Local theta lifts for p-adic unitary dual pairs $U(2) \times U(1)$, $U(2) \times U(3)$ and a p-adic quaternionic dual pair $U(1) \times U(1)$

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Local theta lifts for $p$-adic unitary dual pairs $U(2) \times U(1)$, $U(2) \times U(3)$ and a $p$-adic quaternionic dual pair $U(1) \times U(1)$

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Introduction

The purpose of this paper is to describe the local theta lifts for $p$-adic unitary dual pairs $U(2) \times U(1)$ and $U(2) \times U(3)$ in terms of endoscopy. This is a complement to a result of Gelbart-Rogawski-Soudry [4]. Also Gan-Ichino [2] described the local theta lifts for unitary groups in the almost equal rank case in terms of Vogan $L$-packets. The result in this paper provides another proof of their result in the cases of $U(2) \times U(1)$ and $U(2) \times U(3)$. Our proof is based on some results in [4] and the endoscopic description of the anisotropic unitary group in two variables in Konno-Konno [5]. As an application, we obtain the description of the local theta lift for a $p$-adic quaternionic dual pair $U_D(1) \times U_E(1)$ in terms of endoscopy.

To explain the problem, we recall some results in [4]. Let $F$ be a non-archimedean local field of characteristic 0, and $E$ a quadratic extension of $F$ with associated quadratic character $\omega_{E/F}$. We denote the split (resp. anisotropic) 2-dimensional hermitian space over $E$ by $V_{sp}$ (resp. $V_{an}$) and a three-dimensional skew-hermitian space over $E$ by $W$. We set $V = V_{sp}, V_{an}$. Rogawski gave the endoscopic descriptions of the irreducible admissible representations for $U(V_{sp})$ and $U(W)$ in [15]. Using it, Gelbart-Rogawski-Soudry showed in [4] that the local theta lift for $U(V) \times U(W)$ sends an $L$-packet to an $L$-packet. They also described the local pairing of an endoscopic representation $\pi$ for $U(W)$ by the behavior of the local theta lift of $\pi$ to $U(V_{sp})$ and $U(V_{an})$. Namely, an explicit parametrization of the members of an endoscopic $L$-packet of $U(W)$ is given in terms of the local theta lift to $U(V)$. Moreover they described the local theta lift for $U(V_{sp}) \times U(W)$.

Now we consider the following problem. Since we have the endoscopic description for $U(V_{an})$ by [5], we want to describe the local theta lift for $U(V_{an}) \times U(W)$ in terms of the endoscopic description. Notice that this problem cannot be immediately solved by the result in [4], since the endoscopic description for $U(V_{an})$ is not given in [4].

To explain our result, we prepare a few notation. Take an element $\xi \in E^\times$ such that $\text{Tr}_{E/F}(\xi) = 0$. Then we can define $W$ by

$$(E^{\otimes 3}, \begin{pmatrix} 1 & \xi \\ -1 & 1 \end{pmatrix}).$$

Let $W_F$ be the Weil group of $F$. We denote the local Langlands group of $F$ by $L_F = W_F \times SU_2(\mathbb{R})$. For $G = U(V)$ or $U(W)$, $\Phi(G)$ is the set of
equivalence classes of $L$-parameters of $G$. In [15], Rogawski gave the $L$-packet $\Pi_\phi$ for an $L$-parameter $\phi$ of $U(V_{sp})$ or $U(W)$. On the other hand, the $L$-packet $\Pi_\phi$ for an $L$-parameter $\phi$ of $U(V_{an})$ is given by using the Jacquet-Langlands correspondence (see §2.3 or [4, §1]). Let $\phi_E$ be the restriction of an $L$-parameter $\phi$ of $G$ to $L_E$. We note that $\phi$ is uniquely determined by $\phi_E$. Let $\psi$ be a non-trivial character of $F$. For characters $\mu, \eta$ of $E^\times$ such that $\mu|_{F^\times} = \omega_{E/F}, \eta|_{F^\times} = 1$, we have the Weil representation $\omega_{\psi,V^\mu,W^\eta}$ and the local theta lift $\theta_{\psi,V^\mu,W^\eta}$ for $U(V) \times U(W)$. Set $\epsilon(V) = +$ if $V = V_{sp}$ and $\epsilon(V) = -$ otherwise.

We recall results in [4] in more detail. The unique non-trivial elliptic endoscopic datum $(H, s_0, \xi_0)$ for $U(W)$ up to equivalence is given as follows:

$$H = U(V_{sp}) \times U(1), \ s_0 = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix},$$

$$\xi_0 : L \cong (h, h') \times w \mapsto \begin{cases} (h \mu_0(w), h') \times w, & w \in W_E \\ (-h, h') \times w, & w = w_\sigma. \end{cases}$$

Here we have fixed $w_\sigma \in W_F \setminus W_E$ and a character $\mu_0$ of $E^\times$ such that $\mu_0|_{F^\times} = \omega_{E/F}$. For an $L$-parameter $\phi \in \Phi(U(W))$, define $\Pi_\phi = \{ \rho \in \Phi(H) \mid \xi_0 \circ \rho \sim \phi \}$. For each $\rho \in \Pi_\phi$, the $U(V_{sp})$-part (resp. $U(1)$-part) of $\rho$ is the $L$-parameter of $U(V_{sp})$ (resp. $U(1)$). It is denoted by $\rho_{\phi}$ (resp. $\rho_{\eta}$). Then the character of $E^\times / F^\times \cong U(1)$ corresponding to $\rho_{\eta}$ is written by $\eta_{\rho}$. Now take an irreducible admissible representation $\pi$ of $U(W)$ having an $L$-parameter $\phi$ such that $\Pi_\phi$ is not empty. Then for any $\rho \in \Pi_\phi$, there exists a unique space $V = V_{sp}$ or $V_{an}$ such that $\theta_{\psi,V^\mu,W^\eta}(\pi)$ is non-zero. This $V$ is independent of the choices of $\psi$ and $\mu$. Then we define the local pairing $\epsilon_\rho(\pi) := \epsilon(V)$. In this way we can define $\epsilon_\rho(\pi)$ for any $\pi \in \Pi_\phi$ and $\rho \in \Pi_\phi$. These $\epsilon_\rho(\pi)$ determine each element of $\Pi_\phi$ and coincide with the local pairing defined by Rogawski [15].

Now we can explain our result. Take characters $\mu_1, \mu_2$ of $E^\times$ such that $\mu_1 \neq \mu_2$ and $\mu_1|_{F^\times} = \omega_{E/F}$. Then there exists a unique $L$-parameter $\varphi_{\mu_1,\mu_2}$ of $U(V_{an})$ such that $(\varphi_{\mu_1,\mu_2})_E = \mu_1 \oplus \mu_2$. We denote the $L$-packet of $U(V_{an})$ associated to it by $\Pi_{\mu_1,\mu_2}(V_{an})$. Then the $L$-packet $\Pi_{\mu_1,\mu_2}(V_{an})$ has two elements $\tau(\mu_1, \mu_2)^+_\infty, \tau(\mu_1, \mu_2)^{-}_\infty$, where the signs are specified in §2.3. Then we want to compute the local theta lift $\theta_{\psi,V_{an}^\mu,W^\eta}$ for $U(V_{an}) \times U(W)$.

First we assume that $\mu = \mu_1$. 
Theorem 0.1. We have

\[
\theta_{\psi,V_{\text{an}}^\epsilon,W_n}(\tau(\mu_1,\mu_2)_{\text{an}}^\epsilon) = \begin{cases} 
\pi(\eta, \eta \mu_1 \mu_2^{-1}) & \text{if } \varepsilon = +, \\
0 & \text{if } \varepsilon = -. 
\end{cases}
\]

Here \(\pi(\eta, \eta \mu_1 \mu_2^{-1})\) is the unique irreducible non-generic subrepresentation of \(\text{Ind}_B^{U(W)}(\eta \boxtimes (\eta \mu_1 \mu_2^{-1})_u)\), where \(B\) is the Borel subgroup of all upper triangular matrices in \(U(W)\). Also \((\cdot)_u\) is defined later.

Next we assume that \(\mu \neq \mu_1, \mu_2\). Then by results in [4] we have

\[\pi^\pm := \theta_{\psi,V_{\text{an}}^\epsilon,W_n}(\tau(\mu_1,\mu_2)_{\text{an}}^\pm) \neq 0\]

and the \(L\)-parameter \(\phi\) of \(\pi^\pm\) satisfies \(\phi_E = \mu \eta \mu_1^{-1} \oplus \mu \eta \mu_2^{-1} \oplus \eta\). Thus we have \(\Pi_\phi = \{\rho_0, \rho_1, \rho_2\}\), where

\[
\begin{align*}
\rho_{0,E} &= \mu_0^{-1} \mu \eta (\mu_1^{-1} \oplus \mu_2^{-1}) \times \eta, \\
\rho_{1,E} &= \mu_0^{-1} (\eta \oplus \mu \eta \mu_2^{-1}) \times \mu \eta \mu_1^{-1}, \\
\rho_{2,E} &= \mu_0^{-1} (\mu \eta \mu_1^{-1} \oplus \eta) \times \mu \eta \mu_2^{-1}.
\end{align*}
\]

Theorem 0.2. We have for \(\varepsilon \in \{\pm\}\)

\[
\begin{align*}
\epsilon\rho_0(\pi^\varepsilon) &= -, \\
\epsilon\rho_1(\pi^\varepsilon) &= -\varepsilon, \\
\epsilon\rho_2(\pi^\varepsilon) &= \varepsilon.
\end{align*}
\]

Using Theorem 0.1, we can compute the local theta lift \(\theta_{\psi,V_{\text{an}}^\epsilon,W_1^n}\) for \(U(V_{\text{an}}) \times U(W_1)\), where \(W_1\) is a one-dimensional skew-hermitian space over \(E\) given by \((E, \xi)\).

Theorem 0.3. Take a character \(\eta'\) of \(E^\times\) such that \(\eta'|_{F^\times} = 1_{F^\times}\). Then we have

\[
\theta_{\psi,V_{\text{an}}^\epsilon,W_1^n(\eta'_u)} = \begin{cases} 
\tau(\mu, \mu \eta' \mu^{-1})_{\text{an}}^+ & \text{if } \eta \neq \eta', \\
0 & \text{if } \eta = \eta'. 
\end{cases}
\]

As an application of this theorem, we can compute the local theta lift for a quaternionic unitary dual pair of rank one. Let \(D\) be the quaternion division algebra over \(F\). Fix an embedding \(E \hookrightarrow D\). Let \(V_D, W_D\) be a hermitian space and a skew-hermitian space over \(D\) given by \((D, 1), (D, \xi)\), respectively.
We denote their quaternionic unitary groups by $U_D(1), U_E(1)$, respectively. Then we have the Weil representation $\omega_{\psi,V_D,W_D}$ for $U_D(1) \times U_E(1)$. Here $U_D(1)$ is the kernel of the reduced norm of $D$. Let $\lambda$ be a character of $E^\times$ such that $\text{Ind}_{W_E}^{W_F} \lambda$ is irreducible. Denote the irreducible representation of $D^\times$ with $L$-parameter $\text{Ind}_{W_E}^{W_F} \lambda$ by $\tau_D(\lambda)$. Then there exist irreducible admissible representations $\tau_D(\lambda)^\pm$ of $U_D(1)$ such that $\tau_D(\lambda)|_{U_D(1)} = \tau_D(\lambda)^+ \oplus \tau_D(\lambda)^-$. Here the signs are specified in §7.4.

**Theorem 0.4.** Let $\lambda_\eta$ be a character of $E^\times$ such that $\lambda_\eta|_{U_E(1)} = \eta_u$. Then we have

$$\theta_{\psi,V_D,W_D}(\eta_u) = \begin{cases} \tau_D(\lambda_\eta)^+ & \text{if } \eta \neq 1_F^\times, \\ 0 & \text{if } \eta = 1_F^\times. \end{cases}$$

The paper is organized as follows. In §1.1, we define unitary dual pairs and explain their Weil representations and local theta lifts. In §1.2, we recall the definition of $L$-parameters and describe the $L$-parameters of $U(V_{sp}), U(V_{an})$ and $U(W)$. In §2, we explain the construction and the description of the $L$-packets of $U(V_{sp})$ and $U(V_{an})$ following [5]. Also we prepare some global results for $U(V_{sp})$ and $U(V_{an})$. In §3, we explain the construction and the description of the $L$-packets of $U(W)$ following [15]. Also we state the global multiplicity formula for $U(W)$. In §4.1, we recall the explicit formulas of the mixed model of the Weil representation of a unitary dual pair. In §4.2, we compute the local theta lift for $U(V_{sp}) \times U(W_a)$ using the result in §2. In §5, we recall some results in Gelbart-Rogawski-Soudry [4]. In §6, we prove our theorems. In §7, we compute the local theta lift for $U_D(1) \times U_E(1)$.

**Notation 0.5.** Let $F$ be a non-archimedean local field of characteristic zero and $E$ a quadratic extension of $F$. The map $\sigma$ stands for the non-trivial Galois automorphism of $E$ over $F$. We will often write $\sigma(x) = x^\sigma$. Let $\text{Tr}_{E/F}$ and $N_{E/F}$ be the trace map and norm map of $E$ over $F$, respectively. The symbol $\omega_{E/F}$ stands for the quadratic character of $F^\times$ associated with $E/F$. Let $F$ be an algebraic closure of $E$, $\Gamma = \Gamma_F$ the absolute Galois group of $F$, and $W_F$ the Weil group of $F$. We let $| \cdot |_F$ denote the modulus of $F$. Let $\psi$ be a non-trivial character of $F$. Then $\lambda(E/F, \psi)$ is the Langlands $\lambda$-factor [10]. For each character $\chi$ of $F^\times$, $\Pi(E^\times, \chi)$ stands for the set of characters of $E^\times$ whose restriction to $F^\times$ are $\chi$. For each $\eta \in \Pi(E^\times, 1_{F^\times})$, we define a character $\eta_u$ of $U(1) := \text{Ker} N_{E/F}$ by

$$\eta_u(z/\sigma(z)) := \eta(z), \quad z \in E^\times.$$
Elements of $\Pi(E^\times, 1_{F^\times})$ are denoted by $\eta, \eta', \eta_1, \eta_2$, and so on. Also elements of $\Pi(E^\times, \omega_{E/F})$ are denoted by $\mu, \mu', \mu_1, \mu_2$, and so on.

For a $p$-adic group $G$, we denote the set of equivalence classes of irreducible admissible representations of $G$ by $\text{Irr} G$.

For a topological group $H$, we denote by $H^0$ the component of the neutral element. Put $\pi_0(H) = H/H^0$. Let $Z(H)$ be the center of $H$. If $H$ is abelian, then we denote its Pontryagin dual group by $H^D$.

Fix an element $w_\sigma$ of $W_F \setminus W_E$, a non-zero element $\xi$ of $E$ with $\text{Tr}_{E/F}(\xi) = 0$ and $d_0 \in F^\times \setminus N_{E/F}(E^\times)$.

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1 Preliminaries

1.1 Local theta lift for unitary dual pairs

In this subsection we recall the local theta lift for unitary dual pairs. Let $V$ be a hermitian space and $W$ a skew-hermitian space over $E$. The spaces $V$ and $W$ may be taken as follows:

$$V = (E^\oplus m, R), \quad W = (E^\oplus n, S),$$

where $R = {^tR} := {^t}R^\sigma \in \text{GL}_m(E)$ and $S = {^*}S \in \text{GL}_n(E)$. Then the unitary groups of $V$ and $W$ are given by

$$U(V) = \{ h \in \text{GL}_m(E) \mid {^*}Rh = R \}, \quad U(W) = \{ g \in \text{GL}_n(E) \mid gS^*g = S \}.$$

Here $U(V)$ (resp. $U(W)$) acts on $V$ (resp. $W$) on the left (resp. right).

Let $\mathbb{W} = V \otimes_E W$ be the $F$-vector space equipped with the symplectic form

$$\langle \langle , \rangle \rangle = \text{Tr}_{E/F} (( , ) \otimes ( , )^\sigma).$$

The symplectic group of $\mathbb{W}$ is given by

$$\text{Sp}(\mathbb{W}) = \{ g \in \text{GL}_F(\mathbb{W}) \mid \langle \langle wg, w'g \rangle \rangle = \langle \langle w, w' \rangle \rangle \text{ for all } w, w' \in \mathbb{W} \}.$$

We have a natural homomorphism:

$$U(V) \times U(W) \ni (h, g) \mapsto (v \otimes w \mapsto h^{-1}v \otimes wg) \in \text{Sp}(\mathbb{W}).$$

Then the pair of two groups $(U(V), U(W))$ forms a reductive dual pair in $\text{Sp}(\mathbb{W})$.

Next we explain Weil representations of $U(V) \times U(W)$. The Heisenberg group $H(\mathbb{W})$ of $\mathbb{W}$ is given as follows:

$$H(\mathbb{W}) = \mathbb{W} \times F,$$

$$(w, t) \cdot (w', t') = (w + w', t + t' + \langle \langle w, w' \rangle \rangle / 2).$$
for \( w, w' \in \mathbb{W} \) and \( t, t' \in F \). We take an irreducible unitary representation \((\tau_{\psi, V, W}, S_{V, W})\) of \( H(\mathbb{W}) \) with central character \( \psi \). For any \( g \in \text{Sp}(\mathbb{W}) \), define the representation \((\tau_{\psi, V, W} \cdot g, S_{V, W})\) of \( H(\mathbb{W}) \) by \( \tau_{\psi, V, W}(w, t)g := \tau_{\psi, V, W}(wg, t) \) for \( w \in \mathbb{W}, t \in F \). This is also an irreducible unitary representation of \( H(\mathbb{W}) \) with central character \( \psi \). By Stone-von-Neumann’s theorem, the unitary representation \( \tau_{\psi, V, W} \cdot g \) is isomorphic to \( \tau_{\psi, V, W} \). Thus we have the metaplectic group:

\[
\text{Mp}_\psi(\mathbb{W}) = \left\{ (g, M_g) \mid g \in \text{Sp}(\mathbb{W}), \quad M_g : \tau_{\psi, V, W} \cdot g \cong \tau_{\psi, V, W} \text{ isomorphism} \right\}.
\]

The metaplectic group \( \text{Mp}_\psi(\mathbb{W}) \) has the Weil representation \((\omega_{\psi, V, W}, S_{V, W})\) defined by

\[
\omega_{\psi, V, W}(g, M_g) := M_g.
\]

Now let \( \chi_V \) and \( \chi_W \) be characters of \( E^\times \) such that \( \chi_V \mid_{E^\times} = \omega_{E/F}^{\dim E_V} \) and \( \chi_W \mid_{E^\times} = \omega_{E/F}^{\dim E_W} \), respectively. Then Kudla ([8]) defined the homomorphism \( \iota^{\chi_W}_V \times \iota^{\chi_V}_V \) associated to \( \chi_V \) and \( \chi_W \) so that the following diagram commutes.

\[
\begin{array}{ccc}
U(V) \times U(W) & \xrightarrow{\iota^{\chi_W}_V \times \iota^{\chi_V}_V} & \text{Mp}_\psi(\mathbb{W}) \\
\downarrow & & \downarrow \\
\text{Sp}(\mathbb{W}) & & \\
\end{array}
\]

Thus the Weil representation \((\omega_{\psi, V \times W, W \times V}, S_{V, W})\) of \( U(V) \times U(W) \) associated to \( \chi_V \) and \( \chi_W \) is defined by

\[
\omega_{\psi, V \times W, W \times V} := \omega_{\psi, V, W} \circ \iota^{\chi_W}_V \times \iota^{\chi_V}_V.
\]

In §4.1, we will describe the explicit formulas of the mixed model of the Weil representation \((\omega_{\psi, V \times W, W \times V}, S_{V, W})\) of \( U(V) \times U(W) \).

Next we explain the local theta lift for \( U(V) \times U(W) \). Let \( \pi \) be an irreducible admissible representation of \( U(V) \). If \( H_\pi := \text{Hom}_{U(V)}(\omega_{\psi, V \times W, W \times V}, \pi) \), then

\[
\omega_{\psi, V \times W, W \times V} / \bigcap_{f \in H_\pi} \text{Ker } f
\]

is a smooth representation of \( U(V) \times U(W) \).
Lemma 1.1 ([13]). (i) There exists a smooth representation \( \Theta(\pi) = \Theta_{\psi,\chi_V,\chi_W}(\pi) \) of \( U(W) \) such that
\[
\pi \boxtimes \Theta(\pi) = \omega_{\psi,\chi_V,\chi_W} / \bigcap_{f \in H_\pi} \text{Ker} \, f.
\]
(ii) \( \Theta(\pi) \) is of finite length and hence is admissible.

We call \( \Theta(\pi) \) the big theta lift of \( \pi \). We set
\[
\mathcal{R}_{\psi,\chi_V,\chi_W}(V) := \{ \pi \in \text{Irr} \, U(V) \mid H_\pi \neq 0 \}.
\]
It is clear that \( H_\pi \neq 0 \) if and only if \( \Theta(\pi) \neq 0 \). Then the local Howe duality conjecture is that \( \Theta(\pi) \) has the maximal subrepresentation, that is, the maximal semisimple quotient \( \theta(\pi) = \theta_{\psi,\chi_V,\chi_W}(\pi) \) of \( \Theta(\pi) \) is irreducible. If \( \Theta(\pi) = 0 \), then we set \( \theta(\pi) = 0 \). We call \( \theta(\pi) \) the local theta lift of \( \pi \). Similarly, we can define \( \mathcal{R}_{\psi,\chi_V,\chi_W}(W) \) and the local theta lift \( \theta(\sigma) \) of an irreducible admissible representation \( \sigma \) of \( U(W) \). By the definition of the local theta lift,
\[
\mathcal{R}_{\psi,\chi_V,\chi_W}(V) \ni \pi \mapsto \theta(\pi) \in \mathcal{R}_{\psi,\chi_V,\chi_W}(W)
\]
is bijective and the inverse map is given by \( \sigma \mapsto \theta(\sigma) \).

The local Howe duality conjecture was proven by Waldspurger [17] when the residue characteristic \( p \) of \( F \) is not 2. Recently, Gan and Takeda [3] proved the local Howe duality conjecture for any residual characteristic \( p \). But these results are not used in this paper. Instead the following lemma is used.

Lemma 1.2 ([13]). (i) If \( \pi \) is an irreducible supercuspidal representation, then \( \Theta(\pi) = \theta(\pi) \), that is, \( \Theta(\pi) \) is irreducible.
(ii) If \( \theta(\pi_1) = \theta(\pi_2) \neq 0 \) for two irreducible supercuspidal representations \( \pi_1 \) and \( \pi_2 \), then \( \pi_1 = \pi_2 \).

By this lemma, we can show the local Howe duality conjecture for \( U(V) \times U(W) \), where \( U(V) \) or \( U(W) \) is compact.

Finally we explain the unitary groups which we treat in this paper. For \( a \in F^\times \), we define a one-dimensional skew-hermitian space:
\[
W_a = (E, a\xi)
\]
Also define a three-dimensional skew-hermitian space:

\[ W = (E^\text{B3}, \begin{pmatrix} 1 & \xi \\ -1 & 1 \end{pmatrix}) \]

Let \( V_{sp} \) (resp. \( V_{an} \)) be the two-dimensional split (resp. anisotropic) hermitian space over \( E \) given by

\[ V_{sp} = (E^\text{B2}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}), \]
\[ V_{an} = (E^\text{B2}, \begin{pmatrix} -d_0 & 1 \\ 1 & -d_0 \end{pmatrix}). \]

Thus we have four unitary groups \( U(W_a), U(W), U(V_{sp}) \) and \( U(V_{an}) \). Also we denote the unitary similitude group of \( V = V_{sp}, V_{an} \) by \( GU(V) \).

In [4], the local Howe duality conjecture for \( U(V_{sp}) \times U(W) \) was proved and its local theta lift was computed using endoscopy. Our goal is to compute the local theta lifts for \( U(V_{an}) \times U(W) \) and \( U(V_{an}) \times U(W_a) \). We remark that the local Howe duality conjecture for \( U(V_{an}) \times U(W) \) and \( U(V_{an}) \times U(W_a) \) follow from Lemma 1.2.

1.2 \textbf{L-parameter}

Let \( G \) be \( U(W_a), U(W), U(V_{sp}) \) or \( U(V_{an}) \). We recall the \( L \)-groups for these groups.

\textbf{Lemma 1.3.} The dual group \( \hat{G} \) of \( G \) is \( \text{GL}_n(\mathbb{C}) \), where \( n \) is the dimension of the space associated to \( G \). Also the \( L \)-group \( {}^L G \) is given as follows:

\[ {}^L G = \text{GL}_n(\mathbb{C}) \rtimes W_F, \]
\[ g \rtimes w \cdot g' \rtimes w' = \begin{cases} 
gg' \rtimes w w' & \text{if } w \in W_E, \\
gI_n t g^{-1} I_n^{-1} \rtimes w w' & \text{if } w \notin W_E. \end{cases} \]

Here

\[ I_n = \begin{pmatrix} 1 & & \\ & \ddots & -1 \\ & & 1 \end{pmatrix}. \]
When $G = \text{U}(W)$, the proof of this lemma will be given in §3.1. The other cases are similar.

Next we recall the definition of $L$-parameters. Let $H$ be a connected reductive group defined over $F$.

**Definition 1.4.** We call a homomorphism $\phi : L_F \to {}^L H$ an $L$-parameter of $H$ if the following conditions hold:

1. The image of $W_F$ under $L_F \xrightarrow{\phi} {}^L H \xrightarrow{\text{proj}} \hat{H}$ consists of semi-simple elements of $\hat{H}$.
2. $\phi|_{\text{SU}_2(\mathbb{R})} : \text{SU}_2(\mathbb{R}) \xrightarrow{\phi} {}^L H \xrightarrow{\text{proj}} \hat{H}$ is analytic.
3. $\phi$ is a continuous homomorphism so that the following diagram is commutative:

\[
\begin{array}{ccc}
L_F & \xrightarrow{\phi} & {}^L H \\
\downarrow & & \downarrow \\
W_F & \xrightarrow{} & \hat{H}
\end{array}
\]

Here $L_F = W_F \times \text{SU}_2(\mathbb{R})$ and $\text{SU}_2(\mathbb{R}) = \{g \in \text{SL}_2(\mathbb{C}) \mid g^t g = 1_2\}$.

Two $L$-parameters $\phi$ and $\phi'$ are said to be equivalent if there exists $h \in \hat{H}$ such that $\phi' = \text{Ad}(h) \circ \phi$. Then we write $\phi \sim \phi'$. Also we denote the set of equivalence classes of $L$-parameters of $H$ by $\Phi(H)$.

If $H = \text{GL}_n(F)$, then we can identify an $L$-parameter of $\text{GL}_n(F)$ with a completely reducible $n$-dimensional representation of $L_F$. The following fact is well-known:

**Fact 1.5.** (Local Langlands correspondence for $\text{GL}_n$) There exists a natural bijection from $\text{Irr} \, \text{GL}_n(F)$ to $\Phi(\text{GL}_n(F))$. In particular, if $n = 1$, then its bijection is obtained by the local class field theory.

Similarly we expect that there exists a bijection from $\text{Irr} \, G$ to $\Phi(G)$. However, in general there does not exist such a bijection. But we can describe $\text{Irr} \, G$ in terms of $\Phi(G)$ by the following procedure. For any $\phi \in \Phi(G)$, we construct a finite set $\Pi_{\phi}$ that consists of irreducible admissible representations of $G$. The set $\Pi_{\phi}$ is called an $L$-packet of $G$. Set $S_{\phi} = \text{Cent}_{\hat{G}}(\text{Im} \phi)$ and $S_{\phi} = \pi_0(S_{\phi}/Z(\hat{G})^\Gamma)$. This $S_{\phi}$ is a finite abelian group. Then $\Pi_{\phi}$ is described by the character group of $S_{\phi}$. This procedure is achieved by endoscopy, which is related to the problem of classification of conjugacy classes of $G$. We call such a description of irreducible admissible representations of $G$ endoscopic.
description. We will explain the case of $G = U(V_{sp}), U(V_{an})$ following Konno-Konno [5] in §2. On the other hand, the case of $G = U(W)$ will be explained in §3 following Rogawski [15].

