# p進ユニタリデュアルペアU(2)×U(1), U(2)×U(3)とp進四元数デュアルペアU(1)×U(1)の局所テータリフト

池松, 泰彦

https://doi.org/10.15017/1654662

出版情報：Kyushu University, 2015, 博士（数理学）, 課程博士
バージョン：

# Local theta lifts for $p$-adic unitary dual pairs $\mathrm{U}(2) \times \mathrm{U}(1)$, $\mathrm{U}(2) \times \mathrm{U}(3)$ and a $p$-adic quaternionic dual pair $\mathrm{U}(1) \times \mathrm{U}(1)$

**Yasuhiko Ikematsu**

A dissertation submitted to
Kyushu University
for the degree of
Doctor of Philosophy (Mathematics)
January 2016

# Introduction

The purpose of this paper is to describe the local theta lifts for $p$-adic unitary dual pairs $\mathrm{U}(2) \times \mathrm{U}(1)$ and $\mathrm{U}(2) \times \mathrm{U}(3)$ in terms of endoscopy. This is a complement to a result of Gelbart-Rogawski-Soudry [4]. Also Gan-Ichino [2] described the local theta lifts for unitary groups in the almost equal rank case in terms of Vogan $L$-packets. The result in this paper provides another proof of their result in the cases of $\mathrm{U}(2)\times\mathrm{U}(1)$ and $\mathrm{U}(2)\times\mathrm{U}(3)$. Our proof is based on some results in [4] and the endoscopic description of the anisotropic unitary group in two variables in Konno-Konno [5]. As an application, we obtain the description of the local theta lift for a $p$-adic quaternionic dual pair $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$ in terms of endoscopy.

To explain the problem, we recall some results in [4]. Let $F$ be a non-archimedean local field of characteristic 0, and $E$ a quadratic extension of $F$ with associated quadratic character $\omega_{E/F}$. We denote the split (resp. anisotropic) 2-dimensional hermitian space over $E$ by $V_{sp}$ (resp. $V_{an}$) and a three-dimensional skew-hermitian space over $E$ by $W$. We set $V = V_{sp}, V_{an}$. Rogawski gave the endoscopic descriptions of the irreducible admissible representations for $\mathrm{U}(V_{sp})$ and $\mathrm{U}(W)$ in [15]. Using it, Gelbart-Rogawski-Soudry showed in [4] that the local theta lift for $\mathrm{U}(V) \times \mathrm{U}(W)$ sends an $L$-packet to an $L$-packet. They also described the local pairing of an endoscopic representation $\pi$ for $\mathrm{U}(W)$ by the behavior of the local theta lift of $\pi$ to $\mathrm{U}(V_{sp})$ and $\mathrm{U}(V_{an})$. Namely, an explicit parametrization of the members of an endoscopic $L$-packet of $\mathrm{U}(W)$ is given in terms of the local theta lift to $\mathrm{U}(V)$. Moreover they described the local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W)$.

Now we consider the following problem. Since we have the endoscopic description for $\mathrm{U}(V_{an})$ by [5], we want to describe the local theta lift for $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$ in terms of the endoscopic description. Notice that this problem cannot be immediately solved by the result in [4], since the endoscopic description for $\mathrm{U}(V_{an})$ is not given in [4].

To explain our result, we prepare a few notation. Take an element $\xi \in E^\times$ such that $\mathrm{Tr}_{E/F}(\xi) = 0$. Then we can define $W$ by

$$(E^{\oplus 3}, \begin{pmatrix} & & 1 \\ & \xi & \\ -1 & & \end{pmatrix}).$$

Let $W_F$ be the Weil group of $F$. We denote the local Langlands group of $F$ by $L_F = W_F \times \mathrm{SU}_2(\mathbb{R})$. For $G = \mathrm{U}(V)$ or $\mathrm{U}(W)$, $\Phi(G)$ is the set of equivalence classes of $L$-parameters of $G$. In [15], Rogawski gave the $L$-packet $\Pi_\phi$ for an $L$-parameter $\phi$ of $\mathrm{U}(V_{sp})$ or $\mathrm{U}(W)$. On the other hand, the $L$-packet $\Pi_\phi$ for an $L$-parameter $\phi$ of $\mathrm{U}(V_{an})$ is given by using the Jacquet-Langlands correspondence (see §2.3 or [4, §1]). Let $\phi_E$ be the restriction of an $L$-parameter $\phi$ of $G$ to $L_E$. We note that $\phi$ is uniquely determined by $\phi_E$. Let $\psi$ be a non-trivial character of $F$. For characters $\mu, \eta$ of $E^\times$ such that $\mu|_{F^\times} = \omega_{E/F}, \eta|_{F^\times} = \mathbb{1}$, we have the Weil representation $\omega_{\psi,V^\mu,W^\eta}$ and the local theta lift $\theta_{\psi,V^\mu,W^\eta}$ for $\mathrm{U}(V) \times \mathrm{U}(W)$. Set $\epsilon(V) = +$ if $V = V_{sp}$ and $\epsilon(V) = -$ otherwise.

We recall results in [4] in more detail. The unique non-trivial elliptic endoscopic datum $(H, s_0, \xi_0)$ for $\mathrm{U}(W)$ up to equivalence is given as follows:

$$H = \mathrm{U}(V_{sp}) \times \mathrm{U}(1),\ s_0 = \begin{pmatrix} -1_2 & \\ & 1 \end{pmatrix},$$

$$\xi_0 : {}^L H \ni (h, h') \rtimes w \longmapsto \begin{cases} \begin{pmatrix} h\mu_0(w) & \\ & h' \end{pmatrix} \rtimes w, & w \in W_E \\ \begin{pmatrix} & -h \\ h' & \end{pmatrix} \rtimes w_\sigma, & w = w_\sigma. \end{cases}$$

Here we have fixed $w_\sigma \in W_F \smallsetminus W_E$ and a character $\mu_0$ of $E^\times$ such that $\mu_0|_{F^\times} = \omega_{E/F}$. For an $L$-parameter $\phi \in \Phi(\mathrm{U}(W))$, define $\hat{\Pi}_\phi = \{\rho \in \Phi(H_0) \mid \xi_0 \circ \rho \sim \phi\}$. For each $\rho \in \hat{\Pi}_\phi$, the $\mathrm{U}(V_{sp})$-part (resp. $\mathrm{U}(1)$-part) of $\rho$ is the $L$-parameter of $\mathrm{U}(V_{sp})$ (resp. $\mathrm{U}(1)$). It is denoted by $\rho_\varphi$ (resp. $\rho_\eta$). Then the character of $E^\times/F^\times \cong \mathrm{U}(1)$ corresponding to $\rho_\eta$ is written by $\eta_\rho$. Now take an irreducible admissible representation $\pi$ of $\mathrm{U}(W)$ having an $L$-parameter $\phi$ such that $\hat{\Pi}_\phi$ is not empty. Then for any $\rho \in \hat{\Pi}_\phi$, there exists a unique space $V = V_{sp}$ or $V_{an}$ such that $\theta_{\psi,V^\mu,W^{\eta_\rho}}(\pi)$ is non-zero. This $V$ is independent of the choices of $\psi$ and $\mu$. Then we define the local pairing $\epsilon_\rho(\pi) := \epsilon(V)$. In this way we can define $\epsilon_\rho(\pi)$ for any $\pi \in \Pi_\phi$ and $\rho \in \hat{\Pi}_\phi$. These $\epsilon_\rho(\pi)$ determine each element of $\Pi_\phi$ and coincide with the local pairing defined by Rogawski [15].

Now we can explain our result. Take characters $\mu_1, \mu_2$ of $E^\times$ such that $\mu_1 \neq \mu_2$ and $\mu_i|_{F^\times} = \omega_{E/F}$. Then there exists a unique $L$-parameter $\varphi_{\mu_1,\mu_2}$ of $\mathrm{U}(V_{an})$ such that $(\varphi_{\mu_1,\mu_2})_E = \mu_1 \oplus \mu_2$. We denote the $L$-packet of $\mathrm{U}(V_{an})$ associated to it by $\Pi_{\mu_1,\mu_2}(V_{an})$. Then the $L$-packet $\Pi_{\mu_1,\mu_2}(V_{an})$ has two elements $\tau(\mu_1,\mu_2)^+_{an}, \tau(\mu_1,\mu_2)^-_{an}$, where the signs are specified in §2.3. Then we want to compute the local theta lift $\theta_{\psi,V^\mu_{an},W^\eta}$ for $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$.

First we assume that $\mu = \mu_1$.

**Theorem 0.1.** *We have*

$$\theta_{\psi,V_{an}^{\mu_1},W^\eta}(\tau(\mu_1,\mu_2)_{an}^{\varepsilon}) = \begin{cases} \pi(\eta,\eta\mu_1\mu_2^{-1})^- & \text{if } \varepsilon = +, \\ 0 & \text{if } \varepsilon = -. \end{cases}$$

*Here* $\pi(\eta,\eta\mu_1\mu_2^{-1})^-$ *is the unique irreducible non-generic subrepresentation of* $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta\boxtimes(\eta\mu_1\mu_2^{-1})_u)$, *where* $B$ *is the Borel subgroup of all upper triangular matrices in* $\mathrm{U}(W)$*. Also* $(\cdot)_u$ *is defined later.*

Next we assume that $\mu \neq \mu_1, \mu_2$. Then by results in [4] we have

$$\pi^{\pm} := \theta_{\psi,V_{an}^{\mu},W^\eta}(\tau(\mu_1,\mu_2)_{an}^{\pm}) \neq 0$$

and the $L$-parameter $\phi$ of $\pi^{\pm}$ satisfies $\phi_E = \mu\eta\mu_1^{-1} \oplus \mu\eta\mu_2^{-1} \oplus \eta$. Thus we have $\hat{\Pi}_\phi = \{\rho_0, \rho_1, \rho_2\}$, where

$$\begin{aligned} \rho_{0,E} &= \mu_0^{-1}\mu\eta(\mu_1^{-1} \oplus \mu_2^{-1}) \times \eta, \\ \rho_{1,E} &= \mu_0^{-1}(\eta \oplus \mu\eta\mu_2^{-1}) \times \mu\eta\mu_1^{-1}, \\ \rho_{2,E} &= \mu_0^{-1}(\mu\eta\mu_1^{-1} \oplus \eta) \times \mu\eta\mu_2^{-1}. \end{aligned}$$

**Theorem 0.2.** *We have for* $\varepsilon \in \{\pm\}$

$$\begin{aligned} \epsilon_{\rho_0}(\pi^{\varepsilon}) &= -, \\ \epsilon_{\rho_1}(\pi^{\varepsilon}) &= -\varepsilon, \\ \epsilon_{\rho_2}(\pi^{\varepsilon}) &= \varepsilon. \end{aligned}$$

Using Theorem 0.1, we can compute the local theta lift $\theta_{\psi,V_{an}^{\mu},W_1^\eta}$ for $\mathrm{U}(V_{an}) \times \mathrm{U}(W_1)$, where $W_1$ is a one-dimensional skew-hermitian space over $E$ given by $(E,\xi)$.

**Theorem 0.3.** *Take a character* $\eta'$ *of* $E^\times$ *such that* $\eta'|_{F^\times} = \mathbb{1}_{F^\times}$*. Then we have*

$$\theta_{\psi,V_{an}^{\mu},W_1^\eta}(\eta'_u) = \begin{cases} \tau(\mu,\mu\eta\eta'^{-1})_{an}^{+} & \text{if } \eta \neq \eta', \\ 0 & \text{if } \eta = \eta'. \end{cases}$$

As an application of this theorem, we can compute the local theta lift for a quaternionic unitary dual pair of rank one. Let $D$ be the quaternion division algebra over $F$. Fix an embedding $E \hookrightarrow D$. Let $V_D, W_D$ be a hermitian space and a skew-hermitian space over $D$ given by $(D,1), (D,\xi)$, respectively.

We denote their quaternionic unitary groups by $\mathrm{U}_D(1), \mathrm{U}_E(1)$, respectively. Then we have the Weil representation $\omega_{\psi,V_D,W_D}$ for $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$. Here $\mathrm{U}_D(1)$ is the kernel of the reduced norm of $D$. Let $\lambda$ be a character of $E^\times$ such that $\mathrm{Ind}_{W_E}^{W_F} \lambda$ is irreducible. Denote the irreducible representation of $D^\times$ with $L$-parameter $\mathrm{Ind}_{W_E}^{W_F} \lambda$ by $\tau_D(\lambda)$. Then there exist irreducible admissible representations $\tau_D(\lambda)^\pm$ of $\mathrm{U}_D(1)$ such that $\tau_D(\lambda)|_{\mathrm{U}_D(1)} = \tau_D(\lambda)^+ \oplus \tau_D(\lambda)^-$. Here the signs are specified in §7.4.

**Theorem 0.4.** *Let $\lambda_\eta$ be a character of $E^\times$ such that $\lambda_\eta|_{\mathrm{U}_E(1)} = \eta_u$. Then we have*

$$\theta_{\psi,V_D,W_D}(\eta_u) = \begin{cases} \tau_D(\lambda_\eta)^+ & \text{if } \eta \neq \mathbb{1}_{F^\times}, \\ 0 & \text{if } \eta = \mathbb{1}_{F^\times}. \end{cases}$$

The paper is organized as follows. In §1.1, we define unitary dual pairs and explain their Weil representations and local theta lifts. In §1.2, we recall the definition of $L$-parameters and describe the $L$-parameters of $\mathrm{U}(V_{sp}), \mathrm{U}(V_{an})$ and $\mathrm{U}(W)$. In §2, we explain the construction and the description of the $L$-packets of $\mathrm{U}(V_{sp})$ and $\mathrm{U}(V_{an})$ following [5]. Also we prepare some global results for $\mathrm{U}(V_{sp})$ and $\mathrm{U}(V_{an})$. In §3, we explain the construction and the description of the $L$-packets of $\mathrm{U}(W)$ following [15]. Also we state the global multiplicity formula for $\mathrm{U}(W)$. In §4.1, we recall the explicit formulas of the mixed model of the Weil representation of a unitary dual pair. In §4.2, we compute the local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$ using the result in §2. In §5, we recall some results in Gelbart-Rogawski-Soudry [4]. In §6, we prove our theorems. In §7, we compute the local theta lift for $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$.

**Notation 0.5.** Let $F$ be a non-archimedean local field of characteristic zero and $E$ a quadratic extension of $F$. The map $\sigma$ stands for the non-trivial Galois automorphism of $E$ over $F$. We will often write $\sigma(x) = x^\sigma$. Let $\mathrm{Tr}_{E/F}$ and $\mathrm{N}_{E/F}$ be the trace map and norm map of $E$ over $F$, respectively. The symbol $\omega_{E/F}$ stands for the quadratic character of $F^\times$ associated with $E/F$. Let $\bar{F}$ be an algebraic closure of $E$, $\Gamma = \Gamma_F$ the absolute Galois group of $F$, and $W_F$ the Weil group of $F$. We let $|\cdot|_F$ denote the modulus of $F$. Let $\psi$ be a non-trivial character of $F$. Then $\lambda(E/F, \psi)$ is the Langlands $\lambda$-factor [10]. For each character $\chi$ of $F^\times$, $\Pi(E^\times, \chi)$ stands for the set of characters of $E^\times$ whose restriction to $F^\times$ are $\chi$. For each $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we define a character $\eta_u$ of $\mathrm{U}(1) := \mathrm{Ker}\,\mathrm{N}_{E/F}$ by

$$\eta_u(z/\sigma(z)) := \eta(z), \quad z \in E^\times.$$

Elements of $\Pi(E^\times, \mathbb{1}_{F^\times})$ are denoted by $\eta, \eta', \eta_1, \eta_2$, and so on. Also elements of $\Pi(E^\times, \omega_{E/F})$ are denoted by $\mu, \mu', \mu_1, \mu_2$, and so on.

For a $p$-adic group $G$, we denote the set of equivalence classes of irreducible admissible representations of $G$ by $\operatorname{Irr} G$.

For a topological group $H$, we denote by $H^0$ the component of the neutral element. Put $\pi_0(H) = H/H^0$. Let $Z(H)$ be the center of $H$. If $H$ is abelian, then we denote its Pontryagin dual group by $H^D$.

Fix an element $w_\sigma$ of $W_F \smallsetminus W_E$, a non-zero element $\xi$ of $E$ with $\mathrm{Tr}_{E/F}(\xi) = 0$ and $d_0 \in F^\times \smallsetminus \mathrm{N}_{E/F}(E^\times)$.

## Acknowledgements

The author would like to thank his advisor, Prof. Takuya Konno, for much helpful advice and encouragement.

# Contents

# 1 Preliminaries

## 1.1 Local theta lift for unitary dual pairs

In this subsection we recall the local theta lift for unitary dual pairs.

Let $V$ be a hermitian space and $W$ a skew-hermitian space over $E$. The spaces $V$ and $W$ may be taken as follows:

$$\begin{aligned} V &= (E^{\oplus m}, R), & W &= (E^{\oplus n}, S), \\ (v_1, v_2) &= {}^*v_1 R v_2, & \langle w_1, w_2 \rangle &= w_1 S {}^*w_2. \end{aligned}$$

where $R = {}^*R := {}^tR^\sigma \in \mathrm{GL}_m(E)$ and $S = -{}^*S \in \mathrm{GL}_n(E)$. Then the unitary groups of $V$ and $W$ are given by

$$\begin{aligned} \mathrm{U}(V) &= \{h \in \mathrm{GL}_m(E) \,|\, {}^*hRh = R\}, \\ \mathrm{U}(W) &= \{g \in \mathrm{GL}_n(E) \,|\, gS{}^*g = S\}. \end{aligned}$$

Here $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) acts on $V$ (resp. $W$) on the left (resp. right).

Let

$$\mathbb{W} = V \otimes_E W$$

be the $F$-vector space equipped with the symplectic form

$$\langle\!\langle\ ,\ \rangle\!\rangle = \mathrm{Tr}_{E/F}\left((\ ,\ ) \otimes \langle\ ,\ \rangle^\sigma\right).$$

The symplectic group of $\mathbb{W}$ is given by

$$\mathrm{Sp}(\mathbb{W}) = \{g \in \mathrm{GL}_F(\mathbb{W}) \,|\, \langle\!\langle wg, w'g \rangle\!\rangle = \langle\!\langle w, w' \rangle\!\rangle \text{ for all } w, w' \in \mathbb{W}\}.$$

We have a natural homomorphism:

$$\mathrm{U}(V) \times \mathrm{U}(W) \ni (h, g) \mapsto \big(v \otimes w \mapsto h^{-1}v \otimes wg\big) \in \mathrm{Sp}(\mathbb{W}).$$

Then the pair of two groups $(\mathrm{U}(V), \mathrm{U}(W))$ forms a reductive dual pair in $\mathrm{Sp}(\mathbb{W})$.

Next we explain Weil representations of $\mathrm{U}(V) \times \mathrm{U}(W)$. The Heisenberg group $H(\mathbb{W})$ of $\mathbb{W}$ is given as follows:

$$\begin{aligned} H(\mathbb{W}) &= \mathbb{W} \times F, \\ (w, t) \cdot (w', t') &= \left(w + w', t + t' + \frac{\langle\!\langle w, w' \rangle\!\rangle}{2}\right) \end{aligned}$$

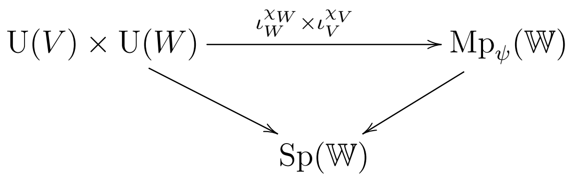
for $w, w' \in \mathbb{W}$ and $t, t' \in F$. We take an irreducible unitary representation $(\tau_{\psi,V,W}, \mathcal{S}_{V,W})$ of $H(\mathbb{W})$ with central character $\psi$. For any $g \in \mathrm{Sp}(\mathbb{W})$, define the representation $(\tau_{\psi,V,W} \cdot g, \mathcal{S}_{V,W})$ of $H(\mathbb{W})$ by $\tau_{\psi,V,W}(w,t)g := \tau_{\psi,V,W}(wg,t)$ for $w \in \mathbb{W}, t \in F$. This is also an irreducible unitary representation of $H(\mathbb{W})$ with central character $\psi$. By Stone-von-Neumann's theorem, the unitary representation $\tau_{\psi,V,W} \cdot g$ is isomorphic to $\tau_{\psi,V,W}$. Thus we have the metaplectic group:

$$\mathrm{Mp}_\psi(\mathbb{W}) = \left\{ (g, M_g) \;\middle|\; \begin{array}{l} g \in \mathrm{Sp}(\mathbb{W}), \\ M_g \colon \tau_{\psi,V,W} \cdot g \cong \tau_{\psi,V,W} \text{ isomorphism} \end{array} \right\}.$$

The metaplectic group $\mathrm{Mp}_\psi(\mathbb{W})$ has the Weil representation $(\omega_{\psi,V,W}, \mathcal{S}_{V,W})$ defined by

$$\omega_{\psi,V,W}(g, M_g) := M_g.$$

Now let $\chi_V$ and $\chi_W$ be characters of $E^\times$ such that $\chi_V|_{F^\times} = \omega_{E/F}^{\dim_E V}$ and $\chi_W|_{F^\times} = \omega_{E/F}^{\dim_E W}$, respectively. Then Kudla ([8]) defined the homomorphism $\iota_W^{\chi_W} \times \iota_V^{\chi_V}$ associated to $\chi_V$ and $\chi_W$ so that the following diagram commutes.




Thus the Weil representation $(\omega_{\psi,V^{\chi_W},W^{\chi_V}}, \mathcal{S}_{V,W})$ of $\mathrm{U}(V) \times \mathrm{U}(W)$ associated to $\chi_V$ and $\chi_W$ is defined by

$$\omega_{\psi,V^{\chi_W},W^{\chi_V}} := \omega_{\psi,V,W} \circ \iota_W^{\chi_W} \times \iota_V^{\chi_V}.$$

In §4.1, we will describe the explicit formulas of the mixed model of the Weil representation $(\omega_{\psi,V^{\chi_W},W^{\chi_V}}, \mathcal{S}_{V,W})$ of $\mathrm{U}(V) \times \mathrm{U}(W)$.

Next we explain the local theta lift for $\mathrm{U}(V) \times \mathrm{U}(W)$. Let $\pi$ be an irreducible admissible representation of $\mathrm{U}(V)$. If $H_\pi := \mathrm{Hom}_{\mathrm{U}(V)}(\omega_{\psi,V^{\chi_W},W^{\chi_V}}, \pi)$, then

$$\omega_{\psi,V^{\chi_W},W^{\chi_V}} \Big/ \bigcap_{f \in H_\pi} \mathrm{Ker}\, f$$

is a smooth representation of $\mathrm{U}(V) \times \mathrm{U}(W)$.

**Lemma 1.1** ([13])**.** (i) *There exists a smooth representation* $\Theta(\pi) = \Theta_{\psi,\chi_V,\chi_W}(\pi)$ *of* $\mathrm{U}(W)$ *such that*

$$\pi \boxtimes \Theta(\pi) = \omega_{\psi,V^{\chi_W},W^{\chi_V}} / \bigcap_{f \in H_\pi} \mathrm{Ker}\, f.$$

(ii) $\Theta(\pi)$ *is of finite length and hence is admissible.*

We call $\Theta(\pi)$ the *big theta lift* of $\pi$. We set

$$\mathcal{R}_{\psi,\chi_V,\chi_W}(V) := \{\pi \in \mathrm{Irr}\,\mathrm{U}(V) \mid H_\pi \neq 0\}.$$

It is clear that $H_\pi \neq 0$ if and only if $\Theta(\pi) \neq 0$. Then the local Howe duality conjecture is that $\Theta(\pi)$ has the maximal subrepresentation, that is, the maximal semisimple quotient $\theta(\pi) = \theta_{\psi,V^{\chi_W},W^{\chi_V}}(\pi)$ of $\Theta(\pi)$ is irreducible. If $\Theta(\pi) = 0$, then we set $\theta(\pi) = 0$. We call $\theta(\pi)$ the *local theta lift* of $\pi$. Similarly, we can define $\mathcal{R}_{\psi,\chi_V,\chi_W}(W)$ and the local theta lift $\theta(\sigma)$ of an irreducible admissible representation $\sigma$ of $\mathrm{U}(W)$. By the definition of the local theta lift,

$$\mathcal{R}_{\psi,\chi_V,\chi_W}(V) \ni \pi \mapsto \theta(\pi) \in \mathcal{R}_{\psi,\chi_V,\chi_W}(W)$$

is bijective and the inverse map is given by $\sigma \mapsto \theta(\sigma)$.

The local Howe duality conjecture was proven by Waldspurger [17] when the residue characteristic $p$ of $F$ is not 2. Recently, Gan and Takeda [3] proved the local Howe duality conjecture for any residual characteristic $p$. But these results are not used in this paper. Instead the following lemma is used.

**Lemma 1.2** ([13])**.** (i) *If* $\pi$ *is an irreducible supercuspidal representation, then* $\Theta(\pi) = \theta(\pi)$*, that is,* $\Theta(\pi)$ *is irreducible.*
(ii) *If* $\theta(\pi_1) = \theta(\pi_2) \neq 0$ *for two irreducible supercuspidal representations* $\pi_1$ *and* $\pi_2$*, then* $\pi_1 = \pi_2$*.*

By this lemma, we can show the local Howe duality conjecture for $\mathrm{U}(V) \times \mathrm{U}(W)$, where $\mathrm{U}(V)$ or $\mathrm{U}(W)$ is compact.

Finally we explain the unitary groups which we treat in this paper. For $a \in F^\times$, we define a one-dimensional skew-hermitian space:

$$W_a = (E, a\xi)$$

Also define a three-dimensional skew-hermitian space:

$$W = (E^{\oplus 3}, \begin{pmatrix} & & 1 \\ & \xi & \\ -1 & & \end{pmatrix}).$$

Let $V_{sp}$ (resp. $V_{an}$) be the two-dimensional split (resp. anisotropic) hermitian space over $E$ given by

$$V_{sp} = (E^{\oplus 2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}),$$
$$V_{an} = (E^{\oplus 2}, \begin{pmatrix} -d_0 & \\ & 1 \end{pmatrix}).$$

Thus we have four unitary groups $\mathrm{U}(W_a), \mathrm{U}(W), \mathrm{U}(V_{sp})$ and $\mathrm{U}(V_{an})$. Also we denote the unitary similitude group of $V = V_{sp}, V_{an}$ by $\mathrm{GU}(V)$.

In [4], the local Howe duality conjecture for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W)$ was proved and its local theta lift was computed using endoscopy. Our goal is to compute the local theta lifts for $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$ and $\mathrm{U}(V_{an}) \times \mathrm{U}(W_a)$. We remark that the local Howe duality conjecture for $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$ and $\mathrm{U}(V_{an}) \times \mathrm{U}(W_a)$ follow from Lemma 1.2.

## 1.2 $L$-parameter

Let $G$ be $\mathrm{U}(W_a), \mathrm{U}(W), \mathrm{U}(V_{sp})$ or $\mathrm{U}(V_{an})$. We recall the $L$-groups for these groups.

**Lemma 1.3.** *The dual group $\hat{G}$ of $G$ is $\mathrm{GL}_n(\mathbb{C})$, where $n$ is the dimension of the space associated to $G$. Also the $L$-group ${}^L G$ is given as follows:*

$$ {}^L G = \mathrm{GL}_n(\mathbb{C}) \rtimes W_F,$$
$$g \rtimes w \cdot g' \rtimes w' = \begin{cases} gg' \rtimes ww' & \text{if } w \in W_E, \\ gI_n {}^t g'^{-1} I_n^{-1} \rtimes ww' & \text{if } w \notin W_E. \end{cases}$$

*Here*

$$I_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

When $G = \mathrm{U}(W)$, the proof of this lemma will be given in §3.1. The other cases are similar.

Next we recall the definition of $L$-parameters. Let $H$ be a connected reductive group defined over $F$.

**Definition 1.4.** We call a homomorphism $\phi : L_F \to {}^L H$ an *$L$-parameter* of $H$ if the following conditions hold:

(1) the image of $W_F$ under $L_F \xrightarrow{\phi} {}^L H \xrightarrow{\text{proj.}} \hat{H}$ consists of semi-simple elements of $\hat{H}$.

(2) $\phi|_{\mathrm{SU}_2(\mathbb{R})} : \mathrm{SU}_2(\mathbb{R}) \xrightarrow{\phi} {}^L H \xrightarrow{\text{proj.}} \hat{H}$ is analytic.

(3) $\phi$ is a continuous homomorphism so that the following diagram is commutative:

$$\begin{array}{ccc} L_F & \xrightarrow{\ \phi\ } & {}^L H \\ & \searrow \quad \swarrow & \\ & W_F & \end{array}$$

Here $L_F = W_F \times \mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{R}) = \{g \in \mathrm{SL}_2(\mathbb{C}) \mid g^t \overline{g} = 1_2\}$.

Two $L$-parameters $\phi$ and $\phi'$ are said to be *equivalent* if there exists $h \in \hat{H}$ such that $\phi' = \mathrm{Ad}(h) \circ \phi$. Then we write $\phi \sim \phi'$. Also we denote the set of equivalence classes of $L$-parameters of $H$ by $\Phi(H)$.

If $H = \mathrm{GL}_n(F)$, then we can identify an $L$-parameter of $\mathrm{GL}_n(F)$ with a completely reducible $n$-dimensional representation of $L_F$. The following fact is well-known:

**Fact 1.5.** (Local Langlands correspondence for $\mathrm{GL}_n$) There exists a natural bijection from $\mathrm{Irr}\,\mathrm{GL}_n(F)$ to $\Phi(\mathrm{GL}_n(F))$. In particular, if $n = 1$, then its bijection is obtained by the local class field theory.

Similarly we expect that there exists a bijection from $\mathrm{Irr}\, G$ to $\Phi(G)$. However, in general there does not exist such a bijection. But we can describe $\mathrm{Irr}\, G$ in terms of $\Phi(G)$ by the following procedure. For any $\phi \in \Phi(G)$, we construct a finite set $\Pi_\phi$ that consists of irreducible admissible representations of $G$. The set $\Pi_\phi$ is called an $L$-packet of $G$. Set $S_\phi = \mathrm{Cent}_{\hat{G}}(\mathrm{Im}\,\phi)$ and $\mathcal{S}_\phi = \pi_0(S_\phi / Z(\hat{G})^\Gamma)$. This $\mathcal{S}_\phi$ is a finite abelian group. Then $\Pi_\phi$ is described by the character group of $\mathcal{S}_\phi$. This procedure is achieved by endoscopy, which is related to the problem of classification of conjugacy classes of $G$. We call such a description of irreducible admissible representations of $G$ *endoscopic* *description*. We will explain the case of $G = \mathrm{U}(V_{sp}), \mathrm{U}(V_{an})$ following Konno-Konno [5] in §2. On the other hand, the case of $G = \mathrm{U}(W)$ will be explained in §3 following Rogawski [15].

