)}\frac{\partial\tilde{\psi}^{(0)}}{\partial r}+\left[\frac{\partial\zeta_0^{(4)}}{\partial r}\psi_{21}^{(2)}-\zeta_0^{(4)}\frac{\partial\psi_{21}^{(2)}}{\partial r}+\zeta_{21}^{(2)}\frac{\partial\psi_{41}^{(4)}}{\partial\theta}\right.\right.$$

$$\left.+\frac{1}{2}\zeta_{21}^{(2)}\frac{\partial\psi_{41}^{(4)}}{\partial r}-\frac{1}{2}\zeta_{41}^{(4)}\frac{\partial\psi_{21}^{(2)}}{\partial\theta}-\zeta_{22}^{(4)}\frac{\partial\psi_{21}^{(4)}}{\partial r}+\frac{1}{3}\left(-\frac{\partial\zeta_{22}^{(4)}}{\partial t}+\hat{\nu}\triangle\zeta_{22}^{(4)}\right)\right]\cos 2\theta+$$

$$\left[\frac{1}{4}\frac{\partial\zeta_{22}^{(4)}}{\partial r}\psi_{21}^{(2)}-\frac{1}{4}\frac{\partial\psi_{21}^{(2)}}{\partial r}\zeta_{22}^{(4)}+\frac{1}{3}\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{41}^{(4)}-\frac{1}{4}\zeta_{21}^{(2)}\frac{\partial\psi_{22}^{(4)}}{\partial r}+\frac{1}{4}\left(-\frac{\partial\zeta_{41}^{(4)}}{\partial t}+\hat{\nu}\triangle\zeta_{41}^{(4)}\right)\right]\sin 2\theta+$$

$$\left[\frac{1}{6}\frac{\partial\zeta_{41}^{(4)}}{\partial r}\psi_{21}^{(2)}-\frac{1}{3}\zeta_{22}^{(4)}\frac{\partial\psi_{21}^{(2)}}{\partial r}+\frac{1}{4}\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{31}^{(3)}-\frac{1}{4}\zeta_{31}^{(3)}\frac{\partial\psi_{31}^{(3)}}{\partial r}+\frac{1}{3}\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{41}^{(4)}\right.$$

$$\left.-\frac{1}{6}\zeta_{21}^{(2)}\frac{\partial\psi_{41}^{(4)}}{\partial r}\right]\cos 6\theta+\left[-\frac{\partial\zeta_{22}^{(4)}}{\partial r}\psi_{21}^{(2)}-\zeta_{22}^{(2)}\frac{\partial\psi_{21}^{(2)}}{\partial r}+\zeta_{21}^{(2)}\frac{\partial\psi_{22}^{(4)}}{\partial r}+\frac{\partial\psi_{22}^{(4)}}{\partial r}\zeta_{21}^{(2)}\right]\theta$$

$$\left.-\frac{1}{2}\frac{\partial\zeta_{41}^{(4)}}{\partial r}\psi_{21}^{(2)}-\frac{\partial\psi_{21}^{(2)}}{\partial r}\zeta_{22}^{(4)}+\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{41}^{(4)}+\frac{1}{2}\frac{\partial\psi_{41}^{(4)}}{\partial r}\zeta_{21}^{(2)}\right\}=-\frac{\partial\zeta_{0}^{(4)}}{\partial t}+\hat{\nu}\triangle\zeta_{0}^{(4)}. \quad (3.113)$$

Integrate respect to θ from $0\to 2\pi$ yields the heat conduction equation

$$-\frac{\partial\zeta_{0}^{(4)}}{\partial t}+\hat{\nu}\triangle\zeta_{0}^{(4)}=0. \quad (3.114)$$

where the same as in section 3.3.3, the well known solution is Oseen vortex for viscous flow. Therefore for inviscid case $\hat{\nu}=0$, an arbitrary solution tells the solution. Moreover we may choose the trivial solution

$$\zeta_{0}^{(4)}=\tilde{\psi}_{0}^{(4)}=\psi_{0}^{(4)}=0. \quad (3.115)$$

We see that after intregration of equation (3.113) with respect to θ over $[0,2\pi)$ we end up with the immediate consequence of absence of components of $\cos\theta$ and $\sin\theta$

$$\dot{X}^{(5)}=0,\ \dot{Y}^{(5)}=0. \quad (3.116)$$

We leave the flow field of $O(\varepsilon^6)$ untouched and will tackle with the solution of $O(\varepsilon^7)$. We will see that the correction to the traveling speed enters at $O(\varepsilon^7)$, the next order.

3.3.8 Seventh-order asymptotic solution

Collecting $O(\varepsilon^5)$ terms in the vorticity equation (3.4) and $O(\varepsilon^7)$ terms in (3.5), we have

$$[\zeta^{(7)},\tilde{\psi}^{(0)}]+[\zeta^{(6)},\tilde{\psi}^{(1)}]+[\zeta^{(5)},\tilde{\psi}^{(2)}]+[\zeta^{(4)},\tilde{\psi}^{(3)}]+[\zeta^{(3)},\tilde{\psi}^{(4)}]+[\zeta^{(2)},\tilde{\psi}^{(5)}]+[\zeta^{(1)},\tilde{\psi}^{(6)}]+[\zeta^{(0)},\tilde{\psi}^{(7)}]$$

$$= -\frac{\partial \zeta^{(5)}}{\partial t} + \hat{\nu}\Delta\zeta^{(5)}, \tag{3.117}$$

$$\zeta^{(7)} = -\triangle\tilde{\psi}^{(7)}, \tag{3.118}$$

where $\tilde{\psi}^{(7)}$ is the streamfunction, of $O(\varepsilon^7)$, for the flow relative to the moving frame as defined by (3.13) with $j = 7$:

$$\tilde{\psi}^{(7)} = \psi^{(7)} - r\left(\dot{X}^{(6)}\sin\theta - \dot{Y}^{(6)}\cos\theta\right). \tag{3.119}$$

Following the same steps as deriving fifth order solution brings the results from first up to fifth orders in the previous subsections. The straemfunctions and the vorticities are

$$\tilde{\psi}^{(1)} = 0,\ \zeta^{(1)} = 0, \tag{3.120}$$

$$\tilde{\psi}^{(2)} = \psi_{21}^{(2)}\cos 2\theta,\ \zeta^{(2)} = \zeta_{21}^{(2)}\cos 2\theta, \tag{3.121}$$

$$\tilde{\psi}^{(3)} = \psi_{31}^{(3)}\cos 3\theta,\ \zeta^{(3)} = \zeta_{31}^{(3)}\cos 3\theta, \tag{3.122}$$

$$\tilde{\psi}^{(4)} = \psi_{22}^{(4)}\sin 2\theta + \psi_{41}^{(4)}\cos 4\theta,\ \zeta^{(4)} = \zeta_{22}^{(4)}\sin 2\theta + \zeta_{41}^{(4)}\cos 4\theta, \tag{3.123}$$

$$\tilde{\psi}^{(5)} = \psi_{0}^{(5)} + \psi_{11}^{(5)}\cos\theta + \psi_{32}^{(5)}\sin 3\theta + \psi_{51}^{(5)}\cos 5\theta, \tag{3.124}$$

$$\zeta^{(5)} = \zeta_{0}^{(5)} + \zeta_{11}^{(5)}\cos\theta + \zeta_{32}^{(5)}\sin 3\theta + \zeta_{51}^{(5)}\cos 5\theta, \tag{3.125}$$

hereinafter by inserting (3.120)-(3.125) into (3.117) and (3.118) concomitantly to give

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{\partial\zeta^{(0)}}{\partial r}\tilde{\psi}^{(7)} - \zeta^{(7)}\frac{\partial\tilde{\psi}^{(0)}}{\partial r} + \left[\frac{\partial\zeta_{11}^{(5)}}{\partial r}\psi_{21}^{(2)} + \frac{1}{2}\zeta_{11}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r} - \frac{3}{2}\frac{\partial\zeta_{41}^{(4)}}{\partial r}\psi_{31}^{(3)}\right.\right.$$

$$\left. -2\zeta_{41}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r} + 2\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{41}^{(4)} + \frac{3}{2}\zeta_{31}^{(3)}\frac{\partial\psi_{41}^{(4)}}{\partial r} + \zeta_{21}^{(2)}\frac{\partial\psi_{0}^{(5)}}{\partial r}\right]\cos\theta + \left[\frac{\partial\zeta_{0}^{(5)}}{\partial r}\psi_{21}^{(2)} - \zeta_{21}^{(2)}\frac{\partial\psi_{0}^{(5)}}{\partial r}\right]\cos 2\theta$$

$$\left[\frac{1}{3}\frac{\partial\zeta_{11}^{(5)}}{\partial r}\psi_{21}^{(2)}-\frac{1}{3}\frac{\partial\zeta_{51}^{(5)}}{\partial r}\psi_{21}^{(2)}-\frac{1}{6}\frac{\partial\psi_{21}^{(2)}}{\partial r}\zeta_{11}^{(5)}-\frac{5}{6}\frac{\partial\psi_{21}^{(2)}}{\partial r}\zeta_{51}^{(5)}+\frac{1}{3}\left(-\frac{\partial\zeta_{32}^{(3)}}{\partial t}+\hat{\nu}\triangle\zeta_{32}^{(3)}\right)\right]\cos 3\theta+$$

$$\left[\frac{1}{7}\frac{\partial\zeta_{51}^{(5)}}{\partial r}\psi_{21}^{(2)}-\frac{5}{14}\zeta_{51}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r}+\frac{3}{14}\frac{\partial\zeta_{41}^{(4)}}{\partial r}\psi_{31}^{(3)}-\frac{2}{7}\zeta_{41}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r}+\frac{2}{7}\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{41}^{(4)}\right.$$

$$\left.-\frac{3}{14}\zeta_{31}^{(3)}\frac{\partial\psi_{41}^{(4)}}{\partial r}+\frac{2}{7}\zeta_{21}^{(2)}\frac{\partial\psi_{51}^{(5)}}{\partial r}-\frac{1}{7}\zeta_{21}^{(2)}\frac{\partial\psi_{51}^{(5)}}{\partial r}\right]\cos 7\theta$$

$$\left[-\frac{\partial\zeta_{32}^{(5)}}{\partial r}\psi_{21}^{(2)}-\frac{3}{2}\zeta_{32}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r}-\frac{3}{2}\frac{\partial\zeta_{22}^{(4)}}{\partial r}\psi_{31}^{(3)}-\zeta_{22}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r}+\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{22}^{(42)}+\frac{3}{2}\zeta_{31}^{(3)}\frac{\partial\psi_{22}^{(4)}}{\partial r}\right.$$

$$\left.-\left(-\frac{\partial\zeta_{11}^{(5)}}{\partial t}+\hat{\nu}\triangle\zeta_{11}^{(5)}\right)\right]\sin\theta+\left[\frac{1}{5}\frac{\partial\zeta_{32}^{(5)}}{\partial r}\psi_{21}^{(2)}-\frac{3}{10}\zeta_{32}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r}+\frac{3}{10}\frac{\partial\zeta_{22}^{(4)}}{\partial r}\psi_{31}^{(3)}-\frac{1}{5}\zeta_{22}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r}\right.$$

$$\left.+\frac{1}{5}\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{22}^{(4)}-\frac{3}{10}\zeta_{31}^{(3)}\frac{\partial\psi_{22}^{(4)}}{\partial r}-\frac{1}{5}\left(-\frac{\partial\zeta_{51}^{(5)}}{\partial t}+\hat{\nu}\triangle\zeta_{51}^{(5)}\right)\right]\sin 5\theta$$

