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Topological idea combined with asymptotic expansions for vortex motion

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1. Introduction

Vortex rings are ubiquitous coherent structures in high-Reynolds-number flows, and are of fundamental importance in fluid mechanics. Vortex rings are used for producing thrust and lift by insects, fishes and animals. Vortex rings are capable of transporting neutrally buoyant materials. Recently they find their utility for creating virtual reality in the field of entertainment. There is an attempt to use an air cannon, as a means of olfactory display, to deliver smells encapsulated in a vortex ring to a targeted person. In a theater, virtual reality contents are created solely by image and sound. Reality is enhanced if we appeal to tactile display. A mini-theater is planned in which air cannons are designed to produce vortex rings, in synchronization with the image and the sound, so that the audience experiences direct impact and freshness. These applications to entertainment necessitate controlled vortex rings, and raise questions pertaining to an inverse problem. When does a vortex ring arrive at a specified point? How far does the ring travel? How large the vortex ring has grown at the moment of impact? This talk addresses these questions.

In 1858, the field of vortex dynamics started with a single piece of paper written by Helmholtz. In his seminal paper [1], Helmholtz proved a distinguishing feature of the vorticity that vortex lines are frozen into the fluid. In the same paper, he studied motion of vortex rings. By an elaboration from the Euler equations, now being widely known through Lamb's textbook [2], Helmholtz had reached an identity for traveling speed of a thin axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent. Helmholtz-Lamb's method is recapitulated in a recent article [3].

By a deep insight into the formation of a columnar vortex along the central line of a rotating tank filled with water, Kelvin [4] envisioned that a columnar vortex should be a state of the maximum of the kinetic energy, with respect to perturbations that maintain the circulation. An almost century passed before Kelvin's variational principle was mathematically formulated and proved. Arnol'd [5] proved that a steady solution of the Euler equations of an incompressible fluid is an extremal of the total kinetic energy with respect to the kinematically accessible perturbations. We mean by *kinematically accessible perturbations* the perturbation flow field for which the perturbed vorticity is frozen into the perturbed flow field. The kinematically accessible perturbations may be alternatively said to be the *isovortical perturbations* or occasionally the *rearrangements*. Mathematical proof for steady isolated vortex as the maximum-energy states uses rearrangement inequalities (see, for example, [6]).

Kelvin's variational principle can be extended to make allowance for motion by adding a constraint of constant impulse [7, 8, 9]; a stationary configuration of vorticity in an inviscid incompressible fluid, in a steadily moving frame, is realizable as an extremal of energy on an isovortical sheet under the constraint of constant impulse.

Our main concern lies in the variational principle for motion of vortex rings. Kelvin-Benjamin's variational principle is adapted to find the traveling speed of steady vortices [9, 10]. This variational principle is applied to motion of vortex rings.

Finite-thickness effect of vortex tubes is a common problem in the dynamics of interacting vortices in two dimensions, and has been intensively studied so far. Compared with the axisymmetric problem, less is known about interaction of (anti-) parallel vortex tubes. For motion of a curved vertex tube, the correction of curvature origin appears at first order in a small parameter, the ratio of the core- to the curvature radii, but, for a planer problem, the finite-size effect of the core makes its appearance at a high order in the ratio of the core radius to the vortex distance. The last section is concerned with motion of a counter-rotating vortex pair [18].

2. Kelvin-Benjamin's variational principle

We assume that the fluid is incompressible, and take the density of fluid to be $\rho_{\mathrm{f}} = 1$. In addition, we assume that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is localized in some finite region in such a way that the velocity decreases sufficiently rapidly. Define the total kinetic energy H and the hydrodynamic impulse $\boldsymbol{P}$, of the fluid filling an unbounded space, by

$$H = \frac{1}{2} \iiint \boldsymbol{u}^2 \mathrm{d}V, \quad \boldsymbol{P} = \frac{1}{2} \iiint \boldsymbol{x} \times \boldsymbol{\omega} \mathrm{d}V. \tag{1}$$

We confine ourselves to steady motion, with constant speed $\boldsymbol{U}$, of a region with vorticity and assume that the flow is stationary in a frame moving with $\boldsymbol{U}$. It is expedient to partition the velocity $\boldsymbol{u}$ as $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{U}$. By the assumption that the relative velocity $\bar{\boldsymbol{u}}$ is steady, it obeys

$$\nabla \times (\bar{\boldsymbol{u}} \times \boldsymbol{\omega}) = \boldsymbol{0}. \tag{2}$$

Suppose that fluid particles undergo an infinitesimal displacement $\delta\boldsymbol{\xi}$ while preserving the volume of an arbitrary fluid element:

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} = \boldsymbol{x} + \delta\boldsymbol{\xi}(\boldsymbol{x})\,; \quad \nabla \cdot \delta\boldsymbol{\xi} = 0. \tag{3}$$

We impose the condition that the flux of vorticity through an arbitrary material surface be unchanged throughout the process of the displacement. Its local representation is [8]

$$\delta\boldsymbol{\omega} = \nabla \times (\delta\boldsymbol{\xi} \times \boldsymbol{\omega})\,. \tag{4}$$

The translation velocity $\boldsymbol{U}$ of a vortex ring is then calculable through the variation

$$\delta H - \boldsymbol{U} \cdot \delta\boldsymbol{P} = 0, \tag{5}$$

under the constraint that, for any smooth Lagrangian displacement of fluid particles, the vorticity is frozen into the fluid. Section 2 touches upon this principle, which is the theme of ref. [9]. Intriguingly, the same principle encompasses motion of a vortex ring ruled by the cubic nonlinear Schrödinger equation, which serves as a model for superfluid liquid helium and a Bose-Einstein condensate, at zero temperature [11].

In the sequel, we restrict this theorem to motion of a steadily moving axisymmetric vortex ring. An isovortical sheet is of infinite dimension. A family of solutions of the Euler equations includes a few parameters. By imposing certain relations among these parameters, we can maintain the solutions on a single isovortical sheet, and the restricted family of the solutions constitutes a finite dimensional set on the sheet.

3. High-Reynolds-number vortex ring

The inner solution for steady motion of a vortex ring, or quasi-steady motion in the presence of viscosity, is found by solving the Euler or the Navier–Stokes equations, subject to the matching condition, in powers of the small parameter ε, the ratio of the core- to the ring-radii [15]. To work out the inner solution, we introduce the relative velocity $\tilde{\boldsymbol{u}}$ in the meridional plane by $\boldsymbol{u} = \tilde{\boldsymbol{u}} + (\dot{R}, \dot{Z})$. Here a dot stands for differentiation with respect to time. Let us non-dimensionalize the inner variables. We introduce, in the core cross-section, local polar coordinates (r, θ) around the core center. The radial coordinate is normalized by the core radius $\varepsilon R_0 (= \sigma)$ and the local velocity (u, v), relative to the moving frame, by the maximum velocity $\Gamma/(\varepsilon R_0)$. The normalization parameter for the ring speed $(\dot{R}(t), \dot{Z}(t))$, the slow dynamics, should be Γ/R_0. The suitable dimensionless inner variables are thus defined as

$$(6)\quad r^* = r/\varepsilon R_0, \quad t^* = t/\frac{R_0}{\Gamma}, \quad \psi^* = \frac{\psi}{\Gamma R_0}, \quad \zeta^* = \zeta/\frac{\Gamma}{R_0^2\varepsilon^2}, \quad \tilde{\boldsymbol{u}}^* = \tilde{\boldsymbol{u}}/\frac{\Gamma}{R_0\varepsilon}, \quad \dot{Z}^* = \dot{Z}/\frac{\Gamma}{R_0}.$$

The difference in normalization between the last two of (6) should be kept in mind. Correspondingly to (6), the kinetic energy H and the hydrodynamic impulse $\boldsymbol{P}$ are normalized as $H^* = H/\Gamma^2 R_0$, $P_z^* = P_z/\Gamma R_0^2$. Hereinafter we drop the superscript * for dimensionless variables. Dimensionless form of the radial position R of the core center is $R = 1 + \varepsilon^2 R^{(2)} + O(\varepsilon^3)$. We can maintain the first term to be unity by adjusting disposable parameters, bearing with the origin of coordinates, in the first-order field [15]. The second-order correction $\varepsilon^2 R^{(2)}$ is tied with the viscous expansion.

A glance at the Euler or the Navier–Stokes equations shows that the dependence, on θ, of the solution in a power series in ε is

$$(7)\qquad \psi = \psi^{(0)}(r) + \varepsilon\psi_{11}^{(1)}(r)\cos\theta + \varepsilon^2\left[\psi_0^{(2)}(r) + \psi_{21}^{(2)}(r)\cos 2\theta\right] + O(\varepsilon^3),$$

$$(8)\qquad \zeta = \zeta^{(0)}(r) + \varepsilon\zeta_{11}^{(1)}(r)\cos\theta + \varepsilon^2\left[\zeta_0^{(2)}(r) + \zeta_{21}^{(2)}(r)\cos 2\theta\right] + O(\varepsilon^3).$$

