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Local subexponentiality of infinitely divisibile distributions

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Abstract. Compound distributions appear in applications to queueing theory and to risk theory. A local property of those distributions on the real line is discussed. The result helps to derive equivalnce conditions to be local subexponential for infinitely divisible distributions on the real line.

Keywords. Local subexponentiality, infinitely divisible distributions, compound distributions

1. Introduction and results

Let ζ, ζ_1, ζ_2 be distributions on $\mathbb{R}$. We write ζ^{n*} for the nth-convolution of ζ with itself, and $\zeta_1 * \zeta_2$ for the convolution of ζ_1 and ζ_2. Furthermore, we denote by $\overline{\zeta}(x) = \zeta((x,\infty))$ the right-tail of ζ. The class $\mathcal{S}$ of distributions on $\mathbb{R}$ is defined by the requirements

$$\lim_{x\to\infty} \frac{\overline{\zeta}(x+s)}{\overline{\zeta}(x)} = 1 \quad \text{for } s \in \mathbb{R}, \tag{1}$$

$$\lim_{x\to\infty} \frac{\overline{\zeta^{2*}}(x)}{\overline{\zeta}(x)} = 2. \tag{2}$$

Then ζ is called subexponential. The condition (1) is not needed in the case of subexponential distributions on $[0,\infty)$. Here we mention that the class $\mathcal{S}$ plays an important role in many applications (for instance, see [3, 4, 8, 9, 10, 14, 15]). Nowadays the study of subexponentiality has a trend toward more detailed properties. Any subexponential distribution η satisfies

$$\overline{\zeta^{n*}}(x) \sim n\overline{\zeta}(x)$$

and thereby we have

$$\zeta^{n*}((x, x+T]) = o(\overline{\zeta}(x))$$

for any T with $0 < T < \infty$. More detailed properties of $\eta^{n*}((x, x+T])$ have really been investigated in some papers [1, 2, 5, 16, 17]. Although the theory was scattered, Asmussen, Foss and Korshunov have recently developed the systematic theory. They introduced the notion of local subexponentiality in [1]: Fix $0 < T < \infty$ and write $\Delta := (0, T]$ and $x + \Delta := \{x + y : y \in \Delta\} = (x, x+T]$.

Definition 1.1. We say that a distribution ζ on $\mathbb{R}$ belongs to the class $\mathcal{L}_\Delta$ if $\zeta(x+\Delta) > 0$ for all sufficiently large x and

$$\frac{\zeta(x+s+\Delta)}{\zeta(x+\Delta)} \to 1 \quad \text{as } x \to \infty, \tag{3}$$

uniformly in $s \in [0, T]$.

Remark 1.1. We can choose a function $h(x) \to \infty$ that (3) holds uniformly in $|s| \le h(x)$. Furthermore, we always take a function $h(x)$ such that $h(x) < x/2$.

Definition 1.2. Let ζ be a distribution on $\mathbb{R}$. We say that ζ is Δ-subexponential if $\zeta \in \mathcal{L}_\Delta$ and

$$\zeta^{2*}(x+\Delta) \sim 2\zeta(x+\Delta). \tag{4}$$

Then we write $\zeta \in \mathcal{S}_\Delta$.

Remark 1.2. It follows from the definition that the class $\mathcal{S}_\Delta$ is included among the class $\mathcal{S}$.

Our aim of the present paper is to investigate local subexponentiality of infinitely divisible distributions. The reason why we focus on those distributions is that they appear in some probabilistic model. An infinitely divisible distribution μ on $\mathbb{R}$ with Lévy measure ν is characterized by its characteristic function

$$\begin{aligned} \varphi(z) &:= \int_{\mathbb{R}} e^{izx}\mu(dx) \\ &= \exp\Big[-2^{-1}az^2 + i\gamma z \\ &\quad + \int_{\mathbb{R}} (e^{izx} - 1 - izx 1_{\{|x|\le 1\}}(x))\nu(dx)\Big], \end{aligned} \tag{5}$$

where $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$, and $a \ge 0$ and $\gamma \in \mathbb{R}$. If μ is an infinitely divisible distribution on $[0,\infty)$, the characteristic function $\varphi(z)$ is represented as

$$\varphi(z) = \exp\left[\int_{[0,\infty)} (e^{izx} - 1)\nu(dx) + i\gamma_0 z\right],$$

where $\nu(\{0\}) = 0$, $\int_{[0,\infty)}(1 \wedge x)\nu(dx) < \infty$ and $\gamma_0 \ge 0$. The normalized Lévy measure $\nu_{(1)}$ is defined by

$$\nu_{(1)}(dx) := \frac{1}{\nu((1,\infty))} 1_{\{x>1\}}(x)\nu(dx).$$

In probability models, we often find a fact of the following type: Let ρ and η are distributions on $\mathbb{R}$. Then $\rho \in \mathcal{S}$ if and only if $\eta \in \mathcal{S}$. Moreover, if $\rho \in \mathcal{S}$, then

$$\lim_{x\to\infty} \frac{\overline{\eta}(x)}{\overline{\rho}(x)} = c \in (0,\infty).$$

The distribution ρ and η is what is called an "input"and an "output". The Cramér-Lundberg model, which is a basic insurance risk model, is such a model. In this case, η is an infinitely divisible distribution and ρ is its normalized Lévy measure. Then Theorem A below is useful. We introduce the early important work by Embrechts et al. as Theorem A. See [7] for details: Let functions $f(x)$ and $g(x)$ be nonnegative but positive for all sufficiently large x. If the functions $f(x)$ and $g(x)$ satisfy

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

then we write

$$f(x) \sim g(x).$$

Theorem A (Embrechts et al. (1979)) *Let μ be an infinitely divisible distribution on $[0,\infty)$ with Lévy measure ν. Then the following assertions are equivalent:*
(i) $\mu \in \mathcal{S}$; (ii) $\nu_{(1)} \in \mathcal{S}$; (iii) $\overline{\mu}(x) \sim \overline{\nu}(x)$.