$$\left.\left[\frac{1}{2}\zeta_{0}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r}\right]\sin 2\theta\right\}=-\frac{\partial\zeta_{0}^{(5)}}{\partial t}+\hat{\nu}\triangle\zeta_{0}^{(5)}. \tag{3.126}$$

furthermore integrate (3.126) respect to θ from $0\to 2\pi$ yields heat conduction equation

$$-\frac{\partial\zeta_{0}^{(5)}}{\partial t}+\hat{\nu}\triangle\zeta_{0}^{(5)}=0, \tag{3.127}$$

where the same way as in subsection 3.3.7, the well known solution is the Oseen vortex for viscous fluid. Therefore for an inviscid case $\hat{\nu}=0$, an arbitrary solution tells the solution. Moreover we may choose the trivial solution

$$\zeta_{0}^{(5)}=\tilde{\psi}_{0}^{(5)}=\psi_{0}^{(5)}=0, \tag{3.128}$$

moreover equation (3.126) is integrated respect to θ from $0 \to 2\pi$ and divided by $-v_0 = \partial\tilde{\psi}^{(0)}/\partial r$ where v_0 is the azimuthal velocity to give rise the vorticity correction at $O(\varepsilon^7)$

$$\zeta^{(7)} = a\tilde{\psi}^{(7)} - \frac{A}{v_0}\cos\theta - \frac{B}{v_0}\cos 3\theta - \frac{C}{v_0}\cos 7\theta - \frac{D}{v_0}\sin\theta - \frac{E}{v_0}\sin 5\theta - \frac{b_7(r,t)}{v_0}, \quad (3.129)$$

where A, B, C, D and E are function of r and t, exhibit the coefficients of $\cos\theta$, $\cos 3\theta$, $\cos 7\theta$, $\sin\theta$ and $\sin 5\theta$ respectively

$$\begin{aligned} A(r,t) &= \frac{\partial\zeta_{11}^{(5)}}{\partial r}\psi_{21}^{(2)} + \frac{1}{2}\zeta_{11}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r} - \frac{3}{2}\frac{\partial\zeta_{41}^{(4)}}{\partial r}\psi_{31}^{(3)} - 2\zeta_{41}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r} + 2\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{41}^{(4)} \\ &\quad + \frac{3}{2}\zeta_{31}^{(3)}\frac{\partial\psi_{41}^{(4)}}{\partial r}, \end{aligned} \quad (3.130)$$

$$B(r,t) = \frac{1}{3}\frac{\partial\zeta_{11}^{(5)}}{\partial r}\psi_{21}^{(2)} - \frac{1}{3}\frac{\partial\zeta_{51}^{(5)}}{\partial r}\psi_{21}^{(2)} - \frac{1}{6}\frac{\partial\psi_{21}^{(2)}}{\partial r}\zeta_{11}^{(5)} - \frac{5}{6}\frac{\partial\psi_{21}^{(2)}}{\partial r}\zeta_{51}^{(5)} + \frac{1}{3}\left(-\frac{\partial\zeta_{32}^{(3)}}{\partial t} + \hat{\nu}\triangle\zeta_{32}^{(3)}\right), \quad (3.131)$$

$$\begin{aligned} C(r,t) &= \frac{1}{7}\frac{\partial\zeta_{51}^{(5)}}{\partial r}\psi_{21}^{(2)} - \frac{5}{14}\zeta_{51}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r} + \frac{3}{14}\frac{\partial\zeta_{41}^{(4)}}{\partial r}\psi_{31}^{(3)} - \frac{2}{7}\zeta_{41}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r} + \frac{2}{7}\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{41}^{(4)} \\ &\quad - \frac{3}{14}\zeta_{31}^{(3)}\frac{\partial\psi_{41}^{(4)}}{\partial r} + \frac{2}{7}\zeta_{21}^{(2)}\frac{\partial\psi_{51}^{(5)}}{\partial r} - \frac{1}{7}\zeta_{21}^{(2)}\frac{\partial\psi_{51}^{(5)}}{\partial r}, \end{aligned} \quad (3.132)$$

$$\begin{aligned} D(r,t) &= -\frac{\partial\zeta_{32}^{(5)}}{\partial r}\psi_{21}^{(2)} - \frac{3}{2}\zeta_{32}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r} - \frac{3}{2}\frac{\partial\zeta_{22}^{(4)}}{\partial r}\psi_{31}^{(3)} - \zeta_{22}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r} + \frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{22}^{(4)} \\ &\quad + \frac{3}{2}\zeta_{31}^{(3)}\frac{\partial\psi_{22}^{(4)}}{\partial r} - \left(-\frac{\partial\zeta_{11}^{(5)}}{\partial t} + \hat{\nu}\triangle\zeta_{11}^{(5)}\right), \end{aligned} \quad (3.133)$$

$$\begin{aligned} E(r,t) &= \frac{1}{5}\frac{\partial\zeta_{32}^{(5)}}{\partial r}\psi_{21}^{(2)} - \frac{3}{10}\zeta_{32}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r} + \frac{3}{10}\frac{\partial\zeta_{22}^{(4)}}{\partial r}\psi_{31}^{(3)} - \frac{1}{5}\zeta_{22}^{(4)}\frac{\partial\psi_{31}^{(3)}}{\partial r} + \frac{1}{5}\frac{\partial\zeta_{31}^{(3)}}{\partial r}\psi_{22}^{(4)} \\ &\quad - \frac{3}{10}\zeta_{31}^{(3)}\frac{\partial\psi_{22}^{(4)}}{\partial r} - \frac{1}{5}\left(-\frac{\partial\zeta_{51}^{(5)}}{\partial t} + \hat{\nu}\triangle\zeta_{51}^{(5)}\right), \end{aligned} \quad (3.134)$$

where $b_7(r,t)$ in unknown function at this approximation level. The terms to the traveling speed $(\dot{X},\dot{Y})$, of the origin of finite core, we have only to handle the components of $\cos\theta$ and $\sin\theta$. Neither the vorticity equation (3.129) nor the inner limit of the Biot-Savart law (3.20) produces terms proportional to $\sin\theta$ at $O(\varepsilon^7)$. Furthermore $\cos\theta$-component of (3.129) combined with (3.118) yields, for $\tilde{\psi}_{11}^{(7)} = \psi_{11}^{(7)} + r\dot{Y}^{(6)}$,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + a\right)\tilde{\psi}_{11}^{(7)} = \frac{A(r,t)}{v_0}. \tag{3.135}$$

The general solution of equation (3.135), non-singular at $r=0$ is

$$\tilde{\psi}_{11}^{(7)} = c_{11}^{(7)} v^{(0)} + v^{(0)} \int^r \left[\frac{1}{v^{(0)2}}\frac{1}{r'}\left\{\int^{r'} -A(r'',t)r''\mathrm{d}r'' + d_{11}^{(5)}\right\}\right]\mathrm{d}r'. \tag{3.136}$$

where $c_{11}^{(7)}$ and $d_{11}^{(7)}$ are arbitrary constants. The same way as deriving the general formula for the traveling speed of a vortex pair at fifth order, since $v^{(0)}$ is the solution of homogeneous second order differential equations then we put the coefficient $c_{11}^{(7)}$ and $d_{11}^{(7)}$ are equal zero. Furthermore noting that generically $v^{(0)} = 1/(2\pi r)$ as $r\to\infty$, the asymptotic behavior of (3.136) at large distances is manipulated as

$$\lim_{r\to\infty}\tilde{\psi}_{11}^{(7)} = \pi\int_0^\infty A(r,t)r\mathrm{d}r, \tag{3.137}$$

where $A(r)$ is calculated by substituting $\zeta_{31}^{(3)} = a\psi_{31}^{(3)}$, $\zeta_{41}^{(4)} = a\psi_{41}^{(4)} + h(r)/v_0$, $\tilde{\psi}_{11}^{(7)} = \psi_{11}^{(7)} + r\dot{Y}^{(6)}$ and $\zeta_{11}^{(5)} = a\tilde{\psi}_{11}^{(5)} + g(r)/v_0$ where $\tilde{\psi}_{11}^{(5)} = \psi_{11}^{(5)} + r\dot{Y}^{(4)}$, $h(r) = 1/4(\partial a/\partial r)(\psi_{21}^{(2)})^2$ and $g(r) = -1/2(\partial a/\partial r)\psi_{21}^{(2)}\psi_{31}^{(3)}$ into (3.130) to give

$$\begin{aligned} A(r,t) \;=\; & \frac{1}{2}\frac{\partial a}{\partial r}\psi_{31}^{(3)}\psi_{41}^{(4)} - \frac{3}{8}\psi_{31}^{(3)}\frac{\partial}{\partial r}\left(\frac{1}{v_0}\right)\frac{\partial a}{\partial r}\left(\psi_{21}^{(2)}\right)^2 - \frac{3}{8v_0}\psi_{31}^{(3)}\frac{\partial^2 a}{\partial r^2}\left(\psi_{21}^{(2)}\right)^2 \\ & -\frac{3}{4v_0}\psi_{21}^{(2)}\psi_{31}^{(3)}\frac{\partial\psi_{21}^{(2)}}{\partial r}\frac{\partial a}{\partial r} - \frac{1}{2v_0}\frac{\partial\psi_{31}^{(3)}}{\partial r}\frac{\partial a}{\partial r}\left(\psi_{21}^{(2)}\right)^2 + \frac{\partial\zeta_{11}^{(5)}}{\partial r}\psi_{21}^{(2)} + \frac{1}{2}\zeta_{11}^{(5)}\frac{\partial\psi_{21}^{(2)}}{\partial r}, \end{aligned} \tag{3.138}$$

the term $\zeta_{11}^{(5)}$ is obtained from the coefficient $\cos\theta$ of vorticity (3.94)

$$\zeta_{11}^{(5)} = a\tilde{\psi}_{11}^{(5)} - \frac{1}{2v_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}, \tag{3.139}$$

and solution $\tilde{\psi}_{11}^{(5)}$ uses (3.97) with choosing an arbitrary constant $c_{11}^{(5)} = 0$

$$\tilde{\psi}_{11}^{(5)} = \frac{v^{(0)}}{2} \int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)\prime}]^2} \left(\int_0^{r'} \frac{\partial a''}{\partial r''} \psi_{21}^{(2)} \psi_{31}^{(3)} r'' \mathrm{d}r'' \right), \tag{3.140}$$

