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Abstract. We discuss about abstract collision systems(ACS) on groups which is an extension of ACS [5, 6]. The ACS is a kind of frameworks of unconventional computing which includes collision based computing, cellular automata (CA), chemical reaction systems and so on. In this paper, we define ACS on groups. When a group G and its subset is given, we create a set of collisions and a local transition function of an ACS from the group G and its operation. First, we describe definitions of components of ACS. Next, we introduce ACS on groups. Finally, we investigate properties of operations of ACS, union, division and composition.

Keywords. collision based computing, cellular automata.

1. Introduction

Recently, there are many investigations about unconventional computing based on the frameworks of the collision-based computing [1], which include cellular automata (CA) and reaction-diffusion systems. Our purpose is to construct computational models and to investigate computational capabilities of those models.

Conway introduced 'The Game of Life' which is one of two dimensional cellular automata [2]. On 'The Game of Life', there are some special patterns called "gliders". He showed that it can simulate any logical circuit by using collisions of gliders. Wolfram and Cook [11, 3] found glider patterns in the one dimensional elementary cellular automaton CA110. Cook introduced cyclic tag systems (CTS) as Turing universal systems. He proved that CTS was simulated by CA110 by using collisions of gliders in CA110. Recently, Martínez et. al. investigated glider phenomena from the view point of regular language [7]. Morita [8] introduced a reversible one dimensional CA which simulated CTS.

We introduced the notion of abstract collision systems (ACS) as tools to discuss about collision phenomena including the phenomena of glider collisions in 'The Game of Life' and 'CA110'. We proved that it is universal for computation [5]. Moreover, we investigated about simulations of ACS by CA. We found some conditions of ACS to be simulated by CA [6].

Notion of automata on groups was first treated as special cases for automata on graphs named Cayley graphs which represent groups [10, 9, 12]. Fujio [4] introduced the composition of CA on groups in order to reduce a complex behaved dynamics into simpler ones. As an example, he showed the rule 90 (3 neighborhood) CA is factorized into the composition of double XORs, which are rule 6 (2 neighborhood) CA.

In this paper, we define ACS on groups, and we investigate properties about this extended systems. Generally, the set of collisions, which is domain of local transition function, is very large set. However, in notion of ACS on groups, we use small set V and a function l named "base function". By using V and l, we induce the set of collisions $\mathcal{C}$ and local transition function f_l.

Next, we consider operation of ACS such as "union", "division" and "composition". We showed one of sufficient conditions that ACS on a group is dividable. Furthermore, we proved that the operation "composition" is right distributive over "union". The operation "composition" is not left distributive. We give a counter-example about this left distributive laws. In addition, we re-formalize CA on groups by using ACS on groups.

This paper consists of the following sections. In Section 2, we introduce abstract collision systems. Let S be a non-empty set. First, we define a set of collisions $\mathcal{C}$ on S. The set $\mathcal{C}$ specifies all combinations of elements which cause collisions. Moreover, we define an abstract collision system.

In Section 3, we define ACS on groups. Let G be a group, and G operates to S. When only $V \subseteq G$ and a map from 2^V to 2^S is given, we construct a set of collisions $\mathcal{C}$ on S and a local transition function $f : \mathcal{C} \to 2^S$ by using the operation of G. Moreover, we investigate behavior of the global transition function of such an ACS.

In Section 4, we define new operations "union" and "division" of ACS. Moreover, we prove that a sufficient condition to divide an ACS on a group.

In Section 5, we discuss about composition of ACS on groups. We define composition of ACS on groups by an operation of base functions of two ACS and we prove that the definition induce the composition of local (resp. global) transition functions.

In Section 6, we prove that composition is right distributive over union. But they are not left distributive. We give a counter-example.

2. Abstract collision systems

In this section, we define an abstract collision system. Let S be a non-empty set. First, we define a set of collisions on S.

Definition 1 (Set of collisions)**.** A set $\mathcal{C} \subseteq 2^S$ is called a **set of collisions** on S iff it satisfies

(SC1) $\{s\} \in \mathcal{C}$ for all $s \in S$,

(SC2) For all $\mathcal{X} \subseteq \mathcal{C}$, $(\cup\mathcal{X}) \in \mathcal{C}$ if $(\cap\mathcal{X}) \neq \phi$,

where $\cap\mathcal{X} = \cap\{X \mid X \in \mathcal{X}\}$ and $\cup\mathcal{X} = \cup\{X \mid X \in \mathcal{X}\}$, respectively.

Proposition 1. *Let $\mathfrak{C}$ be a family of sets of collisions on S. Then a set*

$$\bigcap \mathfrak{C} = \bigcap_{\mathcal{C}\in\mathfrak{C}} C$$

is a set of collisions on S.

Proof. We check conditions (SC1) and (SC2).

We prove (SC1). We have $\{s\} \in \mathcal{C}$ for all $s \in S$ and $\mathcal{C} \in \mathfrak{C}$. Therefore we have $\{s\} \in (\cap\mathfrak{C})$.

We prove (SC2). For all $\mathcal{X} \subseteq (\cap\mathfrak{C})$, and $\mathcal{C} \in \mathfrak{C}$, we assume that $(\cap\mathcal{X}) \neq \phi$. The set $\mathcal{C}$ satisfies the assumption of (SC2), i.e.,

$$\mathcal{X} \subseteq (\cap\mathfrak{C}) \subseteq \mathcal{C},$$
$$(\cap\mathcal{X}) \neq \phi.$$

Therefore we have $(\cup\mathcal{X}) \in \mathcal{C}$ from (SC2). Hence we have $(\cup\mathcal{X}) \in (\cap\mathfrak{C})$. □

Definition 2. For a subset $\widetilde{\mathcal{C}}$ of 2^S, we put

$$(1) \quad \mathfrak{C}(\widetilde{\mathcal{C}}) = \bigcap\left\{\mathcal{C} \mid \mathcal{C} \text{ is a set of collisions on } S, \widetilde{\mathcal{C}} \subseteq \mathcal{C}\right\}.$$

From the above proposition, this set is a set of collisions on S, and it includes the set $\widetilde{\mathcal{C}}$. Moreover, this set is a smallest set in all of sets of collisions on S which includes $\widetilde{\mathcal{C}}$.

Next, we define abstract collision systems. To define a global transition function, we divide a configuration into elements of $\mathcal{C}$.

For all $A \in 2^S$ and $p \in A$, we define

$$(2) \qquad [p]^A_{\mathcal{C}} = \bigcup\left\{X \mid X \in \mathcal{C}, p \in X, X \subseteq A\right\}.$$

Proposition 2. *Let $\mathcal{C}$ be a set of collisions on S. For all $A \in 2^S$ and $p, q \in A$, we have the following:*

(1) $[p]^A_{\mathcal{C}} \neq \phi$.

(2) $[p]^A_{\mathcal{C}} \in \mathcal{C}$

(3) *If $[p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$, then $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$.*

Proof. First, we prove (1). Since $\{p\} \in \mathcal{C}$, $p \in \{p\}$ and $\{p\} \subseteq A$, we have $\{p\} \subseteq [p]^A_{\mathcal{C}}$. Hence $[p]^A_{\mathcal{C}} \neq \phi$.

Next we prove (2). Let

$$\mathcal{X} = \left\{X \mid X \in \mathcal{C}, p \in X, X \subseteq A\right\}.$$

Since $p \in (\cap\mathcal{X})$, we have $(\cap\mathcal{X}) \neq \phi$. Therefore $[p]^A_{\mathcal{C}} = (\cup\mathcal{X}) \in \mathcal{C}$ by (SC2).

Finally, we prove (3). Since $[p]^A_{\mathcal{C}} \in \mathcal{C}$, $[q]^A_{\mathcal{C}} \in \mathcal{C}$ and $[p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$, we see that

$$[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} \in \mathcal{C}.$$

Moreover since $p \in A$ and $[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} \subseteq A$, we have $[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} \subseteq [p]^A_{\mathcal{C}}$. Hence we have

$$[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} = [p]^A_{\mathcal{C}}.$$

Similarly, we have $[p]^A_{\mathcal{C}} \cup [q]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$. Hence $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$. □

Definition 1 is different from the definition of [5]. The original definition is Definition 3. However, from Lemma 1, The statements of these definitions are equivalent. Since Definition 1 is more simple than Definition 3, we use Definition 1 as a definition of the set of collisions in this paper.

Definition 3. A set $\mathcal{C} \subseteq 2^S$ is called a **set of collisions** on S iff it follows that;

(SC1) $\{s\} \in \mathcal{C}$ for all $s \in S$.

(SC'2) For all X_1 and $X_2 \in \mathcal{C}$, $X_1 \cup X_2 \in \mathcal{C}$ if $X_1 \cap X_2 \neq \phi$.

(SC'3) $[p]^A_{\mathcal{C}} \in \mathcal{C}$ for all $A \in 2^S$ and $p \in A$.

Lemma 1. *The statements of Definition 1 and Definition 3 are equivalent, i.e.,*

$$(SC2) \Leftrightarrow ((SC'2) \wedge (SC'3)).$$

Proof. First, We prove (SC'2) from (SC2). Suppose that $X_1, X_2 \in \mathcal{C}$, $X_1 \cap X_2 \neq \phi$. Let $\mathcal{X} = \{X_1, X_2\} \subseteq \mathcal{C}$. Since $(\cap\mathcal{X}) = X_1 \cap X_2 \neq \phi$, we have $X_1 \cup X_2 = (\cup\mathcal{X}) \in \mathcal{C}$ by (SC2). Therefore we have(SC'2).

Next, we see (SC'3) from (SC2) by (2) of Proposition 2.

Finally, we prove (SC2) from (SC'2). For all $\mathcal{X} \subseteq \mathcal{C}$, we assume that $(\cap\mathcal{X}) \neq \phi$. Let $x_0 \in (\cap\mathcal{X})$ and

$$(3) \qquad A = \bigcup \mathcal{X}.$$

Since $x_0 \in A$, we have

$$(4) \qquad [x_0]^A_{\mathcal{C}} \in \mathcal{C}$$

from (SC'3). We see that $[x_0]^A_{\mathcal{C}} \subseteq A$ from the definition of $[x_0]^A_{\mathcal{C}}$. On the other hand, for all $X \in \mathcal{X}$, since $X \in \mathcal{X} \subseteq \mathcal{C}$, we have $X \in \mathcal{C}$. Moreover, since $x_0 \in (\cap\mathcal{X})$ and $A = (\cup\mathcal{X})$, we have $x_0 \in X$ and $X \subseteq A$. Hence we have

$$X \subseteq \bigcup\left\{X \mid X \in \mathcal{C}, x_0 \in X, X \subseteq A\right\} = [x_0]^A_{\mathcal{C}}.$$

Therefore we have

$$(5) \qquad A = [x_0]^A_{\mathcal{C}}.$$

Hence we have

$$\left(\bigcup \mathcal{X}\right) = A = [x_0]^A_{\mathcal{C}} \in \mathcal{C},$$

by (3), (4) and (5). Therefore we have (SC2). □

Definition 4 (An abstract collision system)**.** Let S be a non-empty set and $\mathcal{C}$ be a set of collisions on S. Let $f : \mathcal{C} \to 2^S$. We define an **abstract collision system** M by $M = (S, \mathcal{C}, f)$. We call the function f and the set 2^S a **local transition function** and a **configuration** of M, respectively. We define a **global transition function** $F_M : 2^S \to 2^S$ of M by

$$F_M(A) = \bigcup_{p\in A} \left(f([p]^A_{\mathcal{C}})\right)$$

The following of this paper, for each abstract collision system M, we denote the global transition function of M by F_M.