This result is extended to infinitely divisible distributions on $\mathbb{R}$ by Pakes. See [11, 12]. Now we examine the Cramér-Lundberg model in detail: The model is as follows. The claim sizes $\{X_k\}_{k\in\mathbb{N}}$ are positive i.i.d. random variables having non-lattice distribution ρ with finite mean. The claims occur at the random instants of time

$$0 < T_1 < T_2 < \cdots \quad a.s.$$

Then the inter-arrival times

$$Y_1 = T_1, Y_k = T_k - T_{k-1}, k = 2,3,\cdots$$

are i.i.d. exponentially distributed with finite mean λ^{-1}. In addition, $\{X_k\}$ and $\{Y_k\}$ are independent of each other. The number of claims in the interval $[0,t]$ is denoted by

$$N(t) = \sup\{n \geq 1 : T_n \leq t\},$$

where we understand $\sup\emptyset = 0$. The total claim amount distribution η up to time t is defined by

$$\begin{aligned}\overline{\eta}(x) &= P\left(\sum_{k=1}^{N(t)} X_k > x\right)\\ &= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\overline{\rho^{n*}}(x),\end{aligned}$$

where ρ^{0*} is interpreted as the delta measure at 0. Then ρ is an input, and η is an output whose distribution is compound Poisson, that is, infinitely divisible on $[0,\infty)$. Let ρ be subexponential. Theorem A yields that η is subexponential and

$$\overline{\eta}(x) \sim \lambda t\, \overline{\rho}(x).$$

As seen above, a compound distribution is important from the viewpoint of applications. First, we consider local subexponentiality of the distribution. A compound distribution η on $\mathbb{R}$ is defined by

$$\eta := \sum_{k=0}^{\infty} p_k \rho^{k*}, \tag{6}$$

where ρ is a distribution on $\mathbb{R}$ and $\{p_k\}_{k=0}^{\infty}$ is a discrete probability such that

$$p_0 + p_1 < 1 \quad \text{and} \quad \sum_{k=0}^{\infty} p_k = 1.$$

Our result is as follows:

Theorem 1.1. *Let η be a compound distribution satisfying (6). Suppose that $\rho \in \mathcal{L}_\Delta$ and*

$$\int_{\mathbb{R}} e^{-\epsilon y}\rho(dy) < \infty \quad \text{for some } \epsilon > 0, \tag{7}$$

$$\sum_{k=0}^{\infty} p_k(1+\delta)^k < \infty \quad \text{for some } \delta > 0. \tag{8}$$

Then the following assertions are equivalent:
(i) $\eta \in \mathcal{S}_\Delta$; (ii) $\rho \in \mathcal{S}_\Delta$; (iii) $\eta(x+\Delta) \sim \rho(x+\Delta)\sum_{k=1}^{\infty} kp_k$.

Theorem 1.1 immediately yields the following corollaries. The proofs are omitted. In [16], the corollaries are already delt with but need the assumption corresponding to Theorem B below. If ρ is a distributions on $[0,\infty)$, Corollary 1.2 below is also found in [1].

Corollary 1.1. *Let $p_k = (1-c)c^k$, where $0 < c < 1$. Suppose that ρ is in $\mathcal{L}_\Delta$ and satisfies (7). The following assertions are equivalent:*
(i) $\eta \in \mathcal{S}_\Delta$; (ii) $\rho \in \mathcal{S}_\Delta$; (iii) $\eta(x+\Delta) \sim \dfrac{c}{1-c}\rho(x+\Delta)$.

Corollary 1.2. *Let $p_k = e^{-c}c^k/k!$, where $c > 0$. Suppose that ρ is in $\mathcal{L}_\Delta$ and satisfies (7). The following assertions are equivalent:*
(i) $\eta \in \mathcal{S}_\Delta$; (ii) $\rho \in \mathcal{S}_\Delta$; (iii) $\eta(x+\Delta) \sim c\rho(x+\Delta)$.

Example 1.1. Consider the Cramér-Lundberg model again. By virtue of Corollary 1.2, if the claim size distribution ρ is Δ-subexponential, then so is the total claim amount distribution η and we have

$$\eta(x+\Delta) \sim \lambda t\rho(x+\Delta). \tag{9}$$

The equivalence of (ii) and (iii) in Corollary 1.2 is already proved in Theorem 6 of [1]. So (9) is also shown by the theorem.

The distribution η in Corollary 1.2 is called compound Poisson, and is a typical infinitely divisible distribution. Theorem 1.1 is helpful to obtain the result concerned with local subexponentiality of infinitely divisible distributions. Here we state the fact that have been known. One is pointed out by Asmussen et al. To be precise, it is Theorem 7 of [1]. The assertion is correct, but the proof that (ii) implies (i) (see Theorem B) is lacking. We give the proof as Proposition 3.1 in Sect.3. In what follows, we denote by $\mathbf{ID}_\Delta$ the class of all infinitely divisible distributoins μ on $\mathbb{R}$ such that $\nu(x+\Delta) > 0$ for all sufficiently large x.

Theorem B (Asmussen et al. (2003)) *Let μ be an infinitely divisible distribution on $[0,\infty)$ with Lévy measure ν and let $\mu \in \mathbf{ID}_\Delta$. Furthermore, let $0 < T < \infty$, and assume $\nu_{(1)} \in \mathcal{L}_\Delta$. Then the following assertions are equivalent:*
(i) $\nu_{(1)} \in \mathcal{S}_\Delta$; (ii) $\nu(x+\Delta) \sim \mu(x+\Delta)$.