Finally we explain the classification of the $L$-parameters of $G$. Denote the unique $r$-dimensional irreducible representation of $SU_2(\mathbb{R})$ by $\text{Sym}^{r-1}$. For an $L$-parameter $\phi$ of $G$, we define the homomorphism $\phi_E : L_E \to \text{GL}_n(\mathbb{C})$ by

$$L_E \xrightarrow{\phi|_{L_E}} \text{proj}_{GL_n(\mathbb{C})}.$$

Then $\phi_E$ is an $L$-parameter of $GL_n(E)$ and a completely reducible $n$-dimensional representation of $L_E$. Thus we have

$$\phi_E = \bigoplus_i \varphi_i \boxtimes \text{Sym}^{r_i-1}, \quad \varphi_i \in \text{Irr}_W, \quad (n = \sum_i r_i \dim \varphi_i).$$

Let $\phi$ be an $L$-parameter of $U(W_a)$. Then $\phi_E$ is a character of $E^\times/F^\times$ by the local class field theory. Namely, $\phi_E$ is an element of $\Pi(E^\times, 1_{F^\times})$.

**Lemma 1.6.** We have the bijection:

$$\Phi(U(W_a)) \ni \phi \mapsto \phi_E \in \Pi(E^\times, 1_{F^\times}).$$

Moreover by the bijection $\Pi(E^\times, 1_{F^\times}) \ni \eta \mapsto \eta_a \in \text{Irr} U(W_a)$, we have the bijection:

$$\Phi(U(W_a)) \to \text{Irr} U(W_a).$$

Thus we can identify $\Phi(U(W_a))$ with $\text{Irr} U(W_a)$.

To explain the classification of $\Phi(G)$ where $G = U(V_{sp}), U(V_{an})$ or $U(W)$, we prepare the following lemma:

**Lemma 1.7.** Let $\varphi : W_E \to \text{GL}_n(\mathbb{C})$ be an irreducible $n$-dimensional representation of $W_E$. Assume that $\varphi \circ \text{Ad}(w_\sigma) \cong \chi \varphi^{-1}$.

(i) There exists $X \in \text{GL}_n(\mathbb{C})$ such that $\varphi \circ \text{Ad}(w_\sigma) = X I_n \chi \varphi^{-1} I_n^{-1} X^{-1}$. Also such an $X$ exists unique up to scalar.

(ii) There exists $c(\varphi) \in \{\pm 1\}$ such that $\varphi(w_\sigma^2) = c(\varphi) X I_n \chi X^{-1} I_n^{-1}$.

(iii) $c(\varphi) = 1$ if and only if $\varphi$ extends to an $L$-parameter $\phi$ of $G$ such that $\phi|_{SU_2(\mathbb{R})}$ is trivial. Also such an $L$-parameter $\phi$ is uniquely.
(iv) If \( \varphi \) extends to an L-parameter \( \phi \) of \( G \), then \( S_{\phi} = \{1\} \).

**Proof.** (i) This follows from Schur’s lemma.
(ii) For any element \( w \) of \( W_E \),

\[
\varphi(w_\sigma^2)\varphi(w)\varphi(w_\sigma^{-2})^{-1} = \varphi(w_\sigma^2ww_\sigma^{-2}) = XI_n^t\varphi(w_\sigmaww_\sigma^{-1})^{-1}I_n^{-1}X^{-1} = XI_n^t(XI_n^t\varphi(w)^{-1}I_n^{-1}X^{-1})^{-1}I_n^{-1}X^{-1} = XI_n^tX^{-1}I_n^{-1}\varphi(w)^tI_n^tXI_n^{-1}X^{-1} = (XI_n^tX^{-1}I_n^{-1})\varphi(w)(XI_n^tX^{-1}I_n^{-1})^{-1}.
\]

By Schur’s lemma, there exists a scalar \( c(\varphi) \) such that \( \varphi(w_\sigma^2) = c(\varphi)XI_n^tX^{-1}I_n^{-1} \). Substituting \( w = w_\sigma^2 \) for the equation in (i), we obtain \( c(\varphi)^2 = 1 \).
(iii) Assume that \( \varphi \) extends to an \( L \)-parameter \( \phi \) of \( G \). If we write \( \phi(w_\sigma) = X \rtimes w_\sigma \), then \( X \) is satisfied with the equation in (i). Since

\[
\varphi(w_\sigma^2) \rtimes w_\sigma^2 = \phi(w_\sigma)^2 = XI_n^tX^{-1}I_n^{-1} \rtimes w_\sigma^2,
\]

we have \( c(\varphi) = 1 \). Conversely we assume that \( c(\varphi) = 1 \). If we write \( \phi(w_\sigma) = X \rtimes w_\sigma \), then we can easily prove that \( \phi \) is an \( L \)-parameter of \( G \). Also for any scalar \( \alpha \), we obtain \( \text{Ad}(\alpha) \circ \phi(w_\sigma) = \alpha^2X \rtimes w_\sigma \). Thus the equivalence class of \( \phi \) is independent of the choice of \( X \).
(iv) First we have \( Z(G) = \mathbb{C}^{\times} \) and \( Z(G)^{\Gamma} = \{\pm 1\} \). Also we obtain

\[
S_{\phi} = \text{Cent}_{\hat{G}}(\text{Im} \varphi) \cap \text{Cent}_{\hat{G}}(\phi(w_\sigma)) = Z(\hat{G}) \cap \text{Cent}_{\hat{G}}(\phi(w_\sigma)) = \{z \in Z(\hat{G}) \mid z\phi(w_\sigma)z^{-1} = \phi(w_\sigma)\} = \{\pm 1\}.
\]

Thus we have \( S_{\phi} = \{1\} \). \( \Box \)

By this lemma, we define the following sets:

\[
\text{Irr } W_{E,n,+} := \{ \varphi \in \text{Irr } W_E \mid \text{ dim } \varphi = n, \varphi \circ \text{Ad}(w_\sigma) \cong t^{-1}\varphi^{-1}, c(\varphi) = 1 \};
\]

\[
\text{Irr } W_{E,n,-} := \{ \varphi \in \text{Irr } W_E \mid \text{ dim } \varphi = n, \varphi \circ \text{Ad}(w_\sigma) \cong t^{-1}\varphi^{-1}, c(\varphi) = -1 \}.
\]

Now we explain the classification of the \( L \)-parameters of \( G = U(V_{sp}), U(V_{an}) \).

Here note that \( \Phi(U(V_{sp})) = \Phi(U(V_{an})) \). So we consider only \( \Phi(U(V_{sp})) \). Take
an $L$-parameter $\phi$ of $U(V_{sp})$. $\phi_E$ is a two-dimensional representation of $L_E$. Then we have the following three cases:

Case A: $\phi|_{SU_2(\mathbb{R})} = \text{Sym}^1$;
Case B: $\phi_E$ is an irreducible representation of $W_E$;
Case C: $\phi_E$ is a reducible representation of $W_E$.

The classification of the $L$-parameters of $U(V_{sp})$ is given by the similar argument for that of $U(W)$, which was given in [12].

Proposition 1.8. $\Phi(U(V_{sp})) \ni \phi \mapsto \phi_E \in \Phi(GL_2(E))$ is injective. Also the $L$-parameters $\phi$ of $U(V_{sp})$ are classified as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>$\phi$</th>
<th>$\phi_E$</th>
<th>$S_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>$\eta \boxtimes \text{Sym}^1$</td>
<td>$\eta \in \Pi(E^x, 1_{E^x})$</td>
<td>${1}$</td>
</tr>
<tr>
<td>Case B</td>
<td>$\phi_E$</td>
<td>$\phi_E \in \text{Irr} W_{E,2+}$</td>
<td>${1}$</td>
</tr>
<tr>
<td>Case C1</td>
<td>$\lambda \oplus \lambda^{-1} \circ \sigma$</td>
<td>$\lambda \neq \lambda^{-1} \circ \sigma \in \text{Irr} E^x$</td>
<td>${1}$</td>
</tr>
<tr>
<td>Case C2</td>
<td>$\mu_1 \oplus \mu_2$</td>
<td>$\mu_1 \neq \mu_2 \in \Pi(E^x, \omega_{E/F})$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>Case C3</td>
<td>$\eta \oplus \eta$</td>
<td>$\eta \in \Pi(E^x, 1_{E^x})$</td>
<td>${1}$</td>
</tr>
<tr>
<td>Case C4</td>
<td>$\mu \oplus \mu$</td>
<td>$\mu \in \Pi(E^x, \omega_{E/F})$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
</tbody>
</table>

Next we explain the classification of the $L$-parameters of $G = U(W)$. Take an $L$-parameter $\phi$ of $U(W)$. $\phi_E$ is a three-dimensional representation of $L_E$. Then we have the following five cases:

Case A: $\phi|_{SU_2(\mathbb{R})} = \text{Sym}^2$;
Case B: $\phi|_{SU_2(\mathbb{R})} = \text{Sym}^1 \oplus \text{Sym}^0$;
Case C: $\phi_E$ is an irreducible representation of $W_E$;
Case D: $\phi_E$ has a two-dimensional irreducible representation of $W_E$;
Case E: $\phi_E$ is direct sum of quasi-characters of $W_E$.

Proposition 1.9 ([12]). The map $\Phi(U(W)) \ni \phi \mapsto \phi_E \in \Phi(GL_3(E))$ is injective. Also the $L$-parameters $\phi$ of $U(W)$ are classified as follows:
$\phi$ & $\phi_E$ & $S_\phi$ \\

CaseA $\eta \boxtimes \text{Sym}^2$ & $\eta \in \Pi(E^X, I_{F^x})$ & $\{1\}$ \\

CaseB $\mu \boxtimes \text{Sym}^1 \oplus \eta$ & $\mu \in \Pi(E^X, \omega_{E/F})$, $\eta \in \Pi(E^X, I_{F^x})$ & $\mathbb{Z}/2\mathbb{Z}$ \\

CaseC $\phi_E$ & $\phi_E \in \text{Irr } W_{E,3,+}$ & $\{1\}$ \\

CaseD $\tau \oplus \eta$ & $\tau \in \text{Irr } W_{E,2,-}$, $\eta \in \Pi(E^X, I_{F^x})$ & $\mathbb{Z}/2\mathbb{Z}$ \\

CaseE$_1$ $\eta_1 \oplus \eta_2 \oplus \eta_3$ & $\eta_i \in \Pi(E^X, I_{F^x})$, $\eta_i \neq \eta_j$ & $(\mathbb{Z}/2\mathbb{Z})^3$ \\

CaseE$_2$ $\lambda \oplus \eta \oplus \lambda^{-1} \circ \sigma$ & $\lambda \neq \lambda^{-1} \circ \sigma \in \text{Irr } E^X$, $\eta \in \Pi(E^X, I_{F^x})$ & $\{1\}$ \\

CaseE$_3$ $\eta_1 \oplus \eta_2 \oplus \eta_1$ & $\eta_i \neq \eta_2 \in \Pi(E^X, I_{F^x})$ & $\mathbb{Z}/2\mathbb{Z}$ \\

CaseE$_4$ $\mu \oplus \eta \oplus \mu$ & $\mu \in \Pi(E^X, \omega_{E/F})$, $\eta \in \Pi(E^X, I_{F^x})$ & $\{1\}$ \\

CaseE$_5$ $\eta \oplus \eta \oplus \eta$ & $\eta \in \Pi(E^X, I_{F^x})$ & $\{1\}$ \\ 

We introduce the definition of a tempered $L$-parameter of $G$.

**Definition 1.10.** Let $\phi$ be an $L$-parameter of $G$. We call $\phi$ a **tempered** $L$-parameter if the image of the following homomorphism is relatively compact:

$$L_F \phi \xrightarrow{\oplus} L G \xrightarrow{\text{proj}} \hat{G}.$$ 

**Lemma 1.11.** Let $\phi$ be an $L$-parameter of $G$. Recall that $\phi_E = \bigoplus_i \varphi_i \boxtimes \text{Sym}^{r-1}$. Then $\phi$ is tempered if and only if $\det \circ \varphi$ is a unitary character of $W_E$ for any $i$.

**Proof.** Since the index of $L_E$ in $L_F$ is 2, it is clear that $\phi$ is tempered if and only if $\phi_E$ is. Also $\phi_E$ is tempered if and only if $\varphi_i$ is a unitary representation of $W_E$ for any $i$, since $\text{Sym}^{r-1}$ has a compact image in $G$. Thus we must show that for an irreducible representation $\varphi$ of $W_E$, $\varphi$ is unitary if and only if $\det \circ \varphi$ is a unitary character of $W_E$.

If $\varphi$ is an irreducible unitary representation, it is clear that $\det \circ \varphi$ is a unitary character. Conversely, assume that $\det \circ \varphi$ is a unitary character. Let $I_E$ be the inertia group of $E$ and $Fr_E$ be a Frobenius element. Then $\varphi(Fr_E)$ normalizes $\varphi(I_E)$. Also since $I_E$ is a profinite group, $\varphi(I_E)$ is a finite group. Thus there exists an integer $l$ such that $\varphi(Fr_E^l)$ commutes with $\varphi(W_E)$. Since $\varphi$ is an irreducible representation, by Schur’s Lemma, $\varphi(Fr_E^l)$ is a scalar. By the assumption of $\varphi$, the absolute value of this scalar is one. If we take an unramified character $\chi$ such that $\chi(Fr_E^l) = \varphi(Fr_E^l)^{-1}$, then $\chi \otimes \varphi(W_E)$ is a finite group. Therefore $\chi \otimes \varphi$ is a unitary representation. Since $\chi$ is a unitary character, $\varphi$ is unitary.
By this lemma, we can classify the tempered $L$-parameters of $G$. Any $L$-parameter of $U(W_a)$ is tempered. For an $L$-parameter $\phi$ in the Case $C_1$ for $U(V_{sp})$ or $E_2$ for $U(W)$, $\phi$ is tempered if and only if the corresponding $\lambda$ is a unitary character. The other cases are all tempered $L$-parameters.
2 Endoscopy for $U(2)$

In this section, we explain the local and global endoscopy for $U(V_{sp})$ and $U(V_{an})$ following [5].

2.1 Local endoscopy for $U(V_{sp})$

In this subsection, we may assume that $F$ is a local field of characteristic 0 such that $F \neq \mathbb{C}$. Set $Z = \left( \begin{array}{c} 1 \\ \xi \end{array} \right)$. We recall the following isomorphism:

$$E^\times \times GL_2(F)/\Delta F^\times \ni (x, g) \mapsto x \det(g)^{-1} Z g Z^{-1} \in GU(V_{sp}).$$

Identify $GU(V_{sp})$ with $E^\times \times GL_2(F)/\Delta F^\times$. Then the $L$-group of $GU(V_{sp})$ is given as follows:

$$L GU(V_{sp}) = \hat{GU}(V_{sp}) \rtimes W_E,$$

$$\hat{GU}(V_{sp}) = \left\{ (z_1, z_2, g) \in \mathbb{C}^\times \times \mathbb{C}^\times \times GL_2(\mathbb{C}) \mid z_1 z_2 g = 1 \right\},$$

$$(z_1, z_2, g) \rtimes w \cdot (z'_1, z'_2, g') \rtimes w' = \begin{cases} (z_1 z'_1 z'_2, g g') \rtimes ww' & \text{if } w \in W_E, \\
(z_1 z'_2, z'_1 z'_2, g g') \rtimes ww' & \text{if } w \notin W_E. \end{cases}$$

Also the $L$-homomorphism corresponding to the inclusion $\iota : U(V_{sp}) \hookrightarrow GU(V_{sp})$ is given by

$$L \iota : L GU(V_{sp}) \ni (z_1, z_2, g) \rtimes w \mapsto z_1 g \rtimes w \in L U(V_{sp}).$$

Let $\phi'$ be a two-dimensional representation of $L_F$. By Fact 1.5, there exists an irreducible admissible representation $\tau(\phi')$ of $GL_2(F)$ with $L$-parameter $\phi'$. Let $\chi$ be a quasi-character of $E^\times$ such that $\chi^{-1}|_{F^\times} = \det(\phi')$. Then $\chi \boxtimes \tau(\phi')$ is an irreducible admissible representation of $GU(V_{sp})$. Also the $L$-parameter of its representation is given by

$$L_F \ni (w, g) \mapsto \begin{cases} (\chi(w), \chi(w), \phi'(w, g)) \rtimes w & \text{if } w \in W_E, \\
(1, \chi(w g^2 \sigma), \phi'(w g^2 \sigma, g)) \rtimes w g & \text{if } w = w g. \end{cases}$$

Let $\phi$ be the composite of this $L$-parameter and $L \iota$. Then $\phi_E = \chi \otimes (\phi'|_{L_E})$. Since $\phi$ is an $L$-parameter of $U(V_{sp})$, then define the $L$-packet $\Pi_{\phi}$ associated to $\phi$ by the set of irreducible subrepresentations of $\chi \boxtimes \tau(\phi')|_{U(V_{sp})}$.
Theorem 2.1 ([5]).

(i) \( \chi \boxtimes \tau(\phi') \mid_{U(V_{sp})} \) is multiplicity free.

(ii) \( \Pi_\phi \) has at most two elements.

(iii) \( \Pi_{\phi_1} \cap \Pi_{\phi_2} \) is not empty if and only if \( \phi_1 \sim \phi_2 \). In that case, \( \Pi_{\phi_1} = \Pi_{\phi_2} \).

(iv) Any irreducible admissible representation of \( U(V_{sp}) \) is contained an \( L \)-packet \( \Pi_\phi \).

(v) \( \Pi_\phi \) has two elements if and only if there exist characters \( \mu_1, \mu_2 \in \Pi(E^\times, \omega_{E/F}) \) such that \( \phi_E = \mu_1 \oplus \mu_2 \).

In (v), we denote the \( L \)-packet \( \Pi_\phi \) by \( \Pi_{\mu_1, \mu_2} = \Pi_{\mu_1, \mu_2}(V_{sp}) \) and call it an endoscopic \( L \)-packet. Also its elements are called endoscopic representations. Since we can prove that there exists \( \chi' \in \text{Irr } E^\times \) such that \( \mu_1 = \chi \chi' \) and \( \mu_2 = \chi(\chi' \circ \sigma) \), then we may take \( \phi' = \text{Ind}_{W_E}^{W_F} \chi' \).

The endoscopic description of each endoscopic \( L \)-packet \( \Pi_{\mu_1, \mu_2} \) of \( U(V_{sp}) \) can be given in terms of genericity. We denote by \( B \) the Borel subgroup of \( U(V_{sp}) \) consisting of lower triangular matrices. Let

\[
U = \left\{ u(x) = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \mid x \in E, \text{Tr}_{E/F}(x) = 0 \right\}
\]

be the unipotent radical of \( B \) and

\[
T = \left\{ m(\alpha) = \begin{pmatrix} \alpha & * \\ \alpha^{-1} & \alpha \end{pmatrix} \mid \alpha \in E^\times \right\}
\]

a maximal torus of \( U(V_{sp}) \). Thus we have \( B = TU \). For each \( b \in F^\times \), we define a non-trivial character \( \psi_{U, \xi}^b \) of \( U \) by

\[
\psi_{U, \xi}^b : U \ni u(x) \mapsto \psi(-bx\xi^{-1}) \in \mathbb{C}^1.
\]

An irreducible admissible representation \( \pi \) of \( U(V_{sp}) \) is called \( \psi^b \)-generic, or \( \psi_{U, \xi}^b \)-generic, if it has a non-zero homomorphism into \( \text{Ind}_U^{U(V_{sp})} \psi_{U, \xi}^b \). By the uniqueness of Whittaker model of \( \chi \boxtimes \tau(\phi') \), an endoscopic \( L \)-packet \( \Pi_{\mu_1, \mu_2} \) has the unique \( \psi \)-generic (resp. \( \psi^{d_0} \)-generic) element. We write it as \( \tau(\mu_1, \mu_2)_+ \) (resp. \( \tau(\mu_1, \mu_2)_{-d_0} \)).

Also such a description of \( \Pi_{\mu_1, \mu_2} \) can be characterized by the character identity. For \( \delta \in E^\times \) such that \( \text{Tr}_{E/F}(\delta) = 0 \), we set

\[
T_\delta = \left\{ t_\delta(z, z') = z \begin{pmatrix} x & \xi \xi' y \\ -\delta^2 \xi y & x \end{pmatrix} \in U(V_{sp}) \mid z, z' = x + y\delta \in E^\times, \text{N}_{E/F}(zz') = 1 \right\}.
\]
Proposition 2.2 ([5]). We have the following character identity:

\[
\text{Tr} \tau(\mu_1, \mu_2)(t_\delta(z, z')) - \text{Tr} \tau(\mu_1, \mu_2)(t_\delta(z, z')) = \lambda(E/F, \psi) \omega_{E/F}(\sigma(z'))^{1/2} |z'|^{1/2}.
\]

Also the last equation is independent of the choices of \(\chi, \chi'\) such that \(\mu_1 = \chi\chi'\) and \(\mu_2 = \chi(\chi' \circ \sigma)\).

It is easy to check the following:

Corollary 2.3. (i) For each \(\eta \in \Pi(E^\times, 1_{F^\times})\), we have \(\eta \circ \det \otimes \tau(\mu_1, \mu_2)^{\epsilon_{sp}} = \tau(\eta\mu_1, \eta\mu_2)^{\epsilon_{sp}}\).

(ii) The representation \(\tau(\mu_1, \mu_2)^{-1} \otimes \omega_{E/F}(-1)^{\epsilon_{sp}}\) is the contragredient representation of \(\tau(\mu_1, \mu_2)^{\epsilon_{sp}}\).

(iii) For \(g \in \text{GU}(V_{sp})\) with similitude norm \(a\), we have \(\tau(\mu_1, \mu_2)^{\epsilon_{sp}} \circ \text{Ad}(g) = \tau(\mu_1, \mu_2)^{\epsilon_{sp}}(a)^{\epsilon_{sp}}\).

Finally we explicitly state the \(L\)-packet \(\Pi_\phi\) for each \(L\)-parameter \(\phi\) that is classified in §1.2.

Lemma 2.4. Let \(B^-\) be the Borel subgroup of \(\text{GL}_2(F)\) which consists of the lower-triangular matrices. For any \(\lambda \in \text{Irr} E^\times\) and \(\lambda_1, \lambda_2 \in \text{Irr} F^\times\) such that \(\lambda|_{F^\times} \lambda_1 \lambda_2 = 1\), we have

\[
\lambda \otimes \text{Ind}_{B^{-}}^{\text{GL}_2(F)}(\lambda_1 \otimes \lambda_2)|_{U(V_{sp})} = \text{Ind}_{B}^{U(V_{sp})} \lambda(\lambda_1 \circ N_{E/F}).
\]

Proof. This follows from

\[
(E^\times \times B^-/\Delta F^\times) \cdot U(V_{sp}) = \text{GU}(V_{sp})
\]

\[
(E^\times \times B^-/\Delta F^\times) \cap U(V_{sp}) = B.
\]

Case A: \(\phi_E = \eta \boxtimes \text{Sym}^1\)

If we take \(\chi = \eta\) and \(\phi' = \text{Sym}^1\), then we obtain \(\phi_E = \eta \boxtimes \text{Sym}^1\). Since \(\tau(\phi')\) is the Steinberg representation of \(\text{GL}_2(F)\), \(\eta \boxtimes \tau(\phi')\) is the unique irreducible quotient of \(\eta \boxtimes \text{Ind}_{B^{-}}^{\text{GL}_2(F)}|_{F^\times} \boxtimes |_{F^\times}^{1/2}\). Here \(\text{Ind}_{B^{-}}^{\text{GL}_2(F)}|_{F^\times} \boxtimes |_{F^\times}^{1/2}\) has the unique subrepresentation \(1\) and the unique quotient \(\tau(\phi')\). Similarly
Ind$_B^{U(V_{sp})} \eta| |E|^{1/2}$ has the unique subrepresentation $\eta_0 \circ \text{det}$ and the unique quotient $\text{St}(\eta)$. Thus by Lemma 2.4 we have $\eta \boxtimes \tau(\phi')|_{U(V_{sp})} = \text{St}(\eta)$. Therefore $\Pi_\phi = \{\text{St}(\eta)\}$.

Case B: $\phi_E \in \text{Irr } W_{E,2,+}$
Then we can take a two-dimensional irreducible representation of $W_F$ as $\phi'$. Thus $\tau(\phi')$ is supercuspidal. By Theorem 2.1, $\chi \boxtimes \tau(\phi')|_{U(V_{sp})}$ is an irreducible supercuspidal representation $\tau$ of $U(V_{sp})$.

Case C$_1$: $\phi_E = \lambda \oplus \lambda^{-1} \circ \sigma \quad (\lambda \neq \lambda^{-1} \circ \sigma)$
First assume that $\lambda \circ N_{E/F} \neq |E|^{1/2}$. If we take $\chi = \lambda$ and $\phi' = 1 \oplus \lambda^{-1}|F^\times|$, then $\lambda \boxtimes \tau(\phi') = \lambda \boxtimes \text{Ind}_{B^-}^{\text{GL}_2(F)} 1 \boxtimes (\lambda^{-1}|F^\times|)$. Here $\text{Ind}_{B}^{U(V_{sp})} \lambda$ is irreducible. Thus by Lemma 2.4 we have $\Pi_\phi = \{\text{Ind}_{B}^{U(V_{sp})} \lambda\}$. Next assume that $\lambda \circ N_{E/F} = |E|^{1/2}$ without loss of generality. Then $\lambda = \eta| |E|^{1/2}$ or $\mu| |E|^{1/2}$. If $\lambda = \eta| |E|^{1/2}$, then we may take $\chi = \eta$ and $\phi' = |E|^{1/2} \oplus |F|^{1/2}$. Thus $\tau(\phi') = \text{det} \text{ and } \eta \boxtimes \tau(\phi')|_{U(V_{sp})} = \eta_0 \circ \text{det}$. Therefore $\Pi_\phi = \{\eta_0 \circ \text{det}\}$. If $\lambda = \mu| |E|^{1/2}$, then we may take $\chi = \mu$ and $\phi' = \omega_{E/F}| |E|^{1/2} \oplus |F|^{1/2}$. Thus $\tau(\phi') = \text{Ind}_{B^-}^{\text{GL}_2(F)} \omega_{E/F}| |E|^{1/2} \boxtimes |F|^{1/2} \text{ and } \eta \boxtimes \tau(\phi')|_{U(V_{sp})} = \text{Ind}_{B}^{U(V_{sp})} \mu| |E|^{1/2}$. This representation is irreducible. Therefore $\Pi_\phi = \{\text{Ind}_{B}^{U(V_{sp})} \mu| |E|^{1/2}\}$.

Case C$_2$: $\phi_E = \mu_1 \oplus \mu_2$
By Theorem 2.1, we have $\Pi_\phi = \{\tau(\mu_1, \mu_2)_x^{\pm}\}$.

Case C$_3$: $\phi_E = \eta \oplus \eta$
If we take $\chi = \eta$ and $\phi' = 1 \oplus 1$, then $\eta \boxtimes \tau(\phi') = \eta \boxtimes \text{Ind}_{B^-}^{\text{GL}_2(F)} 1 \boxtimes 1$. Here $\text{Ind}_{B}^{U(V_{sp})} \eta$ is irreducible. Thus we have $\Pi_\phi = \{\text{Ind}_{B}^{U(V_{sp})} \eta\}$

Case C$_4$: $\phi_E = \mu \oplus \mu$
By Theorem 2.1, we have $\Pi_\phi = \{\tau(\mu, \mu)_x^{\pm}\}$. On the other hand, if we take $\chi = \mu$ and $\phi' = \omega_{E/F} \oplus 1$, then $\phi_E = \mu \oplus \mu$. Then $\tau(\phi') = \text{Ind}_{B^-}^{\text{GL}_2(F)} \omega_{E/F} \boxtimes 1$ and $\mu \boxtimes \tau(\phi')|_{U(V_{sp})} = \text{Ind}_{B}^{U(V_{sp})} \mu$. Since this representation is completely reducible, $\text{Ind}_{B}^{U(V_{sp})} \mu = \tau(\mu, \mu)_x^{\pm} \oplus \tau(\mu, \mu)_x^{\pm}$. 

