A set of variant Hermite tetrahedral elements for three-dimensional problems

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Abstract.
We present a set of variant Hermite tetrahedral elements of degree three for three-dimensional problems. A finite element space constructed from these elements has advantages that the degrees of freedom are much smaller than those of the Lagrange element and that it is easily applicable to problems subject to Dirichlet boundary conditions. Applying it to Poisson problems, we prove best possible a priori error estimates. Two numerical examples reflect the theoretical results.

Keywords. variant Hermite elements, tetrahedral elements, a priori error estimates

1. Introduction

The finite element method is one of the most powerful techniques for numerical solutions of partial differential equations. Many kinds of elements have been developed for various problems [2], [3]. Here we focus on the Lagrange element and the Hermite element of degree three. Both elements have a high approximation property. While the former is easily applied to problems subject to Dirichlet boundary conditions, the latter has an advantage that the degrees of freedom (DOF) are much smaller than those of the former. In order to gain both advantages we have presented a set of variant triangular Hermite elements for two-dimensional problems [6]. Those elements connect the Lagrange element and the Hermite element.

In this paper we extend the result to three-dimensional problems and present a set of variant Hermite tetrahedral elements. After discussing the interpolation property, we apply it to Poisson problems and prove best possible a priori error estimates in H^1- and L^2-norms.

The contents of this paper are as follows. In Section 2 we prepare some definitions and assumptions. In Section 3 we present a set of variant Hermite tetrahedral elements and discuss the interpolation error. In Section 4, applying it to Poisson problem, we give a priori error estimates. Numerical results are shown in Section 5 and some remarks are given in Section 6.

2. Preliminaries

Let Ω be a convex polyhedral domain in $\mathbb{R}^3$ with boundary Γ. Let us consider a simplicial decomposition of Ω composed of elements $\{K\}$. Each K is an image of the reference element $\hat{K} = [\hat{a}_1, \cdots, \hat{a}_4] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{0}]$, shown in Fig. 1, by an affine mapping $F_K : \hat{K} \to \mathbb{R}^3$. We set $h_K = \mathrm{diam} K$, $\rho_K = \sup\{\mathrm{diam}\ S;\ S \text{ is a ball contained in } K\}$, and $h = \max_K h_K$. We deal with a family of simplicial decompositions of Ω, $\{\mathcal{T}_h\}_{h\downarrow 0}$. Let us assume the followings:

(T1) All nodes on the boundary of the polyhedron $\Omega_h \equiv (\cup_{K\in\mathcal{T}_h} K)^\circ$ are on Γ and $\Omega = \Omega_h$.

(T2) There exists a constant $\sigma_1 > 0$ such that for all h and all $K \in \mathcal{T}_h$,

$$h_K \le \sigma_1 \rho_K. \tag{1}$$

(T3) There exists a constant $\sigma_2 > 0$ such that for all h and all $K \in \mathcal{T}_h$,

$$h_K \ge \sigma_2 h. \tag{2}$$

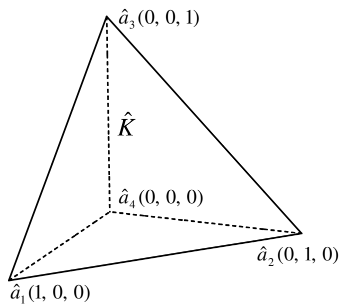

Figure 1: The reference tetrahedral element $\hat{K}$.

Let $L^2(\Omega)$ and $H^m(\Omega)$, $m \ge 1$, be the Sobolev spaces with norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{m,\Omega}$, respectively. We also use the semi-norms $|\cdot|_{m,\Omega}$. The symbol $(\cdot,\cdot)$ is used for the inner product in $L^2(\Omega)$. Similarly the Sobolev spaces $L^2(\Gamma)$ and $H^m(\Gamma)$, $m \ge 1$, and their norms and semi-norms are defined. The symbol $[\cdot,\cdot]$ is used for the inner product in $L^2(\Gamma)$. We denote by $v_{,i}$ the derivative of a function v

with respect to x_i for $i = 1, 2, 3$. Throughout this paper, we us use C and c as generic numerical positive constants independent of h, which may take a different value at each occurrence.

3. A set of variant Hermite elements

In this section, after reviewing the conventional Lagrange and Hermite elements [3], we introduce a set of variant tetrahedral elements for three-dimensional problems. As for a set of variant triangular elements for two-dimensional problems we refer to [6].

3.1. Lagrange and Hermite elements

We denote by LP3 and HP3 the Lagrange element and the Hermite element of degree three, respectively. Let $\mathcal{P}_3$ denote the space of polynomials of degree ≤ 3. Let us set $\hat{a}_{iij} = \frac{1}{3}(2\hat{a}_i + \hat{a}_j)$ $(1 \leq i, j \leq 4,\ i \neq j)$ and $\hat{a}_{ijk} = \frac{1}{3}(\hat{a}_i + \hat{a}_j + \hat{a}_k)$ $(1 \leq i < j < k \leq 4)$. Let us write the components of a point or a vector $a_i \in \mathbb{R}^3$ as $\{a_m^{(i)}\}_{1 \leq m \leq 3}$. Let $\{\lambda_i\}_{i=1}^4$ be the system of barycentric coordinates of $\hat{K}$. We define four index sets by

$$I_V^0 = \{i;\ 1 \leq i \leq 4\}, \tag{3a}$$

$$I_E = \{(i,j);\ 1 \leq i \leq 4,\ 1 \leq j \leq 4,\ i \neq j\}, \tag{3b}$$

$$I_F = \{(i,j,k);\ 1 \leq i < j < k \leq 4\}, \tag{3c}$$

$$I_V^1 = \{(i,m);\ 1 \leq i \leq 4,\ 1 \leq m \leq 3\}. \tag{3d}$$

The numbers of DOF in LP3 and HP3 on $\hat{K}$ are 20, and their $\mathcal{P}_3$-unisolvent sets are

$$\begin{aligned} \hat{\Sigma}^L &= \left\{p(\hat{a}_i);\ i \in I_V^0\right\} \cup \left\{p(\hat{a}_{iij});\ (i,j) \in I_E\right\} \\ &\quad \cup \left\{p(\hat{a}_{ijk});\ (i,j,k) \in I_F\right\}, \\ \hat{\Sigma}^H &= \left\{p(\hat{a}_i);\ i \in I_V^0\right\} \cup \left\{p(\hat{a}_{ijk});\ (i,j,k) \in I_F\right\} \\ &\quad \cup \left\{p_{,m}(\hat{a}_i);\ (i,m) \in I_V^1\right\}. \end{aligned}$$