Lemma 2. *Let F_M be the global transition function of an abstract collision system $M = (S, \mathcal{C}, f)$. If $A \in \mathcal{C}$, we have*

$$F_M(A) = f(A).$$

Proof. Since $A \in \mathcal{C}$, we see that $[p]^A_{\mathcal{C}} = A$ for any $p \in A$. Therefore we have

$$F_M(A) = \bigcup_{p\in A} \left(f([p]^A_{\mathcal{C}})\right) = \bigcup_{p\in A} (f(A)) = f(A)$$

□

Definition 5. Let $M_1 = (S, \mathcal{C}_1, f_1)$ and $M_2 = (S, \mathcal{C}_2, f_2)$ be abstract collision systems. Let F_{M_1} and F_{M_2} be global transition functions of M_1 and M_2, respectively. We say that M_1 and M_2 are **equivalent** if they satisfy

$$F_{M_1}(A) = F_{M_2}(A)$$

for all $A \in 2^S$. When M_1 and M_2 are equivalent, we write $M_1 \equiv M_2$.

Lemma 3. *Let M_1 and M_2 be abstract collision systems*

$$M_1 = (S, \mathcal{C}_1, f_1), M_2 = (S, \mathcal{C}_2, f_2).$$

Suppose that $\mathcal{C}_1 = \mathcal{C}_2$. Moreover, we assume that $f_1 = f_2$. Then we have $M_1 \equiv M_2$.

Proof. Let $\mathcal{C} = \mathcal{C}_1 = \mathcal{C}_2$. Let F_{M_1} and F_{M_2} be global transition functions of M_1 and M_2, respectively. Suppose that $A \in 2^S$. Then we see that

$$\begin{aligned} & F_{M_1}(A) \\ = & \bigcup_{p\in A} f_1\left([p]^A_{\mathcal{C}}\right) \\ = & \bigcup_{p\in A} f_2\left([p]^A_{\mathcal{C}}\right) \\ = & F_{M_2}(A). \end{aligned}$$

Therefore we have $M_1 \equiv M_2$. □

3. Abstract collision systems on groups

In this section, we consider an operation of a group. Let G be a group, and S be a non-empty set.

A map from $G \times S$ to G,

(6) $$G \times S \to S \quad ((g, s) \mapsto gs)$$

is called an **operation** of G to S, iff it satisfies:

(1) $(gh)s = g(hs) \quad (g, h \in G, s \in S)$

(2) $es = s \quad$ (e is an identity element of G).

Then we say that the group G **operates** the set S.

When a group G operates a set S, we define an operation of G to 2^S by

(7) $$gX = \{gx \mid x \in X\} \quad (g \in G, X \in 2^S).$$

We have the following proposition about this operation.

Proposition 3. *For all $g \in G$ and $X, Y \subseteq S$, we have:*

(8) *If $X \subseteq Y$, then $(gX) \subseteq (gY)$.*

(9) $g(X \cup Y) = (gX) \cup (gY)$

(10) $g(X \cap Y) = (gX) \cap (gY)$

Proof. (8) is clear.

Next prove (9). Since $X, Y \subseteq X \cup Y$, we have $gX \subseteq g(X \cup Y)$ and $gY \subseteq g(X \cup Y)$. Therefore we have

$$(gX) \cup (gY) \subseteq g(X \cup Y).$$

On the other hand, for all $z \in g(X \cup Y)$, there exists $w \in X \cup Y$ such that $z = gw$. Then w satisfies $w \in X$ or $w \in Y$. If $w \in X$ (resp. $w \in Y$), we have $z \in gX$ (resp. $z \in gY$). Therefore $z \in (gX) \cup (gY)$. Hence we have

$$g(X \cup Y) \subseteq (gX) \cup (gY).$$

Finally, we prove (10). Since $X \cap Y \subseteq X, Y$, we have $g(X \cap Y) \subseteq (gX)$ and $g(X \cap Y) \subseteq (gY)$. Therefore we have

$$g(X \cap Y) \subseteq (gX) \cap (gY).$$

On the other hand, for all $z \in (gX) \cap (gY)$, there exists $x \in X$ and $y \in Y$ such that $z = gx$ and $z = gy$. Since $g^{-1}z = x = y$, we have $x = y \in X \cap Y$, which implies

$$z = gx = gy \in g(X \cap Y).$$

Hence we have

$$(gX) \cap (gY) \subseteq g(X \cap Y).$$

All claims of Proposition 3 are proved. □

Definition 6. Let S be a non-empty set, V be a non-empty subset of S, G be a group, $l : 2^V \to 2^S$ and the group G operates to the set S. Then let

(11) $$\widetilde{\mathcal{C}}_V = \left\{gX \mid g \in G, X \in 2^V\right\}$$

and $\mathcal{C}$ be a set of collisions on S which includes $\widetilde{\mathcal{C}}_V$. Then we define a local transition function $f_l : \mathcal{C} \to 2^S$ by

$$f_l(X) = \bigcup_{g \in G} gl((g^{-1}X) \cap V). \tag{12}$$

We call an abstract collision system $M = (S, \mathcal{C}, f_l)$ an **abstract collision system on** G made by V and l. Moreover, we call the function l a **base function** of M. In addition, we call f_l a **induced local transition function** by V and l on G, and denoted by $f_l = \mathrm{Ind}(G, V, l)$.

Definition 7. Let $V' \subseteq V$. We call the set V' **essential domain** of l iff it satisfies

$$l(X) = l(X \cap V')$$

for all $X \in 2^V$. We denote the essential domain V' by $V' = \mathrm{ess}\ l$.

We investigate the behavior of the global transition function of the abstract collision system on G. We prove a theorem with respect to the global transition function of an abstract collision system on a group. We prepare the following lemmas.

Lemma 4. *Let $A \in 2^S$ and $g \in G$. If*

$$g^{-1}[p]^A_{\mathcal{C}} \cap V = \phi$$

for all $p \in A$, then we have

$$g^{-1}A \cap V = \phi.$$

Proof. We show by indirect proof. We assume that $g^{-1}A \cap V \neq \phi$. Then there exists $x \in g^{-1}A \cap V$. Let $p = gx$. Since $p = gx \in A$ and $x \in V$, we have

$$x = g^{-1}p \in g^{-1}[p]^A_{\mathcal{C}} \cap V.$$

Hence we have $g^{-1}[p]^A_{\mathcal{C}} \neq \phi$, this contradicts the assumption of the lemma. □

Lemma 5. *Let $A \in 2^S$, $p \in A$ and $g \in G$. If*

$$g^{-1}[p]^A_{\mathcal{C}} \cap V \neq \phi,$$

then we have

$$g^{-1}[p]^A_{\mathcal{C}} \cap V = g^{-1}A \cap V.$$

Proof. From the definition of $[p]^A_{\mathcal{C}}$, it is clear that $[p]^A_{\mathcal{C}} \subseteq A$ Hence we have

$$g^{-1}[p]^A_{\mathcal{C}} \cap V \subseteq g^{-1}A \cap V.$$

Let $x \in g^{-1}A \cap V$ and $q \in [p]^A_{\mathcal{C}} \cap gV$. We show

$$x \in g^{-1}[p]^A_{\mathcal{C}} \cap V.$$

Since $q \in [p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$, we have $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$ by Proposition 2. Let

$$\begin{aligned} X_1 &= [q]^A_{\mathcal{C}}, \\ X_2 &= ([q]^A_{\mathcal{C}} \cap gV) \cup \{gx\}. \end{aligned} \tag{13}$$

Then it is clear that $X_1 \in \mathcal{C}$ by (2) of Proposition 2. Since

$$[q]^A_{\mathcal{C}} \cap gV \subseteq gV, \quad gx \in A \cap gV \subseteq gV,$$

we have $X_2 \subseteq gV$. Hence we have

$$X_1, X_2 \in \widetilde{\mathcal{C}}_V \subseteq \mathcal{C}.$$

Since $q \in [q]^A_{\mathcal{C}}$ and $q \in gV$, we have $q \in X_1$ and $q \in X_2$. Hence $X_1 \cap X_2 \neq \phi$. Therefore we have $X_1 \cup X_2 \in \mathcal{C}$ by (SC2). Moreover, since $x \in g^{-1}A \cap V$, we have $gx \in A$, i.e., $\{gx\} \subseteq A$. Since

$$X_1 \cup X_2 \in \mathcal{C}, \quad q \in X_1 \cup X_2, \quad X_1 \cup X_2 \subseteq A,$$

we have $[q]^A_{\mathcal{C}} \supseteq X_1 \cup X_2$ by (2). Hence $gx \in [q]^A_{\mathcal{C}}$, which implies $x \in g^{-1}[q]^A_{\mathcal{C}}$. Moreover, since $x \in g^{-1}A \cap V$, it is clear that $x \in V$. Therefore we have

$$x \in g^{-1}[q]^A_{\mathcal{C}} \cap V = g^{-1}[p]^A_{\mathcal{C}} \cap V.$$

□

Lemma 6. *For all $g \in G$, $A \in 2^S$, $p, q \in A$, we assume that*

$$(g^{-1}[p]^A_{\mathcal{C}} \cap V) \neq \phi, \quad (g^{-1}[q]^A_{\mathcal{C}} \cap V) \neq \phi.$$

Then we have

$$[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}.$$

Proof. By Lemma 5, we see that

$$(g^{-1}[p]^A_{\mathcal{C}} \cap V) = (g^{-1}[q]^A_{\mathcal{C}} \cap V) = (g^{-1}A \cap V).$$

Hence for all $x \in (g^{-1}[p]^A_{\mathcal{C}} \cap V)$, since $gx \in [p]^A_{\mathcal{C}}, [q]^A_{\mathcal{C}}$, we have $[p]^A_{\mathcal{C}} \cap [q]^A_{\mathcal{C}} \neq \phi$. Therefore we have $[p]^A_{\mathcal{C}} = [q]^A_{\mathcal{C}}$ by Proposition 2. □

By these lemmas, we see the following, immediately.

Lemma 7. *For all $g \in G$ and $A \in 2^S$, suppose that $A \notin \mathcal{C}$. Then we have*

$$\bigcup_{p \in A} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) = g\left(l\left(g^{-1}A \cap V\right) \cup l\left(\phi\right)\right) \tag{14}$$

Proof. Suppose that $g^{-1}[p]^A_{\mathcal{C}} \cap V = \phi$ for all $p \in A$. Then we have $g^{-1}A \cap V = \phi$ by Lemma 4. Hence the left hand side of (14) equals to

$$\bigcup_{p \in A} gl(\phi) = gl(\phi) = g(l(\phi) \cup l(\phi)) = g(l(g^{-1}A \cap V) \cup l(\phi)).$$

This equals to the right hand side.

