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Takeshita, Jun-ichi
Research Institute of Science for Safety and Sustainability (RISS), National Institute of Advanced Industrial Science and Technology (AIST)

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Jun-ichi Takeshita*

Abstract

In this paper, we present a game-theoretical meaning for all vertices in the simple simplex, which are included among the stationary points, of the replicator dynamics of asymmetric two-person games. It is well known that there exists a relationship between a stationary point and a Nash equilibrium. Indeed, the so-called folk theorem of evolutionary game theory claims that a stable stationary point is closely related to a Nash equilibrium in the case of symmetric two-person games. However, for unstable stationary points, its game-theoretical meaning remains unclear. Hence, in this paper, we introduce indices for unstable stationary points by using the Jacobian matrix of the replicator dynamics. We discuss a game-theoretical meaning of the indices, and present an alternative solution concept to the Nash equilibrium of a bimatrix game. Then, any bimatrix game always has this solution if we restrict strategies to pure ones.

Key Words and Phrases: game theory, replicator dynamics, asymmetric two-person game, pure-strategy, Nash equilibrium.

1. Introduction

Replicator dynamics has a central role in evolutionary game theory, originally introduced by Taylor and Jonker (1978) for the special case of two-person games. Stationary points of replicator dynamics are one of the most important research objects in evolutionary game theory. Although there are many studies on stable stationary points, the research on unstable stationary points is not enough. Thus, this paper deals with them.

For stable stationary points, several authors have clarified a game-theoretical meaning. For example, the so-called folk theorem (Proposition 1.1 below) of evolutionary game theory characterizes the stable stationary points for the replicator dynamics of symmetric two-person games.

Proposition 1.1.

(a) *Nash equilibria are stable points.*

(b) *Strict Nash equilibria are asymptotically stable points.*

* Research Institute of Science for Safety and Sustainability (RISS), National Institute of Advanced Industrial Science and Technology (AIST) 16-1, Onogawa, Tsukuba, Ibaraki 305-8569, JAPAN. E-mail: jun-takeshita@aist.go.jp

(c) *If an interior orbit converges to a point, then it is a Nash equilibrium.*

(d) *If a stationary point is Lyapunov stable, then it is a Nash equilibrium.*

Here we note that none of the converse statements hold. Proofs of these claims can be found in, for example, Weibull (1995), Hofbauer and Sigmund (1998), Cressman (2003), and Hofbauer and Sigmund (2003). Proposition 1.1 claims that a stable stationary point is closely related to a Nash equilibrium. Moreover, Cressman (2003), and Hofbauer and Sigmund (2003) showed that a boundary stationary point is a Nash equilibrium if and only if its transversal eigenvalues are non-positive, and Hofbauer and Sigmund (1988), Hofbauer and Sigmund (1998), and Harsanyi (1973) proved that if all Nash equilibria are regular, that is, these Jacobian matrices of the associate replicator dynamics are nonsingular, then their number must be odd.

On the other hand, it is well known that any non-cooperative game has a (mixed-strategy) Nash equilibrium, but not always a pure-strategy Nash equilibrium. Hence we need an additional assumption to get a pure-strategy Nash equilibrium. Indeed, assuming monotonicity of a best response, Topkis (1979), Sato and Kawasaki (2009), and Takeshita and Kawasaki (2012) showed the existence of a pure-strategy Nash equilibrium. Iimura (2003) also showed existence by using another type of hypothesis. Further, Takeshita and Kawasaki (2012) gave a necessary condition for existence in symmetric and asymmetric bimatrix games. However, a solution concept which arbitrary non-cooperative games have in pure strategies does not exist.

The present paper has two aims. One is to present a game-theoretical meaning for the vertices in the simple simplex of a replicator dynamics. In other words, we focus exclusively on the pure strategies of an asymmetric two-person game. Second, applying the created definition of vertices to bimatrix games having no pure-strategy Nash equilibria, our aim is to create a solution concept such that any bimatrix game has at least one solution in pure strategies.

