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A Preliminary Investigation into Dynamics of Lotka–Volterra Equation

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This study was conducted to investigate dynamics of Lotka–Volterra equation by introducing the second derivative for change acceleration in the number of predators and prey. Lotka–Volterra equation gives a simple model to the analysis of predator–prey relationships. The results obtained were as follows. Introducing the second derivative into Lotka–Volterra equation gave the square of change rate of the number of predators and that of prey. Since the square of change rate gave positive value, the following two inequalities were given. (1) Change acceleration of the number of prey per prey was higher than the product of –β and change rate of the number of predators. (2) Change acceleration of the number of predators per predator was higher than the product of δ and change rate of the number of prey. These two might be related to the change in the number of predators and prey. It was suggested from the present study that investigating dynamics of Lotka–Volterra equation gave inequalities showing some change aspects of predator–prey relationships.

INTRODUCTION

Animals require feeds in order to live and grow. There is a predator–prey relationship in the world of wild animals (Iwasa, 2008), and the life of predators depends greatly on the presence of prey. The presence of a large number of prey increases the number of predators and this leads to the decrease in the number of prey, resulting in the decrease in the number of predators. This is followed by the increase in the number of prey again, if there is a large decrease in the number of predators. Therefore, there are oscillations in the number of predators and prey (Iwasa, 2008). It is known that Lotka–Volterra equation gives a simple model to the analysis of these oscillations in predator–prey relationships (Iwasa, 2008). It is of interest to study dynamics of predator–prey relationships, and this may be related to change acceleration in the number of them.

The present study was designed to investigate dynamics of Lotka–Volterra equation by introducing the second derivative for change acceleration in the number of predators and prey.

DYNAMICS OF LOTOKA–VOLTERRA EQUATION SUGGESTED BY CHANGE ACCELERATION IN THE NUMBER OF PREDATORS AND PREY

(A) Introducing the second derivative into Lotka–Volterra equation

Lotka–Volterra equation is given by two differential equations as follows (Iwasa, 2008),

$$\frac{dx}{dt} = \alpha \cdot x - \beta \cdot x \cdot y, \qquad (1)$$

$$\frac{dy}{dt} = -\gamma \cdot y + \delta \cdot y \cdot x. \qquad (2)$$

where x = the number of prey, y = the number of predators, α = parameter related to increase ($\alpha > 0$), β = parameter related to decrease ($\beta > 0$), γ = parameter related to decrease ($\gamma > 0$), δ = parameter related to increase ($\delta > 0$).
Equation (1) describes the change rate in the number of prey, and equation (2) describes that of predator.

The differentiation of equation (1), namely the second derivative, gives

$$\begin{aligned}\frac{d^2x}{dt^2} &= \alpha \cdot \frac{dx}{dt} - \beta \cdot \frac{dx}{dt} \cdot y - \beta \cdot x \cdot \frac{dy}{dt} \\ &= \frac{dx}{dt} \cdot \frac{(\alpha \cdot x - \beta \cdot x \cdot y)}{x} - \beta \cdot x \cdot \frac{dy}{dt} \\ &= \frac{1}{x} \cdot \left(\frac{dx}{dt}\right)^2 - \beta \cdot x \cdot \frac{dy}{dt}. \qquad (3)\end{aligned}$$

Thus,

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$$\left(\frac{dx}{dt}\right)^2 = x \cdot \frac{d^2x}{dt^2} + \beta \cdot x^2 \cdot \frac{dy}{dt}\ . \tag{4}$$

Likewise, the differentiation of equation (2) gives

$$\frac{d^2y}{dt^2} = -\gamma \cdot \frac{dy}{dt} + \delta \cdot \frac{dy}{dt} \cdot x + \delta \cdot y \cdot \frac{dx}{dt}$$

$$= \frac{dy}{dt} \cdot \frac{(-\gamma \cdot y + \delta \cdot y \cdot x)}{y} + \delta \cdot y \cdot \frac{dx}{dt}$$

$$= \frac{1}{y} \cdot \left(\frac{dy}{dt}\right)^2 + \delta \cdot y \cdot \frac{dx}{dt}\ . \tag{5}$$