Next, we assume that there exists $p_1 \in A$ such that $g^{-1}[p_1]^A_{\mathcal{C}} \cap V \neq \phi$. Let

$$\begin{aligned} A' &= \left\{p \in A \mid g^{-1}[p]^A_{\mathcal{C}} \cap V \neq \phi.\right\}, \\ A'' &= \left\{p \in A \mid g^{-1}[p]^A_{\mathcal{C}} \cap V = \phi.\right\}. \end{aligned}$$

Then we have $p_1 \in A'$, which implies $A' \neq \phi$. Since $A \notin \mathcal{C}$, there exists $q_1 \in A$ such that $[p_1]^A_{\mathcal{C}} \neq [q_1]^A_{\mathcal{C}}$. Hence we have

$q_1 \in A''$ by Lemma 6. This implies $A'' \neq \phi$. Therefore the left hand side of (14) equals

$$\begin{aligned}
&\bigcup_{p\in A} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) \\
=&\bigcup_{p\in A'} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) \cup \bigcup_{p\in A''} gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right) \\
=&\bigcup_{p\in A'} gl\left(g^{-1}A \cap V\right) \cup \bigcup_{p\in A'} gl\left(\phi\right) \quad \text{(by Lemma 5)} \\
=&gl\left(g^{-1}A \cap V\right) \cup gl\left(\phi\right).
\end{aligned}$$

This equals the right hand side of (14). □

This lemma induces the following theorem.

Theorem 1. *Let $M = (S, \mathcal{C}, f_l)$ be an abstract collision system on G made by V and l. Let F_l be the global transition function of M. If $A \in \mathcal{C}$, then F_l satisfies*

$$F_l(A) = \bigcup_{g\in G} gl\left(g^{-1}A \cap V\right). \tag{15}$$

If $A \notin \mathcal{C}$, then

$$F_l(A) = \bigcup_{g\in G} g\left(l\left(g^{-1}A \cap V\right) \cup l(\phi)\right). \tag{16}$$

Proof. First, suppose that $A \in \mathcal{C}$. (15) is clear by Lemma 2 and (12). Next, suppose that $A \notin \mathcal{C}$. By Lemma 7, we have

$$\begin{aligned}
F_l(A) &= \bigcup_{p\in A} \left(f_l([p]^A_{\mathcal{C}})\right) \\
&= \bigcup_{p\in A}\bigcup_{g\in G} \left(gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right)\right) \\
&= \bigcup_{g\in G}\bigcup_{p\in A} \left(gl\left(g^{-1}[p]^A_{\mathcal{C}} \cap V\right)\right) \\
&= \bigcup_{g\in G} g\left(l\left(g^{-1}A \cap V\right) \cup l(\phi)\right).
\end{aligned}$$

Hence the theorem follows. □

Corollary 1. *Especially, if $l(\phi) = \phi$, then we have*

$$F_l(A) = \bigcup_{g\in G} gl\left(g^{-1}A \cap V\right)$$

for all $A \in 2^S$.

Corollary 2. *We assume that $(\phi) = \phi$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be sets of collisions on S. Suppose that $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_1$ and $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_2$. We make abstract collision systems $M_1 = (S, \mathcal{C}_1, f_l)$ and $M_2 = (S, \mathcal{C}_2, f_l)$. Then we have*

$$M_1 \equiv M_2.$$

Proof. Let F_{M_1} and F_{M_2} be global transition functions of M_1 and M_2, respectively. By Corollary 2,

$$F_{M_1}(A) = F_{M_2}(A) = \bigcup_{g\in G} gl\left(g^{-1}A \cap V\right)$$

for all $A \in 2^S$. Hence the corollary follows. □

In the following of this paper, we suppose that $l(\phi) = \phi$. Let $M = (S, \mathcal{C}, f_l)$ be an abstract collision system on G made by V and l. From Theorem 1, the abstract collision system M is determined by only G, S, V and l, i.e., M does not depend on the set of collisions $\mathcal{C}$. Therefore we denote the abstract collision system M by $M = GACS(G, S, V, l)$

Then from Corollary 2, we have the following proposition.

Proposition 4. *We have*

$$GACS(G, S, V_1, l_1) \equiv GACS(G, S, V_2, l_2)$$

if $V_1 = V_2$ and $l_1 = l_2$.

By Definition 7 and Theorem 1, we show the following proposition.

Proposition 5. *Let $V' = \text{ess } l$. Suppose that $l(\phi) = \phi$. Then we have*

$$GACS(G, S, V, l) \equiv GACS(G, S, V', l'),$$

where l' is restriction of l onto $2^{V'}$.

Proof. Let M and M' be abstract collision systems on G,

$$M = GACS(G, S, V, l), \quad M' = GACS(G, S, V, l'),$$

respectively. Let F_M and $F_{M'}$ be global transition functions of M and M', respectively. By Definition 7 and Theorem 1, we see that

$$\begin{aligned}
F_M(A) &= \bigcup_{g\in G} gl\left(g^{-1}A \cap V\right) \\
&= \bigcup_{g\in G} gl\left(\left(g^{-1}A \cap V\right) \cap V'\right) \\
&= \bigcup_{g\in G} gl\left(g^{-1}A \cap V'\right), \\
F_{M'}(A) &= \bigcup_{g\in G} gl'\left(g^{-1}A \cap V'\right)
\end{aligned}$$

for all $A \in 2^S$. □

In the following of this section, we will discuss about cellular automata by using an abstract collision system on a group.

Definition 8. Let G be a group, V be a subset of G and l be a map from 2^V to 2^G. We assume that

$$\begin{aligned}
&l(X) \subseteq 2^{\{e\}} \quad (\text{for all } X \in 2^V), \\
&l(\phi) = \phi
\end{aligned}$$

Then we call $GACS(G, G, V, l)$ a **cellular automaton on the group G**.

Let $Q = \{0, 1\}$ and $f_{CA} : Q^{n+1} \to Q$. Suppose that $f_{CA}(0, \ldots, 0) = 0$. We consider the following one-to-one mapping:

$$(x_0, \ldots, x_n) \leftrightarrow \{i \in V \mid x_i = 1\}.$$

We denote a map l by

$$l\left(\{i \in V \mid x_i = 1\}\right) = l\left(x_0, \ldots, x_n\right),$$

for example $l(\{0,1,3\}) = l(1,1,0,1,0,\ldots,0)$. Moreover, we denote ϕ by 0 and $\{0\}$ by 1, i.e,

$$(17) \quad \begin{cases} l\left(x_0, \ldots, x_n\right) = 0 & l\left(\{i \in V \mid x_i = 1\}\right) = \phi \\ l\left(x_0, \ldots, x_n\right) = 1 & l\left(\{i \in V \mid x_i = 1\}\right) = \{0\} \end{cases}$$

Example 1. Let $G = \mathbb{Z}$, $V = \{0, 1\}$. We define l by

$$l(\{0,1\}) = \phi, \quad l(\{0\}) = \{0\},$$
$$l(\{1\}) = \{0\}, \quad l(\phi) = \phi.$$

By using notation of (17), we denote this function by

$$l(1,1) = 0, \quad l(1,0) = 1,$$
$$l(0,1) = 1, \quad l(0,0) = 0,$$

i.e., $l(x_0, x_1) = x_0 \oplus x_1$. Then an abstract collision system $M = GACS(\mathbb{Z}, \mathbb{Z}, \{0,1\}, l)$ is a 1 dimensional, 2 state, 2 neighborhood cellular automaton rule 6.

Example 2. Similarly, we can construct other 1 dimensional, 2 states, n neighborhood cellular automata. Let $Q = \{0,1\}$ and $f_{n\ CA-k} : Q^n \to Q$. Suppose that

$$f_{CA}(0, \ldots, 0) = 0.$$

Let $G = \mathbb{Z}$ and $V = \{0, 1, \ldots, n-1\}$.

We define $l_k^{(n)} : 2^V \to 2^{\mathbb{Z}}$ by

$$l_k^{(n)}\left(x_0, \ldots, x_{n-1}\right) = \begin{cases} \phi & f_{n\ CA-k}\left(x_0, \ldots, x_{n-1}\right) = 0 \\ \{0\} & f_{n\ CA-k}\left(x_0, \ldots, x_{n-1}\right) = 1 \end{cases}$$

for all $(x_0, \ldots, x_{n-1}) \in Q^n$.

By using notation of (17), we denote $l_k^{(n)}$ by

$$l_k^{(n)}\left(x_0, \ldots, x_{n-1}\right) = \begin{cases} 0 & f_{n\ CA-k}\left(x_0, \ldots, x_{n-1}\right) = 0 \\ 1 & f_{n\ CA-k}\left(x_0, \ldots, x_{n-1}\right) = 1 \end{cases}$$
$$= f_{n\ CA-k}(x_0, \ldots, x_{n-1}).$$

Then an abstract collision system on group

$$M_{n\ CA-k} = GACS(\mathbb{Z}, \mathbb{Z}, V, l_k^{(n)})$$

is a 1 dimensional 2 states n neighborhood cellular automaton rule number k.

In the above definition and example, we can construct only 2-state cellular automata. We describe how to make other general cellular automata.

Example 3. Let Q be a non-empty set, $G = \mathbb{Z}$ and $S = \mathbb{Z} \times Q$. We define

$$z_1(z_2, q) = (z_1 + z_2, q)$$

for all $z_1 \in G$ and $(z_2, q) \in S$. We choose a subset $H \subseteq \mathbb{Z}$. and define $V = H \times Q$. Suppose that

$$l\left(X\right) \subseteq \{0\} \times Q$$

for all $X \in 2^V$ and $l(\phi) = \phi$. Then an abstract collision system

$$M = GACS(\mathbb{Z}, \mathbb{Z} \times Q, H \times Q, l)$$

is a 1 dimensional, Q state, H neighborhood cellular automaton.

Since $l(\phi) = \phi$, we note that we can construct any cellular automata which has the rule $f_{CA}(0, 0, \ldots, 0) = 1$.

If $l(\phi) \neq \phi$, we can not construct cellular automata on groups. By Theorem 1, the behavior of the global transition function depends on configurations.

First of all, we describe a theorem with respect to the set $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$. From this theorem, we can evaluate the set $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$.

For all subsets $X, Y \subseteq G$, we define

$$X \otimes Y^{-1} = \left\{xy^{-1} \mid x \in X, y \in Y\right\}.$$

We define a set $\mathcal{C}_V$ by

$$(18) \quad \mathcal{C}_V = \left\{ X \;\middle|\; \begin{array}{c} \text{for all } Y_1 \text{ and } Y_2 \subseteq X, \\ (Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) \neq \phi \\ \text{if } Y_1 \neq \phi, Y_2 \neq \phi \text{ and } Y_1 \cup Y_2 = X. \end{array} \right\}.$$

Then, we can show the following two lemmas.

Lemma 8. *$\mathcal{C}_V$ is a set of collisions on G.*

Proof. We check the condition (SC1). For all $s \in S$, let $X = \{s\}$. For all $Y_1, Y_2 \subseteq X$, we assume that $Y_1 \neq \phi$, $Y_2 \neq \phi$, and $Y_1 \cup Y_2 = X$. Then, since $Y_1 = Y_2 = \{s\} = X$, we have

$$(Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) = X \otimes V^{-1} \neq \phi.$$

Hence $\{s\} \in \mathcal{C}_V$.