Let μ be an infinitely divisible distribution with Lévy measure ν. Through this paper, we decompose μ as $\mu = \mu_1 * \mu_2$, where we put $c = \nu((1,\infty))$ and μ_1 is a compound Poisson distribution with Lévy measure $c\nu_{(1)}$. Then the characteristic function $\varphi_1(z)$ of μ_1 is represented as

$$\varphi_1(z) = \exp\left[\int_{(1,\infty)} (e^{izx}-1)\nu(dx)\right].$$

Another is due to Wang et al. See Theorem 4.2 of [16]. They showed that the equivalence condition $\mu \in \mathcal{S}_\Delta$ is added to Theorem B under a certain condition ((10) below):

Theorem C (Wang et al. (2005)) *Let μ be an infinitely divisible distribution on $[0,\infty)$ with Lévy measure ν and let $\mu \in \mathbf{ID}_\Delta$. Furthermore, let $0 < T < \infty$, and assume that there exists an integer $k = k(c) > 0$ such that*

$$ck^{-1} < \log 2 \quad \text{and} \quad \mu_1^{k^{-1}*} \in \mathcal{L}_\Delta. \tag{10}$$

Then assertions (i) and (ii) of Theorem B and the following assertion are equivalent:
(iii) $\mu \in \mathcal{S}_\Delta$

Here, for $t > 0$, μ_1^{t*} is defined by the distribution having the characteristic function

$$(\varphi_1(z))^t = \exp\left[t\int_{(1,\infty)} (e^{izx}-1)\nu(dx)\right].$$

We have succeeded in eliminating the condition (10) of Theorem C, and have obtained a result in the case of infinitely divisible distributions on $\mathbb{R}$. We find the necessity of the condition (11) below on the left-tails. Our result is as follows:

Theorem 1.2. *Let μ be an infinitely divisible distribution satisfying (5) and let $\mu \in \mathbf{ID}_\Delta$. Suppose that $\nu_{(1)} \in \mathcal{L}_\Delta$. Furthermore, we suppose that*

$$\int_{\mathbb{R}} e^{-\epsilon y}\mu(dy) < \infty \quad \text{for some } \epsilon > 0. \tag{11}$$

Then the following assertions are equivalent:
(i) $\mu \in \mathcal{S}_\Delta$; (ii) $\nu_{(1)} \in \mathcal{S}_\Delta$; (iii) $\nu(x+\Delta) \sim \mu(x+\Delta)$.

Remark 1.3. For any $\epsilon > 0$, $\int_{\mathbb{R}} e^{-\epsilon y}\mu(dy) < \infty$ if and only if $\int_{-\infty}^{-1} e^{-\epsilon y}\nu(dy) < \infty$.

At the end of this section, we introduce two notations which we use in the remaining sections. If the functions $f(x)$ and $g(x)$ satisfy that

$$0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \le \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty,$$

we write

$$f(x) \asymp g(x).$$

For a distribution ζ and a measurable function $f(x)$, we write

$$\int_a^b f(x)\zeta(dx) := \int_{(a,b]} f(x)\zeta(dx)$$

for $-\infty \le a < b < \infty$.

2. Proof of Theorem 1.1

First of all, we mention a fundamental lemma to characterize local subexponentiality. It is analogous to Proposition 2 of [1].

Lemma 2.1 *Let ρ be a distribution on $\mathbb{R}$. Furthermore, let X_1 and X_2 be independent random variables with common distribution ρ. Then the following assertions are equivalent:*
(i) $\rho \in \mathcal{S}_\Delta$
(ii) *There exists a function $h(x)$ such that $h(x) \to \infty$, $h(x) < x/2$ and $\rho(x-y+\Delta) \sim \rho(x+\Delta)$ as $x \to \infty$ uniformly in $|y| \le h(x)$, and*

$$P(X_1+X_2 \in x+\Delta, |X_1| > h(x), |X_2| > h(x)) = o(\rho(x+\Delta)). \tag{12}$$

Proof. Let $h(x)$ be a function satisfying that $h(x) \to \infty$, $h(x) < x/2$ and $\rho(x-y+\Delta) \sim \rho(x+\Delta)$ as $x \to \infty$ uniformly in $|y| \le h(x)$. Put $B := \{X_1+X_2 \in x+\Delta\}$. Now we have

$$\rho^{2*}(x+\Delta) = P(B) = P(B, |X_1| \le h(x)) + P(B, |X_2| \le h(x)) + P(B, |X_1| > h(x), |X_2| > h(x)).$$

Here we have

$$P(B, |X_1| \le h(x)) = \int_{|y|\le h(x)} \rho(x-y+\Delta)\rho(dy) \sim \rho(x+\Delta).$$

In the same way as above, we have

$$P(B, |X_2| \le h(x)) \sim \rho(x+\Delta).$$

These imply the equivalence of (i) and (ii). □

The lemma corresponding to Lemma 1 of [1] is as follows:

Lemma 2.2. *Let η and ρ be distributions on $\mathbb{R}$. Suppose that ρ is in $\mathcal{L}_\Delta$ and satisfies (7). If $\eta \in \mathcal{S}_\Delta$ and*

$$\rho(x+\Delta) \asymp \eta(x+\Delta), \tag{13}$$

then $\rho \in \mathcal{S}_\Delta$.

Proof. Let X_1 and X_2 be independent random variables with common distribution ρ. Furthermore, let Y_1 and Y_2 be independent random variables with common distribution η. By virtue of Lemma 2.1, it suffices to show that

$$P(X_1+X_2 \in x+\Delta, |X_1| > h(x), |X_2| > h(x)) = o(\rho(x+\Delta)),$$

where $h(x)$ is a common function associated with ρ and η. Put $B := \{X_1+X_2 \in x+\Delta\}$ for $x>0$. Since

$$P(B, X_1 < -h(x), X_2 < -h(x)) = 0,$$

we consider only three cases: Let x be sufficiently large. Take $\epsilon > 0$ satisfying (7). Since $\rho \in \mathcal{L}_\Delta$, there is $M>0$ such that

$$\frac{\rho(x-y+\Delta)}{\rho(x+\Delta)} \le Me^{-\epsilon y}$$

for all $y < -h(x)$ (see Theorem 1.3.1 of [6]). Hence we have

$$\begin{aligned} &\frac{P(B, X_1 < -h(x), X_2 > h(x))}{\rho(x+\Delta)} \\ &= \int_{(-\infty,-h(x))} \frac{P(X_2 \in x-y+\Delta, X_2 > h(x))}{\rho(x+\Delta)}\rho(dy) \\ &\le \int_{(-\infty,-h(x))} \frac{\rho(x-y+\Delta)}{\rho(x+\Delta)}\rho(dy) \\ &\to 0 \end{aligned}$$

as $x \to \infty$. In the same way as above, we have

$$\frac{P(B, X_1 > h(x), X_2 < -h(x))}{\rho(x+\Delta)} \to 0 \quad \text{as } x \to \infty.$$

