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The Critical Behavior of 1-dimensional Spin System with Infinite Long-range Interaction Compensating the Effect of Lattice Dimensionality

by

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Abstract

By using the Monte Carlo simulation, we investigate the critical behavior of the 1-dimensional spin system with infinite long-range (LR) interaction $r^{-(1+\sigma)}$. By changing the value σ, we compare the results with those of the d-dimensional spin system with the nearest- neighbor (NN) interaction only.

In the case of XY spin model, we obtain three different type phase transitions, i.e. the mean field type for $0<\sigma<0.5$, the σ- dependent non-trivial one for $0.5<\sigma<1$ and 'Berezinskii-Kosterlitz-Thouless (BKT)-like' transition at $\sigma=1$ as in the NN model of $d>4$, $2<d<4$ and $d=2$, respectively.

In the case of q-state clock spin model with $\sigma=1$, we also confirm the BKT-like transition together with the similar q-dependence of the critical behavior to that of the 2-dimensional NN model.

These results suggest that the infinite long-range interaction can partly compensate the role of the lattice dimensionality by increasing the effective value d from $d=1$ to $d=2$.

Keywords: Phase transition, Dimensionality, Monte Carlo simulation, XY spin model, q-state clock spin model, Long-range interaction, BKT transition

1. Introduction

The lattice dimensionality d of the system has a large effect on the nature of the phase transition. In general, a 1-dimensional system $(d=1)$ does not show any phase transition. On the other hand, the higher dimensional systems $(d>1)$ show various types of phase transition, and the critical behavior changes with the spin symmetry (or the number of spin component D). For

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example in the 2-dimensional system $(d=2)$, in contrast to the usual order-disorder transition of the Ising model $(D=1)$, the XY spin model $(D=2)$ shows the characteristic phase transition without long-range order (LRO), which is called Berezinskii-Kosterlitz-Thouless (BKT) transition.

The critical behavior of the system around the transition temperature T_c is well described by a set of indices defined for various physical quantities. For example, the temperature dependence of the specific heat C, the magnetization M, the susceptibility χ and the correlation length ξ are represented by

$$\begin{aligned} C &\propto t^{-\alpha}, \\ M &\propto t^{\beta}, \\ \chi &\propto t^{-\gamma}, \\ \xi &\propto t^{-\nu}, \end{aligned} \tag{1}$$

where t is the reduced temperature $t = |T - T_c|/T_c$. Also the spin correlation at T_c for the d-dimensional lattice is represented as

$$G(r) \propto \frac{1}{r^{d-2+\eta}}. \tag{2}$$

These indices α, β, γ, ν and η are called critical point exponents (or critical exponents in short). It is well established that the value of the critical exponents only depends on d and D and is not affected by other details of the system. As a result, the spin systems are classified into several groups (universality class) designated by the values of the critical exponents. **Table 1** shows the critical exponents of some typical spin models.

Table 1 The critical exponents of some typical spin models.[1)]

	α	β	γ	ν	η^*
2d-Ising (D=1)	0	1/8	7/4	1	1/4
3d-Ising (D=1)	0.113	0.324	1.239	0.629	0.03
3d-XY (D=2)	-0.01	0.34	1.32	0.70	0.03
Mean field approximation	0	1/2	1	1/2	0

* η is called `anomalous dimension' and shows the deviation from the mean field value.

From **Table 1**, we can see that the lattice dimensionality affects not only the appearance of phase transition but also the intrinsic nature of the phase transition and critical behavior. Then, it is interesting to study whether such important role of the lattice dimensionality can be replaced or compensated by other factors, e.g. by the introduction of the modification of interaction form. One possible candidate is the 1-dimensional spin model with infinite long-range (LR) interaction decaying with power law $r^{-(1+\sigma)}$, where r is the distance between interacting spins. This model has no phase transition for $\sigma > 1$ but shows various types of phase transition characterized by the value of σ for $\sigma \le 1$. The natures of these phase transitions appearing in the model are analogous to those found in the d-dimensional spin model having only the nearest- neighbor (NN) interaction.

In the present paper, we investigate the critical behavior of the 1-dimensional spin model with

LR interaction by using Monte Carlo (MC) simulation and discuss the results comparing with those of the NN model on the 2-dimensional lattice. Calculation is mainly performed for the XY spin model ($D = 2$) and partly for the discrete version of it (q -state clock spin model).

In the next section, we explain the important properties of the 1-dimensional LR interaction model. In chapter 3, we define the XY spin model and its discrete version, i.e. q -state clock spin model with LR interaction. We also explain the outline of the method of MC simulation adopted in the present study. Simulation results for the XY spin model are given and discussed in chapter 4. These calculations are extended to the q -state clock spin model in chapter 5. Chapter 6 is the conclusion of this paper. The explanation of the BKT phase transition in the 2-dimensional NN model is given in the Appendix.