We substitute (3.139) and (3.140) into (3.138) to give

$$\begin{aligned} A(r,t) \;=\; & \frac{1}{2}\frac{\partial a}{\partial r}\psi_{31}^{(3)}\psi_{41}^{(4)} + \psi_{21}^{(2)}\frac{\partial}{\partial r}\left[\frac{av^{(0)}}{2}\int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)\prime}]^2}\left(\int_0^{r'}\frac{\partial a''}{\partial r''}\psi_{21}^{(2)}\psi_{31}^{(3)}r''\mathrm{d}r''\right) - \frac{1}{2v_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3} \\ & + \frac{1}{2}\frac{\partial \psi_{21}^{(2)}}{\partial r}\left[\frac{av^{(0)}}{2}\int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)\prime}]^2}\left(\int_0^{r'}\frac{\partial a''}{\partial r''}\psi_{21}^{(2)}\psi_{31}^{(3)}r''\mathrm{d}r''\right) - \frac{1}{2v_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}\right] \\ & - \frac{3}{8}\psi_{31}^{(3)}\frac{\partial}{\partial r}\left(\frac{1}{v_0}\right)\frac{\partial a}{\partial r}\left(\psi_{21}^{(2)}\right)^2 - \frac{3}{8v_0}\psi_{31}^{(3)}\frac{\partial^2 a}{\partial r^2}\left(\psi_{21}^{(2)}\right)^2 - \frac{3}{4v_0}\psi_{21}^{(2)}\psi_{31}^{(3)}\frac{\partial \psi_{21}^{(2)}}{\partial r}\frac{\partial a}{\partial r} \\ & - \frac{1}{2v_0}\frac{\partial \psi_{31}^{(3)}}{\partial r}\frac{\partial a}{\partial r}\left(\psi_{21}^{(2)}\right)^2, \end{aligned} \tag{3.14}$$

where the second term on the right hand side can be derived easily to yield

$$\begin{aligned} A(r,t) \;=\; & \frac{1}{2}\frac{\partial a}{\partial r}\psi_{31}^{(3)}\psi_{41}^{(4)} + \frac{v^{(0)}}{2}\psi_{21}^{(2)}\frac{\partial a}{\partial r}\int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)\prime}]^2}\left(\int_0^{r'}\frac{\partial a''}{\partial r''}\psi_{21}^{(2)}\psi_{31}^{(3)}r''\mathrm{d}r''\right) \\ & + \frac{a}{2}\psi_{21}^{(2)}\frac{\partial v^{(0)}}{\partial r}\int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)\prime}]^2}\left(\int_0^{r'}\frac{\partial a''}{\partial r''}\psi_{21}^{(2)}\psi_{31}^{(3)}r''\mathrm{d}r''\right) + \frac{a\psi_{21}^{(2)}}{2v^{(0)}r}\left(\int_0^r \frac{\partial a'}{\partial r'}\psi_{21}^{(2)}\psi_{31}^{(3)}r'\mathrm{d}r'\right) \\ & - \psi_{21}^{(2)}\frac{\partial}{\partial r}\left(\frac{1}{2v_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}\right) + \frac{1}{2}\frac{\partial \psi_{21}^{(2)}}{\partial r}\left[\frac{\pi}{2}ar\int_0^\infty \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}r\mathrm{d}r - \frac{1}{2v_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}\right] \\ & - \frac{3}{8}\psi_{31}^{(3)}\frac{\partial}{\partial r}\left(\frac{1}{v_0}\right)\frac{\partial a}{\partial r}\left(\psi_{21}^{(2)}\right)^2 - \frac{3}{8v_0}\psi_{31}^{(3)}\frac{\partial^2 a}{\partial r^2}\left(\psi_{21}^{(2)}\right)^2 - \frac{3}{4v_0}\psi_{21}^{(2)}\psi_{31}^{(3)}\frac{\partial \psi_{21}^{(2)}}{\partial r}\frac{\partial a}{\partial r} \\ & - \frac{1}{2v_0}\frac{\partial \psi_{31}^{(3)}}{\partial r}\frac{\partial a}{\partial r}\left(\psi_{21}^{(2)}\right)^2. \end{aligned} \tag{3.142}$$

For the former example at leading order the Oseen vortex for a viscous fluid, equation (3.137) can be solved by numerical integration in the variable $\xi = r/(\hat{\nu}t)^{1/2}$ so that we have the composer terms of (3.142)

$$\psi_{21}^{(2)} = \frac{\hat{\nu}t}{\pi}\left(\frac{\phi_{21}^{(2)}}{4} - \frac{\xi^2}{16}\right), \tag{3.143}$$

$$\psi_{31}^{(3)} = \frac{(\hat{\nu}t)^{3/2}}{\pi}\left(\frac{\phi_{31}^{(3)}}{48} + \frac{\xi^3}{48}\right), \tag{3.144}$$

$$a(r,t) = \frac{1}{\hat{\nu}t}\frac{\xi^2}{4}\left(e^{\frac{\xi^2}{4}}-1\right)^{-1}, \tag{3.145}$$

$$\frac{\partial a(r,t)}{\partial r} = \frac{1}{(\hat{\nu}t)^{3/2}}\frac{\xi}{8}\left(e^{\frac{\xi^2}{4}}\left(4-\xi^2\right)-4\right)\left(e^{\frac{\xi^2}{4}}-1\right)^{-2}, \tag{3.146}$$

$$\frac{\partial^2 a(r,t)}{\partial r^2} = \frac{1}{(\hat{\nu}t)^2}\frac{\left(8+e^{\frac{\xi^2}{2}}\left(8-10\xi^2+\xi^4\right)+e^{\frac{\xi^2}{4}}\left(-16+10\xi^2+\xi^4\right)\right)}{16\left(e^{\frac{\xi^2}{4}}-1\right)^3}, \tag{3.147}$$

and

$$v_0 = \frac{1}{2\pi\xi\sqrt{\hat{\nu}t}}\left(1-e^{-\frac{\xi^2}{4}}\right). \tag{3.148}$$

Substituting equations (3.143)-(3.148) into the right hand side (3.137) we end up

$$\int_0^\infty A(r,t)r\mathrm{d}r \approx \frac{(\hat{\nu}t)^3}{\pi^2}\left(-16.91349777\ldots\right), \tag{3.149}$$

the matching condition at large distances is manipulated from $\cos\theta$- component, of $O(\varepsilon^7)$, of inner limit of (3.20) as

$$\psi_{11}^{(7)} \sim \frac{3rh_3}{16} + \frac{1}{2r}\int_0^\infty \zeta_{11}^{(7)} r^2 \mathrm{d}r \ \text{ as } r\to\infty, \tag{3.150}$$

where h_3 is the hexapole strength of $O(\varepsilon^3)$ using definition (3.69), and using the definition of streamfunction flow in moving coordinate system equation (3.6), $\tilde{\psi}_{11}^{(7)} = \psi_{11}^{(7)} + r\dot{Y}^{(6)}$, the solution of seventh-order asymptotic expansion then can be written as following

$$\psi_{11}^{(7)} = r\frac{(\hat{\nu}t)^3}{\pi}\left(-16.91349777\ldots\right) - r\dot{Y}^{(6)}, \tag{3.151}$$

furthermore the correction of the traveling speed of a vortex pair at $O(\varepsilon^7)$ is

$$\dot{Y}^{(6)} = \frac{(\hat{\nu}t)^3}{\pi}\left(-16.91349777\ldots\right) - \frac{3h_3}{16}, \tag{3.152}$$

where the second term of the right hand side (3.152) can be obtained from the third order solution represents the hexapole strength $O(\varepsilon^3)$. Finally the velocity of a vortex pair for a

viscous fluid in dimensional variable is

$$\begin{aligned}\dot{Y} &= \frac{\Gamma}{d}\left\{\dot{Y}^{(0)}+\varepsilon^4\dot{Y}^{(4)}+\varepsilon^6\dot{Y}^{(6)}\right\}\\ &= -\frac{\Gamma}{4\pi d}\left\{1-8.73622742\cdots\frac{(\nu t)^2}{d^4}+67.6539911\cdots\frac{(\nu t)^3}{d^6}\right\}.\end{aligned} \tag{3.153}$$

Again, remark $\tilde{\psi}_{11}^{(7)}=\psi_{11}^{(7)}+r\dot{Y}^{(6)}$ with the matching condition using (3.150), then the correction term at $O(\varepsilon^6)$

$$\dot{Y}^{(6)}=\pi\int_0^\infty A(r,t)r\mathrm{d}r-\frac{3h_3}{16}, \tag{3.154}$$

where $A(r,t)$ is the function defined in (3.142). Revisit the general formula (3.100) at $O(\varepsilon^5)$, and combining with (3.154) hence the general formula for the traveling speed of a vortex pair at $O(\varepsilon^7)$ can be written as

$$\dot{Y}=-\frac{1}{4\pi}\left\{1-\pi\varepsilon^4\left(2\pi\int_0^\infty\frac{\partial a(r,t)}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}r\mathrm{d}r-q_2\right)-\pi\varepsilon^6\left(4\pi\int_0^\infty A(r,t)r\mathrm{d}r-\frac{3h_3}{4}\right)\right\}. \tag{3.155}$$

The application of the general formula have been given with initial condition the Oseen vortex for a viscous fluid. Gaifullin and Zubtsov (2004) solved numerically for a viscous fluid using initial condition the Oseen vortex but unfortunately they did not show explicitly the equations for solving the traveling speed of a vortex at $O(\varepsilon^7)$. By comparing the result of Gaifullin and Zubtsov (2004), we have different result as their numerical solution. Until now we do not know whose result is correct so that the proof of the general formula is needed. Since the general formula for the motion of a vortex pair at $O(\varepsilon^7)$ rather complicated, similar analytical proof that we have done at $O(\varepsilon^5)$ in Appendix A becomes very difficult to do because of dependecy of the formula with the asymptotic solutions of each order. Our prediction is at $O(\varepsilon^7)$, the general formula of a counter-rotating of a vortex pair has behavior as the general formula at $O(\varepsilon^5)$ where the higher order formula can be simplified in the term of lower order in terms of the hexapole strength of the vorticity distribution at $O(\varepsilon^3)$.

3.4 Lateral motion of a vortex pair

In the previous section we discuss the velocity of a vortex pair which is the $\cos\theta$ component of the streamfunction for the flow relative to the streamfunction frame (3.13) (recall Chapter 2) contribute to the vertical translation of a vortex pair. The $\sin\theta$ component of the streamfunction for the flow relative to the streamfunction frame (3.13) is responsible to the lateral speed of a vortex pair. In this section we can obtain lateral motion of a vortex pair, especially

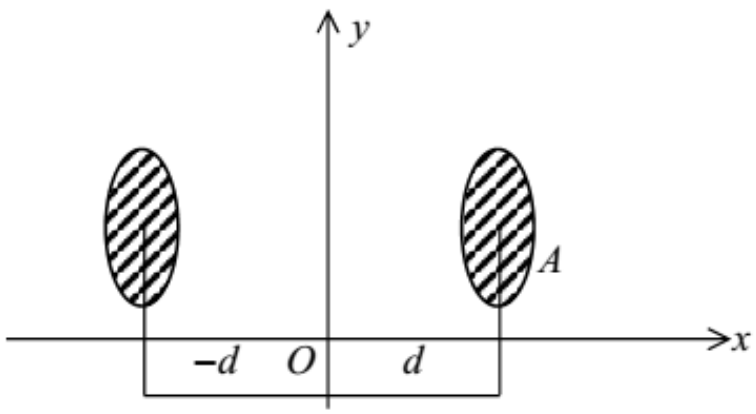

Fig. 3.15 A vortex pair separated with distance $2d$ and core area A

the x-position of the center of vortex core through the hydrodinamics impulse. In the absence of viscosity, a vortex pair makes a steady translation motion, in the direction normal to the line connecting the vorticity centroids, with maintaining the relative distance. The viscosity admits the relative motion, causing change in the distance between the vorticity centroids. This lateral motion occurs even at a higher order, namely, at $O(\varepsilon^7)$.

