Pulse dynamics for reaction-diffusion systems in the neighborhood of codimension two singularity

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Pulse dynamics for reaction-diffusion systems in the neighborhood of codimension two singularity

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Abstract. The dynamics of a pulse for reaction-diffusion systems in 1D is considered in the neighborhood of the bifurcation point with codimension two, at which both of saddle-node and drift bifurcations occur at the same time. It is theoretically shown that when the bifurcation parameter is close to such a bifurcation point, a pulse moves with oscillation, and then starts to split.

Keywords. traveling pulse, self-replicating of pulses, saddle-node bifurcation, drift bifurcation

1. Introduction

In nature, many kinds of spatial and/or temporal patterns are observed in a variety of forms. To understand the dynamics of such patterns, mathematical modeling is one of the powerful approaches and a various types of models have been proposed and analyzed. Among them, reaction-diffusion systems are the most well-studied and display rich dynamics.

Recently, several reaction-diffusion model equations have been known as examples exhibiting various complicated behaviors of spatially localized patterns; self-replicating behavior of pulses ([13], [4] and their references), refleciton of pulses ([3]), the behavior of pulses like elastic objects (e.g. [7], [8], [15], [14]), behaviors on inhomogeneous media ([11], [12]). Through these works, it has been known well that the properties of singlarities have an important role to understand those complicated behaviors. For example, self-replicating behavior of pulses occurs near saddle-node bifurcation point ([13], [4]) and refleciton of pulses does near the drift bifurcation point ([3]). To understand such dynamics theoretically, we will construct invariant manifolds around pulses locally for unstable modes but globally for translation.

In this paper, we will consider the dynamics of pulses near the degenerate bifurcation point with properties both of saddle-node and drift bifurcations. In the case, the bifurcation point is a singularity with codimension two except translation as in the assumption 4) in Section 4. Also we can construct invariant manifolds around pulses and write down the explicit equations of ODE describing the motion of the pulses. Thus the approach allows us to reduce the PDE dynamics to finite dimensional one and analyze various motions of pulses globally in space. Note that similar ODEs near the same codimension 2 singularity were formally derived in [12] to understand the dynamics in heterogeneous media.

The details of mathematically technical parts will be stated in [5]. In this paper, we mainly analyze the reduced ODE in the neighborhood of codimension two bifurcation point and show the occurrence of an oscillatory traveling pulse and self-replication of it. We will also show by numerics that this situation really occurs in the Gray-Scott model on $\boldsymbol{R}^1$.

2. Pulse dynamics in the Gray-Scott model

We consider the following equations called the Gray-Scott model [6]

(2.1)
$$\begin{cases} u_t &= d_1 u_{xx} - uv^2 + A(1-u), \\ v_t &= d_2 v_{xx} + uv^2 - (A+k)v, \end{cases} \quad -\infty < x < \infty.$$

All coefficients d_1, d_2 and A, k in (2.1) are positive constants. It is known that (2.1) show various dynamics of pulses depending on the parameters. Specially, there exist stable stationary pulse solutions for some parameter regions with a profile as in Fig1(e.g. [1]). We note that the stationary solution converges $(1,0)$ as $|x| \to \infty$, where $(1,0)$ is a stable equilibrium of the kinetics of (2.1). When we fix the coefficients d_1, d_2 and A appropriately, the stable stationary pulse solutions exist for $k \in (k_S, k_1)$ with $0 < k_S < k_1$ and at $k = k_S$, there occurs a saddle-node type bifurcation, which has been numerically checked for many cases ([13]). The bifurcation structure causes the splitting dynamics for the parameter $k = k_S - \varepsilon$ with sufficiently small $\varepsilon > 0$ as in Fig 2 ([13], [4]).