Lastly, we have

$$\begin{aligned} &P(B, X_1 > h(x), X_2 > h(x)) \\ &= \int_{h(x)}^{x-h(x)} \rho(x-y+\Delta)\rho(dy) \\ &+ \int_{x-h(x)}^{x-h(x)+T} P(X_1 \in x-y+\Delta, X_1 > h(x))\rho(dy) \\ &\equiv J_1 + J_2. \end{aligned}$$

Here we see

$$J_2 \le \rho(x-h(x)+\Delta)P(X_1 > h(x)) = o(\rho(x+\Delta)).$$

Let x be sufficiently large. There is $c_1 > 0$ such that

$$\rho(x-y+\Delta) \le c_1\eta(x-y+\Delta)$$

for all $y \le x-h(x)$. We see that, for sufficiently large x,

$$\begin{aligned} J_1 &\le c_1 \int_{h(x)}^{x-h(x)} \eta(x-y+\Delta)\rho(dy) \\ &\le c_1 P(X_1+Y_1 \in x+\Delta, X_1 > h(x), Y_1 > h(x)) \\ &= c_1 \int_{h(x)}^{x-h(x)} \rho(x-y+\Delta)\eta(dy) \\ &+ c_1 \int_{x-h(x)}^{x-h(x)+T} P(X_1 \in x-y+\Delta, X_1 > h(x))\eta(dy) \\ &\le c_1^2 \int_{h(x)}^{x-h(x)} \eta(x-y+\Delta)\eta(dy) \\ &\quad +c_1P(X_1 > h(x))\eta(x-h(x)+\Delta) \\ &\le c_1^2 P(Y_1+Y_2 \in x+\Delta, Y_1 > h(x), Y_2 > h(x)) \\ &\quad +o(\eta(x+\Delta)) \\ &= o(\eta(x+\Delta)). \end{aligned}$$

We used Lemma 2.1 in the last equality, because $\eta \in \mathcal{S}_\Delta$. As we have (13), the lemma has been proved. □

Let $\{X_n\}_{n=1}^\infty$ be i.i.d. random variables with a common distribution ρ. Put

$$S_n := \sum_{k=1}^n X_k \quad \text{for } n \ge 1.$$

The lemma corresponding to Proposition 4 of [1] is as follows:

Lemma 2.3. *Let η and ρ be distributions on $\mathbb{R}$. Suppose that ρ is in $\mathcal{L}_\Delta$ and satisfies (7). Furthermore, let η be in $\mathcal{S}_\Delta$ and satisfy that*

$$\int_{\mathbb{R}} e^{-\epsilon y}\eta(dy) < \infty \tag{14}$$

for some $\epsilon > 0$ and

$$\limsup_{x\to\infty} \frac{\rho(x+\Delta)}{\eta(x+\Delta)} < \infty. \tag{15}$$

Then, for any $\delta > 0$, there exist $x_0 = x_0(\delta) > 0$ and $V(\delta) > 0$ such that

$$\rho^{n*}(x+\Delta) \le V(\delta)(1+\delta)^n\eta(x+\Delta) \tag{16}$$

for any $x > x_0$ and any $n \ge 1$.

Proof. Let Y_1 be an independent random variable of X_1 with distribution η. For $x_0 \ge 0$ and $k \ge 1$, we put

$$A_k := \sup_{x>x_0} \frac{\rho^{k*}(x+\Delta)}{\eta(x+\Delta)}.$$

By virtue of the conditions (7) and (14), following the arguments of Lemma 2.2, we have

$$P(Y_1 + X_1 \in x + \Delta, |Y_1| > h(x), |X_1| > h(x)) \\ = o(\eta(x + \Delta)).$$

Take $\epsilon > 0$ satisfying (14). As $\eta \in \mathcal{L}_\Delta$, there is $M_1 > 0$ such that

$$\frac{\eta(x - y + \Delta)}{\eta(x + \Delta)} \le M_1 e^{-\epsilon y}$$

for $y \le 0$ and sufficiently large x. Let $b > 0$ and take sufficiently large x. Then we have

$$\begin{aligned} &P(Y_1 + X_1 \in x + \Delta, X_1 \le x - h(x)) \\ &= \left(\int_{-\infty}^{h(x)} + \int_{h(x)}^{x-h(x)}\right) P(Y_1 \in x - y + \Delta)\rho(dy) \\ &\le M_1 \eta(x + \Delta) \int_{-\infty}^{-b} e^{-\epsilon y}\rho(dy) \\ &+ \int_{-b}^{h(x)} \eta(x - y + \Delta)\rho(dy) \\ &+ P(Y_1 + X_1 \in x + \Delta, Y_1 > h(x), X_1 > h(x)). \end{aligned}$$

Let $\delta_1 > 0$. Take sufficiently large b. Then there is $x_0 > 0$ such that

$$P(Y_1 + X_1 \in x + \Delta, X_1 \le x - h(x)) \le (1 + \delta_1)\eta(x + \Delta)$$

for $x > x_0$. Let $n \ge 2$ and $x > x_0$. Then we have

$$\begin{aligned} P(S_n \in x + \Delta) &= P(S_n \in x + \Delta, X_n \le x - h(x)) \\ &\quad + P(S_n \in x + \Delta, X_n > x - h(x)) \\ &\equiv J_1 + J_2. \end{aligned}$$