We have the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>( \phi )</th>
<th>( \phi_E )</th>
<th>( \Pi_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>( \eta \boxtimes \text{Sym}^1 )</td>
<td>( \eta )</td>
<td>( \text{St}(\eta) )</td>
</tr>
<tr>
<td>Case B</td>
<td>( \phi_E )</td>
<td>( \phi_E )</td>
<td>( \tau )</td>
</tr>
<tr>
<td>Case C1a</td>
<td>( \lambda \oplus \lambda^{-1} \circ \sigma )</td>
<td>( \lambda \circ N_{E/F} \neq \pm 1 )</td>
<td>( \text{Ind}_{B}(\lambda) )</td>
</tr>
<tr>
<td>Case C1b</td>
<td>( \eta \mid E \oplus \eta \mid E )</td>
<td>( \eta \circ \det )</td>
<td></td>
</tr>
<tr>
<td>Case C1c</td>
<td>( \mu \mid E \oplus \mu \mid E )</td>
<td>( \eta \circ \det )</td>
<td></td>
</tr>
<tr>
<td>Case C2</td>
<td>( \mu_1 \oplus \mu_2 )</td>
<td>( \tau(\mu_1, \mu_2)_{sp} )</td>
<td></td>
</tr>
<tr>
<td>Case C3</td>
<td>( \eta \oplus \eta )</td>
<td>( \text{Ind}_{B}(\eta) )</td>
<td></td>
</tr>
<tr>
<td>Case C4</td>
<td>( \mu \oplus \mu )</td>
<td>( \tau(\mu_1, \mu_2)_{sp} )</td>
<td></td>
</tr>
</tbody>
</table>

### 2.2 Global endoscopy for \( U(V_{sp}) \)

Let \( \tilde{E}/\tilde{F} \) be a quadratic extension of a number field. Take a non-zero element \( \tilde{\xi} \) of \( \tilde{E} \) with \( \text{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0 \) and a non-trivial character \( \tilde{\psi} \) of \( \text{A}_{\tilde{F}}^{\times} / \tilde{F}^{\times} \). Denote the quadratic character of \( \text{A}_{\tilde{F}}^{\times} / \tilde{F}^{\times} \) associated with \( \tilde{E}/\tilde{F} \) by \( \omega_{\tilde{E}/\tilde{F}} \). Also for each character \( \chi \) of \( \text{A}_{\tilde{F}}^{\times} / \tilde{F}^{\times} \), \( \Pi(\text{A}_{\tilde{E}}^{\times}, \chi) \) stands for the set of characters of \( \text{A}_{\tilde{E}}^{\times} / \tilde{E}^{\times} \times \) whose restriction to \( \text{A}_{\tilde{F}}^{\times} / \tilde{F}^{\times} \) are \( \chi \).

We define a global endoscopic \( L \)-packet \( \Pi(\tilde{\mu}_1, \tilde{\mu}_2)(\tilde{V}_{sp}) \) of \( U(\tilde{V}_{sp}) \) by

\[
\{ \otimes_v \tau(\tilde{\mu}_1, \tilde{\mu}_2)_{sp} \mid \epsilon_v = + \text{ for almost all } v \}.
\]

We have the following:
Theorem 2.5 ([5]). Let \( \tilde{\tau} = \bigotimes_v \tau(\tilde{\mu}_1,v,\tilde{\mu}_2,v)_{sp} \) be an element of \( \Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V}_{sp}) \). Then \( \tilde{\tau} \) is a cuspidal representation of \( U(\tilde{V}_{sp}) \) if and only if \( \Pi_v \varepsilon_v = + \) and \( \tilde{\mu}_1 \neq \tilde{\mu}_2 \).

Theorem 2.6 ([15]). Let \( \tau = \bigotimes_v \tau_v \) be an irreducible cuspidal representation of \( U(\tilde{V}_{sp}) \). If there exist characters \( \tilde{\mu}_1, \tilde{\mu}_2 \in \Pi(\tilde{A}_E, \omega_{E/F}) \) such that \( \tau_v \in \Pi_{\tilde{\mu}_1,v,\tilde{\mu}_2,v}(\tilde{V}_{sp}) \) for almost all places \( v \), then \( \tau_v \in \Pi_{\tilde{\mu}_1,v,\tilde{\mu}_2,v}(\tilde{V}_{sp}) \) for all places \( v \).

2.3 Local endoscopy for \( U(V_{an}) \)

Let \( D \) be the quaternion division algebra over a non-archimedean local field \( F \). Denote its reduced norm and trace by \( \nu_D, \text{tr}_D \), respectively. We fix an embedding \( E \hookrightarrow D \). Then there exists \( \xi' \in D \) such that \( \text{tr}_D(\xi') = 0 \), \( \xi\xi' = -\xi'\xi \) and \( D = F \oplus F\xi \oplus F\xi' \oplus F\xi'\xi = E \oplus \xi'E \).

Also we may assume that \( \xi'^2 = d_0 \). Then we have the following isomorphism:

\[
E^\times \times D^\times/\Delta F^\times \ni (x,z+\xi'w) \mapsto x\nu_D(z+\xi'w)^{-1} \begin{pmatrix} \sigma(z) & w \\ d_0\sigma(w) & z \end{pmatrix} \in \text{GU}(V_{an}).
\]

We identify \( \text{GU}(V_{an}) \) with \( E^\times \times D^\times/\Delta F^\times \). Then the \( L \)-group of \( \text{GU}(V_{an}) \) is the same as \( \text{GU}(V_{sp}) \).

Let \( \phi' \) be an irreducible two-dimensional representation of \( L_F \). Then there exists an irreducible admissible representation \( \tau_D(\phi') \) of \( D^\times \) with \( L \)-parameter \( \phi' \). Namely, it is the representation of \( D^\times \) associated to \( \tau(\phi') \) by the Jaquet-Langlands correspondence. Let \( \chi \) be a quasi-character of \( E^\times \) such that \( \chi^{-1}|_{F^\times} = \det \phi' \). Then \( \chi \boxtimes \tau_D(\phi') \) is an irreducible admissible representation of \( \text{GU}(V_{an}) \). Its \( L \)-parameter is the same as one of \( \chi \boxtimes \tau(\phi') \) in §2.1. Take \( \phi \) as in §2.1. Define the \( L \)-packet \( \Pi_\phi \) associated to \( \phi \) by the set of irreducible subrepresentations of \( \chi \boxtimes \tau_D(\phi')|_{U(V_{an})} \). If \( \phi' \) is a reducible two-dimensional representation of \( L_F \), we define the \( L \)-packet \( \Pi_\phi \) to be empty.

Theorem 2.7 ([5]).

(i) \( \chi \boxtimes \tau_D(\phi')|_{U(V_{an})} \) is multiplicity free.

(ii) \( \Pi_\phi \) has at most two elements.

(iii) \( \Pi_{\phi_1} \cap \Pi_{\phi_2} \) is not empty if and only if \( \phi_1 \sim \phi_2 \). In that case, \( \Pi_{\phi_1} = \Pi_{\phi_2} \).
(iv) Any irreducible admissible representation of $U(V_{an})$ is contained in an $L$-packet $Π_ϕ$.

(v) $Π_ϕ$ has two elements if and only if there exist characters $μ_1 \neq μ_2 \in Π(E^x, ω_{E/F})$ such that $ϕ_E = μ_1 ⊕ μ_2$.

In (v), we denote the $L$-packet $Π_ϕ$ by $Π_{μ_1, μ_2} = Π_{μ_1, μ_2}(V_{an})$ and call it an endoscopic $L$-packet. Also its elements are called endoscopic representations. Since we can prove that there exists $χ' \in Irr E^x$ such that $μ_1 = χ χ'$ and $μ_2 = χ(χ' \circ σ)$, we may take $ϕ' = Ind_{W_E}^{W_F} χ'$.

We set

$$T = \{ t(z, z') = \begin{pmatrix} zz' & zσ(z') \\ zσ(z') & 1 \end{pmatrix} \in U(V_{an}) \bigg| z, z' ∈ E^x, N_{E/F}(z') = 1 \}.$$  \hspace{1cm} (2)

The endoscopic description of each endoscopic $L$-packet of $U(V_{an})$ is given in terms of character identity as follows:

**Proposition 2.8** ([5]). There exist two irreducible admissible representations $τ(μ_1, μ_2)^{±}_{an} \not\cong τ(μ_1, μ_2)^{−}_{an}$ of $U(V_{an})$ such that $Π_{μ_1, μ_2} = \{ τ(μ_1, μ_2)^{±}_{an} \}$ and the following character identity holds:

$$\text{Tr} τ(μ_1, μ_2)^{±}_{an}(t(z, z')) - \text{Tr} τ(μ_1, μ_2)^{−}_{an}(t(z, z')) = \lambda(E/F, ψ) ω_{E/F} \left( \frac{z' - σ(z')}{ξ} \right) χ(z N_{E/F}(z')) \chi'(σ(z')) - \frac{χ'(z')}{|z' - σ(z')|^1/2} |z'|^{1/2}.$$  \hspace{1cm} \hfill (3)

Also the last equation is independent of the choices of $χ, χ'$ such that $μ_1 = χ χ'$ and $μ_2 = χ(χ' \circ σ)$.

It is easy to check the following:

**Corollary 2.9.** (i) For any $μ_1 \neq μ_2 \in Π(E^x, ω_{E/F})$ and $ε ∈ \{ ± \}$, we have $τ(μ_1, μ_2)^{ε}_{an} = τ(μ_2, μ_1)^{−ε}_{an}$.

(ii) For each $η ∈ Π(E^x, 1_{F^x})$, we have $η_1 ⊗ det ∩ τ(μ_1, μ_2)^ε_{an} = τ(η_μ_1, η_μ_2)^ε_{an}$.

(iii) The representation $τ(μ_1^{-1}, μ_2^{-1})^{ω_{E/F}(−1)^ε}_{an}$ is the contragredient representation of $τ(μ_1, μ_2)^{an}$.

(iv) For $g ∈ GU(V_{an})$ with similitude norm $a$, we have $τ(μ_1, μ_2)^ε_{an} \circ \text{Ad}(g) = τ(μ_1, μ_2)^{ω_{E/F}(a)^ε}_{an}$.

In particular, the description of $Π_{μ_1, μ_2}(V_{an})$ depends on the order of the pair $(μ_1, μ_2)$. 
Finally we explicitly state the $L$-packet $\Pi_\phi$ for each $L$-parameter $\phi$ that is classified in §1.2.

Case A: $\phi_E = \eta \boxtimes \text{Sym}^1$
If we take $\chi = \eta$ and $\phi' = \text{Sym}^1$, then we obtain $\phi_E = \eta \boxtimes \text{Sym}^1$. Since $\tau_D(\phi')$ is the trivial representation of $D^\times$, we have $\eta \boxtimes \tau(\phi')|_{U(V_{an})} = \eta_u \circ \det$. Therefore $\Pi_\phi = \{\eta_u \circ \det\}$.

Case B: $\phi_E \in \text{Irr} W_{E,2,+}$
Then we can take a two-dimensional irreducible representation of $W_F$ as $\phi'$. Thus $\tau(\phi')$ is a supercuspidal representation of $D^\times$. By Theorem 2.7, $\chi \boxtimes \tau(\phi')|_{U(V_{an})}$ is an irreducible supercuspidal representation $\tau$ of $U(V_{an})$.

Case C2: $\phi_E = \mu_1 \oplus \mu_2$
By Theorem 2.7, we have $\Pi_\phi = \{\tau(\mu_1, \mu_2)_{sp}\}$.

Since $\phi'$ is reducible for any $\phi$ in the remaining cases, $\Pi_\phi$ is empty. Thus we have the following table:

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\phi_E$</th>
<th>$\Pi_\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>$\eta \boxtimes \text{Sym}^1$</td>
<td>$\eta_u \circ \det$</td>
</tr>
<tr>
<td>Case B</td>
<td>$\phi_E$</td>
<td>$\tau$</td>
</tr>
<tr>
<td>Case C1</td>
<td>$\lambda \oplus \lambda^{-1} \circ \sigma$</td>
<td>empty</td>
</tr>
<tr>
<td>Case C2</td>
<td>$\mu_1 \oplus \mu_2$</td>
<td>$\tau(\mu_1, \mu_2)_{un}$</td>
</tr>
<tr>
<td>Case C3</td>
<td>$\eta \oplus \eta$</td>
<td>empty</td>
</tr>
<tr>
<td>Case C4</td>
<td>$\mu \oplus \mu$</td>
<td>empty</td>
</tr>
</tbody>
</table>

2.4 Global endoscopy for $U(V_{an})$
In this subsection, we prepare a global result which we need later.

Let $\tilde{E}/\tilde{F}$ be a quadratic extension of a number field. Take an element $\tilde{d}$ of $\tilde{F}^\times$, a non-zero element $\tilde{\xi}$ of $\tilde{E}$ with $\text{Tr}_{E/F}(\tilde{\xi}) = 0$ and a non-trivial character $\tilde{\psi}$ of $\mathbb{A}_{\tilde{F}}$. We define a two-dimensional hermitian space $\tilde{V}$ over $\tilde{E}$:

$$\tilde{V} = (\tilde{E}^\otimes 2, \begin{pmatrix} -\tilde{d} & \cdot \\ \cdot & 1 \end{pmatrix}).$$

Take two characters $\tilde{\mu}_1, \tilde{\mu}_2 \in \Pi(\mathbb{A}_E^\times, \omega_{E/F})$. Assume that if $\tilde{V}_v$ is anisotropic, then $\tilde{\mu}_{1,v} \neq \tilde{\mu}_{2,v}$. Then for any place $v$, we have the local $L$-packet $\Pi_{\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v}}(\tilde{V})$. 
of $U(V_v)$ associated to the $L$-parameter $\varphi_{E_v} = \tilde{\mu}_{1,v} \oplus \tilde{\mu}_{2,v}$. These local $L$-packets are described as follows.

First assume that $V_v \cong V_{sp,v}$. Then we have $U(V_v) \cong U(V_{sp,v})$. We omit $v$ from the notation. If $E$ is not a field, the local $L$-packet $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(V)$ consists of one $\tilde{\psi}$-generic element. It is denoted by $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^+$. If $E$ is a field, we define representations $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm$ of $U(V)$ by the composition of $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm_{sp}$ and an isomorphism $U(V) \cong U(V_{sp})$ induced by $V \cong V_{sp}$. Then $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^+$ is $\tilde{\psi}$-generic and $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(V) = \{ \tau(\tilde{\mu}_1,\tilde{\mu}_2)^+ \}$. Also the following holds:

Lemma 2.10. Define a maximal torus $T$ of $U(V)$ as in §2.3. Then we have the following character identity:

$$
\text{Tr} \tau(\tilde{\mu}_1,\tilde{\mu}_2)^+(t(z,z')) - \text{Tr} \tau(\tilde{\mu}_1,\tilde{\mu}_2)^-(t(z,z')) = \lambda(\tilde{E}/\tilde{F},\tilde{\psi}) \omega_{\tilde{E}/\tilde{F}} \left( \frac{z' - \sigma(z')}{\tilde{\eta}} \right) \chi(z N_{E/F}(z')) \chi'(\sigma(z')) + \chi(z')^{1/2} |z'|^{1/2} \left| \frac{\tilde{E}}{E} \right|
$$

for any $t(z,z') \in T$. Here $\chi, \chi' \in \text{Irr } E^\times$ such that $\tilde{\mu}_1 = \chi \chi'$, $\tilde{\mu}_2 = \chi' \circ \sigma$.

Proof. Now there exists $x \in \tilde{E}^\times$ such that $\tilde{d} = N_{E/F}(x)$. We set

$$
A = \begin{pmatrix} -\tilde{d} & x \\ 1/2 & x \tilde{d}^{-1/2} \end{pmatrix}.
$$

Then the matrix $A$ gives the isomorphism $V \cong V_{sp}$. Namely,

$$\ast A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x & 1 \\ 1/2 & x \tilde{d}^{-1/2} \end{pmatrix} A = \begin{pmatrix} -\tilde{d} \\ 1 \end{pmatrix}.
$$

Thus the group isomorphism $U(V) \cong U(V_{sp})$ is given by $\text{Ad}(A)$. By definition, we have $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm = \tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm_{sp} \circ \text{Ad}(A)$. Also we obtain

$$
\text{Ad}(A)(t(z,z')) = t_{\tilde{d}^{-1}\tilde{\xi}^{-1}/2}(z,z')
$$

for $t(z,z') \in T$. Thus Lemma 2.10 follows from Proposition 2.2 and easy computation.

If $v$ is finite and $V_v$ is anisotropic, we have already explained in §2.3.
Finally, assume that $E_v, \tilde{E}_v = \mathbb{C}/\mathbb{R}$ and $\tilde{V}_v$ is anisotropic. We omit $v$ from the notation. Set

$$
\lambda : \mathbb{C}^\times \ni re^{i\theta} \mapsto e^{i\theta} \in \mathbb{C}^\times \\
\lambda_1 : \mathbb{C}^1 \ni e^{i\theta} \mapsto e^{i\theta} \in \mathbb{C}^1.
$$

Then we may assume $\tilde{\mu}_1 = \lambda^{m+n}$ and $\tilde{\mu}_2 = \lambda^{m-n}$, where $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m - n \equiv 1 \mod 2$. Let $\rho_{m,n}$ be the irreducible $|n|$-dimensional representation of $U(\tilde{V})$ with central character $\lambda_1^n$.

**Lemma 2.11.** Let $r \in \mathbb{R}^\times$ be such that $\tilde{\psi}(x) = e^{irx} (x \in \mathbb{R})$. We set $C_v = \operatorname{sgn}(nr\xi/i)$. Then we have

$$C_v \cdot \operatorname{Tr} \rho_{m,n}(t(z, z')) = \lambda \left( \mathbb{C}/\mathbb{R}, \tilde{\psi} \right) \omega_{\mathbb{C}/\mathbb{R}} \left( \frac{z' - \sigma(z')}{\xi} \right) \chi \left( z N_{\mathbb{C}/\mathbb{R}}(z') \right) \frac{\chi'(z') - \chi'(z') |z'|^{1/2}}{|z' - \sigma(z')|^{1/2}} \chi(\sigma(z') - \chi'(z') |z'|^{1/2} \chi(z'))$$

for any $t(z, z') \in T$. Here $\chi, \chi' \in \operatorname{Irr} \mathbb{C}^\times$ such that $\tilde{\mu}_1 = \chi \chi', \tilde{\mu}_2 = \chi(\chi' \circ \sigma)$.

**Proof.** It is enough to show for $z = e^{i\xi}, z' = e^{i\theta}$. Also we can take $\chi = \lambda^m, \chi' = \lambda^n$. Since $\lambda \left( \mathbb{C}/\mathbb{R}, \tilde{\psi} \right) = \operatorname{sgn}(r)i$, the right hand side is

$$ie^{im\xi} \operatorname{sgn}(r) \frac{e^{i\theta} - e^{-i\theta}}{\xi} \frac{e^{-in\theta} - e^{in\theta}}{2|\sin \theta|}.$$ 

On the other hand,

$$\operatorname{Tr} \rho_{m,n}(t(z, z')) = -ie^{im\xi} \operatorname{sgn}(r) \frac{e^{i\theta} - e^{-i\theta}}{i} \frac{e^{i|n|\theta} - e^{-i|n|\theta}}{2|\sin \theta|}.$$ 

Lemma follows from this.

Thus by this lemma, we set $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2} = \{ \rho_{m,n} \}$ and $\tau(\tilde{\mu}_1, \tilde{\mu}_2)_{C_v} := \rho_{m,n}$.

Now we define a global endoscopic $L$-packet $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V})$ of $U(\tilde{V})$ by

$$\left\{ \bigotimes_v \tau(\tilde{\mu}_1, \tilde{\mu}_2)_{C_v} \bigg| \begin{array}{l}
\tau(\tilde{\mu}_1, \tilde{\mu}_2)_{C_v} \in \Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}), \\
\epsilon_v = + \text{ for almost all } v
\end{array} \right\}.$$ 

**Theorem 2.12.** Let $\tilde{\tau} = \bigotimes_v \tau(\tilde{\mu}_1, \tilde{\mu}_2)_{C_v}$ be an element of $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V})$. Then $\tilde{\tau}$ is a cuspidal representation of $U(\tilde{V})$ if and only if $\Pi_v \epsilon_v = +$.

**Proof.** This follows from [5] and the character identities in Proposition 2.8, Lemma 2.10 and Lemma 2.11.
3 Endoscopy for $U(W)$

In this section, we explain the endoscopy for the unitary group $U(W)$ following [15, Chap.3, 4].

3.1 $L$-group of $U(W)$

In this subsection, we compute the $L$-group of $U(W)$. Set

$$S = \begin{pmatrix} 1 & & \\ & \xi & \\ & & -1 \end{pmatrix}.$$ 

Define the connected reductive group $G$ defined over $F$ as follows:

$$G(R) = \{ g \in \text{GL}_3(E \otimes_F R) \mid gS^*g = S \},$$

where $R$ is any $F$-algebra, $^*g = (\sigma \otimes \text{id}_R)(t^g)$. Then $G(F) = U(W)$.

First we show that $G(\overline{F}) \cong GL_3(\overline{F})$.

Lemma 3.1. (i) $E \otimes_F \overline{F} \ni \sum a \otimes b \mapsto \sum (ab, \sigma(a)b) \in \overline{F} \oplus \overline{F}$ is an isomorphism of $\overline{F}$-algebras.

(ii) The action on $E \otimes_F \overline{F}$ of $\text{Gal}(E/F) \times \Gamma_F$ is transposed on $\overline{F} \oplus \overline{F}$ as follows:

$$(\sigma \otimes \text{id}_F)(x, y) = (y, x),$$

$$(\text{id}_E \otimes \tau)(x, y) = \begin{cases} (\tau(x), \tau(y)) & \text{if } \tau \in \Gamma_E, \\
(\tau(y), \tau(x)) & \text{if } \tau \notin \Gamma_E. \end{cases}$$

Proof. (i) This is trivial.

(ii) Let $\alpha$ be the isomorphism of (i). Take an element $(x, y) \in \overline{F} \oplus \overline{F}$ and write $(x, y) = (ab, \sigma(a)b)$. Then we have

$$\sigma \otimes \text{id}_F(x, y) := \alpha \circ (\sigma \otimes \text{id}_F) \circ \alpha^{-1}(x, y)$$

$$= \alpha \circ (\sigma \otimes \text{id}_F)(a \otimes b)$$

$$= \alpha(\sigma(a) \otimes b)$$

$$= (\sigma(a)b, ab)$$

$$= (y, x)$$

The latter follows from the similar argument. \hfill \Box
By this lemma, we have the following corollary:

**Corollary 3.2.** (i) The isomorphism of Lemma 3.1 (i) induces the following isomorphism:

$$\text{GL}_3(E \otimes_F \bar{F}) \cong \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}).$$

(ii) The action on $\text{GL}_3(E \otimes_F \bar{F})$ of $\text{Gal}(E/F) \times \Gamma_F$ is transposed on $\text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F})$ as in Lemma 3.1 (ii).

By Cor. 3.2, the image in $\text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F})$ of the equation $gS^*g = S$ is

$$(g_1, g_2)(S, S^\sigma) \left( t^g_2, t^g_1 \right) = (S, S^\sigma).$$

Thus the image of $G(\bar{F})$ in $\text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F})$ coincides with the subgroup of $\text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F})$ which consists of the elements of the form

$$\left( g_1, t^S, t^{-1}, t^{-1} \right).$$

This subgroup is isomorphic to $\text{GL}_3(\bar{F})$ by the first projection. Consequently it turns out that

$$G(\bar{F}) \cong \text{GL}_3(\bar{F}).$$

(3)

From this, we see that the Langlands dual group $\hat{G}$ of $G$ is $\text{GL}_3(\mathbb{C})$.

Let $B_3$ be the Borel subgroup of $\text{GL}_3$ which consists of the upper-triangular matrices, and $T$ the maximal torus which consists of the diagonal matrices. Let $X_i$ be the $3 \times 3$-matrix with $(i, i + 1)$-component one and zero elsewhere. Then $(B_3, T, \{X_i\})$ is a splitting of $\text{GL}_3$. We identify $(B_3, T, \{X_i\})$ with a splitting of $G$ by (3). Then the based root datum of $G$ is given as follows:

$$X^* = X^*(T) = \bigoplus_i \mathbb{Z} \epsilon_i,$$

where $\epsilon_i(\text{diag}(t_1, t_2, t_3)) = t_i$,

$$\Delta = \{ \alpha_i := \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq 2 \},$$

$$X_* = X_*(T) = \bigoplus_i \mathbb{Z} \epsilon^\vee_i,$$

where $\epsilon^\vee_i(t) = \text{diag}(1, \ldots, t^i, \ldots, 1)$,

$$\Delta^\vee = \{ \alpha^\vee_i := \epsilon^\vee_i - \epsilon^\vee_{i+1} \mid 1 \leq i \leq 2 \}.$$

On the other hand, take the splitting $(\hat{B}_3, \hat{T}, \{X_i\})$ of $\text{GL}_3(\mathbb{C})$ as that of $\text{GL}_3$. Then the dual based root datum of $G$ is equal to the based root datum of $\text{GL}_3(\mathbb{C})$. Also the action of $\Gamma_F$ on this based root datum is written as follows:

$$\tau(\epsilon_i) = \begin{cases} 
\epsilon_i & \text{if } \tau \in \Gamma_E, \\
-\epsilon_{4-i} & \text{if } \tau \notin \Gamma_E.
\end{cases}$$
This action is induced by the following action of $\Gamma_F$ on $\text{GL}_3(\mathbb{C})$:

$$\tau(g) = \begin{cases} 
  g & \text{if } \tau \in \Gamma_E, \\
  I_3^t g^{-1} I_3^{-1} & \text{if } \tau \notin \Gamma_E.
\end{cases}$$

Since the splitting $(\hat{B}_3, \hat{T}, \{X_i\}_i)$ of $\text{GL}_3(\mathbb{C})$ is stable under this action of $\Gamma_F$ on $\text{GL}_3(\mathbb{C})$, the $L$-group of $G$ is given as follows:

**Lemma 3.3.**

$$L^G = \text{GL}_3(\mathbb{C}) \rtimes W_F,$$

$$g \rtimes w \cdot g' \rtimes w' = \begin{cases} 
  gg' \rtimes ww' & \text{if } w \in W_E, \\
  g I_3^t g'^{-1} I_3^{-1} \rtimes ww' & \text{if } w \notin W_E.
\end{cases}$$

### 3.2 Stable orbital integrals

We recall the definitions of a semi-simple element and a regular semi-simple element of $G(F)$. An element $\gamma \in G(F)$ is said to be semi-simple if the image of $\gamma$ in $\text{GL}_3(\hat{F})$ is diagonalizable. We denote the set of semi-simple elements of $G(F)$ by $G(F)_{ss}$. An element $\gamma \in G(F)$ is said to be regular semi-simple if the characteristic-polynomial of the image of $\gamma$ in $\text{GL}_3(\hat{F})$ has distinct three roots. This condition is equivalent to the condition that $G_\gamma$ is a maximal torus of $G$, where $G_\gamma$ is the centralizer of $\gamma$ in $G$. We denote the set of regular semi-simple elements of $G(F)$ by $G(F)_{reg}$. For any $\gamma \in G(F)_{reg}$, set

$$D_G(\gamma) := \det(\text{Ad}(\gamma) - 1|_{\mathfrak{g}(F) / \mathfrak{a}_\gamma(F)}),$$

where $\mathfrak{g}$ and $\mathfrak{g}_\gamma$ are the Lie algebras of $G$ and $G_\gamma$, respectively. Then we define the normalized orbital integral of $\gamma$ by

$$J(\gamma, f) := |D_G(\gamma)|^{1/2} \int_{G(F) \backslash G(F)} f(g^{-1} \gamma g) \frac{dg}{d\mathcal{I}};$$

where $f \in \mathcal{S}(G(F))$. $J(\gamma, f)$ is convergent ([14]). This $J(\gamma)$ is an invariant distribution on $G(F)$. Here a linear map $A : \mathcal{S}(G(F)) \to \mathbb{C}$ is called an invariant distribution on $G(F)$ if the following condition is satisfied:

$$A(f \circ \text{Ad}(g)) = A(f), \quad f \in \mathcal{S}(G(F)), \quad g \in G(F).$$

We denote the set of invariant distributions on $G(F)$ by $\mathcal{I}(G(F))$. 
Let \( \gamma \) and \( \gamma' \) be two elements of \( G(F) \). The element \( \gamma \) is said to be \emph{conjugate} to \( \gamma' \) if there exists \( g \in G(F) \) such that \( \gamma' = g^{-1}\gamma g \). Then we write \( \gamma \sim \gamma' \). Also \( \gamma \) is said to be \emph{stably conjugate} to \( \gamma' \) if there exists \( g \in G(\overline{F}) \) such that \( \gamma' = g^{-1}\gamma g \). Then we write \( \gamma \sim_{st} \gamma' \). The stable conjugacy is an equivalence relation which is weaker than the conjugacy. Namely, if \( \gamma' \) is conjugate to \( \gamma \), then \( \gamma' \) is stable conjugate to \( \gamma \). Thus we define the following set:

\[
O_{st}(\gamma) := \{ \gamma' \in G(F) \mid \gamma' \sim_{st} \gamma \}/\sim.
\]

We also call this the stable conjugacy class of \( \gamma \). We denote the set of stable conjugacy classes \( O_{st}(\gamma) \) of the semi-simple elements \( \gamma \) of \( G(F) \) by \( O_{st}(G) \).

