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The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

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Abstract

Among Professor Kiyosi Itô's achievements, there is the Itô-Nisio theorem, a completely general theorem relative to the Fourier series decomposition of the Brownian motion. In this paper, some of its applications will be reviewed, and new applications to 1-soliton solutions to the Korteweg-de Vries (KdV in short) equation and Eulerian polynomials will be given.

Key words: Itô-Nisio theorem, quadratic Wiener functional, stochastic area, 1-soliton, Eulerian polynomial
1991 MSC: 60H05, 60H07

1 Introduction

Professor Kiyosi Itô's careful study about the work by N. Wiener on Brownian motion started around 1943, after he moved to Nagoya University; in the foreword to "Kiyosi Itô Selected papers", he said ([21, pp.xiv-xv])

> "Although I had heard much of N. Wiener's great contribution to probability theory, I had not read his work carefully until I went to Nagoya. Even his

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theory of Brownian motion I learned from Lévy's book and Doob's papers. Reading some of his papers I was impressed by the originality with which he initiated not only measure-theoretic probability theory but also path-theoretic process theory as early as the 1920's."

He then learned the Fourier series expansion of Brownian motion by N. Wiener in 1924 ([41]) and R. E. A. C. Paley-N. Wiener in 1934 ([34]). Itô was particularly interested in the latter paper, where the construction of the path of Brownian motion was used to construct the Wiener measure. As Itô wrote in the Commentary in [42, pp.513-519], it is not easy to read Wiener's paper of year 1923 ([40]), in which the existence of the Wiener measure was first proved, because of the heuristic nature of its presentation. In contrast, the Fourier series expansion given in Paley-Wiener [34] is more satisfactory from the logical point of view. Being interested in the method, Itô introduced the Fourier series expansion in detailed and clearly understandable manner in his Japanese book [20, §§38-39] published in 1953. This is an example of Itô's attitudes to devote extraordinary efforts to approaching central problems in probability theory with prior knowledge as little as possible. After the publication of his book, the Japanese mathematicians with interests in probability theory easily understood Chapters IX and X in Paley-Wiener [34].

It should be also mentioned that the sections from 64 to 66 in Itô's book are devoted to the theory of stochastic differential equations, which was created by him in 1942 ([16]). In 1950's, in Japan, many probabilists started the study of probability theory with this book, and then read together the books by J. Doob [3] and W. Feller [5].

Another Fourier series expansion of Brownian motion is the one by P. Lévy in 1940 ([27]), where the Haar wavelet expansion was used. The wavelet expansion is now used widely. For example, see Z. Ciesielski [1], M. Kac [24], and M. Pinsky [36].

Retaining an interest in the Fourier series expansion of Brownian motion, Itô established a theorem in 1968 in the joint paper with Makiko Nisio ([23]), the Itô-Nisio theorem, which is a complete generalization of Wiener's construction of the Brownian motion; they formulated the expansion as a problem on convergence of sums of independent random variables with values in a Banach space, and applied the concept of tightness due to Y. Prohorov [37] to see the convergence. Their result in the most simple case asserts that

Theorem 1 (Itô-Nisio[23], p.45, Theorem 5.2) *Denote by H the Cameron-Martin space of real valued functions over $[0,1]$, that is, the real separable Hilbert space of all real, absolutely continuous functions h on $[0,1]$ with $h(0) = 0$, possessing square integrable derivatives. Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal basis of H, and $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed* *random variables, each of which obeys* $N(0,1)$*, the normal distribution of mean* 0 *and variance* 1*. Then, the sequence*

$$\sum_{n=1}^{\infty} \xi_n u_n(t), \quad t \in [0,1]$$

converges uniformly in t *a.s., and the limit process* $\{S(t)\}_{t\in[0,1]}$ *is a* 1*-dimensional Brownian motion.*

In addition to this, we would like to remark that Itô was very much interested in Prohorov's work [37]. For example, in 1964, 4 years before the above article on the Fourier series expansion appeared, he wrote another paper with M. Nisio ([22]) using Prohorov's result. Itô had been interested in the metric space structure of the space of probability measures. It should be recalled that the metric space structure of $\mathcal{P}(\mathbb{R})$ ($\equiv$ the space of probability measures on $\mathbb{R}$) was already studied in 1930's by Lévy ([26]), and, in 1956, Prohorov introduced a distance between two measures on a complete separable metric space, which is analogous to the Lévy distance on $\mathcal{P}(\mathbb{R})$. Being very much interested in the results of 1930's, in 1943, about two decades before the joint articles with Nisio, Itô wrote an introductory book to probability theory in Japanese, where he gave a concise account of the metric space structure of $\mathcal{P}(\mathbb{R})$ ([17]).

Now returning to our subject, we recall the first section of Lévy [28, pp.171-172], where he discusses the importance of using the Fourier series expansion. After giving two definitions of the Brownian motion: the first one in the fourth line on page 171:

> "Let $X(t)$ be Wiener's well known random function, defined up to an additive constant by the condition
>
> $$X(t) - X(t_0) = \xi\sqrt{t - t_0}, \quad t > t_0, \tag{1.1.1}$$
>
> ξ being a real and normalized Laplacian (often called Gaussian) random variable"

and the second one in the fourteenth line:

> "(1.1.5) $$X(t) = \frac{\xi' t}{\sqrt{2\pi}} + \sum_1^{\infty} \frac{1}{n\sqrt{\pi}}[\xi_n(\cos nt - 1) + \xi_n' \sin nt],$$
>
> the Greek letters indicating normalized Laplacian random variables, all independent of each other"

he wrote

" Thus, the same random function may be defined by (1.1.1) or by (1.1.5). This theorem was proved by N. Wiener [9] in 1924, and ten years later, formula (1.1.5) was used as a definition by Paley and Wiener. Starting from one or the other point of view, it is easy to prove that $X(t)$ is almost surely a well defined and continuous function; $\delta X(t)$ is generally $O(\sqrt{dt}\,)$ $(dt > 0)$, and not $O(dt)$. Thus $X(t)$ is not differentiable.

The explicit representation of $X(t)$, given by (1.1.5), is often very useful. Yet, during more than twenty years, the author and other mathematicians did not have the idea of using it. We shall now apply it to the study of the Brownian plane curve."

