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A MID-POINT THEOREM FOR THE $\bigcup$ TYPE SHAPE OF FUNCTIONS

By

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Abstract

A mid-point theorem is proved in an elementary way for the $\bigcup$ type shape of functions that arise out of exponential quadratic functions. These results are inspired from epidemic patterns and growth over a time period.

Key Words and Phrases: Natural numbers mapping, Mean value theorem.

1. Background and motivation

Quadratic functions can generate variety of sizes of $\bigcup$ shaped and $\bigcap$ shaped functions. Such kind of shapes are generally seen, among other situations, in the growth and decay pattern of a typical epidemic situation over certain period of years. It is often seen, while studying the growth and decay of infections in a population over a period of time, scientists had fitted observed epidemic data using family of exponential or quadratic exponential functions. Quadratic exponential functions are not only helpful in fitting the observed data, but also often used for predicting the future course of the epidemic Pall (1932). These functions consists of parameters or constants which we estimate using the population data. In this paper, we are not concerned various statistical methods of estimation of parameters in the quadratic functions, but concerned in the mathematical properties of quadratic functions, in terms of, especially in relation to the positive integers. Some of these properties are derived while investigating behavior of epidemic over a season in a year, over a decade or more, Rao and Kakehashi (2005), Rao et al. (2006). Typical epidemic data consists of number of incidence or prevalence cases in a population over a regular or irregular time intervals. These observations within a given interval could either be constant or dynamic. Original work in the direction of investigating such functions and establishing a correspondence between natural numbers and sequence of quadratic exponential functions in an elementary approach was inspired by realistic situations in epidemiology, Rao (2003). We have extended these concepts to prove a mid-point theorem on $\bigcup$ shaped functions (see section 3). We can obtain lowest value of function under consideration between two peaks. Suppose an epidemic pattern follows a pattern $\bigcup$, then using this theorem we can time taken to reach lowest value of incidence or prevalence (depending upon the context) before disease numbers to start to grow. Further, one could try to rotate the $\bigcup$-shaped object in a three dimensional space and obtain the volume of such a vessel from the basic principles of Euclidean geometry. In this paper, we have considered $\bigcup$-shaped curves and functions which generate

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such curves in two dimensional space. See Figure 1. Let $f = \exp(ux^2 + vx + w)$ be a quadratic exponential function with (domain) $D(f) = \mathbb{N}$. For a given combination of integer parameters (say, $\mathcal{C}_1$, for first combination of numerical values u, v, w) of f we establish here that the mapping of $1 \in D(f)$ will be same as mapping of some integer $n \in D(f)$ (for the same combination of parameters). We have drawn several curves for a combination of parameters in f. If we change the combination of parameters (say, $\mathcal{C}_2$, for second combination of numerical values u, v, w), then the resultant mapping of 1 will be same as mapping of some $n_1 \in D(f)$. Here $n \neq n_1$ and $f(1)$ for $\mathcal{C}_1$ is not equal to $f(1)$ for $\mathcal{C}_2$ i.e. $f_{\mathcal{C}_1}(1) \neq f_{\mathcal{C}_2}(1)$. See Figure 2 for a general idea. We construct such quadratic exponential sequence of numbers and try to link them to the natural numbers. The readers will also see that for some $n_k \in D(f)$, the distance from 1 to n_k will be equal to $f(n_k)$ in certain conditions. We apply these facts to establish further interesting properties of convergence and derivative of f. Using the principles of mean value theorem we show that $f'(\theta) = 0$ for the mid-point $\theta \in (1, n_k)$. In addition to the application in epidemiology, these results will lead to methods to compute volumes of vessels that are of U shape in a three dimensional space.