On the other hand, for an another pair of parameters (d_1, d_2, A), there exist stable stationary pulse solutions for

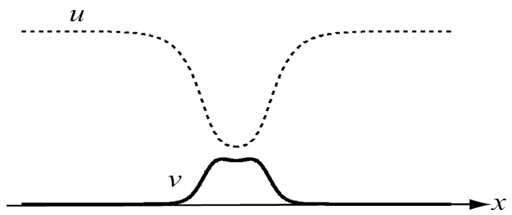

Figure 1: Spatial profile of a stationary solution of (2.1).

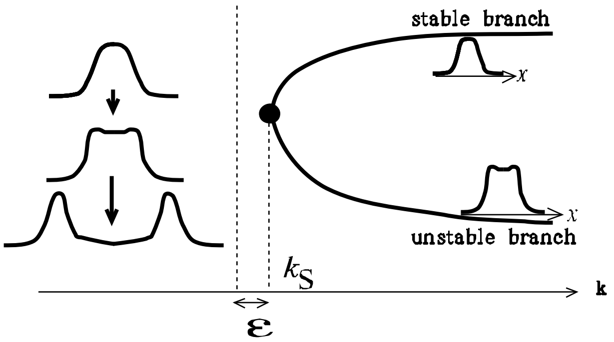

Figure 2: Saddle-node type bifurcation and splitting dynamics.

$k \in (k_T, k_2)$ with $0 < k_T < k_2$ and Jordan block type singularity appears at $k = k_T$. This has been also numerically checked ([13], [10], [3]). It is known that the singularity causes the bifurcation of traveling pulse solutions with pitch-fork type as in Fig 3 ([3]), which is called "drift bifurcation". In this paper, we assume the existence

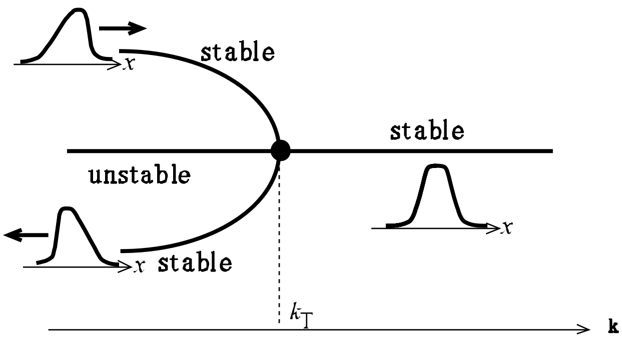

Figure 3: One possibility of the bifurcation diagram of traveling pulses when Jordan block type singularity exists.

of the bifurcation structures both of saddle-node type and the Jordan block type in a general framework and show the splitting dynamics of an oscillating traveling pulse.

3. General framework and results

In this section, we will give the several assumptions for the general settings.

Let us consider general type of reaction-diffusion systems with bifurcation parameters $\boldsymbol{k}$ written by

(3.1)
$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u};\boldsymbol{k}),\ t > 0,\ -\infty < x < \infty,\ \boldsymbol{u} \in \boldsymbol{R}^N.$$

We assume $\boldsymbol{k} \in \boldsymbol{R}^2$.

Let $\mathcal{L}(\boldsymbol{u};\boldsymbol{k}) := D\boldsymbol{u}_{xx} + F(\boldsymbol{u};\boldsymbol{k})$ and $X := \{L^2(\boldsymbol{R}^1)\}^N$. The assumptions are as follows:

1) There exist $\boldsymbol{k} = \boldsymbol{k}_c$ and a stationary pulse solution $S(x)$ of (3.1) such that $\mathcal{L}(S(x);\boldsymbol{k}_c) \equiv 0$ and $S(-x) = S(x)$.

Let L be the linearized operator $\mathcal{L}'(S(x);\boldsymbol{k}_c)$ of (3.1) with respect to $S(x)$.

2) The spectral set of L consists of two sets $\sigma_1 = \{0\}$ and $\sigma_2 \subset \{\lambda \in \boldsymbol{C};\ Re(\lambda) < -\gamma\}$ for a positive constant γ.

Remark 3.1. 0 is necessarily a spectrum of L because (3.1) has a translation invariance and $LS_x = 0$ holds.