Lévy's method has been well known recently, and used widely by many people, for example, [24] and so on. In particular, it is indispensable in the study of quadratic Wiener functionals, which plays an essential role in stochastic analysis. Moreover, as for the Malliavin calculus, another important research field in stochastic analysis, P. Malliavin has taken the advantage of the Itô-Nisio theorem in many situations.

As N. Wiener wrote in his autobiography [43, pp.37-39], when the study on Brownian motion, his first major mathematical work, started, he was strongly influenced by the book of J. Perrin [35]. Perrin noted similarities between on one hand the non-differentiable function, which was constructed by K. Weierstrass in 1872 with the use of sums of trigonometric functions, and on the other hand the irregular movement of physical Brownian motion. The result by Weierstrass was refined by G. Hardy in 1916 ([9]). Having close exchanges with Hardy and Littlewood, Wiener must have had a detailed knowledge of these preceding results. He began to study the Brownian motion in early 1920. There is a mention that makes one guess that, from the very beginning, he had in his mind an idea of using Fourier series expansions. Actually he tackled straightforwardly the problems of Fourier series expansion in the paper of year 1924, and in the joint paper with Paley [34], he discussed systematically about the construction of Wiener measure along this line. See also [28].

As was mentioned before, the studies of the Fourier series expansion of Brownian motion bore as a fruit Lévy's Haar function expansion, and then came up to the Itô-Nisio theorem, the complete generalization. On one hand, if one approaches the study of diffusion processes with heat kernels, then the Fourier series for regular functions play a key role. On the other hand, studying Brownian motion from the point of view of path behaviour (particle movement), then the Itô-Nisio theorem and the Fourier series of irregular functions play a fundamental role. In this paper, we shall see several more concrete examples where Itô-Nisio's Fourier series expansions are indispensable, and we shall give some applications of them.

As shown in his study attitude of deepening the understanding of the Wiener measure by constructing the paths of Brownian motion, Itô had a tendency to dwell on essentials of problems. This can be also seen as he developed the understanding of Kolmogorov's work [25] on diffusion processes by stochastic differential equations. Apart from stochastic topics, we would like to exhibit more episodes of Itô's interests in fundamental subjects. Itô, who returned again and again to logical structures of research subjects, was interested in extensive range of mathematics, not only probability theory. For example, in the late 1940's, he was fascinated by Gödel's theory, and his review [18] of the Japanese translation of Gödel's book received attention of Japanese mathematicians who were interested in Gödel's work. In 1960's, through the lecture by Martin Löf at Aarhus University, he was also very much interested in the concept of a random sequence by Kolmogorov.

All of Itô's mathematical works, including the Itô-Nisio theorem, consistently gave motivations and directions for our researches. Not only in his lectures, but also Itô was willing to tell young mathematicians privately what was behind his research topics. By such private conversations, we learned a lot of things and enriched our understandings over many topics. It is a great honor for us to write a paper for this Tribute to Professor Kiyosi Itô, and on this occasion we would like to express our deepest respect and gratitude to him for all he taught us directly and indirectly.

2 A generalization of Lévy's stochastic area formula

Let W be the space of all continuous functions w on $[0,\infty)$ taking values in $\mathbb{R}^2$ with $w(0)=0$, and let P stand for the Wiener measure on W. For $x \in \mathbb{R}^2$, we set

$$w_x(s) = x + w(s), \quad s \in [0,\infty),\ w \in W.$$

To avoid any confusion, under P, we continue to write $w(s)$ and do not use $w_0(s)$. Take the differential 1-form $\theta = (1/2)(x^1dx^2 - x^2dx^1)$, $x=(x^1,x^2) \in \mathbb{R}^2$, on $\mathbb{R}^2$. Its exterior derivative $d\theta$ is the area element $dx^1 \wedge dx^2$. We define $S(t,w_x)$ to be the stochastic line integral of θ along the curve $[0,t] \ni s \mapsto w_x(s) \in \mathbb{R}^2$;

$$S(t,w_x) = \int_{w_x[0,t]} \theta.$$

For stochastic line integrals, see [12,15]. $S(t,w)$, i.e. $S(t,w_x)$ with $x=0$, is called *Lévy's stochastic area*. It holds:

$$S(t,w_x)=\frac{1}{2}\int_0^t \langle Jw_x(s), dw_x(s)\rangle,$$

where $J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^2$;

$$\langle x,y\rangle = x^1y^1+x^2y^2,\ x=(x^1,x^2),\ y=(y^1,y^2)\in\mathbb{R}^2.$$

From this expression with Itô integral, we see that the stochastic area is a quadratic Wiener functional, i.e., an element of the homogeneous chaos of order 2 of Wiener-Itô, which plays a key role in stochastic analysis. For details, see the second proof of Proposition 2.

While Lévy defined the stochastic area in his own manner in [27], it is now standard to use Itô's stochastic integral to define the area. The discontinuity of $S(t,w)$ in w requires special care in defining the stochastic area ([15,29]).

Let $\alpha,\beta\in\mathbb{R}$, and define

$$\begin{aligned}&p(t,x,y;\alpha,\beta)\\ &=\int_W \exp\Big(\sqrt{-1}\,\alpha S(t,w_x)-\frac{\beta^2}{2}\int_0^t |w_x(s)|^2ds\Big)\delta_y(w_x(t))P(dw),\end{aligned}\tag{1}$$

where $\delta_y(w_x(t))$ stands for Watanabe's pull-back of the Dirac measure δ_y concentrating at $y\in\mathbb{R}^2$ through $w_x(t)$. For the pull-back, see [15]. Then $p(t,x,y;\alpha,\beta)$ is the fundamental solution to the partial differential equation

$$\frac{\partial u}{\partial t}=\Big(\mathcal{L}_\alpha-\frac{\beta^2}{2}|x|^2\Big)u,$$

where

$$\mathcal{L}_\alpha=\frac{1}{2}\Delta+\frac{\alpha\sqrt{-1}}{2}\Big(x^1\frac{\partial}{\partial x^2}-x^2\frac{\partial}{\partial x^1}\Big)-\frac{\alpha^2}{8}|x|^2,\tag{2}$$

Δ being the Laplacian. See [15]. The heat equation was deeply studied by Gaveau [7].