Upon substitution from (7) and (8), we obtain a representation, to $O(\varepsilon^2)$ in dimensionless form, $H = H^{(0)} + \varepsilon^2 H^{(2)}$ as

$$(9)\quad H^{(0)} = -2\pi^2\int_0^\infty r\zeta^{(0)}\psi^{(0)}\mathrm{d}r, \quad H^{(2)} = -2\pi^2\int_0^\infty r\left(\frac{1}{2}\zeta_{11}^{(1)}\psi_{11}^{(1)} + \zeta^{(0)}\psi_0^{(2)} + \zeta_0^{(2)}\psi^{(0)}\right)\mathrm{d}r.$$

The leading-order term $H^{(0)}$ of energy is evaluated with ease as

$$(10)\qquad H_0/\Gamma^2 = \frac{1}{2}R_0\left\{\log\left(\frac{8R_0}{\sigma}\right) + A - 2\right\},$$

where $H_0 = \Gamma^2 R_0 H^{(0)}$ and A is given by

$$(11)\qquad A = \lim_{r\to\infty}\left\{\frac{4\pi^2}{\Gamma^2}\int_0^r r' v_0(r')^2\mathrm{d}r' - \log\left(\frac{r}{\sigma}\right)\right\}.$$

The variation of (10) with respect to an isovortical perturbation is manipulated as

$$(12)\qquad \delta H_0 = \frac{\Gamma^2}{2}\left[\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{1}{2}\right]\delta R_0.$$

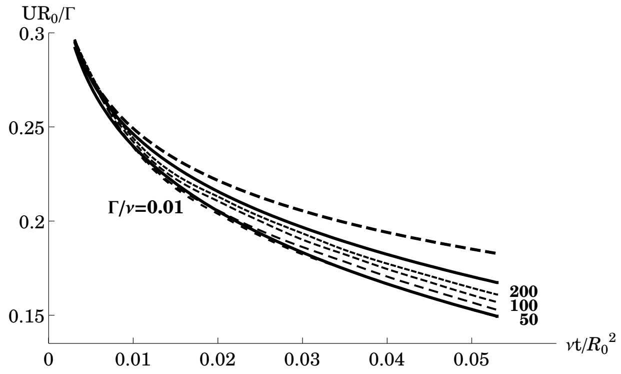

FIGURE 1. Variation of speed of a viscous vortex ring with time.

The variation of the leading term of impulse $P_0 = \Gamma\pi R_0^2$ is $\delta P_0 = 2\pi\Gamma R_0 \delta R_0$, and application of (5) retrieves Fraenkel–Saffman's formula [12, 14]:

$$U_0 = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{1}{2}\right\}. \tag{13}$$

The third-order correction U_2 to the translation speed of the vortex ring requires evaluation of $H^{(2)}$. For an inviscid vortex ring in steady motion, $R_2 = R_0\varepsilon^2 R^{(2)} \equiv 0$ without loss of generality, and, after some manipulations, we arrive at

$$U_2 = \frac{1}{R_0^3}\left\{\frac{d_1}{2}\left[\log\left(\frac{8R_0}{\sigma}\right) - 2\right] - \pi\Gamma B + \frac{\pi}{2\Gamma}\int_0^\infty r^4\zeta_0 v_0 \mathrm{d}r\right\}, \tag{14}$$

where $v_0 = \Gamma v^{(0)}/\sigma$, $\zeta_0 = \Gamma\zeta^{(0)}/\sigma^2$ and d_1 is the dipole strength, and

$$B = \lim_{r\to\infty}\left\{\frac{1}{\Gamma^2}\int_0^r r' v_0 \tilde{\psi}_{11}^{(1)}\mathrm{d}r' + \frac{r^2}{16\pi^2}\left[\log\left(\frac{r}{\sigma}\right) + A\right] + \frac{d_1}{2\pi\Gamma}\log\left(\frac{r}{\sigma}\right)\right\}. \tag{15}$$

This is an extension, to $O(\varepsilon^3)$, of Fraenkel–Saffman's formula (13).

Even if viscosity is switched on, the higher-order asymptotics U_2 is not invalidated at a large Reynolds number. Taking, as the initial condition, a circular line vortex of radius R_0, the leading-order vorticity ζ_0 is given by

$$\zeta_0 = \frac{\Gamma}{4\pi\nu t}e^{-r^2/4\nu t}, \tag{16}$$

where ν is the kinematic viscosity and t is the time measured from the instant at which the core is infinitely thin [12, 16], and the inhomogeneous heat-conduction equation governing $\zeta_0^{(2)}$ becomes tractable, with an introduction of similarity variables. we are eventually led to an extension of Saffman's formula (13) in the form

$$U \approx \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{4R_0}{\sqrt{\nu t}}\right) - 0.55796576 - 3.6715912\frac{\nu t}{R_0^2}\right\}. \tag{17}$$