Our paper is organized as follows. In Section 2, we present the notation for bimatrix games and their replicator dynamics. In Section 3, we define indices for pure strategies, which play a key role in creating a game-theoretical meaning for vertices in the simple simplex for replicator dynamics. Then, in Section 4, we show the game-theoretical meaning of these indices. In Section 5, applying the results of Section 4, we present an alternative solution concept for a Nash equilibrium of a bimatrix game. Finally, Section 6 summarizes the study.

2. Notation

2.1. Non-cooperative two-person games: bimatrix games

Let $G := \{N := \{1,2\}, \{S_k\}_{k\in N}, \{u_k\}_{k\in N}\}$ be a normal-form game, where N is the set of players, S_k is the set of strategies available to player k, and u_k is the payoff function of player k. Throughout this paper, we assume S_1 and S_2 consist of m and n pure strategies, respectively, and we denote by i (resp. j) an element of S_1 (resp. S_2). Further, we denote by a_{ij} (resp. b_{ij}) the payoff of player 1 (resp. 2) when player 1 uses $i \in S_1$ and player 2 uses $j \in S_2$. Thus, the payoffs are given by the $m \times n$-matrices A and B.

The sets of mixed strategies of players 1 and 2 are the $(m-1)$-dimensional unit simplex $\Delta_1 := \{x \in \mathbb{R}^m \colon \sum_{i=1}^m x_i = 1,\ x_i \geq 0\}$ and the $(n-1)$-dimensional unit simplex $\Delta_2 := \{y \in \mathbb{R}^n : \sum_{j=1}^n y_i = 1,\ y_j \geq 0\}$, respectively, and $\Delta = \Delta_1 \times \Delta_2$ is the polyhedron of mixed-strategy pairs (x, y). We identify each pure strategy $i \in S_1$ with the corresponding unit vector $e^i \in \Delta_1$, whose i-th component is 1 and other components are zero. We similarly identify each pure strategy $j \in S_2$ with the corresponding unit vector $e^j \in \Delta_2$. If player 1 uses $x \in \Delta_1$ and player 2 uses $y \in \Delta_2$, then the former has $\langle x, Ay\rangle$ as his expected payoff and the latter $\langle x, By\rangle$, where $\langle\cdot,\cdot\rangle$ is the usual inner product. A pair $(x, y) \in \Delta_1 \times \Delta_2$ is called a *Nash equilibrium* if the following two conditions hold:

$$\begin{aligned} \langle x', Ay\rangle &\leq \langle x, Ay\rangle \quad \forall x' \in \Delta_1, \\ \langle x, By'\rangle &\leq \langle x, By\rangle \quad \forall y' \in \Delta_2. \end{aligned}$$

2.2. The replicator dynamics

For a two-person game, the replicator dynamics is usually defined by the following system of ordinary differential equations in the polyhedron Δ:

$$\text{(RD)} \quad \begin{cases} \dfrac{dx_{i'}}{dt} = x_{i'}(\langle e^{i'}, Ay\rangle - \langle x, Ay\rangle), & i' = 1, 2, \ldots, m, \\ \dfrac{dy_{j'}}{dt} = y_{j'}(\langle x, Be^{j'}\rangle - \langle x, By\rangle), & j' = 1, 2, \ldots, n, \end{cases}$$

see, for example, Taylor (1979), Zeeman (1980), Hofbauer and Sigmund (1988), Friedman (1991), and Samuelson and Zhang (1992) for details. Moreover, the behavior of (RD) is the same as that of $(\mathrm{RD})_{-i,-j}$ defined below because of the restrictions that $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j = 1$.

$$(\mathrm{RD})_{-i,-j} \quad \begin{cases} \dfrac{dx_{i'}}{dt} = x_{i'}(\langle e^{i'}, Ay\rangle - \langle x, Ay\rangle), & i' = 1, \ldots, i-1, i+1, \ldots, m, \\ \dfrac{dy_{j'}}{dt} = y_{j'}(\langle x, Be^{j'}\rangle - \langle x, By\rangle), & j' = 1, \ldots, j-1, j+1, \ldots, n. \end{cases}$$