Let Q and R be projections corresponding to the spectral sets σ_1 and σ_2, respectively. That is, $Q := \frac{1}{2\pi i}\int_\Gamma (\lambda - L)^{-1} d\lambda$ and $R := Id - Q$, where Γ is a circle around 0 inside $\{\lambda \in \boldsymbol{C};\ Re(\lambda) > -\gamma\}$ and Id denotes the identity. Define $E := QX$, $E^\perp := RX$. Assume

3) $E = span\{S_x, \phi_1, \cdots, \phi_m\}$ for $\phi_j \in X$.

Remark 3.2. $\phi_j \in E$ are the generalized eigenfunctions for 0 eigenvalue. If $E = span\{S_x\}$, 0 is called simple, and if all ϕ_j are eigenfunctions for 0 eigenvalue, it is called semi-simple.

Let $\mathcal{M} := \{S(x-p);\ p \in \boldsymbol{R}^1\}$. Then we have

Proposition 3.1. If $\boldsymbol{u}$ is close to $\mathcal{M}$, there exist unique $p \in \boldsymbol{R}^1$, $\boldsymbol{r} = (r_1, \cdots, r_m) \in \boldsymbol{R}^m$ and $\boldsymbol{w} \in E^\perp$ such that

$$\boldsymbol{u} = \tau(p)\{S + \langle\ \boldsymbol{r}, \boldsymbol{\phi}\ \rangle + \boldsymbol{w}\},$$

where τ is the translation operator with $\tau(p)\boldsymbol{u} := \boldsymbol{u}(x-p)$ and $\langle\ \boldsymbol{r}, \boldsymbol{\phi}\ \rangle := \sum_{j=1}^m r_j\phi_j(x)$ with $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_m)$.

Using the transformation in the neighborhood of $\mathcal{M}$, we can rewrite the equation (3.1) of $\boldsymbol{u} \in X$ by the equation of $(p, \boldsymbol{r}, \boldsymbol{w}) \in \boldsymbol{R}^1 \times \boldsymbol{R}^m \times E^\perp$.

Let $\boldsymbol{k} = \boldsymbol{k}_c + \boldsymbol{\eta}$, where $\boldsymbol{\eta} = (\eta_1, \eta_2)$. Then the equation of $(p, \boldsymbol{r}, \boldsymbol{w})$ becomes that of the form of

$$(3.2) \qquad \left\{ \begin{array}{rcl} \dot{p} &=& H_1(\boldsymbol{r}, \boldsymbol{w}; \boldsymbol{\eta}), \\ \dot{\boldsymbol{r}} &=& H_2(\boldsymbol{r}, \boldsymbol{w}; \boldsymbol{\eta}), \\ \boldsymbol{w}_t &=& L\boldsymbol{w} + H_3(\boldsymbol{r}, \boldsymbol{w}; \boldsymbol{\eta}), \end{array} \right.$$

where $\cdot = \dfrac{d}{dt}$ and $H_j(\boldsymbol{r}, \boldsymbol{w}; \boldsymbol{\eta})$ are some functions. We specially note that all $H_j(\boldsymbol{r}, \boldsymbol{w}; \boldsymbol{\eta})$ in (3.2) do not include p variable. Hence we may only consider the equation of $(\boldsymbol{r}, \boldsymbol{w})$

$$(3.3) \qquad \left\{ \begin{array}{rcl} \dot{\boldsymbol{r}} &=& H_2(\boldsymbol{r}, \boldsymbol{w}; \boldsymbol{\eta}), \\ \boldsymbol{w}_t &=& L\boldsymbol{w} + H_3(\boldsymbol{r}, \boldsymbol{w}; \boldsymbol{\eta}). \end{array} \right.$$

Let X^ω be the fractional power space imbedded into $C^1(\boldsymbol{R}^1)$.