Consider the quadratic function, $Q_1(x) = u_1x^2 + v_1x + w_1$ where $u_1 > 0$, $v_1, w_1 \in \mathbb{R}$. Suppose $u_1 = \frac{m_1}{2}$, $v_1 = \frac{n_1}{2}$ and $m_1(>0), n_1, c_1 \in \mathbb{N}$, $m_1 + n_1$ is even, then $Q_1(x)$ is an integer for an integer x, Pall (1932), Pall (1933). The versatile features of quadratic function when its coefficients are positive integers or real numbers have been popular in modeling natural sciences Ojha and Pandey (1989). When quadratic function is taken as an exponent to the irrational number e, then the resultant function is called quadratic exponential function. Functions from such family were widely established tools in modeling biological data Cox and Wermuth (1994), McCullagh (1994). These functions can even mimic properties of Gaussian probability functions McCullagh (1994). Suppose $f(x) = \exp\left(ux^2 + vx\right)$, $v = -mu^{2k-1}$, $u, m, k \in \mathbb{N}$, then, it was proved that $f(1) = f(|A|)$ for $A = \frac{u+v}{u}$, Rao (2003). In fact, this statement was also proved there for $k = 1$ and $u \in \mathbb{R} - \{0\}$, $m \in \mathbb{N}$, Rao (2003). We use these results and establish few interesting properties of such class of exponential function. By using Rolle's theorem we show that the derivative will be zero at the mid-point of the interval $[1, |A|]$.

Note that, $Q(1) = Q(|A|)$ when $Q(x) = ux^2 + vx$. Also note $|A| = 1 - mu^{2k-2}$ if $(1 \geq mu^{2k-2})$ or $|A| = mu^{2k-2} - 1$ if $(1 < mu^{2k-2})$. We show that the above absolute value function is necessary for deriving the main results of this paper. We begin with some simple results.

Observe that$|A|$ is always a positive integer for any combination of $\{k, m, u\} \in \mathbb{N}$. Since, $u \in \mathbb{N}$, then, $u^{2k-2} \in \mathbb{N}$ (because $m, u \in \mathbb{N}$). Therefore $|A| \in \mathbb{N}$.

Additionally, whenever $u = \left(\frac{n+1}{m}\right)^{\frac{1}{2k-2}}$, for $n \in \mathbb{N}$, then $u^{2k-2} = \frac{n+1}{m}$, which means $\left|\frac{u+v}{u}\right| = n$. For that reason $|A|$ could be equal to every natural number for a suitable combination of $\{k, m, u\}$, where $k, m, u \in \mathbb{N}$. For example, if we choose $\{k = 1, m = 2, u = 2\}$, then, we obtain $|A| = 1$, if $\{k = 1, m = 3, u = 2\}$, then $|A| = 2$, if $\{k = 1, m = 4, u = 2\}$, then $|A| = 3$, if $\{k = 3, m = 10, u = 8\}$, then $|A| = 40959$. We know that $f(1) = f(|A|)$. Therefore, we modify the previous result and state that as follows:

Lemma 1.1. *For any combination of* $\{k, m, u\}$, *where* $k, m, u \in \mathbb{N}$, there corresponds a $f(1)$ *such that* $f(1) = f(n_k)$ *for some* $n_k \in \mathbb{N}$.

Proof. For every $|A|$ there corresponds a $n \in \mathbb{N}$ and $f(1) = e^{u\left(1 - mu^{2k-2}\right)} =$