Let $\hat{\Phi}^L$ and $\hat{\Phi}^H$ be sets of functions defined by

$$\begin{aligned} \hat{\Phi}^L &= \left\{\hat{\phi}_i^{0,L};\ i \in I_V^0\right\} \cup \left\{\hat{\phi}_{iij}^{0,L};\ (i,j) \in I_E\right\} \\ &\quad \cup \left\{\hat{\phi}_{ijk}^{0,L};\ (i,j,k) \in I_F\right\}, \\ \hat{\Phi}^H &= \left\{\hat{\phi}_i^{0,H};\ i \in I_V^0\right\} \cup \left\{\hat{\phi}_{ijk}^{0,H};\ (i,j,k) \in I_F\right\} \\ &\quad \cup \left\{\hat{\phi}_{im}^{1,H};\ (i,m) \in I_V^1\right\}, \end{aligned}$$

where

$$\begin{aligned} \hat{\phi}_i^{0,L} &\equiv \frac{1}{2}\lambda_i(3\lambda_i - 1)(3\lambda_i - 2), \\ \hat{\phi}_{iij}^{0,L} &\equiv \frac{9}{2}\lambda_i\lambda_j(3\lambda_i - 1), \quad \hat{\phi}_{ijk}^{0,L} \equiv 27\lambda_i\lambda_j\lambda_k, \\ \hat{\phi}_i^{0,H} &\equiv -2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_i \sum_{\substack{1 \leq j < k \leq 4, \\ j \neq i,\ k \neq i}} \lambda_j\lambda_k, \\ \hat{\phi}_{ijk}^{0,H} &\equiv \hat{\phi}_{ijk}^{0,L}, \quad \hat{\phi}_{im}^{1,H} \equiv \sum_{j=1}^4 \left(\hat{a}_m^{(j)} - \hat{a}_m^{(i)}\right)\hat{\phi}_{iij}^*, \\ \hat{\phi}_{iij}^* &\equiv \lambda_i\lambda_j(2\lambda_i + \lambda_j - 1). \end{aligned}$$

The following lemma has been proven on these elements.

Lemma 1. *(Ciarlet-Raviart [4]) $\hat{\Sigma}^L$ and $\hat{\Sigma}^H$ are $\mathcal{P}_3$-unisolvent, and $\hat{\Phi}^L$ and $\hat{\Phi}^H$ form the basis functions, respectively.*

Let $\Pi_K^L : C^0(K) \to \mathcal{P}_3(K)$ be the Lagrange interpolation operator and $\Pi_K^H : C^1(K) \to \mathcal{P}_3(K)$ be the Hermite one. The interpolation error estimates for general $\mathcal{P}_3$-interpolations Π_K onto $\mathcal{P}_3(K)$ have been given in [3], [4].

Lemma 2. *(Ciarlet-Raviart [4]) Let (K, Σ_K, Φ_K) be a finite element of 3-simplex where Σ_K is $\mathcal{P}_3$-unisolvent. Let $\Pi_K^{(l)} : C^l(K) \to \mathcal{P}_3(K)$ denote its interpolation, where $l = 0, 1$ is the maximum order of derivatives occurring in the definition of Σ_K. Let q be an integer with $2 + l \leq q \leq 4$. Under the assumptions (T1) and (T2), it holds that, for $v \in H^q(K)$,*

$$\left\| v - \Pi_K^{(l)} v \right\|_{m,K} \leq C h_K^{q-m} |v|_{q,K} \quad (m = 0, \cdots, q). \tag{4}$$

Therefore, the interpolation estimates for the Lagrange and Hermite elements are

$$\left\| v - \Pi_K^L v \right\|_{m,K} \leq C h_K^{q-m} |v|_{q,K} \quad (q = 2, 3, 4), \tag{5}$$

$$\left\| v - \Pi_K^H v \right\|_{m,K} \leq C h_K^{q-m} |v|_{q,K} \quad (q = 3, 4) \tag{6}$$

for $v \in H^q(K)$ and $m \in [0, q]$.

3.2. Variant Hermite elements

We consider finite element computations of partial differential equations subject to Dirichlet boundary conditions, employing $\mathcal{P}_3$-elements. In the case of the Hermite element there are four DOF at each vertex, one function value and three derivatives. At vertices on the boundary three of them, one function value and two tangential derivatives, are essential boundary conditions and the last one is a natural boundary condition. (In polyhedral domains the fourth DOF may also become a tangential derivative at corner nodes. Then, all DOF are essential boundary conditions.) In the case of the Lagrange element, all DOF are function values, and they are essential boundary conditions on all boundary nodes, which makes the programming much easier. On the other hand the total number of DOF of the Hermite element is far smaller than that of the Lagrange element as shown in Tables 1 and 3.