2. 1-dimensional System with Long-range Interaction

2.1 Long-range interaction and lattice dimensionality

The 1-dimensional spin model with the finite-range interaction does not have the long-range order (LRO). On the other hand, Dyson proved exactly that a phase transition can occur in the Ising model with power law decaying infinite long-range interaction $r^{-(1+\sigma)}$.[2] After that the renormalization group (RG) calculation was performed by Fisher et al.[3] and by Sak [4], and they predicted that this model would show the mean field type critical behavior for $0 < \sigma < 0.5$ and the non-trivial phase transition, where critical nature varies with the value σ, for $0.5 < \sigma < 1$. For $0.5 < \sigma < 1$, it is also expected that the critical exponent η obeys the relation $\eta = 2 - \sigma$. In order to confirm these predictions, many numerical calculations have been performed.[5-9]

In the case of $\sigma > 1$, the phase transition cannot occur even in the LR model. In this sense, the LR model with $\sigma > 1$ is equivalent to the 1-dimensional NN model. Also in the case of $\sigma < 1$, if we compare the nature of phase transition of LR model with that of d -dimensional NN model, we note that there is an interesting correspondence between σ and d. We compared the nature of the phase transitions of the 1-diensional LR model with that of the d -dimensional NN model in the respective case of the Ising model (**Table 2**) and 3-state Potts model (**Table 3**). We used the well established results that the 3-state Potts model with NN interaction shows continuous second order phase transition in the 2-dimensional lattice while it shows the mean field like first order phase transition in the 3-dimensional lattice. Also it is actually known that the 3-state Potts LR model changes its nature of phase transition from the second order to the first order at $\sigma \approx 0.7$.[10,11]

Table 2 Comparison between the phase transition of the NN model and that of the LR model for Ising model.

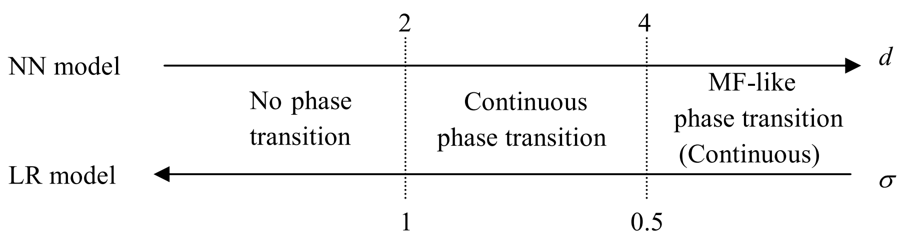

Table 3 Comparison between the phase transition of the NN model and that of the LR model for 3-state Potts model.

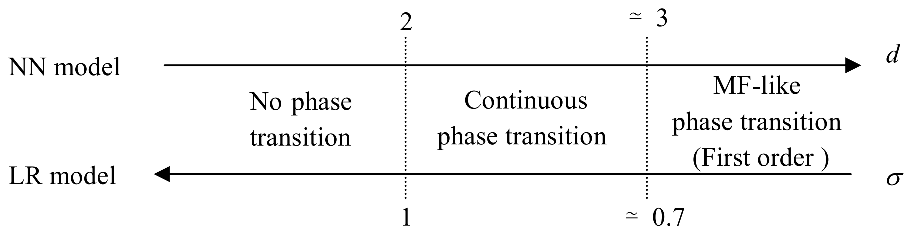

Table 2 (**Table 3**) suggest that in the Ising model (3-state Potts model) the 1-dimensional LR system with $\sigma > 1$, $0.5 < \sigma < 1$ ($0.7 < \sigma < 1$) and $0 < \sigma < 0.5$ ($0 < \sigma < 0.7$) has a similar critical nature to that of the d-dimensional NN models of $2 < d$, $2 < d < 4$ ($2 < d < 3$) and $d > 4$, respectively. That is, the value of σ in the LR model plays the role of lattice dimensionality d in the NN model. Thus we may say that the LR interaction compensates the dimensionality of the lattice.

2.2 The 'BKT-like' transition of the 1-dimensional model

As described in the previous chapters, the 1-dimensional Ising model with LR interaction $r^{-(1+\sigma)}$ shows the mean field type critical behavior for $0 < \sigma < 0.5$, the non-trivial phase transition for $0.5 < \sigma < 1$ and no phase transition for $\sigma > 1$. The situation is very complicated in the case of $\sigma = 1$, since this is the border point of whether the phase transition occurs or not. For the Ising model ($D = 1$) with $\sigma = 1$, the discontinuous jump of the order parameter was expected at T_c by the calculation based on counting the possible spin configuration by Thouless.[12] Later, Anderson explained that the binding-unbinding of the topological defects (kinks) interacting through the Coulomb interaction cause this behavior (kink gas representation).[13]

It is the case of the XY spin model ($D = 2$) that is interesting. The Hamiltonian for the XY spin model is given by

$$H = -\sum_{<ij>} J_{ij} \cos(\theta_i - \theta_j) \quad , \tag{3}$$

where θ_i denotes an angular variable at site i. $J_{ij} = J_0 r_{ij}^{-(1+\sigma)}$ is a coupling constant between site i and site j separated by distance r_{ij}.

As in the Ising model, Fisher et al. predicted that the XY spin model with LR interaction shows the mean field type critical behavior for $0 < \sigma < 0.5$ and the non-trivial phase transition that depends on σ for $0.5 < \sigma < 1$.

However, since the XY spin model has continuous symmetry, this model cannot make the topological defects. Thus the kink gas representation cannot apply to the 1-dimensional XY spin model. Therefore, no attention had been paid to this model even in the case of $\sigma = 1$ for a long time until the study by Simanek. In the spin wave approximation, Simanek[14] showed that, for $\sigma = 1$, the correlation at low temperature is given by

$$G(r_{ij}) = r_{ij}^{-\frac{k_B T}{\pi^2 J_0}} , \tag{4}$$

and the susceptibility at low temperature is given by

$$\chi = \frac{2}{k_B T} \sum_j \left(\frac{1}{r_{0j}} \right)^{\frac{k_B T}{\pi^2 J_0}} , \tag{5}$$

where k_B is the Boltzmann constant.