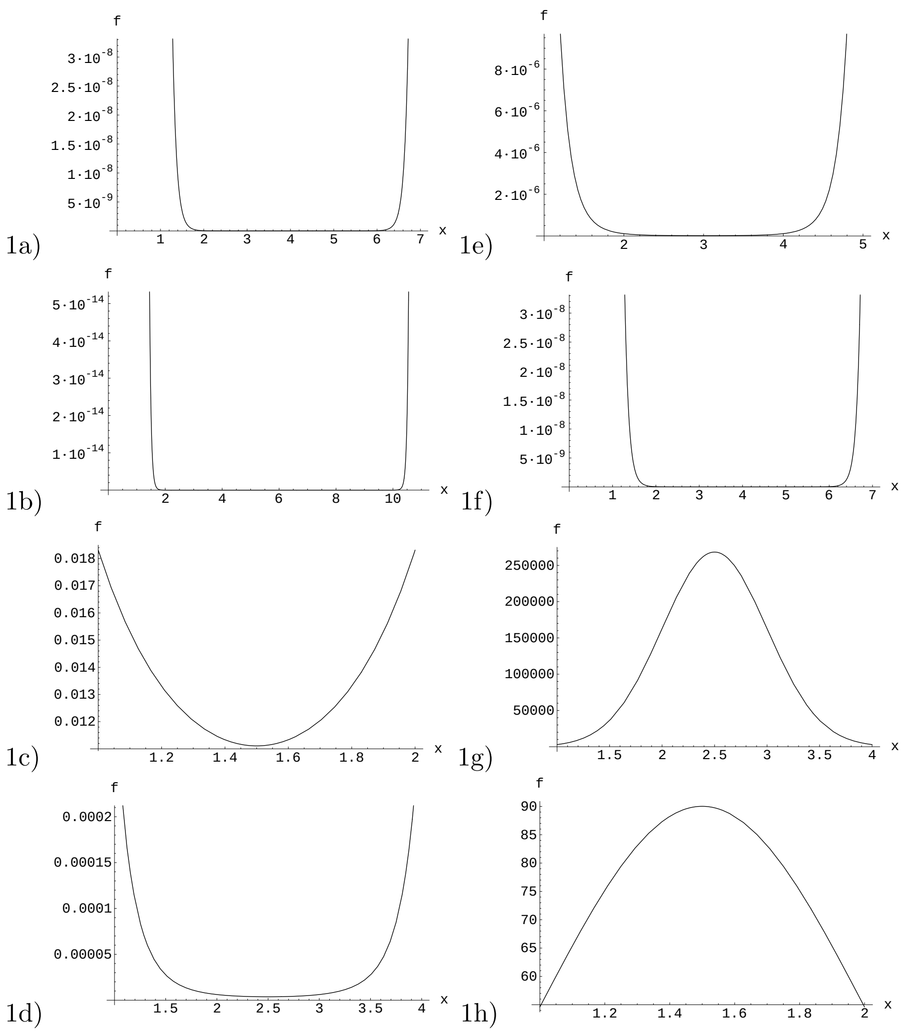

Figure 1: Numerical examples to demonstrate the shape of the function $f(x) = \exp\left(ux^2 + vx\right)$, $v = -mu^{2k-1}$, with some $k, u, m \in \mathbb{N}$. Note that $f(1) = f(|A|)$ for $A = \frac{u+v}{u}$. Following are combinations of k, u, m in each figure 1a) $k = 2$, $u = 2$, $m = 2$; 1b) $k = 2$, $u = 2$, $m = 3$; 1c) $k = 1$, $u = 2$, $m = 3$; 1d) $k = 1$, $u = 2$, $m = 5$; 1e) $k = 1$, $u = 2$, $m = 6$; 1f) $k = 1$, $u = 2$, $m = 8$; 1g) $k = 1$, $u = 2$, $m = 5$ (reciprocal of the function considered); 1h) $k = 1$, $u = 2$, $m = 3$ (reciprocal of the function considered).

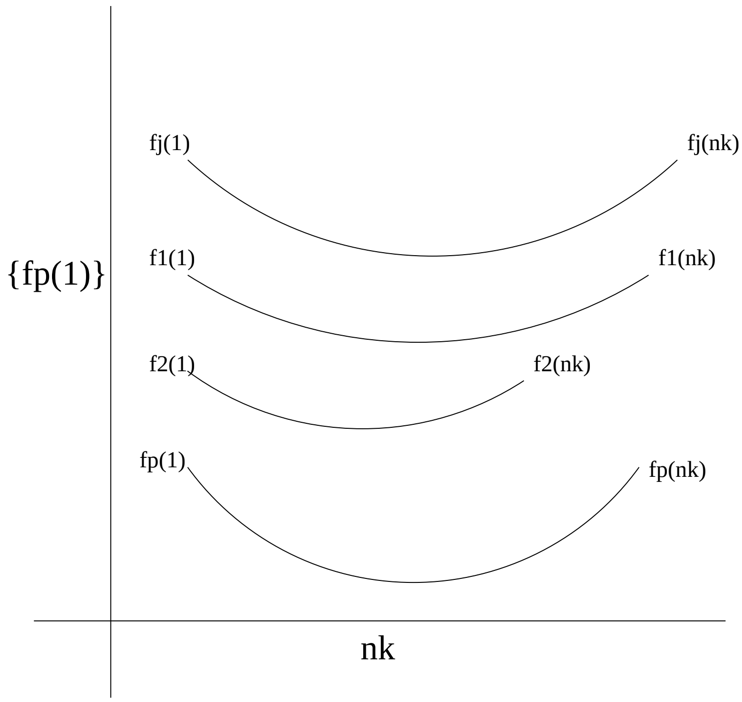

Figure 2: $f_p(1) = f_p(n_k)$ for the p^{th} combination of parameters of f and $n_k \in \mathbb{N}$.

