Finite element analysis of viscoelastic flow problems, with application incorporating the Oldroyd-B model

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出版情報：MI lecture note series. 67, pp.54-55, 2016-02-05. 九州大学マス・フォア・インダストリ研究所
バージョン：
権利関係：

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Viscoelastic flow is an important physical phenomena both from both theoretical and practical points of view. Blood flows in the human body and plastic flows during injection molding are typical examples of viscoelastic flow; for more examples and details, see Larson [2], Öttinger [4], and their references. Therefore, we encounter many research articles on the numerical analysis and simulation of viscoelastic flows; see, for example, Bonito–Clement–Picasso [1], Owens–Phillips [5], and their references.

In order to enhance the mathematical justification of numerical methods for viscoelastic flows, we consider the Oldroyd-B model of viscoelastic flow: Let Ω ($\subset \mathbb{R}^d, d = 2, 3$) be a bounded domain with the $\mathscr{C}^2$-class boundary Γ. Let T (> 0) be a time, $I := (0, T)$ a time interval. Then, find the velocity $u : \Omega \times I \to \mathbb{R}^d$, the pressure $p : \Omega \times I \to \mathbb{R}$, and the non-Newtonian stress $\sigma_{\mathrm{P}} : \Omega \times I \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that

$$
\begin{aligned}
&\rho\big(\partial_t u + (u\cdot\nabla)u\big) - 2\eta_s \nabla\cdot D(u) + \nabla p - \nabla\cdot\sigma_{\mathrm{P}} = f, && \text{in } \Omega\times I,\\
&\nabla\cdot u = 0, && \text{in } \Omega\times I,\\
&\sigma_{\mathrm{P}} + \lambda\big(\partial_t\sigma_{\mathrm{P}} + (u\cdot\nabla)\sigma_{\mathrm{P}} - (\nabla u)\sigma_{\mathrm{P}} - \sigma_{\mathrm{P}}(\nabla u)^T\big) - 2\eta_p D(u) = 0, && \text{in } \Omega\times I,\\
&u = 0 && \text{on } \Gamma\times I,\\
&u = u^0, \qquad \sigma_{\mathrm{P}} = {\sigma_{\mathrm{P}}}^0 && \text{in } \Omega, \ \text{ at } t = 0,
\end{aligned}
$$

where $f : \Omega \times I \to \mathbb{R}^d$ denotes an external force, $u^0 : \Omega \to \mathbb{R}^d$ an initial velocity, ${\sigma_{\mathrm{P}}}^0 : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ an initial non-Newtonian stress; ρ (> 0) the density, η_s the solvent viscosity, η_p (> 0) the polymer viscosity, and λ (> 0) the relaxation time.

First, we establish error estimates of a pressure-stablized finite element method for a three-field Stokes model, which neglects the nonlinear components appearing in the Oldroyd-B model. Under standard assumptions for the finite element analysis of flow problems, we obtain error estimates of finite element solutions (u_h, p_h, $\sigma_{\mathrm{P}h}$) corresponding to (u, p, σ_{P}) as follows: there exists a positive constant c such that

$$
\begin{aligned}
&\|u - u_h\|_{\ell^\infty((L^2(\Omega))^d)}, \quad \|u - u_h\|_{\ell^2((H^1(\Omega))^d)} \le c\,(\Delta t + h),\\
&\|\sigma_{\mathrm{P}} - \sigma_{\mathrm{P}h}\|_{\ell^\infty((L^2(\Omega))^{d\times d})} \le c\,(\Delta t + h),\\
&\|p - p_h\|_{\ell^\infty(L^2(\Omega))} \le c\,\Delta t^{-1/2}(\Delta t + h).
\end{aligned}
$$

Here, Δt and h denote discretized parameters in time and in space, respectively; for the total step number in time N_T, a Banach space X, and an X-valued sequence $v = \{v^{(k)} \in X;\ k = 0, 1, \ldots, N_T\}$, $\|\,.\,\|_{\ell^\infty(X)}$ and $\|\,.\,\|_{\ell^2(X)}$ denote

$$
\begin{aligned}
\|v\|_{\ell^\infty(X)} &:= \max\big\{\|v^{(k)}\|_X;\ k = 0, 1, \ldots, N_T\big\},\\
\|v\|_{\ell^2(X)} &:= \left(\Delta t \sum_{k=0}^{N_T} \|v^{(k)}\|_X^2\right)^2,
\end{aligned}
$$

respectively.

Second, we consider an approximation of the upper-convected derivative along the motion of fluid particles to avoid difficulties of high Deborah number problems.

Finally, we show some numerical examples of contraction flows, and compare numerical results with experimental ones. Figs. 1–2 show some examples of the comparisons; the stream lines of the 4 to 1 contraction flow among the Stokes and the Oldroyd-B models.

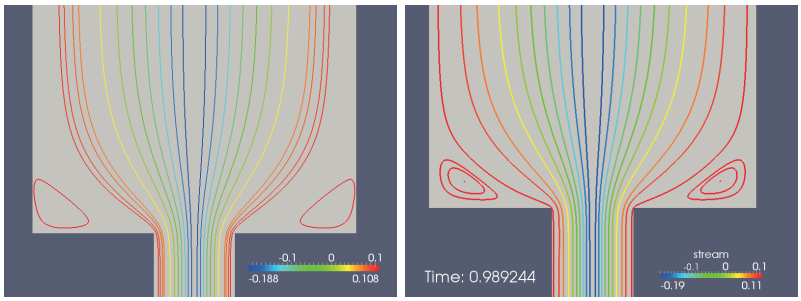

FIG. 1. Stream lines of the 4 to 1 contraction flow (Left: the Stokes model; Right: the Oldroyd-B model)

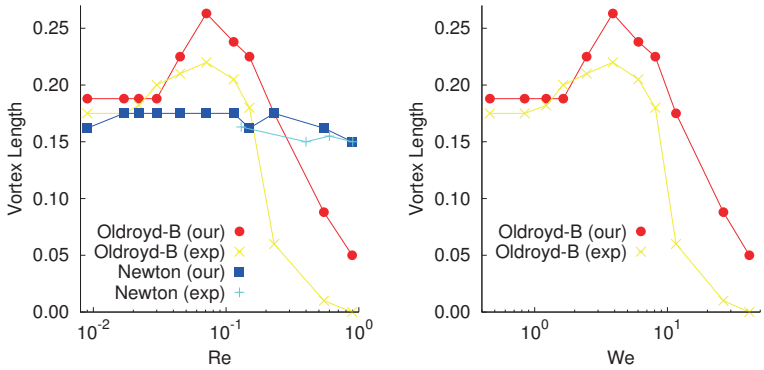

FIG. 2. Comparison of vortex length of the 4 to 1 contraction flow (Left: with respect to the Reynolds number; Right: with respect to the Weissenberg number)

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