These results imply the appearance of the power law decay correlation and infinite susceptibility in the low temperature region of $k_B T / \pi^2 J_0 < 1$. The situation is very similar to the case of ordinary BKT transition found in the 2-dimensioanl XY spin model of NN system (See, Appendix).[15,16]

If the BKT transition occurs in such 1-dimensional system, it means that the infinite long-range interaction can play the role to increase effective lattice dimensionality from $d = 1$ to $d = 2$.

One of our main aims in the present paper is to confirm whether the BKT (or BKT-like) phase transition truly occurs or not in the 1-dimensional XY spin model with LR interaction of $\sigma = 1$.

In the next chapter, we present the spin models and briefly explain the method of MC simulation adopted for the present system with infinite long-range interaction.

3. Definition of the Model and the Method of Calculation

As described in the previous chapter, the Hamiltonian for the XY spin model with LR interaction is given by

$$H = -\sum_{<ij>} J_{ij} \cos(\theta_i - \theta_j) \qquad \left(J_{ij} = J_0 r_{ij}^{-(1+\sigma)}\right). \tag{6}$$

In the calculation, we assumed ferromagnetic coupling $J_0 = 1(> 0)$ and periodic boundary condition.

When we suppose that the angular variable θ_i takes only q discrete values, we obtain the discrete version of the XY spin model that is called as the q-state clock spin model. The Hmiltonian for the 1-dimensional q-state clock spin model with LR interaction is

$$H = -\sum_{<ij>} J_{ij} \cos\frac{2\pi(p_i - p_j)}{q} \qquad (p_i = 1, 2, \cdots, q). \tag{7}$$

This model reduces to the XY spin model in the limit $q \to \infty$. In chapter 5, we also make use of this model.

Because of the infinite long-range interaction, the finite size effect strongly affects the simulation results. Therefore we need a large system size L requiring much calculation time proportional to the square of L.

Recently, many methods based on the cluster algorithm are proposed to shorten the computing time.[17] In this study, we use the simplest discrete update (DUD) MC method.[18] In the DUDMC method, we divide the effective field acting on site i into two parts: $H_i = H_{near} + H_{far}$, where H_{near} is the effective field from spins near the site i, and H_{far} is that from spins at the far sites. When the thermal equilibrium is realized, the field H_{far} does not vary so much during the spin update process. Then we update H_{far} only for every m MC steps (discrete update), while we update H_{near} in every update trial since it is strongly affected by each spin configuration. In this study, we include up to the 8th nearest-neighbor sites in H_{near} and consider the case of m=1 or 2. Furthermore, we make use of the fast Fourier transform algorithm (FFT) to calculate H_{far} with the aid of the convolution theorem in k-space. The FFT can reduce the calculation time from the order L^2 to $L \log L$. The validity of this method is explained in reference 18).

When we need more precise calculation, we also adopt the Swendsen's histogram MC method.

The physical quantities such as the energy E and the magnetization M are estimated from the following thermal average:

$$E = \langle H \rangle,$$
$$M = \frac{1}{L}\sqrt{\left(\sum_i \cos\theta_i\right)^2 + \left(\sum_i \sin\theta_i\right)^2}\ . \tag{8}$$

The specific heat C and the susceptibility χ are calculated by their fluctuation:

$$C = \frac{\langle E^2\rangle - \langle E\rangle^2}{k_B T^2},$$
$$\chi = \frac{\langle M^2\rangle - \langle M\rangle^2}{k_B T}. \tag{9}$$

In the calculation, we assume $k_B = 1$.

4. The Result of the XY Spin Model

4.1 The σ dependence of the critical behavior

It has been predicted that the critical behavior of this system is the mean field type for $0 < \sigma < 0.5$ and non-trivial σ dependent type for $0.5 < \sigma < 1$. At $\sigma = 0.5$, the essentially singular type critical behavior has been predicted. In **Fig. 1**, we showed the temperature dependence of the specific heat C in the cases of $\sigma = 0.4$ and $\sigma = 0.7$. In this calculation, thermal averages were taken 5×10^5 MC steps after equilibrating over 1×10^5 MC steps for system size $L = 2^7 \sim 2^{14}$.

In the mean field approximation, the specific heat jumps from zero to a finite value at $T = T_c$ with the decrease of temperature. For $\sigma = 0.4$, the specific heat shows the tendency of mean field type critical behavior as system size increases. On the other hand, for $\sigma = 0.7$, the specific heat shows no jump, and obviously the system has critical behavior that is different from the mean field type.