Theorem 3.1. ([5]) There are positive constants $\overline{\eta}$ and $\overline{\boldsymbol{r}}$ such that for any $\boldsymbol{\eta}$ and $\boldsymbol{r}$ with $|\boldsymbol{\eta}| < \overline{\eta}$ and $|\boldsymbol{r}| < \overline{r}$ there exists a function $\sigma = \sigma(\boldsymbol{r};\boldsymbol{\eta}) \in E^{\perp}$ with $\|\sigma(\boldsymbol{r};\boldsymbol{\eta})\|_\omega \le C(|\boldsymbol{r}|^2 + |\boldsymbol{\eta}|)$ such that the set $\mathcal{M}_1(\boldsymbol{\eta}) := \{(\boldsymbol{r}, \sigma(\boldsymbol{r};\boldsymbol{\eta})) \in \boldsymbol{R}^m \times E^{\perp};\ |\boldsymbol{r}| < \overline{r}\}$ is locally positively invariant and exponentially attractive with respect to the dynamics of (3.3).

If we define the set $\mathcal{M}(\boldsymbol{\eta}) := \{\tau(p)\{S + \langle\, \boldsymbol{r}, \boldsymbol{\phi}\,\rangle + \sigma(\boldsymbol{r};\boldsymbol{\eta})\};\ p \in \boldsymbol{R}^1,\ |\boldsymbol{r}| < \overline{r}\}$, Theorem 3.1 means $\mathcal{M}(\boldsymbol{\eta})$ is an exponentially attractive positively invariant set for the equation (3.1), which is locally in $\boldsymbol{r}$ but globally defined in the space variable p.

In the next section, we apply the results to the dynamics of pulses in the Gray-Scott model.

4. Dynamics of pulses in the case of multiple degeneracy

As in mentioned in Section 2, the saddle-node type bifurcation causes the splitting behavior and the Jordan block type singularity does the bifurcation of traveling pulses.

In this section, to see the splitting dynamics of traveling pulses, we assume above two types of bifurcation occur at the same time. That is, we assume:

4) For the equation (3.1), there is $\boldsymbol{k} = \boldsymbol{k}_c$ such that at $\boldsymbol{k} = \boldsymbol{k}_c$ there exist eigenfunctions $\xi(x)$ and $\psi(x)$ such that $E = span\{S_x, \xi, \psi\}$ and $L\xi = 0$, $L\psi = -S_x$ are satisfied, and ξ is even while ψ is odd.

Remark 4.1. The existence of an eigenfunction ξ causes the saddle-node type bifurcation ([13], [4]) while ψ does the bifurcation of traveling pulses with pitchfork type ([3]).

Remark 4.2. The assumption 4) is numerically checked for the Gray-Scott model (2.1). In fact, the parameter values at which the assumption 4) holds is found in the parameter space (A, k) as in Fig4.

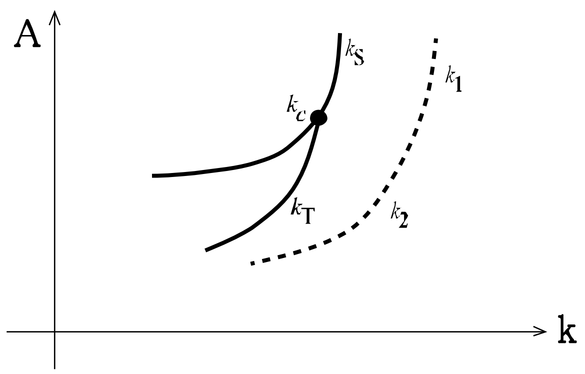

Figure 4: The parameter space (A, k) for the equation (2.1). k_S, k_T and k_j are stated in Section 2.

Let L^* be the adjoint operator of L. Then similar properties to L hold for L^*. That is, there exist $\phi^*(x)$, $\xi^*(x)$ and $\psi^*(x)$ such that $L^*\phi^* = L^*\xi^* = 0$ and $L^*\psi^* = -\phi^*$. ϕ^* and ψ^* are odd and ξ^* is even.