We can describe \( O_{st}(\gamma) \) in terms of Galois cohomology. Let \( \gamma \) and \( \gamma' \) be stable conjugate in \( G(F) \). By definition, there exists an element \( g \) of \( G(\overline{F}) \) such that \( \gamma' = g^{-1}\gamma g \). Then \( \{gs(g^{-1})\}_{s \in \Gamma} \) satisfies the cocycle condition. We denote this 1-cocycle \( \{gs(g^{-1})\}_{s \in \Gamma} \) by \( f_{\gamma'} \).

**Lemma 3.4.**

(i) \( f_{\gamma'} \in Z^1(F,G_\gamma) \).

(ii) The image \([f_{\gamma'}]\) of \( f_{\gamma'} \) in \( H^1(F,G_\gamma) \) is independent of the choice of \( g \).

(iii) \([f_{\gamma'}]\) is contained in \( \text{Ker}(H^1(F,G_\gamma) \to H^1(F,G)) \).

(iv) Set \( \mathfrak{D}(G_\gamma) := \text{Ker}(H^1(F,G_\gamma) \to H^1(F,G)) \). Then the following map is bijective:

\[
O_{st}(\gamma) \ni [\gamma'] \longmapsto [f_{\gamma'}] \in \mathfrak{D}(G_\gamma).
\]

Here the morphism \( H^1(F,G_\gamma) \to H^1(F,G) \) is induced by the inclusion \( G_\gamma \hookrightarrow G \).

**Proof.** These follow from the definition easily. \( \square \)

Since \( H^1(F,G_\gamma) \) is a finite set, so is \( \mathfrak{D}(G_\gamma) \). Thus \( O_{st}(\gamma) \) is a disjoint union of finite number of conjugacy classes. Here note that since \( H^1(F,G) \) is a pointed-set and does not have a group structure in general, \( \mathfrak{D}(G_\gamma) \) is not a group.

We assume that \( \gamma \) is a regular semi-simple element of \( G(F) \). This case is especially important for us. Now \( T = G_\gamma \) is a maximal torus of \( G \). Then we recall Kottwitz’s theorem.
Theorem 3.5 ([7, Theorem 1.2]). There exists the following commutative diagram:

\[
\begin{array}{ccc}
H^1(F,T) & \longrightarrow & H^1(F,G) \\
\downarrow & & \downarrow \\
\pi_0(T^\Gamma)^D & \overset{\text{res. to } Z(G)^\Gamma}{\longrightarrow} & \pi_0(Z(\hat{G})^\Gamma)^D.
\end{array}
\]

Here $H^D$ stands for the Pontryagin dual of $H$.

We set

\[\mathfrak{E}(T) := \ker(\pi_0(T^\Gamma)^D \to \pi_0(Z(\hat{G})^\Gamma)^D).\]

By this theorem, this is isomorphic to $\mathfrak{D}(T)$. In particular, $\mathfrak{D}(T)$ becomes a subgroup of $H^1(F,T)$. Moreover, by Lemma 3.4 there exists a bijective map from $\mathcal{O}_{st}(\gamma)$ to $\mathfrak{E}(T)$. As a result, we get a description of $\mathcal{O}_{st}(\gamma)$ in terms of $L$-groups.

Next we state the following proposition on $\mathfrak{E}(T)$ where $T$ is a maximal torus of $G$.

Proposition 3.6 ([15, §3.4, §3.5, §3.6]).

(i) If two maximal tori $T$ and $T'$ of $G$ are $F$-isomorphic, then they are stable conjugate, that is, there exists $g \in G(\bar{F})$ such that $\text{Ad}(g)$ is an $F$-isomorphism between $T$ and $T'$.

(ii) The maximal tori $T$ of $G$ are divided into the following types up to $F$-isomorphism.

<table>
<thead>
<tr>
<th>maximal torus $T$ of $G$</th>
<th>$\mathfrak{E}(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Res}_{E/F} \mathbb{G}_m \times U(1)$</td>
<td>${1}$</td>
</tr>
<tr>
<td>$U(1) \times U(1) \times U(1)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$T_K \times U(1)$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$T_L$</td>
<td>${1}$</td>
</tr>
</tbody>
</table>

Here, we denote $\text{Res}_{K/F} U_{E[K]/K}(1)$ by $T_K$ and $K$ runs through the quadratic extensions of $F$ except for $E$. Also $T_L$ is $\text{Res}_{L/F} U_{E[L]/L}(1)$ and $L$ runs through the cubic extensions of $F$.

Let $\gamma$ be an element of $G(F)_{\text{reg}}$ and $T = G_{\gamma}$. Then we obtained

$\mathcal{O}_{st}(\gamma) \xrightarrow{\sim} \mathfrak{D}(T) \xrightarrow{\sim} \mathfrak{E}(T)$.

We denote the image of $[\gamma'] \in \mathcal{O}_{st}(\gamma)$ in $\mathfrak{E}(T)$ by $\text{inv}(\gamma, \gamma')$. Also we set

$\mathfrak{R}(T) := \mathfrak{E}(T)^D$.

This is isomorphic to $\text{Coker}(\pi_0(Z(\hat{G})^\Gamma) \to \pi_0(T^\Gamma))$ by Theorem 3.5.
Definition 3.7. For $\kappa \in \mathfrak{K}(T)$ and $f \in \mathcal{S}(G(F))$, define

$$J^\kappa(\gamma, f) := \sum_{[\gamma'] \in O_{st}(\gamma)} \kappa(\text{inv}(\gamma, \gamma')) J(\gamma', f)$$

$k$-orbital integral,

$$J^{st}(\gamma, f) := \sum_{[\gamma'] \in O_{st}(\gamma)} J(\gamma', f)$$

stably orbital integral.

Here, each measure on $G_{\gamma'}(F)$ is the pull back of a Haar measure on $G_{\gamma}(F)$ by $\text{Ad}(g) : G_{\gamma'}(F) \to G_{\gamma}(F)$, where $\gamma' = g^{-1}\gamma g$.

In general, $J^\kappa(\gamma)$ is dependent on the choice of $\gamma$, but $J^{st}(\gamma)$ is not. Namely, if $\gamma$ and $\gamma'$ are stable conjugate, then $J^{st}(\gamma) = J^{st}(\gamma')$.

Definition 3.8. We call an invariant distribution $A$ on $G(F)$ a stable distribution if $\text{Ker} A \supset \bigcap_{\gamma \in G(F)_{reg}} \text{Ker} J^{st}(\gamma)$. Then we denote the space of stable distributions on $G(F)$ by $\mathcal{SI}(G(F))$. This space equals to the closure of the subspace of stable orbital integrals in $\mathcal{I}(G(F))$ with respect to the weak topology.

Finally we consider the relation between $J^\kappa(\gamma)$ and $J^{\kappa'}(\gamma')$. Here $\gamma'$ is stably conjugate to $\gamma$ with centralizer $T'$, $\kappa \in \mathfrak{K}(T)$ and $\kappa' \in \mathfrak{K}(T')$.

Lemma 3.9. Let $g$ be an element of $G(F)$ such that $\gamma' = g^{-1}\gamma g$.

(i) The homomorphism

$$\text{Ad}(g^{-1}) : T \longrightarrow T'$$

is an $F$-isomorphism which maps $\gamma$ to $\gamma'$. Also this isomorphism is independent of the choice of $g$.

(ii) The map $\text{Ad}(g^{-1})$ induces a group-isomorphism from $\mathfrak{D}(T)$ to $\mathfrak{D}(T')$. This isomorphism is independent of the choice of $g$.

Proof. (i) This is trivial.

(ii) By (i), $\text{Ad}(g^{-1})$ induces $H^1(F, T) \cong H^1(F, T')$. To prove the former statement, it is sufficient to show that $\text{Ad}(g^{-1})$ maps $\mathfrak{D}(T)$ to $\mathfrak{D}(T')$. We assume that $a_s = hs(h^{-1})$ is the element of $\mathfrak{D}(T)$ corresponding to $[\gamma''] \in O_{st}(\gamma)$. Then we have

$$\text{Ad}(g^{-1})(a_s) = g^{-1}hs(h^{-1})g = g^{-1}hs(h^{-1}g) \cdot (g^{-1}s(g))^{-1}.$$ 

Since $g^{-1}hs(h^{-1}g)$ and $g^{-1}s(g)$ are elements of $\mathfrak{D}(T')$ and $\mathfrak{D}(T')$ is a group, $\text{Ad}(g^{-1})(a_s)$ is an element of $\mathfrak{D}(T')$. The latter follows from (i).
By this lemma, we get $\mathcal{E}(T) \cong \mathcal{E}(T')$. Also we have $\mathfrak{R}(T) \cong \mathfrak{R}(T')$.

**Lemma 3.10.** Assume that $\kappa \in \mathfrak{R}(T)$ corresponds to $\kappa' \in \mathfrak{R}(T')$ under the above isomorphism. Then we have

(i) $\kappa(\text{inv}(\gamma, \gamma'')) = \kappa'((\text{inv}(\gamma', \gamma'')) \cdot \kappa'((\text{inv}(\gamma', \gamma))^{-1}$ for any $[\gamma''] \in O_{st}(\gamma)$,

(ii) $\kappa(\text{inv}(\gamma, \gamma')) = \kappa'(\text{inv}(\gamma', \gamma))^{-1}$,

(iii) $J^\kappa(\gamma') = \kappa(\text{inv}(\gamma, \gamma'))J^\kappa(\gamma)$.

**Proof.** (i) Identify $\mathcal{D}(T)$ with $\mathcal{E}(T)$. We set $\gamma' = g^{-1}\gamma g$ and $\gamma'' = h^{-1}\gamma h$. Then $\kappa' = \kappa \circ \text{Ad}(g)$, $\text{inv}(\gamma', \gamma) = g^{-1}s(g)$, $\text{inv}(\gamma, \gamma'') = hs(h^{-1})$ and $\text{inv}(\gamma', \gamma'') = g^{-1}hs(h^{-1}g)$. Thus

$$\kappa(\text{inv}(\gamma, \gamma'')) = \kappa(hs(h^{-1}))$$

$$= \kappa'(g^{-1}hs(h^{-1})g)$$

$$= \kappa'(g^{-1}hs(h^{-1})g)s(g^{-1})g$$

$$= \kappa'(g^{-1}hs(h^{-1}g)) \cdot \kappa'(s(g^{-1})g)$$

$$= \kappa'(\text{inv}(\gamma', \gamma'')) \cdot \kappa'(g^{-1}s(g))^{-1}$$

$$= \kappa'(\text{inv}(\gamma', \gamma'')) \kappa'(\text{inv}(\gamma', \gamma))^{-1}.$$  

(ii) This follows from (i).

(iii) This follows from (i) and (ii). \hfill $\square$

### 3.3 Transfer

In this subsection, we explain an endoscopic datum of $G$.

**Definition 3.11.** We call $(H, s, \xi)$ an *endoscopic datum* of $G$ if it satisfies the following conditions:

(i) $H$ is a connected quasi-split group defined over $F$;

(ii) $s$ is a semi-simple element of $\hat{G}$;

(iii) $\xi : {}^LH \to {}^LG$ is a homomorphism over $W_F$;

(iv) $\xi$ induces an isomorphism from $\hat{H}$ to $(\hat{G}_s)^0$;

(v) $\lambda(w) := s\xi(w)s^{-1}\xi(w)^{-1}$ is a 1-coboundary from $W_F$ to $Z(\hat{G})$. 
Moreover an endoscopic datum \((H, s, \xi)\) is said to be elliptic if \(\xi(Z(\hat{H})^\Gamma)^0 \subset Z(\hat{G})^\Gamma\). Furthermore two endoscopic data \((H, s, \xi)\) and \((H', s', \xi')\) are said to be isomorphic if there exists \(g \in \hat{G}\) such that \(g \xi(\overline{L}H)g^{-1} = \xi'(\overline{L}H')\) and \(gs^{-1}s'^{-1} \in Z(\hat{G})\).

**Example 3.12.** (i) If \(H = G,\ s = 1\) and \(\xi = \text{id}_{L^G}\), then \((H, s, \xi)\) is an elliptic endoscopic datum of \(G\). We call it the trivial endoscopic datum.

(ii) Take a quasi-character \(\mu_0\) of \(E^\times\) such that \(\mu_0|_{F^\times} = \omega_{E/F}\). Then the following \((H, s_0, \xi_0)\) is a non-trivial elliptic endoscopic datum of \(G\):

\[
H = U(V_{sp}) \times U(1), \quad s_0 = \begin{pmatrix} -12 \\ 1 \end{pmatrix},
\]

\[
\xi_0 : L^H \ni (h_1, h_2) \mapsto \begin{cases} h_1 \mu_0(w) & w \in W_E \\
 -h_1 & w = w_\sigma. 
\end{cases}
\]

**Lemma 3.13** ([15, §4.6]). The set of isomorphism classes of elliptic endoscopic data \((H, s, \xi)\) of \(G\) is exhausted by the ones in Example 3.12.

Let \((H, s_0, \xi_0)\) be the above non-trivial elliptic endoscopic datum. Next we explain the transfer of orbital integrals.

Take \(h \in \text{GL}_3(E)\) such that \(h \begin{pmatrix} -\xi \\ -\xi \end{pmatrix} h^{-1} = I_3\). We define the following homomorphism:

\[
\text{Ad}(h) : H(F) \ni (z_1, z_2) \mapsto h \begin{pmatrix} *z_1^{-1} \\ z_2 \end{pmatrix} h^{-1} \in G(F)
\]

Then we define

\[
\mathcal{A}_{H/G} : \mathcal{O}_{st}(H) \ni \mathcal{O}_{st}(\gamma_H) \mapsto \mathcal{O}_{st}(\text{Ad}(h)(\gamma_H)) \in \mathcal{O}_{st}(G).
\]

This is independent of the choice of \(h\).

**Definition 3.14.** An element \(\gamma_H\) of \(H(F)_{\text{reg}}\) is said to be \(G\)-regular, if \(\mathcal{A}_{H/G}(\mathcal{O}_{st}(\gamma_H))\) is a stable conjugacy class of a regular semi-simple element of \(G(F)\). Then for any element \(\gamma\) of \(\mathcal{A}_{H/G}(\mathcal{O}_{st}(\gamma_H))\), we call \(\gamma_H\) an image of \(\gamma\). We denote the set of \(G\)-regular elements of \(H(F)_{\text{reg}}\) by \(H(F)_{G-\text{reg}}\).
For a $G$-regular element $\gamma_H$ of $H(F)_{\text{reg}}$, we take an element $\gamma$ of $G(F)$ whose image is $\gamma_H$. Then we explain how to determine $\kappa = \kappa_{\gamma,\gamma_H} \in \mathcal{R}(T_\gamma)$ where $T_\gamma = G_\gamma$. Similarly put $T_{\gamma_H} = H_\gamma$. Since $\gamma' = \text{Ad}(h)(\gamma_H)$ is stably conjugate to $\gamma$, there exists an element $g$ of $G(F)$ such that $\gamma' = g^{-1}\gamma g$.

We denote the Langlands dual group of $T_\gamma$ by $\check{T}_\gamma$. Then $\text{Ad}(g)$ induces an isomorphism from $\check{T}_\gamma$ to $\check{T}_\gamma$. Similarly put $T_{\gamma_H} = H_\gamma$. Since $\gamma' = \text{Ad}(h)(\gamma_H)$ is stably conjugate to $\gamma$, there exists an element $g$ of $G(\bar{F})$ such that $\gamma' = g^{-1}\gamma g$.

We denote the Langlands dual group of $T_\gamma$ by $\check{T}_\gamma$. Then $\text{Ad}(g)$ induces an isomorphism from $\check{T}_\gamma$ to $\check{T}_\gamma$. Similarly put $T_{\gamma_H} = H_\gamma$. Since $\xi^{-1}(s_0)$ is an element $s'$ of $Z(\check{H}) \subset \check{T}_\gamma$, the image of $s'$ under that isomorphism determines an element $t$ of $\check{T}_\gamma$. This $t$ defines an element $\kappa = \kappa_{\gamma,\gamma_H}$ of $\mathcal{R}(T_\gamma)$.

Then we have the following:

**Theorem 3.15** ([15, §4.9]). For any $f \in \mathcal{S}(G(F))$, there exists an $f^H \in \mathcal{S}(H(F))$ such that

$$J_{\text{st}}(\gamma_H, f) = \sum_\gamma \Delta_{H/G}(\gamma_H; \gamma) J(\gamma, f)$$

for any $\gamma_H \in H(F)_{\text{reg}}$. 

Our purpose here is to explain transfer of orbital integrals. It is to construct a linear map

$$\text{Span}\{J_{\text{st}}(\gamma_H) \mid \gamma_H \in H(F)_{G-\text{reg}}\} \rightarrow \mathcal{I}(G(F))$$

which maps $J_{\text{st}}(\gamma_H)$ to a scalar multiple of $J^\kappa(\gamma)$ and whose domain can be extended to $SI(H(F))$. For this, Langlands-Shelstad introduced the map $\Delta_{H/G}$ called the *transfer factor*. See [11] for details. Here we state only properties of $\Delta_{H/G}$:

(i) $\Delta_{H/G}(\gamma_H, \gamma) \neq 0 \Leftrightarrow \gamma_H$ is an image of $\gamma$;

(ii) If $\gamma_H \sim_{\text{st}} \gamma'_H$, then $\Delta_{H/G}(\gamma_H, \cdot) = \Delta_{H/G}(\gamma'_H, \cdot)$;

(iii) If $\gamma \sim \gamma'$, then $\Delta_{H/G}(\cdot, \gamma) = \Delta_{H/G}(\cdot, \gamma')$;

(iv) If $\gamma \sim_{\text{st}} \gamma'$, then $\Delta_{H/G}(\gamma_H, \gamma') = \kappa_{\gamma_H}(\text{inv}(\gamma, \gamma')) \Delta_{H/G}(\gamma_H, \gamma)$.
By this theorem, for each stable invariant distribution \( J \) of \( H \), we can define the invariant distribution \( \text{Tran}_H^G(J) \) of \( G \) by

\[
\text{Tran}_H^G(J)(f) := J(f^H).
\]

Thus we have constructed the following linear map:

\[
\text{Tran}_H^G : \mathcal{SI}(H(F)) \rightarrow \mathcal{I}(G(F)).
\]

### 3.4 Twisted endoscopy for \( U(W) \)

In this subsection, we define the Weil restriction \( \tilde{G} \) of \( G \) and compute its \( L \)-group.

First we define \( \tilde{G} \) as follows:

\[
\tilde{G}(R) := \text{Res}_{E/F} G(R) = \{ g \in \text{GL}_3(E \otimes_F E \otimes_F R) \mid g^* g = S \}.
\]

Here, \( R \) is any \( F \)-algebra and \( ^*g = (\sigma \otimes \text{id}_E \otimes \text{id}_R)(t^g) \). Then we have

\[
\tilde{G} \cong \text{Res}_{E/F} \text{GL}_3.
\]

Also \( \text{id}_E \otimes \sigma \otimes \text{id}_R \) induces an \( F \)-automorphism of \( \tilde{G} \) with order 2. We denote it by \( \varepsilon \).

We consider the structures of \( \tilde{G}(F) \) and \( \tilde{G}(\bar{F}) \). Replacing \( \bar{F} \) by \( E \) in Lemma 3.1, we have \( E \otimes_F E \cong E \oplus E \). Thus we obtain

\[
\text{GL}_3(E \otimes_F E) \cong \text{GL}_3(E) \times \text{GL}_3(E).
\]  \( \tag{4} \)

Shifting the two automorphisms \( \sigma \otimes \text{id}_E \) and \( \text{id}_E \otimes \sigma \) on the left-hand side (4) to the right-hand side, we have the following:

\[
(\sigma \otimes \text{id}_E)(g, g') = (g', g),
\]

\[
(\text{id}_E \otimes \sigma)(g, g') = (\sigma(g'), \sigma(g)).
\]

Thus the image of \( \tilde{G}(F) \) in \( \text{GL}_3(E) \times \text{GL}_3(E) \) coincides with the subgroup which consists of the elements of the form

\[
(g, S^\sigma(t^g)^{-1} S^\sigma^{-1}).
\]

Therefore \( \tilde{G}(F) \) is isomorphic to \( \text{GL}_3(E) \). Then \( \varepsilon = \text{id}_E \otimes \sigma \) can be written on \( \text{GL}_3(E) \) as follows:

\[
\varepsilon(g) = S \cdot (t^g)^{-1} \cdot S^{-1}, \quad g \in \text{GL}_3(E).
\]

As in §3.1, we have the following lemma and corollary:
Lemma 3.16. (i) The following homomorphism is isomorphism:
\[ E \otimes E \otimes \bar{F} \ni \sum a \otimes b \otimes c \mapsto \sum (abc, \sigma(a)bc, a\sigma(b)c, \sigma(ab)c) \in \bar{F} \otimes \bar{F} \otimes \bar{F}. \]

(ii) The action on \( E \otimes E \otimes \bar{F} \) of \( \text{Gal}(E/F) \times \text{Gal}(E/F) \times \Gamma_F \) is transposed on \( \bar{F} \otimes \bar{F} \otimes \bar{F} \) as follows:
\[
(\sigma \otimes \text{id}_E \otimes \text{id}_R)(x, y, z, w) = (y, x, w, z),
\]
\[
(\text{id}_E \otimes \sigma \otimes \text{id}_R)(x, y, z, w) = (z, w, x, y),
\]
\[
(\text{id}_E \otimes \text{id}_E \otimes \tau)(x, y, z, w) = \begin{cases} 
(\tau(x), \tau(y), \tau(z), \tau(w)) & \text{if } \tau \in \text{Gal}(\bar{F}/E), \\
(\tau(w), \tau(z), \tau(y), \tau(x)) & \text{if } \tau \notin \text{Gal}(\bar{F}/E).
\end{cases}
\]

Corollary 3.17. (i) The isomorphism of Lemma 3.16 (i) induces the following isomorphism:
\[ \text{GL}_3(E \otimes E \otimes \bar{F}) \cong \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}). \]

(ii) The action on \( \text{GL}_3(E \otimes E \otimes \bar{F}) \) of \( \text{Gal}(E/F) \times \text{Gal}(E/F) \times \Gamma_F \) is transposed on \( \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \) as in Lemma 3.16 (ii).

By Corollary 3.17, the image in \( \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \) of the equation \( gS^tg = S \) is
\[ (g_1, g_2, g_3, g_4)(S, S^\sigma, S, S^\sigma)(g_1^t, g_2^t, g_3^t, g_3^t) = (S, S^\sigma, S, S^\sigma). \]
Thus the image of \( \tilde{G}(\bar{F}) \) in \( \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \) coincides with the subgroup which consists of the elements of the form
\[ (g_1, S^\sigma(g_1^{-1})S^\sigma^{-1}, g_3, S^\sigma(g_3^{-1})S^\sigma^{-1}). \]
This subgroup is isomorphic to \( \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \) by the first and third projections. Consequently, it turns out that \( \tilde{G}(\bar{F}) \cong \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \).

Next we compute the \( L \)-group of \( \tilde{G} \). Let \( \tilde{B} \) be the Borel subgroup of \( \tilde{G} \) which consists of the upper-triangle matrices, and \( \tilde{T} \) the maximal torus which consists of the diagonal matrices. We fix this Borel pair \((\tilde{B}, \tilde{T})\) of \( \tilde{G} \). We identify \((\tilde{B}(\bar{F}), \tilde{T}(\bar{F}))\) with the standard Borel pair of \( \text{GL}_3(\bar{F}) \times \text{GL}_3(\bar{F}) \).

Then we write the based root datum of \( \tilde{G} \) as follows:
\[ X^*(\tilde{T}) = \bigoplus_i \mathbb{Z}\epsilon_i \oplus \bigoplus_j \mathbb{Z}\epsilon'_j, \]
\[ \epsilon_i(\text{diag}(t_1, t_2, t_3), \text{diag}(t'_1, t'_2, t'_3)) = t_i, \]
\[ \epsilon'_j(\text{diag}(t_1, t_2, t_3), \text{diag}(t'_1, t'_2, t'_3)) = t'_j, \]
\[ \Delta = \{ \alpha_i := \epsilon_i - \epsilon_{i+1}, \beta_j := \epsilon'_j - \epsilon'_{j+1} \mid 1 \leq i, j \leq 2 \}. \]
Here, we omit $X_*(\hat{T})$ and $\Delta^\vee$. From this, the Langlands dual group $\hat{G}$ of $\tilde{G}$ is $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. Also the actions of $\Gamma$ and $\varepsilon$ on this based root datum are written as follows:

$$
\tau(\varepsilon_i) = \begin{cases} 
\varepsilon_i & \text{if } \tau \in \mathrm{Gal}(\bar{F}/E), \\
-\varepsilon'_{i-1} & \text{if } \tau \notin \mathrm{Gal}(\bar{F}/E),
\end{cases}
$$

$$
\tau(\varepsilon'_j) = \begin{cases} 
\varepsilon'_j & \text{if } \tau \in \mathrm{Gal}(\bar{F}/E), \\
-\varepsilon_{i-1} & \text{if } \tau \notin \mathrm{Gal}(\bar{F}/E),
\end{cases}
$$

$$
\varepsilon(\varepsilon_i) = \varepsilon_i, \quad \varepsilon(\varepsilon'_j) = \varepsilon_j.
$$

We fix the standard Borel pair $\hat{(B, \hat{T})}$ of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. Then the dual root datum of $\tilde{G}$ is isomorphic to the root datum of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. By this isomorphism, $\Gamma$ acts on the based root datum of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. Let $X_i$ be a $3 \times 3$-matrix with $(i, i + 1)$-component one and zero elsewhere. Then

$$
\{X_i \times 0, 0 \times X_i\}_i
$$

is a splitting for $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. We have the following lemma by the similar argument in §3.1:

**Lemma 3.18.** The $L$-group of $\tilde{G}$ is as follows:

$$
L\tilde{G} = (\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})) \rtimes W_F,
$$

$$
(g_1, g_2) \rtimes w \cdot (g'_1, g'_2) \rtimes w' = \begin{cases} 
(g_1 g'_1, g_2 g'_2) \rtimes w w' & \text{if } w \in W_E, \\
(g_1 I_3, g_2 g'_2^{-1} I_3^{-1}, g_2 I_3 g'_1^{-1} I_3^{-1}) \rtimes w w' & \text{if } w \notin W_E.
\end{cases}
$$

Moreover we denote the automorphism of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ which is induced by $\varepsilon$ and retains the above splitting of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ by $\hat{\varepsilon}$. Then we have

$$
\hat{\varepsilon}(g_1, g_2) = (g_2, g_1).
$$

Next we explain twisted endoscopy of $\mathrm{GL}_3(E)$ following Ch. 3 and 4 in Rogawski [15].