Hence, in this paper, we deal with the stationary points of $(\mathrm{RD})_{-i,-j}$. Further, we describe equations of $(\mathrm{RD})_{-i,-j}$ associated with A and B as $(\mathrm{RD})^A_{-i}$ and $(\mathrm{RD})^B_{-j}$, respectively, that is,

$$(\mathrm{RD})^A_{-i} \quad \frac{dx_{i'}}{dt} = x_{i'}(\langle e^{i'}, Ay\rangle - \langle x, Ay\rangle) =: f^A_{i'}(x, y), \quad i' = 1, \ldots, i-1, i+1, \ldots, m,$$

$$(\mathrm{RD})^B_{-j} \quad \frac{dy_{j'}}{dt} = y_{j'}(\langle x, Be^{j'}\rangle - \langle x, By\rangle) =: f^B_{j'}(x, y), \quad j' = 1, \ldots, j-1, j+1, \ldots, n.$$

Also, we set

$$\begin{aligned} f(x, y) := \big(&f^A_1(x, y), \ldots, f^A_{i-1}(x, y), f^A_{i+1}(x, y), \ldots, f^A_m(x, y), \\ &f^B_1(x, y), \ldots, f^B_{j-1}(x, y), f^B_{j+1}(x, y), \ldots, f^B_n(x, y)\big)^T \end{aligned}$$

3. The definition of indices

In this section, we first define *indices*, which are indicators of instability for a pure-strategy pair. Hereinafter, we denote the Jacobian matrices of $(\mathrm{RD})_{-i,-j}$, $(\mathrm{RD})^A_{-i}$, and $(\mathrm{RD})^B_{-j}$ by $Df_{-i,-j}(x, y)$, $Df^A_{-i}(x, y)$, and $Df^B_{-j}(x, y)$, respectively.

Definition 3.1. (index) We define the *index* of (i, j) as the number of positive eigenvalues of $Df_{-i,-j}(e^i, e^j)$. Also, we define the *index with respect to P1* and *P2* of (i, j) as the number of positive eigenvalues of $Df^A_{-i}(e^i, e^j)$ and $Df^B_{-j}(e^i, e^j)$, respectively. Further, we denote by $\mathrm{index}(i, j)$, $\mathrm{index}_A(i, j)$, and $\mathrm{index}_B(i, j)$, the index, the index with respect to P1, and the index with respect to P2, respectively, of (i, j).

By the proposition below, we can see that the indices are well defined.

Proposition 3.2. *For any* $i \in S_1$ *and* $j \in S_2$*,* $(e^i, e^j) \in \Delta_1 \times \Delta_2$ *is a stationary point for* $(\mathrm{RD})_{-i,-j}$*,* $(\mathrm{RD})^A_{-i}$*, and* $(\mathrm{RD})^B_{-j}$*.*

Proof. It is enough to show that $(e^i, e^j) \in \Delta_1 \times \Delta_2$ is a stationary point for $(\mathrm{RD})_{-i,-j}$. By $(e^i)_{i'} = 0$ for any $i' \neq i$, we have $f^A_{i'}(e^i, y) = 0$ for any $y \in \Delta_2$, that is, $dx_{i'}/dt|_{x=e^i,y=y} = 0$. Also, by $(e^j)_{j'} = 0$ for any $j' \neq j$, $dy_{j'}/dt|_{x=x,y=e^j} = 0$. Therefore, (e^i, e^j) is a stationary point for $(\mathrm{RD})_{-i,-j}$. □

Proposition 3.3. *For any* $i \in S_1$ *and* $j \in S_2$*, it holds that*

$$\begin{aligned} Df_{-i,-j}(e^i, e^j) &= \mathrm{diag}(a_{1j} - a_{ij}, \ldots, a_{i-1,j} - a_{ij}, a_{i+1,j} - a_{ij}, \ldots, a_{mj} - a_{ij}, \\ &\qquad b_{i1} - b_{ij}, \ldots, b_{i,j-1} - b_{ij}, b_{i,j+1} - b_{ij}, \ldots, b_{in} - b_{ij}), \qquad (1) \\ Df^A_{-i}(e^i, e^j) &= \mathrm{diag}(a_{1j} - a_{ij}, \ldots, a_{i-1,j} - a_{ij}, a_{i+1,j} - a_{ij}, \ldots, a_{mj} - a_{ij}), \qquad (2) \\ Df^B_{-j}(e^i, e^j) &= \mathrm{diag}(b_{i1} - b_{ij}, \ldots, b_{i,j-1} - b_{ij}, b_{i,j+1} - b_{ij}, \ldots, b_{in} - b_{ij}). \qquad (3) \end{aligned}$$

In particular,

$$\mathrm{index}(e^i, e^j) = \mathrm{index}_A(e^i, e^j) + \mathrm{index}_B(e^i, e^j) \qquad (4)$$

holds.