According to the finite size scaling theory, system size dependence of the magnetization M and the susceptibility χ at the critical temperature are given by

$$M \propto L^{-\frac{\beta}{\nu}},$$
$$\chi \propto L^{\frac{\gamma}{\nu}}. \tag{10}$$

We estimated the critical exponent γ/ν from the result of susceptibility χ by using equation (10). In this calculation, we adopted the histogram MC method and we took 8×10^5 MC steps for making an energy histogram. The σ dependence of the critical exponent $\eta = 2 - \gamma/\nu$ is shown in **Fig. 2**. In the region $0.5 < \sigma < 1$, our result fits well with the relation $\eta = 2 - \sigma$ predicted by Fisher et al.[3,4]

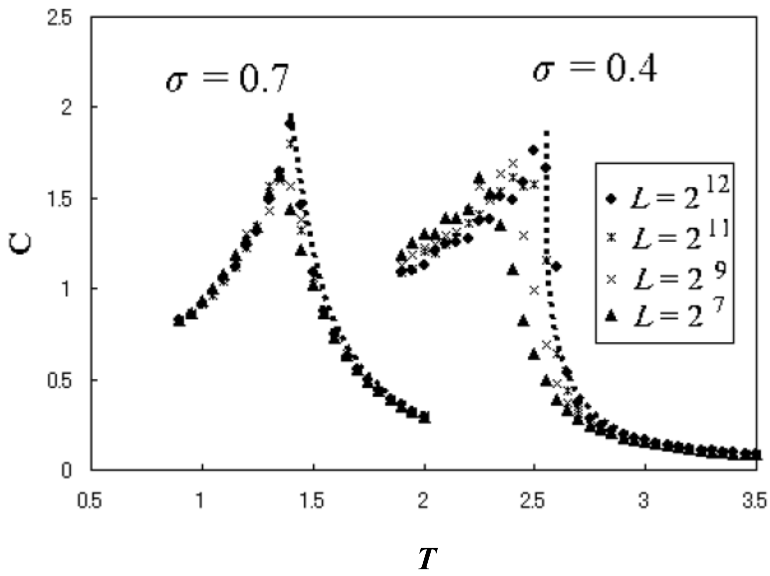

Fig. 1 The temperature dependence of the specific heat for several system size L at mean field region $\sigma = 0.4$ and non-trivial region $\sigma = 0.7$.

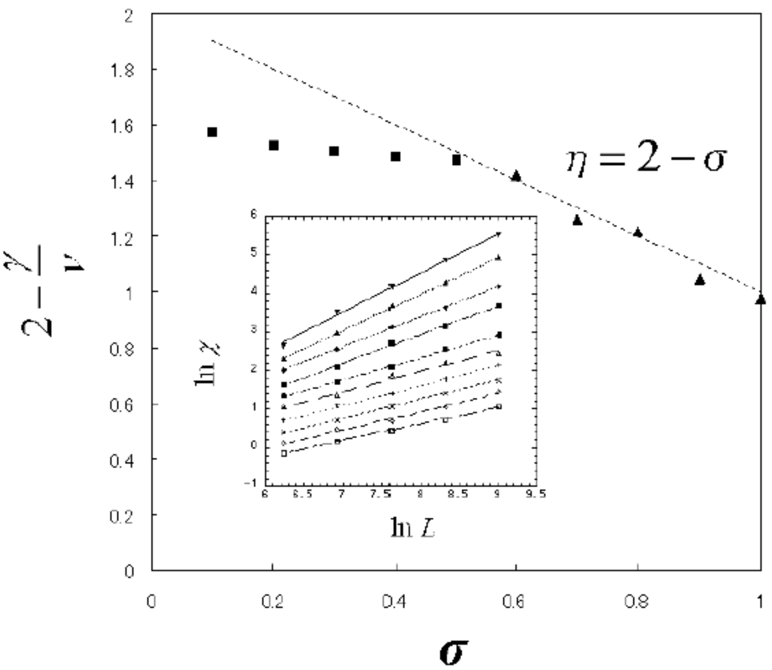

Fig. 2 The σ dependence of the critical exponent $\eta = 2 - \gamma/\nu$. The slope shows the RG prediction of $\eta = 2 - \sigma$. Inset shows $\ln \chi$ vs $\ln L$ for $\sigma = 0.1 \sim \sigma = 1$ from lower curves to upper ones.

The value of γ/ν expected by the mean field calculation is $\gamma/\nu = 2$. Our estimation of γ/ν is almost $\gamma/\nu = 0.5$ in the area of $0 < \sigma < 0.5$. At present we cannot explain this large deviation from the expected mean field value. The smaller σ means the longer range interaction, and so in order to elucidate the situation more careful scaling correction by using larger L systems will be needed for $0 < \sigma < 0.5$.

4.2 The case of $\sigma = 1$

In this section, we show the MC result of the XY spin model with $\sigma = 1$ where the BKT-like transition is expected by the spin wave calculation.

In **Fig. 3**, we showed the temperature dependence of the specific heat C for several system size L . These curves show no system size dependence. On the other hand, the susceptibility χ has the clear system size dependence and does not well defined behavior at low temperature region (not reported here). These behaviors commonly appear on the MC simulations for the 2-dimensional XY spin model with NN interaction that shows BKT transition. In the inset, we show T_c obtained from the maximum of χ , and we got $T_c \approx 0.6$.