An element of $\delta$ in $\tilde{G}(F)$ is said to be $\varepsilon$-semi-simple if $\delta \rtimes \varepsilon$ is semi-simple in $\tilde{G} \rtimes \langle \varepsilon \rangle$. Then we denote the set of $\varepsilon$-semi-simple elements of $\tilde{G}(F)$ by $\tilde{G}(F)_{\mathrm{ss}, \varepsilon}$. Define the $\varepsilon$-centralizer of $\delta$ by

$$
\tilde{G}_{\delta\varepsilon} = \{g \in \tilde{G} \mid g^{-1} \delta \varepsilon(g) = \delta\}.
$$

Then $\delta$ is said to be $\varepsilon$-regular semi-simple if $\tilde{G}_{\delta\varepsilon}$ is a torus. We denote the set of $\varepsilon$-regular semi-simple elements of $\tilde{G}(F)$ by $\tilde{G}(F)_{\mathrm{reg}, \varepsilon}$. For any $\delta \in \tilde{G}(F)_{\mathrm{reg}, \varepsilon}$, set

$$
D_{G, \varepsilon}(\delta) := \det(\Ad(\delta \rtimes \varepsilon) - 1|_{\tilde{g}(F)/\tilde{g}_{\delta}(F)}),
$$
where $\mathfrak{g}$ and $\mathfrak{g}_{\delta e}$ are the Lie algebras of $\tilde{G}$ and $\tilde{G}_{\delta e}$, respectively. Then we define the normalized twisted orbital integral of $\delta$ by

$$J_\varepsilon(\delta, f) := |D_{G,\varepsilon}(\delta)|_F^{1/2} \int_{\tilde{G}_{\delta e}(F) \backslash \tilde{G}(F)} f(g^{-1}\delta \varepsilon(g)) \frac{dg}{dt},$$

where $f \in \mathcal{S}(\tilde{G}(F))$. $J_\varepsilon(\delta, f)$ is convergent ([1]).

An element $\delta$ in $\tilde{G}(F)$ is said to be $\varepsilon$-conjugate to $\delta'$ if there exists $g \in \tilde{G}(F)$ such that $\delta' = g^{-1}\delta \varepsilon(g)$. Then we write $\delta \sim_{\varepsilon} \delta'$. Also $\delta$ in $\tilde{G}(F)$ is said to be stable $\varepsilon$-conjugate to $\delta'$ if there exists $g \in \tilde{G}(F)$ such that $\delta' = g^{-1}\delta \varepsilon(g)$. Then we write $\delta \sim_{\varepsilon-st} \delta'$ and define

$$\mathcal{O}_{\varepsilon-st}(\delta) := \{\delta' \in \tilde{G}(F) \mid \delta' \sim_{\varepsilon-st} \delta\} / \sim_{\varepsilon}.$$

Denote the set of stably $\varepsilon$-conjugacy classes $\mathcal{O}_{\varepsilon-st}(\delta)$ of $\varepsilon$-semi-simple elements $\delta$ by $\mathcal{O}_{\varepsilon-st}(\tilde{G})$.

As Lemma 3.4 (iv), this $\mathcal{O}_{\varepsilon-st}(\delta)$ is classified in terms of Galois cohomology as follows:

**Lemma 3.19.** $\mathcal{O}_{\varepsilon-st}(\delta) \cong \text{Ker}(H^1(F, \tilde{G}_{\delta e}) \to H^1(F, \tilde{G}))$.

**Proof.** Take an element of $[\delta'] \in \mathcal{O}_{\varepsilon-st}(\delta)$. By definition, there exists an element $g \in \tilde{G}(F)$ such that $\delta' = g^{-1}\delta \varepsilon(g)$. Then $f_{\delta'} := \{gs(g^{-1})\}_{s \in \Gamma}$ is an element of $Z^1(F, \tilde{G}_{\delta e})$. The map $[\delta'] \mapsto f_{\delta'}$ induces the bijection we desire. $\square$

By Shapiro’s lemma and Hilbert’s theorem 90, we obtain $H^1(F, \tilde{G}) \cong H^1(E, \text{GL}_n) = \{1\}$. Thus we have $\mathcal{O}_{\varepsilon-st}(\delta) \cong H^1(F, \tilde{G}_{\delta e})$.

Assume that $\delta$ is an $\varepsilon$-regular semi-simple element of $\tilde{G}(F)$. Hence $T = \tilde{G}_{\delta e}$ is a torus. Then $\mathcal{O}_{\varepsilon}(T) := H^1(F, T)$ is a group. We denote the image of $\delta' \in \mathcal{O}_{\varepsilon-st}(\delta)$ in $\mathcal{O}_{\varepsilon}(T)$ by $\text{inv}(\delta, \delta')$. Set $\mathcal{R}_\varepsilon(T) = \mathcal{O}_\varepsilon(T)^T$.

**Definition 3.20.** For $\kappa \in \mathcal{R}_\varepsilon(T)$ and $\varphi \in \mathcal{S}(\tilde{G}(F))$,

$$J_\varepsilon^\kappa(\delta, \varphi) := \sum_{[\delta'] \in \mathcal{O}_{\varepsilon-st}(\delta)} \kappa(\text{inv}(\delta, \delta')) J_\varepsilon(\delta', \varphi) \quad \text{twisted } \kappa \text{ orbital integral}$$

$$J_\varepsilon(\delta, \varphi) := \sum_{[\delta'] \in \mathcal{O}_{\varepsilon-st}(\delta)} J_\varepsilon(\delta', \varphi) \quad \text{twisted stable orbital integral}$$
**Definition 3.21.** We call \((H, s, \xi)\) a **twisted endoscopic datum** of \((\hat{G}, \varepsilon)\) if it is satisfied with the following conditions:

(i) \(H\) is a connected quasi-split group defined over \(F\);
(ii) \(s\) is an \(\hat{\varepsilon}\)-semi-simple element of \(\hat{G}\);
(iii) \(\xi : L^1 H \rightarrow L^1 G\) is a homomorphism over \(WF\);
(iv) \(\xi\) induces an isomorphism from \(\hat{H}\) to \((\hat{\tilde{G}}s\hat{\varepsilon})^0\);
(v) \(\lambda(w) := s\hat{\varepsilon}(\xi(w))s^{-1}\xi(w)^{-1}\) is a 1-coboundary from \(WF\) to \(Z(\hat{G})\).

Moreover we call a twisted endoscopic datum \((H, s, \xi)\) **elliptic** if

\[\xi(Z(\hat{H})^\Gamma)^0 \subset Z(\hat{G})^\Gamma.\]

Also two endoscopic data \((H, s, \xi)\) and \((H', s', \xi')\) are **isomorphic** if there exists \(g \in \hat{G}\) such that

\[g\xi(L^1 H)g^{-1} = \xi'(L^1 H')\]

and

\[gs\hat{\varepsilon}(g)s^{-1} \in Z(\hat{G}).\]

**Example 3.22.** We set \(H = G, s = 1, \xi : L^1 G \ni g \mapsto (g, g) \ni w \in L^1 \tilde{G}\).

Then this is an elliptic datum of \((\tilde{G}, \varepsilon)\).

**Proposition 3.23** ([15, §3.10]). (i) For each \(\delta\) in \(\hat{G}(F)_{ss, \varepsilon}\), we have that

\[N(\delta) := \delta\varepsilon(\delta)\]

belongs to \(G(E)_{ss}\).

(ii) For each \(\delta\) in \(\hat{G}(F)_{ss, \varepsilon}\), there exists an element \(\gamma\) of \(G(F)_{ss}\) such that \(\gamma\) is \(G(F)\)-conjugate to \(N(\delta)\).

(iii) \(\mathcal{N} : \mathcal{O}_{\varepsilon-st}(\hat{G}) \ni \mathcal{O}_{\varepsilon-st}(\delta) \mapsto \mathcal{O}_{st}(N(\delta)) \in \mathcal{O}_{st}(G)\) is a bijective map.

The map \(\mathcal{N} : \mathcal{O}_{\varepsilon-st}(\hat{G}) \rightarrow \mathcal{O}_{st}(G)\) is called the **norm map**. Also the element \(\gamma\) in statement (ii) of the above Proposition is called a **norm** of \(\delta\).

**Theorem 3.24** ([15, §4.10, §4.11]). For each \(\varphi\) in \(\mathcal{S}(\hat{G}(F))\), there exists a \(\varphi^G\) in \(\mathcal{S}(G(F))\) such that the following equation holds:

\[J^G_{\varepsilon}(\delta, \varphi) = J^G(\gamma, \varphi^G)\]

for any \(\delta \in \hat{G}(F)_{reg, \varepsilon}\) and any \(\gamma \in G(F)_{ss}\) such that \(\gamma\) is a norm of \(\delta\).

### 3.5 The endoscopic description for \(U(W)\)

Recall that \(G(F) = U(W)\). Also by §3.4 we identify \(\hat{G}(F)\) with \(GL_3(E)\). Then an irreducible admissible representation \(\tilde{\pi}\) of \(GL_3(E)\) is said to be \(\varepsilon\)-**stable** if \(\tilde{\pi} \circ \varepsilon \cong \tilde{\pi}\). We denote the set of irreducible \(\varepsilon\)-stable representations of
GL_3(E) by Irr GL_3(E). For \( \pi \in \text{Irr GL}_3(E) \), we denote its twisted character by \( \text{Tr}_\varepsilon(\pi) \). By Proposition 1.9 and local Langlands correspondence LLC for GL_3, we have the following injective map:

\[
\Phi(U(W)) \hookrightarrow \Phi(\text{GL}_3(E)) \overset{\text{LLC}}{\cong} \text{Irr GL}_3(E).
\]

The image of the map is contained in \( \text{Irr GL}_3(E) \). Thus we denote it by \( \text{Irr GL}_3(E) \). The first purpose of this subsection is to construct the \( \Lambda \)-packet \( \Pi_\phi \) for each \( \Lambda \)-parameter \( \phi \) of \( U(W) \).

**Definition 3.25.**

(i) For a finite subset \( \Pi \) of \( \text{Irr } U(W) \), we define

\[
\text{Tr}(\Pi) := \sum_{\pi \in \Pi} \text{Tr}(\pi).
\]

(ii) For a finite subset \( \Pi \) of \( \text{Irr } U(W) \) and \( \tilde{\pi} \in \text{Irr GL}_3(E) \), if

\[
\text{Tr}(\Pi)(\varphi^G) = \text{Tr}_\varepsilon(\tilde{\pi})(\varphi)
\]

for any \( \varphi \in \mathcal{S}(\text{GL}_3(E)) \), then we say that the character identity holds for \( \Pi \) and \( \tilde{\pi} \). We denote its relation by \( \Pi \Longleftrightarrow \tilde{\pi} \).

(iii) Let \( \tilde{\pi} \) be an element of \( \text{Irr GL}_3(E) \). If there exists \( \Pi \) such that \( \Pi \Longleftrightarrow \tilde{\pi} \), then we define this \( \Pi \) to be the \( \Lambda \)-packet associated with \( \phi \). Here \( \phi \) is the corresponding \( \Lambda \)-parameter to \( \tilde{\pi} \) under (5). Namely, \( \tilde{\pi} = \text{LLC}(\phi_E) \). We write \( \Pi_\phi := \Pi \).

This construction of an \( \Lambda \)-packet in the above (iii) is possible for each tempered \( \Lambda \)-parameter \( \phi \), that is, \( \tilde{\pi} \) is tempered. But, in general it is not possible for non-tempered case. We will use the following definition for non-tempered case.

Let \( \tilde{\pi} \) be a non-tempered element of \( \text{Irr GL}_3(E) \). Then there exists \( \lambda \in \text{Irr } E^\times \) and \( \eta \in \Pi(E^\times, 1_{F^\times}) \) such that \( \tilde{\pi} \) is the unique non-tempered irreducible subquotient of

\[
\lambda \times \eta \times \lambda^{-1} \circ \sigma := \text{Ind}_{\text{B}_3}^{\text{GL}_3(E)}(\lambda \boxtimes \eta \boxtimes \lambda^{-1} \circ \sigma).
\]

Then \( \text{Ind}_{\text{B}_3}^{U(W)}(\lambda \boxtimes \eta_a) \) has the unique irreducible non-tempered subquotient \( \pi \). Here \( B \) is the Borel subgroup of \( U(W) \) which consists of the upper-triangular matrices. We define \( \Pi_\phi := \{ \pi \} \).

Now we construct the \( \Lambda \)-packets of \( U(W) \). First the following character identity holds from the formula of the character of an induced representation.
Theorem 3.26 ([15]). For $\lambda \in \text{Irr } E^\times$ and $\eta \in \Pi(E^\times, \mathbb{I}_{F^\times})$, we have

$$\text{Ind}^U_W(\lambda \Box \eta_u) \leftrightarrow^C \lambda \times \eta \times \lambda^{-1} \circ \sigma.$$  

With this theorem, we can construct some $L$-packets of $U(W)$.

Case 1: $\text{Ind}^U_W(\lambda \Box \eta_u)$ is reducible and $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ is irreducible

Then we have $\lambda = \eta'$ ($\eta' \neq \eta' \in \Pi(E^\times, \mathbb{I}_{F^\times})$). Thus we obtain the irreducible decomposition:

$$\text{Ind}^G_B(\eta' \Box \eta_u) = \pi(\eta', \eta) \oplus \pi(\eta', \eta)^{-},$$

where $\pi(\eta', \eta)^{\pm}$ is the unique generic irreducible subrepresentation. Since the $L$-parameter of $\tilde{\pi} = \eta' \times \eta \times \eta'$ is $\eta' \oplus \eta' \oplus \eta'$, the corresponding $L$-parameter $\phi$ of $U(W)$ satisfies $\phi_E = \eta' \oplus \eta' \oplus \eta'$, that is, Case E3. By Definition 3.25 (iii), we have

$$\Pi_\phi = \{\pi(\eta', \eta)^{\pm}\}.$$  

Case 2: $\text{Ind}^U_W(\lambda \Box \eta_u)$ is irreducible and $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ is reducible

Then we have $\lambda = \eta'|_{E^{1/2}} \oplus \eta'|_{E^{-1/2}}$ (without loss of generality, we may assume $\lambda = \eta'|_{E^{1/2}}$). Thus we have the Jordan-Hölder composition:

$$\text{JH}(\eta'|_{E^{1/2}} \times \eta \times \eta'|_{E^{-1/2}}) = \left\{\{(\eta'|_{E^{1/2}}, \eta'|_{E^{-1/2}})^t \times \eta, \eta' \circ \det_{GL_2} \times \eta\right\}.$$ 

Here $\tilde{\pi} = \eta' \circ \det_{GL_2} \times \eta$ is the unique non-tempered irreducible subquotient and its $L$-parameter is $\eta'|_{E^{1/2}} \oplus \eta'|_{E^{-1/2}}$. Thus $\phi$ is in Case $E_2$ and non-tempered. Therefore we have

$$\Pi_\phi = \{\text{Ind}^U_W(\eta'|_{E^{1/2}} \Box \eta_u)\}.$$  

Case 3: $\text{Ind}^U_W(\lambda \Box \eta_u)$ is irreducible and $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ is irreducible

Then we have $\lambda \neq \eta|_{E^1}$ and $\lambda \circ N_{E/F} \neq |_{E^{1/2}}$. Also we obtain

$$\Pi_\phi = \{\text{Ind}^U_W(\lambda \Box \eta_u)\},$$

where $\phi_E = \lambda \oplus \eta \oplus \lambda^{-1} \circ \sigma$ in Case $E_2$, $E_4$ or $E_5$. 


Case 4: \( \text{Ind}_B^{U(W)}(\lambda \boxtimes \eta_u) \) is reducible and \( \lambda \times \eta \times \lambda^{-1} \circ \sigma \) is reducible

Then we have the following two sub-cases:

1. \( \lambda = \eta \|_E^1, (\eta \in \Pi(E^x, 1_{F^x})) \),
2. \( \lambda = \mu \|_E^{1/2}, (\mu \in \Pi(E^x, \omega_{E/F})) \).

Case 4-(1)

\[
\begin{align*}
JH(\eta)\|_E^{-1} \times \eta \times \eta |_E = \{ \{\eta|_E^{-1}, \eta, \eta|_E\}^t, \eta \circ \det_{GL_3}\}, \\
JH\left(\text{Ind}_B^{U(W)}(\eta)\|_E^{-1} \boxtimes \eta_u\right) = \left\{ \text{St}^{U(W)}(\eta), \eta_u \circ \det_{U(W)} \right\}.
\end{align*}
\]

Here \( \text{St}^{U(W)}(\eta) \) is the Steinberg representation of \( \text{Ind}_B^{U(W)}(\eta)\|_E^{-1} \boxtimes \eta_u \).

**Theorem 3.27 ([15]).** We have

\[
\begin{align*}
\text{St}^{U(W)}(\eta) & \overset{CI}{\longleftrightarrow} \{ \{\eta|_E^{-1}, \eta, \eta|_E\}^t \}, \\
\eta_u \circ \det_{U(W)} & \overset{CI}{\longleftrightarrow} \eta \circ \det_{GL_3}.
\end{align*}
\]

The \( L \)-parameter of \( \{ \{\eta|_E^{-1}, \eta, \eta|_E\}^t \} \) is \( \eta \boxtimes \text{Sym}^2 \). Thus we have

\[
\Pi_\phi = \{ \text{St}^{U(W)}(\eta) \},
\]

where \( \phi_E = \eta \boxtimes \text{Sym}^2 \) in Case A.

On the other hand, \( \eta_u \circ \det_{U(W)} \) and \( \eta \circ \det_{GL_3} \) are the unique non-tempered irreducible subquotient of \( \text{Ind}_B^{U(W)}(\eta)\|_E^{-1} \boxtimes \eta_u \) and \( \eta\|_E^{-1} \times \eta \times \eta |_E \), respectively. The \( L \)-parameter of \( \eta \circ \det_{GL_3} \) is \( \eta\|_E^{-1} \oplus \eta \oplus \eta |_E \). Thus we have

\[
\Pi_\phi = \{ \eta_u \circ \det_{U(W)} \},
\]

where \( \phi_E = \eta\|_E^{-1} \oplus \eta \oplus \eta |_E \) in Case E2.

Case 4-(2)

\[
\begin{align*}
JH(\text{Ind}_B^{U(W)}(\mu)\|_E^{1/2} \boxtimes \eta_u) = \{ \pi^2(\mu, \eta), \pi^{nt}(\mu, \eta) \}, \\
JH(\mu\|_E^{-1/2} \times \eta \times \mu \|_E^{1/2}) = \{ \{\mu|_E^{-1/2}, \mu|_E^{1/2}\}^t \times \eta, \mu \circ \det_{GL_2} \times \eta \}.
\end{align*}
\]

Here \( \pi^2(\mu, \eta) \) is square-integrable and \( \pi^{nt}(\mu, \eta) \) is non-tempered. Then we have the following:
Theorem 3.28 ([15]). There exists an irreducible supercuspidal representation $\pi^{sc}(\mu, \eta)$ of $G(F)$ such that

$$\{\pi^2(\mu, \eta), \pi^{sc}(\mu, \eta)\} \overset{\text{CL}}{\longleftrightarrow} \{\{\mu|_{E^{-1/2}}, \mu|_{E^{1/2}}\}\}^t \times \eta.$$

The $L$-parameter of $\langle\{\mu|_{E^{-1/2}}, \mu|_{E^{1/2}}\}\rangle^t \times \eta$ is $\mu \boxtimes \text{Sym}^1 \oplus \eta$. Thus we have

$$\Pi_\phi = \{\pi^2(\mu, \eta), \pi^{sc}(\mu, \eta)\},$$

where $\phi_E = \mu \boxtimes \text{Sym}^1 \oplus \eta$ in Case B.

On the other hand, $\pi^{nt}(\mu, \eta)$ and $\mu \circ \det_{GL_2} \times \eta$ are the unique non-tempered irreducible subquotient of $\text{Ind}_{B}^{U(W)}(\mu|_{E^{1/2}} \boxtimes \eta_a)$ and $\mu|_{E^{-1/2}} \times \eta \times \mu|_{E^{1/2}}$, respectively. The $L$-parameter of $\mu \circ \det_{GL_2} \times \eta$ is $\mu|_{E^{-1/2}} \oplus \eta \oplus \mu|_{E^{1/2}}$. Thus we have

$$\Pi_\phi = \{\pi^{nt}(\mu, \eta)\},$$

where $\phi_E = \mu|_{E^{-1/2}} \oplus \eta \oplus \mu|_{E^{1/2}}$ in Case E_2.

We have constructed the $L$-packet for each $L$-parameter of $U(W)$ in Case A, B, E_2, E_3, E_4 or E_5. Thus the remaining cases are C, D and E_1.

Theorem 3.29 ([15]). Let $\phi$ be an $L$-parameter of $U(W)$ in Case C, D or E_1. Then there exists a finite subset $\Pi_\phi$ of $\text{Irr} U(W)$ such that $\Pi_\phi \overset{\text{CL}}{\longleftrightarrow} \text{LLC}(\phi_E)$. Also $\Pi_\phi$ consists of irreducible supercuspidal representation of $U(W)$. Moreover

$$\#\Pi_\phi = \begin{cases} 
1 & \text{Case C}, \\
2 & \text{Case D}, \\
4 & \text{Case E}_1.
\end{cases}$$

Finally we have the following table:
We denote the set of $L$-packets of $U(W)$ by $\Pi(U(W))$. Here note that $\# S_\phi = \# \Pi_\phi$ for any $L$-parameter $\phi$ of $U(W)$.

The $\xi_0$ of the fixed endoscopic datum $(H, s_0, \xi_0)$ gives a map

$$\xi_0 : \Pi(H) \ni \Pi_\phi \mapsto \Pi_{\xi_0 \phi} \in \Pi(U(W)) \uparrow.$$

**Theorem 3.30** ([15]). (i) For $\eta, \eta' \in \Pi(E^\times, 1_{F^\times})$, we have

$$\xi_0(\{\eta_0 \circ \det_{U(V_{sp})} \otimes \eta_a\}) = \{\pi^u(\eta_0, \eta')\}.$$ 

(ii) For $\eta \in \Pi(E^\times, 1_{F^\times})$, we have

$$\xi_0(\{\Ind_{B(V_{sp})}^U(\eta_0^{-1}) | \otimes \eta_a\}) = \{\eta_0 \circ \det_{U(V_{sp})}\}.$$

(iii) Let $\Pi^H \in \Pi(H)$ be an $L$-packet of $H$ except for that in (i) and (ii). If $\Pi = \xi_0(\Pi^H)$, then for each $\tau \in \Pi$ there exists $\langle \Pi^H, \pi \rangle \in \{\pm 1\}$ which satisfies the following:

$$\Tr(\Pi^H) = \sum_{\pi \in \Pi} \langle \Pi^H, \pi \rangle \Tr(\pi).$$

---

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\phi_E$</th>
<th>$\overline{\pi} = \LLC(\phi_E)$</th>
<th>$\Pi_\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
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<td>$\langle \eta, \eta \rangle$</td>
<td>$\St^{U(W)}(\eta)$</td>
</tr>
<tr>
<td>Case B</td>
<td>$\mu \boxtimes \text{Sym}^1 \oplus \eta$</td>
<td>$\langle \mu, \mu \rangle \times \eta$</td>
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</tr>
<tr>
<td>Case C</td>
<td>$\phi_E$</td>
<td>$\text{irred. s.c. repre.}$</td>
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<tr>
<td>Case D</td>
<td>$\tau \oplus \eta$</td>
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</tr>
<tr>
<td>Case E</td>
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<td>$\eta_1 \times \eta_2 \times \eta_3$</td>
<td>$\pi_1, ..., \pi_4$</td>
</tr>
</tbody>
</table>

The second purpose of this subsection is to describe the inner structure of the $L$-packet for each $L$-parameter of $U(W)$. That is achieved by endoscopy.
We describe the inner structure of an $L$-packet of $U(W)$ through an example. We consider an $L$-parameter $\phi$ of $U(W)$ in Case E$_1$. Namely, we put $\phi_E = \eta_1 \oplus \eta_2 \oplus \eta_3$. Then we have

$$S_\phi = \left\{ \begin{pmatrix} \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 \end{pmatrix} \right\}/\{\pm 1_3\} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$ 

Thus the number of elements of $L$-packet $\Pi_\phi$ is four. We take a complete system of representatives of $S_\phi$ as follows:

$s_0 = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, $s_1 = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, $s_2 = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$, $s_3 = 1_3$.

Then we put $\mathcal{H}_i = (\hat{G}_s)^0 \text{Im} \phi$. Assume $i \neq 3$. We take $w_i \in \hat{G}$ such that $\text{Ad}(w_i)\xi(\mathbb{L}^H) \subset \mathcal{H}_i$ and $\text{Ad}(w_i)(s) = s_i$. For example,

$w_0 = 1_3$, $w_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Then for each $i \neq 3$, there exists a unique $L$-parameter $\phi_i^H$ of $H$ such that the following diagram commutes:

If we put $\Pi_i^H := \Pi_{\phi_i^H} \in \Pi(H)$, then we have $\xi(\Pi_i^H) = \Pi_\phi$. Thus by Theorem 3.30 (iii), we have

$$\text{Tran}_H^G(\text{Tr}(\Pi_i^H)) = \sum_{\pi \in \Pi_\phi} \langle \Pi_i^H, \pi \rangle \text{Tr}(\pi).$$

Then we can define the local pairing $\langle s_i, \pi \rangle \in \{\pm 1\}$ by

$\langle s_i, \pi \rangle := \begin{cases} \langle \Pi_i^H, \pi \rangle & \text{if } i = 0, 1, 2, \\ 1 & \text{if } i = 3. \end{cases}$
Theorem 3.31 ([12]). The map $\Pi_{\phi} \ni \pi \mapsto \langle \cdot , \pi \rangle \in \text{Irr } S_{\phi}$ is a bijective map.

Note that this theorem holds for any $L$-parameter $\phi$ of $U(W)$ except for that in Theorem 3.30 (i) and (ii). Now the following corollary is trivial:

**Corollary 3.32.** For any $\pi \in \Pi_{\phi}$, we have $\prod_s \langle s, \pi \rangle = 1$, where $s$ runs the non-trivial elements of $S_\phi$.

Theorem 3.31 was written in [12]. The original form of this theorem in [15], which we will use later, is given as follows. First we define

$$\hat{\Pi}_{\phi} := \{ \rho \in \Phi(H) \mid \xi_0 \circ \rho \sim \phi \}.$$ 

For any $\rho \in \hat{\Pi}_{\phi}$, there exist $s \in S_{\phi} \setminus \{1\}$ and $w \in \text{GL}_3(\mathbb{C})$ such that $\text{Ad}(w) \circ \xi_0 \circ \rho = \phi$ and $\text{Ad}(w)(s_0)$ equals $s$ in $S_{\phi}$. Then $\hat{\Pi}_{\phi}$ corresponds to $S_{\phi} \setminus \{1\}$ bijectively. We set $\langle \rho, \cdot \rangle := \langle s, \cdot \rangle$.

**Theorem 3.33 ([15]).** Let $\pi_1, \pi_2$ be two elements of $\Pi_{\phi}$. If $\langle \rho, \pi_1 \rangle = \langle \rho, \pi_2 \rangle$ for any $\rho \in \hat{\Pi}_{\phi}$, then $\pi_1 = \pi_2$.