Remark 4.3. By the normalization of

$$\langle\, \psi, S_x\,\rangle_{L^2} = \langle\, \psi, \psi^*\,\rangle_{L^2} = 0,\ \langle\, \xi, \xi^*\,\rangle_{L^2} = \langle\, S_x, \psi^*\,\rangle_{L^2} = 1$$

all functions are uniquely determined(e.g. [3]). We note that

$$\langle\, S_x, \xi^*\,\rangle_{L^2} = \langle\, \xi, \phi^*\,\rangle_{L^2} = \langle\, S_x, \phi^*\,\rangle_{L^2} = 0,\ \langle\, \psi, \phi^*\,\rangle_{L^2} = 1$$

hold automatically.

We apply Theorem 3.1. Let $\boldsymbol{k} = \boldsymbol{k}_c + \boldsymbol{\eta}$, where $\boldsymbol{\eta} = (\eta_1, \eta_2)$ and let $\boldsymbol{r} = (q, r) \in \boldsymbol{R}^2$, $\boldsymbol{\phi} = (\xi, \psi)$. Then by investigating the flow on the set $\mathcal{M}(\boldsymbol{\eta})$, we have

Theorem 4.1. ([5]) Under the assumptions 1), 2), 4), there are positive constants $\overline{q}$, $\overline{r}$ and $\overline{\eta}$ such that the solution $\boldsymbol{u} = \boldsymbol{u}(t, x)$ of (3.1) is represented by

(4.1)
$$\boldsymbol{u}(t,x) = S(x-p) + q\psi(x-p) + r\xi(x-p) + O(|q|^2 + |r|^2 + |\boldsymbol{\eta}|)$$

and (p, q, r) satisfy

(4.2)
$$\begin{cases} \dot{p} &=& q + O(|q|^2 + |r|^2 + |\boldsymbol{\eta}|), \\ \dot{q} &=& -\{-N_1 q^2 - N_2 r + n_1\} q + O(|q|^4 + |r|^{\frac{8}{3}} + |\boldsymbol{\eta}|^2), \\ \dot{r} &=& M_1 r^2 - M_2 q^2 - n_2 + O(|q|^3 + |r|^3 + |\boldsymbol{\eta}|^{\frac{3}{2}}) \end{cases}$$

uniformly for $p \in \boldsymbol{R}^1$, $|q| < \overline{q}$ and $|r| < \overline{r}$, $|\boldsymbol{\eta}| < \overline{\eta}$, where N_j, M_j and a_j, b_j are constants and $n_1 := a_1\eta_1 + a_2\eta_2$, $n_2 := b_1\eta_1 + b_2\eta_2$.

Remark 4.4. All constants in (4.2) are determined in explicit ways but we do not give them in this paper while they will be given in detail in the forthcoming paper [5].

Remark 4.5. Constants M_j and N_j are all positive in the Gray-Scott model (2.1) and $M_1 < 1$, which is numerically confirmed under the appropriate coefficients.

From the above remark, we assume:

5) Constants M_j and N_j are all positive and $M_1 < 1$ in (4.2).

Let us consider the dynamics of solutions of the leading terms of (4.2):

$$(4.3)\qquad \begin{cases} \dot{p} &=& q, \\ \dot{q} &=& -\{-N_1 q^2 - N_2 r + (a_1\eta_1 + a_2\eta_2)\} q, \\ \dot{r} &=& M_1 r^2 - M_2 q^2 - (b_1\eta_1 + b_2\eta_2). \end{cases}$$

We can investigate all possible bifurcation diagrams from (4.3), but in this paper we only consider the case $a_1 > 0 > a_2$, $b_1 > 0 > b_2$. In this case, two lines $l_1 := \{(\eta_1, \eta_2);\ a_1\eta_1 + a_2\eta_2 = 0\}$ and $l_2 := \{(\eta_1, \eta_2);\ b_1\eta_1 + b_2\eta_2 = 0\}$ intersect as in Fig5 in the parameter space (η_1, η_2). Specially, when we change the parameter along the arrow $\boldsymbol{l}$ in Fig5, the bifurcation diagram on the equilibria of (4.3) becomes as Fig6. In Fig6, branches with $\pm P_0 = (0, \pm r)$ denote equilibria with $q = 0$ and a positive and negative r component respectively. The branches with $\pm P_f = \pm(q_f, r)$ and