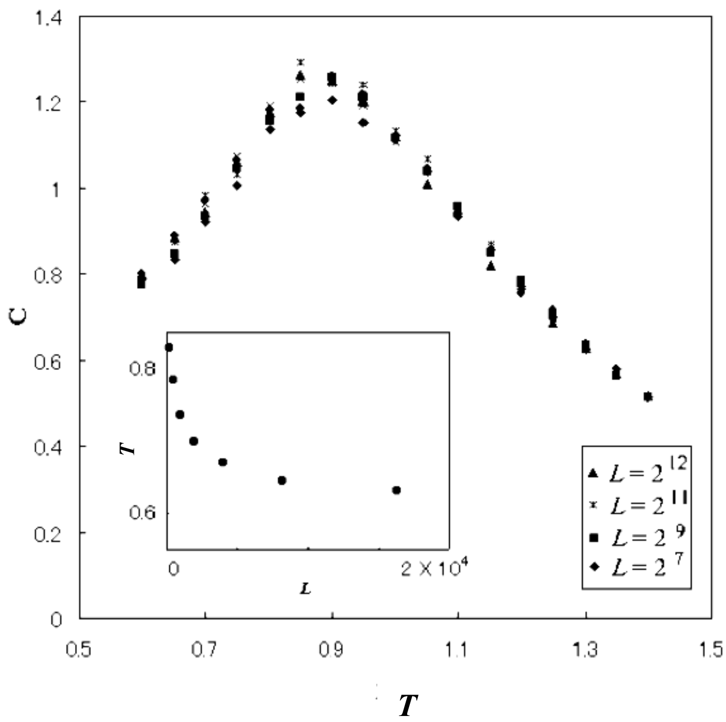

Fig. 3 The temperature dependence of the specific heat for several system size L at $\sigma = 1$. The inset shows the size dependence of the peak temperature of χ calculated by the histogram method.

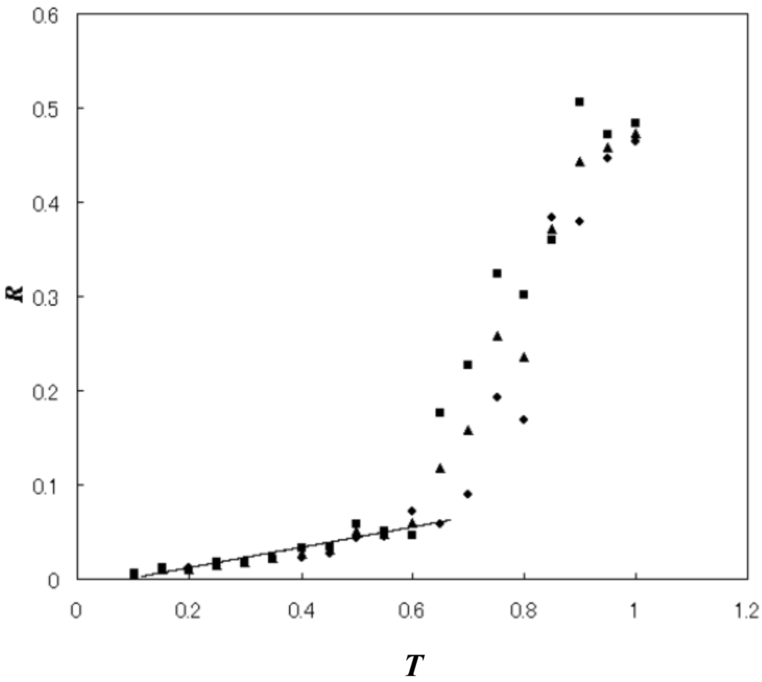

Fig. 4 The phenomenological RG plot of M ($L = 2^{10}, 2^{11}, 2^{12}$) at $\sigma = 1$.

Figure 4 shows the phenomenological renormalization group plot[19] of the magnetization M. We calculated the temperature dependence of the following quantity R:

$$R = -\frac{\ln(M/M')}{\ln(L/L')}, \tag{11}$$

where L and L' denote the different two lattice size $(L, L' = 2^{10}, 2^{11}, 2^{12})$, and M , M' are corresponding magnetizations.

If M is proportional to $L^{-\frac{\beta}{\nu}}$ at critical temperature, R should take the only one value of β/ν independent of L and L'. For ordinary LRO, therefore, the curves of the temperature dependence of R should cross only at critical temperature (point) T_c. On the other hand, in the BKT transition which has the critical line, R should lap over one curve for $T < T_c$.

Our result at lower temperature than $T_c \approx 0.6$ collapse on the one curve and this shows one evidence supporting the BKT transition.

We can estimate the critical exponent η by using finite size scaling and the relation $\gamma = \nu(2-\eta)$ and we got $\eta = 0.977 \approx 1$. This value is different from $\eta = 1/4$ for the ordinary BKT transition found in the 2-dimensional XY spin model with NN interaction. Thus, for the characteristic transition found in the present 1-dimensional LR model, it may be adequate to be called BKT-like transition.

5. The BKT-like Transition of Clock Spin Model

As shown in the Appendix, in the 2-dimensional NN interaction system, it has been also confirmed that the clock spin model shows the BKT transition when $q \geq 5$ [20,21]. In the present chapter, the calculation in the previous section was extended to the 1-dimensional q-state clock spin model in order to confirm the correspondence between σ and d. In the MC simulation, thermal averages were taken 5×10^5 MC steps after equilibrating over 1×10^5 MC steps and the system size L was taken $L = 2^8 \sim 2^{15}$.

In **Fig. 5**, we showed the temperature dependence of the specific heat C, in the case of $q = 6$. The specific heat exhibits two anomalies; the rather sharp peak at low temperature, T_2, and the broad maximum at high temperature, T_1 ($> T_2$). We could estimate the precise positions of T_1 and T_2 by using the histogram MC method, and determine $T_1 = 0.862\pm0.003$ and $T_2 = 0.457\pm0.003$. In the inset, we show the size dependence of the peak height of the specific heat at T_1 and T_2. As expected for the BKT transition, both peaks have no evident size dependence.