Also the following corollary is trivial by Cor. 3.32:

**Corollary 3.34.** For any $\pi \in \Pi_{\phi}$, we have $\prod_{\rho} \langle \rho, \pi \rangle = 1$, where $\rho$ runs the elements of $\hat{\Pi}_{\phi}$.

### 3.6 Global endoscopy for $U(W)$

In this subsection, we explain the global multiplicity formula for certain global endoscopic $L$-packets.

Let $\hat{E}/\hat{F}$ be a quadratic extension of a number field. Take a non-zero element $\hat{\xi}$ of $\hat{E}$ with $\text{Tr}_{\hat{E}/\hat{F}}(\hat{\xi}) = 0$ and a non-trivial character $\hat{\psi}$ of $A_{\hat{F}}$. Let

$$\hat{W} = (\hat{E}^{\mathbb{H}^3}, \begin{pmatrix} \hat{\xi} & 1 \\ -1 & \hat{\xi} \end{pmatrix}).$$

be a three-dimensional skew hermitian space over $\hat{E}$. We denote its unitary group by $U(\hat{W})$. 
Take characters \( \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3 \in \Pi(\mathbb{A}_E^\times, \omega_{\tilde{E}/F}) \) and \( \tilde{\mu}_0 \in \Pi(\mathbb{A}_E^\times, 1) \). For any place \( v \) of \( \tilde{F} \), define an \( L \)-parameter \( \tilde{\phi}_v \) of \( \mathrm{U}(\tilde{W}_v) \) by \( \tilde{\phi}_v,_{E_v} = \tilde{\eta}_{1,v} \oplus \tilde{\eta}_{2,v} \oplus \tilde{\eta}_{3,v} \).

Set \( \tilde{\phi} = \{ \tilde{\phi}_v \}_v \). Then we define the global endoscopic \( L \)-packet associated to \( \tilde{\phi} \) by

\[
\Pi_{\tilde{\phi}} := \left\{ \bigotimes_v \pi_v \bigg| \pi_v \in \Pi_{\tilde{\phi}_v}, \pi_v : \text{unramified for almost all } v \right\}.
\]

We call \( \Pi_{\tilde{\phi}} \) a cuspidal \( L \)-packet if it has an cuspidal representation of \( \mathrm{U}(\tilde{W}) \).

**Theorem 3.35 ([15]).** The set \( \Pi_{\tilde{\phi}} \) is a cuspidal \( L \)-packet of \( \mathrm{U}(\tilde{W}) \) if and only if \( \tilde{\eta}_i \neq \tilde{\eta}_j \) for \( i \neq j \).

To check whether each element of \( \Pi_{\tilde{\phi}} \) is cuspidal, we define \( \hat{\Pi}_{\tilde{\phi}} = \{ \rho_1, \rho_2, \rho_3 \} \), where for any place \( v \), \( \rho_{i,v} \) is an \( L \)-parameter of \( H_v \) and

\[
\begin{align*}
\rho_1 &= \{ \rho_{1,v} \}_v, \\
\rho_{1,v} &= \tilde{\mu}_0^{-1}(\tilde{\eta}_{2,v} \oplus \tilde{\eta}_{3,v}) \times \tilde{\eta}_{1,v}, \\
\rho_2 &= \{ \rho_{2,v} \}_v, \\
\rho_{2,v} &= \tilde{\mu}_0^{-1}(\tilde{\eta}_{1,v} \oplus \tilde{\eta}_{3,v}) \times \tilde{\eta}_{2,v}, \\
\rho_3 &= \{ \rho_{3,v} \}_v, \\
\rho_{3,v} &= \tilde{\mu}_0^{-1}(\tilde{\eta}_{1,v} \oplus \tilde{\eta}_{2,v}) \times \tilde{\eta}_{3,v}.
\end{align*}
\]

Here \( \rho_{i,v} \) is the restriction of \( \rho_{i,v} \) to \( L_{E_v} \). Then set \( \langle \rho, \pi \rangle = \Pi_v(\rho_v, \pi_v) \) for each \( \pi \in \Pi_{\tilde{\phi}} \) and \( \rho \in \hat{\Pi}_{\tilde{\phi}} \).

**Theorem 3.36 ([15]).** Let \( \pi = \bigotimes_v \pi_v \) be an element of \( \Pi_{\tilde{\phi}} \). If \( m(\pi) \) is the multiplicity of \( \pi \) in \( L^2(\mathrm{U}(\tilde{W}) \setminus \mathrm{U}(\tilde{W})(\mathbb{A})) \), then

\[
m(\pi) = \begin{cases} 
1 & \text{if } \langle \rho, \pi \rangle = 1 \text{ for all } \rho \in \hat{\Pi}_{\tilde{\phi}}, \\
0 & \text{otherwise}.
\end{cases}
\]
4 Theta lift

4.1 Mixed model

In this subsection, we retain the notation in §1.1.

To explain the explicit formulas of the mixed model of the Weil representation of a unitary dual pair, we prepare a few notation.

Let \( V = (E^\oplus m, R) \) be a hermitian space and \( W = (E^\oplus n, S) \) a skew-hermitian space over \( E \). We may identify \( W \) with \( M_{m,n}(E) \) by the isomorphism

\[
V \otimes_E W \ni v \otimes w \mapsto v \cdot w \in M_{m,n}(E).
\]

Then the symplectic form on \( W \) is given by

\[
\langle X, X' \rangle = \text{Tr}_{E/F}(X R X' S) \quad X, X' \in W.
\]

Without loss of generality, we may assume that

\[
V = (E^\oplus m, \begin{pmatrix} R_1 & 1_{m_0} \\ 1_{m_0} & \end{pmatrix}).
\]

We put \( V_0 = (E^\oplus 2m_0, \begin{pmatrix} 1_{m_0} & \\ & 1_{m_0} \end{pmatrix}) \) and \( V_1 = (E^\oplus m_1, R_1) \) where \( m_1 = m - 2m_0 \). We take \( X \) as the subspace of \( V \) consisting of the elements of the form

\[
t(x_1, \ldots, x_{m_0}, 0, \ldots, 0)
\]

and \( X' \) as the subspace of \( V \) consisting of the elements of the form

\[
t(0, \ldots, 0, x_{m-m_0+1}, \ldots, x_m).
\]

Thus we have

\[
V = X \oplus V_1 \oplus X'.
\]

Let \( P_{X'} = M_{X'} N_{X'} \) be the maximal parabolic subgroup of \( U(V) \) stabilizing \( X' \), where \( M_{X'} \) is the Levi component of \( P_{X'} \) stabilizing \( X \) and \( N_{X'} \) is the
unipotent radical of $P_X$. We have

$$M_{X'} = \left\{ (m(a, h)) = \begin{pmatrix} a & h \\ -a^{-1} & 1 \end{pmatrix} \right\} \begin{array}{c} a \in \text{GL}_{m_0}(E) \\ h \in U(V_1) \end{array},$$

$$N_{X'} = \left\{ (n(b, c)) = \begin{pmatrix} 1_{m_0} & 1_{m_1} \\ b - \frac{1}{2} bR_1 b & -bR_1 \end{pmatrix} \right\} \begin{array}{c} b \in M_{m_1, m_0}(E) \\ c = -c \in M_{m_0}(E) \end{array}.$$

We denote the space of Schwartz-Bruhat functions on $X = X \otimes_E W = M_{m_0, n}(E)$ by $S(X)$. Take an irreducible unitary representation $(\tau_{\psi, V_1, W}, S_{V_1, W})$ of $H(\mathbb{W}_1)$, where $\mathbb{W}_1 = V_1 \otimes_E W = M_{m_1, m_0}(E)$. Then the Weil representation $\omega_{\psi, V_1, W, V_1}$ is defined on $S(X) \otimes S_{V_1, W}$ and has the following explicit formulas:

**Theorem 4.1.** Let $\chi_{V_0}, \chi_{V_1}$ be characters of $E^\times$ such that $\chi_V = \chi_{V_0} \chi_{V_1}$ and $\chi_{V_1}|_{F^\times} = \omega_{E/F}^{\text{dim}_E V_1}$. For $\phi \otimes \phi' \in S(X) \otimes S_{V_1, W}$, we have

$$\omega_{\psi, V_1, W, V_1} (m(a, h)) \phi(x) \otimes \phi' = \chi_W (\det a) |\det a|_E^{-n/2} \phi(a^{-1} x) \cdot \omega_{\psi, V_1, W, V_1} (h) \phi',$$

$$\omega_{\psi, V_1, W, V_1} (n(b, c)) \phi(x) \otimes \phi' = \psi \left( -\frac{1}{2} \text{Tr}_{E/F}(x c^{\text{ad}} S) \right) \phi(x) \cdot \tau_{\psi, V_1, W} (-bx, 0) \phi',$$

$$\omega_{\psi, V_1, W, V_1} (g) \phi(x) \otimes \phi' = (\chi_{V_0})_u (\det g) \phi(x g) \cdot \omega_{\psi, V_1, W, V_1} (g) \phi'.$$

Here $x \in X$, $m(a, h) \in M_{X'}$, $n(b, c) \in N_{X'}$ and $g \in U(W)$.

Similarly assume that

$$W = (E^{\oplus n}, \begin{pmatrix} S_1 \\ -1_{n_0} \end{pmatrix}).$$

We put $W_0 = (E^{\oplus 2n_0}, \begin{pmatrix} 1_{n_0} \\ -1_{n_0} \end{pmatrix})$ and $W_1 = (E^{\oplus n_1}, S_1)$ where $n_1 = n - 2n_0$. We take $Y$ as the subspace of $W$ consisting of the elements of the form

$$(y_1, \ldots, y_{n_0}, 0, \ldots, 0).$$
and $Y'$ as the subspace of $W$ consisting of the elements of the form

$$(0, \ldots, 0, y_{n-n_0+1}, \ldots, y_n).$$

Thus we have

$$W = Y \oplus W_1 \oplus Y'.$$

Let $P_{Y'} = M_{Y'} N_{Y'}$ be the maximal parabolic subgroup of $U(W)$ stabilizing $Y'$, where $M_{Y'}$ is the Levi component of $P_{Y'}$ stabilizing $Y$ and $N_{Y'}$ is the unipotent radical of $P_{Y'}$. We have

$$M_{Y'} = \left\{ m(a, g) = \begin{pmatrix} a & g \\ g & *a^{-1} \end{pmatrix} \middle| a \in \text{GL}_{n_0}(E), g \in U(W_1) \right\},$$

$$N_{Y'} = \left\{ n(b, c) = \begin{pmatrix} 1_{n_0} & b & c + \frac{1}{2} bS_1^*b \\ 1_{n_1} & S_1^*b \\ 1_{n_0} \end{pmatrix} \middle| b \in \text{M}_{n_0,n_1}(E), c = *c \in \text{M}_{n_0}(E) \right\}.$$

We set $Y = V \otimes_E Y = M_{m,n_0}(E)$. Take an irreducible unitary representations $(\tau_{\psi,V,W_1}, S_{V,W_1})$ of $H(V_1)$, where $V_1 = V \otimes_E W_1 = M_{m,n_1}(E)$.

**Theorem 4.2.** Let $\chi_{W_0}, \chi_{W_1}$ be characters of $E^\times$ such that $\chi_W = \chi_{W_0} \chi_{W_1}$ and $\chi_{W_1}|_{F^x} = \omega_{E^F/W_1}$. For $\phi \otimes \phi' \in S(Y) \otimes S_{V,W_1}$, we have the following explicit formulas:

$$\omega_{\psi,V,W_0,W_1} \chi_W, \chi_{V,W_1}(h) \phi(y) \otimes \phi' = (\chi_{W_0})(\det h) \phi(h^{-1}y) \cdot \omega_{\psi,V,W_1,W_1}(h) \phi',$$

$$\omega_{\psi,V,W_0,W_1}(m(a, g)) \phi(y) \otimes \phi' = \chi_V(\det a) |\det a|^{m/2}_E \phi(ya) \cdot \omega_{\psi,V,W_1,W_1}(g) \phi',$$

$$\omega_{\psi,V,W_0,W_1}(n(b, c)) \phi(y) \otimes \phi' = \psi \left( \frac{1}{2} \text{Tr}_{E/F}(\ast y Ryc) \right) \phi(y) \cdot \tau_{\psi,V,W_1}(yb, 0) \phi'.$$

Here $y \in Y$, $m(a, g) \in M_{Y'}$, $n(b, c) \in N_{Y'}$ and $h \in U(V)$.

Finally we state the following lemma:

**Lemma 4.3.** (i) For $g \in GU(V)$ with similitude norm $d$, we have

$$\omega_{\psi,d,V,W_0,W_1} \approx \omega_{\psi,V,W_0,W_1} \circ \text{Ad}(g) \approx \omega_{\psi,dV,W_0,W_1}.$$
Here $dV = (E^\otimes m, dA)$ and $U(dV) = U(V)$.

(ii) For any character $\eta, \eta' \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we have

$$\omega_{\psi,V_{\eta}^\times W_{\eta'},V_{\eta'}^\times} = (\eta_u \circ \det_{U(V)} \boxtimes \eta'_u \circ \det_{U(W)}) \otimes \omega_{\psi,V_{\eta}^\times W_{\eta'}}.$$

4.2 Local theta lift for $U(V_{sp}) \times U(W_a)$

In this subsection we compute the local theta lift for $U(V_{sp}) \times U(W_a)$.

We denote by $B$ the Borel subgroup of $U(V_{sp})$ consisting of lower triangular matrices. Let

$$U = \left\{ u(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in E, \text{Tr}_{E/F}(x) = 0 \right\}$$

be the unipotent radical of $B$ and

$$T = \left\{ m(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in E^\times \right\}$$

a maximal torus of $U(V_{sp})$. Thus we have $B = TU$.

From Theorem 4.1, the Weil representation $\omega_{\psi,V_{sp}^\times W_a^\eta}$ for $U(V_{sp}) \times U(W_a)$ on the space $S(E)$ of Schwartz-Bruhat functions on $E$ has the following explicit formulas:

$$\omega_{\psi,V_{sp}^\times W_a^\eta} (m(\alpha)) f(t) = \mu(\alpha) |\alpha|_{E}^{-1/2} f(\alpha^{-1} t),$$
$$\omega_{\psi,V_{sp}^\times W_a^\eta} (u(x)) f(t) = \psi(a \xi x N_{E/F}(t)) f(t),$$
$$\omega_{\psi,V_{sp}^\times W_a^\eta} (g) f(t) = \eta_u(g) f(tg).$$

Here $f \in S(E)$, $t \in E$, $m(\alpha) \in T$, $u(x) \in U$ and $g \in U(W_a)$. For brevity, we write $\omega_{\psi} = \omega_{\psi,V_{sp}^\times W_a^\eta}$.

First we compute (twisted) Jacquet module of $\omega_{\psi}$. For each $b \in E^\times$, we set

$$E_b = \left\{ t \in E^\times \mid N_{E/F}(t) = b N_{E/F}(\xi^{-1}) \right\}.$$

One notes that if $b \notin N_{E/F}(E^\times)$, then $E_b$ is empty.

Lemma 4.4. (i) The unnormalized Jacquet module $(\omega_{\psi})_U$ of the Weil representation $\omega_{\psi}$ is isomorphic to $\mu| \cdot |_{E}^{-1/2} \boxtimes \eta_u$ as a $T \times U(W_a)$-module, where

$$\mu| \cdot |_{E}^{-1/2} \boxtimes \eta_u : T \times U(W_a) \ni m(\alpha) \times g \mapsto \mu(\alpha)|\alpha|_{E}^{-1/2} \eta_u(g) \in \mathbb{C}^\times.$$
(ii) The twisted Jacquet module $(\omega_\psi)_{U,\psi_b^U,\xi}$ of the Weil representation $\omega_\psi$ is isomorphic to $\eta_u \otimes S(E_{a-1}b)$ as a $U(W_a)$-module, where the action of $U(W_a)$ on $S(E_{a-1}b)$ is given by right translation.

Proof. (i) If we define
$$\omega_\psi(U) = \{\omega_\psi(u(x))f - f \mid x \in F, f \in S(E)\},$$
then $(\omega_\psi)_U = \omega_\psi / \omega_\psi(U)$. By the above explicit formulas, we have $\omega_\psi(U) = \{f \in S(E) \mid f(0) = 0\}$. Thus the map $f \mapsto f(0)$ induces a $T \times U(W_a)$-isomorphism $(\omega_\psi)_U \cong \mu \cdot |E|^{-1/2} \otimes \eta_u$.

(ii) If we define
$$\omega_\psi(U, \psi_b^U, \xi) = \{\omega_\psi(u(x))f - \psi(-bx\xi^{-1})f \mid x \in F, f \in S(E)\},$$
then $(\omega_\psi)_{U,\psi_b^U,\xi} = \omega_\psi / \omega_\psi(U, \psi_b^U, \xi)$. By the above explicit formulas, we have $\omega_\psi(U, \psi_b^U, \xi) = \{f \in S(E) \mid f|_{E_{a-1}b} = 0\}$. Thus the map $f \mapsto f|_{E_{a-1}b}$ induces a $U(W_a)$-isomorphism $(\omega_\psi)_{U,\psi_b^U,\xi} \cong \eta_u \otimes S(E_{a-1}b)$.

Corollary 4.5. (i) For each $\eta'_u \in \text{Irr} U(W_a)$, the local theta lift $\theta_{\psi,V^u_{sp},W^a_v}(\eta'_u)$ is $\psi^a$-generic and not $\psi^{ad_0}$-generic.

(ii) If $\eta \neq \eta'$, then the local theta lift $\theta_{\psi,V^u_{sp},W^a_v}(\eta'_u)$ is an irreducible supercuspidal representation of $U(V_{sp})$.

(iii) If $\eta = \eta'$, then the local theta lift $\theta_{\psi,V^u_{sp},W^a_v}(\eta_u)$ is the irreducible $\psi^a$-generic subrepresentation $\tau(\mu,\mu)_{\omega_{E/F}(a)}^{\omega_{E/F}(a)} \otimes \text{Ind}_{B_\mu}^{U(V_{sp})} \mu$.

Proof. By the explicit formulas, we have a non-zero $U(W_a)$-homomorphism $\omega_\psi \to \eta'_u$. Thus we have $\tau = \theta_{\psi,V^u_{sp},W^a_v}(\eta'_u) \neq 0$. By the definition of the local theta lift, we have a surjective $U(V_{sp}) \times U(W_a)$-homomorphism
$$\omega_\psi \to \tau \otimes \eta'_u.$$

Taking its twisted Jacquet module with respect to $\psi_b^U,\xi$, we have a surjective $U(W_a)$-homomorphism
$$\eta_u \otimes S(E_{a-1}b) \to \tau_{U,\psi_b^U,\xi} \otimes \eta'_u.$$
Thus if $b = ad_0$, then $\tau_{U,\psi_b^U,\xi}$ is zero. Namely, $\tau$ is not $\psi^{ad_0}$-generic.
We assume that $\eta' \neq \eta$. Taking the Jacquet module of $\omega_\psi$ with respect to $U$, we have $\tau_U = 0$. Thus $\tau$ is supercuspidal. Here, it is well-known that $\tau$ is $\psi^a$-generic or $\psi^{ad_0}$-generic. Thus $\tau$ is $\psi^a$-generic.

Next we assume that $\eta' = \eta$. By Frobenius reciprocity, we have

$$\text{Hom}_{U(V_{sp}) \times U(W_a)}(\omega_\psi, \text{Ind}_B^U(V_{sp}) \mu \boxtimes \eta_u) \cong \text{Hom}_{\mathcal{T} \times U(W_a)}((\omega_\psi)_U, |E^{1/2} \boxtimes \eta_u) \cong \mathbb{C}.$$ 

Thus $\tau$ is an irreducible subrepresentation of $\text{Ind}_B^U(V_{sp}) \mu$, that is, $\tau = \tau(\mu, \mu)_{sp}^{\omega_{E/F}(a)}$. Thus we obtain Corollary 4.5.

**Theorem 4.6.** For each $\eta'_u \in \text{Irr } U(W_a)$, the local theta lift $\theta_{\psi, V_{sp}, W_a}(\eta'_u)$ is the unique $\psi^a$-generic element of $L$-packet $\Pi_{\mu, \mu, \eta, \eta'}^{\omega_{E/F}(a)}$ of $U(V_{sp})$. Namely,

$$\theta_{\psi, V_{sp}, W_a}(\eta'_u) = \tau(\mu, \mu^{\omega_{E/F}(a)}_{sp}).$$

**Proof.** We set $\tau = \theta_{\psi, V_{sp}, W_a}(\eta'_u)$. It is enough to show that $\tau$ is contained in $\Pi_{\mu, \mu, \eta, \eta'}^{\omega_{E/F}(a)}$. We have already proved it for $\eta = \eta'$. Thus we consider the case for $\eta \neq \eta'$. Our proof uses Theorem 2.6. We take the following:

- $\bar{E}/\bar{F}$ is a global quadratic extension;
- finite place $v_0$ such that $\bar{E}_{v_0}/\bar{F}_{v_0} = E/F$;
- $\bar{a} \in \bar{F}^\times$ such that $\bar{a} \equiv a \mod N_{E/F}(E^\times)$;
- $\bar{\xi} \in \bar{E}^\times$ such that $\text{Tr}_{E/F}(\bar{\xi}) = 0$ and $\bar{\xi} \equiv \xi \mod N_{E/F}(E^\times)$;
- $\bar{\psi}$ is a non-trivial character of $\mathbb{A}_F/\bar{F}$ such that $\bar{\psi}_{v_0} = \psi$;
- $\bar{\mu} \in \Pi(\mathbb{A}_{E/F}^\times, \omega_{E/F})$ such that $\bar{\mu}_{v_0} = \mu$;
- $\bar{\eta} \neq \bar{\eta}' \in \Pi(\mathbb{A}_{E/F}^\times, 1)$ such that $\bar{\eta}_{v_0} = \eta, \bar{\eta}'_{v_0} = \eta'$.

Now we define two (skew) hermitian spaces over $\bar{E}$:

$$\bar{V}_{sp} = (\bar{E}^{\otimes 2}, \left( \begin{array}{ccc} 1 & & \\ & 1 & \\
\end{array} \right)) \text{ and } W_{\bar{a}} = (\bar{E}, \bar{a} \bar{\xi}).$$

Then the global theta lift $\bar{\tau} = \theta_{\psi, \bar{V}_{sp}, W_{\bar{a}}}(\bar{\eta}'_u)$ is non-zero and cuspidal, since the constant term of $\bar{\tau}$ is zero and its $\bar{\psi}$-Whittaker model is non-zero by
the explicit formulas of the Weil representation of $\omega_{\tilde{\psi}, \tilde{V}, \tilde{W}, \tilde{a}}$. Thus $\tilde{\tau}$ is an irreducible cuspidal representation. If we show the claim that the local theta lift $\tilde{\tau}_v$ of $\tilde{\eta}_{u,v}$ is contained in $\Pi_{\tilde{\mu}_v, \tilde{\eta}_v, \tilde{\eta}'_{u,v}}$ for almost all places $v$ of $\tilde{F}$, then by Theorem 2.6 we obtain Theorem 4.6.

First suppose that a field extension $\tilde{E}_v/\tilde{F}_v$ and $\tilde{\eta}_v, \tilde{\eta}'_v$ are unramified. Then the claim immediately follows from $\tilde{\eta}_v = \tilde{\eta}'_v$.

Next suppose that $\tilde{E}_v/\tilde{F}_v$ is split. We omit the symbol $v$ from the notation. Then we have $\tilde{E}_v \cong \tilde{F} \oplus \tilde{F}$, $U(V_{sp}) \cong GL_2(\tilde{F})$ and $U(W_{\tilde{a}}) \cong GL_1(\tilde{F})$. Also its Weil representation $(\omega_{\tilde{\psi}}, S(\tilde{F} \oplus \tilde{F}))$ is given as follows:

\[
\omega_{\tilde{\psi}}(h) f(t) = \tilde{\mu}(\det h) |\det h|^{1/2} f(h^{-1} t),
\]
\[
\omega_{\tilde{\psi}}(g) f(t) = \tilde{\eta}(g) |g|_{\tilde{F}} f(t g).
\]

Here $f(t) \in S(\tilde{F} \oplus \tilde{F})$, $h \in GL_2(\tilde{F})$ and $g \in GL_1(\tilde{F})$. Then we want to show that $\tilde{\tau} = Ind(\tilde{\mu} \tilde{\eta}^{-1} \boxtimes \tilde{\mu})$. Take a quasi-character $\chi$ of $\tilde{F}^\times$. For an element $f \in S(\tilde{F} \oplus \tilde{F})$, we define a function $\Phi_f$ on $GL_2(\tilde{F}) \times GL_1(\tilde{F})$ by

\[
\Phi_f(h, g) = \frac{1}{L(0, \chi)} \int_{\tilde{F}^\times} \omega_{\tilde{\psi}}(h, g) f \left( \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right) \chi(t) dt^\times.
\]

By the above explicit formulas, the map $f \mapsto \Phi_f$ induces a non-zero $GL_2(\tilde{F}) \times GL_1(\tilde{F})$-homomorphism

\[
\omega_{\tilde{\psi}} \rightarrow \text{Ind}(\tilde{\mu} \chi | \cdot |^{-1}_{\tilde{F}} \boxtimes \tilde{\mu}) \boxtimes \tilde{\eta}^{-1} \chi | \cdot |_{\tilde{F}}.
\]

If $\chi = \tilde{\eta} \tilde{\eta}'^{-1} | \cdot |_{\tilde{F}}$, then we have

\[
\omega_{\tilde{\psi}} \rightarrow \text{Ind}(\tilde{\mu} \tilde{\eta}^{-1} \boxtimes \tilde{\mu}) \boxtimes \tilde{\eta}'.
\]

Since $\tilde{\mu}, \tilde{\eta}$ and $\tilde{\eta}'$ are unitary, $\text{Ind}(\tilde{\mu} \tilde{\eta}^{-1} \boxtimes \tilde{\mu})$ is irreducible. Thus we have $\tilde{\tau} = \text{Ind}(\tilde{\mu} \tilde{\eta}^{-1} \boxtimes \tilde{\mu})$. 

$\square$
5 Endoscopy and local theta lift

5.1 Endoscopy and local theta lift for $U(V) \times U(W)$

In this subsection we recall some results in [4] about the relation between endoscopy and local theta lift of $U(V) \times U(W)$, where $V = V_{sp}, V_{an}$.

First we define the subset of irreducible representations occurring in Weil representation $\omega_{V', W'}$ of $U(V) \times U(W)$. For $V = V_{sp}$ or $V_{an}$, we set

$$R_{\psi, \mu, \eta}(V) := \{\tau \in \text{Irr } U(V) | \theta_{\psi, V', W'}(\tau) \neq 0\}.$$

On the other hand we set

$$R_{\psi, \mu, \eta}(W) := \{\pi \in \text{Irr } U(W) | \theta_{\psi, V', W'}(\pi) \neq 0, V = V_{sp} \text{ or } V_{an}\}.$$

We remark that the definition of $R_{\psi, \mu, \eta}(W)$ here is slightly different from that in §1.1. The map defined by the local theta lifts $\theta_{\psi, V', W'}$ with respect to $V = V_{sp}, V_{an}$ is denoted as follows:

$$\theta_{\psi, \mu, \eta} : R_{\psi, \mu, \eta}(V_{sp}) \sqcup R_{\psi, \mu, \eta}(V_{an}) \to R_{\psi, \mu, \eta}(W).$$

Theorem 5.1. (i) The map $\theta_{\psi, \mu, \eta}$ is bijective.

(ii) For $\tau \in R_{\psi, \mu, \eta}(V)$ with $L$-parameter $\varphi$, the $L$-parameter of $\theta_{\psi, \mu, \eta}(\tau)$ is $\mu \eta \hat{\varphi}_E \oplus \eta$, where $\hat{\varphi}_E$ is the contragredient representation of $\varphi_E$.

Proof. (i) The injectivity follows from the Dichotomy Theorem [4, Theorem 1.2]. The surjectivity follows from the definition of the map.

(ii) This was proved in [4, §4].