We check the condition (SC2). Let $\mathcal{X} \subseteq \mathcal{C}_V$. We assume that $(\cap \mathcal{X}) \neq \phi$. We show that $(\cup \mathcal{X}) \in \mathcal{C}_V$ by indirect proof. We assume that

$$(Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) = \phi.$$

Then, since $(\cap \mathcal{X}) \neq \phi$, there exists $s_0 \in (\cap \mathcal{X})$. We can assume that $s_0 \in Y_1$ without loss of generality.

Since $(\cup \mathcal{X}) = Y_1 \cup Y_2$ and $Y_2 \neq \phi$, we have

$$\left(\bigcup \mathcal{X}\right) \cap Y_2 = (Y_1 \cup Y_2) \cap Y_2 = Y_2 \neq \phi.$$

Hence there exists $X \in \mathcal{X}$ such that $Y_2 \cap X \neq \phi$.

Since $X \in \mathcal{X} \subseteq \mathcal{C}_V$, we have

$$(19) \qquad X \in \mathcal{C}_V.$$

Since $s_0 \in (\cap \mathcal{X}) \subseteq X$, $s_0 \in Y_1$, we have

$$(20) \qquad s_0 \in Y_1 \cap X \neq \phi.$$

Let $Y_1' = Y_1 \cap X$, $Y_2' = Y_2 \cap X$. Then we have $Y_1 \cap X \neq \phi$, $Y_2 \cap X \neq \phi$. Moreover, we see that

$$\begin{aligned} & Y_1' \cup Y_2' \\ =& (Y_1 \cup Y_2) \cap X \\ =& (\cup \mathcal{X}) \cap X = X, \\ & (Y_1' \otimes V^{-1}) \cap (Y_2' \otimes V) \\ \subseteq& (Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) \\ =& \phi. \end{aligned}$$

Hence $X \notin \mathcal{C}_V$, this contradict (19). □

Lemma 9. *The set $\mathcal{C}_V$ includes the set $\widetilde{\mathcal{C}}_V$, i.e.,*

$$\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_V$$

Proof. For all $g \in G$ and $X \in 2^V$, we show that $gX \in \mathcal{C}_V$. If $X = \phi$ or $\#X = 1$, then we can see easily. We assume that $\#X \geq 2$. Let Y_1 and Y_2 be subsets of gX. Suppose that

$$Y_1 \neq \phi, \quad Y_2 \neq \phi, \quad Y_1 \cup Y_2 = gX.$$

For all $y_1 \in Y_1$, we have

$$y_1 \in Y_1 \subseteq gX \subseteq gV.$$

Therefore there exists $h_1 \in V$ such that $y_1 = gh_1$. Hence we have

$$g = y_1 {h_1}^{-1} \in Y_1 \otimes V^{-1}.$$

Similarly, we have $g \in Y_2 \otimes V^{-1}$. Therefore we have

$$(Y_1 \otimes V^{-1}) \cap (Y_2 \otimes V^{-1}) \neq \phi.$$

Hence $gX \in \mathcal{C}_V$. Therefore we have $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_V$. □

From these two lemmas, we can prove the following proposition, immediately.

Proposition 6. *We have*

$$\mathfrak{C}(V) \subseteq \mathcal{C}_V.$$

Let $V = \{0, 1, 2\}$, $l(\phi) \neq \phi$, Let M be an abstract collision system $M = (S, \mathfrak{C}(\widetilde{\mathcal{C}}_V), f_l)$ made by V and l. Let F_l be the global transition function of M. For example, a configuration $\mathbf{c}_1 = \{0\}$ is an element of $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$. Therefore by Theorem 1, we see that

$$F_l(\mathbf{c}_1) = f_l(\mathbf{c}_1).$$

However, we consider another configuration $\mathbf{c}_2 = \{0, 3\}$. By Proposition 6, we can see $\mathbf{c}_2 \notin \mathfrak{C}(\widetilde{\mathcal{C}}_V)$. Therefore by Theorem 1, we see that

$$F_l(\mathbf{c}_2) = \bigcup_{g \in \mathbb{Z}} g\{0\} = \mathbb{Z}.$$

4. Union and division of abstract collision systems

In this section, we discuss about union and division of abstract collision systems.

Definition 9 (Union)**.** Let M_1 and M_2 be abstract collision systems $M_1 = (S_1, \mathcal{C}_1, f_1)$ and $M_2 = (S_2, \mathcal{C}_2, f_2)$. We define $f_1 \cup f_2$ by

$$(f_1 \cup f_2)(X) = F_{M_1}\left(X \cap 2^{S_1}\right) \cup F_{M_2}\left(X \cap 2^{S_2}\right),$$

where F_{M_1} and F_{M_2} are global transition functions of M_1 and M_2, respectively. We define **union** of M_1 and M_2, which is denoted by $M_1 \cup M_2$, by

$$M_1 \cup M_2 = (S_1 \cup S_2, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_1 \cup f_2). \tag{21}$$

Definition 10 (Division)**.** Let M be an abstract collision system $M = (S, \mathcal{C}, f)$. We say that M is **dividable** iff there exists two abstract collision systems $M_1 \neq M$ and $M_2 \neq M$ such that $M \equiv M_1 \cup M_2$.

Proposition 7.

$$\begin{aligned} &GACS(G, S, V, l_1) \cup GACS(G, S, V, l_2) \\ \equiv &GACS(G, S, V, l_1 \cup l_2), \end{aligned}$$

where

$$(l_1 \cup l_2)(X) = l_1(X) \cup l_2(X)$$

for all $X \in 2^V$.

Proof. We choose arbitrary set of collisions $\mathcal{C}_1$ and $\mathcal{C}_2$ includes $\widetilde{\mathcal{C}}_V$. Let

$$\begin{aligned} f_{l_1} &= \mathrm{Ind}(G, V, l_1), \\ f_{l_2} &= \mathrm{Ind}(G, V, l_2), \\ f_{l_1 \cup l_2} &= \mathrm{Ind}(G, V, l_1 \cup l_2). \end{aligned}$$

Let $M_1 = (S, \mathcal{C}_1, f_{l_1})$ and $M_2 = (S, \mathcal{C}_2, f_{l_2})$. For all $X \in 2^V$, we see that

$$\begin{aligned} &f_{l_1 \cup l_2}(X) \\ &= \bigcup_{g \in G} g\,(l_1 \cup l_2)\left(g^{-1}X \cap V\right) \\ &= \bigcup_{g \in G} g\left\{l_1\left(g^{-1}X \cap V\right) \cup l_2\left(g^{-1}X \cap V\right)\right\} \\ &= \bigcup_{g \in G} gl_1\left(g^{-1}X \cap V\right) \cup \bigcup_{g \in G} gl_2\left(g^{-1}X \cap V\right) \\ &= F_{l_1}(X \cap S) \cup F_{l_2}(X \cap S) \\ &= (f_{l_1} \cup f_{l_2})(X). \end{aligned}$$

Therefore we have

$$\begin{aligned} &M_1 \cup M_2 \\ =&(S, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_{l_1} \cup f_{l_2}) \\ \equiv&(S, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_{l_1 \cup l_2}). \end{aligned}$$

Moreover, since $\widetilde{\mathcal{C}}_V \subseteq \mathcal{C}_1, \mathcal{C}_2$, we have $\widetilde{\mathcal{C}}_V \subseteq \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2)$. Hence we have

$$\begin{aligned} &(S, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_{l_1 \cup l_2}) \\ \equiv&GACS(G, S, V, l_1 \cup l_2) \end{aligned}$$

Therefore we have

$$M_1 \cup M_2 \equiv GACS(G, S, V, l_1 \cup l_2).$$

□

Corollary 3. *Let*

$$M_1 \equiv GACS(G, S, V, l_1), \quad M_2 \equiv GACS(G, S, V, l_2).$$

Let F_{M_1}, F_{M_2} and $F_{M_1 \cup M_2}$ be global transition functions of M_1, M_2 and $M_1 \cup M_2$, respectively. Then we have

$$F_{M_1 \cup M_2}(A) = F_{M_1}(A) \cup F_{M_2}(A)$$

for all $A \in 2^S$

Proof. Let $A \in 2^S$. From Proposition 7, we have

$$M_1 \cup M_2 \equiv GACS(G, S, V, l_1 \cup l_2).$$

Therefore we see that

$$\begin{aligned} & F_{M_1 \cup M_2}(A) \\ = & \bigcup_{g \in G} g(l_1 \cup l_2)(g^{-1}A \cap V) \\ = & \bigcup_{g \in G} gl_1(g^{-1}A \cap V) \cup \bigcup_{g \in G} gl_2(g^{-1}A \cap V) \\ = & F_{M_1}(A) \cup F_{M_2}(A) \end{aligned}$$

by Theorem 1. □

Corollary 4. *Let*

$$\begin{aligned} M_1 &= GACS(G, S, V, l_1), \\ M_2 &= GACS(G, S, V, l_2), \\ M_3 &= GACS(G, S, V, l_3). \end{aligned}$$

We have

$$M_1 \cup M_3 \equiv M_2 \cup M_3$$

if $M_1 \equiv M_2$.

Proof. Let F_{M_1}, F_{M_2}, F_{M_3} $F_{M_1 \cup M_3}$ and $F_{M_1 \cup M_3}$ be global transition functions of M_1, M_2, M_3, $F_{M_1 \cup M_3}$ and $F_{M_2 \cup M_3}$ respectively. Then we have $F_{M_1} = F_{M_2}$. For all $A \in 2^S$, we see that

$$\begin{aligned} & F_{M_1 \cup M_3}(A) \\ = & F_{M_1}(A) \cup F_{M_3}(A) \\ = & F_{M_2}(A) \cup F_{M_3}(A) \\ = & F_{M_2 \cup M_3}(A) \end{aligned}$$

from Corollary 3. Hence we have $M_1 \cup M_3 = M_2 \cup M_3$. □

Next, we consider that divide the set $\mathcal{C}$ into some partitions.

Proposition 8. *Let* $\mathcal{C}$ *be a set of collisions on* S. *The following three conditions are equivalent.*

(a) *There exists two sets of collisions on* S $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *which satisfies:*

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2, \tag{22}$$

$$\text{for all } X_1 \in \mathcal{C}_1, X_2 \in \mathcal{C}_2, \text{ if } \#X_1 \geq 2 \text{ and } \#X_2 \geq 2, \text{ then } X_1 \cap X_2 = \phi. \tag{23}$$

(b) *There exists subsets* $\widetilde{\mathcal{C}_1}$ *and* $\widetilde{\mathcal{C}_2}$ *of* 2^S *which satisfy*

$$\mathcal{C} = \widetilde{\mathcal{C}_1} \cup \widetilde{\mathcal{C}_2}, \tag{24}$$

$$X_1 \in \widetilde{\mathcal{C}_1}, X_2 \in \widetilde{\mathcal{C}_2} \Rightarrow X_1 \cap X_2 = \phi. \tag{25}$$

(c) *There exists* S_1 *and* S_2 *which satisfy*

$$S_1 \cup S_2 = S, \tag{26}$$

$$S_1 \cap S_2 = \phi, \tag{27}$$

$$(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) = \mathcal{C}. \tag{28}$$

Proof. We prove (c) ⇔ (a) and (c) ⇔ (b).

(c) ⇒ (b) Let

$$\widetilde{\mathcal{C}_i} = \mathcal{C} \cap 2^{S_i}.$$

Then we have

$$\widetilde{\mathcal{C}_1} \cup \widetilde{\mathcal{C}_2} = (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C}_2 \cap 2^{S_2}) = \mathcal{C},$$

by (28). Hence we have (24).