Finally we explain the classification of the $L$-parameters of $G$. Denote the unique $r$-dimensional irreducible representation of $\mathrm{SU}_2(\mathbb{R})$ by $\mathrm{Sym}^{r-1}$. For an $L$-parameter $\phi$ of $G$, we define the homomorphism $\phi_E : L_E \to \mathrm{GL}_n(\mathbb{C})$ by

$$L_E \xrightarrow{\phi|_{L_E}} {}^L G \xrightarrow{\text{proj.}} \mathrm{GL}_n(\mathbb{C}).$$

Then $\phi_E$ is an $L$-parameter of $\mathrm{GL}_n(E)$ and a completely reducible $n$-dimensional representation of $L_E$. Thus we have

$$\phi_E = \bigoplus_i \varphi_i \boxtimes \mathrm{Sym}^{r_i - 1}, \quad \varphi_i \in \mathrm{Irr}\, W_E, \quad (n = \sum_i r_i \dim \varphi_i).$$

Let $\phi$ be an $L$-parameter of $\mathrm{U}(W_a)$. Then $\phi_E$ is a character of $E^\times/F^\times$ by the local class field theory. Namely, $\phi_E$ is an element of $\Pi(E^\times, \mathbb{1}_{F^\times})$.

**Lemma 1.6.** *We have the bijection:*

$$\Phi(\mathrm{U}(W_a)) \ni \phi \mapsto \phi_E \in \Pi(E^\times, \mathbb{1}_{F^\times}).$$

*Moreover by the bijection* $\Pi(E^\times, \mathbb{1}_{F^\times}) \ni \eta \mapsto \eta_u \in \mathrm{Irr}\,\mathrm{U}(W_a)$*, we have the bijection:*

$$\Phi(\mathrm{U}(W_a)) \to \mathrm{Irr}\,\mathrm{U}(W_a).$$

*Thus we can identify* $\Phi(\mathrm{U}(W_a))$ *with* $\mathrm{Irr}\,\mathrm{U}(W_a)$.

To explain the classification of $\Phi(G)$ where $G = \mathrm{U}(V_{sp}), \mathrm{U}(V_{an})$ or $\mathrm{U}(W)$, we prepare the following lemma:

**Lemma 1.7.** *Let* $\varphi : W_E \to \mathrm{GL}_n(\mathbb{C})$ *be an irreducible $n$-dimensional representation of* $W_E$*. Assume that* $\varphi \circ \mathrm{Ad}(w_\sigma) \cong {}^t\varphi^{-1}$.

(i) *There exists* $X \in \mathrm{GL}_n(\mathbb{C})$ *such that* $\varphi \circ \mathrm{Ad}(w_\sigma) = X I_n {}^t\varphi^{-1} I_n^{-1} X^{-1}$. *Also such an $X$ exists unique up to scalar.*

(ii) *There exists* $c(\varphi) \in \{\pm 1\}$ *such that* $\varphi(w_\sigma^2) = c(\varphi) X I_n {}^t X^{-1} I_n^{-1}$.

(iii) $c(\varphi) = 1$ *if and only if $\varphi$ extends to an $L$-parameter $\phi$ of $G$ such that* $\phi|_{\mathrm{SU}_2(\mathbb{R})}$ *is trivial. Also such an $L$-parameter $\phi$ is uniquely.*

(iv) *If $\varphi$ extends to an $L$-parameter $\phi$ of $G$, then $\mathcal{S}_\phi = \{1\}$.*

*Proof.* (i) This follows from Schur's lemma.
(ii) For any element $w$ of $W_E$,

$$\begin{aligned}\varphi(w_\sigma^2)\varphi(w)\varphi(w_\sigma^2)^{-1} &= \varphi(w_\sigma^2 w w_\sigma^{-2}) \\ &= XI_n{}^t\varphi(w_\sigma w w_\sigma^{-1})^{-1}I_n^{-1}X^{-1} \\ &= XI_n{}^t(XI_n{}^t\varphi(w)^{-1}I_n^{-1}X^{-1})^{-1}I_n^{-1}X^{-1} \\ &= XI_n{}^tX^{-1t}I_n^{-1}\varphi(w)^tI_n{}^tXI_n^{-1}X^{-1} \\ &= (XI_n{}^tX^{-1}I_n^{-1})\varphi(w)(XI_n{}^tX^{-1}I_n^{-1})^{-1}.\end{aligned}$$

By Schur's lemma, there exists a scalar $c(\varphi)$ such that $\varphi(w_\sigma^2) = c(\varphi)XI_n{}^tX^{-1}I_n^{-1}$. Substituting $w = w_\sigma^2$ for the equation in (i), we obtain $c(\varphi)^2 = 1$.
(iii) Assume that $\varphi$ extends to an $L$-parameter $\phi$ of $G$. If we write $\phi(w_\sigma) = X \rtimes w_\sigma$, then $X$ is satisfied with the equation in (i). Since

$$\varphi(w_\sigma^2) \rtimes w_\sigma^2 = \phi(w_\sigma)^2 = XI_n{}^tX^{-1}I_n^{-1} \rtimes w_\sigma^2,$$

we have $c(\varphi) = 1$. Conversely we assume that $c(\varphi) = 1$. If we write $\phi(w_\sigma) = X \rtimes w_\sigma$, then we can easily prove that $\phi$ is an $L$-parameter of $G$. Also for any scalar $\alpha$, we obtain $\mathrm{Ad}(\alpha) \circ \phi(w_\sigma) = \alpha^2 X \rtimes w_\sigma$. Thus the equivalence class of $\phi$ is independent of the choice of $X$.
(iv) First we have $Z(\hat{G}) = \mathbb{C}^\times$ and $Z(\hat{G})^\Gamma = \{\pm 1\}$. Also we obtain

$$\begin{aligned}S_\phi &= \mathrm{Cent}_{\hat{G}}(\mathrm{Im}\,\varphi) \cap \mathrm{Cent}_{\hat{G}}(\phi(w_\sigma)) \\ &= Z(\hat{G}) \cap \mathrm{Cent}_{\hat{G}}(\phi(w_\sigma)) \\ &= \{z \in Z(\hat{G}) \mid z\phi(w_\sigma)z^{-1} = \phi(w_\sigma)\} \\ &= \{\pm 1\}.\end{aligned}$$

Thus we have $\mathcal{S}_\phi = \{1\}$. □

By this lemma, we define the following sets:

$$\begin{aligned}\mathrm{Irr}\,W_{E,n,+} &:= \left\{\varphi \in \mathrm{Irr}\,W_E \;\middle|\; \dim\varphi = n,\ \varphi \circ \mathrm{Ad}(w_\sigma) \cong {}^t\varphi^{-1},\ c(\varphi) = 1\right\}; \\ \mathrm{Irr}\,W_{E,n,-} &:= \left\{\varphi \in \mathrm{Irr}\,W_E \;\middle|\; \dim\varphi = n,\ \varphi \circ \mathrm{Ad}(w_\sigma) \cong {}^t\varphi^{-1},\ c(\varphi) = -1\right\}.\end{aligned}$$

Now we explain the classification of the $L$-parameters of $G = \mathrm{U}(V_{sp}), \mathrm{U}(V_{an})$. Here note that $\Phi(\mathrm{U}(V_{sp})) = \Phi(\mathrm{U}(V_{an}))$. So we consider only $\Phi(\mathrm{U}(V_{sp}))$. Take an $L$-parameter $\phi$ of $\mathrm{U}(V_{sp})$. $\phi_E$ is a two-dimensional representation of $L_E$. Then we have the following three cases;

Case A: $\phi|_{\mathrm{SU}_2(\mathbb{R})} = \mathrm{Sym}^1$;
Case B: $\phi_E$ is an irreducible representation of $W_E$;
Case C: $\phi_E$ is a reducible representation of $W_E$.

The classification of the $L$-parameters of $\mathrm{U}(V_{sp})$ is given by the similar argument for that of $\mathrm{U}(W)$, which was given in [12].

**Proposition 1.8.** $\Phi(\mathrm{U}(V_{sp})) \ni \phi \mapsto \phi_E \in \Phi(\mathrm{GL}_2(E))$ *is injective. Also the $L$-parameters $\phi$ of* $\mathrm{U}(V_{sp})$ *are classified as follows:*

| $\phi$ | $\phi_E$ | | $\mathcal{S}_\phi$ |
|---|---|---|---|
| *Case A* | $\eta \boxtimes \mathrm{Sym}^1$ | $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\{1\}$ |
| *Case B* | $\phi_E$ | $\phi_E \in \mathrm{Irr}\, W_{E,2,+}$ | $\{1\}$ |
| *Case $C_1$* | $\lambda \oplus \lambda^{-1} \circ \sigma$ | $\lambda \neq \lambda^{-1} \circ \sigma \in \mathrm{Irr}\, E^\times$ | $\{1\}$ |
| *Case $C_2$* | $\mu_1 \oplus \mu_2$ | $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$ | $\mathbb{Z}/2\mathbb{Z}$ |
| *Case $C_3$* | $\eta \oplus \eta$ | $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\{1\}$ |
| *Case $C_4$* | $\mu \oplus \mu$ | $\mu \in \Pi(E^\times, \omega_{E/F})$ | $\mathbb{Z}/2\mathbb{Z}$ |

Next we explain the classification of the $L$-parameters of $G = \mathrm{U}(W)$. Take an $L$-parameter $\phi$ of $\mathrm{U}(W)$. $\phi_E$ is a three-dimensional representation of $L_E$. Then we have the following five cases;

Case A: $\phi|_{\mathrm{SU}_2(\mathbb{R})} = \mathrm{Sym}^2$;
Case B: $\phi|_{\mathrm{SU}_2(\mathbb{R})} = \mathrm{Sym}^1 \oplus \mathrm{Sym}^0$;
Case C: $\phi_E$ is an irreducible representation of $W_E$;
Case D: $\phi_E$ has a two-dimensional irreducible representation of $W_E$;
Case E: $\phi_E$ is direct sum of quasi-characters of $W_E$.

**Proposition 1.9** ([12])**.** *The map* $\Phi(\mathrm{U}(W)) \ni \phi \mapsto \phi_E \in \Phi(\mathrm{GL}_3(E))$ *is injective. Also the $L$-parameters $\phi$ of* $\mathrm{U}(W)$ *are classified as follows:*

| $\phi$ | $\phi_E$ | | $\mathcal{S}_\phi$ |
|---|---|---|---|
| $CaseA$ | $\eta \boxtimes \mathrm{Sym}^2$ | $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\{1\}$ |
| $CaseB$ | $\mu \boxtimes \mathrm{Sym}^1 \oplus \eta$ | $\mu \in \Pi(E^\times, \omega_{E/F})$, $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $CaseC$ | $\phi_E$ | $\phi_E \in \mathrm{Irr}\, W_{E,3,+}$ | $\{1\}$ |
| $CaseD$ | $\tau \oplus \eta$ | $\tau \in \mathrm{Irr}\, W_{E,2,-}$, $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $CaseE_1$ | $\eta_1 \oplus \eta_2 \oplus \eta_3$ | $\eta_i \in \Pi(E^\times, \mathbb{1}_{F^\times})$, $\eta_i \neq \eta_j$ | $(\mathbb{Z}/2\mathbb{Z})^2$ |
| $CaseE_2$ | $\lambda \oplus \eta \oplus \lambda^{-1} \circ \sigma$ | $\lambda \neq \lambda^{-1} \circ \sigma \in \mathrm{Irr}\, E^\times$, $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\{1\}$ |
| $CaseE_3$ | $\eta_1 \oplus \eta_2 \oplus \eta_1$ | $\eta_1 \neq \eta_2 \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $CaseE_4$ | $\mu \oplus \eta \oplus \mu$ | $\mu \in \Pi(E^\times, \omega_{E/F})$, $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\{1\}$ |
| $CaseE_5$ | $\eta \oplus \eta \oplus \eta$ | $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ | $\{1\}$ |

We introduce the definition of a tempered $L$-parameter of $G$.

**Definition 1.10.** Let $\phi$ be an $L$-parameter of $G$. We call $\phi$ a *tempered $L$-parameter* if the image of the following homomorphism is relatively compact:

$$L_F \xrightarrow{\phi} {}^L G \xrightarrow{\mathrm{proj.}} \hat{G}.$$

**Lemma 1.11.** *Let $\phi$ be an $L$-parameter of $G$. Recall that $\phi_E = \bigoplus_i \varphi_i \boxtimes \mathrm{Sym}^{r_i - 1}$. Then $\phi$ is tempered if and only if $\det \circ \varphi_i$ is a unitary character of $W_E$ for any $i$.*

*Proof.* Since the index of $L_E$ in $L_F$ is 2, it is clear that $\phi$ is tempered if and only if $\phi_E$ is. Also $\phi_E$ is tempered if and only if $\varphi_i$ is a unitary representation of $W_E$ for any $i$, since $\mathrm{Sym}^{r_i - 1}$ has a compact image in $\hat{G}$. Thus we must show that for an irreducible representation $\varphi$ of $W_E$, $\varphi$ is unitary if and only if $\det \circ \varphi$ is a unitary character of $W_E$.

If $\varphi$ is an irreducible unitary representation, it is clear that $\det \circ \varphi$ is a unitary character. Conversely, assume that $\det \circ \varphi$ is a unitary character. Let $I_E$ be the inertia group of $E$ and $Fr_E$ be a Frobenius element. Then $\varphi(Fr_E)$ normalizes $\varphi(I_E)$. Also since $I_E$ is a profinite group, $\varphi(I_E)$ is a finite group. Thus there exists an integer $l$ such that $\varphi(Fr_E^l)$ commutes with $\varphi(W_E)$. Since $\varphi$ is an irreducible representation, by Schur's Lemma, $\varphi(Fr_E^l)$ is a scalar. By the assumption of $\varphi$, the absolute value of this scalar is one. If we take an unramified character $\chi$ such that $\chi(Fr_E^l) = \varphi(Fr_E^l)^{-1}$, then $\chi \otimes \varphi(W_E)$ is a finite group. Therefore $\chi \otimes \varphi$ is a unitary representation. Since $\chi$ is a unitary character, $\varphi$ is unitary .

□

By this lemma, we can classify the tempered $L$-parameters of $G$. Any $L$-parameter of $\mathrm{U}(W_a)$ is tempered. For an $L$-parameter $\phi$ in the Case $\mathrm{C}_1$ for $\mathrm{U}(V_{sp})$ or $\mathrm{E}_2$ for $\mathrm{U}(W)$, $\phi$ is tempered if and only if the corresponding $\lambda$ is a unitary character. The other cases are all tempered $L$-parameters.

# 2 Endoscopy for U(2)

In this section, we explain the local and global endoscopy for $\mathrm{U}(V_{sp})$ and $\mathrm{U}(V_{an})$ following [5].

## 2.1 Local endoscopy for $\mathrm{U}(V_{sp})$

In this subsection, we may assume that $F$ is a local field of characteristic 0 such that $F \neq \mathbb{C}$. Set $Z = \begin{pmatrix} 1 & \\ & \xi \end{pmatrix}$. We recall the following isomorphism:

$$E^\times \times \mathrm{GL}_2(F)/\Delta F^\times \ni (x, g) \mapsto x \det(g)^{-1} Z g Z^{-1} \in \mathrm{GU}(V_{sp}).$$

Identify $\mathrm{GU}(V_{sp})$ with $E^\times \times \mathrm{GL}_2(F)/\Delta F^\times$. Then the $L$-group of $\mathrm{GU}(V_{sp})$ is given as follows:

$$
\begin{gathered}
{}^L\,\mathrm{GU}(V_{sp}) = \widehat{\mathrm{GU}}(V_{sp}) \rtimes W_F, \\
\widehat{\mathrm{GU}}(V_{sp}) = \{(z_1, z_2, g) \in \mathbb{C}^\times \times \mathbb{C}^\times \times \mathrm{GL}_2(\mathbb{C}) \mid z_1 z_2 \det g = 1\}, \\
(z_1, z_2, g) \rtimes w \cdot (z_1', z_2', g') \rtimes w' = \begin{cases} (z_1 z_1', z_2 z_2', g g') \rtimes w w' & \text{if } w \in W_E, \\ (z_1 z_2', z_2 z_1', g g') \rtimes w w' & \text{if } w \notin W_E. \end{cases}
\end{gathered}
$$

Also the $L$-homomorphism corresponding to the inclusion $\iota : \mathrm{U}(V_{sp}) \hookrightarrow \mathrm{GU}(V_{sp})$ is given by

$${}^L\iota : {}^L\,\mathrm{GU}(V_{sp}) \ni (z_1, z_2, g) \rtimes w \mapsto z_1 g \rtimes w \in {}^L\,\mathrm{U}(V_{sp}).$$

Let $\phi'$ be a two-dimensional representation of $L_F$. By Fact 1.5, there exists an irreducible admissible representation $\tau(\phi')$ of $\mathrm{GL}_2(F)$ with $L$-parameter $\phi'$. Let $\chi$ be a quasi-character of $E^\times$ such that $\chi^{-1}|_{F^\times} = \det \phi'$. Then $\chi \boxtimes \tau(\phi')$ is an irreducible admissible representation of $\mathrm{GU}(V_{sp})$. Also the $L$-parameter of its representation is given by

$$L_F \ni (w, g) \longmapsto \begin{cases} (\chi(w), \chi(w), \phi'(w, g)) \rtimes w & \text{if } w \in W_E, \\ (1, \chi(w_\sigma^2), \phi'(w_\sigma, g)) \rtimes w_\sigma & \text{if } w = w_\sigma. \end{cases}$$

Let $\phi$ be the composite of this $L$-parameter and ${}^L\iota$. Then $\phi_E = \chi \otimes (\phi'|_{L_E})$. Since $\phi$ is an $L$-parameter of $\mathrm{U}(V_{sp})$, then define the $L$-packet $\Pi_\phi$ associated to $\phi$ by the set of irreducible subrepresentations of $\chi \boxtimes \tau(\phi')|_{\mathrm{U}(V_{sp})}$.

**Theorem 2.1** ([5]). (i) $\chi \boxtimes \tau(\phi')|_{\mathrm{U}(V_{sp})}$ *is multiplicity free.*
(ii) $\Pi_\phi$ *has at most two elements.*
(iii) $\Pi_{\phi_1} \cap \Pi_{\phi_2}$ *is not empty if and only if* $\phi_1 \sim \phi_2$. *In that case,* $\Pi_{\phi_1} = \Pi_{\phi_2}$.
(iv) *Any irreducible admissible representation of* $\mathrm{U}(V_{sp})$ *is contained an L-packet* $\Pi_\phi$.
(v) $\Pi_\phi$ *has two elements if and only if there exist characters* $\mu_1, \mu_2 \in \Pi(E^\times, \omega_{E/F})$ *such that* $\phi_E = \mu_1 \oplus \mu_2$.

In (v), we denote the $L$-packet $\Pi_\phi$ by $\Pi_{\mu_1,\mu_2} = \Pi_{\mu_1,\mu_2}(V_{sp})$ and call it an *endoscopic L-packet*. Also its elements are called *endoscopic representations*. Since we can prove that there exists $\chi' \in \operatorname{Irr} E^\times$ such that $\mu_1 = \chi\chi'$ and $\mu_2 = \chi(\chi' \circ \sigma)$, then we may take $\phi' = \operatorname{Ind}_{W_E}^{W_F} \chi'$.

The endoscopic description of each endoscopic $L$-packet $\Pi_{\mu_1,\mu_2}$ of $\mathrm{U}(V_{sp})$ can be given in terms of genericity. We denote by $B$ the Borel subgroup of $\mathrm{U}(V_{sp})$ consisting of lower triangular matrices. Let

$$U = \left\{ u(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \middle| \, x \in E, \mathrm{Tr}_{E/F}(x) = 0 \right\}$$

be the unipotent radical of $B$ and

$$T = \left\{ m(\alpha) = \begin{pmatrix} \alpha & \\ & {}^*\alpha^{-1} \end{pmatrix} \middle| \, \alpha \in E^\times \right\}$$

a maximal torus of $\mathrm{U}(V_{sp})$. Thus we have $B = TU$. For each $b \in F^\times$, we define a non-trivial character $\psi^b_{U,\xi}$ of $U$ by

$$\psi^b_{U,\xi} : U \ni u(x) \mapsto \psi(-bx\xi^{-1}) \in \mathbb{C}^1. \tag{1}$$

An irreducible admissible representation $\pi$ of $\mathrm{U}(V_{sp})$ is called $\psi^b$-generic, or $\psi^b_{U,\xi}$-generic, if it has a non-zero homomorphism into $\operatorname{Ind}_U^{\mathrm{U}(V_{sp})} \psi^b_{U,\xi}$. By the uniqueness of Whittaker model of $\chi \boxtimes \tau(\phi')$, an endoscopic $L$-packet $\Pi_{\mu_1,\mu_2}$ has the unique $\psi$-generic (resp. $\psi^{d_0}$-generic) element. We write it as $\tau(\mu_1, \mu_2)^+_{sp}$ (resp. $\tau(\mu_1, \mu_2)^-_{sp}$).

Also such a description of $\Pi_{\mu_1,\mu_2}$ can be characterized by the character identity. For $\delta \in E^\times$ such that $\mathrm{Tr}_{E/F}(\delta) = 0$, we set

$$T_\delta = \left\{ t_\delta(z, z') = z \begin{pmatrix} x & -\xi^{-1}y \\ -\delta^2\xi y & x \end{pmatrix} \in \mathrm{U}(V_{sp}) \middle| \, \begin{matrix} z, z' = x + y\delta \in E^\times \\ \mathrm{N}_{E/F}(zz') = 1 \end{matrix} \right\}.$$

**Proposition 2.2** ([5]). *We have the following character identity:*

$$\begin{aligned}&\operatorname{Tr}\tau(\mu_1,\mu_2)^{+}_{sp}(t_\delta(z,z')) - \operatorname{Tr}\tau(\mu_1,\mu_2)^{-}_{sp}(t_\delta(z,z'))\\ &= \lambda\,(E/F,\psi)\,\omega_{E/F}\left(\frac{z'-\sigma(z')}{\sigma(\delta)-\delta}\right)\chi\left(z\,\mathrm{N}_{E/F}(z')\right)\frac{\chi'(\sigma(z'))+\chi'(z')}{|z'-\sigma(z')|_E^{1/2}}|z'|_E^{1/2}.\end{aligned}$$

*Also the last equation is independent of the choices of $\chi, \chi'$ such that $\mu_1 = \chi\chi'$ and $\mu_2 = \chi(\chi' \circ \sigma)$.*

It is easy to check the following:

**Corollary 2.3.** (i) *For each $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we have $\eta_u \circ \det \otimes \tau(\mu_1,\mu_2)^{\varepsilon}_{sp} = \tau(\eta\mu_1, \eta\mu_2)^{\varepsilon}_{sp}$.*
(ii) *The representation $\tau(\mu_1^{-1},\mu_2^{-1})^{\omega_{E/F}(-1)\varepsilon}_{sp}$ is the contragredient representation of $\tau(\mu_1,\mu_2)^{\varepsilon}_{sp}$.*
(iii) *For $g \in \mathrm{GU}(V_{sp})$ with similitude norm $a$, we have $\tau(\mu_1,\mu_2)^{\varepsilon}_{sp} \circ \mathrm{Ad}(g) = \tau(\mu_1,\mu_2)^{\omega_{E/F}(a)\varepsilon}_{sp}$.*

Finally we explicitly state the $L$-packet $\Pi_\phi$ for each $L$-parameter $\phi$ that is classified in §1.2.

**Lemma 2.4.** *Let $B^-$ be the Borel subgroup of $\mathrm{GL}_2(F)$ which consists of the lower-triangular matrices. For any $\lambda \in \operatorname{Irr} E^\times$ and $\lambda_1, \lambda_2 \in \operatorname{Irr} F^\times$ such that $\lambda|_{F^\times}\lambda_1\lambda_2 = 1$, we have*

$$\lambda \boxtimes \operatorname{Ind}_{B^-}^{\mathrm{GL}_2(F)}(\lambda_1 \boxtimes \lambda_2)|_{\mathrm{U}(V_{sp})} = \operatorname{Ind}_B^{\mathrm{U}(V_{sp})}\lambda(\lambda_1 \circ \mathrm{N}_{E/F}).$$

*Proof.* This follows from

$$\begin{aligned}&(E^\times \times B^-/\Delta F^\times)\cdot \mathrm{U}(V_{sp}) = \mathrm{GU}(V_{sp})\\ &(E^\times \times B^-/\Delta F^\times)\cap \mathrm{U}(V_{sp}) = B.\end{aligned}$$

□

Case A: $\phi_E = \eta \boxtimes \mathrm{Sym}^1$

If we take $\chi = \eta$ and $\phi' = \mathrm{Sym}^1$, then we obtain $\phi_E = \eta \boxtimes \mathrm{Sym}^1$. Since $\tau(\phi')$ is the Steinberg representation of $\mathrm{GL}_2(F)$, $\eta \boxtimes \tau(\phi')$ is the unique irreducible quotient of $\eta \boxtimes \operatorname{Ind}_{B^-}^{\mathrm{GL}_2(F)} |\;|_F^{1/2} \boxtimes |\;|_F^{-1/2}$. Here $\operatorname{Ind}_{B^-}^{\mathrm{GL}_2(F)} |\;|_F^{1/2} \boxtimes |\;|_F^{-1/2}$ has the unique subrepresentation $\mathbb{1}$ and the unique quotient $\tau(\phi')$. Similarly

$\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\eta|\ |_E^{1/2}$ has the unique subrepresentation $\eta_u \circ \det$ and the unique quotient $\mathrm{St}(\eta)$. Thus by Lemma 2.4 we have $\eta \boxtimes \tau(\phi')|_{\mathrm{U}(V_{sp})} = \mathrm{St}(\eta)$. Therefore $\Pi_\phi = \{\mathrm{St}(\eta)\}$.

Case B: $\phi_E \in \mathrm{Irr}\, W_{E,2,+}$

Then we can take a two-dimensional irreducible representation of $W_F$ as $\phi'$. Thus $\tau(\phi')$ is supercuspidal. By Theorem 2.1, $\chi \boxtimes \tau(\phi')|_{\mathrm{U}(V_{sp})}$ is an irreducible supercuspidal representation $\tau$ of $\mathrm{U}(V_{sp})$.

Case $\mathrm{C}_1$: $\phi_E = \lambda \oplus \lambda^{-1} \circ \sigma \quad (\lambda \neq \lambda^{-1} \circ \sigma)$

First assume that $\lambda \circ \mathrm{N}_{E/F} \neq |\ |_E^{\pm 1}$. If we take $\chi = \lambda$ and $\phi' = \mathbb{1} \oplus \lambda^{-1}|_{F^\times}$, then $\lambda \boxtimes \tau(\phi') = \lambda \boxtimes \mathrm{Ind}_{B^-}^{\mathrm{GL}_2(F)}\, \mathbb{1} \boxtimes (\lambda^{-1}|_{F^\times})$. Here $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\lambda$ is irreducible. Thus by Lemma 2.4 we have $\Pi_\phi = \{\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\lambda\}$. Next assume that $\chi \circ \mathrm{N}_{E/F} = |\ |_E$ without loss of generality. Then $\lambda = \eta|\ |_E^{1/2}$ or $\mu|\ |_E^{1/2}$. If $\lambda = \eta|\ |_E^{1/2}$, then we may take $\chi = \eta$ and $\phi' = |\ |_F^{1/2} \oplus |\ |_F^{-1/2}$. Thus $\tau(\phi') = \det$ and $\eta \boxtimes \tau(\phi')|_{\mathrm{U}(V_{sp})} = \eta_u \circ \det$. Therefore $\Pi_\phi = \{\eta_u \circ \det\}$. If $\lambda = \mu|\ |_E^{1/2}$, then we may take $\chi = \mu$ and $\phi' = \omega_{E/F}|\ |_F^{1/2} \oplus |\ |_F^{-1/2}$. Thus $\tau(\phi') = \mathrm{Ind}_{B^-}^{\mathrm{GL}_2(F)}\,\omega_{E/F}|\ |_F^{1/2} \boxtimes |\ |_F^{-1/2}$ and $\eta \boxtimes \tau(\phi')|_{\mathrm{U}(V_{sp})} = \mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\mu|\ |_E^{1/2}$. This representation is irreducible. Therefore $\Pi_\phi = \{\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\mu|\ |_E^{1/2}\}$.

Case $\mathrm{C}_2$: $\phi_E = \mu_1 \oplus \mu_2$

By Theorem 2.1, we have $\Pi_\phi = \{\tau(\mu_1, \mu_2)_{sp}^{\pm}\}$.

Case $\mathrm{C}_3$: $\phi_E = \eta \oplus \eta$

If we take $\chi = \eta$ and $\phi' = \mathbb{1} \oplus \mathbb{1}$, then $\eta \boxtimes \tau(\phi') = \eta \boxtimes \mathrm{Ind}_{B^-}^{\mathrm{GL}_2(F)}\,\mathbb{1} \boxtimes \mathbb{1}$. Here $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\eta$ is irreducible. Thus we have $\Pi_\phi = \{\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\eta\}$

Case $\mathrm{C}_4$: $\phi_E = \mu \oplus \mu$

By Theorem 2.1, we have $\Pi_\phi = \{\tau(\mu,\mu)_{sp}^{\pm}\}$. On the other hand, if we take $\chi = \mu$ and $\phi' = \omega_{E/F} \oplus \mathbb{1}$, then $\phi_E = \mu \oplus \mu$. Then $\tau(\phi') = \mathrm{Ind}_{B^-}^{\mathrm{GL}_2(F)}\,\omega_{E/F} \boxtimes \mathbb{1}$ and $\mu \boxtimes \tau(\phi')|_{\mathrm{U}(V_{sp})} = \mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\mu$. Since this representation is completely reducible, $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\mu = \tau(\mu,\mu)_{sp}^{+} \oplus \tau(\mu,\mu)_{sp}^{-}$.