$f(|A|)$. Thus $f(1) = f(n_k)$ for some $n_k \in \mathbb{N}$.

The domain of f is $\mathbb{N}$. The value of $f(1)$ is not same for every combination of $\{k, m, u\}$, where $k, m, u \in \mathbb{N}$. Readers are suggested to keep this in mind for understanding the results presented in this work.

2. Linking natural numbers and exponential function

Theorem 2.1. *Let N be even, $\{m, u, k, N\}$, where $k, m, u, N \in \mathbb{N}$ and $v = (-1)^{Nk-1} \cdot mu^{Nk-1}$. Then, $f(1) = f(|A|)$.*

Proof. $f(1) = e^{u\left(1-mu^{Nk-2}\right)}$ and $|A| = \left|\frac{u+(-1)^{Nk-1}mu^{Nk-1}}{u}\right| = mu^{Nk-2} - 1$. Therefore, $|A| \in \mathbb{N}$ for $\{m, u, k, N\}$, where $k, m, u, N \in \mathbb{N}$. Now,

$$\begin{aligned} f(|A|) &= e^{u\left(mu^{Nk-2}-1\right)^2 + v\left(mu^{Nk-2}-1\right)} \\ &= e^{u\left(m^2u^{2Nk-4}+1-2mu^{Nk-2}\right) - m^2u^{2Nk-3} + mu^{Nk-1}} \\ &= e^{u\left(1-mu^{Nk-2}\right)} \end{aligned}$$

Therefore, $f(1) = f(|A|)$. Since $|A|$ consists of every element of $\mathbb{N}$, it follows that $f(1) = f(n)$.

Theorem 2.2. *When N is odd, k is even, $v = (-1)^{Nk-1}\, mu^{Nk-1}$ and $\{k, m, u, N\}$, where $k, m, u, N \in \mathbb{N}$, then it follows that $f(1) = f(n_k)$, whenever $|A| \in \mathbb{N}$ and for $n_k \in \mathbb{N}$.*

Proof. When N is odd and k is even $|A| = \left|\frac{u+(-1)^{Nk-1}mu^{Nk-1}}{u}\right| = mu^{Nk-2} - 1$. The rest of the proof can be deduced from Theorem 2.1.

Remark. f_1 denotes the 1^{st}, f_2 denotes the 2^{nd}, and so on, f_p denotes the function $f(x) = e^{\left(ux^2+vx\right)}$ associated with the p^{th} combination of parameters $\{m, u, k\}$, i.e. say $\{m_1, u_1, k_1\}$, $\{m_2, u_2, k_2\}$,..., $\{m_p, u_p, k_p\}$, then we can observe following relations:

$$\begin{aligned}
f_1(1) &= f_1(2) \neq f_1(3) \neq ... \neq f_1(n) \neq ... \\
f_2(1) &\neq f_2(2) = f_2(3) \neq ... \neq f_2(n) \neq ... \\
&\vdots \\
f_p(1) &= f_p(2) \neq f_p(3) \neq ... \neq f_p(n) = f_p(n+1) \neq ... \\
&\vdots
\end{aligned}$$

Given Theorem 2.1, suppose we denote the distance from $|A|$ to 1 by $\mathbb{D}$, then $f(|A|) = \mathbb{D}$, if $m = \mathcal{Z}(1-\mathcal{Z})^{Nk-2}/\left\{log(\mathcal{Z}-2)\right\}^{Nk-2}$. Here $\mathcal{Z} = mu^{Nk-2}$.

Suppose $\mathbb{D}_p$ be the logarithmic distance from 1 to $|A|$ for the p^{th} combination of parameters, then

$$\begin{aligned}
\mathbb{D}_p &= \log\{\mathcal{Z}_p - 2\} \\
&= \log\left\{\mathcal{Z}_p\left(1 - \frac{2}{\mathcal{Z}_p}\right)\right\} = \log \mathcal{Z}_p + \log\left(1 - \frac{2}{\mathcal{Z}_p}\right)
\end{aligned}$$

$\mathbb{D}_p$ converges for $\mathcal{Z}_p > 2$.

The relation $f_m(1) = f_m(n)$ is unique for each $n \in \mathbb{N}$ and $\{m_p, u_p, k_p\}$ where $m_p, u_p, k_p \in \mathbb{N}$.