The set $R_{\psi, \mu, \eta}(V)$ is described by the following:

Theorem 5.2 ([4, Lemma 4.2]). For $\tau \in \text{Irr } U(V)$, the local theta lift $\theta_{\psi, \mu, \eta}(\tau)$ is zero if and only if $\theta_{\psi, V', W'}(\tau) \neq 0$, where $W_1 = (E, \xi)$.

To describe $R_{\psi, \mu, \eta}(W)$, we prepare a few notation. Let $\phi$ be an $L$-parameter of $U(W)$. An $L$-packet $\Pi_\phi$ of $U(W)$ is called $(\chi)$-endoscopic if it consists of infinite-dimensional representations and a one-dimensional representation $\chi$ of $W_E$ is a subrepresentation of $\phi_E$. An element of such $\Pi_\phi$ is called $(\chi)$-endoscopic.

Theorem 5.3 ([4, §4]). The set $R_{\psi, \mu, \eta}(W)$ consists of the $\eta$-endoscopic representations and $\eta_\mu \circ \det$. 
The local theta lift \( \theta_{\psi,\mu,\eta} \) has the following property:

**Theorem 5.4 ([4]).** For \( \tau \in \mathcal{R}_{\psi,\mu,\eta}(V) \), \( \theta_{\psi,\mu,\eta}(\tau) \) is generic if and only if \( V = V_{sp}, \tau \) is \( \psi \)-generic and \( \tau \neq \text{Ind}_B U(V_{sp}) \mu \cdot |E|^1 \).

**Remark 5.5.** The proofs of the above theorems in [4] do not require the endoscopic description of \( L \)-packets of \( U(V_{an}) \). But Theorem 2.7 (v) on the cardinality of an \( L \)-packet requires (see [4, p.430]).

Let \( \phi \) be an \( L \)-parameter of \( U(W) \) such that \( \Pi_{\phi} \) is endoscopic. Then Gelbart-Rogawski-Soudry defined a local pairing \( \epsilon_{\rho}(\pi) \) for \( \rho \in \hat{\Pi}_{\phi} \) and \( \pi \in \Pi_{\phi} \) as follows. \( \rho \) is an \( L \)-parameter of \( H_0 = U(V_{sp}) \times U(1) \). Thus we denote its \( U(1) \)-part by \( \rho_\eta \). The character of \( E^{\times}/F^{\times} \) corresponding to \( \rho_\eta \) is denoted by \( \eta_\rho \). Thus \( \Pi_{\phi} \) is an \( \eta_\rho \)-endoscopic \( L \)-packet. Then by Theorem 5.1 and Theorem 5.3, there exists a unique \( V_{\pi,\rho} = V_{sp} \) or \( V_{an} \) for \( \pi \in \Pi_{\phi} \) such that \( \theta_{\psi,V_{sp},W_{\eta\rho}}(\pi) \neq 0 \). Here \( V_{\pi,\rho} \) is independent of the choices of characters \( \psi, \mu \) by Lemma 4.3. Thus we can define the local pairing \( \epsilon_{\rho}(\pi) := + \) if \( V_{\pi,\rho} = V_{sp} \) and \( - \) otherwise. Namely, we have the following definition:

**Definition 5.6.** For \( \rho \in \hat{\Pi}_{\phi} \) and \( \pi \in \Pi_{\phi} \),

\[
\epsilon_{\rho}(\pi) := \begin{cases} + & \text{if } \theta_{\psi,V_{sp},W_{\eta\rho}}(\pi) \neq 0, \\ - & \text{if } \theta_{\psi,V_{an},W_{\eta\rho}}(\pi) \neq 0. \end{cases}
\]

The following theorem gives the relation between the endoscopic description for \( U(W) \) and the local theta lift \( \theta_{\psi,\mu,\eta} \).

**Theorem 5.7 ([4, Theorem 3.1]).** For \( \rho \in \hat{\Pi}_{\phi} \) and \( \pi \in \Pi_{\phi} \), we have \( \langle \rho, \pi \rangle = \epsilon_{\rho}(\pi) \).

Also the following holds:

**Proposition 5.8 ([4]).** Let \( \pi \) be an element of an endoscopic \( L \)-packet \( \Pi_{\phi} \). Then \( \pi \) is generic if and only if \( \epsilon_{\rho}(\pi) = + \) for any \( \rho \in \hat{\Pi}_{\phi} \).

### 5.2 Local theta lift for \( U(V_{sp}) \times U(W) \)

Take an irreducible admissible representation \( \tau \) of \( U(V_{sp}) \) with \( L \)-parameter \( \phi \). Set \( \pi = \theta_{\psi,\mu,\eta}(\tau) \). If \( \pi \) is non-zero, then its \( L \)-parameter \( \phi \) is given by \( \phi_E = \mu \eta \tilde{\phi}_E \oplus \eta \). Then we want to compute the local pairing \( \epsilon_{\rho}(\pi) \) for \( \rho \in \hat{\Pi}_{\phi} \). In the following cases we can deduce this from the above results.
**Proposition 5.9.** Let $\rho_0$ be the element of $\hat{\Pi}_\phi$ such that $\rho_{0,E} = \mu_0^{-1}\eta\mu \hat{\phi}_E \times \eta$.

(i) If $\tau$ is $\psi$-generic, the local pairing $\epsilon_\rho(\pi) = +$ for any $\rho \in \hat{\Pi}_\phi$.

(ii) If $\varphi_E$ is an irreducible representation of $L_E$, we have $\hat{\Pi}_\phi = \{\rho_0\}$ and $\epsilon_{\rho_0}(\pi) = +$.

(iii) If $\tau$ is one dimensional, we have $\hat{\Pi}_\phi = \{\rho_0\}$ and $\epsilon_{\rho_0}(\pi) = +$.

(iv) If $\tau = \tau(\mu_1,\mu_1^-)_{sp}$ $(\mu_1 \in \Pi(E^\times,\omega_{E/F}))$, then $\tau = 0$.

(v) If $\tau = \tau(\mu_1,\mu_2^-)_{sp}$ $(\mu \neq \mu_1,\mu_2 \in \Pi(E^\times,\omega_{E/F}))$, then the cardinality of $\hat{\Pi}_\phi$ is two or three and the local pairing $\epsilon_{\rho_0}(\pi) = +$ and $\epsilon_\rho(\pi) = -$ for any $\rho \neq \rho_0, \in \hat{\Pi}_\phi$.

**Proof.** (i) If $\tau = \text{Ind}_B^U(V_{sp}) \mu \cdot |E|^{-1}$, then $\tau = \eta_u \circ \det$. This case is trivial. Otherwise $\tau$ is generic by Theorem 5.4. Thus this follows from Proposition 4.10.

(ii), (iii) By Theorem 5.1, the first is trivial. The second follows from $V_{\pi,\rho_0} = V$.

(iv) By Theorem 4.6, we have

$$\theta_{\psi,0,V_{sp}^{W_1},W_1^{W_1}}(\tau(\mu_1,\mu)_{sp})$$

$$= \theta_{\psi,V_{sp}^{W_1},W_1^{W_1}}(\tau(\mu_1,\mu)^{sp})$$

$$= (\mu_1\mu)^{-1}_{sp}$$

$$\neq 0.$$  

Thus it follows from Theorem 5.2 that $\tau = 0$.

(v) $\tau$ is not generic by Theorem 5.4. Moreover it is trivial that $\epsilon_{\rho_0}(\pi) = +$.

Assume that $\mu_1 \neq \mu_2$. Then we have $\hat{\Pi}_\phi = \{\rho_0,\rho_1,\rho_2\}$, where

$$\rho_{0,E} = \mu_0^{-1}\eta_\mu(\mu_1^{-1} \oplus \mu_2^{-1}) \times \eta,$$

$$\rho_{1,E} = \mu_0^{-1}(\eta \oplus \mu_1\mu^{-1}_2) \times \mu_1\mu^{-1}_1,$$

$$\rho_{2,E} = \mu_0^{-1}(\mu_1\mu^{-1}_1 \oplus \eta) \times \mu_1\mu^{-1}_2.$$  

By Cor. 3.34, we have $\epsilon_{\rho_0}(\pi)\epsilon_{\rho_1}(\pi)\epsilon_{\rho_2}(\pi) = +$. Since $\epsilon_{\rho_0}(\pi) = +$, we obtain $\epsilon_{\rho_1}(\pi)\epsilon_{\rho_2}(\pi) = +$. If $\epsilon_{\rho_1}(\pi) = +$ and $\epsilon_{\rho_2}(\pi) = +$, then $\tau$ is generic by Proposition 5.8. This is a contradiction. Therefore we have $\epsilon_{\rho_1}(\pi) = \epsilon_{\rho_2}(\pi) = -$.

We can also prove for the case $\mu_1 = \mu_2$ by the similar argument. \[\square\]
6 Main results

6.1 Setting

Take an irreducible admissible representation $\tau$ of $U(V_{an})$ with L-parameter $\varphi$. Set $\pi = \theta_{\psi, \mu, \eta}(\tau)$. If $\pi$ is non-zero, then its L-parameter $\phi$ is given by $\phi_E = \mu \eta \tilde{\varphi}_E \oplus \eta$. Then we want to compute the local pairing $\epsilon_\rho(\pi)$ for $\rho \in \hat{\Pi}_\phi$.

**Lemma 6.1.** Let $\rho_0$ be the element of $\hat{\Pi}_\phi$ such that $\rho_{0,E} = \mu_0^{-1} \eta \mu \tilde{\varphi}_E \times \eta$. If $\varphi_E$ is an irreducible representation of $L_E$, then we have $\hat{\Pi}_\phi = \{\rho_0\}$ and $\epsilon_{\rho_0}(\pi) = +$.

**Proof.** This can be proved by the similar argument in Proposition 5.10 (ii). \hfill \Box

Thus we may assume that $\varphi_E$ is an reducible representation of $L_E$. Namely, there exist $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$ such that $\varphi_E = \mu_1 \oplus \mu_2$. Then $\tau = \tau(\mu_1, \mu_2)$ and the L-parameter $\phi$ of $\pi$ satisfies $\phi_E = \mu \eta \mu_1^{-1} \oplus \mu \eta \mu_2^{-1} \oplus \eta$ by Theorem 5.1.

6.2 Two lemmas

We take the following:

- $\tilde{F}$ is a number field;
- finite place $v_0$ such that $\tilde{F}_{v_0} = F$;
- infinite place $v_1$ such that $\tilde{F}_{v_1} = \mathbb{R}$;
- $\tilde{E}$ is a quadratic extension of $\tilde{F}$ such that $\tilde{E}_{v_0} = E, \tilde{E}_{v_1} = \mathbb{C}$;
- $\tilde{\xi} \in \tilde{E}^\times$ such that $\text{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and $\tilde{\xi} \equiv \xi \mod N_{E/F}(E^\times)$;
- $\tilde{\psi}$ is a non-trivial character of $A_{\tilde{F}/\tilde{F}}$ such that $\tilde{\psi}_{v_0} = \psi$;
- $\tilde{d} \in \tilde{F}$ such that $\tilde{d} \notin N_{E_v/F_v}(\tilde{E}_v^\times)$ at any place $v \neq v_0, v_1$;
- $\tilde{\mu}_1 \neq \tilde{\mu}_2 \in \Pi(A_{\tilde{F}/\tilde{F}}, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{i,v_0} = \mu_i (i = 1, 2)$ and $\tilde{\mu}_{1,v_1} \neq \tilde{\mu}_{2,v_1}$.

Define a two-dimensional hermitian space $\tilde{V}$ over $\tilde{E}$:

$$\tilde{V} = (\tilde{E}^\oplus 2, \begin{pmatrix} -\tilde{d} & \vdots \\ \vdots & 1 \end{pmatrix}).$$
By the above condition, we have \( \tilde{V}_{v_0} \cong V_{an} \) and the signature of \( \tilde{V}_{v_1} \) is \((2, 0)\). Also at any place \( v \neq v_0, v_1 \), the group \( U(\tilde{V}_v) \) is a quasi-split unitary group or general linear group.

The following lemma is used in §6.3.

**Lemma 6.2.** Let the notation be as above. Take \( \tilde{\tau} = \otimes \tilde{\tau}_v \in \Pi_{\mu_1, \mu_2}(\tilde{V}) \) such that \( \tilde{\tau}_v \) is \( \psi_v \)-generic for all \( v \neq v_0, v_1 \). Then \( \tilde{\tau} \) is cuspidal if and only if \( \tilde{\tau}_{v_0} = \tau(\mu_1, \mu_2)_{an} \).

**Proof.** This follows from Theorem 2.12. \( \square \)

**Remark 6.3.** We can take \( \tilde{\psi}, \tilde{\xi}, \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) so that \( C_{v_1} = + \).

To prove our main results, we also prepare the following lemma with respect to the representation \( \rho_{m,n} \) of \( U(\tilde{V}_{v_1}) \):

**Lemma 6.4.** For brevity, we omit \( v_1 \). Then \( \theta_{\tilde{\psi}, \tilde{\mu}_1, \tilde{W}_1}(\rho_{m,n}) \neq 0 \) if and only if \( C_{v_1} = + \), where \( \tilde{W}_1 = (C, \tilde{\xi}) \).

**Proof.** From [6], we recall explicit formulas of Fock model of the Weil representation \( \omega_{\tilde{\psi}} = \omega_{\tilde{\psi}, \tilde{\mu}_1, \tilde{W}_1} \) for \( U(\tilde{V}) \times U(\tilde{W}_1) \). First the signature of \( \tilde{V} \) is \((2, 0)\) so that we may assume that \( \tilde{v}_0 = \chi_{m+n} \). Then the complexification of the Lie algebra of \( U(\tilde{W}_1) \) is \( M_2(C) \). We write \( E_{j,k} \) for the \((j, k)\)-elementary matrix. Also we may assume \( \tilde{\xi} = \pm i \). Since the complexification of the Lie algebra of \( U(\tilde{W}_1) \) is \( C \), we write \( 1_{\tilde{W}_1} \) for its neutral element. Next the Fock space of the Weil representation \( \omega_{\tilde{\psi}} \) is \( C[w_1, w_2] \). Since \( \tilde{\mu}_1 = \chi^{m+n} \), the explicit formulas are given as follows:

1. \( \text{sgn}(r\tilde{\xi}/i) = + \)

\[
\omega_{\tilde{\psi}}(E_{j,k}) = \frac{m + n - 1}{2} \delta_{j,k} - w_k \frac{\partial}{\partial w_j}
\]

\[
\omega_{\tilde{\psi}}(1_{\tilde{W}_1}) = 1 + w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}
\]

2. \( \text{sgn}(r\tilde{\xi}/i) = - \)

\[
\omega_{\tilde{\psi}}(E_{j,k}) = \frac{m + n + 1}{2} \delta_{j,k} + w_j \frac{\partial}{\partial w_k}
\]

\[
\omega_{\tilde{\psi}}(1_{\tilde{W}_1}) = -1 - w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}
\]
By those formulas, we can describe the theta lift \(\theta_{\tilde{\psi}, \tilde{V}, \tilde{W}}\). In the case (1), the theta lift \(\theta_{\tilde{\psi}, \tilde{V}, \tilde{W}}\) is given by

\[
\chi_1^l \mapsto \rho_{m+n-l, l} \quad (l \geq 1).
\]

Here \(\chi_1 : \mathbb{C}^1 \ni z \mapsto z \in \mathbb{C}^1\) is a character of \(U(\tilde{W})\). Thus \(\theta_{\tilde{\psi}, \tilde{V}, \tilde{W}}(\rho_{m,n}) \neq 0\) if and only if \(n \geq 1\). This is equivalent to \(C_{v_1} = +\).

In the case (2), the theta lift is given by

\[
\chi_1^l \mapsto \rho_{m+n-l, l} \quad (l \leq -1).
\]

Thus \(\theta_{\tilde{\psi}, \tilde{V}, \tilde{W}}(\rho_{m,n}) \neq 0\) if and only if \(n \leq -1\). This is equivalent to \(C_{v_1} = +\).

Let 

\[
\tilde{W} = \left( \tilde{E}^{\mathbb{E}, 3}, \begin{pmatrix} \xi & 1 \\ -1 & \xi \end{pmatrix} \right).
\]

be a three-dimensional skew Hermitian space over \(\tilde{E}\). Also we assume that \(C_{v_1} = +\). By Lemma 6.4, the local theta lift \(\theta_{\tilde{\psi}, \tilde{V}, \tilde{W}}(\rho_{m,n})\) of \(\rho_{m,n}\) to \(U(\tilde{W})\) is non-zero. We denote this representation of \(U(\tilde{W})\) by \(\pi_{m,n}\).

### 6.3 Main results

Now we prove our main results. First we consider the local theta lift of some endoscopic representations of \(U(V_{an})\) to \(U(W)\).

**Theorem 6.5.** For \(\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})\), we have

\[
\theta_{\tilde{\psi}, \tilde{V}, \tilde{W}}(\pi(\mu_1, \mu_2)_{an}^+ \circ \tau(\mu_1, \mu_2)_{an}^-) = \pi(\eta, \eta \mu_1 \mu_2^{-1})^{-}. 
\]

Here \(\pi(\eta, \eta \mu_1 \mu_2^{-1})^{-}\) is the unique irreducible non-generic subrepresentation of \(\Ind_{B}^{U(W)}(\eta \otimes (\eta \mu_1 \mu_2^{-1})_{an})\), where \(B\) is the Borel subgroup of all upper triangular matrices in \(U(W)\).

**Proof.** Note that \(\omega_{\psi, \tilde{V}_{an}, \tilde{W}} = \eta_a \circ \det(U(W)) \otimes \omega_{\psi, \tilde{V}_{an}, \tilde{W}}\) by Lemma 4.3. Thus we may assume \(\eta = 1\). Therefore it is enough to show that

\[
\theta_{\psi, \tilde{V}_{an}, \tilde{W}}(\pi(1, \mu_1 \mu_2^{-1})^-) = \tau(\mu_1, \mu_2)_{an}^+.
\]
For brevity, we write \( \theta_{\psi,V} = \theta_{\psi,V^{\rho_1,W'}} \) and \( \pi^- = \pi(1, \mu_1 \mu_2^{-1}) \).

First we show that \( \theta_{\psi,V_{an}}(\pi^-) \in \Pi_{\mu_1, \mu_2}(V_{an}) \). Since \( \pi^- \) is endoscopic, by Theorem 5.1 there exists a unique \( V = V_{sp}, V_{an} \) such that \( \theta_{\psi,V}(\pi^-) \neq 0 \). By Theorems 5.1 and 5.4, \( \theta_{\psi,V}(\pi^-) \) is a non \( \psi \)-generic representation in \( \Pi_{\mu_1, \mu_2}(V) \). If \( V = V_{sp} \), then we have \( \theta_{\psi,V}(\pi^-) = \tau(\mu_1, \mu_2)_{sp} \). But by Proposition 5.10 (iv), \( \theta_{\psi,V_{sp}}(\tau(\mu_1, \mu_2)_{sp}) = 0 \). This is a contradiction. Thus we have \( V = V_{an} \).

Next we show that \( \theta_{\psi,V_{an}}(\pi^-) = \tau(\mu_1, \mu_2)_{an}^+ \). To show it, we use the notation and the assumption in \( \S 6.2 \). We also take a non-trivial character \( \tilde{\eta} \in \Pi(\tilde{A}_E^\times, 1) \) such that \( \tilde{\eta} \neq \tilde{\mu}_1 \tilde{\mu}_2^{-1} \) and \( \tilde{\eta}_v = 1 \) at \( v = v_0, v_1 \). Then we define a global \( L \)-parameter \( \tilde{\phi} \) of \( U(W) \) by \( \tilde{\phi}_E = 1 \otimes \tilde{\eta} \otimes \tilde{\mu}_1 \tilde{\mu}_2^{-1} \), where \( \tilde{\phi}_E \) is the restriction of \( \tilde{\phi} \) to the Weil group of \( E \). Thus we have the local \( L \)-packet \( \Pi_{\tilde{\phi}_v} \) of \( U(W_v) \) for each \( v \). A global \( L \)-packet \( \Pi_{\tilde{\phi}} \) is defined by

\[
\left\{ \bigotimes \pi_v \ \big| \ \pi_v \in \Pi_{\tilde{\phi}_v}, \pi_v \text{ is generic for almost all } v \right\}.
\]

We can take the element \( \tilde{\pi} \in \Pi_{\tilde{\phi}} \) such that \( \tilde{\pi}_{v_0} = \pi^- \), \( \tilde{\pi}_{v_1} = \pi_{m,n} \), and \( \tilde{\pi}_v \) is generic at \( v \neq v_0, v_1 \). It is a cuspidal representation. To see this, we take a character \( \tilde{\mu}_0 \in \Pi(\tilde{A}_E^\times, \omega_{\bar{E}/\bar{F}}) \) such that \( \tilde{\mu}_{0,v} = \mu_0 \), where \( \mu_0 \) is in the introduction. We define \( \tilde{\Pi}_{\tilde{\phi}} \) as in \( \S 3.6 \). Then we have \( \tilde{\Pi}_{\tilde{\phi}} = \{ \rho_0, \rho_1, \rho_2 \} \), where

\[
\rho_{0,E} = \tilde{\mu}_0^{-1}\tilde{\eta} \otimes \tilde{\mu}_1 \tilde{\mu}_2^{-1} \times 1,
\rho_{1,E} = \tilde{\mu}_0^{-1}(1 \otimes \tilde{\mu}_1 \tilde{\mu}_2^{-1}) \times \tilde{\eta},
\rho_{2,E} = \tilde{\mu}_0^{-1}(1 \otimes \tilde{\eta}) \times \tilde{\mu}_1 \tilde{\mu}_2^{-1}.
\]

By Theorem 3.36, we must show that \( \langle \rho_i, \tilde{\pi} \rangle := \Pi_v \langle \rho_i,v, \tilde{\pi}_v \rangle = + \) for each \( i = 0, 1, 2 \). Since \( \tilde{\pi}_v \) is generic for \( v \neq v_0, v_1 \), we have \( \langle \rho_i,v, \tilde{\pi}_v \rangle = + \) for \( i = 0, 1, 2 \) ([4, Prop. 3.3]). Also for \( v = v_0, v_1 \), we have \( \langle \rho_0,v, \tilde{\pi}_v \rangle = \epsilon_{\rho_0,v}(\tilde{\pi}_v) = - \). Thus we obtain \( \langle \rho_0, \tilde{\pi} \rangle = + \). Since \( \rho_{0,v} = \rho_1,v \) for \( v = v_0, v_1 \), we also obtain \( \langle \rho_1, \tilde{\pi} \rangle = + \). Moreover since \( \langle \rho_0, \tilde{\pi} \rangle \langle \rho_1, \tilde{\pi} \rangle \langle \rho_2, \tilde{\pi} \rangle = + \) by [4, p.454], we have \( \langle \rho_2, \tilde{\pi} \rangle = + \). Thus \( \tilde{\pi} \) is a cuspidal representation.

By [4, Theorem 1.1 (b)], there exists a unique two-dimensional hermitian space \( V \) over \( \tilde{E} \) such that the global theta lift \( \tilde{\tau} = \theta_{\psi,V^{\rho_1,W'},\tilde{\pi}}(\tilde{\pi}) \neq 0 \). Also the global theta lift \( \tilde{\tau} \) is an irreducible cuspidal representation of \( U(\tilde{V}) \) and is included in \( \Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}) \). Since \( \tilde{\pi}_v \) is generic at \( v \neq v_0 \) and \( v_1 \), \( V_v \) is split, that is, \( V_v \cong V_v \). On the other hand, we have already proved above that \( V_{v_0} \cong V_{an} \). Moreover it follows from the definition of \( \pi_{m,n} \) that \( V_{v_1} \cong V_{v_1} \).
Thus we have $V = \tilde{V}$. Here $\tilde{\tau}_v$ is $\tilde{\psi}_v$-generic for $v \neq v_0, v_1$ and $\tilde{\tau}_{v_1} = \rho_{m,n}$. By Lemma 6.2, we obtain $\tilde{\tau}_{v_0} = \tau(\mu_1, \mu_2)_{\tilde{\mu}_an}^+$. Namely, $\theta_{\psi, V_{an}}(\tau^-) = \tau(\mu_1, \mu_2)_{an}^{-}$.

To compute the local theta lift for $U(V_{an}) \times U(W_a)$ from Theorem 6.5, we need the following lemma:

**Lemma 6.6.** Let $N$ be the unipotent radical of the Borel subgroup $B$ of $U(W)$. Take a characters $\eta' \in \Pi(E^\times, \mathbb{1}_F^\times)$ such that $\eta' \neq \eta$.

(i) The Jacquet module $(\omega_{\psi, V^\mu_{an}, W^n})_N$ of the Weil representation $\omega_{\psi, V^\mu_{an}, W^n}$ of $U(V_{an}) \times U(W)$ is isomorphic to $\eta|_E \boxtimes \omega_{\psi, V^\mu_{an}, W^n}$ as an $E^\times \times U(V_{an}) \times U(W^1)$-module.

(ii) If $\tau_{\eta'} := \theta_{\psi, V^\mu_{an}, W^n}(\eta'') \neq 0$, then $\theta_{\psi, V^\mu_{an}, W^n}(\tau_{\eta'}) = \pi(\eta, \eta')^-$.

(iii) We have $\tau_{\eta} = 0$, that is, $\theta_{\psi, V^\mu_{an}, W^n}(\eta_{an}) = 0$.

**Proof.** (i) By Theorem 4.2, we have the explicit formulas of the mixed model of the Weil representation $\omega_{\psi, V^\mu_{an}, W^n}$ of $U(V_{an}) \times U(W)$. Then (i) follows from the similar argument in the proof of Lemma 4.4.

(ii) Since $\tau_{\eta'} \neq 0$, we have a surjective homomorphism

$$\omega_{\psi, V^\mu_{an}, W^n} \to \tau_{\eta'} \boxtimes \eta'_{an}.$$

By (i) and Frobenius reciprocity, we obtain a non-zero $U(V_{an}) \times U(W)$-homomorphism

$$\omega_{\psi, V^\mu_{an}, W^n} \to \tau_{\eta'} \boxtimes \text{Ind}_B^{U(W)}(\eta \boxtimes \eta'_{an}).$$

Thus $\theta_{\psi, V^\mu_{an}, W^n}(\tau_{\eta'})$ is an irreducible subquotient of $\text{Ind}_B^{U(W)}(\eta \boxtimes \eta'_{an})$. Since $\eta' \neq \eta$, $\text{Ind}_B^{U(W)}(\eta \boxtimes \eta'_{an})$ has the unique non-generic irreducible subrepresentation $\frac{\pi(\eta, \eta')^-}{\pi(\eta, \eta')}$. Also by Theorem 5.4, $\theta_{\psi, V^\mu_{an}, W^n}(\tau_{\eta'})$ is non-generic. Thus (ii) holds.

(iii) Assume $\tau_{\eta} \neq 0$. By the similar argument, we have $\theta_{\psi, V^\mu_{an}, W^n}(\tau_{\eta})$ is an irreducible non-generic subquotient of $\text{Ind}_B^{U(W)}(\eta \boxtimes \eta_{an})$. But $\text{Ind}_B^{U(W)}(\eta \boxtimes \eta_{an})$ is a generic irreducible representation. This is a contradiction. Thus we have $\tau_{\eta} = 0$.

Now we can compute the local theta lift for $U(V_{an}) \times U(W_a)$.

**Theorem 6.7.** For $\eta' \in \Pi(E^\times, \mathbb{1}_F^\times)$, we have

$$\theta_{\psi, V^\mu_{an}, W^n}(\eta'_{an}) = \begin{cases} \tau(\mu_1, \mu_2)_{\tilde{\mu}_an}^{\omega_{E/F}(a)} & \text{if } \eta \neq \eta', \\ 0 & \text{if } \eta = \eta'. \end{cases}$$
Proof. Take $g \in \mathrm{GU}(V_{an})$ with similitude norm $a$. Then by Lemma 4.3, we have

$$\theta_{\psi, V_{an}}(\eta'_u) = \theta_{\psi, V_{an}}(\eta'_u) \circ \mathrm{Ad}(g).$$

Thus we may assume $a = 1$ by Corollary 2.8 (iv).