Here we introduce variant Hermite elements, which connect the Lagrange element and the Hermite element. Thus we can use the Lagrange element in the simplices intersecting Γ and the Hermite element in an interior part of Ω. Let a_i be an internal vertex of $\mathcal{T}_h$ ($a_i \notin \Gamma$), and a_j be any vertex connected to a_i. We collect all DOF, $\{p(a_{iij})\}_j$, of the Lagrange type at $a_{iij} \equiv (2a_i + a_j)/3$, and replace them by Hermitian type DOF, $\{p_{,m}(a_i)\}_{m=1}^3$ (see Fig. 2). We call the present finite element HP3V(Hermite interpolation with Polynomials of degree 3 of Variant type). We note that the DOF around the vertex on Γ are of Lagrangian type $\big($Fig. 2 (a)$\big)$, while the ones at the vertex in Ω are of Hermitian type $\big($Fig. 2 (b)$\big)$

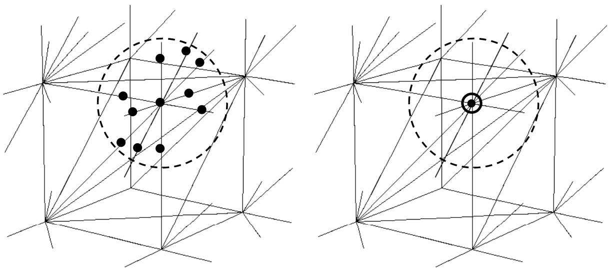

Figure 2: The nodes around a vertex.
(a) left: Lagrangian type (b) right: Hermitian type

For each simplex K we define by s_K the number of vertices in Ω, $s_K = \#\{a_i : \text{vertex of } K;\ a_i \in \Omega\}$. We define two index sets depending on s $(\in [0,4])$,

$$I_E(s) = \{(i,j);\ 1 \le i \le 4-s,\ 1 \le j \le 4,\ i \ne j\},$$
$$I_V^1(s) = \{(i,m);\ 5-s \le i \le 4,\ 1 \le m \le 3\}.$$

We note that $I_E(0) = I_E$ and $I_V^1(4) = I_V^1$, where I_E and I_V^1 are index sets defined in (3b) and (3d), respectively. We number local indices of the vertices so that $a_i \in \Gamma$ $(1 \le i \le 4-s)$ and $a_i \in \Omega$ $(5-s \le i \le 4)$. Let us define a set of variant elements $\mathrm{L}(1-\frac{s}{4})\mathrm{H}(\frac{s}{4})$ $(0 \le s \le 4)$ on $\hat{K}$, where $s=0$ and $s=4$ correspond to the Lagrange element and the Hermite one, respectively. The pairs of DOF and basis functions are defined by

(7)
$$\begin{aligned}\Big(\hat\Sigma(\tfrac{s}{4}), \hat\Phi(\tfrac{s}{4})\Big) \equiv & \Big\{\big(p(\hat a_i), \hat\phi_i^0\big);\ i \in I_V^0\Big\}\\ & \cup \Big\{\Big(p(\hat a_{iij}), \hat\phi_{iij}^0\Big);\ (i,j) \in I_E(s)\Big\}\\ & \cup \Big\{\Big(p(\hat a_{ijk}), \hat\phi_{ijk}^0\Big);\ (i,j,k) \in V_F\Big\}\\ & \cup \Big\{\Big(p_{,m}(\hat a_i), \hat\phi_{im}^0\Big);\ (i,m) \in I_V^1(s)\Big\}\end{aligned}$$

where

$$(8)\qquad \hat\phi_i^0 \equiv \begin{cases} \hat\phi_i^{0,L} - \frac{1}{9}\sum_{j=5-s}^4 \hat\phi_{jji}^{0,L} & (1 \le i \le 4-s)\,, \\ \hat\phi_i^{0,H} - \frac{7}{4}\sum_{j=1}^{4-s} \hat\phi_{jji}^{*} & (5-s \le i \le 4)\,, \end{cases}$$

$$(9)\qquad \hat\phi_{iij}^0 \equiv \begin{cases} \hat\phi_{iij}^{0,L} & (1 \le i,\ j \le 4-s,\ i \ne j)\,, \\ \frac{27}{4}\hat\phi_{iij}^{*} & (1 \le i \le 4-s,\ 5-s \le j \le 4)\,, \end{cases}$$

$$(10)\qquad \hat\phi_{ijk}^0 \equiv \hat\phi_{ijk}^{0,L},$$

$$(11)\qquad \hat\phi_{im}^1 \equiv \hat\phi_{im}^{1,H} - \frac{1}{2}\sum_{j=1}^{4-s}\big(\hat a_m^{(j)} - \hat a_m^{(i)}\big)\hat\phi_{jji}^{*}.$$

The tetrahedral elements $\mathrm{L}(1-\frac{s}{4})\mathrm{H}(\frac{s}{4})$ are shown in Fig. 3. We have the following property whose proof is given

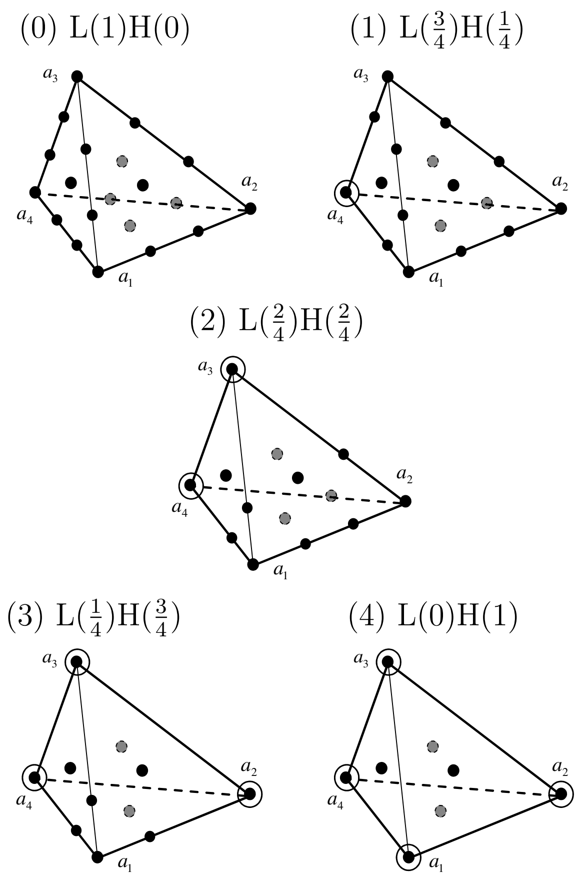

Figure 3: The family of the variant tetrahedral elements.

in Appendix A.1.