Moreover, for all $X_1 \in \widetilde{\mathcal{C}_1}, X_2 \in \widetilde{\mathcal{C}_2}$, since

$$X_1 \in 2^{S_1}, X_2 \in 2^{S_2}$$

and (27) of (c), we have

$$X_1 \cap X_2 \subseteq S_1 \cap S_2 = \phi.$$

Hence we have (25).

(b) ⇒ (c) Let

$$S_i = \bigcup \widetilde{\mathcal{C}_i}.$$

We prove (26), i.e., $S_1 \cup S_2 = S$. It is clear that $S_1 \cap S_2 \subseteq S$. On the other hand, for all $s \in S$, we have $\{s\} \in \mathcal{C}$. Therefore we have

$$s \in (\cup \widetilde{\mathcal{C}_1}) \cup (\cup \widetilde{\mathcal{C}_2}) = S_1 \cup S_2,$$

by (24). This implies $S \subseteq S_1 \cup S_2$. Therefore we have (26).

Next, we prove (27), i.e., $S_1 \cap S_2 = \phi$. We suppose that $S_1 \cap S_2 \neq \phi$. Then, there exists $s \in S_1 \cap S_2$. Therefore, there exists $X_1 \in \widetilde{\mathcal{C}_1}$ and $X_2 \in \widetilde{\mathcal{C}_2}$ such that $s \in X_1, s \in X_2$. This implies $s \in X_1 \cap X_2 \neq \phi$. This contradicts (25). Hence we have (27). Finally, we prove (28). It is clear that

$$(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \subseteq \mathcal{C}.$$

On the other hand, since $S_1 = (\cup \widetilde{\mathcal{C}_1})$, we have $X \subseteq (\cup \widetilde{\mathcal{C}_1}) = S_1$ for all $X \in \widetilde{\mathcal{C}_1}$. This implies $X \in 2^{S_1}$. Therefore $\widetilde{\mathcal{C}_1} \subseteq 2^{S_1}$. Hence we have

$$\widetilde{\mathcal{C}_1} \subseteq \mathcal{C} \cap 2^{S_1}.$$

Similarly, we have $\widetilde{\mathcal{C}_2} \subseteq \mathcal{C} \cap 2^{S_2}$. Therefore we have

$$\mathcal{C} = \widetilde{\mathcal{C}_1} \cup \widetilde{\mathcal{C}_2} \subseteq (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

Hence we have (28).

(c) ⇒ (a) Let

$$\mathcal{C}_i = (\mathcal{C} \cap 2^{S_i}) \cup \{\{s\} \mid s \in S_{3-i}\}, \quad (i = 1, 2). \tag{29}$$

For all $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$, suppose that $\#X_1 \geq 2$ and $\#X_2 \geq 2$. Since

$$X_i \notin \{\{s\} \mid s \in S_{3-i}\},$$

we have $X_i \in (\mathcal{C} \cap 2^{S_i})$. By (27), we have (23) as following:

$$X_1 \cap X_2 \subseteq S_1 \cap S_2 = \phi.$$

Moreover, we have (22) as following:

$$\begin{aligned}\mathcal{C}_1 \cup \mathcal{C}_2 &= (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \cup \{\{s\} \mid s \in S\} \\ &= \mathcal{C} \cup \{\{s\} \mid s \in S\} \\ &= \mathcal{C}.\end{aligned}$$

Finally, we show that $\mathcal{C}_1$ and $\mathcal{C}_2$ are sets of collisions on S. By (29), it is easy to show that $\mathcal{C}_i$ satisfies the condition (SC1). We check the condition (SC2). We assume $\mathcal{X} \subseteq \mathcal{C}_1$ without loss of generality. Suppose that $(\cap\mathcal{X}) \neq \phi$.

We suppose that $\#\mathcal{X} = 1$. Then there exists $X \in \mathcal{C}_1$ such that $\mathcal{X} = \{X\}$. Therefore $(\cup\mathcal{X}) = X \in \mathcal{C}_1$.

We suppose that $\#\mathcal{X} \geq 2$. We assume that there exists $X \in \mathcal{X}$ such that $X \notin \mathcal{C} \cap 2^{S_1}$. Then there exists $s_2 \in S_2$ such that $X = \{s_2\}$. Therefore $(\cap\mathcal{X}) \supseteq \{s_2\}$. However, by (29), we have

$$X \in \mathcal{C}_1, X \neq \{s_2\} \Rightarrow s_2 \notin X.$$

Therefore, $(\cap\mathcal{X}) = \phi$. This contradicts $(\cap\mathcal{X}) \neq \phi$. Hence we have $X \in \mathcal{C} \cap 2^{S_1}$ for all $X \in \mathcal{X}$. This implies $\mathcal{X} \subseteq \mathcal{C} \cap 2^{S_1}$. Since $\mathcal{C}$ is a set of collisions on S, we have $(\cup\mathcal{X}) \in \mathcal{C}$ from $\mathcal{X} \subseteq \mathcal{C}$ and $(\cap\mathcal{X}) \neq \phi$. Moreover, we have $(\cup\mathcal{X}) \in 2^{S_1}$ from $\mathcal{X} \subseteq 2^{S_1}$. Therefore, $(\cup\mathcal{X}) \in \mathcal{C} \cap 2^{S_1}$. Hence the set $\mathcal{C}_1$ satisfies the condition (SC2).

(a) ⇒ (c) We assume there exists $s \in S$ such that

$$X \in \mathcal{C}, s \in X \Rightarrow X = \{s\}.$$

Then we can easily prove (c), by putting $S_1 = \{s\}$ and $S_2 = S \setminus S_1$. In fact, it is clear that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \phi$. Let $X \in \mathcal{C}$. If $s \in X$ then $X = \{s\} \subseteq S_1$. If $s \notin X$ then $X \subseteq S_2$. This implies that $X \in 2^{S_1} \cup 2^{S_2}$. Therefore we have

$$X \in \mathcal{C} \cap (2^{S_1} \cup 2^{S_2}) = (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

Hence we have $(\mathcal{C} \cap 2^{S_1}) \cap (\mathcal{C} \cap 2^{S_2}) = \mathcal{C}$.

In the following, suppose that for all $s \in S$, there exists $X \in \mathcal{C}$ such that

$$\text{(30)} \qquad s \in X, \quad \#X \geq 2.$$

Let

$$\text{(31)} \qquad S_i = \bigcup \{X \mid X \in \mathcal{C}_i, \#X \geq 2\}, \quad (i = 1, 2).$$

First, it is clear that $S_i \neq \phi$ and $S_1 \cup S_2 \subseteq S$. We show that $S_1 \cup S_2 \supseteq S$. For all $s \in S$, there exists $X \in \mathcal{C}$ such that $s \in X$, $\#X \geq 2$. Since $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, we have $X \in \mathcal{C}_1$ or $X \in \mathcal{C}_2$. If $X \in \mathcal{C}_1$ (resp. $\mathcal{C}_2$), we have $X \subseteq S_1$ (resp. $X \subseteq S_2$) by $\#X \geq 2$. Therefore we can conclude that $s \in X \subseteq S_1 \cup S_2$. Next, we prove (27). We assume that $S_1 \cap S_2 \neq \phi$. There exist $X_1 \in \mathcal{C}_1(\#X_1 \geq 2)$ and $X_2 \in \mathcal{C}_2(\#X_2 \geq 2)$ such that $s \in X_1$, $s \in X_2$. This implies $X_1 \cap X_2 \neq \phi$. This contradicts (23). Finally, we prove (28). Let $X \in \mathcal{C}$. Suppose that $\#X = 1$. Since $S = S_1 \cup S_2$, we have $X \subseteq S_1$ or $X \subseteq S_2$. Therefore $X \in 2^{S_1} \cup 2^{S_2}$. We suppose that $\#X \geq 2$. Then we have $X \subseteq S_1$ or $X \subseteq S_2$ by (31). This implies $X \in 2^{S_1} \cup 2^{S_2}$. Therefore we have

$$X \in \mathcal{C} \cap (2^{S_1} \cup 2^{S_2}) = (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

Hence we see that

$$\mathcal{C} \subseteq (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}).$$

On the other hand, it is clear that

$$(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \subseteq \mathcal{C}.$$

Hence we have (28). □

Definition 11. Let $\mathcal{C}$ be a set of collisions on S. We call the set $\mathcal{C}$ is **dividable** iff it satisfies conditions of Proposition 8.

Proposition 9. *Let $M = (S, \mathcal{C}, f)$ be an abstract collision system. If the set $\mathcal{C}$ is dividable, then M is dividable.*

Proof. Since the set $\mathcal{C}$ is dividable, it satisfies the condition (c). Therefore, there exists S_1 and S_2 such that

$$\text{(32)} \quad S_1 \cup S_2 = S, \quad S_1 \cap S_2 = \phi, \quad (\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) = \mathcal{C}.$$

Let

$$\text{(33)} \qquad M_1 = (S_1, \mathcal{C} \cap 2^{S_1}, f_1), \quad M_2 = (S_2, \mathcal{C} \cap 2^{S_2}, f_2),$$

where f_1 and f_2 is restriction of f onto $\mathcal{C} \cap 2^{S_1}$ and $\mathcal{C} \cap 2^{S_2}$, respectively. Since $\mathcal{C}$ is a set of collision on S, we have

$$\mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2) = \mathfrak{C}(\mathcal{C}) = \mathcal{C}.$$

Therefore for all $X \in \mathcal{C}$, we have $X \in (\mathcal{C} \cap 2^{S_1})$ or $X \in (\mathcal{C} \cap 2^{S_2})$. We suppose that $X \in (\mathcal{C} \cap 2^{S_1})$. Since $X \subseteq S_1$, we have $X \cap S_2 = \phi$. Hence we have

$$\text{(34)} \qquad (f_1 \cup f_2)(X) = f_1(X) \cup \phi = f_1(X) = f(X).$$

We can also prove (34) in the same way for $X \in (\mathcal{C} \cap 2^{S_2})$.

Hence we have

$$M_1 \cup M_2 = (S_1 \cup S_2, \mathfrak{C}(\mathcal{C}_1 \cup \mathcal{C}_2), f_1 \cup f_2) = (S, \mathcal{C}, f_1 \cup f_2).$$

Therefore we have $M_1 \cup M_2 \equiv M$ by Lemma 3 and (34). □

The converse of Proposition 9 does not hold. We show that there exists an abstract collision system $M = (S, \mathcal{C}, f)$ such that $\mathcal{C}$ is not dividable but M is dividable.

Proposition 10. *Let G be a cyclic group and its generator be an element a, i.e.,*

$$G = < a > = \{a^n \mid n \in \mathbb{Z}\}.$$

We assume that $V \supseteq \{a^0, a^1\}$. Then any set of collisions $\mathcal{C}$ which includes $\widetilde{\mathcal{C}}_V$ is not dividable.