3.4.1 General formula of lateral position

The hydrodynamic impulse $\boldsymbol{P}$ is a constant vector of the motion even in the presence of viscosity (Lamb 1945, Saffman 1992), and is defined as integrals, over the whole plane, including the vorticity $\boldsymbol{\omega} = \zeta(x,y,t)\boldsymbol{e}_z$ and the position vector $\boldsymbol{x} = (x,y,z)$,

$$\boldsymbol{P} = \int \boldsymbol{\omega} \times \boldsymbol{x} \mathrm{d}A, \tag{3.156}$$

where $\boldsymbol{e}_z$ is the unit vector in the z-direction, A is vortex-core area, $\mathbf{r}$ is core-radius vector and ω is vorticity vector where can be seen in Fig. 3.15. Vector position of a vortex pair can be written as following through Fig. 3.1

$$\begin{aligned}\mathbf{x} &= x\mathbf{e}_x + y\mathbf{e}_y, \\ &= (X + r\cos\theta)\,\mathbf{e}_x + (Y + r\sin\theta)\,\mathbf{e}_y,\end{aligned} \tag{3.157}$$

Cross product of vorticity and vector position is

$$\boldsymbol{\omega} \times \mathbf{x} = -\zeta y\mathbf{e}_x + \zeta x\mathbf{e}_y. \tag{3.158}$$

substituting (3.158) into (3.156) then the impulse equation (3.156) becomes

$$\mathbf{P} = -\mathbf{e}_x \int \zeta y \mathrm{d}A + \mathbf{e}_y \int \zeta x \mathrm{d}A. \tag{3.159}$$

By the symmetry of the solution, the x-component P_x of the impulse is identically zero.

$$P_x = \int \zeta y \mathrm{d}A = 0, \tag{3.160}$$

and y-component impulse with changing coordinates from Cartesian coordinates to polar coordinates systems

$$\begin{aligned}P_y &= \int \zeta x dA, \\ &= 2\left(X\int \zeta r \mathrm{d}r\mathrm{d}\theta + \iint \zeta r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta\right), \\ &= 2\left(X\Gamma + \int \zeta r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta\right),\end{aligned} \tag{3.161}$$

where $\Gamma = \int \zeta \mathrm{d}A$ is the circulation or the strength of each vortex. The integral on the rightmost side is taken over the right vortex. We substitute series form for the vorticity ζ using (3.9)

$$\zeta = \zeta^{(0)} + \varepsilon\zeta^{(1)} + \varepsilon^2\zeta^{(2)} + \varepsilon^3\zeta^{(3)} + \varepsilon^4\zeta^{(4)} + \varepsilon^5\zeta^{(5)} + \varepsilon^6\zeta^{(6)} + \varepsilon^7\zeta^{(7)} + \varepsilon^8\zeta^{(8)} + \cdots, \tag{3.162}$$

and set non-dimensional variables of equation (3.161) to give

$$\begin{aligned}P_y &= 2d\Gamma\left(X + \varepsilon\int \zeta^{(0)} r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta + \varepsilon^2 \int \zeta^{(1)} r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta\right. \\ &\quad \left. + \varepsilon^3 \int \zeta^{(2)} r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta + \varepsilon^4 \int \zeta^{(3)} r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta + \cdots\right),\end{aligned} \tag{3.163}$$

after we omit star for dimensional variables. Furthermore by inserting series form for position X correspondingly to (3.11)

$$X = X^{(0)} + \varepsilon X^{(1)} + \varepsilon^2 X^{(2)} + \varepsilon^3 X^{(3)} + \varepsilon^4 X^{(4)} + \varepsilon^5 X^{(5)} + \varepsilon^6 X^{(6)} + \varepsilon^7 X^{(7)} + \varepsilon^8 X^{(8)} + \cdots, \tag{3.164}$$

into (3.163) we end up

$$\begin{aligned} P_y &= 2d\Gamma\left(X^{(0)} + \varepsilon\left(X^{(1)} + \int \zeta^{(0)} r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta\right) + \varepsilon^2\left(X^{(2)} + \int \zeta^{(1)} r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta\right)\right. \\ &\quad \left. + \cdots + \varepsilon^n\left(X^{(n)} + \int \zeta^{(n-1)} r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta\right) + \cdots\right), \end{aligned} \tag{3.165}$$

with a natural choice $X^{(0)} = d$. Only the dipole-component of ζ contributes the second integral of (3.165), and in dimensionless variable we may write $y-$ component impulse

$$P_y = 2d\Gamma\left\{X^{(0)} + \varepsilon^6\left(X^{(6)} + \pi\int_0^\infty \zeta_{11}^{(5)} r^2 dr\right) + \varepsilon^8\left(X^{(8)} + \pi\int_0^\infty \zeta_{11}^{(7)} r^2 dr\right)\cdots\right\}, \tag{3.166}$$

hereafter stars dropped off, where we read off from (3.94) and (3.129)

$$\zeta_{11}^{(5)} = a\tilde{\psi}_{11}^{(5)} - \frac{1}{2\nu_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}, \tag{3.167}$$

and

$$\zeta_{11}^{(7)} = a\tilde{\psi}_{11}^{(7)} - \frac{A(r,t)}{\nu_0}, \tag{3.168}$$

where $A(r,t)$ employs (3.142). The impulse is an invariant of the motion and by the constancy of P_y in (3.166) dictates that

$$X^{(6)} + \pi\int_0^\infty \zeta_{11}^{(5)} r^2 dr = const, \tag{3.169}$$

it is also valid for $X^{(8)}$ that is

$$X^{(8)} + \pi\int_0^\infty \zeta_{11}^{(7)} r^2 dr = const. \tag{3.170}$$

We have thus reached a desired formula for the lateral motion of the vorticity centroid, which is expressed, in terms of dimensional variables, as

$$X = d - \frac{\pi\nu^{5/2}}{\Gamma^{7/2}}\left(\int_0^\infty \zeta_{11}^{(5)} r^2 dr + \frac{\nu}{\Gamma}\int_0^\infty \zeta_{11}^{(7)} r^2 dr + C^{(0)}\right), \tag{3.171}$$

where $C^{(0)}$ is a constant to be set at initial instant. In the next subsection, the illustration of (3.171) for the motion of a viscous vortex pair evolving from the delta function core will be given.

3.4.2 Example to calculate lateral position

We give an example to calculate lateral position of a viscous vortex that we concentrate on the vorticity distribution starting from a delta-function cores, where at time $t = 0$, y-component impulse is $P_y = 2\Gamma d$, the x-positions of vortex center started at $O(\varepsilon^0)$ with natural choice $X^{(0)} = d$, of by normalization we may write $X^{(0)} = 1$. Moreover the sixth order of ε is

$$X^{(6)} = -\pi \int_0^\infty \zeta_{11}^{(5)} r^2 \mathrm{d}r. \tag{3.172}$$

where we may simply take $C^{(0)} = 0$ and the vorticity $\zeta_{11}^{(5)}$ is obtained from fifth-order inner solution

$$\zeta_{11}^{(5)} = a\tilde{\psi}_{11}^{(5)} - \frac{1}{2v_0}\frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)}, \tag{3.173}$$

we write (3.173) for a function $\zeta_{11}^{(5)} = \zeta_{11}^{(5)}(\xi)$ of $\xi = r/\sqrt{\hat{\nu}t}$ such that can be written as following

$$\zeta_{11}^{(5)} = (\hat{\nu}t)^{3/2} h(\xi), \tag{3.174}$$

using (3.57) and (3.71) we may write $\psi_{21}^{(2)}$ and $\psi_{31}^{(3)}$ as following

$$\psi_{21}^{(2)} = \frac{\hat{\nu}t}{\pi}\left(\frac{1}{4}\phi_{21}^{(2)}(\xi) - \frac{\xi^2}{16}\right), \tag{3.175}$$

and

$$\psi_{31}^{(3)} = \frac{(\hat{\nu}t)^{3/2}}{\pi}\left(\frac{1}{48}\phi_{31}^{(3)}(\xi) + \frac{\xi^3}{48}\right). \tag{3.176}$$

The derivation of $a(r,t)$ can be written the same way as (3.175) and (3.176). Since $a(r,t) = -1/v^{(0)}(\partial\zeta^{(0)}/\partial r)$, from (3.35), where $\zeta^{(0)}$ and $v^{(0)}$ use (3.52) and (3.53) respectively, it is easily to get

$$a(r,t) = \frac{r^2}{4\hat{\nu}^2 t^2}\left(e^{\frac{r^2}{4\hat{\nu}t}} - 1\right)^{-1}, \tag{3.177}$$

furthermore by taking derivation (3.177) respect to r where $\xi = r/\sqrt{\hat{\nu}t}$ yields

$$\frac{\partial a(r,t)}{\partial r} = \frac{1}{(\hat{\nu}t)^{3/2}}\frac{\xi}{8}\left(e^{\frac{\xi^2}{4}}\left(4-\xi^2\right) - 4\right)\left(e^{\frac{\xi^2}{4}} - 1\right)^{-2}. \tag{3.178}$$

Now $\tilde{\psi}_{11}^{(5)}$ can be regarded from (3.97) with choosing an arbitrary constant $c_{11}^{(5)} = 0$. This choice amounts to choosing the origin of the moving coordinate at the stagnation point in the moving with the vortex core (Fukumoto and Moffatt 2000). Furthermore we have $\tilde{\psi}_{11}^{(5)} = \tilde{\psi}_{11}^{(5)}(\xi)$

$$\tilde{\psi}_{11}^{(5)} = \frac{(\hat{\nu}t)^{5/2}}{\pi}\frac{1}{\xi}\int_0^{\xi\sqrt{\hat{\nu}t}} \frac{\xi' d\xi'}{(1-e^{-\xi'^2/4})^2}\left[\int_0^{\xi'\sqrt{\hat{\nu}t}} Q(\xi'')\xi'' d\xi''\right] \tag{3.179}$$

where

$$Q(\xi) = \frac{\xi}{8}\left(e^{\frac{\xi^2}{4}}\left(4-\xi^2\right)-4\right)(e^{\xi^2/4}-1)^{-2}\left(\frac{1}{4}\phi_{21}^{(2)}(\xi)-\frac{\xi^2}{16}\right)\left(\frac{1}{48}\phi_{31}^{(3)}(\xi)+\frac{\xi^3}{48}\right). \tag{3.180}$$

Inserting all terms from (3.175) to (3.180) with

$$v_0 = \frac{1}{2\pi\xi\sqrt{\hat{\nu}t}}\left(1-e^{-\frac{\xi^2}{4}}\right), \tag{3.181}$$

into (3.173) we have

$$\zeta_{11}^{(5)} = (\hat{\nu}t)^{3/2}h(\xi), \tag{3.182}$$

where

$$\begin{aligned} h(\xi) &= \frac{\xi}{4\pi}\frac{(1-e^{-\xi'^2/4})}{(e^{\xi'^2/4}-1)}\int_0^{\xi\sqrt{\hat{\nu}t}} \frac{\xi' d\xi'}{(1-e^{-\xi'^2/4})^2}\left[\int_0^{\xi'\sqrt{\hat{\nu}t}} Q(\xi'')\xi'' d\xi''\right] \\ &\quad -\frac{\xi}{\pi(1-e^{-\xi'^2/4})}Q(\xi). \end{aligned} \tag{3.183}$$

Using (3.169), carefully numerical calculation results the lateral position of the vortex center at $O(\varepsilon^6)$

$$X^{(6)} \approx 40.1358688(\hat{\nu}t)^3, \tag{3.184}$$

where from the result of (3.184), the x-position of the vortex center in dimensional form is

$$X = d\left(1+40.1358688\frac{(\nu t)^3}{d^6}\right), \tag{3.185}$$

This result recovers that of Nakagawa (2004) and refines that of Gaifullin and Zubtsov (2004). The author leaves the numerical calculation of the lateral position of the vortex center at $O(\varepsilon^8)$.