In addition, we take $x_1 > 0$ such that $x_1 > x_0$ and $h(x) > x_0$ for $x > x_1$. For $x > x_1$,

$$\begin{aligned} J_1 &= \int_{-\infty}^{x-h(x)} P(S_{n-1} \in x - y + \Delta)P(X_n \in dy) \\ &\le A_{n-1} \int_{-\infty}^{x-h(x)} \eta(x - y + \Delta)P(X_n \in dy) \\ &= A_{n-1} P(Y_1 + X_n \in x + \Delta, X_n \le x - h(x)) \\ &\le (1 + \delta_1)A_{n-1}\eta(x + \Delta). \end{aligned}$$

Here, if necessary, we take x_0 and x_1 such that

$$L_1 := \sup_{x_0 < x \le x_1} (\eta(x + \Delta))^{-1} < \infty.$$

Then, for any x with $x_0 < x \le x_1$, we have

$$J_1 \le 1 \le L_1 \eta(x + \Delta).$$

Let $b > 0$. We have

$$\begin{aligned} J_2 &= P(S_{n-1} + X_n \in x + \Delta, S_{n-1} \le h(x) + T, \\ &\qquad X_n > x - h(x)) \\ &\le \int_{-\infty}^{-b} P(X_n \in x - y + \Delta)P(S_{n-1} \in dy) \\ &\quad + \int_{-b}^{h(x)+T} P(X_n \in x - y + \Delta, \\ &\qquad X_n > x - h(x))P(S_{n-1} \in dy) \\ &\equiv J_{21} + J_{22}. \end{aligned}$$

Let $\delta_1 > 0$. Here we can take sufficiently small $\epsilon_1 > 0$ such that

$$\int_{\mathbb{R}} e^{-\epsilon_1 y}\rho(dy) < 1 + \delta_1.$$

There is $M_2 > 0$ such that

$$\begin{aligned} \frac{\rho(x - y + \Delta)}{\eta(x + \Delta)} &= \frac{\rho(x - y + \Delta)}{\eta(x - y + \Delta)} \cdot \frac{\eta(x - y + \Delta)}{\eta(x + \Delta)} \\ &\le M_2 e^{-\epsilon_1 y} \end{aligned}$$

for $y \le 0$ and sufficiently large x. Hence we obtain that, for sufficiently large b,

$$\begin{aligned} J_{21} &\le M_2 \eta(x + \Delta) \int_{-\infty}^{-b} e^{-\epsilon_1 y} P(S_{n-1} \in dy) \\ &\le M_2 \eta(x + \Delta) \left(\int_{\mathbb{R}} e^{-\epsilon_1 y}\rho(dy)\right)^{n-1} \\ &\le M_2 \eta(x + \Delta)(1 + \delta_1)^{n-1}. \end{aligned}$$

Here, if necessary, we take $x_0 > 0$ such that

$$\begin{aligned} L_2 &:= \sup_{\substack{-b < y \le h(x)+T \\ x > x_0}} \frac{\rho(x - y + \Delta)}{\eta(x + \Delta)} \\ &= \sup_{\substack{-b < y \le h(x)+T \\ x > x_0}} \frac{\rho(x - y + \Delta)}{\eta(x - y + \Delta)} \cdot \frac{\eta(x - y + \Delta)}{\eta(x + \Delta)} \\ &< \infty. \end{aligned}$$

Hence,

$$\begin{aligned} J_{22} &\le \sup_{-b < y \le h(x)+T} P(X_n \in x - y + \Delta) \\ &\le L_2 \eta(x + \Delta) \end{aligned}$$

for $x > x_0$. We consequently obtain that

$$A_n \le (1 + \delta_1)A_{n-1} + (L_1 + L_2) + M_2(1 + \delta_1)^{n-1}$$

for $x > x_0$. By induction, there is $V_1 > A_1$ such that

$$A_n \le V_1 n(1 + \delta_1)^{n-1} \quad \text{for } x > x_0 \text{ and } n \ge 2.$$

Here there is a positive integer $n_0 \ge 2$ such that $n < (1 + \delta_1)^{n+1}$ for $n > n_0$. Taking $\delta = 2\delta_1 + \delta_1^2$ and $V(\delta) = n_0 V_1$, we obtain (16). $\square$

The lemma corresponding to Proposition 3 of [1] is as follows:

Lemma 2.4. *Suppose that ρ is a distribution on $\mathbb{R}$ in $\mathcal{S}_\Delta$. Let ρ_j for $j = 1, 2$ be a distribution on $\mathbb{R}$ such that*

$$\frac{\rho_j(x+\Delta)}{\rho(x+\Delta)} \to c_j \quad \text{as } x \to \infty, \tag{17}$$

for some constant $c_j \geq 0$. Furthermore, we suppose that for $j = 1, 2$, ρ_j satisfies

$$\int_{\mathbb{R}} e^{-\epsilon_j y} \rho_j(dy) < \infty \quad \text{for some } \epsilon_j > 0. \tag{18}$$

Then

$$\frac{\rho_1 * \rho_2(x+\Delta)}{\rho(x+\Delta)} \to c_1 + c_2 \quad \text{as } x \to \infty. \tag{19}$$

Proof. Let $h(x)$ be a function satisfying Lemma 2.1 (ii). Furthermore, let X_1 and X_2 be independent random variables with distributions ρ_1 and ρ_2, respectively. Put $B := \{X_1 + X_2 \in x + \Delta\}$. Then we have

$$\begin{aligned}
&\rho_1 * \rho_2(x+\Delta) \\
&= P(B, |X_1| \leq h(x)) + P(B, |X_2| \leq h(x)) \\
&\quad + P(B, |X_1| > h(x), |X_2| > h(x)).
\end{aligned}$$