It follows from the result by H. Matsumoto ([31]) that

Proposition 2 *Let $\alpha, \beta \in \mathbb{R}$, and set $m_1 = (\alpha^2 + 4\beta^2)^{1/2}$. Then it holds that*

$$\int_W \exp\Big(\sqrt{-1}\,\alpha S(t,w) - \frac{\beta^2}{2}\int_0^t |w(s)|^2 ds\Big)\delta_y(w(t))P(dw)$$
$$= \frac{1}{2\pi t}\frac{m_1 t/2}{\sinh(m_1 t/2)}\exp\Big(-\frac{1}{2t}\frac{m_1 t/2}{\tanh(m_1 t/2)}|y|^2\Big), \quad y \in \mathbb{R}^2. \tag{3}$$

Matsumoto's result is based on the Van Vleck formula, the formula established in [11], which is a Wiener integral counterpart to the application in the Feynman path integral theory of the result due to Van Vleck [39] on fundamental solutions.

PROOF. The assertion is a special case of the observation made by Matsumoto in [31, pp.172-173]. We shall give the proof after briefly revisiting his result.

Let $k_1, k_2 > 0$ and $B \in \mathbb{R}$. Denote by $q(t,a,b)$, $t \geq 0, a, b \in \mathbb{R}^2$, the heat kernel associated with the differential operator

$$-\Big\{\frac{1}{2}\sum_{j=1}^{2}\Big(\sqrt{-1}\frac{\partial}{\partial x^j} - B\theta_j\Big)^2 + \frac{1}{2}\{k_1^2(x^1)^2 + k_2^2(x^2)^2\}\Big\}$$
$$= \mathcal{L}_B - \frac{1}{2}\{k_1^2(x^1)^2 + k_2^2(x^2)^2\},$$

where $\theta_1 = x_2/2$ and $\theta_2 = -x_1/2$, and $\mathcal{L}_B$ is the operator defined in (2). Setting

$$m_1 = \{(k_1+k_2)^2 + B^2\}, \quad m_2 = \{(k_1-k_2)^2 + B^2\},$$

$$s_1 = \frac{m_1+m_2}{2}, \quad s_2 = \frac{-m_1+m_2}{2},$$

$$K(t) = 2k_1k_2B^2(\cosh(s_1t)\cosh(s_2t) - 1)$$
$$-\{B^2(k_1^2+k_2^2) + (k_1-k_2)^2\}\sinh(s_1t)\sinh(s_2t),$$

$$\alpha_i(t) = s_1(s_2^2 - k_i^2)\cosh(s_1t)\sinh(s_2t)$$
$$-s_2(s_1^2 - k_i^2)\sinh(s_1t)\cosh(s_2t),$$

$$\beta_i(t) = s_2(s_1^2 - k_i^2)\sinh(s_1 t) - s_1(s_2^2 - k_i^2)\sinh(s_2 t), \quad i = 1, 2,$$

$$\begin{aligned}\gamma(t) = &\ 2k_1k_2\{\cosh(s_1 t)\cosh(s_2 t) - 1\} \\ &+(m_1^2 - 2k_1k_2)\sinh(s_1 t)\sinh(s_2 t),\end{aligned}$$

$$\begin{aligned}\widetilde{S}_{cl}(t,a,b) = &\ \frac{m_1m_2}{K(t)}\alpha_1(t)\{(a^1)^2 + (b_1)^2\} + \frac{m_1m_2}{K(t)}\beta_1(t)a^1b^1 \\ &+\frac{m_1m_2}{K(t)}\alpha_2(t)\{(a^2)^2 + (b^2)^2\} + \frac{m_1m_2}{K(t)}\beta_2(t)a^2b^2 \\ &+\frac{\sqrt{-1}\,B(k_1^2 - k_2^2)}{2K(t)}\gamma(t)(a^1a^2 - b^1b^2) \\ &-\frac{\sqrt{-1}\,k_1k_2m_1m_2B}{K(t)}\{\cosh(s_1 t) - \cosh(s_2 t)\}(a^1b^2 - a^2b^1),\end{aligned}$$

Matsumoto [31] showed that

$$q(t,a,b) = \frac{1}{2\pi}\left(\frac{k_1k_2m_1^2m_2^2}{K(t)}\right)^{1/2}\exp(-\widetilde{S}_{cl}(t,a,b)). \tag{4}$$

In our situation, $B = \alpha$, $k_1 = k_2 = \beta$, $a = 0$, and $b = y$. Hence $m_1 = (\alpha^2 + 4\beta^2)^{1/2}$ and $m_2 = |\alpha|$. Using the identities: $\cosh x \cosh y - \sinh x \sinh y = \cosh(x-y)$ and $\cosh x - 1 = 2\sinh^2(x/2)$, we see that

$$K(t) = 4\alpha^2\beta^2\sinh^2(m_1 t/2),$$

which implies:

$$\frac{k_1k_2m_1^2m_2^2}{K(t)} = \frac{(m_1/2)^2}{\sinh^2(m_1 t/2)}.$$

Since

$$s_1(s_2^2 - \beta^2) = s_2(s_1^2 - \beta^2) = -\beta^2|\alpha|$$

and $\sinh x \cosh y - \cosh x \sinh y = \sinh(x-y)$, it holds that

$$\alpha_1(t) = \alpha_2(t) = |\alpha|\beta^2\sinh(m_1 t).$$

Then

$$\widetilde{S}_{cl}(t,0,y) = \frac{m_1m_2}{2K(t)}\alpha_1(t)|y|^2 = \frac{1}{2t}\frac{m_1t/2}{\tanh(m_1t/2)}|y|^2.$$

Plugging these into (4), we obtain the desired identity. $\square$

Remark 3 *The identity in the case when $\alpha = 0$ can be shown as an application of the Feynman-Kac formula. If $\beta = 0$ and $y = 0$, then the identity is well-known Lévy's formula ([28]).*

Remark 4 *Observe that*

$$\alpha \int_0^t w^1(s)dw^2(s) + \beta \int_0^t w^2(s)dw^1(s) = \frac{\alpha - \beta}{2} S(t, w) + \frac{\alpha + \beta}{2} w^1(t)w^2(t).$$

This identity was used by M. Yor [44] to study the joint distribution

$$\Big(\int_0^1 w^1(s)dw^2(s), \int_0^1 w^2(s)dw^1(s) \Big).$$

Using this expression, we obtain a variant of Matsumoto's result to the operator

$$\Big(\sqrt{-1}\frac{\partial}{\partial x^1} + \frac{B_1 x^2}{2}\Big)^2 + \Big(\sqrt{-1}\frac{\partial}{\partial x^2} + \frac{B_2 x^1}{2}\Big)^2 + \frac{1}{2}\{k_1^2(x^1)^2 + k_2^2(x^2)^2\}$$

with $B_1, B_2 \in \mathbb{R}$.