Chapter 4

Case study: A vortex pair motion in an inviscid fluid

In this chapter we give an example of applying the general formula of the motion a vortex pair for an inviscid fluid $\nu = 0$ to the Rankine vortex, a vortex patch of uniform vorticity at leading order $O(\varepsilon^0)$. The assumption is made that the core is very thin, that is the core radius σ is much smaller than the vortex radius d such that $\varepsilon = \sigma/d \ll 1$. With this condition, the method of matched asymptotic expansion is employed.

The flow region is devided into two regions, inner and outer regions. The inner region is the core and its surrounding with thickness or order of the the core radius σ, and the outer region is the surrounding region whose characteristic length is the vortex radius d. The outer solution is an irrotational flow given by the Biot-Savart law. The inner solution is obtained by solving the Navier-Stokes equations. An asymptotic expansion of the Biot-Savart law near the vortex core provides with the matching condition for an asymptotic expansion for limiting the Navier-Stokes equations for large radius r. For the inner field, the Euler equations are solved in a moving coordinate system in the form of a power series in ε. Using the general formula of the traveling speed of a vortex pair as in Chapter 3, we can directly obtain the velocity of a vortex pair in an inviscid fluid by substituting the solutions into the general formula for the traveling speed of a vortex pair. Furthermore we shall show that the traveling speed of a vortex pair in an inviscid fluid coincides with the result of Yang and Kubota (1994).

4.1 Inner solution

In this section, asymptotic expansions of the inner solution will be derived starting from zeroth order up to fifth order.

4.1.1 Zeroth-order solution

Equations governing the leading-order vorticity $\zeta^{(0)}$ are supplied by the $O(\varepsilon^{-2})$ terms in (3.4) and the $O(\varepsilon^{0})$ terms in (3.5). Zeroth-order solution for ζ of the Navier-Stokes equation reduces to the Jacobian form of the Euler equations

$$\left[\zeta^{(0)}, \tilde{\psi}^{(0)}\right] = 0, \tag{4.1}$$

$$\zeta^{(0)} = -\triangle\tilde{\psi}^{(0)}, \tag{4.2}$$

resulting in $\zeta^{(0)} = \mathscr{F}(\tilde{\psi}^{(0)})$, for some function $\mathscr{F}$. Suppose that the flow $\tilde{\psi}^{(0)}$ has a single stagnation point at $r = 0$, all the streamlines being closed around that point, then it is probable that the solution (4.1), coupled with the $\zeta - \psi$ relation at $O(\varepsilon^{(0)})$ must be radial $\tilde{\psi}^{(0)} = \tilde{\psi}^{(0)}(r)$; streamfunction are then necessarily circles. The function $\tilde{\psi}^{(0)}$ and $\zeta^{(0)}$ can be found by solving the solution of the heat conduction equation with $\nu = 0$

$$\frac{\partial\zeta^{(0)}}{\partial t} = 0, \tag{4.3}$$

where in this case an arbitrary function, though steady, satisfies equation (4.3). We may take the Rankine vortex solution at the leading order $O(\varepsilon^{0})$. The vorticity and the velocity of the Rankine vortex are respectively defined by

$$\zeta^{(0)} = \begin{cases} \frac{\Gamma}{\pi\sigma^2} & \text{for } r < \sigma, \\ 0 & \text{for } r > \sigma, \end{cases} \tag{4.4}$$

$$v^{(0)} = \begin{cases} \frac{\Gamma}{2\pi\sigma^2} r & \text{for } r < \sigma, \\ \frac{\Gamma}{2\pi r} & \text{for } r > \sigma, \end{cases} \tag{4.5}$$

where σ is the core radius. For dimensionless variables, the circulation is $\Gamma = 1$ and the core radius is $\sigma = 1$. Furthermore the velocity and the vorticity of the Rankine vortex are given by

$$\zeta^{(0)} = \begin{cases} \frac{1}{\pi} & \text{for } r < \sigma, \\ 0 & \text{for } r > \sigma, \end{cases} \tag{4.6}$$

$$v^{(0)} = \begin{cases} \frac{r}{2\pi} & \text{for } r < \sigma, \\ \frac{1}{2\pi r} & \text{for } r > \sigma, \end{cases} \tag{4.7}$$

from equation (4.6), the vorticity $\zeta^{(0)}$ is the Heaviside step function (2.25) as in Chapter 2 where in general the function is undefined at $r = 1$ because of discontinuity. It is invalid and unacceptable in the real physical phenomena. Moreover, to handle this problem, setting of the boundary condition is important, and will be provided to the next order solution.

4.1.2 First-order solution

To obtain first-order asymptotic solution in an inviscid fluid, a much simpler procedure is used. Recall section 3.3.2. Collecting terms of $O(\varepsilon^1)$ in the vorticity equation 3.4 and $O(\varepsilon^1)$ in the streamfunction equation (3.5) gives (3.28) and (3.29). The streamfunction $\tilde{\psi}^{(1)}$ in $O(\varepsilon)$ is defined by (3.13) with $j = 1$, the flow relative to the moving frame. The $O(\varepsilon)$ terms of (3.20) provides the matching condition on $\tilde{\psi}^{(1)}$. Integrating (3.28) with respect to θ over $[0, 2\pi)$ and dividing by $v^{(0)} = -\partial\psi^{(0)}/\partial r$ being the azimuthal velocity field results in the monopole and dipole $\cos\theta$-component of streamfunctions (3.31). Furthermore equalizing the vorticity (3.34) with (3.29) we obtain governing equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + a\right)\tilde{\psi}_{11}^{(1)} = 0, \tag{4.8}$$

where $\tilde{\psi}_{11}^{(1)}(r,t) = \psi_{11}^{(1)} + r\dot{Y}^{(0)}$, and

$$a = -\frac{1}{v^{(0)}}\frac{\partial\zeta^{(0)}}{\partial r}. \tag{4.9}$$

The derivative of the vorticity $\zeta^{(0)}$ is taken to be a Dirac delta function (2.26) which is given by

$$\frac{\partial\zeta^{(0)}}{\partial r} = -\frac{1}{\pi}\delta(r-1), \tag{4.10}$$

furthermore, inserting (4.10) into equation (4.9) yields

$$a = 2\delta(r-1), \tag{4.11}$$

where $a = 2\delta(r-1)$ at $r = 1$ and $a = 0$ at everywhere else (see Appendix C). The solution of (4.8) is $\tilde{\psi}_{11}^{(1)} = cv^{(0)}$ for an arbitrary constant c. The Euler-type equation has the general solution $\tilde{\psi}_{11}^{(1)} = Ar^p$ where A is any constant and p is the order of r so that (4.8) can be written as the Euler type as following

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\tilde{\psi}_{11}^{(1)} = 0. \tag{4.12}$$

The solution of equation (4.12) is then

$$\tilde{\psi}_{11}^{(1)} = \begin{cases} c_1 r & \text{for } r < 1, \\ \frac{c_2}{r} & \text{for } r > 1, \end{cases} \tag{4.13}$$

where c_1 and c_2 are any constant. In the solution (4.13) we can see that there is discontinuity at $r = 1$, which means that there is velocity jump $\partial\tilde{\psi}_{11}^{(1)}/\partial r$ at $r = 1$. To handle the singularity at $r = 1$, it is important to enforce the connecting condition. The boundary condition at $r = 1$ must satisfy

$$\tilde{\psi}_{11}^{(1)}\Big|_{r=1+\eta} = \tilde{\psi}_{11}^{(1)}\Big|_{r=1-\eta}, \tag{4.14}$$

and

$$\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\Bigg|_{r=1+\eta} - \frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\Bigg|_{r=1-\eta} + 2\tilde{\psi}_{11}^{(1)}\Bigg|_{r=1} = 0, \tag{4.15}$$

where $\eta \to 0$. The boundary condition (4.15) is obtained by integrating equation (4.8) with respect to r and by taking limit $\eta \to 0$

$$\lim_{\eta\to 0}\int_{1-\eta}^{1+\eta}\left(\frac{\partial^2\tilde{\psi}_{11}^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r} - \frac{\tilde{\psi}_{11}^{(1)}}{r^2} + 2\delta(r-1)\,\tilde{\psi}_{11}^{(1)}\right)\mathrm{d}r = 0, \tag{4.16}$$

where the second and third terms vanish after integration, and applying a delta function property (2.28) to the fourth term. Therefore applying boundary conditions (4.14) and (4.15) yields $c_1 = c_2$, hence the solution (4.13) now can be written as

$$\tilde{\psi}_{11}^{(1)} = \begin{cases} c_1 r & \text{for } r < 1, \\ \frac{c_1}{r} & \text{for } r > 1. \end{cases} \tag{4.17}$$

An arbitrary constant c_1 corresponds to the freedom of choosing the origin of the local comoving coordinates $(\tilde{x},\tilde{y})$ within the accuracy of $O(\varepsilon^2)$ (Fukumoto and Moffatt 2000). Since the solution has the fore-and-aft symmetry, the origin should be maintained at the core center in the y direction by selecting $c_1 = 0$. Remark that $\tilde{\psi}_{11}^{(1)}(r,t) = \psi_{11}^{(1)} + r\dot{Y}^{(0)}$, thus we may conveniently take an inner solution for $\tilde{\psi}_{11}^{(1)}$ is equal to zero. The matching condition at $O(\varepsilon)$ is

$$\psi^{(1)} \sim \left(\frac{r}{4\pi} + \frac{1}{2r} \int_0^\infty \zeta_{11}^{(1)} r^2 \mathrm{d}r \right) \cos\theta \quad \text{as } r \to \infty, \tag{4.18}$$

so that the traveling speed of a vortex pair in an inviscid fluid is

$$\dot{Y}^{(0)} = -\frac{1}{4\pi}, \; \dot{X}^{(0)} = 0. \tag{4.19}$$

The velocity of a vortex pair in an inviscid fluid starts with the induced velocity of a point vortex. Here we have the same result as the first-order solution of the motion of a vortex pair in a viscous fluid.

4.1.3 Second-order solution

Recall section 3.3.3. Collecting $O(\varepsilon^2)$ terms in the vorticity equation (3.4) and $O(\varepsilon^2)$ terms in (3.5) gives (3.43) and (3.44). The streamfunction $\tilde{\psi}^{(2)}$ in $O(\varepsilon)$ is defined by (3.13) with $j = 2$, the flow relative to the moving frame. The $O(\varepsilon)$ terms of (3.20) provides the matching condition for $\tilde{\psi}^{(1)}$. Integrating (3.43) with respect to θ over $[0, 2\pi)$ and dividing by $v^{(0)} = -\partial\psi^{(0)}/\partial r$ being the azimuthal velocity field results in the monopole and quadrupole $\cos 2\theta$-component of streamfunctions (3.46). Furthermore equalizing for the vorticity (3.54) with (3.44), the quadrupole component is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2} + a \right) \psi_{21}^{(2)} = 0. \tag{4.20}$$