Here we have that

$$\begin{aligned}
\frac{P(B, |X_1| \leq h(x))}{\rho(x+\Delta)} &= \int_{|y| \leq h(x)} \frac{P(X_2 \in x - y + \Delta)}{\rho(x-y+\Delta)} \\
&\qquad \times \frac{\rho(x-y+\Delta)}{\rho(x+\Delta)} \rho_1(dy) \\
&\to c_2
\end{aligned}$$

as $x \to \infty$. In the same way as above, we have

$$\frac{P(B, |X_2| \leq h(x))}{\rho(x+\Delta)} \to c_1 \quad \text{as } x \to \infty.$$

Let x be sufficiently large. As $\rho \in \mathcal{L}_\Delta$, there is $M > 0$ such that

$$\frac{\rho(x-y+\Delta)}{\rho(x+\Delta)} \leq M e^{-\epsilon_1 y}$$

for all $y < -h(x)$. Let $\delta > 0$. This yields that

$$\begin{aligned}
&\frac{P(X_1 + X_2 \in x + \Delta, X_1 < -h(x), X_2 > h(x))}{\rho(x+\Delta)} \\
&\leq \int_{(-\infty, -h(x))} \frac{\rho_2(x-y+\Delta)}{\rho(x-y+\Delta)} \cdot \frac{\rho(x-y+\Delta)}{\rho(x+\Delta)} \rho_1(dy) \\
&\leq M(c_2 + \delta) \int_{(-\infty, -h(x))} e^{-\epsilon_1 y} \rho_1(dy)
\end{aligned}$$

for sufficiently large $x > 0$. Hence,

$$\frac{P(X_1 + X_2 \in x + \Delta, X_1 < -h(x), X_2 > h(x))}{\rho(x+\Delta)} \to 0$$

as $x \to \infty$. In the same way as above, we have

$$\frac{P(X_1 + X_2 \in x + \Delta, X_1 > h(x), X_2 < -h(x))}{\rho(x+\Delta)} \to 0$$

as $x \to \infty$. The remaining part follows from the arguments of Lemma 2.2. We conclude that

$$P(B, |X_1| > h(x), |X_2| > h(x)) = o(\rho(x+\Delta)).$$

The lemma has been proved. □

In the case where ρ is a distribution on $[0, \infty)$, the following lemma is showed in [16]. Hence we prove it in the case where ρ is a distiburion on $\mathbb{R}$ but not on $[0, \infty)$.

Lemma 2.5. *Let ρ be a distribution on $\mathbb{R}$ satisfying (7). Let N be a positive integer. If $\rho \in \mathcal{L}_\Delta$ and $\rho^{N*} \in \mathcal{S}_\Delta$, then $\rho \in \mathcal{S}_\Delta$.*

Proof. Suppose that $N \geq 2$ and $\rho((-\infty, 0)) > 0$. Put

$$\rho_+ := c_1^{-1} 1_{\{x \geq 0\}}(x)\rho \quad \text{and} \quad \rho_- := c_2^{-1} 1_{\{x<0\}}(x)\rho.$$

Here $c_1 = \rho([0, \infty))$ and $c_2 = \rho((-\infty, 0))$. Then

$$\rho = c_1\rho_+ + c_2\rho_-.$$

It is obvious that $\rho^{N*} \geq c_1^N \rho_+^{N*}$ and, by Proposition 1 of [1], that $\rho_+^{n*} \in \mathcal{L}_\Delta$ for $n \geq 1$. Suppose that there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = \infty$ and

$$\lim_{n\to\infty} \frac{\rho_+^{N*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} = 0.$$

Let $1 \leq k \leq N-1$ and let n be sufficiently large. Take $\epsilon > 0$ satisfying (7). As $\rho_+^{k*} \in \mathcal{L}_\Delta$, there is $M > 0$ such that

$$\frac{\rho_+^{k*}(x_n - y + \Delta)}{\rho_+^{k*}(x_n+\Delta)} \leq M e^{-\epsilon y}$$

for $y \leq 0$. Hence,

$$\begin{aligned}
&\frac{\rho_+^{k*} * \rho_-^{(N-k)*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \\
&= \frac{\rho_+^{k*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \int_{-\infty}^0 \frac{\rho_+^{k*}(x_n-y+\Delta)}{\rho_+^{k*}(x_n+\Delta)} \rho_-^{(N-k)*}(dy) \\
&\leq \frac{\rho_+^{k*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \int_{-\infty}^0 M e^{-\epsilon y} \rho_-^{(N-k)*}(dy) \\
&\leq M \frac{\rho_+^{k*}(x_n+\Delta)}{\rho_+^{N*}(x_n+\Delta)} \\
&\qquad \times \frac{\rho_+^{N*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \left(\int_{\mathbb{R}} e^{-\epsilon y} \rho_-(dy) \right)^{N-k}.
\end{aligned} \tag{20}$$

Here, by Fatou's lemma, we have

$$\liminf_{x\to\infty} \frac{\rho_+^{N*}(x+\Delta)}{\rho_+^{k*}(x+\Delta)} \geq 1.$$

The right-hand side of the last inequality in (20) goes to 0 as $n \to \infty$. Hence we obtain that

$$\begin{aligned}1 &= \lim_{n\to\infty} \frac{\rho^{N*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \\ &\le c_1^N \limsup_{n\to\infty} \frac{\rho_+^{N*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \\ &\quad + \sum_{k=1}^{N-1} \binom{N}{k} c_1^k c_2^{N-k} \limsup_{n\to\infty} \frac{\rho_+^{k*} * \rho_-^{(N-k)*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \\ &= 0.\end{aligned}$$

This is a contradiction. Hence

$$\liminf_{x\to\infty} \frac{\rho_+^{N*}(x+\Delta)}{\rho^{N*}(x+\Delta)} > 0$$

and thereby

$$\rho_+^{N*}(x+\Delta) \asymp \rho^{N*}(x+\Delta).$$

As $\rho_+^{N*} \in \mathcal{L}_\Delta$ and $\rho^{N*} \in \mathcal{S}_\Delta$, it follows from Lemma 2.2 that $\rho_+^{N*} \in \mathcal{S}_\Delta$. By virtue of Corollary 2.1 of [16], we have $\rho_+ \in \mathcal{S}_\Delta$. By using Lemma 2.2 again, we have $\rho \in \mathcal{S}_\Delta$. □