In the remainder of this section, we shall give an alternative proof of Proposition 2 with $\beta = 0$ by using the general Fourier series expansion of Brownian motion due to Itô-Nisio. The method was first used by Lévy [28] in the case when $y = 0$. The following observation was essentially made by Ikeda-Watanabe [15, pp.476-478], and we are aiming to make clear which specified trigonometric functions are involved in this computation.

PROOF. (An alternative proof of Proposition 2 for $\beta = 0$)

Suppose $\beta = 0$. Due to the scaling property of Brownian motion, it suffices to show (3) for $t = 1$. Hence we work on the space $\{w|_{[0,1]} | w \in W\}$ of restrictions of elements in W on $[0, 1]$. For the sake of simplicity, we use the same letter W to indicate the space of restrictions. Let H be the corresponding Cameron-Martin space. Then:

$$L^2(W; P) = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$$

denotes the decomposition of $L^2(W;P)$, the Hilbert space of square integrable variables with respect to P, in terms of the homogeneous chaos of Wiener-Itô ([19]). A *quadratic Wiener functional* is an element of $\mathcal{C}_2$. By the result of Itô on multiple Wiener integrals ([19]), each quadratic Wiener functional $F \in \mathcal{C}_2$ admits a kernel representation;

$$F(w) = \sum_{i,j=1}^{2} \left\{ \int_0^1 \int_0^t F^{ij}(t,s) dw^j(s) dw^i(t) + \int_0^1 \int_0^s F^{ij}(t,s) dw^i(t) dw^j(s) \right\},$$

where F^{ij} is square integrable on $[0,1]^2$ with respect to the Lebesgue measure and $F^{ij}(t,s) = F^{ji}(s,t)$, $t, s \in [0,1]$, $i, j = 1, 2$. Setting the 2×2 matrix $F(t,s)$ to be $(F^{ij}(t,s))_{1 \le i,j \le 2}$, we define a symmetric Hilbert-Schmidt operator B of H into H by

$$(Bh)'(t) = \int_0^1 F(t,s) h'(s) ds, \quad h \in H,$$

where h' denotes the derivative of h. Such a correspondence between quadratic Wiener functionals and symmetric Hilbert-Schmidt operators of H into H is bijective ([13]).

It is easily seen that $S(1,w)$ is a quadratic Wiener functional. The corresponding kernel $F(t,s) = (F^{ij}(t,s))_{1 \le i,j \le 2}$ is given by

$$F^{11}(t,s) = F^{22}(t,s) = 0,$$
$$F^{12}(t,s) = F^{21}(s,t) = \begin{cases} \frac{1}{4}, & 0 \le t \le s \le 1, \\ \\ -\frac{1}{4}, & 0 \le s < t \le 1. \end{cases}$$

Moreover, if we set

$$\mathcal{I}[h] = \begin{pmatrix} \mathcal{I}[h^1] \\ \mathcal{I}[h^2] \end{pmatrix}, \quad \mathcal{I}[h^i](s) = \int_0^s h^i(u) du, \quad h \in H,$$

then the Hilbert-Schmidt operator B associated with $S(1,w)$ is represented as

$$B = B_V + B_F, \tag{5}$$

where

$$(B_V h)(s) = \frac{1}{2}\mathcal{I}[Jh](s), \quad (B_F h)(s) = -\frac{1}{4}(Jh)(1)s, \quad s \in [0,1].$$

Let $H_0 = \{h \in H \,|\, h(1) = 0\}$ and set $B^{\#} = \pi_0 B \pi_0$, π_0 being the orthogonal projection of H onto H_0. Then the above decomposition (5) implies that

$$(B^{\#}h)(s) = \frac{1}{2}\mathcal{I}[Jh](s) - \frac{1}{2}\mathcal{I}[Jh](1)s, \quad s \in [0,1].$$

It is then easily checked that $B^{\#}$ admits the eigenfunction expansion:

$$B^{\#} = \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{2}\frac{1}{2n\pi}\{k_n \otimes k_n + \widetilde{k}_n \otimes \widetilde{k}_n\},$$

where $k \otimes k : H \to H$ is defined by $(k \otimes k)(h) = \langle k, h\rangle_H k$, $h \in H$, and

$$k_n(s) = \frac{1}{2n\pi}\begin{pmatrix} \cos(2n\pi s) - 1 \\ \sin(2n\pi s) \end{pmatrix}, \ s \in [0,1], \quad \widetilde{k}_n = J k_n.$$

See [13,32]. This leads us to the orthonormal basis $\{\phi_n^1, \psi_{n+1}^1, \phi_n^2, \psi_{n+1}^2\}_{n=0}^{\infty}$ of H defined by

$$\phi_0^1(s) = \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad \phi_0^2(s) = \begin{pmatrix} 0 \\ s \end{pmatrix},$$

$$\phi_n^1(s) = \frac{\sqrt{2}}{2n\pi}\begin{pmatrix} \cos(2n\pi s) - 1 \\ 0 \end{pmatrix}, \quad \phi_n^2(s) = \frac{\sqrt{2}}{2n\pi}\begin{pmatrix} 0 \\ \cos(2n\pi s) - 1 \end{pmatrix},$$

$$\psi_n^1(s) = \frac{\sqrt{2}}{2n\pi}\begin{pmatrix} \sin(2n\pi s) \\ 0 \end{pmatrix}, \quad \psi_n^2(s) = \frac{\sqrt{2}}{2n\pi}\begin{pmatrix} 0 \\ \sin(2n\pi s) \end{pmatrix}.$$

Applying the Itô-Nisio theorem, we obtain the expansion:

$$w(s) = \sum_{i=1}^{2}\Big\{\sum_{n=0}^{\infty} \xi_n^{(i)}(w)\phi_n^i(s) + \sum_{m=1}^{\infty} \eta_m^{(i)}(w)\psi_m^i(s)\Big\}$$

where

$$\xi_n^{(i)}(w)=\int_0^1\langle(\phi_n^i)'(s),dw(s)\rangle,\quad \eta_m^{(i)}(w)=\int_0^1\langle(\psi_m^i)'(s),dw(s)\rangle.$$