Figure 1 displays the comparison of the asymptotic formula (17) with a direct numerical simulation of the axisymmetric Navier–Stokes equations [13]. The normalized speed UR_0/Γ of the ring is drawn as a function of normalized time $\nu t/R_0^2$ for its small values. The upper thick solid line is our formula (17), and the thick broken line is the

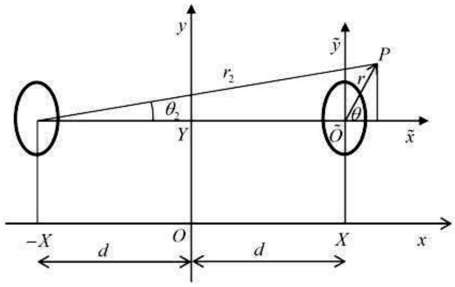

FIGURE 2. The Cartesian coordinates system (x, y) fixed in space and the polar coordinates system (r, θ) centered on (X, Y) in moving frame.

first-order truncation (13). The dashed lines are the results of the numerical simulations, attached with the circulation Reynolds number Γ/ν, ranging from 0.01 to 200. Augmented only with a single correction term, (14) appears to furnish a close upper bound on the translation speed. The lowermost solid line is the low-Reynolds-number limit obtained in a different manner [17]. Notably, the large-Reynolds-number asymptotic formula (17) compares fairly well with the numerical result of even moderate and small Reynolds numbers.

4. High-Reynolds-number motion of a vortex pair

The rest of this paper is concerned with motion of a counter-rotating vortex pair at very low Reynolds numbers [18].

4.1. **Inner and outer expansions.** Consider a counter-rotating vortex pair with circulations $\pm\Gamma$ moving in an inviscid fluid or a viscous fluid with the kinematic viscosity ν. The core radius σ of the two vortices is assumed to be much smaller than the distance $2d$ between the centroids of the two vortices. The outer solution is provided by the Biot–Savart law, though the distribution of vorticity remains to be calculated. The latter is found by the solution to the inner problem. The behavior of the Biot–Savart law valid near one of the vortex provides the matching condition on the inner solution.

4.2. **Inner solution and traveling speed of a vortex pair.** The inner solution is obtained by integrating the Navier–Stokes equation. We introduce the Cartesian coordinates (x, y), fixed in space, with the x axis parallel to the direction of the line connecting the centroids. At the same time, we introduce local polar coordinates (r, θ), centered at the centroid (X, Y) of one of the vortices, moving with it. The angle is measured from the direction parallel to the x-axis, and therefore the laboratory and the moving frames are viewed with each other through $x = X + r\cos\theta$ and $y = Y + r\sin\theta$ (figure 2). The radial coordinate r is non-dimensionalized by εd where $\varepsilon = \sigma/d = \sqrt{\nu/\Gamma}$ is a small parameter. The solution for the streamfunction ψ is sought in a power series in ε as

$$\psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \varepsilon^3\psi^{(3)} + \varepsilon^4\psi^{(4)} + \varepsilon^5\psi^{(5)} + \dots. \tag{18}$$

The Navier–Stokes equation dictates that $\psi^{(2)} = \psi_{21}^{(2)}(r, t)\cos 2\theta$ and $\psi^{(3)} = \psi_{31}^{(3)}(r, t)\cos 3\theta$. The matching condition at $O(\varepsilon^5)$ yields the correction, of $O(\varepsilon^4)$, to the traveling speed

$\dot{Y}$,

(19) $$\dot{Y}^{(4)} = \frac{\pi}{2}\int_0^\infty \frac{\partial a}{\partial r}\psi_{21}^{(2)}\psi_{31}^{(3)} r\mathrm{d}r - \frac{q_2}{4}, \quad \text{where } a = -\frac{1}{v^{(0)}}\frac{\partial \zeta^{(0)}}{\partial r};\ \zeta^{(0)} = \frac{1}{r}\frac{\partial}{\partial r}\left(rv^{(0)}\right).$$

Here $v^{(0)}$ and $\zeta^{(0)}$ are respectively the local azimuthal velocity and the axial vorticity at $O(\varepsilon^0)$, the leading order, and q_2 is the strength of quadrupole of $O(\varepsilon^2)$.

Moreover, we notice that two terms in $\dot{Y}^{(4)}$ gives the same contribution, and as a consequence, $\dot{Y}^{(4)} = -q_2/2$. The eventual formula for the translation speed of a vortex pair includes the strength of the second-order quadrupole field only and is expressed, in terms of the dimensional variables,

(20) $$\dot{Y} \approx -\frac{\Gamma}{4\pi d}\left(1 + \frac{2\pi}{\Gamma d^2}q\right); \quad q = \varepsilon^2 q_2; \quad \psi_{\text{right}} = -\frac{\Gamma}{2\pi}\log r + q\frac{\cos 2\theta}{r^2} + \cdots.$$

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