Proof. We start with the following relation.

$$Df_{-i,-j}(x, y) = \left(\begin{array}{c|c} \left(\dfrac{\partial f^A_k}{\partial x_{i'}}(x, y) \right)_{k,i'=1,\ldots,i-1,i+1,\ldots,m} & \left(\dfrac{\partial f^A_k}{\partial y_{j'}}(x, y) \right)_{k,j'=1,\ldots,j-1,j+1,\ldots,n} \\ \hline \left(\dfrac{\partial f^B_k}{\partial x_{i'}}(x, y) \right)_{k,i'=1,\ldots,i-1,i+1,\ldots,m} & \left(\dfrac{\partial f^B_k}{\partial y_{j'}}(x, y) \right)_{k,j'=1,\ldots,j-1,j+1,\ldots,n} \end{array} \right).$$

When $k \neq i'$, we have

$$\frac{\partial f^A_k}{\partial x_{i'}}(x, y) = x_k \frac{\partial}{\partial x_{i'}} \left(\langle e^k, Ay \rangle - \langle x, Ay \rangle \right).$$

Since $(e^i)_k = 0$, we have $\partial f^A_k / \partial x_{i'}(e^i, e^j) = 0$. Hence, $Df^A_{-i}(e^i, e^j)$ is a diagonal matrix. Next, when $k = i'$, we have

$$\frac{\partial f^A_{i'}}{\partial x_{i'}}(x, y) = \left(\langle e^{i'}, Ay \rangle - \langle x, Ay \rangle \right) + x_{i'} \frac{\partial}{\partial x_{i'}} \left(\langle e^{i'}, Ay \rangle - \langle x, Ay \rangle \right).$$

Since $(e^i)_{i'} = 0$, we have

$$\frac{\partial f^A_{i'}}{\partial x_{i'}}(e^i, e^j) = \langle e^{i'}, Ae^j \rangle - \langle e^i, Ae^j \rangle = a_{i'j} - a_{ij},$$

which implies (2). By a similar argument, we obtain (3).

In order to show (1), we shall show that

$$\left(\frac{\partial f_k^A}{\partial y_{j'}}(e^i, e^j)\right)_{k,j'=1,\ldots,j-1,j+1,\ldots,n} = \left(\frac{\partial f_k^B}{\partial x_{i'}}(e^i, e^j)\right)_{k,i'=1,\ldots,i-1,i+1,\ldots,m} = O.$$

Since

$$\frac{\partial f_k^A}{\partial y_{j'}}(x, y) = x_k \frac{\partial}{\partial y_{j'}} \left(\langle e^k, Ay\rangle - \langle x, Ay\rangle\right)$$

and $(e^j)_k = 0$, we have $\partial f_k^A/\partial y_{j'}(e^i, e^j) = 0$ for any $k, j' = 1, \ldots, j-1, j+1, \ldots, n$, i.e.,

$$\left(\frac{\partial f_k^A}{\partial y_{j'}}(e^i, e^j)\right)_{k,j'=1,\ldots,j-1,j+1,\ldots,n} = O.$$

Also, by a similar argument, we have

$$\left(\frac{\partial f_k^B}{\partial x_{i'}}(e^i, e^j)\right)_{k,i'=1,\ldots,i-1,i+1,\ldots,m} = O.$$

Therefore, we obtain (1). Last claim (4) is a direct consequence of (1) to (3) and the definition of the index. $\square$

Here we give an example of the indices of pure-strategy pairs.