Since $X_0^i(w)=w^i(1)$, in terms of components, this can be rewritten as

$$w^i(s)=w^i(1)\phi_0^{i,i}+\sum_{n=1}^\infty\xi_n^{(i)}(w)\phi_n^{i,i}(s)+\sum_{n=1}^\infty\eta_n^{(i)}(w)\psi_n^{i,i}(s),\quad i=1,2,$$

where $\phi_n^i=\begin{pmatrix}\phi_n^{i,1}\\ \phi_n^{i,2}\end{pmatrix}$ and $\psi_n^i=\begin{pmatrix}\psi_n^{i,1}\\ \psi_n^{i,2}\end{pmatrix}$. It is easily seen that $\{w^i(1),\xi_n^{(i)},\eta_n^{(i)};i=1,2,n=1,2,\dots\}$ are independent random variables, each of which obeys $N(0,1)$. These expansions are exactly the same as the ones used in [15, pp.476-477]. In particular, we have that, P-a.s.,

$$S(1,w)=\sum_{n=1}^\infty\frac{1}{2\pi n}\Big\{(\eta_n^{(1)}(w)-\sqrt{2}\,w^1(1))\xi_n^{(2)}-(\eta_n^{(2)}(w)-\sqrt{2}\,w^2(1))\xi_n^{(1)}\Big\}.$$

As was seen by Ikeda-Watanabe, it holds that

$$\begin{aligned}&E[e^{\sqrt{-1}\alpha S(1)}|w(1)=y]\\&=\prod_{n=1}^\infty\Big(1+\frac{\alpha^2}{(2n\pi)^2}\Big)^{-1}\exp\Big(-\sum_{n=1}^\infty\frac{(\alpha/2\pi n)^2}{1+(\alpha/2\pi n)^2}|y|^2\Big),\end{aligned}\tag{6}$$

where $E[\cdot|w(1)=y]$ stands for the conditional expectation given $w(1)=y$. In conjunction with the well-known formulas

$$\sinh x=x\prod_{n=1}^\infty\Big(1+\frac{x^2}{\pi^2n^2}\Big),\quad \coth x=\frac{1}{x}+2x\sum_{n=1}^\infty\frac{1}{x^2+\pi^2n^2}\tag{7}$$

and the identity

$$\int_W e^{\sqrt{-1}\alpha S(1,w)}\delta_y(w(1))P(dw)=\frac{1}{2\pi}e^{-|y|^2/2}E[e^{\sqrt{-1}\alpha S(1)}|w(1)=y]$$

we arrive at the desired identity. [2] $\square$

[2] In [15, p.478], the multiplication by $1/2$ is missing when the infinite series expression of cot is applied.

Remark 5 *The first product in (6) can be rewritten as*

$$\prod_{n=1}^{\infty}\left(1+\frac{\alpha^2}{(2n\pi)^2}\right)^{-1}=\{\det_2(I-2\sqrt{-1}\,\alpha B^{\#})\}^{-1/2}.$$

Remark 6 *It is well known ([13,28,32]) that the Hilbert-Schmidt operator B is decomposed as*

$$B=\sum_{n\in\mathbb{Z}}\frac{1}{2}\,\frac{1}{(2n+1)\pi}\{h_n^B\otimes h_n^B+\tilde{h}_n^B\otimes\tilde{h}_n^B\},$$

where

$$h_n^B(s)=\frac{1}{(2n+1)\pi}\begin{pmatrix}\cos((2n+1)\pi t)-1,\\ \sin((2n+1)\pi t)\end{pmatrix},\quad \tilde{h}_n^B=Jh_n^B.$$

In particular, B has the eigenvalues $1/\{2(2n\pi)\pi\}$, $n\in\mathbb{Z}$, the multiplicity of each being two, and:

$$\int_W e^{\sqrt{-1}\,\alpha S(1)}dP=\frac{1}{\cosh(\alpha/2)}=\{\det_2(I-2\sqrt{-1}\alpha B)\}^{-1/2}.$$

Remark 7 *The formulas in (7) are the well known Euler formulas. The latter proof relies upon these formulas. Conversely, the Euler formulas can be shown by combining Proposition 2 and the identity (6).*

In [24, Sections 2 and 7], M. Kac also gave two such directions to approach Lévy's formula in the case when $\alpha=0$, i.e., the case of harmonic oscillator. The importance of proving Lévy's formula from two directions is discussed there. Two such types of computation are widely known in the theory of Feynman path integrals. See for example [6, Problems 2-2, 3-8, pp.71-73].

3 Lévy's stochastic area formula for Ornstein-Uhlenbeck processes

In [14,38], the authors gave a probabilistic approach to reflectionless potentials, soliton solutions and the τ-function to the KdV equation. Their observation is based on quadratic Wiener functionals obtained as the square norms on time interval for Ornstein-Uhlenbeck processes. In this section, we investigate stochastic areas determined by Ornstein-Uhlenbeck processes, and establish a similar result to the one in [14]. In the case of the Brownian motion, i.e., the Ornstein-Uhlenbeck process with $p=0$, the result is well known ([8,28]). We continue to work on the Wiener space (W,P) of 2-dimensional Brownian motion. To apply the KdV equation, real numbers x, y, z instead of s, t, u are used to indicate time parameters.

For $p \in \mathbb{R}$, define the 2-dimensional Ornstein-Uhlenbeck process $\{\xi^p(z)\}_{z\geq 0}$ to be the solution to the stochastic differential equation

$$d\xi^p(y) = dw(y) + p\xi^p(y)dy, \quad \xi^p(0) = 0,$$

that is, if we represent: $\xi^p(y) = (\xi^{p,1}(y), \xi^{p,2}(y))$ and $w(y) = (w^1(y), w^2(y))$, then

$$d\xi^{p,i}(y) = dw^i(y) + p\xi^{p,i}(y)dy, \quad \xi^{p,i}(0) = 0, \quad i = 1, 2.$$