We have the following table:

| $\phi$ | $\phi_E$ | $\Pi_\phi$ |
|---|---|---|
| Case A | $\eta \boxtimes \mathrm{Sym}^1$ | $\mathrm{St}(\eta)$ |
| Case B | $\phi_E$ | $\tau$ |
| Case $\mathrm{C}_{1a}$ | $\lambda \oplus \lambda^{-1} \circ \sigma$ <br> $(\lambda \circ \mathrm{N}_{E/F} \neq \mid \mid^{\pm 1})$ | $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}(\lambda)$ |
| Case $\mathrm{C}_{1b}$ | $\eta\mid \mid_E^{1/2} \oplus \eta\mid \mid_E^{-1/2}$ | $\eta_u \circ \det$ |
| Case $\mathrm{C}_{1c}$ | $\mu\mid \mid_E^{1/2} \oplus \mu\mid \mid_E^{-1/2}$ | $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}(\mu\mid \mid_E^{1/2})$ |
| Case $\mathrm{C}_2$ | $\mu_1 \oplus \mu_2$ | $\tau(\mu_1, \mu_2)_{sp}^{\pm}$ |
| Case $\mathrm{C}_3$ | $\eta \oplus \eta$ | $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}(\eta)$ |
| Case $\mathrm{C}_4$ | $\mu \oplus \mu$ | $\tau(\mu, \mu)_{sp}^{\pm}$ |

## 2.2 Global endoscopy for $\mathrm{U}(V_{sp})$

Let $\tilde{E}/\tilde{F}$ be a quadratic extension of a number field. Take a non-zero element $\tilde{\xi}$ of $\tilde{E}$ with $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and a non-trivial character $\tilde{\psi}$ of $\mathbb{A}_{\tilde{F}}/\tilde{F}$. Denote the quadratic character of $\mathbb{A}_{\tilde{F}}^\times/\tilde{F}^\times$ associated with $\tilde{E}/\tilde{F}$ by $\omega_{\tilde{E}/\tilde{F}}$. Also for each character $\chi$ of $\mathbb{A}_{\tilde{F}}^\times/\tilde{F}^\times$, $\Pi(\mathbb{A}_{\tilde{E}}^\times, \chi)$ stands for the set of characters of $\mathbb{A}_{\tilde{E}}^\times/\tilde{E}^\times$ whose restriction to $\mathbb{A}_{\tilde{F}}^\times$ are $\chi$.

We define a two-dimensional hermitian space $\tilde{V}_{sp}$ over $\tilde{E}$:

$$\tilde{V}_{sp} = (\tilde{E}^{\oplus 2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}).$$

Also take two characters $\tilde{\mu}_1, \tilde{\mu}_2 \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$. Then for any place $v$ of $\tilde{F}$, we have the local $L$-packet $\Pi_{\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v}}(\tilde{V}_{sp})$ of $\mathrm{U}(\tilde{V}_{sp,v})$ associated to the $L$-parameter $\varphi_{\tilde{E}_v} = \tilde{\mu}_{1,v} \oplus \tilde{\mu}_{2,v}$. If $\tilde{E}_v$ is a field, the local $L$-packet $\Pi_{\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v}}(\tilde{V}_{sp})$ consists of two elements and has only one $\tilde{\psi}_v$-generic element. Here $\tilde{\psi}_v$-genericity is defined by using $\tilde{\xi}$ as in §2.1 (1). Then we denote the $\tilde{\psi}_v$-generic element by $\tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})_{sp}^+$. The other element is denoted by $\tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})_{sp}^-$. If $\tilde{E}_v$ is not a field, then $\mathrm{U}(\tilde{V}_{sp,v}) \cong \mathrm{GL}_2(\tilde{F}_v)$. Thus the local $L$-packet $\Pi_{\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v}}(\tilde{V}_{sp})$ consists of one $\tilde{\psi}_v$-generic element. It is denoted by $\tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})_{sp}^+$.

We define a global endoscopic $L$-packet $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}_{sp})$ of $\mathrm{U}(\tilde{V}_{sp})$ by

$$\left\{ \otimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})_{sp}^{\epsilon_v} \,\middle|\, \epsilon_v = + \text{ for almost all } v \right\}.$$

We have the following:

**Theorem 2.5** ([5]). *Let* $\tilde{\tau} = \otimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})_{sp}^{\epsilon_v}$ *be an element of* $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V}_{sp})$. *Then* $\tilde{\tau}$ *is a cuspidal representation of* $\mathrm{U}(\tilde{V}_{sp})$ *if and only if* $\Pi_v \epsilon_v = +$ *and* $\tilde{\mu}_1 \neq \tilde{\mu}_2$.

**Theorem 2.6** ([15]). *Let* $\tau = \otimes \tau_v$ *be an irreducible cuspidal representation of* $\mathrm{U}(\tilde{V}_{sp})$. *If there exist characters* $\tilde{\mu}_1, \tilde{\mu}_2 \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ *such that* $\tau_v \in \Pi_{\tilde{\mu}_{1,v},\tilde{\mu}_{2,v}}(\tilde{V}_{sp})$ *for almost all places* $v$, *then* $\tau_v \in \Pi_{\tilde{\mu}_{1,v},\tilde{\mu}_{2,v}}(\tilde{V}_{sp})$ *for all places* $v$.

## 2.3 Local endoscopy for $\mathrm{U}(V_{an})$

Let $D$ be the quaternion division algebra over a non-archimedean local field $F$. Denote its reduced norm and trace by $\nu_D, \mathrm{tr}_D$, respectively. We fix an embedding $E \hookrightarrow D$. Then there exists $\xi' \in D$ such that $\mathrm{tr}_D(\xi') = 0$, $\xi\xi' = -\xi'\xi$ and

$$\begin{aligned} D &= F \oplus F\xi \oplus F\xi' \oplus F\xi'\xi \\ &= E \oplus \xi' E. \end{aligned}$$

Also we may assume that $\xi'^{,2} = d_0$. Then we have the following isomorphism:

$$E^\times \times D^\times/\Delta F^\times \ni (x, z+\xi' w) \mapsto x\nu_D(z+\xi' w)^{-1} \begin{pmatrix} \sigma(z) & w \\ d_0\sigma(w) & z \end{pmatrix} \in \mathrm{GU}(V_{an}).$$

We identify $\mathrm{GU}(V_{an})$ with $E^\times \times D^\times/\Delta F^\times$. Then the $L$-group of $\mathrm{GU}(V_{an})$ is the same as $\mathrm{GU}(V_{sp})$.

Let $\phi'$ be an irreducible two-dimensional representation of $L_F$. Then there exists an irreducible admissible representation $\tau_D(\phi')$ of $D^\times$ with $L$-parameter $\phi'$. Namely, it is the representation of $D^\times$ associated to $\tau(\phi')$ by the Jaquet-Langlands correspondence. Let $\chi$ be a quasi-character of $E^\times$ such that $\chi^{-1}|_{F^\times} = \det\phi'$. Then $\chi \boxtimes \tau_D(\phi')$ is an irreducible admissible representation of $\mathrm{GU}(V_{an})$. Its $L$-parameter is the same as one of $\chi \boxtimes \tau(\phi')$ in §2.1. Take $\phi$ as in §2.1. Define the $L$-packet $\Pi_\phi$ associated to $\phi$ by the set of irreducible subrepresentations of $\chi \boxtimes \tau_D(\phi')|_{\mathrm{U}(V_{an})}$. If $\phi'$ is a reducible two-dimensional representation of $L_F$, we define the $L$-packet $\Pi_\phi$ to be empty.

**Theorem 2.7** ([5]). (i) $\chi \boxtimes \tau_D(\phi')|_{\mathrm{U}(V_{an})}$ *is multiplicity free.*
(ii) $\Pi_\phi$ *has at most two elements.*
(iii) $\Pi_{\phi_1} \cap \Pi_{\phi_2}$ *is not empty if and only if* $\phi_1 \sim \phi_2$. *In that case,* $\Pi_{\phi_1} = \Pi_{\phi_2}$.

(iv) *Any irreducible admissible representation of* $\mathrm{U}(V_{an})$ *is contained an* $L$-*packet* $\Pi_\phi$.
(v) $\Pi_\phi$ *has two elements if and only if there exist characters* $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$ *such that* $\phi_E = \mu_1 \oplus \mu_2$.

In (v), we denote the $L$-packet $\Pi_\phi$ by $\Pi_{\mu_1,\mu_2} = \Pi_{\mu_1,\mu_2}(V_{an})$ and call it an *endoscopic* $L$*-packet*. Also its elements are called *endoscopic representations*. Since we can prove that there exists $\chi' \in \mathrm{Irr}\, E^\times$ such that $\mu_1 = \chi\chi'$ and $\mu_2 = \chi(\chi' \circ \sigma)$, we may take $\phi' = \mathrm{Ind}_{W_E}^{W_F} \chi'$.

We set

$$T = \left\{ t(z, z') = \begin{pmatrix} zz' & \\ & z\sigma(z') \end{pmatrix} \in \mathrm{U}(V_{an}) \,\middle|\, \begin{array}{c} z, z' \in E^\times, \\ N_{E/F}(zz') = 1 \end{array} \right\}. \qquad (2)$$

The endoscopic description of each endoscopic $L$-packet of $\mathrm{U}(V_{an})$ is given in terms of character identity as follows:

**Proposition 2.8** ([5])**.** *There exist two irreducible admissible representations* $\tau(\mu_1, \mu_2)_{an}^+ \not\cong \tau(\mu_1, \mu_2)_{an}^-$ *of* $\mathrm{U}(V_{an})$ *such that* $\Pi_{\mu_1,\mu_2} = \{\tau(\mu_1, \mu_2)_{an}^\pm\}$ *and the following character identity holds:*

$$\begin{aligned} &\mathrm{Tr}\,\tau(\mu_1, \mu_2)_{an}^+(t(z, z')) - \mathrm{Tr}\,\tau(\mu_1, \mu_2)_{an}^-(t(z, z')) \\ &= \lambda\,(E/F, \psi)\, \omega_{E/F}\left(\frac{z' - \sigma(z')}{\xi}\right) \chi\left(z\, \mathrm{N}_{E/F}(z')\right) \frac{\chi'(\sigma(z')) - \chi'(z')}{|z' - \sigma(z')|_E^{1/2}} |z'|_E^{1/2}. \end{aligned}$$

*Also the last equation is independent of the choices of* $\chi, \chi'$ *such that* $\mu_1 = \chi\chi'$ *and* $\mu_2 = \chi(\chi' \circ \sigma)$.

It is easy to check the following:

**Corollary 2.9.** (i) *For any* $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$ *and* $\varepsilon \in \{\pm\}$, *we have* $\tau(\mu_1, \mu_2)_{an}^\varepsilon = \tau(\mu_2, \mu_1)_{an}^{-\varepsilon}$.
(ii) *For each* $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, *we have* $\eta_u \circ \det \otimes \tau(\mu_1, \mu_2)_{an}^\varepsilon = \tau(\eta\mu_1, \eta\mu_2)_{an}^\varepsilon$.
(iii) *The representation* $\tau(\mu_1^{-1}, \mu_2^{-1})_{an}^{\omega_{E/F}(-1)\varepsilon}$ *is the contragredient representation of* $\tau(\mu_1, \mu_2)_{an}^\varepsilon$.
(iv) *For* $g \in \mathrm{GU}(V_{an})$ *with similitude norm* $a$, *we have* $\tau(\mu_1, \mu_2)_{an}^\varepsilon \circ \mathrm{Ad}(g) = \tau(\mu_1, \mu_2)_{an}^{\omega_{E/F}(a)\varepsilon}$.

In particular, the description of $\Pi_{\mu_1,\mu_2}(V_{an})$ depends on the order of the pair $(\mu_1, \mu_2)$.

Finally we explicitly state the $L$-packet $\Pi_\phi$ for each $L$-parameter $\phi$ that is classified in §1.2.

Case A: $\phi_E = \eta \boxtimes \mathrm{Sym}^1$

If we take $\chi = \eta$ and $\phi' = \mathrm{Sym}^1$, then we obtain $\phi_E = \eta \boxtimes \mathrm{Sym}^1$. Since $\tau_D(\phi')$ is the trivial representation of $D^\times$, we have $\eta \boxtimes \tau(\phi')|_{\mathrm{U}(V_{an})} = \eta_u \circ \det$. Therefore $\Pi_\phi = \{\eta_u \circ \det\}$.

Case B: $\phi_E \in \mathrm{Irr}\, W_{E,2,+}$

Then we can take a two-dimensional irreducible representation of $W_F$ as $\phi'$. Thus $\tau(\phi')$ is a supercuspidal representation of $D^\times$. By Theorem 2.7, $\chi \boxtimes \tau(\phi')|_{\mathrm{U}(V_{an})}$ is an irreducible supercuspidal representation $\tau$ of $\mathrm{U}(V_{an})$.

Case $C_2$: $\phi_E = \mu_1 \oplus \mu_2$

By Theorem 2.7, we have $\Pi_\phi = \{\tau(\mu_1, \mu_2)^\pm_{sp}\}$.

Since $\phi'$ is reducible for any $\phi$ in the remaing cases, $\Pi_\phi$ is empty. Thus we have the following table:

| $\phi$ | $\phi_E$ | $\Pi_\phi$ |
|---|---|---|
| CaseA | $\eta \boxtimes \mathrm{Sym}^1$ | $\eta_u \circ \det$ |
| CaseB | $\phi_E$ | $\tau$ |
| Case$C_1$ | $\lambda \oplus \lambda^{-1} \circ \sigma$ | empty |
| Case$C_2$ | $\mu_1 \oplus \mu_2$ | $\tau(\mu_1, \mu_2)^\pm_{an}$ |
| Case$C_3$ | $\eta \oplus \eta$ | empty |
| Case$C_4$ | $\mu \oplus \mu$ | empty |

## 2.4 Global endoscopy for $\mathrm{U}(V_{an})$

In this subsection, we prepare a global result which we need later.

Let $\tilde{E}/\tilde{F}$ be a quadratic extension of a number field. Take an element $\tilde{d}$ of $\tilde{F}^\times$, a non-zero element $\tilde{\xi}$ of $\tilde{E}$ with $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and a non-trivial character $\tilde{\psi}$ of $\mathbb{A}_{\tilde{F}}$. We define a two-dimensional hermitian space $\tilde{V}$ over $\tilde{E}$:

$$\tilde{V} = (\tilde{E}^{\oplus 2}, \begin{pmatrix} -\tilde{d} & \\ & 1 \end{pmatrix}).$$

Take two characters $\tilde{\mu}_1, \tilde{\mu}_2 \in \Pi(\mathbb{A}^\times_{\tilde{E}}, \omega_{\tilde{E}/\tilde{F}})$. Assume that if $\tilde{V}_v$ is anisotropic, then $\tilde{\mu}_{1,v} \neq \tilde{\mu}_{2,v}$. Then for any place $v$, we have the local $L$-packet $\Pi_{\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v}}(\tilde{V})$

of $\mathrm{U}(\tilde{V}_v)$ associated to the $L$-parameter $\varphi_{\tilde{E}_v} = \tilde{\mu}_{1,v} \oplus \tilde{\mu}_{2,v}$. These local $L$-packets are described as follows.

First assume that $\tilde{V}_v \cong \tilde{V}_{sp,v}$. Then we have $\mathrm{U}(\tilde{V}_v) \cong \mathrm{U}(\tilde{V}_{sp,v})$. We omit $v$ from the notation. If $\tilde{E}$ is not a field, the local $L$-packet $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$ consists of one $\tilde{\psi}$-generic element. It is denoted by $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^+$. If $\tilde{E}$ is a field, we define representations $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm$ of $\mathrm{U}(\tilde{V})$ by the composition of $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm_{sp}$ and an isomorphism $\mathrm{U}(\tilde{V}) \cong \mathrm{U}(\tilde{V}_{sp})$ induced by $\tilde{V} \cong \tilde{V}_{sp}$. Then $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^+$ is $\tilde{\psi}$-generic and $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V}) = \{\tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm\}$. Also the following holds:

**Lemma 2.10.** *Define a maximal torus $T$ of $\mathrm{U}(\tilde{V})$ as in §2.3 (2). Then we have the following character identity:*

$$\begin{aligned}&\mathrm{Tr}\,\tau(\tilde{\mu}_1,\tilde{\mu}_2)^+(t(z,z')) - \mathrm{Tr}\,\tau(\tilde{\mu}_1,\tilde{\mu}_2)^-(t(z,z'))\\ &= \lambda\left(\tilde{E}/\tilde{F},\tilde{\psi}\right)\omega_{\tilde{E}/\tilde{F}}\left(\frac{z'-\sigma(z')}{\tilde{\xi}}\right)\chi\left(z\,\mathrm{N}_{\tilde{E}/\tilde{F}}(z')\right)\frac{\chi'(\sigma(z'))+\chi'(z')}{|z'-\sigma(z')|_{\tilde{E}}^{1/2}}|z'|_{\tilde{E}}^{1/2}\end{aligned}$$

*for any $t(z,z') \in T$. Here $\chi, \chi' \in \mathrm{Irr}\,\tilde{E}^\times$ such that $\tilde{\mu}_1 = \chi\chi'$, $\tilde{\mu}_2 = \chi(\chi'\circ\sigma)$.*

*Proof.* Now there exists $x \in \tilde{E}^\times$ such that $\tilde{d} = \mathrm{N}_{\tilde{E}/\tilde{F}}(x)$. We set

$$A = \begin{pmatrix} -\tilde{d} & x \\ 1/2 & x\tilde{d}^{-1}/2 \end{pmatrix}.$$

Then the matrix $A$ gives the isomorphism $\tilde{V} \cong \tilde{V}_{sp}$. Namely,

$${}^*A\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}A = \begin{pmatrix} -\tilde{d} & \\ & 1 \end{pmatrix}.$$

Thus the group isomorphism $\mathrm{U}(\tilde{V}) \cong \mathrm{U}(\tilde{V}_{sp})$ is given by $\mathrm{Ad}(A)$. By definition, we have $\tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm = \tau(\tilde{\mu}_1,\tilde{\mu}_2)^\pm_{sp}\circ\mathrm{Ad}(A)$. Also we obtain

$$\mathrm{Ad}(A)(t(z,z')) = t_{\tilde{d}^{-1}\tilde{\xi}^{-1}/2}(z,z')$$

for $t(z,z') \in T$. Thus Lemma 2.10 follows from Proposition 2.2 and easy computation. □

If $v$ is finite and $\tilde{V}_v$ is anisotropic, we have already explained in §2.3.

Finally, assume that $\tilde{E}_v/\tilde{F}_v = \mathbb{C}/\mathbb{R}$ and $\tilde{V}_v$ is anisotropic. We omit $v$ from the notation. Set

$$\lambda : \mathbb{C}^\times \ni re^{i\theta} \mapsto e^{i\theta} \in \mathbb{C}^\times$$
$$\lambda_1 : \mathbb{C}^1 \ni e^{i\theta} \mapsto e^{i\theta} \in \mathbb{C}^1.$$

Then we may assume $\tilde{\mu}_1 = \lambda^{m+n}$ and $\tilde{\mu}_2 = \lambda^{m-n}$, where $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m - n \equiv 1 \mod 2$. Let $\rho_{m,n}$ be the irreducible $|n|$-dimensional representation of $\mathrm{U}(\tilde{V})$ with central character $\lambda_1^m$.

**Lemma 2.11.** *Let $r \in \mathbb{R}^\times$ be such that $\tilde{\psi}(x) = e^{irx}$ $(x \in \mathbb{R})$. We set $C_v = \mathrm{sgn}(nr\tilde{\xi}/i)$. Then we have*

$$\begin{aligned} &C_v \cdot \mathrm{Tr}\, \rho_{m,n}(t(z,z')) \\ =&\lambda\left(\mathbb{C}/\mathbb{R}, \tilde{\psi}\right) \omega_{\mathbb{C}/\mathbb{R}}\left(\frac{z' - \sigma(z')}{\tilde{\xi}}\right) \chi\left(z\, \mathrm{N}_{\mathbb{C}/\mathbb{R}}(z')\right) \frac{\chi'(\sigma(z')) - \chi'(z')}{|z' - \sigma(z')|_{\mathbb{C}}^{1/2}} |z'|_{\mathbb{C}}^{1/2} \end{aligned}$$

*for any $t(z,z') \in T$. Here $\chi, \chi' \in \mathrm{Irr}\,\mathbb{C}^\times$ such that $\tilde{\mu}_1 = \chi\chi', \tilde{\mu}_2 = \chi(\chi' \circ \sigma)$.*

*Proof.* It is enough to show for $z = e^{i\zeta}, z' = e^{i\theta}$. Also we can take $\chi = \lambda^m, \chi' = \lambda^n$. Since $\lambda\left(\mathbb{C}/\mathbb{R}, \tilde{\psi}\right) = \mathrm{sgn}(r)i$, the right hand side is

$$ie^{im\zeta}\,\mathrm{sgn}(r\frac{e^{i\theta} - e^{-i\theta}}{\tilde{\xi}})\frac{e^{-in\theta} - e^{in\theta}}{2|\sin\theta|}.$$

On the other hand,

$$\mathrm{Tr}\, \rho_{m,n}(t(z,z')) = -ie^{im\zeta}\,\mathrm{sgn}(\frac{e^{i\theta} - e^{-i\theta}}{i})\frac{e^{i|n|\theta} - e^{-i|n|\theta}}{2|\sin\theta|}.$$

Lemma follows from this. □

Thus by this lemma, we set $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2} = \{\rho_{m,n}\}$ and $\tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})^{C_v} := \rho_{m,n}$.

Now we define a global endoscopic $L$-packet $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$ of $\mathrm{U}(\tilde{V})$ by

$$\left\{ \bigotimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})^{\epsilon_v} \;\middle|\; \begin{array}{l} \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})^{\epsilon_v} \in \Pi_{\tilde{\mu}_{1,v},\tilde{\mu}_{2,v}}(\tilde{V}), \\ \epsilon_v = + \text{ for almost all } v \end{array} \right\}.$$

**Theorem 2.12.** *Let $\tilde{\tau} = \bigotimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})^{\epsilon_v}$ be an element of $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$. Then $\tilde{\tau}$ is a cuspidal representation of $\mathrm{U}(\tilde{V})$ if and only if $\Pi_v \epsilon_v = +$.*

*Proof.* This follows from [5] and the character identities in Proposition 2.8, Lemma 2.10 and Lemma 2.11. □

# 3 Endoscopy for $\mathrm{U}(W)$

In this section, we explain the endoscopy for the unitary group $\mathrm{U}(W)$ following [15, Chap.3, 4].

## 3.1 $L$-group of $\mathrm{U}(W)$

In this subsection, we compute the $L$-group of $\mathrm{U}(W)$. Set

$$S = \begin{pmatrix} & & 1 \\ & \xi & \\ -1 & & \end{pmatrix}.$$

Define the connected reductive group $G$ defined over $F$ as follows:

$$G(R) = \{g \in \mathrm{GL}_3(E \otimes_F R) \mid gS^*g = S\},$$

where $R$ is any $F$-algebra, ${}^*g = (\sigma \otimes \mathrm{id}_R)({}^tg)$. Then $G(F) = \mathrm{U}(W)$.

First we show that $G(\bar{F}) \cong \mathrm{GL}_3(\bar{F})$.

**Lemma 3.1.** (i) *$E \otimes_F \bar{F} \ni \sum a \otimes b \mapsto \sum(ab, \sigma(a)b) \in \bar{F} \oplus \bar{F}$ is an isomorphism of $\bar{F}$-algebras.*
(ii) *The action on $E \otimes_F \bar{F}$ of $\mathrm{Gal}(E/F) \times \Gamma_F$ is transposed on $\bar{F} \oplus \bar{F}$ as follows:*

$$(\sigma \otimes \mathrm{id}_{\bar{F}})(x, y) = (y, x),$$
$$(\mathrm{id}_E \otimes \tau)(x, y) = \begin{cases} (\tau(x), \tau(y)) & \text{if } \tau \in \Gamma_E, \\ (\tau(y), \tau(x)) & \text{if } \tau \notin \Gamma_E. \end{cases}$$

*Proof.* (i) This is trivial.

(ii) Let $\alpha$ be the isomorphism of (i). Take an element $(x, y) \in \bar{F} \oplus \bar{F}$ and write $(x, y) = (ab, \sigma(a)b)$. Then we have

$$\begin{aligned} \sigma \otimes \mathrm{id}_{\bar{F}}(x, y) &:= \alpha \circ (\sigma \otimes \mathrm{id}_{\bar{F}}) \circ \alpha^{-1}(x, y) \\ &= \alpha \circ (\sigma \otimes \mathrm{id}_{\bar{F}})(a \otimes b) \\ &= \alpha(\sigma(a) \otimes b) \\ &= (\sigma(a)b, ab) \\ &= (y, x) \end{aligned}$$

The latter follows from the similar argument. □

By this lemma, we have the following corollary:

**Corollary 3.2.** (i) *The isomorphism of Lemma 3.1* (i) *induces the following isomorphism:*

$$\mathrm{GL}_3(E \otimes_F \bar{F}) \cong \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}).$$

(ii) *The action on* $\mathrm{GL}_3(E \otimes_F \bar{F})$ *of* $\mathrm{Gal}(E/F) \times \Gamma_F$ *is transposed on* $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ *as in Lemma 3.1* (ii).

By Cor. 3.2, the image in $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ of the equation $gS^*g = S$ is

$$(g_1, g_2)\,(S, S^\sigma)\left({}^t g_2, {}^t g_1\right) = (S, S^\sigma)\,.$$

Thus the image of $G(\bar{F})$ in $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ coincides with the subgroup of $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ which consists of the elements of the form

$$\left(g_1,\ {}^t S \cdot {}^t g_1^{-1} \cdot {}^t S^{-1}\right).$$

This subgroup is isomorphic to $\mathrm{GL}_3(\bar{F})$ by the first projection. Consequently it turns out that

$$G(\bar{F}) \cong \mathrm{GL}_3(\bar{F}). \tag{3}$$

From this, we see that the Langlands dual group $\hat{G}$ of $G$ is $\mathrm{GL}_3(\mathbb{C})$.

Let $B_3$ be the Borel subgroup of $\mathrm{GL}_3$ which consists of the upper-triangular matrices, and $T$ the maximal torus which consists of the diagonal matrices. Let $X_i$ be the $3 \times 3$-matrix with $(i, i+1)$-component one and zero elsewhere. Then $(B_3, T, \{X_i\}_i)$ is a splitting of $\mathrm{GL}_3$. We identify $(B_3, T, \{X_i\}_i)$ with a splitting of $G$ by (3). Then the based root datum of $G$ is given as follows:

$$\begin{aligned}
&X^* = X^*(T) = \bigoplus_i \mathbb{Z}\epsilon_i, \qquad \text{where } \ \epsilon_i(\mathrm{diag}(t_1, t_2, t_3)) = t_i,\\
&\Delta = \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid 1 \le i \le 2\},\\
&X_* = X_*(T) = \bigoplus_i \mathbb{Z}\epsilon_i^\vee, \qquad \text{where } \ \epsilon_i^\vee(t) = \mathrm{diag}(1, ..., \overset{i}{t}, ..., 1),\\
&\Delta^\vee = \{\alpha_i^\vee := \epsilon_i^\vee - \epsilon_{i+1}^\vee \mid 1 \le i \le 2\}.
\end{aligned}$$

On the other hand, take the splitting $(\hat{B}_3, \hat{T}, \{X_i\}_i)$ of $\mathrm{GL}_3(\mathbb{C})$ as that of $\mathrm{GL}_3$. Then the dual based root datum of $G$ is equal to the based root datum of $\mathrm{GL}_3(\mathbb{C})$. Also the action of $\Gamma_F$ on this based root datum is written as follows:

$$\tau(\epsilon_i) = \begin{cases} \epsilon_i & \text{if } \tau \in \Gamma_E, \\ -\epsilon_{4-i} & \text{if } \tau \notin \Gamma_E. \end{cases}$$

This action is induced by the following action of $\Gamma_F$ on $\mathrm{GL}_3(\mathbb{C})$:

$$\tau(g) = \begin{cases} g & \text{if } \tau \in \Gamma_E, \\ I_3{}^t g^{-1} I_3^{-1} & \text{if } \tau \notin \Gamma_E. \end{cases}$$

Since the splitting $(\hat{B}_3, \hat{T}, \{X_i\}_i)$ of $\mathrm{GL}_3(\mathbb{C})$ is stable under this action of $\Gamma_F$ on $\mathrm{GL}_3(\mathbb{C})$, the $L$-group of $G$ is given as follows:

**Lemma 3.3.**

$$^L G = \mathrm{GL}_3(\mathbb{C}) \rtimes W_F,$$

$$g \rtimes w \cdot g' \rtimes w' = \begin{cases} gg' \rtimes ww' & \text{if } w \in W_E, \\ gI_3{}^t g'^{-1} I_3^{-1} \rtimes ww' & \text{if } w \notin W_E. \end{cases}$$

## 3.2 Stable orbital integrals

We recall the definitions of a semi-simple element and a regular semi-simple element of $G(F)$. An element $\gamma \in G(F)$ is said to be *semi-simple* if the image of $\gamma$ in $\mathrm{GL}_3(\bar{F})$ is diagonalizable. We denote the set of semi-simple elements of $G(F)$ by $G(F)_{ss}$. An element $\gamma \in G(F)$ is said to be *regular semi-simple* if the characteristic-polynomial of the image of $\gamma$ in $\mathrm{GL}_3(\bar{F})$ has distinct three roots. This condition is equivalent to the condition that $G_\gamma$ is a maximal torus of $G$, where $G_\gamma$ is the centralizer of $\gamma$ in $G$. We denote the set of regular semi-simple elements of $G(F)$ by $G(F)_{reg}$. For any $\gamma \in G(F)_{reg}$, set

$$D_G(\gamma) := \det(\mathrm{Ad}(\gamma) - 1|_{\mathfrak{g}(F)/\mathfrak{g}_\gamma(F)}),$$

where $\mathfrak{g}$ and $\mathfrak{g}_\gamma$ are the Lie algebras of $G$ and $G_\gamma$, respectively. Then we define the normalized orbital integral of $\gamma$ by

$$J(\gamma, f) := |D_G(\gamma)|_F^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g) \frac{dg}{dt},$$

where $f \in \mathcal{S}(G(F))$. $J(\gamma, f)$ is convergent ([14]). This $J(\gamma)$ is an invariant distribution on $G(F)$. Here a linear map $A : \mathcal{S}(G(F)) \to \mathbb{C}$ is called an *invariant distribution* on $G(F)$ if the following condition is satisfied:

$$A(f \circ \mathrm{Ad}(g)) = A(f), \quad f \in \mathcal{S}(G(F)), \quad g \in G(F).$$

We denote the set of invariant distributions on $G(F)$ by $\mathcal{I}(G(F))$.