We may write the solution (4.20) as

$$\psi_{21}^{(2)} = -\frac{r^2}{16\pi} + \hat{\psi}_{21}^{(2)}, \tag{4.21}$$

where the first term is the induced motion from the companion vortex and the second term $\hat{\psi}_{21}^{(2)}$ is the self-induced motion of the right vortex as regarded. The boundary condition is

$$\tilde{\psi}_{21}^{(2)}\Big|_{r=1+\eta} = \tilde{\psi}_{21}^{(2)}\Big|_{r=1-\eta}, \tag{4.22}$$

and

$$\left.\frac{\partial \tilde{\psi}_{21}^{(2)}}{\partial r}\right|_{r=1+\eta} - \left.\frac{\partial \tilde{\psi}_{21}^{(2)}}{\partial r}\right|_{r=1-\eta} + 2\tilde{\psi}_{21}^{(2)}\Big|_{r=1} = 0, \tag{4.23}$$

where $\eta \to 0$. The same way as deriving the first-order solution, the Euler type of second-order differential equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tilde{\psi}_{21}^{(2)} = 0, \tag{4.24}$$

where the solution of (4.24) is $\tilde{\psi}_{21}^{(2)} = Ar^p$ where A is any constant and p is the order of r so that the solution of equation (4.24) is

$$\tilde{\psi}_{21}^{(2)} = \begin{cases} -\frac{r^2}{16\pi} + d_1 r^2 & \text{for } r < 1, \\ -\frac{r^2}{16\pi} + \frac{d_2}{r^2} & \text{for } r > 1, \end{cases} \tag{4.25}$$

where d_1 and d_2 are any constant. Applying boundary condition (4.22) into (4.25) yields $d_1 = d_2$, then using boundary condition (4.23) yields $d_1 = d_2 = -1/16\pi$. Therefore the solution of equation (4.20) is given by

$$\psi_{21}^{(2)} = \begin{cases} -\frac{1}{8\pi} r^2 & \text{for } r < 1, \\ -\frac{1}{16\pi} r^2 - \frac{1}{16\pi}\frac{1}{r^2} & \text{for } r > 1, \end{cases} \tag{4.26}$$

with $\tilde{\psi}_{21}^{(2)} = \psi_{21}^{(2)}$ kept in view from (3.13) because of the absence of the correction to the travelling speed at this order. The matching condition of second-order solution to be imposed on $\psi^{(2)}$ is

$$\psi^{(2)} \sim \left(-\frac{r^2}{16\pi} + \frac{q_2}{r^2}\right)\cos 2\theta \text{ as } r \to \infty, \tag{4.27}$$

where

$$q_2 = \frac{1}{4}\int_0^\infty \zeta_{21}^{(2)} r^3 dr. \tag{4.28}$$

Then the solution (4.26) and the matching condition (4.27) give rise to

$$q_2 = -\frac{1}{16\pi}. \tag{4.29}$$

Since we do not have the correction to the translation of a vortex pair in an inviscid fluid, we continue to solve this pertubation method to the next order, and the solution (4.29) will be used to calculate the velocity of a vortex pair in an inviscid fluid.

4.1.4 Third-order solution

Recall section 3.3.4. Collecting $O(\varepsilon^3)$ terms in the vorticity equation (3.4) and $O(\varepsilon^3)$ terms in (3.5) gives (3.61) and (3.62). The streamfunction $\tilde{\psi}^{(3)}$ in $O(\varepsilon)$ is defined by (3.13) with $j = 3$, the flow relative to the moving frame. The $O(\varepsilon^3)$ terms of (3.20) provide the matching condition for $\tilde{\psi}^{(3)}$. Integrating (3.61) with respect to θ over $[0, 2\pi)$ and dividing by $v^{(0)} = -\partial\psi^{(0)}/\partial r$ being the azimuthal velocity field results in the monopole and hexapole $\cos 3\theta$-component of streamfunctions (3.64). Furthermore equalizing for the vorticity (3.66) with (3.62) gives the hexapole component

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{9}{r^2} + a\right)\psi_{31}^{(3)} = 0. \tag{4.30}$$

We may write the solution of equation (4.30) as

$$\psi_{31}^{(3)} = \frac{r^3}{48\pi} + \hat{\psi}_{31}^{(3)}, \tag{4.31}$$

where the first term is the induced motion from the companion vortex and the second term $\hat{\psi}_{31}^{(3)}$ is the self-induced motion of the right vortex as regarded. The boundary conditions are

$$\tilde{\psi}_{31}^{(3)}\Big|_{r=1+\eta} = \tilde{\psi}_{31}^{(3)}\Big|_{r=1-\eta}, \tag{4.32}$$

and

$$\frac{\partial\tilde{\psi}_{31}^{(3)}}{\partial r}\Big|_{r=1+\eta} - \frac{\partial\tilde{\psi}_{31}^{(3)}}{\partial r}\Big|_{r=1-\eta} + 2\tilde{\psi}_{31}^{(3)}\Big|_{r=1} = 0, \tag{4.33}$$

where $\eta \to 0$. The same way as deriving second-order inner solution, the Euler-type equation of third-order asymptotic expansion is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{9}{r^2}\right)\tilde{\psi}_{31}^{(3)} = 0, \tag{4.34}$$

where the solution of (4.34) is $\tilde{\psi}_{31}^{(3)} = Ar^p$ where A is any constant and p is the order of r so that the solution of equation (4.34) is

$$\tilde{\psi}_{31}^{(3)} = \begin{cases} \frac{r^3}{48\pi} + e_1 r^3 & \text{for } r < 1, \\ \frac{r^3}{48\pi} + \frac{e_2}{r^3} & \text{for } r > 1, \end{cases} \tag{4.35}$$

where e_1 and e_2 are any constant. Using boundary conditions (4.32) into (4.35) yields $e_1 = e_2$, and applying boundary conditions (4.33) yields $e_1 = e_2 = 1/96\pi$. The solution of equation (4.30) is

$$\psi_{31}^{(3)} = \begin{cases} \frac{1}{32\pi} r^3 & \text{for } r < 1, \\ \frac{1}{48\pi} r^3 + \frac{1}{96\pi}\frac{1}{r^3} & \text{for } r > 1, \end{cases} \tag{4.36}$$

with $\tilde{\psi}_{31}^{(3)} = \psi_{31}^{(3)}$ kept in view from (3.13) because of the absence of the correction to the travelling speed at this order. The matching condition of the third-order solution to be imposed on $\psi^{(3)}$ is

$$\psi^{(3)} \sim \left(\frac{r^3}{48\pi} + \frac{h_3}{r^3} \right) \cos 3\theta, \text{ as } r \to \infty, \tag{4.37}$$

where

$$h_3 = \frac{1}{6} \int_0^\infty \zeta_{31}^{(3)} r^4 dr. \tag{4.38}$$

The solution (4.36) with the matching condition (4.37) yields

$$h_3 = \frac{1}{96\pi}. \tag{4.39}$$

As in Chapter 3, the traveling speed of a counter-rotating vortex pair at $O(\varepsilon^5)$ was calculated from the contribution of the second and third-order solutions, and we will leave the fourth-order solution for the motion of a vortex pair in an inviscid fluid and directly go to fifth-order solution.

4.1.5 Fifth-order inner solution

To obtain fifth-order inner solution, we solve nonhomogeneous second order differential equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + a \right) \tilde{\psi}_{11}^{(5)} = -\frac{F(r)}{v^{(0)}}, \tag{4.40}$$

where $\tilde{\psi}_{11}^{(5)} = \psi_{11}^{(5)} + r\dot{Y}^{(4)}$, and

$$F(r) = -\frac{1}{2}\frac{\partial a}{\partial r} \psi_{21}^{(2)} \psi_{31}^{(3)}, \tag{4.41}$$

where $\psi_{21}^{(2)}$ and $\psi_{31}^{(3)}$ are the second and third-order solutions (4.26) and (4.36) respectively. The same way as deriving the first-order solution, and remark $a = 2\delta(r-1)$ at $r = 1$ and

$a = 0$ at everywhere else, the second order differential equation (4.40) becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\tilde{\psi}_{11}^{(5)} = -\frac{F(r)}{v^{(0)}}, \tag{4.42}$$

where (4.42) is nonhomogeneous the Euler-type equation. The solution of (4.42) is $\tilde{\psi}_{11}^{(5)} = \tilde{\psi}_{11H}^{(5)} + \tilde{\psi}_{11P}^{(5)}$ where subscripts H and P represent homogeneous and particular solutions respectively. The homogeneous solution $\tilde{\psi}_{11H}^{(5)}$ is

$$\tilde{\psi}_{11H}^{(5)} = \begin{cases} y_1 \equiv c_1 r & \text{for } r < 1, \\ y_2 \equiv \frac{c_1}{r} & \text{for } r > 1. \end{cases} \tag{4.43}$$

where this solution is obtained with similar way as solving the first-order asymptotic expansions. Then the particular solution can be obtained using the method of variational parameter which is given by

$$\tilde{\psi}_{11P}^{(5)} = \begin{cases} -y_1 \int \frac{y_2 F(r)}{W(y_1,y_2)} \mathrm{d}r & \text{for } r < 1, \\ y_2 \int \frac{y_1 F(r)}{W(y_1,y_2)} \mathrm{d}r & \text{for } r > 1, \end{cases} \tag{4.44}$$

where $W(y_1,y_2) = y_1 y_2' - y_2 y_1' = -2c_1^2/r$ is the Wronskian. We calculate (4.44) by substituting (4.26) for $\psi_{21}^{(2)}$, (4.36) for $\psi_{31}^{(3)}$ and using the Dirac delta function property (2.30) to yield

$$\tilde{\psi}_{11P}^{(5)} = \begin{cases} \frac{1}{64\pi} r & \text{for } r < 1, \\ -\frac{1}{64\pi}\frac{1}{r} & \text{for } r > 1, \end{cases} \tag{4.45}$$

so that the solution of equation (4.42) is given by the coupled of the homogeneous and the particular solutions as following

$$\tilde{\psi}_{11}^{(5)} = \begin{cases} c_1 r + \frac{1}{64\pi} r & \text{for } (r < 1), \\ \frac{c_1}{r} - \frac{1}{64\pi}\frac{1}{r} & \text{for } (r > 1). \end{cases} \tag{4.46}$$

We may choose an arbitrary constant $c_1 = 0$ such that by maching condition

$$\psi_{11}^{(5)} \sim q_2 \frac{r}{4} + \frac{1}{2r}\left(\int_0^\infty \zeta_{11}^{(5)} r^2 \mathrm{d}r\right) \quad \text{as } r \to \infty, \tag{4.47}$$

remark $\tilde{\psi}_{11}^{(5)} = \psi_{11}^{(5)} + r\dot{Y}^{(4)}$, and substituting (4.29) into (4.47) provides the correction of the traveling speed of a vortex pair at $O(\varepsilon^5)$

$$\dot{Y}^{(4)} = \frac{1}{32\pi}. \tag{4.48}$$

Furthermore the traveling speed of a vortex pair in an inviscid fluid is served as

$$\dot{Y} = \frac{\Gamma}{d}\left\{\dot{Y}^{(0)} + \varepsilon^4\dot{Y}^{(4)}\right\} = -\frac{\Gamma}{4\pi d}\left\{1 - \frac{\sigma^4}{8d^4}\right\}, \tag{4.49}$$

where this result recovers the result of Yang and Kubota (1994). The next section will be shown that our general formula can be tested using the solutions of the motion of an anti-parallel vortex pair in an inviscid fluid with initial condition of the Rankine vortex.