In the sequel, let $x > 0$. By the Maruyama-Girsanov theorem [30,15], the process $\left\{w(z) - \int_0^z pw(y)dy\right\}_{z\in[0,x]}$ is a 2-dimensional Brownian motion under the probability measure $\exp(p\int_0^x \langle w(y), dw(y)\rangle - (p^2/2)\int_0^x |w(y)|^2 dy)\, dP$. Since $\int_0^x \langle w(y), dw(y)\rangle = \frac{1}{2}|w(x)|^2 - x$, we have:

$$\begin{aligned}
&\int_W \Phi(\{\xi^p(z)\}_{z\in[0,x]})dP \\
&= \int_W \Phi(\{w(z)\}_{z\in[0,x]}) \exp\left(\frac{p}{2}|w(x)|^2 - \frac{p^2}{2}\int_0^x |w(y)|^2 dy\right) P(dw) e^{-px}
\end{aligned}$$

for every $\Phi \in C_b(W_x)$, where $W_x = \{w|_{[0,x]} \,|\, w \in W\}$. In particular, for $\Psi \in C_b(W_x)$ such that $\Psi(\{\xi^p(z)\}_{z\in[0,x]})$ is smooth in the sense of the Malliavin calculus, it also holds that, for any $b \in \mathbb{R}^2$,

$$\begin{aligned}
&\int_W \Psi(\{\xi^p(z)\}_{z\in[0,x]})\delta_b(\xi^p(x))dP \\
&= \int_W \Psi(\{w(z)\}_{z\in[0,x]}) \exp\left(-\frac{p^2}{2}\int_0^x |w(y)|^2 dy\right)\delta_b(w(x))P(dw)e^{p\{(|b|^2/2)-x\}}.
\end{aligned}$$

Thus we arrive at

Proposition 8 *For $b \in \mathbb{R}^2$, $\alpha, C \in \mathbb{R}$, it holds that*

$$\begin{aligned}
&\int_W \exp\left(\frac{\sqrt{-1}\,\alpha}{2}\int_0^x \langle J\xi^p(y), d\xi^p(y)\rangle - C|\xi^p(x)|^2\right)\delta_b(\xi^p(x))dP \\
&= \int_W \exp\left(\sqrt{-1}\,\alpha S(x,w) - \frac{p^2}{2}\int_0^x |w(y)|^2 dy\right)\delta_b(w(x))P(dw)
\end{aligned}$$

$$\times \exp\biggl(\Bigl\{\frac{p}{2} - C\Bigr\}|b|^2 - px\biggr). \tag{8}$$

It should be mentioned that the above expression is meaningful for any C, while the integrand $\exp(\cdots)$ on the left hand side is not smooth in the sense of the Malliavin calculus for $C < 0$ with large absolute value. Namely, by the localization via $\delta_b(\xi^p(x))$, one may replace $|\xi^p(x)|^2$ by $\psi(|\xi^p(x)|^2)$ with smooth $\psi : \mathbb{R} \to \mathbb{R}$ which is compactly supported and equal to the identity function on the interval $[0, 2|b|^2)$. After this replacement, one can take the pairing with Watanabe's pull-back.

In conjunction with Proposition 2, this proposition implies that

Theorem 9 *Let $\alpha, C \in \mathbb{R}$ and set $m_1 = (\alpha^2 + 4p^2)^{1/2}$. Then it holds that*

$$\int_W \exp\biggl(\frac{\sqrt{-1}\,\alpha}{2}\int_0^x \langle J\xi^p(y), d\xi^p(y)\rangle - C|\xi^p(y)|^2\biggr)\delta_b(\xi^p(y))dP$$
$$= \frac{1}{2\pi x}\frac{m_1 x/2}{\sinh(m_1 x/2)}\exp\biggl(-\biggl\{\frac{1}{2x}\frac{m_1 x/2}{\tanh(m_1 x/2)} + \Bigl(C - \frac{p}{2}\Bigr)\biggr\}|b|^2 - px\biggr). \tag{9}$$

Moreover, if $C \geq p/2$, then it holds that

$$\int_W \exp\biggl(\frac{\sqrt{-1}\,\alpha}{2}\int_0^x \langle J\xi^p(y), d\xi^p(y)\rangle - C|\xi^p(x)|^2\biggr)dP$$
$$= \frac{m_1 e^{-px}}{m_1\cosh(m_1 x/2) + (4C - 2p)\sinh(m_1 x/2)}. \tag{10}$$

PROOF. The identity (9) follows immediately from Propositions 2 and 8.

If $C \geq p/2$, then

$$\frac{1}{2x}\frac{m_1 x/2}{\tanh(m_1 x/2)} + \Bigl(C - \frac{p}{2}\Bigr) > 0.$$

Integrating (9) in b over $\mathbb{R}^2$, we come to the identity:

$$\int_W \exp\biggl(\frac{\sqrt{-1}\,\alpha}{2}\int_0^x \langle J\xi^p(y), d\xi^p(y)\rangle - C|\xi^p_x|^2\biggr)dP$$
$$= \frac{m_1/2}{\sinh(m_1 x/2)}\biggl\{2\biggl(\frac{1}{2x}\frac{m_1 x/2}{\tanh(m_1 x/2)} + \Bigl(C - \frac{p}{2}\Bigr)\biggr)\biggr\}^{-1} e^{-px},$$

which implies (10). $\square$

We first apply Theorem 9 to reflectionless potentials. For $\eta, m > 0$, the reflectionless potential with scattering data (η, m) is the function on $\mathbb{R}$ of the form

$$-2(d/dx)^2 \log\left(1 + \frac{m}{2\eta}\, e^{-2\eta x}\right).$$

See [33]. We have that

Corollary 10 *We continue to use the same notation as in Theorem 9. Let $C \in [p/2, (p/2) + (m_1/4))$, and put*

$$Q(x) = \log\left(\int_W \exp\left(\frac{\sqrt{-1}\,\alpha}{2}\int_0^x \langle J\xi^p(y), d\xi^p(y)\rangle - C|\xi^p(x)|^2\right) dP\right).$$

Then the function $q = 2(d/dx)^2 Q$ is the reflectionless potential with scattering data

$$\left(\frac{m_1}{2}, \frac{m_1(m_1 - 4C + 2p)}{m_1 + 4C - 2p}\right).$$

If $p \le 0$, then we can take $C = 0$ in the above equation and

$$2(d/dx)^2 \log\left(\int_W \exp\left(\frac{\sqrt{-1}\,\alpha}{2}\int_0^x \langle J\xi^p(y), d\xi^p(y)\rangle\right) dP\right)$$

is a reflectionless potential.