Lemma 3. *For $s \in [0,4]$, $\hat\Sigma(\frac{s}{4})$ is $\mathcal{P}_3$-unisolvent and $\hat\Phi(\frac{s}{4})$ forms the basis functions.*

3.3. Interpolation error analysis

We consider the $\mathrm{L}(1-\frac{s}{4})\mathrm{H}(\frac{s}{4})$ finite element, $s \in [1,3]$. Let N_0^s and N_1^s be the numbers of nodes in $\hat K$, where function values and derivative values are imposed,

$$(12)\qquad \begin{aligned}(\hat\Sigma_s, \hat\Phi_s) \equiv & \big\{\big(p(\hat a_i^0), \hat\phi_i^0\big);\ 1 \le i \le N_0^s\big\}\\ & \cup \big\{\big(p_{,m}(\hat a_i^1), \hat\phi_{im}^1\big);\ 1 \le i \le N_1^s,\ 1 \le m \le 3\big\}.\end{aligned}$$

Let $\hat\Pi : C^1(\hat K) \to \mathcal{P}_3(\hat K)$ be the interpolation operator defined by, for $\hat v \in C^1(\hat K)$,

$$(13)\qquad \hat\Pi \hat v = \sum_{i=1}^{N_0^s} \hat v(\hat a_i^0)\hat\phi_i^0 + \sum_{i=1}^{N_1^s}\sum_{m=1}^{3} \hat v_{,m}(\hat a_i^1)\hat\phi_{im}^1.$$

The set of DOF and bases in an element K is denoted by

$$(14)\qquad \begin{aligned}\big(\Sigma_K, \Phi_K\big) \equiv & \big\{\big(p(a_i^{0,K}), \phi_i^{0,K}\big);\ 1 \le i \le N_0^s\big\}\\ & \cup \big\{\big(p_{,m}(a_i^{1,K}), \phi_{im}^{1,K}\big);\ 1 \le i \le N_1^s,\ 1 \le m \le 3\big\},\end{aligned}$$

where the nodes and basis functions are defined by

$$a_i^{0,K} = F_K(\hat{a}_i^0), \quad a_i^{1,K} = F_K(\hat{a}_i^1), \tag{15}$$

$$\phi_i^{0,K} = \hat{\phi}_i^0 \circ F_K^{-1}, \tag{16}$$

$$\phi_{im}^{1,K} = \sum_{l=1}^{3} (F_K)_{m,l}(\hat{a}_i^1)\, \hat{\phi}_{il}^1 \circ F_K^{-1}. \tag{17}$$

The local interpolation in K, $\Pi_K : C^1(K) \to \mathcal{P}_3(K)$ is defined by, for $v \in C^1(K)$,

$$\Pi_K v = \sum_{i=1}^{N_0^s} v(a_i^{0,K})\phi_i^{0,K} + \sum_{i=1}^{N_1^s}\sum_{m=1}^{3} v_{,m}(a_i^{1,K})\phi_{im}^{1,K}. \tag{18}$$

Then it holds that, for $v \in C^1(K)$,

$$(\Pi_K v) \circ F_K = \hat{\Pi}(v \circ F_K), \tag{19}$$

which is easily proven by substituting (15)–(17) into (18). From Lemma 3, we have interpolation error estimates corresponding to (4):

Corollary 1. *Let $1 \le s \le 3$ be an integer and K be an* L$(1-\frac{s}{4})$H$(\frac{s}{4})$ *element. Then, under the assumptions (T1) and (T2) it holds that, for $v \in H^q(K)$ and $q = 3, 4$,*

$$\|v - \Pi_K v\|_{m,K} \le C h_K^{q-m} |v|_{q,K} \quad (m = 0, \cdots, q). \tag{20}$$

Let $N_D \equiv N_D^{(0)} + 3N_D^{(1)}$ be the total numbers of DOF, where $N_D^{(0)}$ and $N_D^{(1)}$ are the number of function value DOF and the number of sets of derivative value DOF, respectively.We denote the nodes in $\mathcal{T}_h$ by $\{a_j^{0,h}\}_{j=1}^{N_D^{(0)}}$ (of function values) and by $\{a_j^{1,h}\}_{j=1}^{3N_D^{(1)}}$ (of derivatives), where $a_{j+mN_D^1}^{1,h} = a_j^{1,h}$ is satisfied for $m = 1, 2$ and $j = 1, \cdots, N_D^{(1)}$. Let $\hat{j}$ be the local number of the node such that $a_j^{0,h} = a_{\hat{j}}^{0,K}$ or $a_j^{1,h} = a_{\hat{j}}^{1,K}$. The corresponding basis functions $\{\phi_j^{0,h}\}_{j=1}^{N_D^{(0)}}$ and $\{\phi_j^{1,h}\}_{j=1}^{3N_D^{(1)}}$ are defined by

$$\phi_j^{0,h} = \begin{cases} \phi_{\hat{j}}^{0,K} & (a_j^{0,h} \in \exists K), \\ 0 & (\text{otherwise}) \end{cases} \tag{21}$$

for $j = 1, \cdots, N_D^{(0)}$, and by

$$\phi_{j+(l-1)N_D^{(1)}}^{1,h} = \begin{cases} \phi_{\hat{j}l}^{1,K} & \left(a_j^{1,h} \in \exists K\right), \\ 0 & (\text{otherwise}) \end{cases} \tag{22}$$

for $j = 1, \cdots, N_D^{(1)}$ and $l = 1, 2, 3$. It can be easily shown that, for $j = 1, \cdots, N_D^{(0)}$ and $m = 1, 2, 3$,

$$\phi_j^{0,h}(a_i^{0,h}) = \delta_{ij}, \quad (i = 1, \cdots, N_D^{(0)}),$$
$$\phi_{j,m}^{0,h}(a_i^{1,h}) = 0, \quad (i = 1, \cdots, N_D^{(1)})$$

and that, for $j = 1, \cdots, N_D^{(1)}$ and $l, m = 1, 2, 3$,

$$\phi_{j+(l-1)N_D^{(1)}}^{1,h}(a_i^{0,h}) = 0, \ (i = 1, \cdots, N_D^{(0)}),$$
$$\left(\phi_{j+(l-1)N_D^{(1)}}^{1,h}\right)_{,m}(a_i^{1,h}) = \delta_{ij}\delta_{lm}, \ (i = 1, \cdots, N_D^{(1)}).$$

We define the HP3V element space X_h by

$$X_h = \left\{ \sum_{j=1}^{N_D^{(0)}} c_j^0 \phi_j^{0,h} + \sum_{j=1}^{3N_D^{(1)}} c_j^1 \phi_j^{1,h};\ c_j^0, c_j^1 \in \mathbb{R} \right\}, \tag{23}$$

which is shown to be H^1-conforming.