4.2 Testing general formula with an inviscid case

In this section, we can test the general formula (3.104)

$$\int_0^\infty \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r = -\frac{q_2}{2\pi}, \tag{4.50}$$

using the solutions of the motion of counter-rotating a vortex pair in an inviscid fluid with initial condition of the Rankine vortex. We calculate the left hand side of (4.50) as following

$$\begin{aligned}
\int_0^\infty \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r &= \int_0^1 \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r + \int_1^\infty \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r && (4.51)\\
&= \frac{1}{2}\left(\int_0^1 \frac{\partial(2\delta(r-1))}{\partial r}\left(-\frac{r^2}{8\pi}\right)\left(\frac{r^3}{32\pi}\right) r\mathrm{d}r\right.\\
&\quad \left. + \int_1^\infty \frac{\partial(2\delta(r-1))}{\partial r}\left(-\frac{r^2}{16\pi} - \frac{1}{16\pi r^2}\right)\left(\frac{r^3}{48\pi} + \frac{1}{96\pi r^3}\right) r\mathrm{d}r\right) && (4.52)\\
&= \left(-\int_0^1 \delta(r-1)\frac{\partial}{\partial r}\left\{\left(-\frac{r^2}{8\pi}\right)\left(\frac{r^3}{32\pi}\right) r\right\} dr\right.\\
&\quad \left. - \int_1^\infty \delta(r-1)\frac{\partial}{\partial r}\left\{\left(-\frac{r^2}{16\pi} - \frac{1}{16\pi r^2}\right)\left(\frac{r^3}{48\pi} + \frac{1}{96\pi r^3}\right) r\right\} dr\right) && (4.53)\\
&= \frac{3}{128\pi^2} + \frac{1}{128\pi^2}\\
&= \frac{1}{32\pi^2}, && (4.54)
\end{aligned}$$

where $\psi_{21}^{(2)}$ and $\psi_{31}^{(3)}$ are solutions of the second and third-oder asymptotic expansions, (4.26) and (4.36) respectively; and a uses (4.11). The half value appears in (4.52) because of the symmetry of the core region so that by partial integration or using a delta function property (2.5) we have (4.53). Simple calculations are used to obtain (4.54). Furthermore the right hand side of (4.50) is calculated by inserting q_2 using (4.29) which gives the same result as (4.54). Here we can see that the general formula (4.50) is valid for both a viscous and an inviscid fluids. Then recall the general formula of the traveling speed of a counter-rotating vortex pair in dimensionless form (3.105). Substituting the second-order solution (4.29) for q_2 into (4.50) so that we can easily obtain the traveling speed of a counter-rotating vortex in an inviscid fluid

$$\dot{Y} = \frac{\Gamma}{d}\left\{\dot{Y}^{(0)} + \varepsilon^4 \dot{Y}^{(4)}\right\} = -\frac{\Gamma}{4\pi d}\left\{1 - \frac{\sigma^4}{8d^4}\right\}, \tag{4.55}$$

where this result coincides with the result of Yang and Kubota (1994).

Chapter 5

Conclusion

The general formula for an arbitrary vorticity distribution of the motion of a counter-rotating vortex pair in an inviscid as well as a viscous fluids has been established and we completely solved the problem which was initiated by Ting and Tung (1965). The matched asymptotic expansions have been extended to a higher order in ε to calculate a higher-order traveling speed of a vortex pair in incompressible fluid. The inner solutions have been obtained by solving the Navier-Stokes equation pertubatively in a small parameter $\varepsilon = 1/\sqrt{Re} = \sqrt{\nu/\Gamma}$ where $Re = \Gamma/\nu$ is the Reynolds number, ν is the kinematic viscosity of the fluid and Γ is the circulation of each vortex. The parameter ε is a measure for the typical-length ratio between core radius σ to the distance d between the vortices. The radius of vortex core is assumed to be much smaller than the distance between two centroids $(\varepsilon \ll 1)$. The Biot-Savart law is applied as the inner limit of the outer solution with negligible vorticity, and by adapting Dyson's technique the self-induced motion of a field near vortex core can be evaluated. The asymptotic expansions start with the general circular flow and the first-order solution gives the traveling speed of a vortex pair in an inviscid and a viscous fluids starting with a pair of point vortices. The vortex core is deformed in an elliptical form by the pure shear induced by yhe companion vortex at $O(\varepsilon^2)$ and $O(\varepsilon^3)$. A highly tidy and powerful formula for the traveling speed of a vortex pair has been obtained in which the strength of the quadrupole of $O(\varepsilon^2)$ suffices to calculate the $O(\varepsilon^5)$ correction to the traveling speed. The general formula for the traveling speed of a vortex pair includes the strengh of the second-order quadrupole field only and is given in term of the dimensional form as

$$\dot{Y} = -\frac{1}{4\pi}\left\{1 - \pi\varepsilon^4\left(2\pi\int_0^\infty \frac{\partial a(r,t)}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r - q_2\right) - \pi\varepsilon^6\left(4\pi\int_0^\infty A(r,t) r\mathrm{d}r - \frac{3h_3}{4}\right)\right\}. \tag{5.1}$$

where the traveling speed of a vortex pair at $O(\varepsilon^5)$ can be written as

$$\dot{Y} = -\frac{\Gamma}{4\pi d}\left\{1 + \frac{2\pi}{\Gamma d^2} q\right\}; \quad q = \varepsilon^2 q_2, \tag{5.2}$$

where q_2 is the strength of the quadrupole field of self induced origin. The formula is remarkably simple in the sense that the correction of the translation speed is expressed in terms of the quadrupole strength of $O(\varepsilon^2)$. Furthermore, the general formula is reinforced with the analytical proof and study case with initial condition of the Rankine vortex. This general formula is valid for a viscous and an inviscid fluids where for an inviscid fluid, and the formula is in good agreement with the result of the translation speed of a counter-rotating vortex pair with symmetric cores in ideal fluid by Yang and Kubota (1994).

Futhermore the solution for the translation of a vortex pair is continued up to $O(\varepsilon^7)$ and the general formula for the motion of a vortex pair has been obtained. The correction of the velocity of a vortex pair at this order is given by

$$\dot{Y}^{(6)} = \pi \int_0^\infty A(r,t) r \mathrm{d}r - \frac{3h_3}{16}, \tag{5.3}$$

where h_3 is hexapole strength at $O(\varepsilon^3)$, and

$$\begin{aligned} A(r) \;=\; & \frac{\partial \zeta_{11}^{(5)}}{\partial r}\psi_{21}^{(2)} + \frac{1}{2}\zeta_{11}^{(5)}\frac{\partial \psi_{21}^{(2)}}{\partial r} - \frac{3}{2}\frac{\partial \zeta_{41}^{(4)}}{\partial r}\psi_{31}^{(3)} \\ & -2\zeta_{41}^{(4)}\frac{\partial \psi_{31}^{(3)}}{\partial r} + 2\frac{\partial \zeta_{31}^{(3)}}{\partial r}\psi_{41}^{(4)} + \frac{3}{2}\zeta_{31}^{(3)}\frac{\partial \psi_{41}^{(4)}}{\partial r}. \end{aligned} \tag{5.4}$$

Gaifullin and Zubtsov (2004) investigated the translation speed of a vortex pair with special case of Oseen vortex as initial condition. Our general formula can be applied to an arbitrary initial condition, for example to a delta function. Through this example to the general formula at $O(\varepsilon^5)$ we have good agreement with the result of Gaifullin and Zubtsov (2004). But for $O(\varepsilon^7)$ Gaifullin and Zubtsov (2004) did not show explicitly the equations for solving the traveling speed of a vortex at this stage, and we have different result as their numerical result with a delta function as initial condition. For future work, the general formula for the translation of vortex pair needs to be proved. Since the general formula for the motion of a vortex pair at $O(\varepsilon^7)$ is rather complicated, a similar analytical proof that we have done at $O(\varepsilon^5)$ becomes very difficult to do. Our expetation is at $O(\varepsilon^7)$, the general formula of a counter-rotating of a vortex pair has behavior as the general formula at $O(\varepsilon^5)$ where the higher order formula can be simplified in terms of the lower order hexapole strength in terms of the vorticity distribution at $O(\varepsilon^3)$.

Finally we have completed this dissertation with the general formula of the lateral motion of a vortex pair where in an inviscid fluid results in a steady translation motion in the direction normal to the line connecting the vorticity centroids. However, for a viscous fluid, the viscosity exhibits the relative motion by the viscous diffusion of vorticity since this vortex core diffusion makes the distance between the vorticity centroids relatively changes. The formula for the lateral position of the centroids of the vortex is

$$X = d - \frac{\pi \nu^{5/2}}{\Gamma^{7/2}} \left(\int_0^\infty \zeta_{11}^{(5)} r^2 dr + \frac{\nu}{\Gamma} \int_0^\infty \zeta_{11}^{(7)} r^2 dr + C^{(0)} \right), \tag{5.5}$$

where $C^{(0)}$ is a constant to be set at initial instant and this formula is in dimensional variables.

Appendix A

Analytical proof

In this Appendix, we shall prove (3.104) or

$$\int_0^\infty \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r = -\frac{q_2}{2\pi}, \tag{A.1}$$

whereby (3.99) drastically simplifies to

$$\dot{Y}^{(4)} = -\frac{q_2}{2}. \tag{A.2}$$

Recall equations (3.55) and (3.67) of the second- and third-order asymptotics of the streamfunction supplemented with their boundary conditions (3.56) and (3.68)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2} + a\right)\psi_{21}^{(2)} = 0, \tag{A.3}$$

$$\begin{cases} \psi_{21}^{(2)} \propto r^2 & \text{as } r \to 0, \\ \psi_{21}^{(2)} \sim -\frac{r^2}{16\pi} + \frac{q_2}{r^2} & \text{as } r \to \infty, \end{cases} \tag{A.4}$$

and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{9}{r^2} + a\right)\psi_{31}^{(3)} = 0, \tag{A.5}$$

$$\begin{cases} \psi_{31}^{(3)} \propto r^3 & \text{as } r \to 0, \\ \psi_{31}^{(3)} \sim \frac{r^3}{48\pi} + \frac{\tau_3}{r^3} & \text{as } r \to \infty. \end{cases} \tag{A.6}$$

We start with multiplying (A.3) by $r(\partial\psi_{31}^{(3)}/\partial r)$

$$r\frac{\partial\psi_{31}^{(3)}}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{21}^{(2)}}{\partial r}\right) - \frac{4}{r^2}\frac{\psi_{21}^{(2)}}{r}\right\} + ar\frac{\partial\psi_{31}^{(3)}}{\partial r}\psi_{21}^{(2)} = 0. \tag{A.7}$$

This is rearranged as

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{21}^{(2)}}{\partial r}\frac{\partial\psi_{31}^{(3)}}{\partial r}-r\psi_{21}^{(2)}\frac{\partial^2\psi_{31}^{(3)}}{\partial r^2}\right)+\psi_{21}^{(2)}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial^2\psi_{31}^{(3)}}{\partial r^2}\right)-\frac{4}{r}\frac{\partial\psi_{31}^{(3)}}{\partial r}\right.$$
$$\left.+r\frac{\partial}{\partial r}\left(a\psi_{31}^{(3)}\right)\right\}-\frac{\partial a}{\partial r}r\psi_{21}^{(2)}\psi_{31}^{(3)}=0. \tag{A.8}$$

Next, multipling, by $r\psi_{21}^{(2)}$, the derivative of (A.5) respect to r, we have

$$\psi_{21}^{(2)}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial^2\psi_{31}^{(3)}}{\partial r^2}\right)-\frac{10}{r}\frac{\partial\psi_{31}^{(2)}}{\partial r}+\frac{18}{r^2}\psi_{31}^{(3)}+r\frac{\partial}{\partial r}\left(a\psi_{31}^{(3)}\right)\right\}=0. \tag{A.9}$$

Subtracting (A.9) from (A.8), we eliminate the term including $\partial(a\psi_{31}^{(3)})/\partial r)$, leaving