PROOF. Define $\gamma \ge 0$ so that

$$m_1 \tanh\gamma = 4C - 2p, \quad \text{i.e., } e^{-2\gamma} = \frac{m_1 - 4C + 2p}{m_1 + 4C - 2p}. \tag{11}$$

Since $\cosh(t+s) = \cosh t \cosh s + \sinh t \sinh s$, we have:

$$\begin{aligned} &m_1 \cosh(m_1 x/2) + (4C - 2p)\sinh(m_1 x/2) \\ &\quad = \frac{m_1}{\cosh\gamma}\frac{e^{(m_1 x/2)+\gamma}}{2}(1 + e^{-2\gamma}e^{-m_1 x}). \end{aligned}$$

By Theorem 9, it then holds:

$$q(x) = -2(d/dx)^2 \log(1 + e^{-2\gamma}\, e^{-m_1 x}).$$

Thus q is the reflectionless potential with scattering data $(m_1/2, m_1 e^{-2\gamma})$. □

As an application of the corollary, we consider a stochastic representation of 1-soliton solutions to the KdV equation. After deforming the Brownian motion into Ornstein-Uhlenbeck processes with parameter p, we shall obtain a 1-parameter family of soliton solutions of the KdV equation which varies according to the initial condition.

To see this, for $p \in \mathbb{R}, t \geq 0$, we set

$$V^p(x,t) = \log\Bigg(\int_W \exp\Bigg(\frac{\sqrt{-1}\,\alpha}{2}\int_0^x \langle J\xi^p(y), d\xi^p(y)\rangle - \left\{\frac{p}{2} + \frac{m_1}{4}\tanh(m_1^3 t/8)\right\}|\xi^p(x)|^2\Bigg)dP\Bigg)$$

and

$$v^p(x,t) = 2(\partial/\partial x)^2 V(x,t).$$

The 1-parameter family $\{v^p; p \in \mathbb{R}\}$ satisfies:

Corollary 11 $\{v^p; p \in \mathbb{R}\}$ *is a* 1*-parameter family of* 1*-soliton solutions* v^p *to the KdV equation*

$$\frac{\partial u}{\partial t} = \frac{3}{2}\,u\,\frac{\partial u}{\partial x} + \frac{1}{4}\,\frac{\partial^3 u}{\partial x^3} \quad \text{with } u(x,0) = -\frac{(\alpha^2+4p^2)^{1/2}}{2\cosh^2((\alpha^2+4p^2)^{1/2}x)}.$$

PROOF. Due to Corollary 10 and (11), for each $t \geq 0$, $v^p(\cdot, t)$ is the reflectionless potential with scattering data $(m_1/2, m_1 e^{-2(m_1/2)^3 t})$. Then it is well known that v^p is a 1-soliton solution (cf. [33]). The initial value $v^p(x,0)$ is easily computed. □

4 Eulerian polynomials and stochastic area

Recently F. Hirzebruch [10] and A. Cohen [2] began to study several types of Eulerian polynomials from a new point of view. In [28], Lévy pointed out that the stochastic area is related to Euler and Bernoulli numbers. In this section, we shall see that the stochastic area is also related to Eulerian polynomials introduced by Euler [4] (also see [2,10]).

Define the Eulerian polynomials $P_n(\xi)$, $n = 0, 1, \ldots$, by

$$\sum_{k=0}^{\infty}(k+1)^n \xi^k = \frac{P_n(\xi)}{(1-\xi)^{n+1}}, \quad |\xi| < 1.$$

For the convenience of the reader, let us recall that

$$\frac{1}{1-\xi} = \sum_{k=0}^{\infty} \xi^k, \quad |\xi| < 1,$$

so that $P_0(\xi) = 1$. Differentiating both sides of the previous identity, we get:

$$P_1(\xi) = 1.$$

Further differentiating both sides of the identity, thus defining P_n, by induction, we see that P_n is of degree $n-1$ (cf. [10]).

From the definition, we easily obtain the exponential generating function for the Eulerian polynomials ([10]):

$$\sum_{n=0}^{\infty} P_n(\xi)\frac{\lambda^n}{n!} = \frac{(1-\xi)e^{(1-\xi)\lambda}}{1-\xi e^{(1-\xi)\lambda}} \quad \text{for } \lambda \in \mathbb{R} \text{ with } |\xi| e^{(1-\xi)\lambda} < 1. \tag{12}$$

We shall show that

Proposition 12 *For $-1 < \xi \le 0$ and $\lambda \in \mathbb{R}$ with $|\xi| e^{(1-\xi)\lambda} < 1$ and $|\lambda| < 1$, it holds:*

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(\xi)\frac{\lambda^n}{n!} = &\int_W e^{\sqrt{-1}\,(1-\xi)\lambda S(1,w) + (1-\xi)\lambda |w(1)|^2/4} P(dw) \\ &\times \int_W e^{\sqrt{-1}\,(1-\xi)\lambda S(1,w) + (1+\xi)\lambda |w(1)|^2/4} P(dw).\end{aligned} \tag{13}$$

The above product may be gathered into one integration with respect to 4-dimensional Brownian motion. Before proceeding to the proof, we see that exact representations of Eulerian polynomials follow from (13). Namely, the identity (13) yields:

$$\begin{aligned} P_n(\xi) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \int_W \left\{ \sqrt{-1}\,(1-\xi)S(1,w) + \frac{(1-\xi)|w(1)|^2}{4} \right\}^k P(dw) \\ \times \int_W \left\{ \sqrt{-1}\,(1-\xi)S(1,w) + \frac{(1+\xi)|w(1)|^2}{4} \right\}^{n-k} P(dw). \qquad (14) \end{aligned}$$

For example, we can compute $P_1(\xi)$, $P_2(\xi)$, and $P_3(\xi)$ as follows. Let

$$C_{j,k} = \int_W S(1,w)^j |w(1)|^{2k} P(dw).$$

Applying Itô's formula, we see:

$$C_{1,0} = C_{1,1} = C_{1,2} = C_{3,0} = 0, \qquad (15)$$

$$C_{0,k} = 2^k k!, \quad k = 1,2,3, \quad C_{2,0} = \frac{1}{4}, \quad C_{2,1} = \frac{5}{6}. \qquad (16)$$