Lemma 4. *It holds that*

$$X_h \subset C^0(\bar{\Omega}) \cap H^1(\Omega).$$

Proof. It suffices to prove the continuity of $v_h \in X_h$ on each face. Let K be an L$(1-\frac{s}{4})$H$(\frac{s}{4})$ element, $s \in [0, 4]$, with vertices $\{a_i\}_{i=1}^4$. Let $L \subset \partial K$ be a surface of K and a_{i_*} be a vertex of K such that $a_{i_*} \notin L$. We take a set of orthonormal vectors $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ such that $\boldsymbol{e}'_3 \perp L$. For $i \in [5-s, 4]$, $i \ne i_*$, and $m = 1, 2, 3$ we replace the DOF $p_{,m}(a_i^{1,K})$ by $Dp(a_i^{1,K})\boldsymbol{e}'_m$, and the basis functions by

$$\phi'^{1,K}_{im} = \sum_{l=1}^{3} \boldsymbol{e}'^{(m)}_l \phi_{il}^{1,K}. \tag{24}$$

On the surface L it holds that, for $i \ne i_*$,

$$\phi_{i_*}^{0,K} = \phi_{i_*i_*j}^{0,K} = \phi_{iii_*}^{0,K} = \phi_{i_*jk}^{0,K} = \phi_{ii_*k}^{0,K} = \phi_{iji_*}^{0,K}$$
$$= \phi'^{1,K}_{i_*m} = \phi'^{1,K}_{i3} = 0.$$

Here, the total number of bases vanishing on L is 10. Since any function in the space $\mathcal{P}_3(L)$ is uniquely determined by the rest 10 DOF, $v_{h|L}$ is determined by them. This shows the continuity of the function v_h on L. □

The HP3V interpolation $\Pi_h : C^1(\Omega) \cap C^0(\bar{\Omega}) \to X_h$ is defined by

(25)
$$\Pi_h v = \sum_{i=1}^{N_D^{(0)}} v(a_i^{0,h})\phi_i^{0,h} + \sum_{i=1}^{N_D^{(1)}}\sum_{m=1}^{3} v_{,m}(a_i^{1,h})\phi_{i+(m-1)N_D^{(1)}}^{1,h}.$$

Remark 1. We note that $\left(\Pi_h v\right)_{|K} = \Pi_K(v_{|K})$ for $v \in H^3(\Omega)$ and $K \in \mathcal{T}_h$, and $\Pi_h v = 0$ on Γ for $v \in H^3(\Omega) \cap H_0^1(\Omega)$.

4. Application to the Poisson problem

We apply the HP3V finite element to Poisson problems subject to Dirichlet boundary conditions. We establish a priori error estimates of the finite element solution in Theorem 1.

The problem we consider is to find a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases} \tag{26}$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$, and $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ are given functions.

4.1. Finite element scheme and a priori error estimates

Let us define the space V_h and its affine set $V_h(g)$ as follows:

(27) $V_h(g) \equiv \{v_h \in X_h;\ v_h(a_j) = g(a_j),\ \forall a_j : \text{node on } \Gamma\}$,
(28) $V_h \equiv V_h(0)$.

We solve the Poisson problem (26) by an HP3V finite element scheme. The discrete problem we deal with is to find $u_h \in V_h(g)$ which satisfies

$$a(u_h, v_h) = \langle f_h, v_h \rangle, \quad \forall v_h \in V_h, \tag{29}$$

where

$$a(u_h, v_h) \equiv \int_\Omega \nabla u_h \cdot \nabla v_h dx, \tag{30}$$

$$\langle f_h, v_h \rangle \equiv \int_\Omega (\Pi_h f) v_h dx. \tag{31}$$

Theorem 1. *Let Ω be a convex polyhedron. Let u_h be the HP3V finite element solution of* (29). *Under the assumptions (T1)–(T3), $u \in H^4(\Omega)$, and $f \in H^{3+m}(\Omega)$ $(m = 0, 1)$, it follows that*

$$\|u - u_h\|_{1,\Omega} \le Ch^3 \left(|u|_{4,\Omega} + h^m |f|_{3+m,\Omega} \right). \tag{32}$$

Moreover, if $f \in H^4(\Omega)$ and $g \in H^4(\Gamma)$, then the following holds:

$$\|u - u_h\|_{0,\Omega} \le Ch^4 \left(|u|_{4,\Omega} + |f|_{4,\Omega} + |g|_{4,\Gamma} \right). \tag{33}$$

Since the H^1-error estimate is proven by the standard theory (for instance, see [3]), we omit the proof. The L^2-error estimate (33) is proven by the Aubin-Nitsche process [1], which is shown here for completeness.

Let us consider the following Poisson problem. Find $w : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta w = u - u_h & \text{in } \Omega, \\ \quad w = 0 & \text{on } \Gamma. \end{cases} \tag{34}$$

Since Ω is a convex polyhedron, we have $w \in H^2(\Omega)$ and

$$\|w\|_{2,\Omega} \le C \|u - u_h\|_{0,\Omega}. \tag{35}$$

We use the Clément $\mathcal{P}_3$-interpolation $\Pi^C_{h0} : H^1_0(\Omega) \to V_h$.