Let N_{p+i} be even for $\left\{\begin{array}{l} p = 1, 2, 3, ... \\ i = 0, 1, 2, ... \end{array}\right.$. If $|A_p| > |A_{p-1}|$ then $f_p(1) < f_{p-1}(1)$ for all $p = 2, 3, 4, ...$ This fact is demonstrated through Figure 3.

The number of pairs $\{f_m(1),\, f_m(n)\}$ that satisfy Remark 2. are countable.

Theorem 2.3. *Let $\mathcal{B}_\sigma(0) = \{b \in \mathbb{R}^+ : |b - 0| < \sigma\}$ for $\sigma > 0$ and N_{p+i} is even for $\left\{\begin{array}{l} p = 1, 2, 3, ... \\ i = 0, 1, 2, ... \end{array}\right.$, if $|A_{p+1}| > |A_p|$ then the sequence $\{f_p(1)\}_{p=1,2,3,...} \in \mathcal{B}_\sigma(0)$ for $p > M \in \mathbb{N}$.*

Proof. $|A_{p+1}| > |A_p| \Rightarrow m_{p+1}u^{N_{p+1}k_{p+1}-2} - 1 > m_p u^{N_p k_p - 2} - 1$.

$$\begin{aligned}
\Rightarrow \left(1 - m_{p+1}u^{N_{p+1}k_{p+1}-2}\right) &< \left(1 - m_p u^{N_p k_p - 2}\right) \\
\Rightarrow u_{p+1}\left(1 - m_{p+1}u^{N_{p+1}k_{p+1}-2}\right) &< u_p\left(1 - m_p u^{N_p k_p - 2}\right) \\
\Rightarrow e^{u_{p+1}\left(1 - m_{p+1}u^{N_{p+1}k_{p+1}-2}\right)} &< e^{u_p\left(1 - m_p u^{N_p k_p - 2}\right)}
\end{aligned}$$

This implies $f_{p+1}(1) < f_p(1)$. We know that $e^{-p} \to 0$ as $p \to \infty$. Thus $f_p(1) \in \mathcal{B}_\sigma(0)$.

COROLLARY 2.4. *Since* $f_p(1) \in \mathcal{B}_\sigma(0)$, *it follows from Remark 2. that* $\{f_1(2), f_2(3), ..., f_p(n)\ , f_{p+1}(n+1), ...\}$ *is a convergent sequence.*

3. Mid-point Theorem

THEOREM 3.1. *Suppose* $\theta \in (1, |A|)$ *such that* $f'(\theta) = 0$. *Then this* θ *is the mid-point of the interval* $I = [1, |A|]$.

PROOF. It is easy to verify that f is continuous on $[1, |A|]$ and differentiable on $(1, |A|)$, and from Lemma 1.1 we have $f(1) = f(|A|)$, so by Rolle's theorem there exists a $\theta \in (1, |A|)$ such that $f'(\theta) = 0$. We have $|A| = mu^{2k-2} - 1$. Mid-point of the interval I is $\frac{m}{2}u^{2k-2}$.

$f'(\theta) = e^{u\theta^2+v\theta}(2u\theta + v) = f(\theta)(2u\theta + v)$. Since $f'(\theta) = 0$, this means obviously $2u\theta+v = 0$, because $f(\theta) \neq 0$. Therefore $\theta = \frac{-v}{2u} = \frac{m}{2}u^{2k-2}$. Hence θ is the mid-point of the interval I. For a numerical example, consider $\{k = 3,\ m = 10, u = 8\}$ as in section 1. For this combination $|A| = 40959$, mid-point of the interval is 20480 and $f'(20480) = 0$.

THEOREM 3.2. *Suppose* $\delta_1 < \delta_2 < ...\delta_n$, *where* $\delta_i(> 0) \in I$. *Then for a given combination of* $\{k, m, u\}$, $f(1 + \delta_1) > f(1 + \delta_2) > ...f(1 + \delta_n)= f(|A| - \delta_n) < ... < f(|A|-\delta_2) < f(|A|-\delta_1)$ *if and only if* $(1+\delta_n)$ *is a mid-point of* I, *where* $\delta_n = \frac{m}{2}u^{2k-2}-1$.