Then it follows from (14) and (15) that

$$\begin{aligned} P_1(\xi) &= \frac{1}{2} C_{0,1}, \\ P_2(\xi) &= \left\{ -2C_{2,0} + \frac{C_{0,2}}{2^3} + \frac{C_{0,1}^2}{2^3} \right\} + 4C_{2,0}\xi + \left\{ -2C_{2,0} + \frac{C_{0,2}}{2^3} - \frac{C_{0,1}^2}{2^3} \right\} \xi^2, \\ P_3(\xi) &= \left\{ -\frac{3}{2} C_{2,1} + \frac{1}{2^5} C_{0,3} - \frac{3}{2} C_{0,1} C_{2,0} + \frac{3}{2^5} C_{0,1} C_{0,2} \right\} \\ &\quad + \{3C_{2,1} + 3C_{0,1}C_{2,0}\}\xi \\ &\quad + \left\{ -\frac{3}{2} C_{2,1} + \frac{3}{2^5} C_{0,3} - \frac{3}{2} C_{0,1} C_{2,0} - \frac{3}{2^5} C_{0,1} C_{0,2} \right\} \xi^2. \end{aligned}$$

Substituting (16) into these, we arrive at the well known expressions ([10]):

$$P_1(\xi) = 1, \quad P_2(\xi) = 1 + \xi, \quad P_3(\xi) = 1 + 4\xi + \xi^2.$$

For the proof of Proposition 12, we prepare a lemma.

Lemma 13 *For $a \in \mathbb{R}$ and $\beta \in \mathbb{R}$ with $|a| \le 1/2$ and $-a\beta < 1$, it holds:*

$$\int_W e^{\sqrt{-1}\beta\{S(1,w) + \sqrt{-1}a|w(1)|^2/2\}} P(dw) = \left\{ \left(\frac{1}{2} + a \right) e^{\beta/2} + \left(\frac{1}{2} - a \right) e^{-\beta/2} \right\}^{-1}.$$

PROOF. Since $-a\beta < 1$, $e^{\sqrt{-1}\beta\{S(1,w)+\sqrt{-1}a|w(1)|^2/2\}}$ is smooth in the sense of the Malliavin calculus. Then we have:

$$\int_W e^{\sqrt{-1}\beta\{S(1,w)+\sqrt{-1}a|w(1)|^2/2\}} P(dw)$$
$$= \int_{\mathbb{R}^2} dy \int_W e^{\sqrt{-1}\beta\{S(1,w)+\sqrt{-1}a|w(1)|^2/2\}} \delta_y(w(1)) P(dw).$$

By Proposition 2, this is equal to

$$\frac{1}{2\pi} \frac{\beta/2}{\sinh(\beta/2)} \int_{\mathbb{R}^2} dy \exp\left(-\frac{\beta/2}{\sinh(\beta/2)} \{\cosh(\beta/2) + 2a\sinh(\beta/2)\} \frac{|y|^2}{2} \right).$$

Notice that

$$\cosh(\beta/2) + 2a\sinh(\beta/2) = \left(\frac{1}{2} + a\right) e^{\beta/2} + \left(\frac{1}{2} - a\right) e^{-\beta/2} > 0.$$

Then, by an elementary change of variables, we obtain the desired identity. □

We now proceed to the:

PROOF of Proposition 12. Suppose that $-1 < \xi \leq 0$, $|\xi| e^{(1-\xi)\lambda} < 1$, and $|\lambda| < 1$. Observe that

$$\sum_{n=0}^{\infty} P_n(\xi) \frac{\lambda^n}{n!} = e^{(1-\xi)\lambda/2} \left\{ \frac{1}{1-\xi} e^{-(1-\xi)\lambda/2} - \frac{\xi}{1-\xi} e^{(1-\xi)\lambda/2} \right\}^{-1}. \tag{17}$$

Applying Lemma 13 with $a = -1/2$ and $\beta = (1-\xi)\lambda$, we have:

$$\int_W e^{\sqrt{-1}\,(1-\xi)\lambda S(1,w) + (1-\xi)\lambda|w(1)|^2/4} P(dw) = e^{(1-\xi)\lambda/2}. \tag{18}$$

If $a = -(1+\xi)/\{2(1-\xi)\}$ and $\beta = (1-\xi)\lambda$, then $|a| \leq 1/2$ and $-a\beta < 1$. Applying Lemma 13 again, since $(1/2) + a = -\xi/(1-\xi)$ and $(1/2) - a = 1/(1-\xi)$, we obtain:

$$\int_W e^{\sqrt{-1}\,(1-\xi)\lambda S(1,w) + (1+\xi)\lambda|w(1)|^2/4} P(dw)$$
$$= \left\{ \frac{1}{1-\xi} e^{-(1-\xi)\lambda/2} - \frac{\xi}{1-\xi} e^{(1-\xi)\lambda/2} \right\}^{-1}.$$

Plugging this and (18) into (17), we obtain the desired identity. □

Let $P(B_n;\xi)$ be the Eulerian polynomial of type B_n, where we have borrowed the notation from Cohen [2]. Then it holds ([2,10]):

$$\sum_{k=0}^{\infty}(2k+1)^n\xi^k = \frac{P(B_n;\xi)}{(1-\xi)^{n+1}}.$$

A similar result as above can be shown for $P(B_n;\xi)$, $n=0,1,\dots$ Namely, the exponential generating function satisfies:

$$\sum_{n=0}^{\infty} P(B_n;\xi)\frac{\lambda^n}{n!} = \frac{(1-\xi)e^{(1-\xi)\lambda}}{1-\xi e^{2(1-\xi)\lambda}} \quad \text{for } |\xi|e^{2(1-\xi)\lambda} < 1.$$

It follows from this expression that

$$P(B_n;-1) = 2^n E_n,$$

where the Euler number E_n is the n-th derivative of $1/\cosh x$ at $x=0$ ([10]). Suppose that $-1<\xi\le 0$, $|\xi|e^{2(1-\xi)\lambda}<1$, and $|\lambda|<1$. Applying Lemma 13 with $a=-(1+\xi)/\{2(1-\xi)\}$ and $\beta = 2(1-\xi)\lambda$, we obtain:

$$\sum_{n=0}^{\infty} P(B_n;\xi)\frac{\lambda^n}{n!} = \int_W e^{2\sqrt{-1}\,(1-\xi)\lambda S(1,w)+(1+\xi)\lambda|w(1)|^2/2}P(dw),$$

and

$$P(B_n;\xi) = \int_W \left\{2\sqrt{-1}\,(1-\xi)S(1,w) + \frac{(1+\xi)|w(1)|^2}{2}\right\}^n P(dw).$$

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