Let $\gamma$ and $\gamma'$ be two elements of $G(F)$. The element $\gamma$ is said to be *conjugate* to $\gamma'$ if there exists $g \in G(F)$ such that $\gamma' = g^{-1}\gamma g$. Then we write $\gamma \sim \gamma'$. Also $\gamma$ is said to be *stably conjugate* to $\gamma'$ if there exists $g \in G(\bar{F})$ such that $\gamma' = g^{-1}\gamma g$. Then we write $\gamma \sim_{st} \gamma'$. The stable conjugacy is an equivalence relation which is weaker than the conjugacy. Namely, if $\gamma'$ is conjugate to $\gamma$ , then $\gamma'$ is stable conjugate to $\gamma$. Thus we define the following set:

$$\mathcal{O}_{st}(\gamma) := \{\gamma' \in G(F) \mid \gamma' \sim_{st} \gamma\}/\sim .$$

We also call this the stable conjugacy class of $\gamma$. We denote the set of stable conjugacy classes $\mathcal{O}_{st}(\gamma)$ of the semi-simple elements $\gamma$ of $G(F)$ by $\mathcal{O}_{st}(G)$.

We can describe $\mathcal{O}_{st}(\gamma)$ in terms of Galois cohomology. Let $\gamma$ and $\gamma'$ be stable conjugate in $G(F)$. By definition, there exists an element $g$ of $G(\bar{F})$ such that $\gamma' = g^{-1}\gamma g$. Then $\{gs(g^{-1})\}_{s\in\Gamma}$ satisfies the cocycle condition. We denote this 1-cocycle $\{gs(g^{-1})\}_{s\in\Gamma}$ by $f_{\gamma'}$.

**Lemma 3.4.** (i) $f_{\gamma'} \in Z^1(F, G_\gamma)$.

(ii) *The image* $[f_{\gamma'}]$ *of* $f_{\gamma'}$ *in* $H^1(F, G_\gamma)$ *is independent of the choice of* $g$.

(iii) $[f_{\gamma'}]$ *is contained in* $\mathrm{Ker}(H^1(F, G_\gamma) \to H^1(F, G))$.

(iv) *Set* $\mathfrak{D}(G_\gamma) := \mathrm{Ker}(H^1(F, G_\gamma) \to H^1(F, G))$. *Then the following map is bijective:*

$$\mathcal{O}_{st}(\gamma) \ni [\gamma'] \longmapsto [f_{\gamma'}] \in \mathfrak{D}(G_\gamma).$$

*Here the morphism* $H^1(F, G_\gamma) \to H^1(F, G)$ *is induced by the inclusion* $G_\gamma \hookrightarrow G$.

*Proof.* These follow from the definition easily. □

Since $H^1(F, G_\gamma)$ is a finite set, so is $\mathfrak{D}(G_\gamma)$. Thus $\mathcal{O}_{st}(\gamma)$ is a disjoint union of finite number of conjugacy classes. Here note that since $H^1(F, G)$ is a pointed-set and does not have a group structure in general, $\mathfrak{D}(G_\gamma)$ is not a group.

We assume that $\gamma$ is a regular semi-simple element of $G(F)$. This case is especially important for us. Now $T = G_\gamma$ is a maximal torus of $G$. Then we recall Kottwitz's theorem.

**Theorem 3.5** ([7, Theorem 1.2]). *There exists the following commutative diagram:*

$$\begin{array}{ccc} H^1(F,T) & \longrightarrow & H^1(F,G) \\ \wr\downarrow & & \wr\downarrow \\ \pi_0(\hat{T}^\Gamma)^D & \xrightarrow{\text{res. to } Z(\hat{G})^\Gamma} & \pi_0(Z(\hat{G})^\Gamma)^D. \end{array}$$

*Here $H^D$ stands for the Pontryagin dual of $H$.*

We set

$$\mathfrak{E}(T) := \mathrm{Ker}(\pi_0(\hat{T}^\Gamma)^D \to \pi_0(Z(\hat{G})^\Gamma)^D).$$

By this theorem, this is isomorphic to $\mathfrak{D}(T)$. In particular, $\mathfrak{D}(T)$ becomes a subgroup of $H^1(F,T)$. Moreover, by Lemma 3.4 there exists a bijective map from $\mathcal{O}_{st}(\gamma)$ to $\mathfrak{E}(T)$. As a result, we get a description of $\mathcal{O}_{st}(\gamma)$ in terms of $L$-groups.

Next we state the following proposition on $\mathfrak{E}(T)$ where $T$ is a maximal torus of $G$.

**Proposition 3.6** ([15, §3.4, §3.5, §3.6]). (i) *If two maximal tori $T$ and $T'$ of $G$ are $F$-isomorphic, then they are stable conjugate, that is, there exists $g \in G(\bar{F})$ such that $\mathrm{Ad}(g)$ is an $F$-isomorphism between $T$ and $T'$.*
(ii) *The maximal tori $T$ of $G$ are divided into the following types up to $F$-isomorphism.*

| *maximal torus $T$ of $G$* | $\mathfrak{E}(T)$ |
|---|---|
| $\mathrm{Res}_{E/F}\,\mathbb{G}_m \times \mathrm{U}(1)$ | $\{1\}$ |
| $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ |
| $T_K \times \mathrm{U}(1)$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $T_L$ | $\{1\}$ |

*Here, we denote $\mathrm{Res}_{K/F}\,\mathrm{U}_{EK/K}(1)$ by $T_K$ and $K$ runs through the quadratic extensions of $F$ except for $E$. Also $T_L$ is $\mathrm{Res}_{L/F}\,\mathrm{U}_{EL/L}(1)$ and $L$ runs through the cubic extensions of $F$.*

Let $\gamma$ be an element of $G(F)_{reg}$ and $T = G_\gamma$. Then we obtained

$$\mathcal{O}_{st}(\gamma) \xrightarrow{\sim} \mathfrak{D}(T) \xrightarrow{\sim} \mathfrak{E}(T).$$

We denote the image of $[\gamma'] \in \mathcal{O}_{st}(\gamma)$ in $\mathfrak{E}(T)$ by $\mathrm{inv}(\gamma, \gamma')$. Also we set

$$\mathfrak{K}(T) := \mathfrak{E}(T)^D.$$

This is isomorphic to $\mathrm{Coker}(\pi_0(Z(\hat{G})^\Gamma) \to \pi_0(\hat{T}^\Gamma))$ by Theorem 3.5.

**Definition 3.7.** For $\kappa \in \mathfrak{K}(T)$ and $f \in \mathcal{S}(G(F))$, define

$$J^{\kappa}(\gamma, f) := \sum_{[\gamma'] \in \mathcal{O}_{st}(\gamma)} \overline{\kappa(\mathrm{inv}(\gamma, \gamma'))} J(\gamma', f) \quad \textit{κ-orbital integral},$$

$$J^{st}(\gamma, f) := \sum_{[\gamma'] \in \mathcal{O}_{st}(\gamma)} J(\gamma', f) \quad \textit{stably orbital integral}.$$

Here, each measure on $G_{\gamma'}(F)$ is the pull back of a Haar measure on $G_{\gamma}(F)$ by $\mathrm{Ad}(g) : G_{\gamma'}(F) \to G_{\gamma}(F)$, where $\gamma' = g^{-1}\gamma g$.

In general, $J^{\kappa}(\gamma)$ is dependent on the choice of $\gamma$, but $J^{st}(\gamma)$ is not. Namely, if $\gamma$ and $\gamma'$ are stable conjugate, then $J^{st}(\gamma) = J^{st}(\gamma')$.

**Definition 3.8.** We call an invariant distribution $A$ on $G(F)$ a *stable distribution* if $\mathrm{Ker}\, A \supset \bigcap_{\gamma \in G(F)_{reg}} \mathrm{Ker}\, J^{st}(\gamma)$. Then we denote the space of stable distributions on $G(F)$ by $\mathcal{SI}(G(F))$. This space equals to the closure of the subspace of stable orbital integrals in $\mathcal{I}(G(F))$ with respect to the weak topology.

Finally we consider the relation between $J^{\kappa}(\gamma)$ and $J^{\kappa'}(\gamma')$. Here $\gamma'$ is stably conjugate to $\gamma$ with centralizer $T'$, $\kappa \in \mathfrak{K}(T)$ and $\kappa' \in \mathfrak{K}(T')$.

**Lemma 3.9.** *Let $g$ be an element of $\in G(\bar{F})$ such that $\gamma' = g^{-1}\gamma g$.*

(i) *The homomorphism*
$$\mathrm{Ad}(g^{-1}) : T \longrightarrow T'$$
*is an $F$-isomorphism which maps $\gamma$ to $\gamma'$. Also this isomorphism is independent of the choice of $g$.*

(ii) *The map $\mathrm{Ad}(g^{-1})$ induces a group-isomorphism from $\mathfrak{D}(T)$ to $\mathfrak{D}(T')$. This isomorphism is independent of the choice of $g$.*

*Proof.* (i) This is trivial.

(ii) By (i), $\mathrm{Ad}(g^{-1})$ induces $H^1(F, T) \cong H^1(F, T')$. To prove the former statement, it is sufficient to show that $\mathrm{Ad}(g^{-1})$ maps $\mathfrak{D}(T)$ to $\mathfrak{D}(T')$. We assume that $a_s = hs(h^{-1})$ is the element of $\mathfrak{D}(T)$ corresponding to $[\gamma''] \in \mathcal{O}_{st}(\gamma)$. Then we have

$$\mathrm{Ad}(g^{-1})(a_s) = g^{-1}hs(h^{-1})g = g^{-1}hs(h^{-1}g) \cdot (g^{-1}s(g))^{-1}.$$

Since $g^{-1}hs(h^{-1}g)$ and $g^{-1}s(g)$ are elements of $\mathfrak{D}(T')$ and $\mathfrak{D}(T')$ is a group, $\mathrm{Ad}(g^{-1})(a_s)$ is an element of $\mathfrak{D}(T')$. The latter follows from (i). □

By this lemma, we get $\mathfrak{E}(T) \cong \mathfrak{E}(T')$. Also we have $\mathfrak{K}(T) \cong \mathfrak{K}(T')$.

**Lemma 3.10.** *Assume that $\kappa \in \mathfrak{K}(T)$ corresponds to $\kappa' \in \mathfrak{K}(T')$ under the above isomorphism. Then we have*

(i) $\kappa(\mathrm{inv}(\gamma, \gamma'')) = \kappa'(\mathrm{inv}(\gamma', \gamma'')) \cdot \kappa'(\mathrm{inv}(\gamma', \gamma))^{-1}$ *for any* $[\gamma''] \in \mathcal{O}_{st}(\gamma)$,

(ii) $\kappa(\mathrm{inv}(\gamma, \gamma')) = \kappa'(\mathrm{inv}(\gamma', \gamma))^{-1}$,

(iii) $J^{\kappa'}(\gamma') = \kappa(\mathrm{inv}(\gamma, \gamma'))J^{\kappa}(\gamma)$.

*Proof.* (i) Identify $\mathfrak{D}(T)$ with $\mathfrak{E}(T)$. We set $\gamma' = g^{-1}\gamma g$ and $\gamma'' = h^{-1}\gamma h$. Then $\kappa' = \kappa \circ \mathrm{Ad}(g)$, $\mathrm{inv}(\gamma', \gamma) = g^{-1}s(g)$, $\mathrm{inv}(\gamma, \gamma'') = hs(h^{-1})$ and $\mathrm{inv}(\gamma', \gamma'') = g^{-1}hs(h^{-1}g)$. Thus

$$\begin{aligned}
\kappa(\mathrm{inv}(\gamma, \gamma'')) &= \kappa(hs(h^{-1})) \\
&= \kappa'(g^{-1}hs(h^{-1})g) \\
&= \kappa'(g^{-1}hs(h^{-1}g)s(g^{-1})g) \\
&= \kappa'(g^{-1}hs(h^{-1}g)) \cdot \kappa'(s(g^{-1})g) \\
&= \kappa'(\mathrm{inv}(\gamma', \gamma'')) \cdot \kappa'(g^{-1}s(g))^{-1} \\
&= \kappa'(\mathrm{inv}(\gamma', \gamma''))\kappa'(\mathrm{inv}(\gamma', \gamma))^{-1}.
\end{aligned}$$

(ii) This follows from (i).
(iii) This follows from (i) and (ii). □

## 3.3 Transfer

In this subsection, we explain an endoscopic datum of $G$.

**Definition 3.11.** We call $(H, s, \xi)$ an *endoscopic datum* of $G$ if it satisfies the following conditions:

(i) $H$ is a connected quasi-split group defined over $F$;

(ii) $s$ is a semi-simple element of $\hat{G}$;

(iii) $\xi : {}^LH \to {}^LG$ is a homomorphism over $W_F$;

(iv) $\xi$ induces an isomorphism from $\hat{H}$ to $(\hat{G}_s)^0$;

(v) $\lambda(w) := s\xi(w)s^{-1}\xi(w)^{-1}$ is a 1-coboundary from $W_F$ to $Z(\hat{G})$.

Moreover an endoscopic datum $(H, s, \xi)$ is said to be *elliptic* if $\xi(Z(\hat{H})^\Gamma)^0 \subset Z(\hat{G})^\Gamma$. Furthermore two endoscopic data $(H, s, \xi)$ and $(H', s', \xi')$ are said to be *isomorphic* if there exists $g \in \hat{G}$ such that $g\xi({}^LH)g^{-1} = \xi'({}^LH')$ and $gsg^{-1}s'^{-1} \in Z(\hat{G})$.

**Example 3.12.** (i) If $H = G$, $s = 1$ and $\xi = \mathrm{id}_{{}^LG}$, then $(H, s, \xi)$ is an elliptic endoscopic datum of $G$. We call it the *trivial endoscopic datum.*
(ii) Take a quasi-character $\mu_0$ of $E^\times$such that $\mu_0|_{F^\times} = \omega_{E/F}$. Then the following $(H, s_0, \xi_0)$ is a non-trivial elliptic endoscopic datum of $G$:

$$H = \mathrm{U}(V_{sp}) \times \mathrm{U}(1),\ s_0 = \begin{pmatrix} -1_2 & \\ & 1 \end{pmatrix},$$

$$\xi_0 : {}^LH \ni (h_1, h_2) \rtimes w \longrightarrow \begin{cases} \begin{pmatrix} h_1\mu_0(w) & \\ & h_2 \end{pmatrix} \rtimes w, & w \in W_E \\ \begin{pmatrix} & -h_1 \\ h_2 & \end{pmatrix} \rtimes w_\sigma, & w = w_\sigma. \end{cases}$$

**Lemma 3.13** ([15, §4.6])**.** *The set of isomorphism classes of elliptic endoscopic data $(H, s, \xi)$ of $G$ is exhausted by the ones in Example 3.12.*

Let $(H, s_0, \xi_0)$ be the above non-trivial elliptic endoscopic datum. Next we explain the transfer of orbital integrals.

Take $h \in \mathrm{GL}_3(E)$ such that $h \begin{pmatrix} & -\xi & \\ -\xi & & \\ & & -\xi \end{pmatrix} {}^*h = I_3$. We define the following homomorphism:

$$\mathrm{Ad}(h) : H(F) \ni (z_1, z_2) \mapsto h \begin{pmatrix} {}^*z_1^{-1} & \\ & z_2 \end{pmatrix} h^{-1} \in G(F)$$

Then we define

$$\mathcal{A}_{H/G} : \mathcal{O}_{st}(H) \ni \mathcal{O}_{st}(\gamma_H) \longmapsto \mathcal{O}_{st}(\mathrm{Ad}(h)(\gamma_H)) \in \mathcal{O}_{st}(G).$$

This is independent of the choice of $h$.

**Definition 3.14.** An element $\gamma_H$ of $H(F)_{reg}$ is said to be *$G$-regular*, if $\mathcal{A}_{H/G}(\mathcal{O}_{st}(\gamma_H))$ is a stable conjugacy class of a regular semi-simple element of $G(F)$. Then for any element $\gamma$ of $\mathcal{A}_{H/G}(\mathcal{O}_{st}(\gamma_H))$, we call $\gamma_H$ an *image* of $\gamma$. We denote the set of $G$-regular elements of $H(F)_{reg}$ by $H(F)_{G-reg}$.

For a $G$-regular element $\gamma_H$ of $H(F)_{reg}$, we take an element $\gamma$ of $G(F)$ whose image is $\gamma_H$. Then we explain how to determine $\kappa = \kappa_{\gamma,\gamma_H} \in \mathfrak{K}(T_\gamma)$ where $T_\gamma = G_\gamma$. Similarly put $T_{\gamma_H} = H_{\gamma_H}$. Since $\gamma' = \mathrm{Ad}(h)(\gamma_H)$ is stably conjugate to $\gamma$, there exists an element $g$ of $G(\bar{F})$ such that $\gamma' = g^{-1}\gamma g$. We denote the Langlands dual group of $T_\gamma$ by $\hat{T}_\gamma$. Then $\mathrm{Ad}(g)$ induces an isomorphism from $\hat{T}_{\gamma'}$ to $\hat{T}_\gamma$. Also $\mathrm{Ad}(h)$ induces an isomorphism from $\hat{T}_{\gamma_H}$ to $\hat{T}_{\gamma'}$. Thus there exists an isomorphism from $\hat{T}_{\gamma_H}$ to $\hat{T}_\gamma$. Since $\xi^{-1}(s_0)$ is an element $s'$ of $Z(\hat{H}) \subset \hat{T}_{\gamma_H}$, the image of $s'$ under that isomorphism determines an element $t$ of $\hat{T}_\gamma$. This $t$ defines an element $\kappa = \kappa_{\gamma,\gamma_H}$ of $\mathfrak{K}(T_\gamma)$. Then $J^\kappa(\gamma)$ depends on the choice of $\gamma$, but, it is independent of the choice of $\gamma$ up to scalar by Lemma 3.10. Also this is independent of the choice of $\delta$ and of $h$.

Our purpose here is to explain transfer of orbital integrals. It is to construct a linear map

$$\mathrm{Span}\{J^{st}(\gamma_H) \mid \gamma_H \in H(F)_{G-reg}\} \longrightarrow \mathcal{I}(G(F))$$

which maps $J^{st}(\gamma_H)$ to a scalar multiple of $J^\kappa(\gamma)$ and whose domain can be extended to $\mathcal{SI}(H(F))$. For this, Langlands-Shelstad introduced the map $\Delta_{H/G}$ called the *transfer factor*. See [11] for details. Here we state only properties of $\Delta_{H/G}$:

$$\Delta_{H/G} : H(F)_{G-reg} \times G(F)_{reg} \longrightarrow \mathbb{C}$$

(i) $\Delta_{H/G}(\gamma_H, \gamma) \neq 0 \Leftrightarrow \gamma_H$ is an image of $\gamma$;

(ii) If $\gamma_H \sim_{st} \gamma'_H$, then $\Delta_{H/G}(\gamma_H, \cdot\,) = \Delta_{H/G}(\gamma'_H, \cdot\,)$;

(iii) If $\gamma \sim \gamma'$, then $\Delta_{H/G}(\,\cdot\,, \gamma) = \Delta_{H/G}(\,\cdot\,, \gamma')$;

(iv) If $\gamma \sim_{st} \gamma'$, then $\Delta_{H/G}(\gamma_H, \gamma') = \kappa_{\gamma,\gamma_H}(\mathrm{inv}(\gamma, \gamma'))\Delta_{H/G}(\gamma_H, \gamma)$.

Then we have the following:

**Theorem 3.15** ([15, §4.9]). *For any $f \in \mathcal{S}(G(F))$, there exists an $f^H \in \mathcal{S}(H(F))$ such that*

$$J^{st}(\gamma_H, f^H) = \sum_\gamma \Delta_{H/G}(\gamma_H, \gamma) J(\gamma, f)$$

*for any $\gamma_H \in H(F)_{G-reg}$.*

By this theorem, for each stable invariant distribution $J$ of $H$, we can define the invariant distribution $\mathrm{Tran}_H^G(J)$ of $G$ by

$$\mathrm{Tran}_H^G(J)(f) := J(f^H).$$

Thus we have constructed the following linear map:

$$\mathrm{Tran}_H^G : \mathcal{SI}(H(F)) \longrightarrow \mathcal{I}(G(F)).$$

## 3.4 Twisted endoscopy for $\mathrm{U}(W)$

In this subsection, we define the Weil restriction $\tilde{G}$ of $G$ and compute its $L$-group.

First we define $\tilde{G}$ as follows:

$$\tilde{G}(R) := \mathrm{Res}_{E/F}\, G(R) = \{g \in \mathrm{GL}_3(E \otimes_F E \otimes_F R) \mid gS^*g = S\}.$$

Here, $R$ is any $F$-algebra and ${}^*g = (\sigma \otimes \mathrm{id}_E \otimes \mathrm{id}_R)({}^tg)$. Then we have

$$\tilde{G} \cong \mathrm{Res}_{E/F}\, \mathrm{GL}_3\,.$$

Also $\mathrm{id}_E \otimes \sigma \otimes \mathrm{id}_R$ induces an $F$-automorphism of $\tilde{G}$ with order 2. We denote it by $\varepsilon$.

We consider the structures of $\tilde{G}(F)$ and $\tilde{G}(\bar{F})$. Replacing $\bar{F}$ by $E$ in Lemma 3.1, we have $E \otimes_F E \cong E \oplus E$. Thus we obtain

$$\mathrm{GL}_3(E \otimes_F E) \cong \mathrm{GL}_3(E) \times \mathrm{GL}_3(E). \tag{4}$$

Shifting the two automorphisms $\sigma \otimes \mathrm{id}_E$ and $\mathrm{id}_E \otimes \sigma$ on the left-hand side (4) to the right-hand side, we have the following:

$$(\sigma \otimes \mathrm{id}_E)(g, g') = (g', g),$$
$$(\mathrm{id}_E \otimes \sigma)(g, g') = (\sigma(g'), \sigma(g)).$$

Thus the image of $\tilde{G}(F)$ in $\mathrm{GL}_3(E) \times \mathrm{GL}_3(E)$ coincides with the subgroup which consists of the elements of the form

$$(g, S^{\sigma}({}^tg^{-1})S^{\sigma,-1}).$$

Therefore $\tilde{G}(F)$ is isomorphic to $\mathrm{GL}_3(E)$. Then $\varepsilon = \mathrm{id}_E \otimes \sigma$ can be written on $\mathrm{GL}_3(E)$ as follows:

$$\varepsilon(g) = S \cdot \sigma({}^tg)^{-1} \cdot S^{-1}, \quad g \in \mathrm{GL}_3(E).$$

As in §3.1, we have the following lemma and corollary:

**Lemma 3.16.** (i) *The following homomorphism is isomorphism:*

$$E\otimes E\otimes \bar{F} \ni \sum a\otimes b\otimes c \mapsto \sum (abc, \sigma(a)bc, a\sigma(b)c, \sigma(ab)c) \in \bar{F}\oplus\bar{F}\oplus\bar{F}\oplus\bar{F}.$$

(ii) *The action on* $E \otimes E \otimes \bar{F}$ *of* $\mathrm{Gal}(E/F) \times \mathrm{Gal}(E/F) \times \Gamma_F$ *is transposed on* $\bar{F} \oplus \bar{F} \oplus \bar{F} \oplus \bar{F}$ *as follows:*

$$(\sigma \otimes \mathrm{id}_E \otimes \mathrm{id}_R)(x, y, z, w) = (y, x, w, z),$$
$$(\mathrm{id}_E \otimes \sigma \otimes \mathrm{id}_R)(x, y, z, w) = (z, w, x, y),$$
$$(\mathrm{id}_E \otimes \mathrm{id}_E \otimes \tau)(x, y, z, w) = \begin{cases} (\tau(x), \tau(y), \tau(z), \tau(w)) & \text{if } \tau \in \mathrm{Gal}(\bar{F}/E), \\ (\tau(w), \tau(z), \tau(y), \tau(x)) & \text{if } \tau \notin \mathrm{Gal}(\bar{F}/E). \end{cases}$$

**Corollary 3.17.** (i) *The isomorphism of Lemma 3.16* (i) *induces the following isomorphism:*

$$\mathrm{GL}_3(E \otimes E \otimes \bar{F}) \cong \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}).$$

(ii) *The action on* $\mathrm{GL}_3(E\otimes E\otimes\bar{F})$ *of* $\mathrm{Gal}(E/F)\times\mathrm{Gal}(E/F)\times\Gamma_F$ *is transposed on* $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ *as in Lemma 3.16* (ii).

By Corollary 3.17, the image in $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ of the equation $gS^*g = S$ is

$$(g_1, g_2, g_3, g_4)(S, S^\sigma, S, S^\sigma)({}^tg_2, {}^tg_1, {}^tg_4, {}^tg_3) = (S, S^\sigma, S, S^\sigma).$$

Thus the image of $\tilde{G}(\bar{F})$ in $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ coincides with the subgroup which consists of the elements of the form

$$(g_1, S^\sigma({}^tg_1^{-1})S^{\sigma,-1}, g_3, S^\sigma({}^tg_3^{-1})S^{\sigma,-1}).$$

This subgroup is isomorphic to $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$ by the first and third projections. Consequently, it turns out that $\tilde{G}(\bar{F}) \cong \mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$.

Next we compute the $L$-group of $\tilde{G}$. Let $\tilde{B}$ be the Borel subgroup of $\tilde{G}$ which consists of the upper-triangle matrices, and $\tilde{T}$ the maximal torus which consists of the diagonal matrices. We fix this Borel pair $(\tilde{B}, \tilde{T})$ of $\tilde{G}$. We identify $(\tilde{B}(\bar{F}), \tilde{T}(\bar{F}))$ with the standard Borel pair of $\mathrm{GL}_3(\bar{F}) \times \mathrm{GL}_3(\bar{F})$. Then we write the based root datum of $\tilde{G}$ as follows:

$$X^*(\tilde{T}) = \bigoplus_i \mathbb{Z}\epsilon_i \oplus \bigoplus_j \mathbb{Z}\epsilon'_j,$$
$$\epsilon_i(\mathrm{diag}(t_1, t_2, t_3), \mathrm{diag}(t'_1, t'_2, t'_3)) = t_i,$$
$$\epsilon'_j(\mathrm{diag}(t_1, t_2, t_3), \mathrm{diag}(t'_1, t'_2, t'_3)) = t'_j,$$
$$\Delta = \{\alpha_i := \epsilon_i - \epsilon_{i+1}\ ,\ \beta_j := \epsilon'_j - \epsilon'_{j+1} \mid 1 \le i, j \le 2\}.$$

Here, we omit $X_*(\tilde{T})$ and $\Delta^\vee$. From this, the Langlands dual group $\hat{\tilde{G}}$ of $\tilde{G}$ is $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. Also the actions of $\Gamma$ and $\varepsilon$ on this based root datum are written as follows:

$$\tau(\epsilon_i) = \begin{cases} \epsilon_i & \text{if } \tau \in \mathrm{Gal}(\bar{F}/E), \\ -\epsilon'_{4-i} & \text{if } \tau \notin \mathrm{Gal}(\bar{F}/E), \end{cases}$$
$$\tau(\epsilon'_j) = \begin{cases} \epsilon'_j & \text{if } \tau \in \mathrm{Gal}(\bar{F}/E), \\ -\epsilon_{4-j} & \text{if } \tau \notin \mathrm{Gal}(\bar{F}/E) \end{cases}$$
$$\varepsilon(\epsilon_i) = \epsilon'_i, \ \ \varepsilon(\epsilon'_j) = \epsilon_j.$$

We fix the standard Borel pair $(\hat{B}, \hat{T})$ of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. Then the dual root datum of $\tilde{G}$ is isomorphic to the root datum of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. By this isomorphism, $\Gamma$ acts on the based root datum of $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. Let $X_i$ be a $3 \times 3$-matrix with $(i, i+1)$-component one and zero elsewhere. Then $\{X_i \times 0, 0 \times X_i\}_i$ is a splitting for $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$. We have the following lemma by the similar argument in §3.1:

**Lemma 3.18.** *The L-group of $\tilde{G}$ is as follows:*

$$^L\tilde{G} = (\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})) \rtimes W_F,$$
$$(g_1, g_2) \rtimes w \cdot (g'_1, g'_2) \rtimes w' = \begin{cases} (g_1 g'_1, g_2 g'_2) \rtimes ww' & \text{if } w \in W_E, \\ (g_1 I_3 {}^t g_2'^{-1} I_3^{-1}, g_2 I_3 {}^t g_1'^{-1} I_3^{-1}) \rtimes ww' & \text{if } w \notin W_E. \end{cases}$$

*Moreover we denote the automorphism of* $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ *which is induced by $\varepsilon$ and retains the above splitting of* $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ *by $\hat{\varepsilon}$. Then we have $\hat{\varepsilon}(g_1, g_2) = (g_2, g_1)$.*

Next we explain twisted endoscopy of $\mathrm{GL}_3(E)$ following Ch. 3 and 4 in Rogawski [15].

An element of $\delta$ in $\tilde{G}(F)$ is said to be $\varepsilon$-semi-simple if $\delta \rtimes \varepsilon$ is semi-simple in $\tilde{G} \rtimes \langle \varepsilon \rangle$. Then we denote the set of $\varepsilon$-semi-simple elements of $\tilde{G}(F)$ by $\tilde{G}(F)_{ss,\varepsilon}$. Define the $\varepsilon$-centralizer of $\delta$ by

$$\tilde{G}_{\delta\varepsilon} = \{g \in \tilde{G} \mid g^{-1}\delta\varepsilon(g) = \delta\}.$$

Then $\delta$ is said to be $\varepsilon$-regular semi-simple if $\tilde{G}_{\delta\varepsilon}$ is a torus. We denote the set of $\varepsilon$-regular semi-simple elements of $\tilde{G}(F)$ by $\tilde{G}(F)_{reg,\varepsilon}$. For any $\delta \in \tilde{G}(F)_{reg,\varepsilon}$, set

$$D_{G,\varepsilon}(\delta) := \det(\mathrm{Ad}(\delta \rtimes \varepsilon) - 1|_{\tilde{\mathfrak{g}}(F)/\tilde{\mathfrak{g}}_{\delta\varepsilon}(F)}),$$

where $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_{\delta\varepsilon}$ are the Lie algebras of $\tilde{G}$ and $\tilde{G}_{\delta\varepsilon}$, respectively. Then we define the normalized twisted orbital integral of $\delta$ by

$$J_\varepsilon(\delta, f) := |D_{G,\varepsilon}(\delta)|_F^{1/2} \int_{\tilde{G}_{\delta\varepsilon}(F)\backslash \tilde{G}(F)} f(g^{-1}\delta\varepsilon(g)) \frac{dg}{dt},$$

where $f \in \mathcal{S}(\tilde{G}(F))$. $J_\varepsilon(\delta, f)$ is convergent ([1]).