Example 3.4. We consider the bimatrix game with the following payoff matrices:

$$A = \left(\begin{array}{c|c|c} -1 & -7 & -5 \\ -5 & 1 & -8 \end{array}\right), \quad B = \left(\begin{array}{ccc} -4 & 5 & 7 \\ \hline -5 & 4 & 1 \end{array}\right).$$

Then the Jacobian matrices are easily computed as follows:

$$Df_{-1,-1}(e^1, e^1) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 11 \end{pmatrix}, \quad Df_{-1}^A(e^1, e^1) = -4, \quad Df_{-1}^B(e^1, e^1) = \begin{pmatrix} 9 & 0 \\ 0 & 11 \end{pmatrix},$$

$$Df_{-1,-2}(e^1, e^2) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Df_{-1}^A(e^1, e^2) = 8, \quad Df_{-2}^B(e^1, e^2) = \begin{pmatrix} -9 & 0 \\ 0 & 2 \end{pmatrix},$$

$$Df_{-1,-3}(e^1, e^3) = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad Df_{-1}^A(e^1, e^3) = -3, \quad Df_{-3}^B(e^1, e^3) = \begin{pmatrix} -11 & 0 \\ 0 & -2 \end{pmatrix},$$

$$Df_{-2,-1}(e^2, e^1) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad Df_{-2}^A(e^2, e^1) = 4, \quad Df_{-1}^B(e^2, e^1) = \begin{pmatrix} 9 & 0 \\ 0 & 6 \end{pmatrix},$$

$$Df_{-2,-2}(e^2, e^2) = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad Df_{-2}^A(e^2, e^2) = -8, \quad Df_{-2}^B(e^2, e^2) = \begin{pmatrix} -9 & 0 \\ 0 & -3 \end{pmatrix},$$

$$Df_{-2,-3}(e^2, e^3) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad Df_{-2}^A(e^2, e^3) = 3, \quad Df_{-3}^B(e^2, e^3) = \begin{pmatrix} -6 & 0 \\ 0 & 3 \end{pmatrix}.$$

Since $\mathrm{index}(i,j) = \mathrm{index}_A(i,j) + \mathrm{index}_B(i,j)$,

$$\begin{aligned}
\mathrm{index}(1,1) &= 0 + 2 = 2,\\
\mathrm{index}(1,2) &= 1 + 1 = 2,\\
\mathrm{index}(1,3) &= 0 + 0 = 0,\\
\mathrm{index}(2,1) &= 1 + 2 = 3,\\
\mathrm{index}(2,2) &= 0 + 0 = 0,\\
\mathrm{index}(2,3) &= 1 + 1 = 2.
\end{aligned}$$

4. A game-theoretical meaning of the indices

In this section, we show a game-theoretical meaning of the indices.

THEOREM 4.1. *The following hold:*

$$\mathrm{index}(i,j) = \#\{i' \in S_1 \colon u_1(i',j) > u_1(i,j)\} + \#\{j' \in S_2 \colon u_2(i,j') > u_2(i,j)\}, \tag{5}$$

$$\mathrm{index}_A(i,j) = \#\{i' \in S_1 \colon u_1(i',j) > u_1(i,j)\}, \tag{6}$$

$$\mathrm{index}_B(i,j) = \#\{j' \in S_2 \colon u_2(i,j') > u_2(i,j)\}. \tag{7}$$

PROOF. We prove only (6) and (7) since (5) can be shown by using (4) in Proposition 3.3, along with (6) and (7). By Proposition 3.3, the eigenvalues of $Df^A_{-i}(e^i, e^j)$ are

$$a_{1j} - a_{ij}, \dots, a_{i-1,j} - a_{ij}, a_{i+1,j} - a_{ij}, \dots, a_{mj} - a_{ij}.$$

Thus, the number of positive eigenvalues is $i' \in S_1$ such that $a_{i'j} - a_{ij} > 0$. Since $u_1(i,j) = a_{ij}$ for any $(i,j) \in S_1 \times S_2$, it follows that

$$\mathrm{index}_A(i,j) = \#\{i' \in S_1 \colon u_1(i',j) > u_1(i,j)\},$$

which is (6). The proof of (7) is similar. □

REMARK. The theorem above implies that if $\mathrm{index}_A(i,j) = k_1$ and $\mathrm{index}_B(i,j) = k_2$, then $i \in S_1$ is the (k_1+1)-th best pure strategy for $j \in S_2$ and $j \in S_2$ is the (k_2+1)-th best pure strategy for $i \in S_1$.