By Lemma 6.6, we consider only $\eta \neq \eta'$. We show that

$$\theta_{\psi, V_{an}}(\tau(\mu, \mu \eta')^{-1}) = \eta'_u.$$

By Theorem 6.5, we obtain

$$\theta_{\psi, V_{an}}(\tau(\mu, \mu \eta')^{-1}) = \pi(\eta, \eta').$$

Since $\tau(\mu, \mu \eta')^{-1}$ is supercuspidal and $\pi(\eta, \eta')$ is not supercuspidal, we have $\theta_{\psi, V_{an}}(\tau(\mu, \mu \eta')^{-1}) = 0$. Thus there exists a unique character $\eta'' \in \Pi(E, \mathbb{I}_{E/F})$ such that $\theta_{\psi, V_{an}}(\tau(\mu, \mu \eta')^{-1}) = \eta''$. By Lemma 6.6 (ii), we obtain $\eta'' = \eta'$. Therefore $\eta'' = \eta'$.

Corollary 6.8. We have

$$\theta_{\psi, V_{an}}(\tau(\mu_1, \mu_2)) = 0.$$

Proof. By Theorem 6.4,

$$\theta_{\psi, V_{an}}(\tau(\mu_1, \mu_2)) = \pi(\eta, \eta') = \pi(\eta, \eta').$$

Thus this corollary follows from Theorem 5.2.

Finally, we compute the local theta lift of the remaining endoscopic representations of $U(V_{an})$ to $U(W)$. Take characters $\mu_1, \mu_2 \in \Pi(E, \mathbb{I}_{E/F})$. We assume that $\mu, \mu_1, \mu_2$ are distinct. Then by Theorem 5.2 we have

$$\pi^\pm := \theta_{\psi, V_{an}}(\tau(\mu_1, \mu_2)^\pm) = 0.$$

Also the $L$-parameter $\phi$ of $\pi^\pm$ satisfies $\phi_E = \mu \eta \mu_1^{-1} \oplus \mu \eta \mu_2^{-1} \oplus \eta$. Thus we have $\Pi_\phi = \{\rho_0, \rho_1, \rho_2\}$, where

$$\rho_0 = \mu_0^{-1}(\mu \eta \mu_1^{-1} \oplus \mu \eta \mu_2^{-1} \oplus \eta),$$

$$\rho_1 = \mu_0^{-1}(\eta \oplus \mu \eta \mu_2^{-1}) \times \mu \eta \mu_1^{-1},$$

$$\rho_2 = \mu_0^{-1}(\mu \eta \mu_1^{-1} \oplus \eta) \times \mu \eta \mu_2^{-1}.$$
\textbf{Theorem 6.9.} We have for \( \varepsilon \in \{ \pm \} \),
\[
\begin{align*}
\epsilon_{\rho_0}(\pi^\varepsilon) &= -, \\
\epsilon_{\rho_1}(\pi^\varepsilon) &= -\varepsilon, \\
\epsilon_{\rho_2}(\pi^\varepsilon) &= \varepsilon.
\end{align*}
\]

\textit{Proof.} By Definition 5.7, we have \( \epsilon_{\rho_0}(\pi^\varepsilon) = - \). By Cor. 3.34, we also have \( \epsilon_{\rho_1}(\pi^\varepsilon) \epsilon_{\rho_2}(\pi^\varepsilon) = + \).

Thus we obtain \( \epsilon_{\rho_1}(\pi^\varepsilon) \epsilon_{\rho_2}(\pi^\varepsilon) = - \). Since these \( \epsilon_{\rho_i} \) distinguish \( \pi^\pm \), it is enough to show the theorem for \( \varepsilon = + \).

Let \( \tilde{\tau} \) be the cuspidal representation of \( U(\tilde{V}) \) in Lemma 6.2. Take the following:

- \( \tilde{\eta} \in \Pi(A_E^\times, \mathbb{I}) \) such that \( \tilde{\eta}_0 = \eta, \tilde{\eta}_v = 1 \);
- \( \tilde{\mu} \in \Pi(A_E^\times, \omega_E/F) \) such that \( \tilde{\mu}_v = \mu, \tilde{\mu}_v = \tilde{\mu}_1, \).

Since we assume \( C_{v_1} = + \), the local theta lift \( \theta_{\tilde{\psi}, \tilde{V}_v, \tilde{W}_{\tilde{V}_v}}(\tilde{\tau}_v) \neq 0 \) for each \( v \).

Thus by [4, Theorem 5.1], the global theta lift \( \tilde{\pi} := \theta_{\tilde{\psi}, \tilde{V}, \tilde{W}^V}(\tilde{\tau}) \) is a non-zero cuspidal representation of \( U(\tilde{V}) \). Note that \( \tilde{\pi}_{v_0} = \pi^+ \) and \( \tilde{\pi}_{v_1} = \pi_{m,n} \).

We take a character \( \tilde{\mu}_0 \in \Pi(A_E^\times, \omega_E/F) \) such that \( \tilde{\mu}_{v_0} = \mu_0 \). Then we define \( \tilde{\Pi}_\phi = \{ \tilde{\rho}_0, \tilde{\rho}_1, \tilde{\rho}_2 \} \) as follows:

\[
\begin{align*}
\rho_{0,E} &= \tilde{\mu}_0^{-1} \tilde{\mu} \tilde{\eta}(\tilde{\mu}_1^{-1} \oplus \tilde{\mu}_2^{-1}) \times \tilde{\eta}, \\
\rho_{1,E} &= \tilde{\mu}_0^{-1}(\tilde{\eta} \oplus \tilde{\mu} \tilde{\eta}^{-1}) \times \tilde{\mu} \tilde{\eta}^{-1}, \\
\rho_{2,E} &= \tilde{\mu}_0^{-1}(\tilde{\mu} \tilde{\eta}^{-1} \oplus \tilde{\eta}) \times \tilde{\mu} \tilde{\eta}^{-1}.
\end{align*}
\]

Since \( \tilde{\pi} \) is cuspidal, we have \( \langle \rho_i, \tilde{\pi} \rangle = + \) for any \( i = 0, 1, 2 \).

Now \( \tilde{\tau}_v \) is \( \tilde{\psi}_v \)-generic for \( v \neq v_0, v_1 \). Thus \( \tilde{\pi}_v \) is generic. By Proposition 5.8, we obtain \( \langle \rho_{i,v}, \tilde{\tau}_v \rangle = \epsilon_{\rho_{i,v}}(\tilde{\pi}_v) = + \) for any \( i \).

By the choices of \( \tilde{\eta}, \tilde{\mu} \), we have \( \rho_{0,v_1} = \rho_{1,v_1} \). Thus we obtain \( \langle \rho_{0,v_1}, \tilde{\pi}_{v_1} \rangle = \langle \rho_{0,v_1}, \tilde{\pi}_{v_1} \rangle = \epsilon_{\rho_{0,v_1}}(\pi_{m,n}) = - \).

Thus we have \( \epsilon_{\rho_{1,v_1}}(\tilde{\pi}_{v_1}) = - \) and \( \epsilon_{\rho_{2,v_1}}(\pi^+) = + \). \( \square \)
7 Local theta lift for a quaternionic dual pair
\[ U_D(1) \times U_E(1) \]

7.1 Quaternionic unitary groups

Let \( D \) be the quaternion division algebra over \( F \) with the main involution \( \iota \). We denote its reduced norm and trace by \( \nu_D, \text{tr}_D \), respectively.

First we consider (skew) hermitian spaces over \( D \) and the quaternion unitary groups. Let \( V_D \) be a hermitian space over \( D \). Namely, \( V_D \) is a right \( D \)-vector space with a hermitian form \( \langle , \rangle \). Let \( W_D \) be a skew-hermitian space over \( D \). Then \( W_D \) is a left \( D \)-vector space with a skew-hermitian form \( \langle , \rangle \). The spaces \( V_D \) and \( W_D \) may be taken as follows:

\[
V_D = (D^{\oplus m}, A), \quad W_D = (D^{\oplus n}, B),
\]

\[
(v_1, v_2) = *v_1 Av_2, \quad \langle w_1, w_2 \rangle = w_1 B^* w_2.
\]

Here \( A = *A := {}^t A^\iota \in \text{GL}_m(D) \) and \( B = -*B \in \text{GL}_n(D) \). Then the quaternion unitary groups of \( V_D \) and \( W_D \) are given by

\[
U(V_D) = \{ h \in \text{GL}_m(D) \mid {}^* h Ah = A \},
\]

\[
U(W_D) = \{ g \in \text{GL}_n(D) \mid gB^* g = B \}.
\]

Here \( U(V_D) \) (resp. \( U(W_D) \)) acts on \( V_D \) (resp. \( W_D \)) on the left (resp. right).

In this paper, we assume that \( \dim V_D = 1 \) and \( \dim W_D = 1 \). Then we have the following results:

**Lemma 7.1** ([16]). (i) Any one-dimensional hermitian space \( V_D \) is isomorphic to \( (D, 1) \).

(ii) For any one-dimensional skew-hermitian space \( W_D \), there exist a quadratic extension \( E/F \) in \( D \) and \( \xi \in E^\times \) such that \( \text{Tr}_{E/F}(\xi) = 0 \) and \( W_D \cong (D, \xi) \).

Thus we assume that \( V_D = (D, 1) \) and \( W_D = (D, \xi) \). Then we write

\[
U_D(1) := U(V_D) \quad \text{and} \quad U_E(1) := U(W_D).
\]

It is easy to check that

\[
U_D(1) = \text{Ker} \nu_D \supset U_E(1) = \text{Ker} N_{E/F}.
\]
Finally we consider a relation between $V_D$ and $V_{an}$. Since $D$ is a quaternion division algebra, there exists $\xi' \in D$ such that $\text{tr}_D(\xi') = 0$, $\xi' \xi = -\xi' \xi$ and

$$D = F \oplus F \xi \oplus F \xi' \oplus F \xi' \xi = E \oplus \xi' E.$$

Then we may assume that $\xi'^2 = d_0$. We can consider $V_D$ as a two-dimensional hermitian space over $E$. The following lemma follows from easy computations.

**Lemma 7.2.** (i) The $E$-linear map $i : V_D \ni z + \xi' w \mapsto \langle w, z \rangle \in V_{an}$ is an isomorphism as hermitian spaces over $E$.

(ii) We denote the homomorphism induced from $i$ by $I : U_D(1) \to U(V_{an})$. Then

$$I : U_D(1) \ni z + \xi' w \mapsto \begin{pmatrix} \sigma(z) \\ d_0 \sigma(w) \end{pmatrix} \in U(V_{an})$$

is an injective homomorphism and its image is $SU(V_{an})$.

By this lemma, we identify $U_D(1)$ with $SU(V_{an})$. Then $U_E(1)$ is identified with a subgroup consisting of diagonal matrices of $U(V_{an})$ by $U_E(1) \ni \gamma \mapsto \begin{pmatrix} \gamma^{-1} \\ \gamma \end{pmatrix} \in U(V_{an})$.

### 7.2 The Weil representation for $U_D(1) \times U_E(1)$

For $V_D$ and $W_D$, we have the Weil representation $\omega_{\psi, V_D, W_D}$ for $U_D(1) \times U_E(1)$ by Kudla ([8]). In this subsection we prove the following:

**Proposition 7.3.** Let $\mu$ be an element of $\Pi(E^\times, \omega_{E/F})$. Then we have

$$\omega_{\psi, V_D, W_D} \cong \omega_{\psi, V_{an}^2, W_1^2} \circ I \times id_{U_E(1)}$$

as $U_D(1) \times U_E(1)$-modules. Note that $U_E(1) = U(W_1)$.

We must recall the construction of the Weil representation $\omega_{\psi, V_D, W_D}$ for $U_D(1) \times U_E(1)$. Let

$$\mathcal{W}_D = V_D \otimes_D W_D$$

(6)
be the $F$-vector space equipped with the symplectic form
\[ \langle \langle \ , \ \rangle \rangle = \text{tr}_D \left( \langle \ , \ \rangle \otimes \langle \ , \ \rangle^t \right). \]

We take an irreducible unitary representation $(\tau, \psi, \mathcal{S}_D)$ of the Heisenberg group $H(\mathbb{W}_D)$ with central character $\psi$. Then we have the metaplectic group:
\[ \text{Mp}_\psi(\mathbb{W}_D) = \left\{ (g, M_g) \mid g \in \text{Sp}(\mathbb{W}_D), M_g : \tau_{\psi,D} \cdot g \cong \tau_{\psi,D} \text{ isomorphism} \right\}. \]
The metaplectic group $\text{Mp}_\psi(\mathbb{W}_D)$ has the Weil representation $(\omega_{\psi,D}, \mathcal{S}_D)$ defined by
\[ \omega_{\psi,D}(g, M_g) := M_g. \]
By the doubling method, Kudla ([8]) defined the splitting
\[ \iota_{W_D} \times \iota_{V_D} : U_D(1) \times U_E(1) \to \text{Mp}_\psi(\mathbb{W}_D). \]
Namely, we have the commutative diagram:
\[ U_D(1) \times U_E(1) \xrightarrow{\iota_{W_D} \times \iota_{V_D}} \text{Mp}_\psi(\mathbb{W}_D) \]
\[ \xrightarrow{\text{Sp}(\mathbb{W}_D)} \]

Then the Weil representation $(\omega_{\psi,V_D,W_D}, \mathcal{S}_D)$ for $U_D(1) \times U_E(1)$ is defined by
\[ \omega_{\psi,V_D,W_D} := \omega_{\psi,D} \circ \iota_{W_D} \times \iota_{V_D}. \]

Now we define the symplectic space $\mathbb{W} = V_{an} \otimes_F W_1$ over $F$ as in §2.1. Then the following lemma follows from easy computations:

**Lemma 7.4.** (i) The map
\[ \mathbb{W}_D = D \ni x_0 + x_1 \xi + x_2 \xi' + x_3 \xi' \xi \mapsto (x_2 + \xi x_3, x_0 + \xi x_1) \in E^\oplus_2 = \mathbb{W} \]
is an isomorphism as symplectic spaces over $F$.

(ii) We have the commutative diagram:
\[ U_D(1) \times U_E(1) \xrightarrow{I \times \text{id}} U(V_{an}) \times U(W_1) \]
\[ \xrightarrow{\text{Sp}(\mathbb{W}_D)} \text{Sp}(\mathbb{W}) \]

*Here the bottom map is one induced by (i).*
Since $H(\mathbb{W}_D) \cong H(\mathbb{W})$, we can consider $(\tau_{\psi,D}, S_D)$ as an irreducible unitary representation of $H(\mathbb{W})$. Then by §1.1, we have the metaplectic group $\text{Mp}_\psi(\mathbb{W})$ and the Weil representation

$$\omega_{\psi,V_D,1} := \omega_{\psi,V_{an},W_1} \circ \iota_{W_1}^\mu \times \iota_{V_{an}}^1$$
onumber

on $S_D$ for $U(V_{an}) \times U(W_1)$.

To prove Proposition 7.3, it is enough to show that the following diagram is commutative:

$$\begin{align*}
U_D(1) \times U_E(1) & \xrightarrow{I \times \text{id}} U(V_{an}) \times U(W_1) \\
\iota_{W_D} \times \iota_{V_D} & \downarrow \iota_{W_1}^\mu \times \iota_{V_{an}}^1 \\
\text{Mp}_\psi(\mathbb{W}_D) & \xrightarrow{\Phi} \text{Mp}_\psi(\mathbb{W})
\end{align*}$$

(7)

Here $\Phi$ is the isomorphism induced by Lemma 7.4 (i).

First we consider $U_D(1)$. The homomorphism $\Phi^{-1} \circ \iota_{W_1}^\mu \circ I$ is another splitting of $U_D(1)$ by Lemma 7.4 (ii). The difference of two splittings of $U_D(1)$ is a character of $U_D(1)$. Since $U_D(1)$ has no non-trivial character, we obtain $\Phi^{-1} \circ \iota_{W_1}^\mu \circ I = \iota_{W_D}$. Namely, the above diagram is commutative for $U_D(1)$.

Next we consider $U_E(1)$. We must recall the doubling method. Take two skew-hermitian spaces over $D$:

$$W_{D,sp} = (D^{\oplus 2}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}),$$

$$W_{-D} = (D, -\xi).$$

Then we have

$$W_D \oplus W_{-D} \cong W_{D,sp}.$$  

(8)

We write $U_E(-1) := U(W_{-D})$. On the other hand, if we define a skew-hermitian space over $E$:

$$W_{sp} = (E^{\oplus 2}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}),$$

then we have

$$W_1 \oplus W_{-1} \cong W_{sp}.$$
We can choose the isomorphism (8) so that the following diagram is commutative:

\[
\begin{array}{c}
W_1 \oplus W_{-1} \xrightarrow{\cong} W_{sp} \\
\downarrow \hspace{2cm} \downarrow j \\
W_D \oplus W_{-D} \xrightarrow{\cong} W_{D,sp}
\end{array}
\]

Here the vertical maps are induced by \(E \hookrightarrow D\). Then we obtain the commutative diagram:

\[
\begin{array}{c}
U(W_1) \times U(W_{-1}) \longrightarrow U(W_{sp}) \\
\hspace{2cm} \downarrow J \\
U_E(1) \times U_E(-1) \longrightarrow U(W_{D,sp})
\end{array}
\]

Here \(J : U(W_{sp}) \ni g \mapsto g \in U(W_{D,sp})\).

Define the symplectic spaces \(V_{an} \otimes_E W_{sp}\) and \(V_D \otimes_D W_{D,sp}\) as in \(\S 1.1\) and (6), respectively. Then \(i\) and \(j\) give the isomorphism

\[V_{an} \otimes_E W_{sp} \cong V_D \otimes_D W_{D,sp}.\]

Also we have the commutative diagram:

\[
\begin{array}{c}
U(W_1) \times U(W_{-1}) \longrightarrow U(W_{sp}) \longrightarrow \text{Sp}(V_{an} \otimes_E W_{sp}) \\
\hspace{2cm} \downarrow J \\
U_E(1) \times U_E(-1) \longrightarrow U(W_{D,sp}) \longrightarrow \text{Sp}(V_D \otimes_D W_{D,sp})
\end{array}
\]

Now from \([8]\) the Weil representation \(\omega_{\psi, V_D, W_{D,sp}}\) of \(U_D(1) \times U(W_{D,sp})\) on the space \(\mathcal{S}(V_D)\) of Schwartz-Bruhat functions on \(V_D = D\) has the following explicit formulas:

\[
\begin{align*}
\omega_{\psi, V_D, W_{D,sp}}(h) f(x) &= f(h^{-1}x), \\
\omega_{\psi, V_D, W_{D,sp}} \left( a \begin{array}{c} t(a) \\ b \end{array} \right) f(x) &= |\nu_D(a)| f(xa), \\
\omega_{\psi, V_D, W_{D,sp}} \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) f(x) &= \psi(b\nu_D(x)) f(x).
\end{align*}
\]

Here \(f \in \mathcal{S}(V_D), x \in V_D, h \in U_D(1), a \in D^\times\) and \(b \in F\). On the other hand, from Theorem 4.2 the Weil representation \(\omega_{\psi, V_{an} W_{sp}}\) of \(U(V_{an}) \times U(W_{sp})\) on
\( S(V_{an}) \) has the following explicit formulas:

\[
\omega_{\psi, V_{an}, W_{sp}}(h) f(y) = f(h^{-1} y),
\]

\[
\omega_{\psi, V_{an}, W_{sp}} \left( \begin{pmatrix} a & \sigma(a)^{-1} \\ b & 1 \end{pmatrix} \right) f(y) = |a| E f(ya),
\]

\[
\omega_{\psi, V_{an}, W_{sp}} \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) f(y) = \psi \left( b \left( -d_0 N_{E/F}(y_1) + N_{E/F}(y_2) \right) \right) f(y).
\]

Here \( f \in S(V_{an}) \), \( y = t(y_1, y_2) \in V_{an} \), \( h \in U(V_{sp}) \), \( a \in E \times F \), and \( b \in F \).

By these explicit formulas, we can check that if we identify \( V_{D} \) with \( V_{an} \) by Lemma 7.2 (i), then

\[
\omega_{\psi, V_{D}, W_{D, sp}} \circ J = \omega_{\psi, V_{an}, W_{sp}}
\]
as \( U(W_{sp}) \)-modules on \( S(V_{an}) \). This implies that the following diagram is commutative:

\[
\begin{array}{ccc}
U(W_1) \times U(W_{-1}) & \longrightarrow & U(W_{sp}) \\
\| & & \downarrow J \\
U_{E}(1) \times U_{E}(-1) & \longrightarrow & U_{D, sp} \times U_{D, sp} \\
\end{array}
\]

Here \( \dagger, \dagger \) are the splittings of \( U(W_{sp})\), \( U(W_{D, sp}) \) that define the Weil representations \( \omega_{\psi, V_{D}, W_{D, sp}}, \omega_{\psi, V_{an}, W_{sp}} \) of \( U(W_{sp}), U(W_{D, sp}) \), respectively.

This shows that the diagram (7) for \( U_{E}(1) \) is commutative.

### 7.3 Endoscopy for \( U_{D}(1) \)

Let \( \lambda \) be a quasi-character of \( E \times F \) such that \( \lambda \neq \lambda \circ \sigma \). Then there exists an irreducible supercuspidal representation \( \tau_{D}(\lambda) \) of \( D \times F \) with \( L \)-parameter \( \phi' = \text{Ind}_{W_{F}}^{W_{E}} \lambda \). Note that this representation is written as \( \tau_{D}(\phi') \) in §2.3.

**Theorem 7.5** ([9]). *There exist two irreducible representations \( \tau_{D}(\lambda)^{+}, \tau_{D}(\lambda)^{-} \) of \( U_{D}(1) \) such that \( \tau_{D}(\lambda)|_{U_{D}(1)} = \tau_{D}(\lambda)^{+} \oplus \tau_{D}(\lambda)^{-} \) and the following character identity holds:*

\[
\text{Tr} \tau_{D}(\lambda)^{+}(\gamma) - \text{Tr} \tau_{D}(\lambda)^{-}(\gamma) = \lambda(E/F, \psi) \omega_{E/F} \left( \frac{\gamma^{-1} - \gamma}{\xi} \right) \frac{\lambda(\gamma) - \lambda(\gamma^{-1})}{|\gamma - \gamma^{-1}|_{E}^{1/2}}
\]

*for any \( \gamma \in U_{E}(1) \). Also \( \tau_{D}(\lambda)^{\pm} \) are determined by the restriction \( \lambda|_{U_{E}(1)} \). Moreover \( \tau_{D}(\lambda)^{+} \cong \tau_{D}(\lambda)^{-} \) if and only if \( \lambda^{2}|_{U_{E}(1)} = 1 \).*
7.4 Main theorem

**Theorem 7.6.** For \( \eta \in \Pi(E^\times, 1_{F^\times}) \), let \( \lambda_\eta \) be a character of \( E^\times \) such that \( \lambda_\eta|_{U_E(1)} = \eta_u \). Then we have

\[
\theta_{\psi, V_D, W_D}(\eta_u) = \begin{cases} 
\tau_D(\lambda_\eta)^+ & \text{if } \eta \neq 1_{F^\times}, \\
0 & \text{if } \eta = 1_{F^\times}.
\end{cases}
\]

**Proof.** By Theorem 6.4, we have

\[
\omega_{\psi, V_{an}, W_D} = \bigoplus_{\eta \neq 1} \tau(\mu, \mu\eta^{-1})_{an}^+ \boxtimes \eta_u.
\]

Here \( \eta \) runs the non-trivial elements of \( \Pi(E^\times, 1_{F^\times}) \). Thus by Proposition 7.3, we obtain

\[
\omega_{\psi, V_D, W_D} = \bigoplus_{\eta \neq 1} \tau(\mu, \mu\eta^{-1})_{an}^+|_{U_D(1)} \boxtimes \eta_u.
\]

Therefore we must show that \( \tau(\mu, \mu\eta^{-1})_{an}^+|_{U_D(1)} = \tau_D(\lambda_\eta)^+ \). This is shown in the next lemma.

**Lemma 7.7.** We have \( \tau(\mu, \mu\eta^{-1})_{an}^+|_{U_D(1)} = \tau_D(\lambda_\eta)^+ \).

**Proof.** First we recall the following isomorphism:

\[
E^\times \times D^\times / \Delta F^\times \ni (x, z + \xi' w) \mapsto x\nu_D(z + \xi' w)^{-1}\left( \begin{array}{cc} \sigma(z) & w \\ d_0\sigma(w) & z \end{array} \right) \in GU(V_{an}).
\]

Then we have the commutative diagram:

\[
\begin{array}{ccc}
U_D(1) & \xrightarrow{I} & U(V_{an}) \\
\downarrow & & \downarrow \\
E^\times \times D^\times / \Delta F^\times & \xrightarrow{\cong} & GU(V_{an})
\end{array}
\]

Here the vertical maps are natural inclusions.

Take quasi-characters \( \chi, \chi' \) of \( E^\times \) such that \( \chi\chi' = \mu, \chi(\chi' \circ \sigma) = \mu\eta^{-1} \).

Then \( \hat{\tau} = \chi \boxtimes \tau_D(\chi') \) is an irreducible admissible representation of \( E^\times \times D^\times / \Delta F^\times \). By \( \S 2.3 \), we have

\[
\hat{\tau}|_{U(V_{an})} = \tau(\mu, \mu\eta^{-1})_{an}^+ \oplus \tau(\mu, \mu\eta^{-1})_{an}^-.
\]
On the other hand, by Theorem 7.5 we obtain
\[ \tilde{\tau}|_{U_D(1)} = \tau_D(\chi'|_{U_D(1)}) = \tau_D(\chi')^+ \oplus \tau_D(\chi')^- . \]
Thus we have
\[ \tau(\mu, \mu \eta^{-1})^+|_{U_D(1)} = \tau_D(\lambda_\eta)^+ \text{ or } \tau_D(\lambda_\eta)^-. \]
Here note that \( \tau_D(\chi')^\pm = \tau_D(\lambda_\eta)^\pm \), since \( \chi'|_{U_E(1)} = \eta_u \).

Next note that \( U_E(1) \) is identified with a subgroup of \( U(V_{an}) \) by
\[ U_E(1) \ni \gamma \mapsto t(1, \gamma^{-1}) = \left( \begin{array}{c} \gamma^{-1} \\ \gamma \end{array} \right) \in U(V_{an}) . \]

Then by Theorem 2.8, the restriction of the character identity of \( \tau(\mu, \mu \eta^{-1})^\pm_{an} \) to \( U_E(1) \) is given as follows:
\[
\begin{align*}
\text{Tr} \tau(\mu, \mu \eta^{-1})^\pm_{an}(\gamma) - \text{Tr} \tau(\mu, \mu \eta^{-1})^-_{an}(\gamma) \\
= \lambda(E/F, \psi) \omega_{E/F} \left( \frac{\gamma^{-1} - \sigma(\gamma^{-1})}{\xi} \right) \chi \left( N_{E/F}(\gamma^{-1}) \right) \frac{\chi'(\sigma(\gamma^{-1})) - \chi'(\gamma^{-1})}{|\gamma^{-1} - \sigma(\gamma^{-1})|^{1/2}} |\gamma^{-1}|^{1/2} \\
= \lambda(E/F, \psi) \omega_{E/F} \left( \frac{\gamma^{-1} - \gamma}{\xi} \right) \eta_u(\gamma) - \eta_u(\gamma^{-1}) |\gamma^{-1} - \gamma|^{1/2} \\
= \text{Tr} \tau_D(\lambda_\eta)^+(\gamma) - \text{Tr} \tau_D(\lambda_\eta)^-(\gamma) .
\end{align*}
\]
The last equation follows from Theorem 7.5. Thus we obtain
\[ \tau(\mu, \mu \eta^{-1})^\pm_{an}|_{U_D(1)} = \tau_D(\lambda_\eta)^+ . \]

\[ \square \]

References


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