Lemma 5. *(Clément [5]) Let q and m be integers, $q \in [0, 4]$ and $m \in [0, q]$. Under the assumptions (T1)–(T3), it holds that, for $v \in H^q(\Omega) \cap H^1_0(\Omega)$,*

$$\left(\sum_{K \in \mathcal{T}_h} \left| v - \Pi^C_{h0} v \right|^2_{m,K} \right)^{\frac{1}{2}} \le Ch^{q-m} |v|_{q,\Omega}. \tag{36}$$

4.2. Proof of (33) in Theorem 1

Since $w \in H^2(\Omega)$ and $\Pi^C_{h0} w \in H^1_0(\Omega)$, we deduce that

$$\begin{aligned}
&\|u - u_h\|^2_{0,\Omega} \\
&= (u - u_h, -\Delta w)_\Omega \\
&= (\nabla(u - u_h), \nabla w)_\Omega - \left[u - u_h, \frac{\partial w}{\partial n} \right]_\Gamma \\
&= \left(\nabla(u - u_h), \nabla(w - \Pi^C_{h0} w) \right)_\Omega \\
&\quad + \left(\nabla(u - u_h), \nabla \Pi^C_{h0} w \right)_\Omega - \left[u - \Pi_h u, \frac{\partial w}{\partial n} \right]_\Gamma \\
&= \left(\nabla(u - u_h), \nabla(w - \Pi^C_{h0} w) \right)_\Omega \\
&\quad + \left(f - \Pi_h f, \Pi^C_{h0} w \right)_\Omega - \left[u - \Pi_h u, \frac{\partial w}{\partial n} \right]_\Gamma \\
&\le \|\nabla(u - u_h)\|_{0,\Omega} \left\| \nabla(w - \Pi^C_{h0} w) \right\|_{0,\Omega} \\
&\quad + \|f - \Pi_h f\|_{0,\Omega} \left\| \Pi^C_{h0} w \right\|_{0,\Omega} + \|u - \Pi_h u\|_{0,\Gamma} \left\| \frac{\partial w}{\partial n} \right\|_{0,\Gamma} \\
&\equiv I_1 + I_2 + I_3.
\end{aligned} \tag{37}$$

Here, in the third equality, we have used the fact that $u_h = \Pi_h u$ on Γ. At first, we observe

$$\left\| \nabla(w - \Pi^C_{h0} w) \right\|_{0,\Omega} \le ch |w|_{2,\Omega}, \tag{38}$$

$$\left\| \Pi^C_{h0} w \right\|_{0,\Omega} \le c \|w\|_{0,\Omega}, \tag{39}$$

$$\left\| \frac{\partial w}{\partial n} \right\|_{0,\Gamma} \le c \|w\|_{2,\Omega}, \tag{40}$$

where (38) and (39) are derived from Lemma 5 and (40) is the consequence of the trace theorem. The first term I_1 is estimated by using H^1-error estimate (32):

$$\begin{aligned}
I_1 &\le ch^3 \left(|u|_{4,\Omega} + h |f|_{4,\Omega} \right) h |w|_{2,\Omega} \\
&\le ch^4 \left(|u|_{4,\Omega} + h |f|_{4,\Omega} \right) \|u - u_h\|_{0,\Omega}.
\end{aligned} \tag{41}$$

As for I_2, we use Corollary 1 to get

$$\begin{aligned}
I_2 &\le ch^4 |f|_{4,\Omega} \|w\|_{0,\Omega} \\
&\le ch^4 |f|_{4,\Omega} \|u - u_h\|_{0,\Omega}.
\end{aligned} \tag{42}$$

In I_3, each Π_K is equivalent to the Lagrange $\mathcal{P}_3$-interpolation on $L \equiv K \cap \Gamma$: $H^2(L) \to \mathcal{P}_3(L)$. Hence, we obtain

$$\begin{aligned}
I_3 &\le ch^4 |g|_{4,\Gamma} \|w\|_{2,\Omega} \\
&\le ch^4 |g|_{4,\Gamma} \|u - u_h\|_{0,\Omega}.
\end{aligned} \tag{43}$$

Combining (37) with (41)–(43), we obtain the inequality (33).

5. Numerical examples

We show two numerical results to observe the effectiveness of our scheme by solving (26). Given a domain Ω and an exact solution u, we calculate f and g. Solving (26) by the HP3V scheme (29), we obtain the finite element solution

u_h. We compute the relative errors in H_0^1- and L^2-norms defined by

$$\text{(a)} \ \frac{\|\nabla(\Pi_h u - u_h)\|_{0,\Omega}}{\|\nabla(\Pi_h u)\|_{0,\Omega}}, \quad \text{(b)} \ \frac{\|\Pi_h u - u_h\|_{0,\Omega}}{\|\Pi_h u\|_{0,\Omega}}.$$

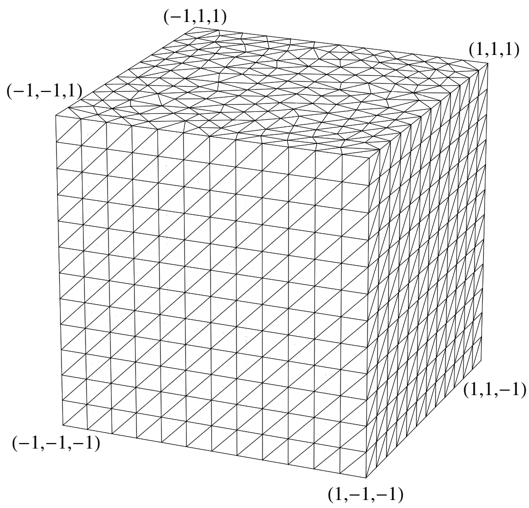

Figure 4: A simplicial decomposition $\mathcal{T}_h$ when $N = 12$ in Example 1.

Table 1: The numbers of DOF in HP3, HP3V and LP3 in Example 1.

N	HP3	HP3V	LP3	HP3V/LP3
6	4,894	6,346	8,056	0.788
12	36,526	42,627	60,976	0.700
24	275,524	300,612	462,601	0.650
48	2,134,282	2,235,544	3,592,810	0.622

Table 2: Relative errors and slopes in Example 1.