Proof. First we prove that

$$X_n = \left\{a^0, a^1, \ldots, a^n\right\}$$

is an element of $\mathcal{C}$ for all $n \in \mathbb{N}$. We prove this by using mathematical induction. When $n = 1$, since

$$X_1 = \left\{a^0, a^1\right\} \in 2^V,$$

we have $X_1 \in \widetilde{\mathcal{C}}_V \subseteq \mathcal{C}$. Let $k \geq 1$ and we assume that $X_k \in \mathcal{C}$. Since $\left\{a^0, a^1\right\} \in 2^V$ and $a^k \in G$, we have

$$X'_{k+1} = \left\{a^k, a^{k+1}\right\} = a^k \left\{a^0, a^1\right\} \in \widetilde{\mathcal{C}}_V \subseteq \mathcal{C}.$$

Therefore we have

$$X_k \in \mathcal{C}, X'_{k+1} \in \mathcal{C}, X_k \cap X'_{k+1} = \{a^k\} \neq \phi.$$

Hence we have

$$X_k \cup X'_{k+1} = X_{k+1} \in \mathcal{C},$$

by (SC2).

Next, we show that $\mathcal{C}$ is not dividable. We assume that $\mathcal{C}$ is dividable. Then there exist two set S_1 and S_2 such that they satisfy 3 conditions of (c) in Proposition 8.

We assume $a^0 \in S_1$ without loss of generality. Since $S_2 \neq \phi$, we can take an element $a^n \in S_2$.

Then the set

$$Y_n = \left\{a^{-|n|}, \ldots, a^0, \ldots, a^n\right\}$$

is an element of $\mathcal{C}$. Since $a^0 \in S_1$, $a^n \in S_2$ and $S_1 \cap S_2 \neq \phi$, we have

$$Y_n \notin 2^{S_1} \cup 2^{S_2}.$$

This implies

$$(\mathcal{C} \cap 2^{S_1}) \cup (\mathcal{C} \cap 2^{S_2}) \neq \mathcal{C}.$$

This contradicts (28). Hence $\mathcal{C}$ is not dividable. □

Example 4. We consider a 1 dimensional, 2 states, 2 neighborhood cellular automaton rule number 6:

$$f_{CA6}(x_0, x_1) = x_0 \oplus x_1.$$

We note that

$$x_0 \oplus x_1 = (x_0 \wedge \neg x_1) \vee (\neg x_0 \wedge x_1).$$

Let $G = \mathbb{Z}$. We define $l_6^{(2)}$ by

$$l_6^{(2)}(\{0,1\}) = \phi, \quad l_6^{(2)}(\{0\}) = \{0\},$$
$$l_6^{(2)}(\{1\}) = \{0\}, \quad l_6^{(2)}(\phi) = \phi$$

By using notation of (17), we denote $l_6^{(2)}$ by

$$l_6^{(2)}(1,1) = 0, \quad l_6^{(2)}(1,0) = 1,$$
$$l_6^{(2)}(0,1) = 1, \quad l_6^{(2)}(0,0) = 0,$$

i.e., $l_6^{(2)}(x_0, x_1) = x_0 \oplus x_1$. Let $V = \mathrm{ess}\ l_6^{(2)} = \{0, 1\}$. Then we see that the set of collisions $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$ is not dividable from Proposition 10.

Moreover, we define two functions $l_2^{(2)}$ and $l_4^{(2)}$ by

$$l_2^{(2)}(1,1) = 0, \quad l_2^{(2)}(1,0) = 0,$$
$$l_2^{(2)}(0,1) = 1, \quad l_2^{(2)}(0,0) = 0,$$
$$l_4^{(2)}(1,1) = 0, \quad l_4^{(2)}(1,0) = 1,$$
$$l_4^{(2)}(0,1) = 0, \quad l_4^{(2)}(0,0) = 0,$$

i.e.,

$$l_2^{(2)}(x_0, x_1) = \neg x_0 \vee x_1,$$
$$l_4^{(2)}(x_0, x_1) = x_0 \vee \neg x_1.$$

Let

$$\begin{aligned} M_{2CA-6} &= GACS(\mathbb{Z}, \mathbb{Z}, V, l_6^{(2)}), \\ M_{2CA-2} &= GACS(\mathbb{Z}, \mathbb{Z}, V, l_2^{(2)}), \\ M_{2CA-4} &= GACS(\mathbb{Z}, \mathbb{Z}, V, l_4^{(2)}). \end{aligned}$$

Then have $M_{2CA-6} \equiv M_{2CA-2} \cup M_{2CA-4}$.

The results of 1 dimensional 2 states 2 neighborhood cellular automata are listed in Table 1. From this table, we see that cellular automata which is dividable are rule 6, 10, 12 and 14.

Example 5. We consider a 1 dimensional 2 state 3 neighborhood cellular automaton CA 222, i.e.,

$$V = \{0, 1, 2\},$$
$$l_{222}^{(3)}(x_0, x_1, x_2) = (x_0 \oplus x_2) \vee x_1,$$
$$M_{3CA-222} = GACS(\mathbb{Z}, \mathbb{Z}, V, l_{222}^{(3)}).$$

Then we see that $\mathrm{ess}\ l_{222}^{(3)} = \{0, 1, 2\}$ and $\mathcal{C}$ is not dividable.

On the other hand, we define two functions

$$l_{90}^{(3)}(x_0, x_1, x_2) = x_0 \oplus x_2,$$
$$l_{204}^{(3)}(x_0, x_1, x_2) = x_1.$$

We make abstract collision systems

$$M_{3CA-90} = GACS(\mathbb{Z}, \mathbb{Z}, \{0, 2\}, l_{90}^{(3)}),$$
$$M_{3CA-204} = GACS(\mathbb{Z}, \mathbb{Z}, \{1\}, l_{204}^{(3)}).$$

Then we can easily prove that

$$M_{3CA-222} \equiv M_{3CA-90} \cup M_{3CA-204}.$$

Finally, we show a sufficient condition with which ACS is dividable.

Theorem 2. *Let G be a group. We consider an abstract collision system on G, $GACS(G, G, V, l)$. We assume that there exists a normal subgroup H of G and $d \in G$ such that $H \neq G$ and $dV \subseteq H$. Then the set $\mathfrak{C}(\widetilde{\mathcal{C}}_V)$ is dividable.*

Proof. Without loss of generality, we can assume that $d = e$ (e is the identity element of G), and the index $\#(G/H)$ is

Table 1: union of two 2 neighborhood CA

$l_2 \backslash l_1$	0	2	4	6	8	10	12	14
0	0	2	4	6	8	10	12	14
2	2	2	6	6	10	10	14	14
4	4	6	4	6	12	14	12	14
6	6	6	6	6	14	14	14	14
8	8	10	12	14	8	10	12	14
10	10	10	14	14	10	10	14	14
12	12	14	12	14	12	14	12	14
14	14	14	14	14	14	14	14	14

2. In other cases, we can prove similarly. We prove (c) of Proposition 8.

Let $h \in G \setminus H$, and

$$S_1 = H, \quad S_2 = hH.$$

It is clear that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$.

Next, we prove that $\mathfrak{C}(\widetilde{\mathcal{C}}) \subseteq 2^H \cup 2^{hH}$. To prove above, we show that $2^H \cup 2^{hH}$ is a set of collisions on S and

$$\widetilde{\mathcal{C}}_V = \left\{gX \mid g \in G, X \in 2^V\right\} \subseteq (2^H \cup 2^{hH}). \tag{35}$$

It is clear that $2^H \cup 2^{hH}$ is a set of collisions on S. We prove (35). Let $Y \in \widetilde{\mathcal{C}}_V$. There exists $g \in G$, $X \in 2^V$ such that $Y = gX$. Since $V \subseteq H$, we have $X \in 2^H$, i.e., $X \subseteq H$.

Hence $Y = gX \subseteq gH$. Since gH equals to H or hH, 2^{gH} equals to 2^H or 2^{hH}. Therefore we have

$$Y \in 2^{gH} \subseteq 2^H \cup 2^{hH}.$$

Hence we have $Y \in (2^H \cup 2^{hH})$ for all $Y \in \widetilde{\mathcal{C}}_V$. This implies (35). Let $C_1 = \mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^H$ and $C_2 = \mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^{hH}$. Then we see that

$$\begin{aligned} C_1 \cup C_2 &= (\mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^H) \cup (\mathfrak{C}(\widetilde{\mathcal{C}}) \cap 2^{hH}) \\ &= \mathfrak{C}(\widetilde{\mathcal{C}}) \cap (2^H \cup 2^{hH}) \\ &= \mathfrak{C}(\widetilde{\mathcal{C}}). \end{aligned}$$

Hence we have (28). □

Example 6. We consider a 1 dimensional 2 state 3 neighborhood cellular automata CA 90. Let $l_{90}^{(3)}$ be

$$l_{90}^{(3)}(x_0, x_1, x_2) = x_0 \oplus x_2.$$

First,it seems to be able to divide cells into cells which position are even and odd. We see intuitively that this division is able if the number of cells is infinite or even. We describe this facts by using Theorem 2.

First, we suppose that the number of cells is infinite, i.e., $G = \mathbb{Z}$. Let $V = \{0, 1, 2\}$, then we can see that

$$\operatorname{ess} l_{90}^{(3)} = \{0, 2\}.$$

Therefore we choose $H = 2\mathbb{Z}$, we see that $\operatorname{ess} l_{90}^{(3)} \subseteq H$. Hence the abstract collision system $GACS(\mathbb{Z}, \mathbb{Z}, \operatorname{ess} l_{90}^{(3)})$ is dividable.

Next, we suppose that the number of cells is finite and even i.e., $G = \mathbb{Z}/(2n)\mathbb{Z}$. Similarly, we choose $H = \{2n \mid n \in G\}$. Then H is a normal subgroup and we have $H \neq G$ and $\operatorname{ess} l_{90}^{(3)} \subseteq H$.

5. Composition of abstract collision systems on groups

In this section, we discuss composition of abstract collision systems.

Definition 12. Let $l : 2^V \to 2^S$. The **range** of l, which is denoted by Range l, is defined by

$$\text{Range } l = \bigcup \left\{l(X) \mid X \in 2^V\right\} \subseteq S.$$

Let $S = G$.