Thus,

$$\left(\frac{dy}{dt}\right)^2 = y \cdot \frac{d^2y}{dt^2} - \delta \cdot y^2 \cdot \frac{dx}{dt}\ . \tag{6}$$

The relationship between the left– hand side and the first term in the right–hand side in each of equations (4) and (6) comes from the same relationship in each of equations (1) and (2) for Lotka–Volterra equation. This is caused by the property of exponential functions with base *e*. The same phenomenon is also observed in basic growth mechanics (Shimojo, 2007; Shimojo *et al.*, 2006, 2007a, 2007b, 2008, 2009a, 2009b, 2009c, 2009d). Thus,

$$W = W_0 \cdot \exp(r \cdot t), \tag{7}$$

where W = weight, t = time, W_0 = W at t = 0, r = relative growth rate,
and then introducing the first and second derivatives into basic growth function (7) gives

$$\left(\frac{dW}{dt}\right)^2 = W \cdot \frac{d^2W}{dt^2}\ . \tag{8}$$

Shimojo (2007) and Shimojo *et al.* (2006, 2007b, 2008, 2009d) suggest that the form of equation (8) shows an analogy with that of Newton's equation of motion that is given as follows (Kawabe, 2006),

$$\frac{dp}{dt} = m \cdot \frac{d^2r}{dt^2}\ , \tag{9}$$

where p = momentum, m = mass of an object, r = position, t = time.
Equations (4) and (6) also include the part showing an analogy with equation (9). However, the presence of nonlinear term in equations (4) and (6), namely the second term of the right–hand side, shows that Lotka–Volterra equation takes more complex form.

(B) Inequalities derived from equations (4) and (6)

The inequality (4–3) is derived from equation (4) according to the following procedures,

$$\left(\frac{dx}{dt}\right)^2 = x \cdot \frac{d^2x}{dt^2} + \beta \cdot x^2 \cdot \frac{dy}{dt}\ . \tag{4}$$

$$\left(\frac{dx}{dt}\right)^2 > 0, \tag{4–1}$$

$$x \cdot \frac{d^2x}{dt^2} + \beta \cdot x^2 \cdot \frac{dy}{dt} > 0, \tag{4–2}$$

therefore,

$$\frac{1}{x} \cdot \frac{d^2x}{dt^2} > -\beta \cdot \frac{dy}{dt}\ . \tag{4–3}$$

Inequality (4–3) shows that change acceleration of the number of prey per prey is higher than the product of –β and change rate of the number of predators.

Likewise, the inequality (6–3) is derived from equation (6),

$$\left(\frac{dy}{dt}\right)^2 = y \cdot \frac{d^2y}{dt^2} - \delta \cdot y^2 \cdot \frac{dx}{dt}\ . \tag{6}$$

$$\left(\frac{dy}{dt}\right)^2 > 0, \tag{6–1}$$

$$y \cdot \frac{d^2y}{dt^2} - \delta \cdot y^2 \cdot \frac{dx}{dt} > 0, \tag{6–2}$$

therefore,

$$\frac{1}{y} \cdot \frac{d^2y}{dt^2} > \delta \cdot \frac{dx}{dt}\ . \tag{6–3}$$

Inequality (6–3) shows that change acceleration of the number of predators per predator is higher than the product of δ and change rate of the number of prey.

(C) Inequalities in Lotka–Volterra equation

The outline of what Loka–Volterra equation shows is as follows. (1) If there are not predators, then the prey shows an exponential increase. (2) If predators are present, then the prey shows a decrease in number. (3) If there is a decrease in the number of prey, then predators show a decrease in number. (4) If the number of predators decreases greatly, then prey shows an increase. (5) If the prey is not present, then predators show an exponential decrease.

The present preliminary investigation into inequalities that are derived from dynamics of Loka–Volterra equation shows the following two. (6) Change acceleration of the number of prey per prey is higher than the product of $-\beta$ and change rate of the number of predators. (7) Change acceleration of the number of predators per predator is higher than the product of δ and change rate of the number of prey. Phenomena (6) and (7) suggested in the present study may be related to phenomena (1) ~ (5).

(D) Conclusions

It is suggested from the present study that investigating dynamics of Lotka–Volterra equation gives inequalities showing some change aspects of predator–prey relationships.

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