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{21}^{(2)}}{\partial r}\frac{\partial\psi_{31}^{(3)}}{\partial r}-r\psi_{21}^{(2)}\frac{\partial^2\psi_{31}^{(3)}}{\partial r^2}\right)+6\psi_{21}^{(2)}\left(\frac{1}{r}\frac{\partial\psi_{31}^{(3)}}{\partial r}-\frac{3}{r^2}\psi_{31}^{(3)}\right)$$
$$-\frac{\partial a}{\partial r}r\psi_{21}^{(2)}\psi_{31}^{(3)}=0. \tag{A.10}$$

We repeat the above procedure by interchanging $\psi_{21}^{(2)}$ and $\psi_{31}^{(3)}$ and correspondingly (A.3) and (A.5). We multiply (A.5) by $r(\partial\psi_{21}^{(2)}/\partial r)$ and rearrange the resulting terms in a desired way. Then by subtracting the derivative of (A.3) respect to r, multiplied by $r\psi_{31}^{(3)}$, we are left with

$$\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{21}^{(2)}}{\partial r}\frac{\partial\psi_{31}^{(3)}}{\partial r}-r\psi_{31}^{(3)}\frac{\partial^2\psi_{21}^{(2)}}{\partial r^2}-\frac{4}{r}\psi_{21}^{(2)}\psi_{31}^{(3)}\right)$$
$$+4\psi_{21}^{(2)}\left(\frac{1}{r}\frac{\partial\psi_{31}^{(3)}}{\partial r}-\frac{3}{r^2}\psi_{31}^{(3)}\right)-\frac{\partial a}{\partial r}r\psi_{21}^{(2)}\psi_{31}^{(3)}=0 \tag{A.11}$$

Multiplying (A.10) by 2 and (A.11) by 3, and taking their diffrence, we eventually reach

$$\frac{\partial a}{\partial r}r\psi_{21}^{(2)}\psi_{31}^{(3)}=\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{21}^{(2)}}{\partial r}\frac{\partial\psi_{31}^{(3)}}{\partial r}+2r\psi_{21}^{(2)}\frac{\partial^2\psi_{31}^{(3)}}{\partial r^2}\right.$$
$$\left.-3r\frac{\partial^2\psi_{21}^{(2)}}{\partial r^2}\psi_{31}^{(3)}-\frac{12}{r}\psi_{21}^{(2)}\psi_{31}^{(3)}\right). \tag{A.12}$$

At this stage, integrate (A.12) with respect to r from 0 to a large number, say L, and thereafter take the limit of $L\to\infty$. With the help of the boundary conditions (A.4) and (A.6), we finally reach (A.1) and (A.2).

Appendix B

Calculating $\log r_2$

From Figure 3.1, $\tilde{x} = r\cos\theta$ and $\tilde{y} = r\sin\theta$; $\cos\theta_2 = (2d+\tilde{x})/r_2$ and $\sin\theta_2 = \tilde{y}/r_2$ then we obtain

$$r_2^2 = (2d+\tilde{x})^2 + \tilde{y}^2,$$

$$r_2 = \left(4d^2 + 4d\tilde{x} + r^2\right)^{\frac{1}{2}}, \tag{B.1}$$

now we insert it into logarithm simply we get

$$\log r_2 = \frac{1}{2}\log\left(4d^2 + 4dr\cos\theta + r^2\right). \tag{B.2}$$

By simple algebra and applying the Taylor series into equation (B.2), and using the series property of logarithm

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}, \tag{B.3}$$

and using dimensionless variables $r = \varepsilon d r^*$, we end up

$$\log r_2 = \log(2d) + \frac{\varepsilon r\cos\theta}{2} - \frac{\varepsilon^2 r^2\cos 2\theta}{8} + \frac{\varepsilon^3 r^3\cos 3\theta}{24}$$

$$- \frac{\varepsilon^4 r^4\cos 4\theta}{64} + \frac{\varepsilon^5 r^5\cos 5\theta}{160} - \frac{\varepsilon^6 r^6\cos 6\theta}{384} + \frac{\varepsilon^7 r^7\cos 7\theta}{896} + \cdots, \tag{B.4}$$

hereafter we omit the star.

Note: The formula for calculating denomenator is klm where k, l and m are $k = 2$, $l = 0, 1, 2, \ldots$ and $m = 2^j$ respectively where $j = 0, 1, 2, \ldots$.

Appendix C

Calculating Calculation of $a(r,t)$

A. Calculation of $a(r,t)$ using the Oseen vortex

The vorticity $\zeta^{(0)}$ and an azimuthal velocity $v^{(0)}$ of the Oseen vortex respectively are

$$\zeta^{(0)} = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right), \tag{C.1}$$

$$v^{(0)} = \frac{\Gamma}{2\pi r}\left[1-\exp\left(-\frac{r^2}{4\nu t}\right)\right], \tag{C.2}$$

since we have definition

$$a(r,t) = -\frac{1}{v^{(0)}}\frac{\partial\zeta^{(0)}}{\partial r}, \tag{C.3}$$

it is easily to get

$$a(r,t) = \frac{r^2}{4\hat{\nu}^2 t^2}\left(e^{\frac{r^2}{4\hat{\nu}t}}-1\right)^{-1}. \tag{C.4}$$

B. Calculation of $a(r,t)$ using the Rankine vortex

The calculation of $a(r)$ for Rankine vortex is similar as calculating using the Oseen vortex. Recall chapter 3. The vorticity $\zeta^{(0)}$ and an azimuthal velocity $v^{(0)}$ of the Rankine vortex are

$$\zeta^{(0)} = \begin{cases} \frac{1}{\pi} & \text{for } r<1, \\ 0 & \text{for } r>1, \end{cases} \tag{C.5}$$

$$v^{(0)} = \begin{cases} \frac{r}{2\pi} & \text{for } r < 1, \\ \frac{1}{2\pi r} & \text{for } r > 1, \end{cases} \tag{C.6}$$

substitute the vorticity $\zeta^{(0)}$ (C.5) and an azimuthal velocity $v^{(0)}$ (C.6) into equation (C.3) to yield

$$a(r) = \begin{cases} 2\delta(r-1) & \text{for } r = 1, \\ 0 & \text{for } r < 1 \text{ and } r > 1, \end{cases} \tag{C.7}$$

where

$$\frac{\partial \zeta^{(0)}}{\partial r} = -\frac{1}{\pi}\delta(r-1). \tag{C.8}$$

Appendix D

The first term of matching condition of 5th-order and 7th-order solutions

In this Appendix we will derive matching condition of 5th and 7th order solutions

A. The first term of matching condition of 5th-order

The matching condition of the fifth-order asymptotic expansions can be obtained from equation (3.20) after we insert series expansion for the solutions ψ and $\zeta_{ij}(i = 1,2,\ldots,\ j = 1,2,\ldots)$ in ε using (3.9) and (3.10) to give the coefficient ε^5

$$\psi^{(5)} \sim \left(\int_0^\infty \zeta_{21}^{(n)} r^3 \mathrm{d}r\right) \frac{\cos 2\theta_2}{4r_2^2} + \frac{1}{2r}\left(\int_0^\infty \zeta_{11}^{(5)} r^2 \mathrm{d}r\right) \cos\theta, \tag{D.1}$$

recall appendix B, using the geometry of Fig. 3.1 that $\tilde{x} = r\cos\theta$ and $\tilde{y} = r\sin\theta$; $\cos\theta_2 = (2d + \tilde{x})/r_2$ and $\sin\theta_2 = \tilde{y}/r_2$ we can write

$$\cos 2\theta_2 = 2\left(\cos\theta_2\right)^2 - 1 = 2\left(\frac{2d+y}{r_2}\right)^2 - 1 = \frac{4d^2 + 4dy + y^2 - x^2}{r_2^2}. \tag{D.2}$$

We expand $\cos 2\theta_2/r_2^2$ using the Taylor expansions for small r and use non-dimensional variable $r = \varepsilon d r^*$ to get

$$\frac{\cos 2\theta_2}{r_2^2} = \frac{1}{4d^2} - \frac{1}{4d^2}\varepsilon r^* \cos\theta + O(\varepsilon^3). \tag{D.3}$$

B. The first term of matching condition of 7th-order

The same as deriving the matching condition for 5th-order solution, the matching condition of the 7th-order solution can be obtained from equation (3.20) after we insert series expansion for the solutions ψ and $\zeta_{ij}(i = 1,2,\ldots,\ j = 1,2,\ldots)$ in ε using (3.9) and (3.10) to give the coefficient ε^7

$$\psi^{(7)} \sim \left(\frac{r}{32} \int_0^\infty \zeta_{31}^{(3)} r^4 \mathrm{d}r + \frac{1}{2r} \int_0^\infty \zeta_{11}^{(7)} r^2 \mathrm{d}r \right) \cos\theta. \tag{D.4}$$

recall appendix B, using the geometry of Fig. 3.1 that $\tilde{x} = r\cos\theta$ and $\tilde{y} = r\sin\theta$; $\cos\theta_2 = (2d + \tilde{x})/r_2$ and $\sin\theta_2 = \tilde{y}/r_2$ we can write

$$\begin{aligned} \cos 3\theta_2 &= 4\left(\cos\theta_2\right)^3 - 3\left(\cos\theta_2\right) = 4\left(\frac{2d+y}{r_2}\right)^3 - 3\left(\frac{2d+y}{r_2}\right) \\ &= \frac{4(8d^3 + 12d^2y + 6dy^2 + y^3 - x^2)}{r_2^3} - 3\left(\frac{2d+y}{r_2}\right). \end{aligned} \tag{D.5}$$

We expand $\cos 3\theta_2 / r_2^3$ using the Taylor expansions for small r and use non-dimensional variable $r = \varepsilon d r^*$ to give

$$\frac{\cos 3\theta_2}{r_2^3} = \frac{1}{8d^3} - \frac{3}{16d^3}\varepsilon r^* \cos\theta + O(\varepsilon^4). \tag{D.6}$$

Appendix E

Proof $b_1(r,t) \equiv 0$

Since $\psi^{(0)}$ and $\zeta^{(0)}$ does not depend on θ, (3.61) is written as

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\zeta^{(0)}}{\partial r}\tilde{\psi}^{(3)} - \zeta^{(3)}\frac{\partial\psi^{(0)}}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial\zeta^{(2)}}{\partial r}\frac{\partial\tilde{\psi}^{(1)}}{\partial\theta} - \frac{\partial\zeta^{(2)}}{\partial\theta}\frac{\partial\tilde{\psi}^{(1)}}{\partial r}\right.$$
$$\left. + \frac{\partial\zeta^{(1)}}{\partial r}\frac{\partial\tilde{\psi}^{(2)}}{\partial\theta} - \frac{\partial\zeta^{(1)}}{\partial\theta}\frac{\partial\tilde{\psi}^{(2)}}{\partial r}\right) = -\frac{\partial\zeta^{(1)}}{\partial t} + \hat{\nu}\Delta\zeta^{(1)}. \tag{E.1}$$

Substitute the functional forms (3.31), (3.34) and (3.46), (3.54) into (E.2), with $\tilde{\psi}^{(2)} = \psi^{(2)}$ and $\tilde{\psi}^{(3)} = \psi^{(3)}$ kept in view from (3.13), owing to the absence of corrections to the traveling speed at these orders. We see that there are no terms independent of θ on the left-hand side. On the right-hand side, the terms including $b_1(r,t)$ are only independent of θ. Integration of (E.1) with respect to θ over $[0,2\pi)$ leaves

$$\frac{\partial b_1}{\partial t} - \hat{\nu}\Delta b_1 = 0. \tag{E.2}$$

This heat-conduction equation yields, under the zero initial condition $b(r,0) = 0$ and zero boundary condition at $r \to \infty$, $b_1(r,t) \equiv 0$ identically.

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