An element $\delta$ in $\tilde{G}(F)$ is said to be *$\varepsilon$-conjugate* to $\delta'$ if there exists $g \in \tilde{G}(F)$ such that $\delta' = g^{-1}\delta\varepsilon(g)$. Then we write $\delta \sim_\varepsilon \delta'$. Also $\delta$ in $\tilde{G}(F)$ is said to be *stable $\varepsilon$-conjugate* to $\delta'$ if there exists $g \in \tilde{G}(\bar{F})$ such that $\delta' = g^{-1}\delta\varepsilon(g)$. Then we write $\delta \sim_{\varepsilon-st} \delta'$ and define

$$\mathcal{O}_{\varepsilon-st}(\delta) := \{\delta' \in \tilde{G}(F) \,|\, \delta' \sim_{\varepsilon-st} \delta\}/\sim_\varepsilon .$$

Denote the set of stably $\varepsilon$-conjugacy classes $\mathcal{O}_{\varepsilon-st}(\delta)$ of $\varepsilon$-semi-simple elements $\delta$ by $\mathcal{O}_{\varepsilon-st}(\tilde{G})$.

As Lemma 3.4 (iv), this $\mathcal{O}_{\varepsilon-st}(\delta)$ is classified in terms of Galois cohomology as follows:

**Lemma 3.19.** $\mathcal{O}_{\varepsilon-st}(\delta) \cong \mathrm{Ker}(H^1(F, \tilde{G}_{\delta\varepsilon}) \to H^1(F, \tilde{G}))$.

*Proof.* Take an element of $[\delta'] \in \mathcal{O}_{\varepsilon-st}(\delta)$. By definition, there exists an element $g \in \tilde{G}(\bar{F})$ such that $\delta' = g^{-1}\delta\varepsilon(g)$. Then $f_{\delta'} := \{gs(g^{-1})\}_{s\in\Gamma}$ is an element of $Z^1(F, \tilde{G}_{\delta\varepsilon})$. The map $[\delta'] \mapsto f_{\delta'}$ induces the bijection we desire. □

By Shapiro's lemma and Hilbert's theorem 90, we obtain $H^1(F, \tilde{G}) \cong H^1(E, \mathrm{GL}_n) = \{1\}$. Thus we have $\mathcal{O}_{\varepsilon-st}(\delta) \cong H^1(F, \tilde{G}_{\delta\varepsilon})$.

Assume that $\delta$ is an $\varepsilon$-regular semi-simple element of $\tilde{G}(F)$. Hence $T = \tilde{G}_{\delta\varepsilon}$ is a torus. Then $\mathfrak{D}_\varepsilon(T) := H^1(F, T)$ is a group. We denote the image of $\delta' \in \mathcal{O}_{\varepsilon-st}(\delta)$ in $\mathfrak{D}_\varepsilon(T)$ by $\mathrm{inv}(\delta, \delta')$. Set $\mathfrak{K}_\varepsilon(T) = \mathfrak{D}_\varepsilon(T)^D$.

**Definition 3.20.** For $\kappa \in \mathfrak{K}_\varepsilon(T)$ and $\varphi \in \mathcal{S}(\tilde{G}(F))$,

$$J_\varepsilon^\kappa(\delta, \varphi) := \sum_{[\delta'] \in \mathcal{O}_{st}(\delta)} \overline{\kappa(\mathrm{inv}(\delta, \delta'))} J_\varepsilon(\delta', \varphi) \quad \textit{twisted } \kappa \textit{ orbital integral}$$

$$J_\varepsilon^{st}(\delta, \varphi) := \sum_{[\delta'] \in \mathcal{O}_{\varepsilon-st}(\delta)} J_\varepsilon(\delta', \varphi) \quad \textit{twisted stable orbital integral}$$

**Definition 3.21.** We call $(H, s, \xi)$ a *twisted endoscopic datum* of $(\tilde{G}, \varepsilon)$ if it is satisfied with the following conditions:

(i) $H$ is a connected quasi-split group defined over $F$;

(ii) $s$ is an $\hat{\varepsilon}$-semi-simple element of $\hat{\tilde{G}}$;

(iii) $\xi : {}^L H \to {}^L G$ is a homomorphism over $W_F$;

(iv) $\xi$ induces an isomorphism from $\hat{H}$ to $(\hat{\tilde{G}}_{s\hat{\varepsilon}})^0$;

(v) $\lambda(w) := s\hat{\varepsilon}(\xi(w))s^{-1}\xi(w)^{-1}$ is a 1-coboundary from $W_F$ to $Z(\hat{\tilde{G}})$.

Moreover we call a twisted endoscopic datum $(H, s, \xi)$ *elliptic* if

$$\xi(Z(\hat{H})^\Gamma)^0 \subset Z(\hat{G})^\Gamma.$$

Also two endoscopic data $(H, s, \xi)$ and $(H', s', \xi')$ are *isomorphic* if there exists $g \in \hat{G}$ such that $g\xi({}^L H)g^{-1} = \xi'({}^L H')$ and $gs\hat{\varepsilon}(g)^{-1}s'^{-1} \in Z(\hat{G})$.

**Example 3.22.** We set $H = G,\ s = 1,\ \xi : {}^L G \ni g \rtimes w \mapsto (g, g) \rtimes w \in {}^L\tilde{G}$. Then this is an elliptic datum of $(\tilde{G}, \varepsilon)$.

**Proposition 3.23** ([15, §3.10])**.** (i) *For each $\delta$ in $\tilde{G}(F)_{ss,\varepsilon}$, we have that $N(\delta) := \delta\varepsilon(\delta)$ belongs to $G(E)_{ss}$.*
(ii) *For each $\delta$ in $\tilde{G}(F)_{ss,\varepsilon}$, there exists an element $\gamma$ of $G(F)_{ss}$ such that $\gamma$ is $G(\bar{F})$-conjugate to $N(\delta)$.*
(iii) $\mathcal{N} : \mathcal{O}_{\varepsilon-st}(\tilde{G}) \ni \mathcal{O}_{\varepsilon-st}(\delta) \mapsto \mathcal{O}_{st}(N(\delta)) \in \mathcal{O}_{st}(G)$ *is a bijective map.*

The map $\mathcal{N} : \mathcal{O}_{\varepsilon-st}(\tilde{G}) \to \mathcal{O}_{st}(G)$ is called the *norm map*. Also the element $\gamma$ in statement (ii) of the above Proposition is called a *norm* of $\delta$.

**Theorem 3.24** ([15, §4.10, §4.11])**.** *For each $\varphi$ in $\mathcal{S}(\tilde{G}(F))$, there exists a $\varphi^G$ in $\mathcal{S}(G(F))$ such that the following equation holds:*

$$J^{st}_\varepsilon(\delta, \varphi) = J^{st}(\gamma, \varphi^G)$$

*for any $\delta \in \tilde{G}(F)_{reg,\varepsilon}$ and any $\gamma \in G(F)_{ss}$ such that $\gamma$ is a norm of $\delta$.*

## 3.5 The endoscopic description for $\mathrm{U}(W)$

Recall that $G(F) = \mathrm{U}(W)$. Also by §3.4 we identify $\tilde{G}(F)$ with $\mathrm{GL}_3(E)$. Then an irreducible admissible representation $\tilde{\pi}$ of $\mathrm{GL}_3(E)$ is said to be $\varepsilon$-*stable* if $\tilde{\pi} \circ \varepsilon \cong \tilde{\pi}$. We denote the set of irreducible $\varepsilon$-stable representations of

$\mathrm{GL}_3(E)$ by $\mathrm{Irr}\,\mathrm{GL}_3(E)_\varepsilon$. For $\tilde{\pi} \in \mathrm{Irr}\,\mathrm{GL}_3(E)_\varepsilon$, we denote its twisted character by $\mathrm{Tr}_\varepsilon(\tilde{\pi})$. By Proposition 1.9 and local Langlands correspondence LLC for $\mathrm{GL}_3$, we have the following injective map:

$$\Phi(\mathrm{U}(W)) \hookrightarrow \Phi(\mathrm{GL}_3(E)) \overset{\mathrm{LLC}}{\cong} \mathrm{Irr}\,\mathrm{GL}_3(E). \tag{5}$$

The image of the map is contained in $\mathrm{Irr}\,\mathrm{GL}_3(E)_\varepsilon$. Thus we denote it by $\mathrm{Irr}\,\mathrm{GL}_3(E)_{\varepsilon,+}$.

The first purpose of this subsection is to construct the $L$-packet $\Pi_\phi$ for each $L$-parameter $\phi$ of $\mathrm{U}(W)$.

**Definition 3.25.** (i) For a finite subset $\Pi$ of $\mathrm{Irr}\,\mathrm{U}(W)$, we define

$$\mathrm{Tr}(\Pi) := \sum_{\pi \in \Pi} \mathrm{Tr}(\pi).$$

(ii) For a finite subset $\Pi$ of $\mathrm{Irr}\,\mathrm{U}(W)$ and $\tilde{\pi} \in \mathrm{Irr}\,\mathrm{GL}_3(E)_{\varepsilon,+}$, if

$$\mathrm{Tr}(\Pi)(\varphi^G) = \mathrm{Tr}_\varepsilon(\tilde{\pi})(\varphi) \quad \text{for any } \varphi \in \mathcal{S}(\mathrm{GL}_3(E)),$$

then we say that the character identity holds for $\Pi$ and $\tilde{\pi}$. We denote its relation by $\Pi \overset{CI}{\longleftrightarrow} \tilde{\pi}$.

(iii) Let $\tilde{\pi}$ be an element of $\mathrm{Irr}\,\mathrm{GL}_3(E)_{\varepsilon,+}$. If there exists $\Pi$ such that $\Pi \overset{CI}{\longleftrightarrow} \tilde{\pi}$, then we define this $\Pi$ to be the $L$-packet associated with $\phi$. Here $\phi$ is the corresponding $L$-parameter to $\tilde{\pi}$ under (5). Namely, $\tilde{\pi} = \mathrm{LLC}(\phi_E)$. We write $\Pi_\phi := \Pi$.

This construction of an $L$-packet in the above (iii) is possible for each tempered $L$-parameter $\phi$, that is, $\tilde{\pi}$ is temperd. But, in general it is not possible for non-tempered case. We will use the following definition for non-tempered case.

Let $\tilde{\pi}$ be a non-tempered element of $\mathrm{Irr}\,\mathrm{GL}_3(E)_{\varepsilon,+}$. Then there exists $\lambda \in \mathrm{Irr}\,E^\times$ and $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ such that $\tilde{\pi}$ is the unique non-tempered irreducible subquotient of

$$\lambda \times \eta \times \lambda^{-1} \circ \sigma := \mathrm{Ind}_{B_3}^{\mathrm{GL}_3(E)}(\lambda \boxtimes \eta \boxtimes \lambda^{-1} \circ \sigma).$$

Then $\mathrm{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u)$ has the unique irreducible non-tempered subquotient $\pi$. Here $B$ is the Borel subgroup of $\mathrm{U}(W)$ which consists of the upper-triangular matrices. We define $\Pi_\phi = \{\pi\}$.

Now we construct the $L$-packets of $\mathrm{U}(W)$. First the following character identity holds from the formula of the character of an induced representation.

**Theorem 3.26** ([15]). *For* $\lambda \in \operatorname{Irr} E^\times$ *and* $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, *we have*

$$\operatorname{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u) \overset{CI}{\longleftrightarrow} \lambda \times \eta \times \lambda^{-1} \circ \sigma.$$

With this theorem, we can construct some $L$-packets of $\mathrm{U}(W)$.

Case 1: $\operatorname{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u)$ is reducible and $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ is irreducible
Then we have $\lambda = \eta'$ $(\eta \neq \eta' \in \Pi(E^\times, \mathbb{1}_{F^\times}))$. Thus we obtain the irreducible decomposition:

$$\operatorname{Ind}_B^G(\eta' \boxtimes \eta_u) = \pi(\eta', \eta)^+ \oplus \pi(\eta', \eta)^-,$$

where $\pi(\eta', \eta)^+$ is the unique generic irreducible subrepresentation. Since the $L$-parameter of $\tilde{\pi} = \eta' \times \eta \times \eta'$ is $\eta' \oplus \eta \oplus \eta'$, the corresponding $L$-parameter $\phi$ of $\mathrm{U}(W)$ satisfies $\phi_E = \eta' \oplus \eta \oplus \eta'$, that is, Case$\mathrm{E}_3$. By Definition 3.25 (iii), we have

$$\Pi_\phi = \{\pi(\eta', \eta)^\pm\}.$$

Case 2: $\operatorname{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u)$ is irreducible and $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ is reducible
Then we have $\lambda = \eta' |\ |_E^{\pm 1/2}$ $(\eta' \in \Pi(E^\times, \mathbb{1}_{F^\times}))$. Without loss of generality, we may assume $\lambda = \eta' |\ |_E^{1/2}$. Thus we have the Jordan-Hölder composition:

$$\mathrm{JH}(\eta' |\ |_E^{1/2} \times \eta \times \eta' |\ |_E^{-1/2}) = \left\{ \langle \{\eta' |\ |_E^{-1/2}, \eta' |\ |_E^{1/2}\} \rangle^t \times \eta,\ \ \eta' \circ \det_{\mathrm{GL}_2} \times \eta \right\}.$$

Here $\tilde{\pi} = \eta' \circ \det_{\mathrm{GL}_2} \times \eta$ is the unique non-tempered irreducible subquotient and its $L$-parameter is $\eta' |\ |_E^{1/2} \oplus \eta \oplus \eta' |\ |_E^{-1/2}$. Thus $\phi$ is in Case $\mathrm{E}_2$ and non-tempered. Therefore we have

$$\Pi_\phi = \{\operatorname{Ind}_B^{\mathrm{U}(W)}(\eta' |\ |_E^{1/2} \boxtimes \eta_u)\}.$$

Case 3: $\operatorname{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u)$ is irreducible and $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ is irreducible
Then we have $\lambda \neq \eta |\ |_E^{\pm 1}$ and $\lambda \circ \mathrm{N}_{E/F} \neq |\ |_E^{\pm 1}$. Also we obtain

$$\Pi_\phi = \{\operatorname{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u)\},$$

where $\phi_E = \lambda \oplus \eta \oplus \lambda^{-1} \circ \sigma$ in Case $\mathrm{E}_2$, $\mathrm{E}_4$ or $\mathrm{E}_5$.

Case 4: $\mathrm{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u)$ is reducible and $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ is reducible
Then we have the following two sub-cases:

$$\begin{aligned} &(1) \quad \lambda = \eta| \, |_E^{\pm 1}, (\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})), \\ &(2) \quad \lambda = \mu| \, |_E^{\pm 1/2}, (\mu \in \Pi(E^\times, \omega_{E/F})). \end{aligned}$$

Case 4-(1)

$$\begin{aligned} \mathrm{JH}\left(\eta| \, |_E^{-1} \times \eta \times \eta| \, |_E\right) &= \left\{\langle\{\eta| \, |_E^{-1}, \eta, \eta| \, |_E\}\rangle^t, \ \eta \circ \det{}_{\mathrm{GL}_3}\right\}. \\ \mathrm{JH}\left(\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta| \, |_E^{-1} \boxtimes \eta_u)\right) &= \left\{\mathrm{St}^{\mathrm{U}(W)}(\eta), \ \eta_u \circ \det{}_{\mathrm{U}(W)}\right\}. \end{aligned}$$

Here $\mathrm{St}^{\mathrm{U}(W)}(\eta)$ is the Steinberg representation of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta| \, |_E^{-1} \boxtimes \eta_u)$.

**Theorem 3.27** ([15])**.** *We have*

$$\begin{aligned} \mathrm{St}^{\mathrm{U}(W)}(\eta) &\overset{CI}{\longleftrightarrow} \langle\{\eta| \, |_E^{-1}, \eta, \eta| \, |_E\}\rangle^t, \\ \eta_u \circ \det{}_{\mathrm{U}(W)} &\overset{CI}{\longleftrightarrow} \eta \circ \det{}_{\mathrm{GL}_3}. \end{aligned}$$

The $L$-parameter of $\langle\{\eta| \, |_E^{-1}, \eta, \eta| \, |_E\}\rangle^t$ is $\eta \boxtimes \mathrm{Sym}^2$. Thus we have

$$\Pi_\phi = \{\mathrm{St}^{\mathrm{U}(W)}(\eta)\},$$

where $\phi_E = \eta \boxtimes \mathrm{Sym}^2$ in Case A.

On the other hand, $\eta_u \circ \det_{\mathrm{U}(W)}$ and $\eta \circ \det_{\mathrm{GL}_3}$ are the unique non-tempered irreducible subquotient of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta| \, |_E^{-1} \boxtimes \eta_u)$ and $\eta| \, |_E^{-1} \times \eta \times \eta| \, |_E$, respectively. The $L$-parameter of $\eta \circ \det_{\mathrm{GL}_3}$ is $\eta| \, |_E^{-1} \oplus \eta \oplus \eta| \, |_E$. Thus we have

$$\Pi_\phi = \{\eta_u \circ \det{}_{\mathrm{U}(W)}\},$$

where $\phi_E = \eta| \, |_E^{-1} \oplus \eta \oplus \eta| \, |_E$ in Case $\mathrm{E}_2$.

Case 4-(2)

$$\begin{aligned} \mathrm{JH}(\mathrm{Ind}_B^{\mathrm{U}(W)}(\mu| \, |_E^{1/2} \boxtimes \eta_u)) &= \{\pi^2(\mu, \eta), \pi^{nt}(\mu, \eta)\}, \\ \mathrm{JH}(\mu| \, |_E^{-1/2} \times \eta \times \mu| \, |_E^{1/2}) &= \{\langle\{\mu| \, |_E^{-1/2}, \mu| \, |_E^{1/2}\}\rangle^t \times \eta, \ \mu \circ \det{}_{\mathrm{GL}_2} \times \eta\}. \end{aligned}$$

Here $\pi^2(\mu, \eta)$ is square-integrable and $\pi^{nt}(\mu, \eta)$ is non-tempered. Then we have the following:

**Theorem 3.28** ([15])**.** *There exists an irreducible supercuspidal representation* $\pi^{sc}(\mu, \eta)$ *of* $G(F)$ *such that*

$$\{\pi^2(\mu,\eta),\ \pi^{sc}(\mu,\eta)\} \xleftrightarrow{CI} \langle\{\mu|\ |_E^{-1/2}, \mu|\ |_E^{1/2}\}\rangle^t \times \eta.$$

The $L$-parameter of $\langle\{\mu|\ |_E^{-1/2}, \mu|\ |_E^{1/2}\}\rangle^t \times \eta$ is $\mu \boxtimes \mathrm{Sym}^1 \oplus \eta$. Thus we have

$$\Pi_\phi = \{\pi^2(\mu,\eta),\ \pi^{sc}(\mu,\eta)\},$$

where $\phi_E = \mu \boxtimes \mathrm{Sym}^1 \oplus \eta$ in Case B.

On the other hand, $\pi^{nt}(\mu,\eta)$ and $\mu \circ \det_{\mathrm{GL}_2} \times \eta$ are the unique non-tempered irreducible subquotient of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\mu|\ |_E^{1/2} \boxtimes \eta_u)$ and $\mu|\ |_E^{-1/2} \times \eta \times \mu|\ |_E^{1/2}$, respectively. The $L$-parameter of $\mu \circ \det_{\mathrm{GL}_2} \times \eta$ is $\mu|\ |_E^{-1/2} \oplus \eta \oplus \mu|\ |_E^{1/2}$. Thus we have

$$\Pi_\phi = \{\pi^{nt}(\mu,\eta)\},$$

where $\phi_E = \mu|\ |_E^{-1/2} \oplus \eta \oplus \mu|\ |_E^{1/2}$ in Case $\mathrm{E}_2$.

We have constructed the $L$-packet for each $L$-parameter of $\mathrm{U}(W)$ in Case A, B, $\mathrm{E}_2$, $\mathrm{E}_3$, $\mathrm{E}_4$ or $\mathrm{E}_5$. Thus the remaining cases are C, D and $\mathrm{E}_1$.

**Theorem 3.29** ([15])**.** *Let* $\phi$ *be an* $L$*-parameter of* $\mathrm{U}(W)$ *in Case C,D or* $E_1$*. Then there exists a finite subset* $\Pi_\phi$ *of* $\mathrm{Irr}\,\mathrm{U}(W)$ *such that* $\Pi_\phi \xleftrightarrow{CI} \mathrm{LLC}(\phi_E)$*. Also* $\Pi_\phi$ *consists of irreducible supercuspidal representation of* $\mathrm{U}(W)$*. Moreover*

$$\#\Pi_\phi = \begin{cases} 1 & \textit{Case } C, \\ 2 & \textit{Case } D, \\ 4 & \textit{Case } E_1. \end{cases}$$

*Finally we have the following table:*

| $\phi$ | $\phi_E$ | $\tilde{\pi} = \mathrm{LLC}(\phi_E)$ | $\Pi_\phi$ |
|---|---|---|---|
| *Case A* | $\eta \boxtimes \mathrm{Sym}^2$ | $\langle\{\eta\|\ \|_E^{-1}, \eta, \eta\|\ \|_E\}\rangle^t$ | $\mathrm{St}^{\mathrm{U}(W)}(\eta)$ |
| *Case B* | $\mu \boxtimes \mathrm{Sym}^1 \oplus \eta$ | $\langle\{\mu\|\ \|_E^{-1/2}, \mu\|\ \|_E^{1/2}\}\rangle^t \times \eta$ | $\pi^2(\mu,\eta), \pi^{sc}(\mu,\eta)$ |
| *Case C* | $\phi_E$ | *irred. s.c. repre.* | $\pi$ |
| *Case D* | $\tau \oplus \eta$ | $\mathrm{LLC}(\tau) \times \eta$ | $\pi_1, \pi_2$ |
| *Case* $E_1$ | $\eta_1 \oplus \eta_2 \oplus \eta_3$ | $\eta_1 \times \eta_2 \times \eta_3$ | $\pi_1, \ldots, \pi_4$ |
| *Case* $E_{2a}$ | $\lambda \oplus \eta \oplus \lambda^{-1} \circ \sigma$, $\lambda \neq \eta\|\ \|_E^{\pm 1}$ $\lambda \circ \mathrm{N}_{E/F} \neq \|\ \|_E^{\pm 1}$ | $\lambda \times \eta \times \lambda^{-1} \circ \sigma$ | $\mathrm{Ind}_B^{\mathrm{U}(W)}(\lambda \boxtimes \eta_u)$ |
| *Case* $E_{2b}$ | $\eta\|\ \|_E^{-1} \oplus \eta \oplus \eta\|\ \|_E$ | $\eta \circ \det_{\mathrm{GL}_3}$ | $\eta_u \circ \det_{\mathrm{U}(W)}$ |
| *Case* $E_{2c}$ | $\eta'\|\ \|_E^{-1/2} \oplus \eta \oplus \eta'\|\ \|_E^{1/2}$ | $\eta' \circ \det_{\mathrm{GL}_2} \times \eta$ | $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta'\|\ \|_E^{1/2} \boxtimes \eta_u)$ |
| *Case* $E_{2d}$ | $\mu\|\ \|_E^{-1/2} \oplus \eta \oplus \mu\|\ \|_E^{1/2}$ | $\mu \circ \det_{\mathrm{GL}_2} \times \eta$ | $\pi^{nt}(\mu,\eta)$ |
| *Case*$E_3$ | $\eta' \oplus \eta \oplus \eta'$ | $\eta' \times \eta \times \eta'$ | $\pi(\eta',\eta)^{\pm}$ |
| *Case* $E_4$ | $\mu \oplus \eta \oplus \mu$ | $\mu \times \eta \times \mu$ | $\mathrm{Ind}_B^{\mathrm{U}(W)}(\mu \boxtimes \eta_u)$ |
| *Case* $E_5$ | $\eta \oplus \eta \oplus \eta$ | $\eta \times \eta \times \eta$ | $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta_u)$ |

The second purpose of this subsection is to describe the inner structure of the $L$-packet for each $L$-parameter of $\mathrm{U}(W)$. That is achieved by endoscopy. We denote the set of $L$-packets of $\mathrm{U}(W)$ by $\Pi(\mathrm{U}(W))$. Here note that $\#\mathcal{S}_\phi = \#\Pi_\phi$ for any $L$-parameter $\phi$ of $\mathrm{U}(W)$.

The $\xi_0$ of the fixed endoscopic datum $(H, s_0, \xi_0)$ gives a map

$$\xi_0 : \Pi(H) \ni \Pi_\phi \mapsto \Pi_{\xi_0 \circ \phi} \in \Pi(\mathrm{U}(W))^1.$$

**Theorem 3.30** ([15])**.** (i) *For $\eta, \eta' \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we have*

$$\xi_0(\{\eta'_u \circ \det_{\mathrm{U}(V_{sp})} \boxtimes \eta_u\}) = \{\pi^{nt}(\eta'\mu_0, \eta)\}.$$

(ii) *For $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we have*

$$\xi_0(\{\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}(\eta\mu_0^{-1}|\ |_E) \boxtimes \eta_u\}) = \{\eta_u \circ \det_{\mathrm{U}(V_{sp})}\}.$$

(iii) *Let $\Pi^H \in \Pi(H)$ be an $L$-packet of $H$ except for that in* (i) *and* (ii)*. If $\Pi = \xi_0(\Pi^H)$, then for each $\pi \in \Pi$ there exists $\langle \Pi^H, \pi \rangle \in \{\pm 1\}$ which satisfies the following:*

$$\mathrm{Tran}_H^G(\mathrm{Tr}(\Pi^H)) = \sum_{\pi \in \Pi} \langle \Pi^H, \pi \rangle \, \mathrm{Tr}(\pi).$$

[1] We set $\Pi(H) = \Pi(\mathrm{U}(V_{sp})) \times \Pi(\mathrm{U}(1))$.

We describe the inner structure of an $L$-packet of $\mathrm{U}(W)$ through an example. We consider an $L$-parameter $\phi$ of $\mathrm{U}(W)$ in Case $\mathrm{E}_1$. Namely, we put $\phi_E = \eta_1 \oplus \eta_2 \oplus \eta_3$. Then we have

$$\mathcal{S}_\phi = \{ \begin{pmatrix} \pm1 & & \\ & \pm1 & \\ & & \pm1 \end{pmatrix} \}/\{\pm 1_3\} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Thus the number of elements of $L$-packet $\Pi_\phi$ is four. We take a complete system of representatives of $\mathcal{S}_\phi$ as follows:

$$s_0 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},\ s_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix},\ s_2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}\ s_3 = 1_3.$$

Then we put $\mathcal{H}_i = (\hat{G}_{s_i})^0 \operatorname{Im} \phi$. Assume $i \neq 3$. We take $w_i \in \hat{G}$ such that $\mathrm{Ad}(w_i)\xi({}^L H) \subset \mathcal{H}_i$ and $\mathrm{Ad}(w_i)(s) = s_i$. For example,

$$w_0 = 1_3,\ w_1 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix},\ w_2 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Then for each $i \neq 3$, there exists a unique $L$-parameter $\phi_i^H$ of $H$ such that the following diagram commutes:

$$\begin{array}{ccc} L_E & \xrightarrow{\phi|_{L_E}} & {}^L G \\ & {\scriptstyle \phi_i^H|_{L_E}} \searrow \quad \nearrow {\scriptstyle \mathrm{Ad}(w_i)\circ\xi} & \\ & {}^L H & \end{array}$$

If we put $\Pi_i^H := \Pi_{\phi_i^H} \in \Pi(H)$, then we have $\xi(\Pi_i^H) = \Pi_\phi$. Thus by Theorem 3.30 (iii), we have

$$\mathrm{Tran}_H^G(\mathrm{Tr}(\Pi_i^H)) = \sum_{\pi \in \Pi_\phi} \langle \Pi_i^H, \pi \rangle \,\mathrm{Tr}(\pi).$$

Then we can define the local pairing $\langle s_i, \pi \rangle \in \{\pm 1\}$ by

$$\langle s_i, \pi \rangle := \begin{cases} \langle \Pi_i^H, \pi \rangle & \text{if } i = 0, 1, 2, \\ 1 & \text{if } i = 3. \end{cases}$$

**Theorem 3.31** ([12]). *The map* $\Pi_\phi \ni \pi \mapsto \langle\ \cdot\ ,\pi\rangle \in \mathrm{Irr}\,\mathcal{S}_\phi$ *is a bijective map.*

Note that this theorem holds for any $L$-parameter $\phi$ of $\mathrm{U}(W)$ except for that in Theorem 3.30 (i) and (ii). Now the following corollary is trivial:

**Corollary 3.32.** *For any* $\pi \in \Pi_\phi$*, we have* $\prod_s \langle s, \pi\rangle = 1$*, where* $s$ *runs the non-trivial elements of* $\mathcal{S}_\phi$*.*

Theorem 3.31 was written in [12]. The original form of this theorem in [15], which we will use later, is given as follows. First we define

$$\hat{\Pi}_\phi := \{\rho \in \Phi(H) \,|\, \xi_0 \circ \rho \sim \phi\}.$$

For any $\rho \in \hat{\Pi}_\phi$, there exist $s \in \mathcal{S}_\phi \setminus \{1\}$ and $w \in \mathrm{GL}_3(\mathbb{C})$ such that $\mathrm{Ad}(w) \circ \xi_0 \circ \rho = \phi$ and $\mathrm{Ad}(w)(s_0)$ equals $s$ in $\mathcal{S}_\phi$. Then $\hat{\Pi}_\phi$ correspons to $\mathcal{S}_\phi \setminus \{1\}$ bijectively. We set $\langle \rho, \cdot\rangle := \langle s, \cdot\rangle$.

**Theorem 3.33** ([15]). *Let* $\pi_1, \pi_2$ *be two elements of* $\Pi_\phi$*. If* $\langle \rho, \pi_1\rangle = \langle \rho, \pi_2\rangle$ *for any* $\rho \in \hat{\Pi}_\phi$*, then* $\pi_1 = \pi_2$*.*

Also the following corollary is trivial by Cor. 3.32:

**Corollary 3.34.** *For any* $\pi \in \Pi_\phi$*, we have* $\prod_\rho \langle \rho, \pi\rangle = 1$*, where* $\rho$ *runs the elements of* $\hat{\Pi}_\phi$*.*

## 3.6 Global endoscopy for $\mathrm{U}(W)$

In this subsection, we explain the global multiplicity formula for certain global endoscopic $L$-packets.