Definition 13 (Composition)**.** Let $V_1 \subseteq G$, $V_2 \subseteq G$, $l_1 : 2^{V_1} \to 2^S$ and $l_2 : 2^{V_2} \to 2^S$. We define a set $V_2(l_1)$ by

$$V_2(l_1) = \begin{cases} \{v_2 \in G \mid (v_2(\text{Range } l_1)) \cap V_2 \neq \phi\}, \\ \qquad\qquad \text{if Range } l_1 \neq \phi \\ V_2, \qquad \text{if Range } l_1 = \phi. \end{cases} \tag{36}$$

Moreover, we define a set $V_2(l_1) \otimes V_1$ and a function $l_2 \Diamond l_1 : 2^{V_2(l_1) \otimes V_1} \to 2^S$ by

$$V_2(l_1) \otimes V_1 = \{v_2 v_1 \mid v_2 \in V_2(l_1), v_1 \in V_1\}, \tag{37}$$

$$l_2 \Diamond l_1 (X) = l_2 \left(\bigcup_{v \in V_2(l_1)} v l_1((v^{-1}X) \cap V_1) \cap V_2 \right). \tag{38}$$

Lemma 10. *The two sets in the Definition 13 satisfy*

$$V_2(l_1) \neq \phi, \quad V_2(l_1) \otimes V_1 \neq \phi.$$

Proof. We prove $V_2(l_1) \neq \phi$. If Range $l_1 = \phi$, we have $V(l_1) = V_2 \neq \phi$ from (36). Suppose that Range $l_1 \neq \phi$. For all $x \in$ Range l_1 and $y \in V_2$, let $v_2 = yx^{-1}$. Then $y = v_2 x$. Since $v_2 x \in v_2(\text{Range } l_1)$ and $y \in V_2$, we have

$$y \in (v_2(\text{Range } l_1)) \cap V_2.$$

This implies

$$(v_2(\text{Range } l_1)) \cap V_2 \neq \phi.$$

Hence $v_2 \in V_2(l_1)$. Therefore we have $V_2(l_1) \neq \phi$. □

Lemma 11. *For all $v_2 \in V_2(l_1)$, we have*

$$V_1 \subset {v_2}^{-1}(V_2(l_1) \otimes V_1).$$

Especially, we have

$$({v_2}^{-1}(V_2(l_1) \otimes V_1)) \cap V_1 = V_1. \tag{39}$$

Proof. Let $v_1 \in V_1$. We have $v_1 = {v_2}^{-1}(v_2 v_1)$. Since $v_2 \in V_2(l_1)$, we have

$$(v_2 v_1) \in V_2(l_1) \otimes V_1.$$

Therefore we have

$$v_1 = (v_2)^{-1}(v_2 v_1) \in {v_2}^{-1}(V_2(l_1) \otimes V_1).$$

Hence we have

$$V_1 \subset {v_2}^{-1}(V_2(l_1) \otimes V_1)$$

□

Lemma 12. *Let $h \in G$. For all $g \in G \setminus h(V_2(l_1))$ and $X \subset V_1$, we have*

$$(h^{-1} g l_1(X)) \cap V_2 = \phi. \tag{40}$$

Proof. Suppose that Range $l_1 = \phi$. Since $l_1(X) = \phi$ for all $X \subseteq V_1$, our claim is clear. Suppose that Range $l_1 \neq \phi$. We show by indirect proof. We assume that

$$(h^{-1} g l_1(X)) \cap V_2 \neq \phi. \tag{41}$$

Since $l_1(X) \subseteq$ Range l_1, we have

$$(h^{-1} g l_1(X)) \cap V_2 \subset (h^{-1} g(\text{Range } l_1)) \cap V_2.$$

Therefore we have

$$(h^{-1} g(\text{Range } l_1)) \cap V_2 \neq \phi$$

from (41). Hence we can conclude that $h^{-1} g \in V_2(l_1)$. This implies $g \in hV_2(l_1)$. This contradicts $g \in G \setminus (hV_2(l_1))$. □

Theorem 3. *Let f_{l_1}, f_{l_2} and $f_{l_2 \Diamond l_1}$ be induced local transition functions by V_1 and l_1, V_2 and l_2, $V_2(l_1) \otimes V_1$ and $l_2 \Diamond l_1$, respectively, i.e.,*

$$\begin{aligned} f_{l_1} &= \mathrm{Ind}(G, V_1, l_1), \\ f_{l_2} &= \mathrm{Ind}(G, V_2, l_2), \\ f_{l_2 \Diamond l_1} &= \mathrm{Ind}(G, V_2(l_1) \otimes V_1, l_2 \Diamond l_1). \end{aligned}$$

Then we have

$$f_{l_2 \Diamond l_1} = f_{l_2} \circ f_{l_1}. \tag{42}$$

Proof. First, we assume that Range $l_1 \neq \phi$. For all $X \in 2^{V_2(l_1) \otimes V_1}$, we compute $f_{l_2 \Diamond l_1}$ and $f_{l_2} \circ f_{l_1}$.

$$\begin{aligned} & f_{l_2 \Diamond l_1}(X) \\ &= \bigcup_{g_2 \in G} g_2 \, (l_2 \Diamond l_1) \left({g_2}^{-1} X \cap (V_2(l_1) \otimes V_1) \right) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(\bigcup_{v \in V_2(l_1)} v \\ &\qquad l_1 \left(v^{-1} ({g_2}^{-1} X \cap (V_2(l_1) \otimes V_1)) \cap V_1 \right) \cap V_2 \Bigg) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(\bigcup_{v \in V_2(l_1)} v \\ &\qquad l_1 \left((g_2 v)^{-1} X \cap v^{-1} (V_2(l_1) \otimes V_1) \cap V_1 \right) \cap V_2 \Bigg) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg(\bigcup_{v \in V_2(l_1)} {g_2}^{-1} (g_2 v) \\ &\qquad l_1 \left((g_2 v)^{-1} X \cap V_1 \right) \cap V_2 \Bigg), \end{aligned} \tag{43}$$

and

$$\begin{aligned} & f_{l_2} \circ f_{l_1}(X) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \left({g_2}^{-1} f_{l_1}(X) \cap V_2 \right) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \Bigg({g_2}^{-1} \\ &\qquad \left(\bigcup_{g_1 \in G} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) \right) \cap V_2 \Bigg) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \left(\bigcup_{g_1 \in G} \left({g_2}^{-1} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) \right) \cap V_2 \right) \end{aligned} \tag{44}$$

To show that they are equal, we prove that

$$\begin{aligned} & \bigcup_{v \in V_2(l_1)} {g_2}^{-1} (g_2 v) l_1 \left((g_2 v)^{-1} X \cap V_1 \right) \cap V_2 \\ &= \bigcup_{g_1 \in G} {g_2}^{-1} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) \\ &= \bigcup_{g_1 \in g_2 V_2(l_1)} {g_2}^{-1} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) \\ &\quad \cup \bigcup_{g_1 \in G \setminus g_2 V_2(l_1)} {g_2}^{-1} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) \end{aligned} \tag{45}$$

for all $g_2 \in G$. Since we have

$$\bigcup_{g_1 \in G \setminus g_2 V_2(l_1)} {g_2}^{-1} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) = \phi$$

by Lemma 12, we show

$$\bigcup_{v \in V_2(l_1)} {g_2}^{-1}(g_2 v) l_1 \left((g_2 v)^{-1} X \cap V_1 \right) \cap V_2$$

$$= \bigcup_{g_1 \in g_2 V_2(l_1)} {g_2}^{-1} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) \tag{46}$$

instead of (45).

However, $v \in V_2(l_1)$ and $g_1 \in g_2(V_2(l_2))$ is one-to-one with $g_1 = g_2 v$. Hence we have (46).

Next, we assume that Range $l_1 = \phi$. For all $X \in 2^{V_2 \otimes V_1}$, (38) becomes

$$l_2 \Diamond l_1(X) = l_2(\phi). \tag{47}$$

On the other hand, f_{l_1} satisfies $f_{l_1}(Y) = \phi$ for all $Y \in 2^S$. Therefore we have

$$f_{l_2} \circ f_{l_1}(Y) = f_{l_2}(\phi).$$

Hence for all $Y \in 2^{V_2 \otimes V_1}$, we have

$$\begin{aligned} f_{l_2 \Diamond l_1}(Y) &= \bigcup_{g \in G} g l_2 \Diamond l_1((g^{-1} Y) \cap (V_2 \otimes V_1)) \\ &= \bigcup_{g \in G} g l_2(\phi) \\ &= \bigcup_{g \in G} g l_2((g^{-1} \phi) \cap V_2) \\ &= f_{l_2}(\phi) \\ &= f_{l_2} \circ f_{l_1}(Y). \end{aligned}$$

That's our claim. □

Definition 14. Let

$$\begin{aligned} M_1 &= GACS(G, G, V_1, l_1), \\ M_2 &= GACS(G, G, V_2, l_2). \end{aligned}$$

We define an abstract collision system $M_2 \Diamond M_1$ by

$$M_2 \Diamond M_1 = GACS(G, G, V_2(l_1) \otimes V_1, l_2 \Diamond l_1).$$

Theorem 4. *Let*

$$\begin{aligned} M_1 &= GACS(G, G, V_1, l_1), \\ M_2 &= GACS(G, G, V_2, l_2). \end{aligned}$$

Let F_{M_1}, F_{M_2} and $F_{M_2 \Diamond M_1}$ be global transition functions of M_1, M_2 and $M_2 \Diamond M_1$, respectively. Then we have

$$F_{M_2 \Diamond M_1}(A) = F_{M_2} \circ F_{M_1}(A)$$

for all $A \in 2^G$.

Proof. We see that

$$\begin{aligned} & F_{M_2 \Diamond M_1}(A) \\ &= \bigcup_{g_2 \in G} g_2 (l_2 \Diamond l_1) \left({g_2}^{-1} A \cap (V_2(l_1) \otimes V_1) \right), \\ & F_{M_2} \circ F_{M_1}(A) \\ &= \bigcup_{g_2 \in G} g_2 l_2 \left({g_2}^{-1} \left(\bigcup_{g_1 \in G} g_1 l_1 \left({g_1}^{-1} X \cap V_1 \right) \right) \cap V_2 \right) \end{aligned}$$

from Theorem 1. The right hand sides of these formulae are appeared in (43) and (44) in the proof of Theorem 3, and we proved they are equal. Hence we have

$$F_{M_2 \Diamond M_1}(A) = F_{M_2} \circ F_{M_1}(A).$$

□

Corollary 5. *Let*

$$\begin{aligned} M_1 &= GACS(G, G, V, l_1), \\ M_2 &= GACS(G, G, V, l_2), \\ M_3 &= GACS(G, G, V', l_3). \end{aligned}$$

Then we have

$$\begin{aligned} M_3 \Diamond M_1 &\equiv M_3 \Diamond M_2, \\ M_1 \Diamond M_3 &\equiv M_2 \Diamond M_3 \end{aligned}$$

if $M_1 \equiv M_2$.

Proof. Let F_{M_1}, F_{M_2}, F_{M_3} $F_{M_3 \Diamond M_1}$ and $F_{M_3 \Diamond M_2}$ be global transition functions of M_1, M_2, M_3, $M_3 \Diamond M_1$ and $M_3 \Diamond M_2$, respectively. Then we have $F_{M_1} = F_{M_2}$. Therefore for all $A \in 2^S$, we have

$$\begin{aligned} & F_{M_3 \Diamond M_1}(A) \\ =& F_{M_3} \circ F_{M_1}(A) \\ =& F_{M_3} \circ F_{M_2}(A) \\ =& F_{M_3 \Diamond M_2}(A) \end{aligned}$$

from Theorem 4. Hence we have $M_3 \Diamond M_1 \equiv M_3 \Diamond M_2$. Similarly, we have $M_1 \Diamond M_3 \equiv M_2 \Diamond M_3$. □

Example 7. Let M_{2CA-i}, M_{2CA-j} and M_{3CA-k} be cellular automata on groups

$$\begin{aligned} M_{2CA-i} &= GACS(\mathbb{Z}, \mathbb{Z}, \{0,1\}, l_i^{(2)}), \\ M_{2CA-j} &= GACS(\mathbb{Z}, \mathbb{Z}, \{0,1\}, l_j^{(2)}), \\ M_{3CA-k} &= GACS(\mathbb{Z}, \mathbb{Z}, \{0,1,2\}, l_k^{(3)}). \end{aligned}$$

Table 2: The composition of 2 neighborhood CA, $l_i \Diamond l_j$.