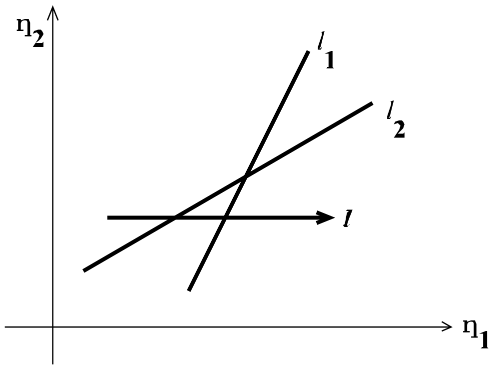

Figure 5: Two lines l_1 and l_2 in the parameter space (η_1, η_2).

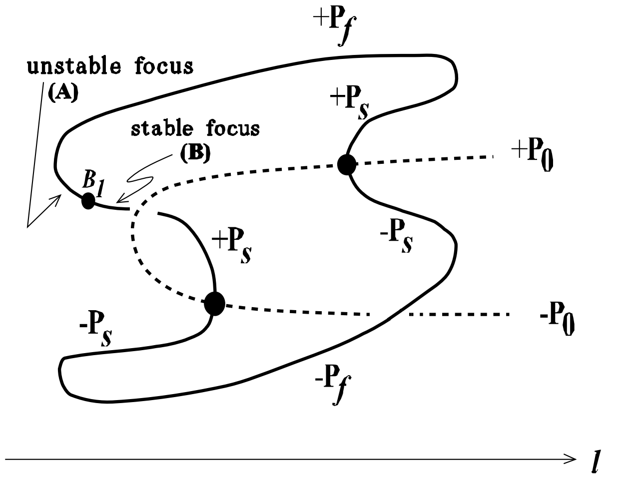

Figure 6: The bifurcation diagram on the equilibria of (4.3) when the parameter moves along the arrow $\boldsymbol{l}$ in Fig5.

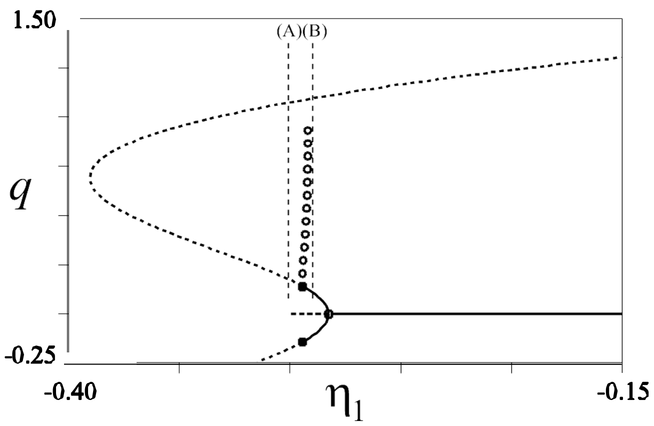

Figure 7: The bifurcation diagram on the equilibria of (4.3) near the point B_1 by numerics with parameters $N_1 = 0.8$, $N_2 = 1.0$, $M_1 = 0.1$, $M_2 = 0.1$, $a_1 = 1.0$, $a_2 = -0.5$, $b_1 = 0.1$, $b_2 = -0.1$ and $\eta_2 = -0.3$.