Let $\tilde{E}/\tilde{F}$ be a quadratic extension of a number field. Take a non-zero element $\tilde{\xi}$ of $\tilde{E}$ with $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and a non-trivial character $\tilde{\psi}$ of $\mathbb{A}_{\tilde{F}}$. Let

$$\tilde{W} = (\tilde{E}^{\oplus 3}, \begin{pmatrix} & & 1 \\ & \tilde{\xi} & \\ -1 & & \end{pmatrix}).$$

be a three-dimensional skew hermitian space over $\tilde{E}$. We denote its unitary group by $\mathrm{U}(\tilde{W})$.

Take characters $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3 \in \Pi(\mathbb{A}_{\tilde{E}}^{\times}, \omega_{\tilde{E}/\tilde{F}})$ and $\tilde{\mu}_0 \in \Pi(\mathbb{A}_{\tilde{E}}^{\times}, \mathbb{1})$. For any place $v$ of $\tilde{F}$, define an $L$-parameter $\tilde{\phi}_v$ of $\mathrm{U}(\tilde{W}_v)$ by $\tilde{\phi}_{v,\tilde{E}_v} = \tilde{\eta}_{1,v} \oplus \tilde{\eta}_{2,v} \oplus \tilde{\eta}_{3,v}$. Set $\tilde{\phi} = \{\tilde{\phi}_v\}_v$. Then we define the global endoscopic $L$-packet associated to $\tilde{\phi}$ by

$$\Pi_{\tilde{\phi}} := \left\{ \bigotimes_v \pi_v \,\middle|\, \begin{array}{l} \pi_v \in \Pi_{\tilde{\phi}_v}, \\ \pi_v : \text{unramified for almost all } v \end{array} \right\}.$$

We call $\Pi_{\tilde{\phi}}$ a cuspidal $L$-packet if it has an cuspidal representation of $\mathrm{U}(\tilde{W})$.

**Theorem 3.35** ([15])**.** *The set $\Pi_{\tilde{\phi}}$ is a cuspidal $L$-packet of $\mathrm{U}(\tilde{W})$ if and only if $\tilde{\eta}_i \neq \tilde{\eta}_j$ for $i \neq j$.*

To check whether each element of $\Pi_{\tilde{\phi}}$ is cuspidal, we define $\hat{\Pi}_{\tilde{\phi}} = \{\rho_1, \rho_2, \rho_3\}$, where for any place $v$, $\rho_{i,v}$ is an $L$-parameter of $H_v$ and

$$\begin{aligned} \rho_1 &= \{\rho_{1,v}\}_v, & \rho_{1,\tilde{E}_v} &= \tilde{\mu}_{0,v}^{-1}(\tilde{\eta}_{2,v} \oplus \tilde{\eta}_{3,v}) \times \tilde{\eta}_{1,v}, \\ \rho_2 &= \{\rho_{2,v}\}_v, & \rho_{2,\tilde{E}_v} &= \tilde{\mu}_{0,v}^{-1}(\tilde{\eta}_{1,v} \oplus \tilde{\eta}_{3,v}) \times \tilde{\eta}_{2,v}, \\ \rho_3 &= \{\rho_{3,v}\}_v, & \rho_{3,\tilde{E}_v} &= \tilde{\mu}_{0,v}^{-1}(\tilde{\eta}_{1,v} \oplus \tilde{\eta}_{2,v}) \times \tilde{\eta}_{3,v}. \end{aligned}$$

Here $\rho_{i,\tilde{E}_v}$ is the restriction of $\rho_{i,v}$ to $L_{\tilde{E}_v}$. Then set $\langle \rho, \pi \rangle = \Pi_v \langle \rho_v, \pi_v \rangle$ for each $\pi \in \Pi_{\tilde{\phi}}$ and $\rho \in \hat{\Pi}_{\tilde{\phi}}$.

**Theorem 3.36** ([15])**.** *Let $\pi = \bigotimes_v \pi_v$ be an element of $\Pi_{\tilde{\phi}}$. If $m(\pi)$ is the multiplicity of $\pi$ in $L^2(\mathrm{U}(\tilde{W}) \backslash \mathrm{U}(\tilde{W})(\mathbb{A}))$, then*

$$m(\pi) = \begin{cases} 1 & \text{if } \langle \rho, \pi \rangle = 1 \text{ for all } \rho \in \hat{\Pi}_{\phi}, \\ 0 & \text{otherwise}. \end{cases}$$

# 4 Theta lift

## 4.1 Mixed model

In this subsection, we retain the notation in §1.1.

To explain the explicit formulas of the mixed model of the Weil representation of a unitary dual pair, we prepare a few notation.

Let $V = (E^{\oplus m}, R)$ be a hermitian space and $W = (E^{\oplus n}, S)$ a skew-hermitian space over $E$. We may identify $\mathbb{W}$ with $\mathrm{M}_{m,n}(E)$ by the isomorphism

$$V \otimes_E W \ni v \otimes w \mapsto v \cdot w \in \mathrm{M}_{m,n}(E).$$

Then the symplectic form on $\mathbb{W}$ is given by

$$\langle\!\langle X, X' \rangle\!\rangle = \mathrm{Tr}_{E/F}({}^*XRX'^*S) \quad X, X' \in \mathbb{W}.$$

Without loss of generality, we may assume that

$$V = (E^{\oplus m}, \begin{pmatrix} & & 1_{m_0} \\ & R_1 & \\ 1_{m_0} & & \end{pmatrix}).$$

We put $V_0 = (E^{\oplus 2m_0}, \begin{pmatrix} & 1_{m_0} \\ 1_{m_0} & \end{pmatrix})$ and $V_1 = (E^{\oplus m_1}, R_1)$ where $m_1 = m - 2m_0$. We take $X$ as the subspace of $V$ consisting of the elements of the form

$${}^t(x_1, \dots, x_{m_0}, 0, \dots, 0)$$

and $X'$ as the subspace of $V$ consisting of the elements of the form

$${}^t(0, \dots, 0, x_{m-m_0+1}, \dots, x_m).$$

Thus we have

$$V = X \oplus V_1 \oplus X'.$$

Let $P_{X'} = M_{X'}N_{X'}$ be the maximal parabolic subgroup of $\mathrm{U}(V)$ stabilizing $X'$, where $M_{X'}$ is the Levi component of $P_{X'}$ stabilizing $X$ and $N_{X'}$ is the unipotent radical of $P_{X'}$. We have

$$M_{X'} = \left\{ m(a,h) = \begin{pmatrix} a & & \\ & h & \\ & & {}^*a^{-1} \end{pmatrix} \middle| \begin{array}{c} a \in \mathrm{GL}_{m_0}(E) \\ h \in \mathrm{U}(V_1) \end{array} \right\},$$

$$N_{X'} = \left\{ n(b,c) = \begin{pmatrix} 1_{m_0} & & \\ b & 1_{m_1} & \\ c - \frac{1}{2}{}^*bR_1b & -{}^*bR_1 & 1_{m_0} \end{pmatrix} \middle| \begin{array}{c} b \in \mathrm{M}_{m_1,m_0}(E) \\ c = -{}^*c \in \mathrm{M}_{m_0}(E) \end{array} \right\}.$$

We denote the space of Schwartz-Bruhat functions on $\mathbb{X} = X \otimes_E W = \mathrm{M}_{m_0,n}(E)$ by $\mathcal{S}(\mathbb{X})$. Take an irreducible unitary representation $(\tau_{\psi,V_1,W}, \mathcal{S}_{V_1,W})$ of $H(\mathbb{W}_1)$, where $\mathbb{W}_1 = V_1 \otimes_E W = \mathrm{M}_{m_1,n}(E)$. Then the Weil representation $\omega_{\psi,V^{\chi_W},W^{\chi_V}}$ is defined on $\mathcal{S}(\mathbb{X}) \otimes \mathcal{S}_{V_1,W}$ and has the following explicit formulas:

**Theorem 4.1.** *Let $\chi_{V_0}, \chi_{V_1}$ be characters of $E^\times$ such that $\chi_V = \chi_{V_0}\chi_{V_1}$ and $\chi_{V_i}|_{F^\times} = \omega_{E/F}^{\dim_E V_i}$. For $\phi \otimes \phi' \in \mathcal{S}(\mathbb{X}) \otimes \mathcal{S}_{V_1,W}$, we have*

$$\begin{aligned}
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}\,(m(a,h))\,\phi(x) \otimes \phi' \\
&\qquad = \chi_W(\det a)|\det a|_E^{-n/2}\phi(a^{-1}x) \cdot \omega_{\psi,V_1^{\chi_W},W^{\chi_{V_1}}}(h)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}\,(n(b,c))\,\phi(x) \otimes \phi' \\
&\qquad = \psi\left(-\frac{1}{2}\,\mathrm{Tr}_{E/F}({}^*xcx^*S)\right)\phi(x) \cdot \tau_{\psi,V_1,W}(-bx,0)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}\,(g)\,\phi(x) \otimes \phi' \\
&\qquad = (\chi_{V_0})_u(\det g)\phi(xg) \cdot \omega_{\psi,V_1^{\chi_W},W^{\chi_{V_1}}}(g)\phi'.
\end{aligned}$$

*Here $x \in \mathbb{X}$, $m(a,h) \in M_{X'}$, $n(b,c) \in N_{X'}$ and $g \in \mathrm{U}(W)$.*

Similarly assume that

$$W = (E^{\oplus n}, \begin{pmatrix} & & 1_{n_0} \\ & S_1 & \\ -1_{n_0} & & \end{pmatrix}).$$

We put $W_0 = (E^{\oplus 2n_0}, \begin{pmatrix} & 1_{n_0} \\ -1_{n_0} & \end{pmatrix})$ and $W_1 = (E^{\oplus n_1}, S_1)$ where $n_1 = n - 2n_0$. We take $Y$ as the subspace of $W$ consisting of the elements of the form

$$(y_1, \ldots, y_{n_0}, 0, \ldots, 0)$$

and $Y'$ as the subspace of $W$ consisting of the elements of the form

$$(0, \ldots, 0, y_{n-n_0+1}, \ldots, y_n).$$

Thus we have

$$W = Y \oplus W_1 \oplus Y'.$$

Let $P_{Y'} = M_{Y'} N_{Y'}$ be the maximal parabolic subgroup of $\mathrm{U}(W)$ stabilizing $Y'$, where $M_{Y'}$ is the Levi component of $P_{Y'}$ stabilizing $Y$ and $N_{Y'}$ is the unipotent radical of $P_{Y'}$. We have

$$M_{Y'} = \left\{ m(a,g) = \begin{pmatrix} a & & \\ & g & \\ & & {}^*a^{-1} \end{pmatrix} \middle| \begin{array}{c} a \in \mathrm{GL}_{n_0}(E) \\ g \in \mathrm{U}(W_1) \end{array} \right\},$$

$$N_{Y'} = \left\{ n(b,c) = \begin{pmatrix} 1_{n_0} & b & c + \frac{1}{2} b S_1 {}^*b \\ & 1_{n_1} & S_1 {}^*b \\ & & 1_{n_0} \end{pmatrix} \middle| \begin{array}{c} b \in \mathrm{M}_{n_0,n_1}(E) \\ c = {}^*c \in \mathrm{M}_{n_0}(E) \end{array} \right\}.$$

We set $\mathbb{Y} = V \otimes_E Y = \mathrm{M}_{m,n_0}(E)$. Take an irreducible unitary representations $(\tau_{\psi,V,W_1}, \mathcal{S}_{V,W_1})$ of $H(\mathbb{W}_1)$, where $\mathbb{W}_1 = V \otimes_E W_1 = \mathrm{M}_{m,n_1}(E)$.

**Theorem 4.2.** *Let $\chi_{W_0}, \chi_{W_1}$ be characters of $E^\times$ such that $\chi_W = \chi_{W_0}\chi_{W_1}$ and $\chi_{W_i}|_{F^\times} = \omega_{E/F}^{\dim_E W_i}$. For $\phi \otimes \phi' \in \mathcal{S}(\mathbb{Y}) \otimes \mathcal{S}_{V,W_1}$, we have the following explicit formulas:*

$$\begin{aligned}
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}(h)\, \phi(y) \otimes \phi' \\
&\qquad = (\chi_{W_0})_u(\det h)\phi(h^{-1}y) \cdot \omega_{\psi,V^{\chi_{W_1}},W_1^{\chi_V}}(h)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}(m(a,g))\, \phi(y) \otimes \phi' \\
&\qquad = \chi_V(\det a)|\det a|_E^{m/2}\phi(ya) \cdot \omega_{\psi,V^{\chi_{W_1}},W_1^{\chi_V}}(g)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}(n(b,c))\, \phi(y) \otimes \phi' \\
&\qquad = \psi\left(\frac{1}{2}\mathrm{Tr}_{E/F}({}^*yRyc)\right)\phi(y) \cdot \tau_{\psi,V,W_1}(yb,0)\phi'.
\end{aligned}$$

*Here $y \in \mathbb{Y}$, $m(a,g) \in M_{Y'}$, $n(b,c) \in N_{Y'}$ and $h \in \mathrm{U}(V)$.*

Finally we state the following lemma:

**Lemma 4.3.** (i) *For $g \in \mathrm{GU}(V)$ with similitude norm $d$, we have*

$$\begin{aligned}
\omega_{\psi^d,V^{\chi_W},W^{\chi_V}} &\cong \omega_{\psi,V^{\chi_W},W^{\chi_V}} \circ \mathrm{Ad}(g) \\
&\cong \omega_{\psi,dV^{\chi_W},W^{\chi_V}}
\end{aligned}$$

*Here* $dV = (E^{\oplus m}, dA)$ *and* $\mathrm{U}(dV) = \mathrm{U}(V)$.
(ii) *For any character* $\eta, \eta' \in \Pi(E^\times, \mathbb{1}_{F^\times})$, *we have*

$$\omega_{\psi, V^{\eta\chi_W}, W^{\eta'\chi_V}} = (\eta_u \circ \det_{\mathrm{U}(V)} \boxtimes \eta'_u \circ \det_{\mathrm{U}(W)}) \otimes \omega_{\psi, V^\mu, W^\eta}.$$

## 4.2 Local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$

In this subsection we compute the local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$.

We denote by $B$ the Borel subgroup of $\mathrm{U}(V_{sp})$ consisting of lower triangular matrices. Let

$$U = \left\{ u(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \middle| x \in E, \mathrm{Tr}_{E/F}(x) = 0 \right\}$$

be the unipotent radical of $B$ and

$$T = \left\{ m(\alpha) = \begin{pmatrix} \alpha & \\ & {}^*\alpha^{-1} \end{pmatrix} \middle| \alpha \in E^\times \right\}$$

a maximal torus of $\mathrm{U}(V_{sp})$. Thus we have $B = TU$.

From Theorem 4.1, the Weil representation $\omega_{\psi, V_{sp}^\mu, W_a^\eta}$ for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$ on the space $\mathcal{S}(E)$ of Schwartz-Bruhat functions on $E$ has the following explicit formulas:

$$\begin{aligned} \omega_{\psi, V_{sp}^\mu, W_a^\eta} (m(\alpha))\, f(t) &= \mu(\alpha)|\alpha|_E^{-1/2} f(\alpha^{-1} t), \\ \omega_{\psi, V_{sp}^\mu, W_a^\eta} (u(x))\, f(t) &= \psi(a\xi x \,\mathrm{N}_{E/F}(t)) f(t), \\ \omega_{\psi, V_{sp}^\mu, W_a^\eta} (g)\, f(t) &= \eta_u(g) f(tg). \end{aligned}$$

Here $f \in \mathcal{S}(E)$, $t \in E$, $m(\alpha) \in T$, $u(x) \in U$ and $g \in \mathrm{U}(W_a)$. For brevity, we write $\omega_\psi = \omega_{\psi, V_{sp}^\mu, W_a^\eta}$.

First we compute (twisted) Jacquet module of $\omega_\psi$. For each $b \in F^\times$, we set

$$E_b = \left\{ t \in E^\times \,\middle|\, \mathrm{N}_{E/F}(t) = b \,\mathrm{N}_{E/F}(\xi^{-1}) \right\}.$$

One notes that if $b \notin \mathrm{N}_{E/F}(E^\times)$, then $E_b$ is empty.

**Lemma 4.4.** (i) *The unnormalized Jacquet module* $(\omega_\psi)_U$ *of the Weil representation* $\omega_\psi$ *is isomorphic to* $\mu |\cdot|_E^{-1/2} \boxtimes \eta_u$ *as a* $T \times \mathrm{U}(W_a)$*-module, where*

$$\mu |\cdot|_E^{-1/2} \boxtimes \eta_u : T \times \mathrm{U}(W_a) \ni m(\alpha) \times g \mapsto \mu(\alpha)|\alpha|_E^{-1/2} \eta_u(g) \in \mathbb{C}^\times.$$

(ii) *The twisted Jacquet module* $(\omega_\psi)_{U,\psi^b_{U,\xi}}$ *of the Weil representation* $\omega_\psi$ *is isomorphic to* $\eta_u \otimes \mathcal{S}(E_{a^{-1}b})$ *as a* $\mathrm{U}(W_a)$*-module, where the action of* $\mathrm{U}(W_a)$ *on* $\mathcal{S}(E_{a^{-1}b})$ *is given by right translation.*

*Proof.* (i) If we define

$$\omega_\psi(U) = \{\omega_\psi(u(x))f - f \mid x \in F, f \in \mathcal{S}(E)\},$$

then $(\omega_\psi)_U = \omega_\psi/\omega_\psi(U)$. By the above explicit formulas, we have $\omega_\psi(U) = \{f \in \mathcal{S}(E) \mid f(0) = 0\}$. Thus the map $f \mapsto f(0)$ induces a $T \times \mathrm{U}(W_a)$-isomorphism $(\omega_\psi)_U \cong \mu|\cdot|_E^{-1/2} \boxtimes \eta_u$.

(ii) If we define

$$\omega_\psi(U, \psi^b_{U,\xi}) = \left\{\omega_\psi(u(x))f - \psi(-bx\xi^{-1})f \,\middle|\, x \in F, f \in \mathcal{S}(E)\right\},$$

then $(\omega_\psi)_{U,\psi^b_{U,\xi}} = \omega_\psi/\omega_\psi(U, \psi^b_{U,\xi})$. By the above explicit formulas, we have $\omega_\psi(U, \psi^b_{U,\xi}) = \left\{f \in \mathcal{S}(E) \,\middle|\, f|_{E_{a^{-1}b}} = 0\right\}$. Thus the map $f \mapsto f|_{E_{a^{-1}b}}$ induces a $\mathrm{U}(W_a)$-isomorphism $(\omega_\psi)_{U,\psi^b_{U,\xi}} \cong \eta_u \otimes \mathcal{S}(E_{a^{-1}b})$. □

**Corollary 4.5.** (i) *For each* $\eta'_u \in \mathrm{Irr}\,\mathrm{U}(W_a)$*, the local theta lift* $\theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u)$ *is* $\psi^a$*-generic and not* $\psi^{ad_0}$*-generic.*
(ii) *If* $\eta \neq \eta'$*, then the local theta lift* $\theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u)$ *is an irreducible supercuspidal representation of* $\mathrm{U}(V_{sp})$*.*
(iii) *If* $\eta = \eta'$*, then the local theta lift* $\theta_{\psi,V^\mu,W^\eta_a}(\eta_u)$ *is the irreducible* $\psi^a$*-generic subrepresentation* $\tau(\mu,\mu)^{\omega_{E/F}(a)}_{sp}$ *of* $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\mu$*.*

*Proof.* By the explicit formulas, we have a non-zero $\mathrm{U}(W_a)$-homomorphism $\omega_\psi \to \eta'_u$. Thus we have $\tau = \theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u) \neq 0$. By the definition of the local theta lift, we have a surjective $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$-homomorphism

$$\omega_\psi \to \tau \boxtimes \eta'_u.$$

Taking its twisted Jacquet module with respect to $\psi^b_{U,\xi}$, we have a surjective $\mathrm{U}(W_a)$-homomorphism

$$\eta_u \otimes \mathcal{S}(E_{a^{-1}b}) \to \tau_{U,\psi^b_{U,\xi}} \boxtimes \eta'_u.$$

Thus if $b = ad_0$, then $\tau_{U,\psi^b_{U,\xi}}$ is zero. Namely, $\tau$ is not $\psi^{ad_0}$-generic.

We assume that $\eta' \neq \eta$. Taking the Jacquet module of $\omega_\psi$ with respect to $U$, we have $\tau_U = 0$. Thus $\tau$ is supercuspidal. Here, it is well-known that $\tau$ is $\psi^a$-generic or $\psi^{ad_0}$-generic. Thus $\tau$ is $\psi^a$-generic.

Next we assume that $\eta' = \eta$. By Frobenius reciprocity, we have

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{U}(V_{sp})\times \mathrm{U}(W_a)}(\omega_\psi, \mathrm{Ind}_B^{\mathrm{U}(V_{sp})} \mu \boxtimes \eta_u) \\ \cong\ &\mathrm{Hom}_{T\times \mathrm{U}(W_a)}((\omega_\psi)_U, \mu|\cdot|_E^{-1/2} \boxtimes \eta_u) \\ \cong\ &\mathbb{C}. \end{aligned}$$

Thus $\tau$ is an irreducible subrepresentation of $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})} \mu$, that is, $\tau = \tau(\mu,\mu)_{sp}^{\pm}$. Since $\tau$ is not $\psi^{ad_0}$-generic, we have $\tau = \tau(\mu,\mu)_{sp}^{\omega_{E/F}(a)}$. Thus we obtain Corollary 4.5. □

**Theorem 4.6.** *For each $\eta'_u \in \mathrm{Irr}\,\mathrm{U}(W_a)$, the local theta lift $\theta_{\psi, V_{sp}^\mu, W_a^\eta}(\eta'_u)$ is the unique $\psi^a$-generic element of L-packet $\Pi_{\mu,\mu\eta\eta'^{-1}}$ of $\mathrm{U}(V_{sp})$. Namely,*

$$\theta_{\psi, V_{sp}^\mu, W_a^\eta}(\eta'_u) = \tau(\mu, \mu\eta\eta'^{-1})_{sp}^{\omega_{E/F}(a)}.$$

*Proof.* We set $\tau = \theta_{\psi, V_{sp}^\mu, W_a^\eta}(\eta'_u)$. It is enough to show that $\tau$ is contained in $\Pi_{\mu,\mu\eta\eta'^{-1}}$. We have already proved it for $\eta = \eta'$. Thus we consider the case for $\eta \neq \eta'$. Our proof uses Theorem 2.6. We take the following:

- $\tilde{E}/\tilde{F}$ is a global quadratic extension;
- finite place $v_0$ such that $\tilde{E}_{v_0}/\tilde{F}_{v_0} = E/F$;
- $\tilde{a} \in \tilde{F}^\times$ such that $\tilde{a} \equiv a \mod \mathrm{N}_{E/F}(E^\times)$;
- $\tilde{\xi} \in \tilde{E}^\times$ such that $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and $\tilde{\xi} \equiv \xi \mod \mathrm{N}_{E/F}(E^\times)$;
- $\tilde{\psi}$ is a non-trivial character of $\mathbb{A}_{\tilde{F}}/\tilde{F}$ such that $\tilde{\psi}_{v_0} = \psi$;
- $\tilde{\mu} \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{v_0} = \mu$;
- $\tilde{\eta} \neq \tilde{\eta}' \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \mathbb{1})$ such that $\tilde{\eta}_{v_0} = \eta, \tilde{\eta}'_{v_0} = \eta'$.

Now we define two (skew) hermitian spaces over $\tilde{E}$:

$$\tilde{V}_{sp} = (\tilde{E}^{\oplus 2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}) \text{ and } W_{\tilde{a}} = (\tilde{E}, \tilde{a}\tilde{\xi}).$$

Then the global theta lift $\tilde{\tau} = \theta_{\tilde{\psi}, \tilde{V}_{sp}^{\tilde{\mu}}, W_{\tilde{a}}^{\tilde{\eta}}}(\tilde{\eta}'_u)$ is non-zero and cuspidal, since the constant term of $\tilde{\tau}$ is zero and its $\tilde{\psi}$-Whittaker model is non-zero by the explicit formulas of the Weil representation of $\omega_{\tilde{\psi},\tilde{V}_{sp}^{\tilde{\mu}},W_{\tilde{a}}^{\tilde{\eta}}}$. Thus $\tilde{\tau}$ is an irreducible cuspidal representation. If we show the claim that the local theta lift $\tilde{\tau}_v$ of $\tilde{\eta}'_{u,v}$ is contained in $\Pi_{\tilde{\mu}_v,\tilde{\mu}_v\tilde{\eta}_v\tilde{\eta}_v'^{-1}}$ for almost all places $v$ of $\tilde{F}$, then by Theorem 2.6 we obtain Theorem 4.6.

First suppose that a field extension $\tilde{E}_v/\tilde{F}_v$ and $\tilde{\eta}_v$, $\tilde{\eta}'_v$ are unramified. Then the claim immediately follows from $\tilde{\eta}_v = \tilde{\eta}'_v$.

Next suppose that $\tilde{E}_v/\tilde{F}_v$ is split. We omit the symbol $v$ from the notation. Then we have $\tilde{E} \cong \tilde{F} \oplus \tilde{F}, \mathrm{U}(\tilde{V}_{sp}) \cong \mathrm{GL}_2(\tilde{F})$ and $\mathrm{U}(W_{\tilde{a}}) \cong \mathrm{GL}_1(\tilde{F})$. Also its Weil representation $(\omega_{\tilde{\psi}}, \mathcal{S}(\tilde{F} \oplus \tilde{F}))$ is given as follows:

$$\begin{aligned}
\omega_{\tilde{\psi}}\,(h)\,f(t) &= \tilde{\mu}(\det h)|\det h|_{\tilde{F}}^{-1/2} f(h^{-1}t),\\
\omega_{\tilde{\psi}}\,(g)\,f(t) &= \tilde{\eta}(g)|g|_{\tilde{F}} f(tg).
\end{aligned}$$

Here $f(t) \in \mathcal{S}(\tilde{F} \oplus \tilde{F})$, $h \in \mathrm{GL}_2(\tilde{F})$ and $g \in \mathrm{GL}_1(\tilde{F})$. Then we want to show that $\tilde{\tau} = \mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu})$. Take a quasi-character $\chi$ of $\tilde{F}^\times$. For an element $f \in \mathcal{S}(\tilde{F} \oplus \tilde{F})$, we define a function $\Phi_f$ on $\mathrm{GL}_2(\tilde{F}) \times \mathrm{GL}_1(\tilde{F})$ by

$$\Phi_f(h,g) = \frac{1}{L(0,\chi)} \int_{\tilde{F}^\times} \omega_{\tilde{\psi}}(h,g) f \begin{pmatrix} t \\ 0 \end{pmatrix} \chi(t) dt^\times.$$

By the above explicit formulas, the map $f \mapsto \Phi_f$ induces a non-zero $\mathrm{GL}_2(\tilde{F}) \times \mathrm{GL}_1(\tilde{F})$-homomorphism

$$\omega_{\tilde{\psi}} \to \mathrm{Ind}(\tilde{\mu}\chi|\cdot|_{\tilde{F}}^{-1} \boxtimes \tilde{\mu}) \boxtimes \tilde{\eta}\chi^{-1}|\cdot|_{\tilde{F}}.$$

If $\chi = \tilde{\eta}\tilde{\eta}'^{-1}|\cdot|_{\tilde{F}}$, then we have

$$\omega_{\tilde{\psi}} \to \mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu}) \boxtimes \tilde{\eta}'.$$

Since $\tilde{\mu}, \tilde{\eta}$ and $\tilde{\eta}'$ are unitary, $\mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu})$ is irreducible. Thus we have $\tilde{\tau} = \mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu})$. □

# 5 Endoscopy and local theta lift

## 5.1 Endoscopy and local theta lift for $\mathrm{U}(V) \times \mathrm{U}(W)$

In this subsection we recall some results in [4] about the relation between endoscopy and local theta lift of $\mathrm{U}(V) \times \mathrm{U}(W)$, where $V = V_{sp}, V_{an}$.

First we define the subset of irreducible representations occurring in Weil representation $\omega_{\psi,V^\mu,W^\eta}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$. For $V = V_{sp}$ or $V_{an}$, we set

$$\mathcal{R}_{\psi,\mu,\eta}(V) := \{\tau \in \operatorname{Irr}\mathrm{U}(V) \mid \theta_{\psi,V^\mu,W^\eta}(\tau) \neq 0\}.$$

On the other hand we set

$$\mathcal{R}_{\psi,\mu,\eta}(W) := \{\pi \in \operatorname{Irr}\mathrm{U}(W) \mid \theta_{\psi,V^\mu,W^\eta}(\pi) \neq 0, V = V_{sp} \text{ or } V_{an}\}.$$

We remark that the defintion of $\mathcal{R}_{\psi,\mu,\eta}(W)$ here is slightly different from that in §1.1. The map defined by the local theta lifts $\theta_{\psi,V^\mu,W^\eta}$ with respect to $V = V_{sp}, V_{an}$ is denoted as follows:

$$\theta_{\psi,\mu,\eta} \colon \mathcal{R}_{\psi,\mu,\eta}(V_{sp}) \sqcup \mathcal{R}_{\psi,\mu,\eta}(V_{an}) \to \mathcal{R}_{\psi,\mu,\eta}(W).$$

**Theorem 5.1.** (i) *The map $\theta_{\psi,\mu,\eta}$ is bijective.*
(ii) *For $\tau \in \mathcal{R}_{\psi,\mu,\eta}(V)$ with $L$-parameter $\varphi$, the $L$-parameter of $\theta_{\psi,\mu,\eta}(\tau)$ is $\mu\eta\check{\varphi}_E \oplus \eta$, where $\check{\varphi}_E$ is the contragredient representation of $\varphi_E$.*

*Proof.* (i) The injectivity follows from the Dichotomy Theorem [4, Theorem 1.2]. The surjectivity follows from the definition of the map.

(ii) This was proved in [4, §4]. □

The set $\mathcal{R}_{\psi,\mu,\eta}(V)$ is described by the following:

**Theorem 5.2** ([4, Lemma 4.2])**.** *For $\tau \in \operatorname{Irr}\mathrm{U}(V)$, the local theta lift $\theta_{\psi,\mu,\eta}(\tau)$ is zero if and only if $\theta_{\psi^{d_0},V^\mu,W_1^\eta}(\tau) \neq 0$, where $W_1 = (E, \xi)$.*

To describe $\mathcal{R}_{\psi,\mu,\eta}(W)$, we prepare a few notation. Let $\phi$ be an $L$-parameter of $\mathrm{U}(W)$. An $L$-packet $\Pi_\phi$ of $\mathrm{U}(W)$ is called ($\chi$-)*endoscopic* if it consists of infinite-dimensional representations and a one-dimensional representation $\chi$ of $W_E$ is a subrepresentation of $\phi_E$. An element of such $\Pi_\phi$ is called ($\chi$-)*endoscopic*.