Here we recall Example 3.4.

EXAMPLE 4.2. (Example 3.4)We consider the bimatrix game with the following payoff matrices:

$$A = \left(\begin{array}{c|c|c} -1 & -7 & -5 \\ \boxed{-5} & 1 & -8 \end{array}\right), \quad B = \left(\begin{array}{ccc} -4 & 5 & 7 \\ \hline \boxed{-5} & 4 & 1 \end{array}\right).$$

We have already shown the following:

$$\begin{aligned}
&\mathrm{index}(1,1) = 2, && \mathrm{index}_A(1,1) = 0, && \mathrm{index}_B(1,1) = 2,\\
&\mathrm{index}(1,2) = 2, && \mathrm{index}_A(1,2) = 1, && \mathrm{index}_B(1,2) = 1,\\
&\mathrm{index}(1,3) = 0, && \mathrm{index}_A(1,3) = 0, && \mathrm{index}_B(1,3) = 0,\\
&\mathrm{index}(2,1) = 3, && \mathrm{index}_A(2,1) = 1, && \mathrm{index}_B(2,1) = 2,\\
&\mathrm{index}(2,2) = 0, && \mathrm{index}_A(2,2) = 0, && \mathrm{index}_B(2,2) = 0,\\
&\mathrm{index}(2,3) = 2, && \mathrm{index}_A(2,3) = 1, && \mathrm{index}_B(2,3) = 1.
\end{aligned}$$

Here we focus on the pure-strategy pair $(2, 1)$. The tables below show that the payoff and rank for each player when player 1 chooses strategy 2 and player 2 chooses strategy 1.

player 1		
strategy i	payoff	rank
1	-1	1
2	-5	**2**

player 2		
strategy j	payoff	rank
1	-5	**3**
2	4	1
3	1	2

These table show that if we take $(i, j) = (2, 1)$, then $i = 2$ is the second best pure strategy for $j = 1$ and $j = 1$ is the third best pure strategy for $i = 2$. Since $\mathrm{index}_A(2, 1) = 1$ and $\mathrm{index}_B(2, 1) = 2$, these show the claim of Remark 4.

The index characterizes a pure-strategy Nash equilibrium.

THEOREM 4.3. $\mathrm{index}(i^*, j^*) = 0$ *is equivalent to that* (i^*, j^*) *is a pure-strategy Nash equilibrium of the bimatrix game with payoff matrices* A *and* B.

PROOF. ("only if" part) Suppose that $\mathrm{index}(i^*, j^*) = 0$. Then we have

$$\mathrm{index}_A(i^*, j^*) = \mathrm{index}_B(i^*, j^*) = 0.$$

By $\mathrm{index}_A(i^*, j^*) = 0$, we obtain

$$\#\{i \in S_1 \colon u_1(i, j^*) > u_1(i^*, j^*)\} = 0.$$

In other words, $u_1(i^*, j^*) \geq u_1(i, j^*)$ for any $i \in S_1$. This implies that $i^* \in S_1$ is the best response for $j^* \in S_2$. By a similar argument, it follows that $j^* \in S_2$ is the best response for $i^* \in S_1$. Therefore, (i^*, j^*) is a Nash equilibrium.

("if" part) Suppose that (i^*, j^*) is a Nash equilibrium. Then, by the definition of a Nash equilibrium, the following hold:

$$\begin{aligned} u_1(i^*, j^*) &\geq u_1(i, j^*) \quad \forall i \in S_1, \\ u_2(i^*, j^*) &\geq u_2(i^*, j) \quad \forall j \in S_2. \end{aligned}$$

These imply that

$$\{i \in S_1 \colon u_1(i, j^*) > u_1(i^*, j^*)\} = \{j \in S_2 \colon u_2(i^*, j) > u_2(i^*, j^*)\} = \emptyset.$$

Therefore, it follows that

$$\mathrm{index}_A(i^*, j^*) = \mathrm{index}_B(i^*, j^*) = 0,$$

which implies the desired conclusion. $\square$

5. Application to bimatrix games

In this section, we use the indices as the next best solution concept of non-cooperative two-person games that have no pure-strategy Nash equilibria. It is well known that a pure-strategy Nash equilibrium does not always exist. With this in mind, we suggest an alternative solution concept for a Nash equilibrium.