$\pm P_s = \pm(q_s, r)$ denote equilibria with $0 < q_s < q_f$. Stabilities of equilibria are also calculated easily. Specially the stability along the branch $+P_s$ change from stable focuses to unstable focuses at a point, say B_1. In the parameter region of unstable focuses, amplitudes of almost all solutions become large, which means the beginning of split. In fact, as discussed in Section 8.5 of [9], when M_j and N_j are positive and $M_1 < 1$, the stable traveling pulses P_f lose the stability by subcritical Hopf bifurcation. It is the case of this paper by the assumption 5).

Fig7 is a typical example of the bifurcation diagram in the neighborhood of B_1 by numerics. In the figure, solid lines denote stable equilibria, broken lines do unstable ones, white circles correspond to periodic solutions, white boxes do pitchfork bifurcation points, black boxes do Hopf bifurcation points.

Let us come back to the Gray-Scott model. In the Gray-Scott model, the spatial profiles of eigenfunctions $\xi(x)$ and $\psi(x)$ are as in Fig8 (c.f. [13], [4], [3]).

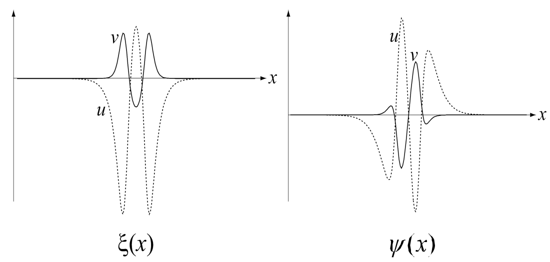

Figure 8: Spatial profiles of eigenfunctions $\xi(x)$ and $\psi(x)$.

Since $\xi(x)$ has a negative dent in the middle, it will accelerate the splitting of solutions $\boldsymbol{u}$ if r increases in the representation (4.1). On the other hand, q denotes the velocity of pulses because $\dot{p} = q$ in (4.3). Therefore, the branch $+P_0$ corresponds to stationary solutions with dimple and $-P_0$ does non-dimply solutions. P_f corresponds to faster traveling pulses and P_s does slower ones. Specially, equilibria $+P_s$ change the stability from stable focuses to unstable ones at B_1. Unstable focuses appearing in the neighborhood of B_1 cause the splitting of traveling pulses. In fact, we can numerically observe in Fig9 that solutions asysmptotically converge stable traveling pulses with oscillation under parameters corresponding to stable focuses ((B) in Fig6) and that solutions split with oscillation under parameters of unstable focuses ((A) in Fig6). Thus, we can analyze the dynamics of moving pulses by constructing invariant manifolds globally in spacial variables.

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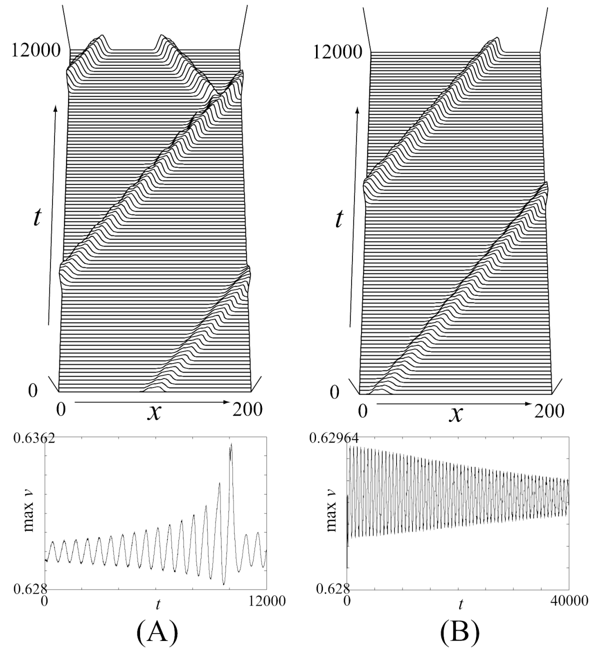

Figure 9: Numerical simulations of oscillatory dynamics of traveling pulses at the positions (A), (B) in the neighborhood of the point B_1 for the Gray-Scott model (2.1).

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