**Theorem 5.3** ([4, §4])**.** *The set $\mathcal{R}_{\psi,\mu,\eta}(W)$ consists of the $\eta$-endoscopic representations and $\eta_u \circ \det$.*

The local theta lift $\theta_{\psi,\mu,\eta}$ has the following property:

**Theorem 5.4** ([4]). *For $\tau \in \mathcal{R}_{\psi,\mu,\eta}(V)$, $\theta_{\psi,\mu,\eta}(\tau)$ is generic if and only if $V = V_{sp}$, $\tau$ is $\psi$-generic and $\tau \neq \mathrm{Ind}_B^{\mathrm{U}(V_{sp})} \mu|\cdot|_E^{-1}$.*

**Remark 5.5.** The proofs of the above theorems in [4] do not require the endoscopic description of $L$-packets of $\mathrm{U}(V_{an})$. But Theorem 2.7 (v) on the cardinality of an $L$-packet requires (see [4, p.430]).

Let $\phi$ be an $L$-parameter of $\mathrm{U}(W)$ such that $\Pi_\phi$ is endoscopic. Then Gelbart-Rogawski-Soudry defined a local pairing $\epsilon_\rho(\pi)$ for $\rho \in \hat{\Pi}_\phi$ and $\pi \in \Pi_\phi$ as follows. $\rho$ is an $L$-parameter of $H_0 = \mathrm{U}(V_{sp}) \times \mathrm{U}(1)$. Thus we denote its $\mathrm{U}(1)$-part by $\rho_\eta$. The character of $E^\times/F^\times$ corresponding to $\rho_\eta$ is denoted by $\eta_\rho$. Thus $\Pi_\phi$ is an $\eta_\rho$-endoscopic $L$-packet. Then by Theorem 5.1 and Theorem 5.3, there exists a unique $V_{\pi,\rho} = V_{sp}$ or $V_{an}$ for $\pi \in \Pi_\phi$ such that $\theta_{\psi, V^\mu_{\pi,\rho}, W^{\eta_\rho}}(\pi) \neq 0$. Here $V_{\pi,\rho}$ is independent of the choices of characters $\psi, \mu$ by Lemma 4.3. Thus we can define the local pairing $\epsilon_\rho(\pi) := +$ if $V_{\pi,\rho} = V_{sp}$ and $-$ otherwise. Namely, we have the following definition:

**Definition 5.6.** For $\rho \in \hat{\Pi}_\phi$ and $\pi \in \Pi_\phi$,

$$\epsilon_\rho(\pi) := \begin{cases} + & \text{if } \theta_{\psi, V^\mu_{sp}, W^{\eta_\rho}}(\pi) \neq 0, \\ - & \text{if } \theta_{\psi, V^\mu_{an}, W^{\eta_\rho}}(\pi) \neq 0. \end{cases}$$

The following theorem gives the relation between the endoscopic description for $\mathrm{U}(W)$ and the local theta lift $\theta_{\psi,\mu,\eta}$.

**Theorem 5.7** ([4, Theorem 3.1]). *For $\rho \in \hat{\Pi}_\phi$ and $\pi \in \Pi_\phi$, we have $\langle \rho, \pi \rangle = \epsilon_\rho(\pi)$.*

Also the following holds:

**Proposition 5.8** ([4]). *Let $\pi$ be an element of an endoscopic $L$-packet $\Pi_\phi$. Then $\pi$ is generic if and only if $\epsilon_\rho(\pi) = +$ for any $\rho \in \hat{\Pi}_\phi$.*

## 5.2 Local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W)$

Take an irreducible admissible representation $\tau$ of $\mathrm{U}(V_{sp})$ with $L$-parameter $\varphi$. Set $\pi = \theta_{\psi,\mu,\eta}(\tau)$. If $\pi$ is non-zero, then its $L$-parameter $\phi$ is given by $\phi_E = \mu\eta\check{\varphi}_E \oplus \eta$. Then we want to compute the local pairing $\epsilon_\rho(\pi)$ for $\rho \in \hat{\Pi}_\phi$. In the following cases we can deduce this from the above results.

**Proposition 5.9.** *Let $\rho_0$ be the element of $\hat{\Pi}_\phi$ such that $\rho_{0,E} = \mu_0^{-1}\eta\mu\check{\varphi}_E \times \eta$.*

(i) *If $\tau$ is $\psi$-generic, the local pairing $\epsilon_\rho(\pi) = +$ for any $\rho \in \hat{\Pi}_\phi$.*

(ii) *If $\varphi_E$ is an irreducible representation of $L_E$, we have $\hat{\Pi}_\phi = \{\rho_0\}$ and $\epsilon_{\rho_0}(\pi) = +$.*

(iii) *If $\tau$ is one dimensional, we have $\hat{\Pi}_\phi = \{\rho_0\}$ and $\epsilon_{\rho_0}(\pi) = +$.*

(iv) *If $\tau = \tau(\mu, \mu_1)^-_{sp}$ ($\mu_1 \in \Pi(E^\times, \omega_{E/F})$), then $\pi = 0$.*

(v) *If $\tau = \tau(\mu_1, \mu_2)^-_{sp}$ ($\mu \neq \mu_1, \mu_2 \in \Pi(E^\times, \omega_{E/F})$), then the cardinality of $\hat{\Pi}_\phi$ is two or three and the local pairing $\epsilon_{\rho_0}(\pi) = +$ and $\epsilon_\rho(\pi) = -$ for any $\rho \neq \rho_0, \in \hat{\Pi}_\phi$.*

*Proof.* (i) If $\tau = \mathrm{Ind}_B^{\mathrm{U}(V_{sp})} \mu| \cdot |_E^{-1}$, then $\pi = \eta_u \circ \det$. This case is trivial. Otherwise $\pi$ is generic by Theorem 5.4. Thus this follows from Proposition 4.10.

(ii), (iii) By Theorem 5.1, the first is trivial. The second follows from $V_{\pi,\rho_0} = V$.

(iv) By Theorem 4.6, we have

$$\begin{aligned}
&\theta_{\psi^{d_0}, V_{sp}^{\mu_1}, W_1^1}(\tau(\mu_1, \mu)^-_{sp}) \\
=&\theta_{\psi, V_{sp}^{\mu_1}, W_1^1}(\tau(\mu_1, \mu)^+_{sp}) \\
=&(\mu_1\mu^{-1})_u \\
\neq& 0.
\end{aligned}$$

Thus it follows from Theorem 5.2 that $\pi = 0$.

(v) $\pi$ is not generic by Theorem 5.4. Moreover it is trivial that $\epsilon_{\rho_0}(\pi) = +$.

Assume that $\mu_1 \neq \mu_2$. Then we have $\hat{\Pi}_\phi = \{\rho_0, \rho_1, \rho_2\}$, where

$$\begin{aligned}
\rho_{0,E} &= \mu_0^{-1}\mu\eta(\mu_1^{-1} \oplus \mu_2^{-1}) \times \eta, \\
\rho_{1,E} &= \mu_0^{-1}(\eta \oplus \mu\eta\mu_2^{-1}) \times \mu\eta\mu_1^{-1}, \\
\rho_{2,E} &= \mu_0^{-1}(\mu\eta\mu_1^{-1} \oplus \eta) \times \mu\eta\mu_2^{-1}.
\end{aligned}$$

By Cor. 3.34, we have $\epsilon_{\rho_0}(\pi)\epsilon_{\rho_1}(\pi)\epsilon_{\rho_2}(\pi) = +$. Since $\epsilon_{\rho_0}(\pi) = +$, we obtain $\epsilon_{\rho_1}(\pi)\epsilon_{\rho_2}(\pi) = +$. If $\epsilon_{\rho_1}(\pi) = +$ and $\epsilon_{\rho_2}(\pi) = +$, then $\pi$ is generic by Proposition 5.8. This is a contradiction. Therefore we have $\epsilon_{\rho_1}(\pi) = \epsilon_{\rho_2}(\pi) = -$.

We can also prove for the case $\mu_1 = \mu_2$ by the similar argument. □

# 6 Main results

## 6.1 Setting

Take an irreducible admissible representation $\tau$ of $\mathrm{U}(V_{an})$ with $L$-parameter $\varphi$. Set $\pi = \theta_{\psi,\mu,\eta}(\tau)$. If $\pi$ is non-zero, then its $L$-parameter $\phi$ is given by $\phi_E = \mu\eta\check{\varphi}_E \oplus \eta$. Then we want to compute the local pairing $\epsilon_\rho(\pi)$ for $\rho \in \hat{\Pi}_\phi$.

**Lemma 6.1.** *Let $\rho_0$ be the element of $\hat{\Pi}_\phi$ such that $\rho_{0,E} = \mu_0^{-1}\eta\mu\check{\varphi}_E \times \eta$. If $\varphi_E$ is an irreducible representation of $L_E$, then we have $\hat{\Pi}_\phi = \{\rho_0\}$ and $\epsilon_{\rho_0}(\pi) = +$.*

*Proof.* This can be proved by the similar argument in Proposition 5.10 (ii). □

Thus we may assume that $\varphi_E$ is an reducible representation of $L_E$. Namely, there exist $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$ such that $\varphi_E = \mu_1 \oplus \mu_2$. Then $\tau = \tau(\mu_1, \mu_2)^{\pm}_{an}$ and the $L$-parameter $\phi$ of $\pi$ satisfies $\phi_E = \mu\eta\mu_1^{-1} \oplus \mu\eta\mu_2^{-1} \oplus \eta$ by Theorem 5.1.

## 6.2 Two lemmas

We take the following:

- $\tilde{F}$ is a number field;
- finite place $v_0$ such that $\tilde{F}_{v_0} = F$;
- infinite place $v_1$ such that $\tilde{F}_{v_1} = \mathbb{R}$;
- $\tilde{E}$ is a quadratic extension of $\tilde{F}$ such that $\tilde{E}_{v_0} = E, \tilde{E}_{v_1} = \mathbb{C}$;
- $\tilde{\xi} \in \tilde{E}^\times$ such that $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and $\tilde{\xi} \equiv \xi \mod \mathrm{N}_{E/F}(E^\times)$;
- $\tilde{\psi}$ is a non-trivial character of $\mathbb{A}_{\tilde{F}}/\tilde{F}$ such that $\tilde{\psi}_{v_0} = \psi$;
- $\tilde{d} \in \tilde{F}$ such that $\tilde{d} \notin \mathrm{N}_{\tilde{E}_v/\tilde{F}_v}(\tilde{E}_v^\times)$ at $v = v_0, v_1$ and $\tilde{d} \in \mathrm{N}_{\tilde{E}_v/\tilde{F}_v}(\tilde{E}_v^\times)$ at any place $v \neq v_0, v_1$;
- $\tilde{\mu}_1 \neq \tilde{\mu}_2 \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{i,v_0} = \mu_i$ $(i = 1, 2)$ and $\tilde{\mu}_{1,v_1} \neq \tilde{\mu}_{2,v_1}$.

Define a two-dimensional hermitian space $\tilde{V}$ over $\tilde{E}$:

$$\tilde{V} = (\tilde{E}^{\oplus 2}, \begin{pmatrix} -\tilde{d} & \\ & 1 \end{pmatrix}).$$

By the above condition, we have $\tilde{V}_{v_0} \cong V_{an}$ and the signature of $\tilde{V}_{v_1}$ is $(2,0)$. Also at any place $v \neq v_0, v_1$, the group $\mathrm{U}(\tilde{V}_v)$ is a quasi-split unitary group or general linear group.

The following lemma is used in §6.3.

**Lemma 6.2.** *Let the notation be as above. Take* $\tilde{\tau} = \otimes \tilde{\tau}_v \in \Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V})$ *such that* $\tilde{\tau}_v$ *is* $\tilde{\psi}_v$*-generic for all* $v \neq v_0, v_1$*. Then* $\tilde{\tau}$ *is cuspidal if and only if* $\tilde{\tau}_{v_0} = \tau(\mu_1, \mu_2)_{an}^{C_{v_1}}$*.*

*Proof.* This follows from Theorem 2.12. □

**Remark 6.3.** We can take $\tilde{\psi}, \tilde{\xi}, \tilde{\mu}_1$ and $\tilde{\mu}_2$ so that $C_{v_1} = +$.

To prove our main results, we also prepare the following lemma with respect to the representation $\rho_{m,n}$ of $\mathrm{U}(\tilde{V}_{v_1})$:

**Lemma 6.4.** *For brevity, we omit* $v_1$*. Then* $\theta_{\tilde{\psi}, \tilde{V}^{\tilde{\mu}_1}, \tilde{W}_1^{\mathbb{1}}}(\rho_{m,n}) \neq 0$ *if and only if* $C_{v_1} = +$*, where* $\tilde{W}_1 = (\mathbb{C}, \tilde{\xi})$*.*

*Proof.* From [6], we recall explicit formulas of Fock model of the Weil representation $\omega_{\tilde{\psi}} = \omega_{\tilde{\psi}, \tilde{V}^{\tilde{\mu}_1}, \tilde{W}_1^{\mathbb{1}}}$ for $\mathrm{U}(\tilde{V}) \times \mathrm{U}(\tilde{W}_1)$. First the signature of $\tilde{V}$ is $(2,0)$ so that we may assume that $\tilde{V} = (\mathbb{C}^{\oplus 2}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})$. Then the complexification of the Lie algebra of $\mathrm{U}(\tilde{V})$ is $\mathrm{M}_2(\mathbb{C})$. We write $E_{j,k}$ for the $(j,k)$-elementary matrix. Also we may assume $\tilde{\xi} = \pm i$. Since the complexification of the Lie algebra of $\mathrm{U}(\tilde{W}_1)$ is $\mathbb{C}$, we write $1_{\tilde{W}_1}$ for its neutral element. Next the Fock space of the Weil representation $\omega_{\tilde{\psi}}$ is $\mathbb{C}[w_1, w_2]$. Since $\tilde{\mu}_1 = \chi^{m+n}$, the explicit formulas are given as follows:

(1) $\mathrm{sgn}(r\tilde{\xi}/i) = +$

$$\omega_{\tilde{\psi}}(E_{j,k}) = \frac{m+n-1}{2}\delta_{j,k} - w_k \frac{\partial}{\partial w_j}$$
$$\omega_{\tilde{\psi}}(1_{\tilde{W}_1}) = 1 + w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}$$

(2) $\mathrm{sgn}(r\tilde{\xi}/i) = -$

$$\omega_{\tilde{\psi}}(E_{j,k}) = \frac{m+n+1}{2}\delta_{j,k} + w_j \frac{\partial}{\partial w_k}$$
$$\omega_{\tilde{\psi}}(1_{\tilde{W}_1}) = -1 - w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}$$

By those formulas, we can describe the theta lift $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^{\mathbb{1}}}$. In the case (1), the theta lift $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^{\mathbb{1}}}$ is given by

$$\chi_1^l \longmapsto \rho_{m+n-l,l} \quad (l \geq 1).$$

Here $\chi_1 : \mathbb{C}^1 \ni z \mapsto z \in \mathbb{C}^1$ is a character of $\mathrm{U}(\tilde{W}_1)$. Thus $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^{\mathbb{1}}}(\rho_{m,n}) \neq 0$ if and only if $n \geq 1$. This is equivalent to $C_{v_1} = +$.

In the case (2), the theta lift is given by

$$\chi_1^l \longmapsto \rho_{m+n-l,l} \quad (l \leq -1).$$

Thus $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^{\mathbb{1}}}(\rho_{m,n}) \neq 0$ if and only if $n \leq -1$. This is equivalent to $C_{v_1} = +$. □

Let

$$\tilde{W} = (\tilde{E}^{\oplus 3}, \begin{pmatrix} & & 1 \\ & \tilde{\xi} & \\ -1 & & \end{pmatrix}).$$

be a three-dimensional skew hermitian space over $\tilde{E}$. Also we assume that $C_{v_1} = +$. By Lemma 6.4, the local theta lift $\theta_{\tilde{\psi}_{v_1},\tilde{V}_{v_1}^{\tilde{\mu}_1,v_1},\tilde{W}_{v_1}^{\mathbb{1}}}(\rho_{m,n})$ of $\rho_{m,n}$ to $\mathrm{U}(\tilde{W}_{v_1})$ is non-zero. We denote this representation of $\mathrm{U}(\tilde{W}_{v_1})$ by $\pi_{m,n}$.

## 6.3 Main results

Now we prove our main results. First we consider the local theta lift of some endoscopic representations of $\mathrm{U}(V_{an})$ to $\mathrm{U}(W)$.

**Theorem 6.5.** *For $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$, we have*

$$\theta_{\psi,V_{an}^{\mu_1},W^{\eta}}(\tau(\mu_1,\mu_2)_{an}^+) = \pi(\eta, \eta\mu_1\mu_2^{-1})^-.$$

*Here $\pi(\eta,\eta\mu_1\mu_2^{-1})^-$ is the unique irreducible non-generic subrepresentation of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes (\eta\mu_1\mu_2^{-1})_u)$, where $B$ is the Borel subgroup of all upper triangular matrices in $\mathrm{U}(W)$.*

*Proof.* Note that $\omega_{\psi,V_{an}^{\mu_1},W^{\eta}} = \eta_u \circ \det_{\mathrm{U}(W)} \otimes \omega_{\psi,V_{an}^{\mu_1},W^{\mathbb{1}}}$ by Lemma 4.3. Thus we may assume $\eta = \mathbb{1}$. Therefore it is enough to show that

$$\theta_{\psi,V_{an}^{\mu_1},W^{\mathbb{1}}}(\pi(\mathbb{1},\mu_1\mu_2^{-1})^-) = \tau(\mu_1,\mu_2)_{an}^+.$$

For brevity, we write $\theta_{\psi,V} = \theta_{\psi,V^{\mu_1},W^{\mathbb{1}}}$ and $\pi^- = \pi(\mathbb{1}, \mu_1\mu_2^{-1})^-$.

First we show that $\theta_{\psi,V_{an}}(\pi^-) \in \Pi_{\mu_1,\mu_2}(V_{an})$. Since $\pi^-$ is $\mathbb{1}$-endoscopic, by Theorem 5.1 there exists a unique $V = V_{sp}, V_{an}$ such that $\theta_{\psi,V}(\pi^-) \neq 0$. By Theorems 5.1 and 5.4, $\theta_{\psi,V}(\pi^-)$ is a non $\psi$-generic representation in $\Pi_{\mu_1,\mu_2}(V)$. If $V = V_{sp}$, then we have $\theta_{\psi,V}(\pi^-) = \tau(\mu_1,\mu_2)_{sp}^-$. But by Proposition 5.10 (iv), $\theta_{\psi,V_{sp}}(\tau(\mu_1,\mu_2)_{sp}^-) = 0$. This is a contradiction. Thus we have $V = V_{an}$.

Next we show that $\theta_{\psi,V_{an}}(\pi^-) = \tau(\mu_1,\mu_2)_{an}^+$. To show it, we use the notation and the assumption in §6.2. We also take a non-trivial character $\tilde{\eta} \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \mathbb{1})$ such that $\tilde{\eta} \neq \tilde{\mu}_1\tilde{\mu}_2^{-1}$ and $\tilde{\eta}_v = \mathbb{1}$ at $v = v_0, v_1$. Then we define a global $L$-parameter $\tilde{\phi}$ of $\mathrm{U}(\tilde{W})$ by $\tilde{\phi}_{\tilde{E}} = \mathbb{1} \oplus \tilde{\eta} \oplus \tilde{\mu}_1\tilde{\mu}_2^{-1}$, where $\tilde{\phi}_{\tilde{E}}$ is the restriction of $\tilde{\phi}$ to the Weil group of $\tilde{E}$. Thus we have the local $L$-packet $\Pi_{\tilde{\phi}_v}$ of $\mathrm{U}(\tilde{W}_v)$ for each $v$. A global $L$-packet $\Pi_{\tilde{\phi}}$ is defined by

$$\left\{ \bigotimes \pi_v \,\middle|\, \pi_v \in \Pi_{\tilde{\phi}_v}, \pi_v \text{ is generic for almost all } v \right\}.$$

We can take the element $\tilde{\pi} \in \Pi_{\tilde{\phi}}$ such that $\tilde{\pi}_{v_0} = \pi^-$, $\tilde{\pi}_{v_1} = \pi_{m,n}$ and $\tilde{\pi}_v$ is generic at $v \neq v_0, v_1$. It is a cuspidal representation. To see this, we take a character $\tilde{\mu}_0 \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{0,v_0} = \mu_0$, where $\mu_0$ is in the introduction. We define $\hat{\Pi}_{\tilde{\phi}}$ as in §3.6. Then we have $\hat{\Pi}_{\tilde{\phi}} = \{\rho_0, \rho_1, \rho_2\}$, where

$$\begin{aligned}
\rho_{0,\tilde{E}} &= \tilde{\mu}_0^{-1}(\tilde{\eta} \oplus \tilde{\mu}_1\tilde{\mu}_2^{-1}) \times \mathbb{1}, \\
\rho_{1,\tilde{E}} &= \tilde{\mu}_0^{-1}(\mathbb{1} \oplus \tilde{\mu}_1\tilde{\mu}_2^{-1}) \times \tilde{\eta}, \\
\rho_{2,\tilde{E}} &= \tilde{\mu}_0^{-1}(\mathbb{1} \oplus \tilde{\eta}) \times \tilde{\mu}_1\tilde{\mu}_2^{-1}.
\end{aligned}$$

By Theorem 3.36, we must show that $\langle \rho_i, \tilde{\pi} \rangle := \Pi_v \langle \rho_{i,v}, \tilde{\pi}_v \rangle = +$ for each $i = 0, 1, 2$. Since $\tilde{\pi}_v$ is generic for $v \neq v_0, v_1$, we have $\langle \rho_{i,v}, \tilde{\pi}_v \rangle = +$ for $i = 0, 1, 2$ ([4, Prop. 3.3]). Also for $v = v_0, v_1$, we have $\langle \rho_{0,v}, \tilde{\pi}_v \rangle = \epsilon_{\rho_{0,v}}(\tilde{\pi}_v) = -$. Thus we obtain $\langle \rho_0, \tilde{\pi} \rangle = +$. Since $\rho_{0,v} = \rho_{1,v}$ for $v = v_0, v_1$, we also obtain $\langle \rho_1, \tilde{\pi} \rangle = +$. Moreover since $\langle \rho_0, \tilde{\pi} \rangle \langle \rho_1, \tilde{\pi} \rangle \langle \rho_2, \tilde{\pi} \rangle = +$ by [4, p.454], we have $\langle \rho_2, \tilde{\pi} \rangle = +$. Thus $\tilde{\pi}$ is a cuspidal representation.

By [4, Theorem 1.1 (b)], there exists a unique two-dimensional hermitian space $V$ over $\tilde{E}$ such that the global theta lift $\tilde{\tau} = \theta_{\tilde{\psi},V^{\tilde{\mu}_1},\tilde{W}^{\mathbb{1}}}(\tilde{\pi}) \neq 0$. Also the global theta lift $\tilde{\tau}$ is an irreducible cuspidal representation of $\mathrm{U}(\tilde{V})$ and is included in $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$. Since $\tilde{\pi}_v$ is generic at $v \neq v_0$ and $v_1$, $V_v$ is split, that is, $V_v \cong \tilde{V}_v$. On the other hand, we have already proved above that $V_{v_0} \cong V_{an}$. Moreover it follows from the definition of $\pi_{m,n}$ that $V_{v_1} \cong \tilde{V}_{v_1}$.

Thus we have $V = \tilde{V}$. Here $\tilde{\tau}_v$ is $\tilde{\psi}_v$-generic for $v \neq v_0, v_1$ and $\tilde{\tau}_{v_1} = \rho_{m,n}$. By Lemma 6.2, we obtain $\tilde{\tau}_{v_0} = \tau(\mu_1, \mu_2)^+_{an}$. Namely, $\theta_{\psi,V_{an}}(\pi^-) = \tau(\mu_1, \mu_2)^+_{an}$. □

To compute the local theta lift for $\mathrm{U}(V_{an}) \times \mathrm{U}(W_a)$ from Theorem 6.5, we need the following lemma:

**Lemma 6.6.** *Let $N$ be the unipotent radical of the Borel subgroup $B$ of $\mathrm{U}(W)$. Take a characters $\eta' \in \Pi(E^\times, \mathbb{1}_{F^\times})$ such that $\eta' \neq \eta$.*
*(i) The Jacquet module $(\omega_{\psi,V^\mu_{an},W^\eta})_N$ of the Weil representation $\omega_{\psi,V^\mu_{an},W^\eta}$ of $\mathrm{U}(V_{an})\times\mathrm{U}(W)$ is isomorphic to $\eta|\cdot|_E \boxtimes \omega_{\psi,V^\mu_{an},W^\eta_1}$ as an $E^\times \times \mathrm{U}(V_{an}) \times \mathrm{U}(W_1)$-module.*
*(ii) If $\tau_{\eta'} := \theta_{\psi,V^\mu_{an},W^\eta_1}(\eta'_u) \neq 0$, then $\theta_{\psi,V^\mu_{an},W^\eta}(\tau_{\eta'}) = \pi(\eta, \eta')^-$.*
*(iii) We have $\tau_\eta = 0$, that is, $\theta_{\psi,V^\mu_{an},W^\eta_1}(\eta_u) = 0$.*

*Proof.* (i) By Theorem 4.2, we have the explicit formulas of the mixed model of the Weil representation $\omega_{\psi,V^\mu_{an},W^\eta}$ of $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$. Then (i) follows from the similar argument in the proof of Lemma 4.4.

(ii) Since $\tau_{\eta'} \neq 0$, we have a surjective homomorphism

$$\omega_{\psi,V^\mu_{an},W^\eta_1} \to \tau_{\eta'} \boxtimes \eta'_u.$$

By (i) and Frobenius reciprocity, we obtain a non-zero $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$-homomorphism

$$\omega_{\psi,V^\mu_{an},W^\eta} \to \tau_{\eta'} \boxtimes \mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta'_u).$$

Thus $\theta_{\psi,V^\mu_{an},W^\eta}(\tau_{\eta'})$ is an irreducible subquotient of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta'_u)$. Since $\eta' \neq \eta$, $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta'_u)$ has the unique non-generic irreducible subrepresentation $\pi(\eta, \eta')^-$. Also by Theorem 5.4, $\theta_{\psi,V^\mu_{an},W^\eta}(\tau_{\eta'})$ is non-generic. Thus (ii) holds.

(iii) Assume $\tau_\eta \neq 0$. By the similar argument, we have $\theta_{\psi,V^\mu_{an},W^\eta}(\tau_\eta)$ is an irreducible non-generic subquotient of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta_u)$. But $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta_u)$ is a generic irreducible representation. This is a contradiction. Thus we have $\tau_\eta = 0$. □

Now we can compute the local theta lift for $\mathrm{U}(V_{an}) \times \mathrm{U}(W_a)$.

**Theorem 6.7.** *For $\eta' \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we have*

$$\theta_{\psi,V^\mu_{an},W^\eta_a}(\eta'_u) = \begin{cases} \tau(\mu, \mu\eta\eta'^{-1})^{\omega_{E/F}(a)}_{an} & \text{if } \eta \neq \eta', \\ 0 & \text{if } \eta = \eta'. \end{cases}$$

*Proof.* Take $g \in \mathrm{GU}(V_{an})$ with similitude norm $a$. Then by Lemma 4.3, we have

$$\theta_{\psi,V_{an}^{\mu},W_a^{\eta}}(\eta'_u) = \theta_{\psi,V_{an}^{\mu},W_1^{\eta}}(\eta'_u) \circ \mathrm{Ad}(g).$$

Thus we may assume $a = 1$ by Corollary 2.8 (iv).

By Lemma 6.6, we consider only $\eta \neq \eta'$. We show that

$$\theta_{\psi,V_{an}^{\mu},W_1^{\eta}}(\tau(\mu, \mu\eta\eta'^{-1})_{an}^{+}) = \eta'_u.$$

By Theorem 6.5, we obtain

$$\theta_{\psi,V_{an}^{\mu},W^{\eta}}(\tau(\mu, \mu\eta\eta'^{-1})_{an}^{+}) = \pi(\eta, \eta')^{-}.$$

Since $\tau(\mu, \mu\eta\eta'^{-1})_{an}^{+}$ is supercuspidal and $\pi(\eta, \eta')^{-}$ is not supercuspidal, we have $\theta_{\psi,V_{an}^{\mu},W_1^{\eta}}(\tau(\mu, \mu\eta\eta'^{-1})_{an}^{+}) \neq 0$. Thus there exists a unique character $\eta'' \in \Pi(E^{\times}, \mathbb{1}_{F^{\times}})$ such that $\theta_{\psi,V_{an}^{\mu},W_1^{\eta}}(\tau(\mu, \mu\eta\eta'^{-1})_{an}^{+}) = \eta''_u$. By Lemma 6.6 (ii), we obtain $\pi(\eta, \eta'')^{-} = \pi(\eta, \eta')^{-}$. Therefore $\eta'' = \eta'$. □

**Corollary 6.8.** *We have*

$$\theta_{\psi,V_{an}^{\mu_1},W^{\eta}}(\tau(\mu_1, \mu_2)_{an}^{-}) = 0.$$

*Proof.* By Theorem 6.4,

$$\begin{aligned} &\theta_{\psi^{d_0},V_{an}^{\mu_1},W_1^{\eta}}(\tau(\mu_1, \mu_2)_{an}^{-}) \\ =&\theta_{\psi,V_{an}^{\mu_1},W_1^{\eta}}(\tau(\mu_1, \mu_2)_{an}^{+}) \\ =&(\mu_1\mu_2^{-1}\eta)_u \\ \neq& 0 \end{aligned}$$

Thus this corollary follows from Theorem 5.2. □

Finally, we compute the local theta lift of the remaining endoscopic representations of $\mathrm{U}(V_{an})$ to $\mathrm{U}(W)$. Take characters $\mu_1, \mu_2 \in \Pi(E^{\times}, \omega_{E/F})$. We assume that $\mu, \mu_1, \mu_2$ are distinct. Then by Theorem 5.2 we have

$$\pi^{\pm} := \theta_{\psi,V_{an}^{\mu},W^{\eta}}(\tau(\mu_1, \mu_2)_{an}^{\pm}) \neq 0.$$

Also the $L$-parameter $\phi$ of $\pi^{\pm}$ satisfies $\phi_E = \mu\eta\mu_1^{-1} \oplus \mu\eta\mu_2^{-1} \oplus \eta$. Thus we have $\hat{\Pi}_{\phi} = \{\rho_0, \rho_1, \rho_2\}$, where

$$\begin{aligned} \rho_{0,E} &= \mu_0^{-1}\mu\eta(\mu_1^{-1} \oplus \mu_2^{-1}) \times \eta, \\ \rho_{1,E} &= \mu_0^{-1}(\eta \oplus \mu\eta\mu_2^{-1}) \times \mu\eta\mu_1^{-1}, \\ \rho_{2,E} &= \mu_0^{-1}(\mu\eta\mu_1^{-1} \oplus \eta) \times \mu\eta\mu_2^{-1}. \end{aligned}$$

**Theorem 6.9.** *We have for $\varepsilon \in \{\pm\}$,*

$$\begin{aligned}\epsilon_{\rho_0}(\pi^\varepsilon) &= -,\\ \epsilon_{\rho_1}(\pi^\varepsilon) &= -\varepsilon,\\ \epsilon_{\rho_2}(\pi^\varepsilon) &= \varepsilon.\end{aligned}$$

*Proof.* By Definition 5.7, we have $\epsilon_{\rho_0}(\pi^\varepsilon) = -$. By Cor. 3.34, we also have

$$\epsilon_{\rho_0}(\pi^\varepsilon)\epsilon_{\rho_1}(\pi^\varepsilon)\epsilon_{\rho_2}(\pi^\varepsilon) = +.$$

Thus we obtain $\epsilon_{\rho_1}(\pi^\varepsilon)\epsilon_{\rho_2}(\pi^\varepsilon) = -$. Since these $\epsilon_{\rho_i}$ distinguish $\pi^\pm$, it is enough to show the theorem for $\varepsilon = +$.