PROOF. Verify easily that $f(1 + \delta) = f(|A| - \delta)$ for $\delta(> 0) \in I$. Consider

$$f(1 + \delta_1) > ... > f(1 + \delta_n) \quad = \quad f(|A| - \delta_n) < ... < f(|A| - \delta_1) \tag{1}$$

By Theorem 3.1 we know $f'(\theta) = 0$ for $\theta \in I$. Hence $(1 + \delta_n)$ is a mid-point.

To prove converse we begin as follows. Since mid-point of the interval I is $\frac{m}{2}u^{2k-2}$, we have

$$\begin{aligned} f(\frac{m}{2}u^{2k-2}) &= e^{4(\frac{m}{2}u^{2k-2})^2 - mu^{2k-1}\frac{m}{2}u^{2k-2}} \\ &= e^{-\frac{m^2}{4}u^{4k-3}} \end{aligned} \tag{2}$$

Now for given $\delta_n = \frac{m}{2}u^{2k-2} - 1$, we can verify that

$$f(1 + \delta_n) = f(|A| - \delta_n) \quad = \quad e^{-\frac{m^2}{4}u^{4k-3}} \tag{3}$$

Since $\delta_1 < \delta_2 < ...\delta_n$ and by equations 2 and 3, the result 1 is straightforward.

For large value of the distance function $\mathbb{D}$ defined, the shape of f look like the alphabet U, Rao (2003). Suppose instead of positive integer, let $u \in \mathbb{Z}^-$and other parameters k, m remain as before, and if we denote resulting function as g, then the shape of g was shown to have mirror image of U, Rao (2003). Based on this information and from Theorem 3.2, we state the following corollary.

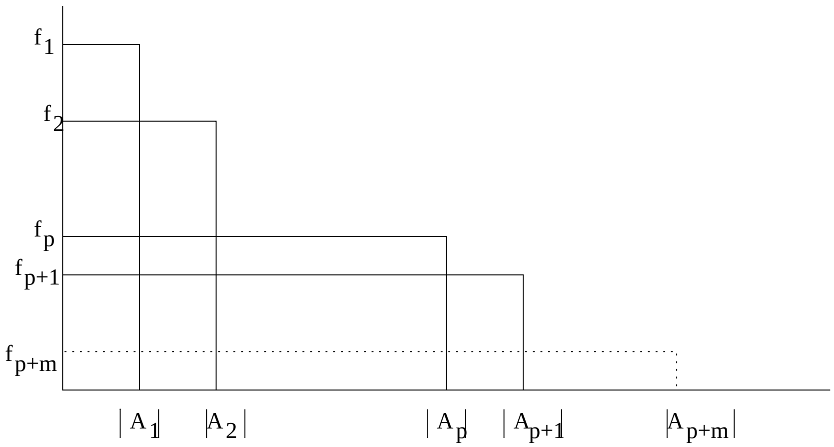

Figure 3: Relation between $|A_p|$ and $f_p(1)$. Vertical lines corresponding to $|A_p|$ are lengths of $f_p(1)$ for each p.

COROLLARY 3.3. *Suppose $\delta_1 < \delta_2 < ...\delta_n$, where $\delta_i(> 0) \in I$. Then for a given combination of $\{k, m, u\}$, $g(1 + \delta_1) < g(1 + \delta_2) < ...g(1 + \delta_n) = g(|A| - \delta_n) > ... > g(|A| - \delta_2) > g(|A| - \delta_1)$ if and only if $(1+\delta_n)$ is a mid-point of I, where $\delta_n = \frac{m}{2}u^{2k-2} - 1$.*

Readers can also verify Darboux's theorem for f on the intervals $[1, \theta]$ and $[\theta, |A|]$ for some $f'(1) > \beta_1 > f'(\theta)$ or $f'(\theta) < \beta_2 < f'(|A|)$ such that $f'(\beta_1) = \alpha_1$ or $f'(\beta_2) = \alpha_2$ for $\alpha_1 \in [1, \theta]$ and $\alpha_2 \in [\theta, |A|]$.

In general, results on dynamics and periodic properties for the quadratic function of the form $x^2 + K$, Walde and Russo (1994) and periodic properties of natural numbers Fine (1958) can be found. However, this present note is basically deals with a correspondence between natural numbers, quadratic exponential function, and the convergence of such functions mapped on natural numbers constructed using $|A|$.

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