We have prepared for the proof of the theorem. Now we prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $\eta \in \mathcal{S}_\Delta$. Let N be a positive integer such that $N \ge 2$ and $p_N > 0$. Then we have

$$p_N^{-1} \ge \frac{\rho^{N*}(x+\Delta)}{\eta(x+\Delta)}.$$

Suppose that there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = \infty$ and

$$\lim_{n\to\infty} \frac{\rho^{N*}(x_n+\Delta)}{\eta(x_n+\Delta)} = 0.$$

By virtue of Fatou's lemma, we have

$$\liminf_{x\to\infty} \frac{\rho^{N*}(x+\Delta)}{\rho^{k*}(x+\Delta)} \ge 1$$

for $1 \le k \le N-1$. Here,

$$\begin{aligned}\limsup_{x\to\infty} \frac{\rho(x+\Delta)}{\eta(x+\Delta)} &= \limsup_{x\to\infty} \frac{\rho^{N*}(x+\Delta)}{\eta(x+\Delta)} \cdot \frac{\rho(x+\Delta)}{\rho^{N*}(x+\Delta)} \\ &\le p_N^{-1} < \infty.\end{aligned}$$

Hence it follows from Lemma 2.3 that for $\delta > 0$,

$$\begin{aligned}1 &= \lim_{n\to\infty} \frac{\eta(x_n+\Delta)}{\eta(x_n+\Delta)} \\ &\le \limsup_{n\to\infty} \sum_{k=1}^{N} p_k \frac{\rho^{k*}(x_n+\Delta)}{\rho^{N*}(x_n+\Delta)} \cdot \frac{\rho^{N*}(x_n+\Delta)}{\eta(x_n+\Delta)} \\ &\quad + \limsup_{n\to\infty} \sum_{k=N+1}^{\infty} p_k \frac{\rho^{k*}(x_n+\Delta)}{\eta(x_n+\Delta)} \\ &\le \sum_{k=N+1}^{\infty} p_k V(\delta)(1+\delta)^k.\end{aligned}$$

Here, from (8), we can take sufficiently large N such that

$$\sum_{k=N+1}^{\infty} p_k V(\delta)(1+\delta)^k < 1.$$

This is a contradiction. Thus

$$\liminf_{x\to\infty} \frac{\rho^{N*}(x+\Delta)}{\eta(x+\Delta)} > 0$$

and thereby

$$\rho^{N*}(x+\Delta) \asymp \eta(x+\Delta).$$

Here $\rho \in \mathcal{L}_\Delta$ implies that $\rho^{N*} \in \mathcal{L}_\Delta$. As $\rho^{N*} \in \mathcal{L}_\Delta$ and $\eta \in \mathcal{S}_\Delta$, it follows from Lemma 2.2 that $\rho^{N*} \in \mathcal{S}_\Delta$. By Lemma 2.5, we have $\rho \in \mathcal{S}_\Delta$.

Suppose that $\rho \in \mathcal{S}_\Delta$. By virtue of Lemma 2.3, we can use the dominated convergence theorem. Hence it follows from Lemma 2.4 that

$$\begin{aligned}\lim_{x\to\infty} \frac{\eta(x+\Delta)}{\rho(x+\Delta)} &= \sum_{k=1}^{\infty} p_k \lim_{x\to\infty} \frac{\rho^{k*}(x+\Delta)}{\rho(x+\Delta)} \\ &= \sum_{k=1}^{\infty} p_k k.\end{aligned}$$

Then $\rho \in \mathcal{L}_\Delta$ implies that $\eta \in \mathcal{L}_\Delta$. As $\eta \in \mathcal{L}_\Delta$ and $\rho \in \mathcal{S}_\Delta$, we see from Lemma 2.2 that $\eta \in \mathcal{S}_\Delta$, too.

Suppose that (iii) holds. Put

$$\rho_+(dx) := c_1^{-1} 1_{[0,\infty)}(x)\rho(dx),$$

where $c_1 = \rho([0,\infty))$. Then we have

$$\frac{\eta(x+\Delta)}{\rho_+(x+\Delta)} \to c_1 \sum_{k=1}^{\infty} k p_k \quad \text{as } x \to \infty.$$

Theorem 2 (ii) of [1] holds for any ditribution G on $\mathbb{R}$. Here G appears in the statetment of the theorem. We can use Theorem 2 (ii) of [1] and thereby $\rho_+ \in \mathcal{S}_\Delta$. By Lemma 2.2, we have $\rho \in \mathcal{S}_\Delta$. □

3. Proof of Theorem 1.2

Put $c := \nu((1,\infty))$. We decompose μ as $\mu = \mu_1 * \mu_2$, where μ_1 is a compound Poisson distribution with Lévy measure $c\nu_{(1)}$.

Proposition 3.1. *Let μ be an infinitely divisible distribution satisfying (5) and let $\mu \in \mathbf{ID}_\Delta$. Suppose that $\nu_{(1)} \in \mathcal{L}_\Delta$. If $\mu(x+\Delta) \sim \nu(x+\Delta)$, then $\nu_{(1)} \in \mathcal{S}_\Delta$.*