Let $\tilde{\tau}$ be the cuspidal representation of $\mathrm{U}(\tilde{V})$ in Lemma 6.2. Take the following:

- $\tilde{\eta} \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \mathbb{1})$ such that $\tilde{\eta}_{v_0} = \eta, \tilde{\eta}_{v_1} = \mathbb{1}$;
- $\tilde{\mu} \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{v_0} = \mu, \tilde{\mu}_{v_1} = \tilde{\mu}_{1,v_1}$.

Since we assume $C_{v_1} = +$, the local theta lift $\theta_{\tilde{\psi}_v, \tilde{V}_v^{\tilde{\mu}_v}, \tilde{W}_v^{\tilde{\eta}_v}}(\tilde{\tau}_v) \neq 0$ for each $v$. Thus by [4, Theorem 5.1], the global theta lift $\tilde{\pi} := \theta_{\tilde{\psi}, \tilde{V}^{\tilde{\mu}}, \tilde{W}^{\tilde{\eta}}}(\tilde{\tau})$ is a non-zero cuspidal representation of $\mathrm{U}(\tilde{W})$. Note that $\tilde{\pi}_{v_0} = \pi^+$ and $\tilde{\pi}_{v_1} = \pi_{m,n}$.

We take a character $\tilde{\mu}_0 \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{0,v_0} = \mu_0$. Then we define $\hat{\Pi}_\phi = \{\tilde{\rho}_0, \tilde{\rho}_1, \tilde{\rho}_2\}$ as follows:

$$\begin{aligned}\rho_{0,\tilde{E}} &= \tilde{\mu}_0^{-1}\tilde{\mu}\tilde{\eta}(\tilde{\mu}_1^{-1} \oplus \tilde{\mu}_2^{-1}) \times \tilde{\eta},\\ \rho_{1,\tilde{E}} &= \tilde{\mu}_0^{-1}(\tilde{\eta} \oplus \tilde{\mu}\tilde{\eta}\tilde{\mu}_2^{-1}) \times \tilde{\mu}\tilde{\eta}\tilde{\mu}_1^{-1},\\ \rho_{2,\tilde{E}} &= \tilde{\mu}_0^{-1}(\tilde{\mu}\tilde{\eta}\tilde{\mu}_1^{-1} \oplus \tilde{\eta}) \times \tilde{\mu}\tilde{\eta}\tilde{\mu}_2^{-1}.\end{aligned}$$

Since $\tilde{\pi}$ is cuspidal, we have $\langle \rho_i, \tilde{\pi} \rangle = +$ for any $i = 0, 1, 2$.

Now $\tilde{\tau}_v$ is $\tilde{\psi}_v$-generic for $v \neq v_0, v_1$. Thus $\tilde{\pi}_v$ is generic. By Proposition 5.8, we obtain $\langle \rho_{i,v}, \tilde{\pi}_v \rangle = \epsilon_{\rho_{i,v}}(\tilde{\pi}_v) = +$ for any $i$.

By the choices of $\tilde{\eta}, \tilde{\mu}$, we have $\rho_{0,v_1} = \rho_{1,v_1}$. Thus we obtain $\langle \rho_{1,v_1}, \tilde{\pi}_{v_1} \rangle = \langle \rho_{0,v_1}, \tilde{\pi}_{v_1} \rangle = \epsilon_{\rho_{0,v_1}}(\pi_{m,n}) = -$.

Thus we have $\epsilon_{\rho_1}(\pi^+) = \langle \rho_{1,v_1}, \tilde{\pi}_{v_1} \rangle = -$ and $\epsilon_{\rho_2}(\pi^+) = +$. □

# 7 Local theta lift for a quaternionic dual pair $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$

## 7.1 Quaternionic unitary groups

Let $D$ be the quaternion division algebra over $F$ with the main involution $\iota$. We denote its reduced norm and trace by $\nu_D, \mathrm{tr}_D$, respectively.

First we consider (skew) hermitian spaces over $D$ and the quaternion unitary groups. Let $V_D$ be a hermitian space over $D$. Namely, $V_D$ is a right $D$-vector space with a hermitian form $(\ ,\ )$. Let $W_D$ be a skew-hermitian space over $D$. Then $W_D$ is a left $D$-vector space with a skew-hermitian form $\langle\ ,\ \rangle$. The spaces $V_D$ and $W_D$ may be taken as follows:

$$\begin{aligned} V_D &= (D^{\oplus m}, A), & W_D &= (D^{\oplus n}, B), \\ (v_1, v_2) &= {}^*v_1 A v_2, & \langle w_1, w_2 \rangle &= w_1 B^* w_2. \end{aligned}$$

Here $A = {}^*A := {}^tA^\iota \in \mathrm{GL}_m(D)$ and $B = -{}^*B \in \mathrm{GL}_n(D)$. Then the quaternion unitary groups of $V_D$ and $W_D$ are given by

$$\begin{aligned} \mathrm{U}(V_D) &= \{h \in \mathrm{GL}_m(D) \mid {}^*hAh = A\}, \\ \mathrm{U}(W_D) &= \{g \in \mathrm{GL}_n(D) \mid gB^*g = B\}. \end{aligned}$$

Here $\mathrm{U}(V_D)$ (resp. $\mathrm{U}(W_D)$) acts on $V_D$ (resp. $W_D$) on the left (resp. right).

In this paper, we assume that $\dim V_D$ and $\dim W_D = 1$. Then we have the following results:

**Lemma 7.1** ([16])**.** *(i) Any one-dimensional hermitian space $V_D$ is isomorphic to $(D, 1)$.*
*(ii) For any one-dimensional skew-hermitian space $W_D$, there exist a quadratic extension $E/F$ in $D$ and $\xi \in E^\times$ such that $\mathrm{Tr}_{E/F}(\xi) = 0$ and $W_D \cong (D, \xi)$.*

Thus we assume that $V_D = (D, 1)$ and $W_D = (D, \xi)$. Then we write

$$\mathrm{U}_D(1) := \mathrm{U}(V_D) \quad \text{and} \quad \mathrm{U}_E(1) := \mathrm{U}(W_D).$$

It is easy to check that

$$\mathrm{U}_D(1) = \mathrm{Ker}\, \nu_D \supset \mathrm{U}_E(1) = \mathrm{Ker}\, \mathrm{N}_{E/F}\,.$$

Finally we consider a relation between $V_D$ and $V_{an}$. Since $D$ is a quaternion division algebra, there exists $\xi' \in D$ such that $\mathrm{tr}_D(\xi') = 0, \xi\xi' = -\xi'\xi$ and

$$\begin{aligned} D &= F \oplus F\xi \oplus F\xi' \oplus F\xi'\xi \\ &= E \oplus \xi' E. \end{aligned}$$

Then we may assume that $\xi'^{,2} = d_0$. We can consider $V_D$ as a two-dimensional hermitian space over $E$. The following lemma follows from easy computations.

**Lemma 7.2.** (i) *The $E$-linear map $i : V_D \ni z + \xi' w \mapsto {}^t(w, z) \in V_{an}$ is an isomorphism as hermitian spaces over $E$.*
(ii) *We denote the homomorphism induced from $i$ by $I : \mathrm{U}_D(1) \to \mathrm{U}(V_{an})$. Then*

$$I : \mathrm{U}_D(1) \ni z + \xi' w \mapsto \begin{pmatrix} \sigma(z) & w \\ d_0\sigma(w) & z \end{pmatrix} \in \mathrm{U}(V_{an})$$

*is an injective homomorphism and its image is $\mathrm{SU}(V_{an})$.*

By this lemma, we identify $\mathrm{U}_D(1)$ with $\mathrm{SU}(V_{an})$. Then $\mathrm{U}_E(1)$ is identified with a subgroup consisting of diagonal matrices of $\mathrm{U}(V_{an})$ by

$$\mathrm{U}_E(1) \ni \gamma \mapsto \begin{pmatrix} \gamma^{-1} & \\ & \gamma \end{pmatrix} \in \mathrm{U}(V_{an}).$$

## 7.2 The Weil representation for $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$

For $V_D$ and $W_D$, we have the Weil representation $\omega_{\psi,V_D,W_D}$ for $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$ by Kudla ([8]). In this subsection we prove the following:

**Proposition 7.3.** *Let $\mu$ be an element of $\Pi(E^\times, \omega_{E/F})$. Then we have*

$$\omega_{\psi,V_D,W_D} \cong \omega_{\psi,V_{an}^{\mu},W_1^{\mathbb{1}}} \circ I \times \mathrm{id}_{\mathrm{U}_E(1)}$$

*as $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$-modules. Note that $\mathrm{U}_E(1) = \mathrm{U}(W_1)$.*

We must recall the construction of the Weil representation $\omega_{\psi,V_D,W_D}$ for $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$. Let

$$\mathbb{W}_D = V_D \otimes_D W_D \tag{6}$$

be the $F$-vector space equipped with the symplectic form

$$\langle\!\langle\ ,\ \rangle\!\rangle = \mathrm{tr}_D\left( (\ ,\ ) \otimes \langle\ ,\ \rangle^{\iota} \right).$$

We take an irreducible unitary representation $(\tau_{\psi,D}, \mathcal{S}_D)$ of the Heisenberg group $H(\mathbb{W}_D)$ with central character $\psi$. Then we have the metaplectic group:

$$\mathrm{Mp}_{\psi}(\mathbb{W}_D) = \left\{ (g, M_g) \;\middle|\; \begin{array}{l} g \in \mathrm{Sp}(\mathbb{W}_D), \\ M_g \colon \tau_{\psi,D} \cdot g \cong \tau_{\psi,D} \text{ isomorphism} \end{array} \right\}.$$

The metaplectic group $\mathrm{Mp}_{\psi}(\mathbb{W}_D)$ has the Weil representation $(\omega_{\psi,D}, \mathcal{S}_D)$ defined by

$$\omega_{\psi,D}(g, M_g) := M_g.$$

By the doubling method, Kudla ([8]) defined the splitting

$$\iota_{W_D} \times \iota_{V_D} : \mathrm{U}_D(1) \times \mathrm{U}_E(1) \to \mathrm{Mp}_{\psi}(\mathbb{W}_D).$$

Namely, we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{U}_D(1) \times \mathrm{U}_E(1) & \xrightarrow{\iota_{W_D} \times \iota_{V_D}} & \mathrm{Mp}_{\psi}(\mathbb{W}_D) \\ & \searrow \quad \swarrow & \\ & \mathrm{Sp}(\mathbb{W}_D) & \end{array}$$

Then the Weil representation $(\omega_{\psi,V_D,W_D}, \mathcal{S}_D)$ for $\mathrm{U}_D(1) \times \mathrm{U}_E(1)$ is defined by

$$\omega_{\psi,V_D,W_D} := \omega_{\psi,D} \circ \iota_{W_D} \times \iota_{V_D}.$$

Now we define the symplectic space $\mathbb{W} = V_{an} \otimes_E W_1$ over $F$ as in §2.1. Then the following lemma follows from easy computations:

**Lemma 7.4.** (i) *The map*

$$\mathbb{W}_D = D \ni x_0 + x_1\xi + x_2\xi' + x_3\xi'\xi \mapsto {}^t(x_2 + \xi x_3, x_0 + \xi x_1) \in E^{\oplus 2} = \mathbb{W}$$

*is an isomorphism as symplectic spaces over $F$.*
(ii) *We have the commutative diagram:*

$$\begin{array}{ccc} \mathrm{U}_D(1) \times \mathrm{U}_E(1) & \xrightarrow{I \times \mathrm{id}} & \mathrm{U}(V_{an}) \times \mathrm{U}(W_1) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(\mathbb{W}_D) & \xrightarrow{\cong} & \mathrm{Sp}(\mathbb{W}) \end{array}$$

*Here the bottom map is one induced by (i).*

Since $H(\mathbb{W}_D) \cong H(\mathbb{W})$, we can consider $(\tau_{\psi,D}, \mathcal{S}_D)$ as an irreducible unitary representation of $H(\mathbb{W})$. Then by §1.1, we have the metaplectic group $\mathrm{Mp}_\psi(\mathbb{W})$ and the Weil representation

$$\omega_{\psi,V_{an}^\mu,W_1^{\mathbb{1}}} := \omega_{\psi,V_{an},W_1} \circ \iota_{W_1}^\mu \times \iota_{V_{an}}^{\mathbb{1}}$$

on $\mathcal{S}_{\mathcal{D}}$ for $\mathrm{U}(V_{an}) \times \mathrm{U}(W_1)$.

To prove Proposition 7.3, it is enough to show that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{U}_D(1) \times \mathrm{U}_E(1) & \xrightarrow{I\times\mathrm{id}} & \mathrm{U}(V_{an}) \times \mathrm{U}(W_1) \\ {\scriptstyle \iota_{W_D}\times\iota_{V_D}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{W_1}^\mu\times\iota_{V_{an}}^{\mathbb{1}}} \\ \mathrm{Mp}_\psi(\mathbb{W}_D) & \xrightarrow{\Phi} & \mathrm{Mp}_\psi(\mathbb{W}) \end{array} \tag{7}$$

Here $\Phi$ is the isomorphism induced by Lemma 7.4 (i).

First we consider $\mathrm{U}_D(1)$. The homomorphism $\Phi^{-1} \circ \iota_{W_1}^\mu \circ I$ is another splitting of $\mathrm{U}_D(1)$ by Lemma 7.4 (ii). The difference of two splittings of $\mathrm{U}_D(1)$ is a character of $\mathrm{U}_D(1)$. Since $\mathrm{U}_D(1)$ has no non-trivial character, we obtain $\Phi^{-1} \circ \iota_{W_1}^\mu \circ I = \iota_{W_D}$. Namely, the above diagram is commutative for $\mathrm{U}_D(1)$.

Next we consider $\mathrm{U}_E(1)$. We must recall the doubling method. Take two skew-hermitian spaces over $D$:

$$W_{D,sp} = (D^{\oplus 2}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}),$$
$$W_{-D} = (D, -\xi).$$

Then we have

$$W_D \oplus W_{-D} \cong W_{D,sp}. \tag{8}$$

We write $\mathrm{U}_E(-1) := \mathrm{U}(W_{-D})$. On the other hand, if we define a skew-hermitian space over $E$:

$$W_{sp} = (E^{\oplus 2}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}),$$

then we have

$$W_1 \oplus W_{-1} \cong W_{sp}.$$

We can choose the isomorphism (8) so that the following diagram is commutative:

$$\begin{array}{ccc} W_1 \oplus W_{-1} & \xrightarrow{\cong} & W_{sp} \\ \downarrow & & \downarrow j \\ W_D \oplus W_{-D} & \xrightarrow{\cong} & W_{D,sp} \end{array}$$

Here the vertical maps are induced by $E \hookrightarrow D$. Then we obtain the commutative diagram:

$$\begin{array}{ccc} \mathrm{U}(W_1) \times \mathrm{U}(W_{-1}) & \longrightarrow & \mathrm{U}(W_{sp}) \\ \| & & \downarrow J \\ \mathrm{U}_E(1) \times \mathrm{U}_E(-1) & \longrightarrow & \mathrm{U}(W_{D,sp}) \end{array}$$

Here $J : \mathrm{U}(W_{sp}) \ni g \mapsto g \in \mathrm{U}(W_{D,sp})$.

Define the symplectic spaces $V_{an} \otimes_E W_{sp}$ and $V_D \otimes_D W_{D,sp}$ as in §1.1 and (6), respectively. Then $i$ and $j$ give the isomorphism

$$V_{an} \otimes_E W_{sp} \cong V_D \otimes_D W_{D,sp}.$$

Also we have the commutative diagram:

$$\begin{array}{ccccc} \mathrm{U}(W_1) \times \mathrm{U}(W_{-1}) & \longrightarrow & \mathrm{U}(W_{sp}) & \longrightarrow & \mathrm{Sp}(V_{an} \otimes_E W_{sp}) \\ \| & & \downarrow J & & \downarrow \\ \mathrm{U}_E(1) \times \mathrm{U}_E(-1) & \longrightarrow & \mathrm{U}(W_{D,sp}) & \longrightarrow & \mathrm{Sp}(V_D \otimes_D W_{D,sp}) \end{array}$$

Now from [8] the Weil representation $\omega_{\psi,V_D,W_{D,sp}}$ of $\mathrm{U}_D(1) \times \mathrm{U}(W_{D,sp})$ on the space $\mathcal{S}(V_D)$ of Schwartz-Bruhat functions on $V_D = D$ has the following explicit formulas:

$$\begin{aligned} &\omega_{\psi,V_D,W_{D,sp}}(h)\, f(x) = f(h^{-1}x), \\ &\omega_{\psi,V_D,W_{D,sp}} \begin{pmatrix} a & \\ & \iota(a)^{-1} \end{pmatrix} f(x) = |\nu_D(a)|_F f(xa), \\ &\omega_{\psi,V_D,W_{D,sp}} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} f(x) = \psi(b\nu_D(x)) f(x). \end{aligned}$$

Here $f \in \mathcal{S}(V_D)$, $x \in V_D$, $h \in \mathrm{U}_D(1)$, $a \in D^\times$ and $b \in F$. On the other hand, from Theorem 4.2 the Weil representation $\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}$ of $\mathrm{U}(V_{an}) \times \mathrm{U}(W_{sp})$ on

$\mathcal{S}(V_{an})$ has the following explicit formulas:

$$\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}(h)\,f(y) = f(h^{-1}y),$$
$$\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}\begin{pmatrix} a & \\ & \sigma(a)^{-1}\end{pmatrix} f(y) = |a|_E f(ya),$$
$$\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}\begin{pmatrix} 1 & b \\ & 1\end{pmatrix} f(y) = \psi\left(b\left(-d_0\,\mathrm{N}_{E/F}(y_1) + \mathrm{N}_{E/F}(y_2)\right)\right) f(y).$$

Here $f \in \mathcal{S}(V_{an})$, $y = {}^t(y_1, y_2) \in V_{an}$, $h \in \mathrm{U}(V_{sp})$, $a \in E^\times$ and $b \in F$. By these explicit formulas, we can check that if we identify $V_D$ with $V_{an}$ by Lemma 7.2 (i), then

$$\omega_{\psi,V_D,W_{D,sp}} \circ J = \omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}$$

as $\mathrm{U}(W_{sp})$-modules on $\mathcal{S}(V_{an})$. This implies that the following diagram is commutative:

$$\begin{array}{ccccc}
\mathrm{U}(W_1)\times\mathrm{U}(W_{-1}) & \longrightarrow & \mathrm{U}(W_{sp}) & \xrightarrow{\dagger} & \mathrm{Mp}_\psi(V_{an}\otimes_E W_{sp}) \\
\| & & \downarrow J & & \downarrow \\
\mathrm{U}_E(1)\times\mathrm{U}_E(-1) & \longrightarrow & \mathrm{U}(W_{D,sp}) & \xrightarrow{\dagger\dagger} & \mathrm{Mp}_\psi(V_D\otimes_D W_{D,sp})
\end{array}$$

Here $\dagger, \dagger\dagger$ are the splittings of $\mathrm{U}(W_{sp}), \mathrm{U}(W_{D,sp})$ that define the Weil representations $\omega_{\psi,V_D,W_{D,sp}}, \omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}$ of $\mathrm{U}(W_{sp}), \mathrm{U}(W_{D,sp})$, respectively.

This shows that the diagram (7) for $\mathrm{U}_E(1)$ is commutative.

## 7.3 Endoscopy for $\mathrm{U}_D(1)$

Let $\lambda$ be a quasi-character of $E^\times$ such that $\lambda \neq \lambda\circ\sigma$. Then there exists an irreducible supercuspidal representation $\tau_D(\lambda)$ of $D^\times$ with $L$-parameter $\phi' = \mathrm{Ind}_{W_E}^{W_F}\lambda$. Note that this representation is written as $\tau_D(\phi')$ in §2.3.

**Theorem 7.5** ([9])**.** *There exist two irreducible representations $\tau_D(\lambda)^+, \tau_D(\lambda)^-$ of $\mathrm{U}_D(1)$ such that $\tau_D(\lambda)|_{\mathrm{U}_D(1)} = \tau_D(\lambda)^+ \oplus \tau_D(\lambda)^-$ and the following character identity holds:*

$$\begin{aligned}&\mathrm{Tr}\,\tau_D(\lambda)^+(\gamma) - \mathrm{Tr}\,\tau_D(\lambda)^-(\gamma)\\ &= \lambda\,(E/F,\psi)\,\omega_{E/F}\left(\frac{\gamma^{-1}-\gamma}{\xi}\right)\frac{\lambda(\gamma)-\lambda(\gamma^{-1})}{|\gamma-\gamma^{-1}|_E^{1/2}}\end{aligned}$$

*for any $\gamma \in \mathrm{U}_E(1)$. Also $\tau_D(\lambda)^\pm$ are determined by the restriction $\lambda|_{\mathrm{U}_E(1)}$. Moreover $\tau_D(\lambda)^+ \cong \tau_D(\lambda)^-$ if and only if $\lambda^2|_{\mathrm{U}_E(1)} = \mathbb{1}$.*

## 7.4 Main theorem

**Theorem 7.6.** *For $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, let $\lambda_\eta$ be a character of $E^\times$ such that $\lambda_\eta|_{\mathrm{U}_E(1)} = \eta_u$. Then we have*

$$\theta_{\psi,V_D,W_D}(\eta_u) = \begin{cases} \tau_D(\lambda_\eta)^+ & \text{if } \eta \neq \mathbb{1}_{F^\times}, \\ 0 & \text{if } \eta = \mathbb{1}_{F^\times}. \end{cases}$$

*Proof.* By Theorem 6.4, we have

$$\omega_{\psi,V_{an}^\mu,W_1^{\mathbb{1}}} = \bigoplus_{\eta\neq\mathbb{1}} \tau(\mu,\mu\eta^{-1})_{an}^+ \boxtimes \eta_u.$$

Here $\eta$ runs the non-trivial elements of $\Pi(E^\times, \mathbb{1}_{F^\times})$. Thus by Proposition 7.3, we obtain

$$\omega_{\psi,V_D,W_D} = \bigoplus_{\eta\neq\mathbb{1}} \tau(\mu,\mu\eta^{-1})_{an}^+|_{\mathrm{U}_D(1)} \boxtimes \eta_u.$$

Therefore we must show that $\tau(\mu,\mu\eta^{-1})_{an}^+|_{\mathrm{U}_D(1)} = \tau_D(\lambda_\eta)^+$. This is shown in the next lemma. □

**Lemma 7.7.** *We have $\tau(\mu,\mu\eta^{-1})_{an}^+|_{\mathrm{U}_D(1)} = \tau_D(\lambda_\eta)^+$.*

*Proof.* First we recall the following isomorphism:

$$E^\times \times D^\times/\Delta F^\times \ni (x, z+\xi' w) \mapsto x\nu_D(z+\xi' w)^{-1} \begin{pmatrix} \sigma(z) & w \\ d_0\sigma(w) & z \end{pmatrix} \in \mathrm{GU}(V_{an}).$$

Then we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{U}_D(1) & \xrightarrow{I} & \mathrm{U}(V_{an}) \\ \downarrow & & \downarrow \\ E^\times \times D^\times/\Delta F^\times & \xrightarrow{\cong} & \mathrm{GU}(V_{an}) \end{array}$$

Here the vertical maps are natural inclusions.

Take quasi-characters $\chi, \chi'$ of $E^\times$ such that $\chi\chi' = \mu$, $\chi(\chi'\circ\sigma) = \mu\eta^{-1}$. Then $\tilde{\tau} = \chi \boxtimes \tau_D(\chi')$ is an irreducible admissible representation of $E^\times \times D^\times/\Delta F^\times$. By §2.3, we have

$$\tilde{\tau}|_{\mathrm{U}(V_{an})} = \tau(\mu,\mu\eta^{-1})_{an}^+ \oplus \tau(\mu,\mu\eta^{-1})_{an}^-.$$

On the other hand, by Theorem 7.5 we obtain

$$\tilde{\tau}|_{\mathrm{U}_D(1)} = \tau_D(\chi')|_{\mathrm{U}_D(1)} = \tau_D(\chi')^+ \oplus \tau_D(\chi')^-.$$

Thus we have

$$\tau(\mu, \mu\eta^{-1})^+_{an}|_{\mathrm{U}_D(1)} = \tau_D(\lambda_\eta)^+ \text{ or } \tau_D(\lambda_\eta)^-.$$

Here note that $\tau_D(\chi')^\pm = \tau_D(\lambda_\eta)^\pm$, since $\chi'|_{\mathrm{U}_E(1)} = \eta_u$.

Next note that $\mathrm{U}_E(1)$ is identified with a subgroup of $\mathrm{U}(V_{an})$ by

$$\mathrm{U}_E(1) \ni \gamma \mapsto t(1, \gamma^{-1}) = \begin{pmatrix} \gamma^{-1} & \\ & \gamma \end{pmatrix} \in \mathrm{U}(V_{an}).$$

Then by Theorem 2.8, the restriction of the character identity of $\tau(\mu, \mu\eta^{-1})^\pm_{an}$ to $\mathrm{U}_E(1)$ is given as follows:

$$\begin{aligned}
&\operatorname{Tr} \tau(\mu, \mu\eta^{-1})^+_{an}(\gamma) - \operatorname{Tr} \tau(\mu, \mu\eta^{-1})^-_{an}(\gamma) \\
=& \lambda\left(E/F, \psi\right) \omega_{E/F}\left(\frac{\gamma^{-1} - \sigma(\gamma^{-1})}{\xi}\right) \chi\left(\mathrm{N}_{E/F}(\gamma^{-1})\right) \frac{\chi'(\sigma(\gamma^{-1})) - \chi'(\gamma^{-1})}{|\gamma^{-1} - \sigma(\gamma^{-1})|_E^{1/2}} |\gamma^{-1}|_E^{1/2} \\
=& \lambda\left(E/F, \psi\right) \omega_{E/F}\left(\frac{\gamma^{-1} - \gamma}{\xi}\right) \frac{\eta_u(\gamma) - \eta_u(\gamma^{-1})}{|\gamma^{-1} - \gamma|_E^{1/2}} \\
=& \operatorname{Tr} \tau_D(\lambda_\eta)^+(\gamma) - \operatorname{Tr} \tau_D(\lambda_\eta)^-(\gamma).
\end{aligned}$$

The last equation follows from Theorem 7.5. Thus we obtain

$$\tau(\mu, \mu\eta^{-1})^+_{an}|_{\mathrm{U}_D(1)} = \tau_D(\lambda_\eta)^+.$$

□

# References


[1] James Arthur, *The local behavior of weighted orbital integrals*, Duke Math. J. **56** (1988), 223–293.

[2] Wee Teck Gan and Atsushi Ichino, *The Gross-Prasad conjecture and local theta correspondance*, (2014).

[3] Wee Teck Gan and Shuichiro Takeda, *A Proof of Howe Duality Conjecture*, (2015).

[4] Stephen Gelbart, Jonathan Rogawski, and David Soudry, *Endoscopy, theta-liftings, and period integrals for the unitary group in three variables*, Ann. of Math. (2) **145** (1997), no. 3, 419–476. MR 98m:11125

[5] Kazuko Konno and Takuya Konno, *Lecture on endoscopy for unitary groups in two variables*, november, 2005, under revision.

[6] Takuya Konno and Kazuko Konno, *On doubling construction for real unitary dual pairs*, Kyushu J. Math. **61** (2007), no. 1, 35–82.

[7] Robert E. Kottwitz, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR 88d:22027

[8] Stephen S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math. **87** (1994), no. 1-3, 361–401. MR 95h:22019

[9] J.-P. Labesse and R. P. Langlands, *L-indistinguishability for* $SL(2)$, Canad. J. Math. **31** (1979), no. 4, 726–785. MR 81b:22017

[10] R. P. Langlands, *On Artin's L-function*, Rice Univ. Studies **56** (1970), 23–28.

[11] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271. MR 89c:11172

[12] Robert P. Langlands and Dinakar Ramakrishnan (eds.), *The zeta functions of Picard modular surfaces*, Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1992. MR 92m:14001

[13] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin Heidelberg, 1987. MR 91f:11040

[14] R. Rao, *Orbital integrals on reductive groups*, Ann. Math. **96** (1972), 505–510.

[15] Jonathan D. Rogawski, *Automorphic representations of unitary groups in three variables*, Princeton University Press, Princeton, NJ, 1990. MR 91k:22037

[16] Winfried Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 270, Springer-Verlag, Berlin, 1985. MR MR770063 (86k:11022)

[17] J.-L. Waldspurger, *D'emonstration d'une conjecture de dualit'e de Howe dans le cas $p$-adique, $p \neq 2$*, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), Weizmann, Jerusalem, 1990, pp. 267–324. MR 93h:22035

Graduate School of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

*E-mail address*: y-ikematsu@math.kyushu-u.ac.jp