All those behaviors of C are very similar to those found in the 2-dimensional 6-state clock spin model with the NN interaction (See Appendix). Thus, it is considered that two-step phase transition occurs, i.e. BKT at T_1 and ordinary LRO at T_2 leading intermediate BKT phase for temperatures $T_2 < T < T_1$.

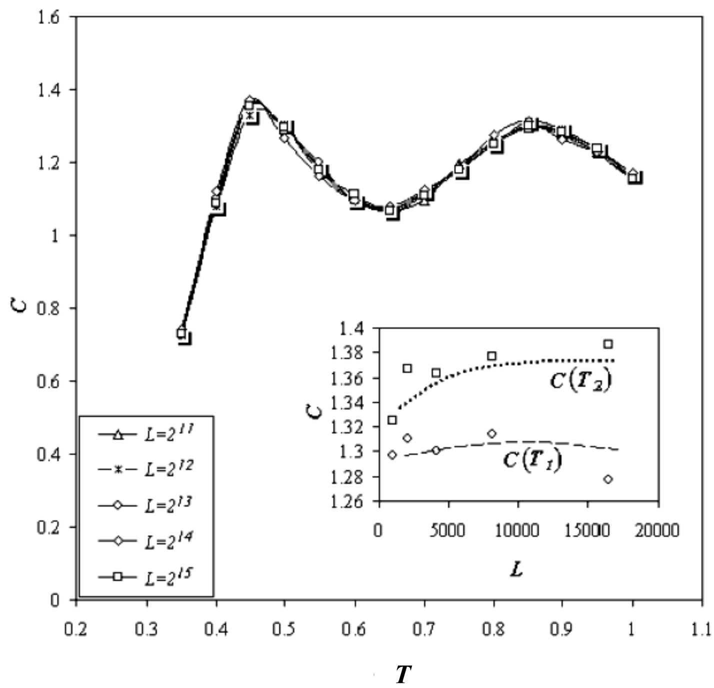

Fig. 5 The temperature dependence of the specific heat for several system size L ($q = 6$). The inset shows the size dependence of the peak height at T_1 and T_2 calculated by the histogram method.

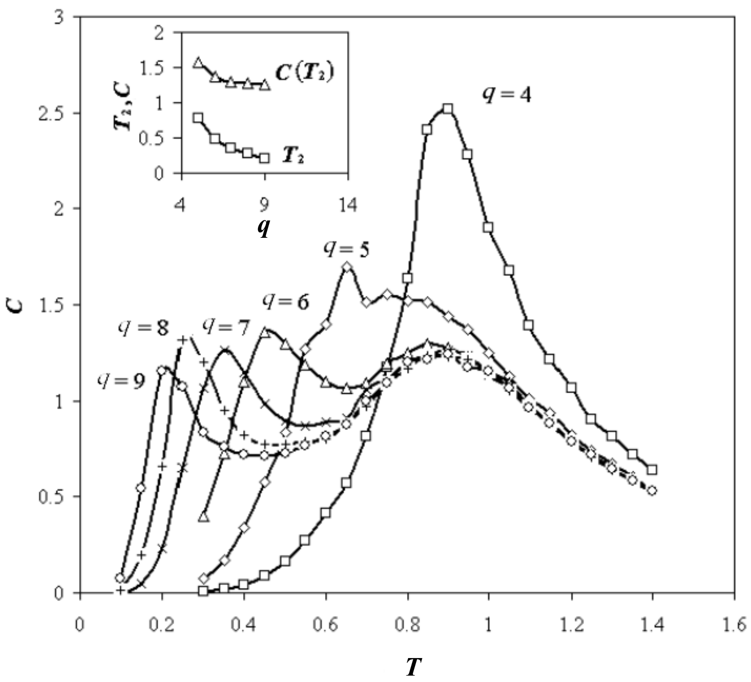

Fig. 6 The temperature dependence of the specific heat for several values of q ($L = 2^{14}$). The inset shows the q dependence of the temperature T_2 and its peak height calculated by the histogram method.

The temperature dependences of the specific heat in the case of q = 4～9 are shown in **Fig. 6**. In the case of q = 4, the curve exhibits only one peak probably corresponding to T_2. This behavior is similar to that of the 2-dimensional 4-state clock spin model with the NN interaction. In the case of q = 5, the peak begins to split into two peaks, and there appear two separate peaks in the case of q = 6. With further increase of q, T_2 shift to the lower temperature. By using the histogram MC method we estimated the q dependence of the temperature T_2 and the peak height of C at T_2. The results are shown in the inset. We may be able to expect that the sharp peak at T_2 shift to the zero temperature and disappears in the limit of $q \to \infty$. The position of the peak at higher temperature does not depend on the value of q and consistently coincides with that of the single peak found for the 1-dimensional XY spin model with LR interaction.[22,23)]

In **Fig. 7**, we showed the temperature dependence of the magnetization M, in the case of q = 6. As discussed before, the low temperature phase at $T < T_2$ is expected to be the ordinary LRO phase with finite magnetization and the phase at $T_2 < T < T_1$ is the intermediate BKT phase where total magnetization is expected to disappear. However, a comparatively large magnetization is observed even for the intermediate region $T_2 < T < T_1$. This is explained by the strong finite size effect of the BKT system having long-range correlation. Then, in order to get more precise conclusion is needed to extrapolate the present results into the infinite size system. The finite size scaling theory for the BKT phase transition predicts that the size dependence of M should be as follows

$$M \propto L^{-(d-2+\eta)}, \tag{12}$$

where d is the lattice dimensionality.