DEFINITION 5.1. (Quasi Nash pair) A pair $(i^\star, j^\star)$ is called a *quasi Nash pair* of a bimatrix game if the following conditions hold:

$$(i^\star, j^\star) \in \operatorname*{argmin}_{(i,j)\in S_1\times S_2} \operatorname{index}(i,j) =: M \tag{8}$$

$$(i^\star, j^\star) \in \operatorname*{argmin}_{(i,j)\in M} \left|\operatorname{index}_A(i,j) - \operatorname{index}_B(i,j)\right|. \tag{9}$$

REMARK. When $(i^\star, j^\star)$ is a quasi Nash pair, $\operatorname{index}_A(i^\star, j^\star) = k_1 \geq 1$, and $\operatorname{index}_B(i^\star, j^\star) = k_2 \geq 1$, players 1 and 2 make compromises by choosing the (k_1+1)-th and (k_2+1)-th, respectively, best pure strategy since if both player do not make a compromise, then the game breaks down in pure strategies. More precisely, (8) means that the degree of the compromise is a minimum, and (9) means that the degree of bias of the compromise between the two players is a minimum. For example, we consider two possible solutions. One is $\operatorname{index}(i,j) = 2$, $\operatorname{index}_A(i,j) = 2$, and $\operatorname{index}_B(i,j) = 0$, and the other is $\operatorname{index}(i,j) = 2$, $\operatorname{index}_A(i,j) = 1$, and $\operatorname{index}_B(i,j) = 1$. Then we choose the latter, since if we choose the former, then only player 1 makes a major compromise and he/she will complain.

By the definition of a quasi Nash pair, we immediately get the following theorem.

THEOREM 5.2. *Any bimatrix game has at least one quasi Nash pair.*

Here we give an example of quasi Nash pairs of a game.

EXAMPLE 5.3. We consider the bimatrix game with the following payoff matrices:

$$A = \left(\begin{array}{c|c|c|c} 1 & 2 & 5 & -3 \\ -4 & 7 & 2 & 9 \\ 6 & -3 & -4 & 6 \end{array}\right), \quad B = \left(\begin{array}{cccc} -2 & -1 & -4 & 7 \\ \hline 9 & -2 & 8 & -5 \\ \hline 1 & 4 & 9 & 4 \end{array}\right).$$

Then the Jacobi matrices of $(\mathrm{RD})_{-i,-j}$ $(i = 1,2,3,\ j = 1,2,3,4)$ are

$$\begin{aligned}
&Df_{-1,-1}(e^1, e^1) = \operatorname{diag}(-5, 5, 1, -2, 9), && Df_{-1,-2}(e^1, e^2) = \operatorname{diag}(5, -5, -1, -3, 8),\\
&Df_{-1,-3}(e^1, e^3) = \operatorname{diag}(-3, -9, 2, 3, 11), && Df_{-1,-4}(e^1, e^4) = \operatorname{diag}(12, 9, -9, -8, -11),\\
&Df_{-2,-1}(e^2, e^1) = \operatorname{diag}(5, 10, -11, -1, -14), && Df_{-2,-2}(e^2, e^2) = \operatorname{diag}(-5, -10, 11, 10, -3),\\
&Df_{-2,-3}(e^2, e^3) = \operatorname{diag}(3, -6, 1, -10, -13), && Df_{-2,-4}(e^2, e^4) = \operatorname{diag}(-12, -3, 14, 3, 13),\\
&Df_{-3,-1}(e^3, e^1) = \operatorname{diag}(-5, -10, 3, 8, 3), && Df_{-3,-2}(e^3, e^2) = \operatorname{diag}(5, 10, -3, 5, 0),\\
&Df_{-3,-3}(e^3, e^3) = \operatorname{diag}(9, 6, -8, -5, -5), && Df_{-3,-4}(e^3, e^4) = \operatorname{diag}(-9, 3, -3, 0, 5).
\end{aligned}$$