N	h	(a)	m	(b)	m
6	6.26E-01	3.81E-02		7.04E-03	
12	3.00E-01	5.99E-03	2.52	6.91E-04	3.19
24	1.52E-01	8.71E-04	2.83	5.55E-05	3.70
48	8.12E-02	1.18E-04	3.19	3.82E-06	4.27

Example 1. The domain Ω and u are

$$\Omega = \{x \in \mathbb{R}^3; |x_i| < 1 \ (i = 1, 2, 3)\},$$
$$u(x_1, x_2, x_3) = \sin \pi x_1 \cos \pi x_2 \exp x_3. \tag{44}$$

We make use of simplicial decompositions $\mathcal{T}_h$ with

$$N = 6, 12, 24, 48,$$

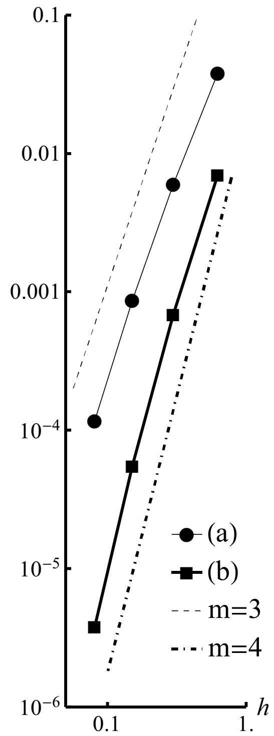

Figure 5: Graphs of relative errors in (a)H_0^1- and (b)L^2-norms in Example 1.

where N indicates the number of division of each edge. A mesh is shown in Fig. 4. The numbers of DOF in HP3, HP3V and LP3 are shown in Table 1. Fig. 5 shows graphs of the relative errors vs. the mesh size h in logarithmic scale, and Table 2 shows the slopes m of the segments of the graphs. Those results reflect the theoretical convergence orders, $m = 3, 4$, given in (32) and (33).

Table 3: The numbers of DOF in HP3, HP3V and LP3 in Example 2.

N	HP3	HP3V	LP3	HP3V/LP3
9	67,900	77,510	113,590	0.682
18	534,808	574,940	898,849	0.640
36	4,170,352	4,332,012	7,023,655	0.617

Example 2. The domain Ω is a prism shown in Fig. 6, and u is given as

$$u(x_1, x_2, x_3) = \sin \pi x_1 \cos \pi x_2 \exp x_3. \tag{45}$$

We make use of simplicial decompositions $\mathcal{T}_h$ with

$$N = 9, 18, 36,$$

where N is the number of division of the shortest edge between the points $(0, 2, 3)$ and $(1, 4, 3)$. A decomposition

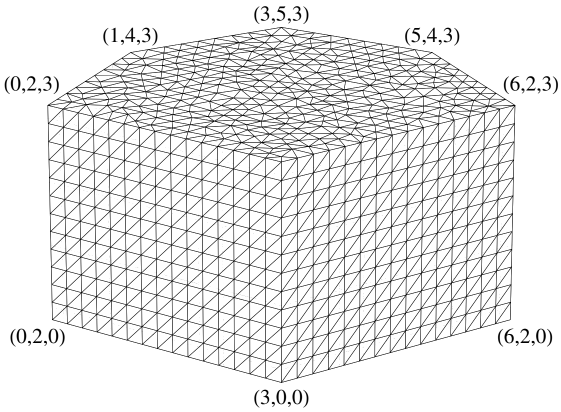

Figure 6: A simplicial decomposition $\mathcal{T}_h$ when $N = 9$ in Example 2.

Table 4: Relative errors and slopes in Example 2.

N	h	(a)	m	(b)	m
9	5.06E-01	2.07E-02		2.96E-03	
18	2.43E-01	2.91E-03	2.67	2.62E-04	3.30
36	1.21E-01	3.86E-04	2.89	1.87E-05	3.77

is shown in Fig. 6. The numbers of DOF in HP3, HP3V and LP3 are shown in Table 3. Fig. 7 shows graphs of the relative errors vs. the mesh size h in logarithmic scale, and Table 2 shows the slopes m of the segments of the graphs. Those results reflect the theoretical convergence orders, $m = 3, 4$, given in (32) and (33).

6. Concluding remarks

We have introduced a set of variant Hermite tetrahedral elements of degree three, which connect the Hermite element and the Lagrange element. We have discussed the approximation property of the HP3V element constructed from the elements. Applying the element to Poisson problems, we have proven a priori convergence error estimates (32) and (33) in the H^1- and L^2-norms with rates of $O(h^3)$ and $O(h^4)$, respectively. The HP3V element can be applied not only to Poisson problems but also to various problems. In order to maintain the high convergence order for domains of general shape the isoparamteric element procedure is required. We will discuss isoparametric HP3V elements for curved domains in forthcoming papers.

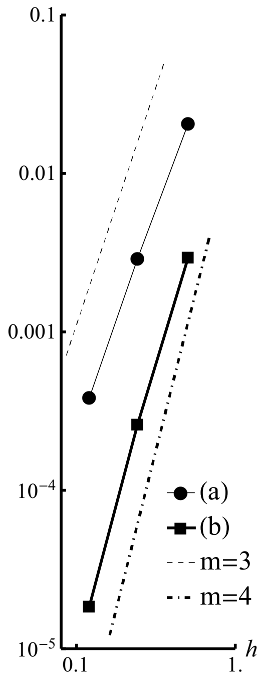

Figure 7: Graphs of relative errors in (a)H_0^1- and (b)L^2-norms in Example 2.