$l_i \backslash l_j$	0	2	4	6	8	10	12	14
0	0	0	0	0	0	0	0	0
2	0	34	68	66	8	34	12	2
4	0	12	48	24	64	68	48	16
6	0	46	116	90	72	102	60	18
8	0	0	0	36	128	136	192	236
10	0	34	68	102	136	170	204	238
12	0	12	48	60	192	204	240	252
14	0	46	116	126	200	238	252	254

Then we have

$$V(l_j^{(2)}) \otimes V = \{0,1,2\},$$
$$l_i^{(2)} \Diamond l_j^{(2)}(x_0,x_1,x_2) = l_i^{(2)} \left(l_j^{(2)}(x_0,x_1), l_j^{(2)}(x_1,x_2) \right),$$

by Theorem 3. This means that we can construct a 3 neighborhood cellular automaton by composing two 2 neighborhood cellular automata.

The result of compositions of 2 neighborhood cellular automata are listed in Table 2. For example,

$$M_{2CA-6} \Diamond M_{2CA-6} = M_{3CA-90},$$
$$M_{2CA-8} \Diamond M_{2CA-4} = M_{3CA-0}.$$

Since

$$l_6^{(2)}(x_0,x_1) = x_0 \oplus x_1,$$
$$l_{90}^{(3)}(x_0,x_1,x_2) = x_0 \oplus x_2,$$

the farmer example shows

$$\begin{aligned} &(x_0 \oplus x_1) \oplus (x_1 \oplus x_2) \\ =&l_6^{(2)} \left(l_6^{(2)}(x_0,x_1), l_6^{(2)}(x_1,x_2) \right) \\ =&l_{90}^{(3)}(x_0,x_1,x_2) \\ =&x_0 \oplus x_2. \end{aligned}$$

Similarly, the latter example shows

$$\begin{aligned} &(x_0 \wedge \neg x_1) \wedge (x_1 \wedge \neg x_2) \\ =&l_8^{(2)} \left(l_4^{(2)}(x_0,x_1), l_4^{(2)}(x_1,x_2) \right) \\ =&l_0^{(3)}(x_0,x_1,x_2) \\ =&0. \end{aligned}$$

We assumed that $S = G$ in order to simplify the discussion. The following of this section, we extend the definition of composition in the case of $S \neq G$.

In Definition 13, we would like to reset $V_1, V_2 \subseteq G$ by $V_1, V_2 \subseteq S$. However, since the set S has no operation, (37) is not well-defined. We would like to define (37) by using the operation of G to S. First, let $V \subseteq S$ and $H \subseteq G$, we define

$$HV = \{hv \mid h \in H, v \in V\}. \tag{48}$$

Next, we take $H_1, H_2 \subseteq G$. We replace V_1 and V_2 by H_1V and H_2V, respectively.

Definition 15. Let $V \subseteq S$, $H_1 \subseteq G$, $H_2 \subseteq G$, $l_1 : 2^{H_1V} \to 2^S$ and $l_2 : 2^{H_2V} \to 2^S$. We define a set $H_2(l_1)$ by

$$H_2(l_1) = \begin{cases} \{h \in G \mid (h(\text{Range } l_1)) \cap H_2V \neq \phi\}, & \\ \qquad\qquad \text{if Range } l_1 \neq \phi & \\ H_2, \qquad \text{if Range } l_1 = \phi & \end{cases} \tag{49}$$

Moreover, we define a sets $H_2(l_1) \otimes H_1$ and a function $l_2 \Diamond l_1 : 2^{H_2(l_1) \otimes H_1V} \to 2^S$ by

$$\begin{aligned} &H_2(l_1) \otimes H_1 \\ =&\{h_2h_1 \mid h_2 \in H_2(l_1), h_1 \in H_1\}, \end{aligned} \tag{50}$$

$$\begin{aligned} &l_2 \Diamond l_1(X) \\ =&l_2 \left(\bigcup_{h \in H_2(l_1)} hl_1((h^{-1}X) \cap H_1V) \cap H_2V \right) \end{aligned} \tag{51}$$

Theorem 5. *Let f_{l_1}, f_{l_2} and $f_{l_2 \Diamond l_1}$ be induced local transition function by H_1V and l_1 H_2V and l_2, $(H_2(l_1) \otimes H_1)V$ and $l_2 \Diamond l_1$, respectively, i.e.,*

$$f_{l_1} = \text{Ind}(G, H_1V, l_1),$$
$$f_{l_2} = \text{Ind}(G, H_2V, l_2),$$
$$f_{l_2 \Diamond l_1} = \text{Ind}(G, (H_2(l_1) \otimes H_1)V, l_2 \Diamond l_1).$$

Then we have

$$f_{l_2 \Diamond l_1} = f_{l_2} \circ f_{l_1}. \tag{52}$$

Example 8. Let $H = \{0,1\}$ Let M_1 and M_2 be 1 dimensional Q state, H neighborhood cellular automata, defined by Example 3. Then we have

$$l_2 \Diamond l_1(x_0,x_1,x_2) = l_2\left(l_1(x_0,x_1), l_1(x_1,x_2)\right) \tag{53}$$

by composing M_1 and M_2.

6. Distributive Law

In this section, we consider that two operations, union and composition of abstract collision systems on groups, and check the distribution law. We consider the most easy case, cellular automata on groups.

Example 9. Let M_{2CA-i} and M_{3CA-j} be cellular automata on groups

$$M_{2CA-i} = GACS(\mathbb{Z}, \mathbb{Z}, \{0,1\}, l_i^{(2)}),$$
$$M_{3CA-j} = GACS(\mathbb{Z}, \mathbb{Z}, \{0,1,2\}, l_j^{(3)}),$$

respectively. From Table 1 and Table 2, we have $M_{2CA-2} \cup M_{2CA-4} = M_{2CA-6}$ and $M_{2CA-6} \Diamond M_{2CA-6} = M_{3CA-90}$. Moreover, we have

$$\begin{aligned} M_{2CA-6} \Diamond M_{2CA-2} &= M_{3CA-46}, \\ M_{2CA-6} \Diamond M_{2CA-4} &= M_{3CA-116}, \\ M_{2CA-2} \Diamond M_{2CA-6} &= M_{3CA-66}, \\ M_{2CA-4} \Diamond M_{2CA-6} &= M_{3CA-24} \end{aligned}$$

from Table 2. Furthermore, we can compute easily

$$\begin{aligned} M_{3CA-46} \cup M_{3CA-116} &= M_{3CA-126}, \\ M_{3CA-66} \cup M_{3CA-24} &= M_{3CA-90}. \end{aligned}$$

Therefore we see that

$$\begin{aligned} & M_{2CA-6} \Diamond (M_{2CA-2} \cup M_{2CA-4}) \\ =& (M_{2CA-2} \cup M_{2CA-4}) \Diamond M_{2CA-6} \\ =& M_{2CA-6} \Diamond M_{2CA-6} \\ =& M_{3CA-90}, \\ & (M_{2CA-6} \Diamond M_{2CA-2}) \cup (M_{2CA-6} \Diamond M_{2CA-4}) \\ =& M_{3CA-46} \cup M_{3CA-116} \\ =& M_{3CA-126}, \\ & (M_{2CA-2} \Diamond M_{2CA-6}) \cup (M_{2CA-4} \Diamond M_{2CA-6}) \\ =& M_{3CA-66} \cup M_{3CA-24} \\ =& M_{3CA-90}. \end{aligned}$$

Hence we have

$$\begin{aligned} & M_{2CA-6} \Diamond (M_{2CA-2} \cup M_{2CA-4}) \\ \neq & (M_{2CA-6} \Diamond M_{2CA-2}) \cup (M_{2CA-6} \Diamond M_{2CA-4}), \\ & (M_{2CA-2} \cup M_{2CA-4}) \Diamond M_{2CA-6} \\ =& (M_{2CA-2} \Diamond M_{2CA-6}) \cup (M_{2CA-4} \Diamond M_{2CA-6}). \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} & M_{2CA-k} \Diamond (M_{2CA-i} \cup M_{2CA-j}) \\ \neq & (M_{2CA-k} \Diamond M_{2CA-i}) \cup (M_{2CA-k} \Diamond M_{2CA-j}), \\ & (M_{2CA-j} \cup M_{2CA-k}) \Diamond M_{2CA-i} \\ =& (M_{2CA-j} \Diamond M_{2CA-i}) \cup (M_{2CA-k} \Diamond M_{2CA-i}) \end{aligned}$$

for all rule number i, j and k.

Theorem 6. *Let*

$$\begin{aligned} M_1 &= GACS(G, G, V, l_1), \\ M_2 &= GACS(G, G, V, l_2), \\ M_3 &= GACS(G, G, V', l_3). \end{aligned}$$

Then we have

$$(M_1 \cup M_2) \Diamond M_3 = (M_1 \Diamond M_3) \cup (M_2 \Diamond M_3). \tag{54}$$

Proof. We see that

$$\begin{aligned} & (M_1 \cup M_2) \Diamond M_3 \\ \equiv\ & GACS(G, G, V, l_1 \cup l_2) \Diamond GACS(G, G, V_3, l_3) \\ & \qquad \text{(by Cor. 5)} \\ \equiv\ & GACS(G, G, V(l_3) \otimes V_3, (l_1 \cup l_2) \Diamond l_3) \quad \text{(by Def. 14)}, \\ & (M_1 \Diamond M_3) \cup (M_2 \Diamond M_3) \\ \equiv\ & GACS(G, G, V_3(l_3) \otimes V, l_1 \Diamond l_3) \\ \cup\ & GACS(G, G, V_3(l_3) \otimes V, l_2 \Diamond l_3) \qquad \text{(by Cor. 4)} \\ \equiv\ & GACS(G, G, V_3(l_3) \otimes V, (l_1 \Diamond l_3) \cup (l_2 \Diamond l_3)) \quad \text{(by Prop. 7)}. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} & (l_1 \cup l_2) \Diamond l_3 (X) \\ =\ & (l_1 \cup l_2) \left(\bigcup_{v \in V(l_3)} v l_3 \left(v^{-1} X \cap V_3 \right) \cap V \right) \\ =\ & l_1 \left(\bigcup_{v \in V(l_3)} v l_3 \left(v^{-1} X \cap V_3 \right) \cap V \right) \\ \cup\ & l_2 \left(\bigcup_{v \in V(l_3)} v l_3 \left(v^{-1} X \cap V_3 \right) \cap V \right) \\ =\ & l_1 \Diamond l_3 (X) \cup l_2 \Diamond l_3 (X) \end{aligned}$$

for all $X \in 2^{V(l_3) \otimes V_3}$. Hence we have (54). □

This theorem says that the operation $\diamond$ is **right distributive** over $\cup$, but $\diamond$ is **not left distributive** over $\cup$.

7. Conclusion

We defined abstract collision systems on groups, and investigated their properties. First, we see the behavior of the global transition functions of ACS on groups. Second, we consider union and division of ACS. In the case of ACS on groups, we proved a sufficient condition that ACS is dividable. Finally, by using operations of groups, we discussed a composition of ACS on groups.

The union of cellular automata on groups is to take "logical or" of local transition rules. The composition of cellular automata with ACS matches the composition of local transition rules of cellular automata in Fujio's work [4]. Then the operations "composition" and "logical or" of local transition rules of cellular automata satisfy the right distribution law, but they do not satisfy the left distribution law.

As future works, we would like to show the necessary condition of division. Moreover, we would like to consider division there are no operations on groups.

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