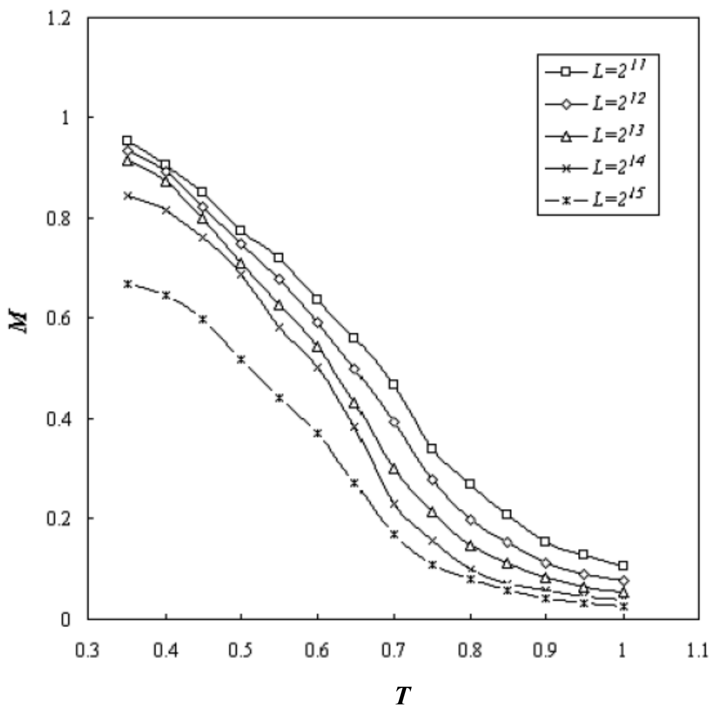

Fig. 7 The temperature dependence of the magnetization for several system size L $(q=6)$.

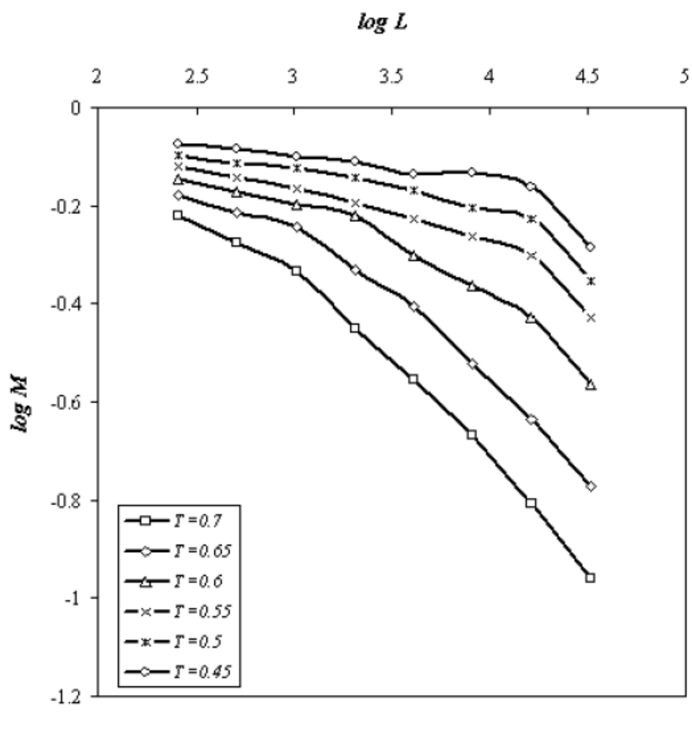

Fig. 8 Log-Log plot of the relation between M and L for some selected temperatures between T_1 and T_2 $(q=6)$.

We showed the size dependence of magnetization for various temperature between T_1 and T_2 in **Fig. 8** in the case of q = 6. As expected in this temperature region, M seems to decrease and disappear with increase of the system size L. Because of the large finite size effect in the long-range system, there are large deviations from the liner behavior even in the case of a fairly large value of L. Then, for the estimation of the slope, $d-2+\eta$, we used the largest two sizes available in the present calculation. The results are shown in **Table 4** for some selected values of temperatures in the region of $T_2 < T < T_1$. The value of η varies with temperature.

Though the value of η differs from that obtained for the 2-dimensional clock spin model with NN interaction, the tendency of the temperature dependence is quite similar to each other. Thus we may conclude that there exists intermediate phase (BKT-like phase) in 1-dimensional 6-state clock spin model with LR interaction with $\sigma = 1$.

Table 4 The critical exponent η estimated from **Fig. 8.**

	$T = 0.45$	$T = 0.50$	$T = 0.55$	$T = 0.60$	$T = 0.65$	$T = 0.70$
$d-2+\eta$	0.345	0.406	0.418	0.448	0.452	0.503
η ($d=1$)	1.345	1.406	1.418	1.448	1.452	1.503

6. Conclusion

We have discussed the critical behavior of the 1-dimensional spin model with LR interaction decaying as $r^{-(1+\sigma)}$.