Appendix

A.1. Proof of Lemma 3

Proof is required only for $s \in [1, 3]$. The number of DOF is 20 in each variant element and is equal to $\dim \mathcal{P}_3$. It is sufficient to check for $\hat{p} \in \mathcal{P}_3(\hat{K})$,

$$\hat{p} = \sum_{i \in I_V^0} \hat{p}(\hat{a}_i)\hat{\phi}_i^0 + \sum_{(i,j)\in I_E(s)} \hat{p}(\hat{a}_{iij})\hat{\phi}_{iij}^0 + \sum_{(i,j,k)\in I_F} \hat{p}(\hat{a}_{ijk})\hat{\phi}_{ijk}^0 + \sum_{(i,m)\in I_V^1(s)} \hat{p}_{,m}(\hat{a}_i)\hat{\phi}_{im}^1. \tag{46}$$

Since the functions $\hat{\phi}_i^{0,L}$, $\hat{\phi}_{iij}^{0,L}$ and $\hat{\phi}_{ijk}^{0,L}$ are base functions of the Lagrange element, and since $\hat{\phi}_i^{0,H}$, $\hat{\phi}_{im}^{1,H}$ and $\hat{\phi}_{ijk}^{0,L}$ are those of the Hermite element, the followings are obtained: For $i \in I_V^0$,

$$\begin{cases} \hat{\phi}_i^0(\hat{a}_p) = \delta_{ip} & \big(p \in I_V^0\big), \\ \hat{\phi}_i^0(\hat{a}_{ppq}) = 0 & \big((p,q) \in I_E(s)\big), \\ \hat{\phi}_i^0(\hat{a}_{pqr}) = 0 & \big((p,q,r) \in I_F\big), \\ \hat{\phi}_{i,l}^0(\hat{a}_p) = 0 & \big((p,l) \in I_V^1(s)\big). \end{cases} \tag{47}$$

For $(i,j) \in I_E(s)$,

$$\begin{cases} \hat{\phi}_{iij}^0(\hat{a}_p) = 0 & \big(p \in I_V^0\big), \\ \hat{\phi}_{iij}^0(\hat{a}_{ppq}) = \delta_{ip}\delta_{jq} & \big((p,q) \in I_E(s)\big), \\ \hat{\phi}_{iij}^0(\hat{a}_{pqr}) = 0 & \big((p,q,r) \in I_F\big), \\ \hat{\phi}_{iij,l}^0(\hat{a}_p) = 0 & \big((p,l) \in I_V^1(s)\big). \end{cases} \tag{48}$$

For $(i,j,k) \in I_F$,

$$
\begin{cases}
\hat{\phi}^0_{ijk}(\hat{a}_p) = 0 \quad \big(p \in I_V^0\big), \\
\hat{\phi}^0_{ijk}(\hat{a}_{ppq}) = 0 \quad \big((p,q) \in I_E(s)\big), \\
\hat{\phi}^0_{ijk}(\hat{a}_{pqr}) = \delta_{ip}\delta_{jq}\delta_{kr} \ \big((p,q,r) \in I_F\big), \\
\hat{\phi}^0_{ijk,l}(\hat{a}_p) = 0 \quad \big((p,l) \in I_V^1(s)\big).
\end{cases} \tag{49}
$$

For $(i,m) \in I_V^1(s)$,

$$
\begin{cases}
\hat{\phi}^1_{im}(\hat{a}_p) = 0 \quad \big(p \in I_V^0\big), \\
\hat{\phi}^1_{im}(\hat{a}_{ppq}) = 0 \quad \big((p,q) \in I_E(s)\big), \\
\hat{\phi}^1_{im}(\hat{a}_{pqr}) = 0 \quad \big((p,q,r) \in I_F\big), \\
\hat{\phi}^1_{im,l}(\hat{a}_p) = \delta_{ip}\delta_{ml} \ \big((p,l) \in I_V^1(s)\big).
\end{cases} \tag{50}
$$

Let us prove (47). From (8) we get

$$
\begin{aligned}
\hat{\phi}^0_i(\hat{a}_p) &= \hat{\phi}^{0,L}_i(\hat{a}_p) - \frac{1}{9}\sum_{j=5-s}^{4} \hat{\phi}^0_{jji,L}(\hat{a}_p) \\
&= \delta_{ip} - 0 = \delta_{ip} \ (p \in I_V^0), \\
\hat{\phi}^0_i(\hat{a}_{ppq}) &= \hat{\phi}^{0,L}_i(\hat{a}_{ppq}) - \frac{1}{9}\sum_{j=5-s}^{4} \hat{\phi}^0_{jji,L}(\hat{a}_{ppq}) \\
&= 0 - \frac{1}{9}\sum_{j=5-s}^{4} \delta_{jp}\delta_{iq} = 0 \ \big((p,q) \in I_E(s)\big), \\
\hat{\phi}^0_i(\hat{a}_{pqr}) &= \hat{\phi}^{0,L}_i(\hat{a}_{pqr}) - \frac{1}{9}\sum_{j=5-s}^{4} \hat{\phi}^0_{jji,L}(\hat{a}_{pqr}) \\
&= 0 - 0 = 0 \ \big((p,q,r) \in I_F\big), \\
\hat{\phi}^0_{i,l}(\hat{a}_p) &= \frac{1}{2}\big(\lambda_i(3\lambda_i - 1)(3\lambda_i - 2)\big)_{,l}(\hat{a}_p) \\
&\quad - \frac{1}{2}\sum_{j=5-s}^{4} \big(\lambda_i\lambda_j(3\lambda_j - 1)\big)_{,l}(\hat{a}_p) \\
&= \frac{1}{2}\lambda_{i,l}(27\lambda_i^2 - 18\lambda_i + 2)(\hat{a}_p) \\
&- \frac{1}{2}\sum_{j=5-s}^{4} \big(\lambda_{i,l}\lambda_j(3\lambda_j - 1) + \lambda_{j,l}\lambda_i(6\lambda_j - 1)\big)(\hat{a}_p) \\
&= \frac{1}{2}(9\delta_{ip} + 2)\lambda_{i,l} \\
&\quad - \frac{1}{2}\sum_{j=5-s}^{4} \big(\lambda_{i,l}\delta_{jp}(3\delta_{jp} - 1) + \lambda_{j,l}\delta_{ip}(6\delta_{jp} - 1)\big) \\
&= \lambda_{i,l} - \frac{1}{2}(3-1)\lambda_{i,l} = 0 \quad \big((p,l) \in I_V^1(s)\big).
\end{aligned}
$$

The others are proven similarly. (47)–(50) imply the identity (46).

Acknowledgments

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