Since the first two elements and the last three elements of each $Df_{-i,-j}(e^i, e^j)$ are eigenvalues of $Df^A_{-i}(e^i, e^j)$ and $Df^B_{-j}(e^i, e^j)$, respectively, we have

$$\begin{array}{lll}
\mathrm{index}(1,1) = 3, & \mathrm{index}_A(1,1) = 1, & \mathrm{index}_B(1,1) = 2, \\
\mathrm{index}(1,2) = 2, & \mathrm{index}_A(1,2) = 1, & \mathrm{index}_B(1,2) = 1, \\
\mathrm{index}(1,3) = 3, & \mathrm{index}_A(1,3) = 0, & \mathrm{index}_B(1,3) = 3, \\
\mathrm{index}(1,4) = 2, & \mathrm{index}_A(1,4) = 2, & \mathrm{index}_B(1,4) = 0, \\
\mathrm{index}(2,1) = 2, & \mathrm{index}_A(2,1) = 2, & \mathrm{index}_B(2,1) = 0, \\
\mathrm{index}(2,2) = 2, & \mathrm{index}_A(2,2) = 0, & \mathrm{index}_B(2,2) = 2, \\
\mathrm{index}(2,3) = 2, & \mathrm{index}_A(2,3) = 1, & \mathrm{index}_B(2,3) = 1, \\
\mathrm{index}(2,4) = 3, & \mathrm{index}_A(2,4) = 0, & \mathrm{index}_B(2,4) = 3, \\
\mathrm{index}(3,1) = 3, & \mathrm{index}_A(3,1) = 0, & \mathrm{index}_B(3,1) = 3, \\
\mathrm{index}(3,2) = 3, & \mathrm{index}_A(3,2) = 2, & \mathrm{index}_B(3,2) = 1, \\
\mathrm{index}(3,3) = 2, & \mathrm{index}_A(3,3) = 2, & \mathrm{index}_B(3,3) = 0, \\
\mathrm{index}(3,4) = 2, & \mathrm{index}_A(3,4) = 1, & \mathrm{index}_B(3,4) = 1.
\end{array}$$

Thus, we obtain

$$\underset{(i,j)\in S_1\times S_2}{\mathrm{argmin}}\ \mathrm{index}(i,j) = \{(1,2),\ (1,4),\ (2,1),\ (2,2),\ (2,3),\ (3,3),\ (3,4)\},$$

and

$$\begin{array}{ll}
|\,\mathrm{index}_A(1,2) - \mathrm{index}_B(1,2)| = 0, & |\,\mathrm{index}_A(1,4) - \mathrm{index}_B(1,4)| = 2, \\
|\,\mathrm{index}_A(2,1) - \mathrm{index}_B(2,1)| = 2, & |\,\mathrm{index}_A(2,2) - \mathrm{index}_B(2,2)| = 2, \\
|\,\mathrm{index}_A(2,3) - \mathrm{index}_B(2,3)| = 0, & |\,\mathrm{index}_A(3,3) - \mathrm{index}_B(3,3)| = 2, \\
|\,\mathrm{index}_A(3,4) - \mathrm{index}_B(3,4)| = 0. &
\end{array}$$

Therefore, the quasi Nash pairs of this game are $(i, j) = (1, 2), (2, 3), (3, 4)$.

6. Conclusions

In this paper, we have presented a game-theoretical meaning for all vertices in the simple simplex of the replicator dynamics of asymmetric two-person games. For this purpose, we have defined an *index* for a pure-strategy pair by using the Jacobian matrix of the replicator dynamics, and we have then considered the game-theoretical meaning of this index. As a result, we have concluded that the number of the positive eigenvalues of the Jacobian matrices of the replicator dynamics with respect to players i and j are a measure of each player's climb-down; see Theorem 4.1. Taking this viewpoint, we say that this paper's main result is a kind of extension of the folk theorem of evolution game theory. More put mathematically, the dimension of the unstable manifold of vertices in the simple simplex for the replicator dynamics with respect to a player is a measure of the player's climb-down if the Jacobian matrices of the replicator dynamics do not have zero as an eigenvalue.

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