For the XY spin model, we confirmed the appearance of the mean field type phase transition for $0 < \sigma < 0.5$ and non-trivial σ-dependent phase transition for $0.5 < \sigma < 1$. Those critical behaviors coincide with the results of the d-dimensional XY spin model with NN interaction in the cases of $d > 4$ and $2 < d < 4$, respectively. This supports our speculation that the role of the lattice dimensionality can be replaced or compensated by the LR interaction. From this viewpoint, the σ-dependent non-trivial phase transition is well understood by successive changes of universality class induced by the change of effective lattice dimensionality d. In the extreme case of $\sigma = 1$, we found the characteristic phase in the low temperature region where correlation has power law decay similar to the BKT phase in the ordinary 2-dimensional XY spin model with NN interaction. The transition to this phase has different critical exponent with the ordinary BKT, and so may be called a BKT-like transition.

A similar calculation is also performed for the q-state clock spin model restricted to the case of $\sigma = 1$. In this calculation, we could observe the two transitions having different characters and confirm the nature of the intermediate phase is BKT-like. These results are very similar to the results of the 2-dimensional q-state clock spin model with NN interaction.

In conclusion, we could confirm that the infinite long-range interaction can partly compensate the effect of the lattice dimensionality and plays a very important role in affecting the nature of the phase transition.

Appendix The BKT Transition of the 2-dimensional XY and Clock Spin Model

The theorem of Mermin-Wagner[24)] proves exactly that the 2-dimensional XY spin model does not have LRO for $T > 0$. However, it was expected that a certain phase that is different from a paramagnet phase would appear at low temperature. This phase transition is called BKT transition.

According to the Kosterlitz-Thouless scenario, as shown in **Fig. 9**, the BKT transition is caused by the shift from a plasma state at high temperature to dielectric state at low temperature. In the plasma phase, spin vortices move freely by heat fluctuation. On the other hand, the spin vortices make pairs in the dielectric phase.

All the energy of the system is

$$U = -\pi J \sum_{i \neq j} n_i n_j \log \frac{\left|\vec{r}_i - \vec{r}_j\right|}{r_0} + E_c, \tag{13}$$

where r_0 is the radius of the vortex and E_c is the energy of the vortex core. n_i denotes the sign of the vortex (for the vortex $n_i = 1$ and the anti-vortex $n_i = -1$). Equation (13) shows that many vortices interact with the Coulomb force in 2-dimensional plane.

The free energy is given by

$$f \approx \exp\left(\frac{A}{\sqrt{t}}\right), \tag{14}$$

where A is a constant. Since f is infinitely differentiable by temperature, it turns out that specific heat does not diverge at T_c.

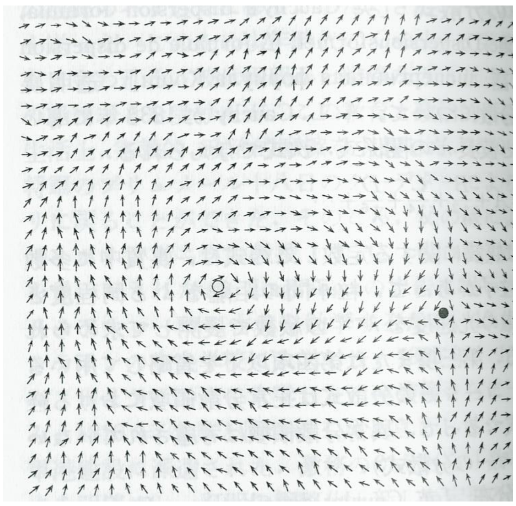

Fig. 9 The example of the pair of spin vortices.[25)]

The correlation function at low temperature is given by the power law decay of the form

$$G(r) = r^{-\frac{k_B T}{2\pi J}}. \tag{15}$$

The RG theory of the BKT transition shows that these conclusions are qualitatively correct and also predicts that the critical exponent η for $T > T_c$ is given by

$$\eta(T_c) = \frac{1}{4}. \tag{16}$$

The BKT transition is seen not only in the 2-dimensional XY spin model but also in the 2-dimensional q-state clock spin model with $q \geq 5$. Although this is the discrete version of the XY spin model, unlike ordinary XY spin model, it shows two steps of phase transitions.

Figure 10 shows the temperature dependence of the specific heat of q-state clock spin model with $q \geq 5$. It shows the phase transitions from the paramagnetic phase at high temperature to LRO phase at low temperature through the intermediate BKT phase. The critical exponent η is found to vary with temperature and takes between $\eta(T_1) = \frac{1}{4}$ and $\eta(T_2) = \frac{4}{q^2}$.

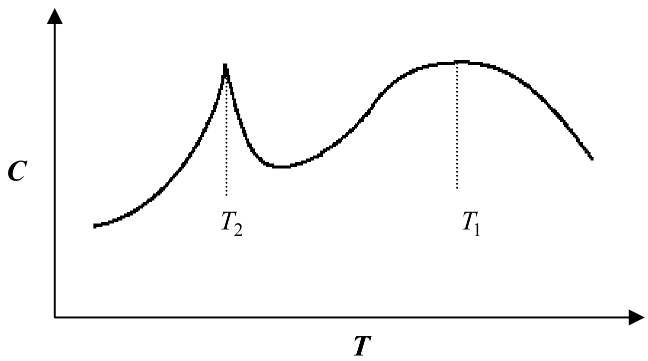

Fig. 10 The temperature dependence of the specific heat of q-state clock spin model ($q \geq 5$).

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