Proof. Notice that

$$\mu_1 = e^{-c} \sum_{k=1}^{\infty} \frac{c^k}{k!} (\nu_{(1)})^{k*}.$$

Let $A > 0$. Hence we have

$$c = \lim_{x\to\infty} \frac{\mu(x+\Delta)}{\nu_{(1)}(x+\Delta)}$$

$$\geq e^{-c}\limsup_{x\to\infty}\frac{c^2}{2!}\int_{\mathbb{R}}\frac{(\nu_{(1)})^{2*}(x-y+\Delta)}{\nu_{(1)}(x+\Delta)}\mu_2(dy)$$
$$+e^{-c}\liminf_{x\to\infty}\sum_{\substack{k\neq 2\\ k\geq 1}}\frac{c^k}{k!}\int_{\mathbb{R}}\frac{(\nu_{(1)})^{k*}(x-y+\Delta)}{\nu_{(1)}(x+\Delta)}\mu_2(dy)$$
$$\geq e^{-c}\frac{c^2}{2!}\limsup_{x\to\infty}\frac{(\nu_{(1)})^{2*}(x+\Delta)}{\nu_{(1)}(x+\Delta)}$$
$$\times\int_{-A}^{A}\frac{(\nu_{(1)})^{2*}(x-y+\Delta)}{(\nu_{(1)})^{2*}(x+\Delta)}\mu_2(dy)$$
$$+e^{-c}\sum_{\substack{k\neq 2\\ k\geq 1}}\frac{c^k}{k!}\int_{\mathbb{R}}\liminf_{x\to\infty}\frac{(\nu_{(1)})^{k*}(x-y+\Delta)}{\nu_{(1)}(x+\Delta)}\mu_2(dy)$$
$$\equiv I.$$

By Proposition 1 of [1], if $\nu_{(1)}\in\mathcal{L}_\Delta$, then $(\nu_{(1)})^{2*}\in\mathcal{L}_\Delta$. Hence we obtain from Corollary 1 of [1] that

$$I \geq e^{-c}\frac{c^2}{2!}\limsup_{x\to\infty}\frac{(\nu_{(1)})^{2*}(x+\Delta)}{\nu_{(1)}(x+\Delta)}\mu_2((-A,A]) +e^{-c}\sum_{\substack{k\neq 2\\ k\geq 1}}\frac{c^k}{k!}k.$$

This implies that

$$c-e^{-c}(ce^c-c^2) \geq e^{-c}\frac{c^2}{2!}\limsup_{x\to\infty}\frac{(\nu_{(1)})^{2*}(x+\Delta)}{\nu_{(1)}(x+\Delta)}\mu_2((-A,A]).$$

As $A\to\infty$, we have

$$2\geq\limsup_{x\to\infty}\frac{(\nu_{(1)})^{2*}(x+\Delta)}{\nu_{(1)}(x+\Delta)}.$$

Recall that $\nu_{(1)}\in\mathcal{L}_\Delta$. Using Corollary 1 of [1] again, we have $\nu_{(1)}\in\mathcal{S}_\Delta$. □

Proposition 3.2 *Let μ be an infinitely divisible distribution satisfying (5) and let $\mu\in\mathbf{ID}_\Delta$. Suppose that $\nu_{(1)}\in\mathcal{L}_\Delta$ and μ satisfies (11). If $\mu\in\mathcal{S}_\Delta$, then $\nu_{(1)}\in\mathcal{S}_\Delta$.*

Proof. Recall that

$$\mu_1=e^{-c}\sum_{k=0}^{\infty}\frac{c^k}{k!}(\nu_{(1)})^{k*}.$$

Take a positive integer N such that $e^{-c}2^N>1$. As we have $(\nu_{(1)})^{N*}\in\mathcal{L}_\Delta$, it follows that

$$\liminf_{x\to\infty}\frac{\mu(x+\Delta)}{(\nu_{(1)})^{N*}(x+\Delta)} \geq\int_{\mathbb{R}}\liminf_{x\to\infty}e^{-c}\frac{c^N}{N!}\cdot\frac{(\nu_{(1)})^{N*}(x-y+\Delta)}{(\nu_{(1)})^{N*}(x+\Delta)}\mu_2(dy) \geq e^{-c}\frac{c^N}{N!}. \tag{21}$$

Suppose that there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty}x_n=\infty$ and

$$\lim_{n\to\infty}\frac{(\nu_{(1)})^{N*}(x_n+\Delta)}{\mu(x_n+\Delta)}=0.$$

By Fatou's lemma, we have

$$\liminf_{x\to\infty}\frac{\mu*\mu_2(x+\Delta)}{\mu^{2*}(x+\Delta)} \geq\int_{\mathbb{R}}\liminf_{x\to\infty}\frac{\mu(x-y+\Delta)}{\mu(x+\Delta)}\cdot\frac{\mu(x+\Delta)}{\mu^{2*}(x+\Delta)}\mu_2(dy) =2^{-1}. \tag{22}$$

Let $1\leq k\leq N$. By Fatou's lemma again, we have

$$\liminf_{n\to\infty}\frac{\mu(x_n+\Delta)}{(\nu_{(1)})^{k*}(x_n+\Delta)}$$
$$\geq\lim_{n\to\infty}\frac{\mu(x_n+\Delta)}{(\nu_{(1)})^{N*}(x_n+\Delta)}$$
$$\times\int_{\mathbb{R}}\liminf_{n\to\infty}\frac{(\nu_{(1)})^{k*}(x_n-y+\Delta)}{(\nu_{(1)})^{k*}(x_n+\Delta)}(\nu_{(1)})^{(N-k)*}(dy)$$
$$=\infty.$$

Here we used $(\nu_{(1)})^{k*}\in\mathcal{L}_\Delta$. Hence,

$$\lim_{n\to\infty}\frac{(\nu_{(1)})^{k*}(x_n+\Delta)}{\mu(x_n+\Delta)}=0.$$

Take $\epsilon>0$ satisfying (11). As $(\nu_{(1)})^{k*}\in\mathcal{L}_\Delta$, there is $M>0$ such that

$$\frac{(\nu_{(1)})^{k*}(x_n-y+\Delta)}{(\nu_{(1)})^{k*}(x_n+\Delta)}\leq Me^{\epsilon|y|}$$

for $y\in\mathbb{R}$ and all sufficiently large x_n. Notice that (11) yields

$$\int_{-\infty}^{0}e^{-\epsilon y}\mu_2(dy)<\infty.$$

Hence, using Theorem 26.8 of [13], we see

$$\int_{\mathbb{R}}e^{\epsilon|y|}\mu_2^{2*}(dy)\leq\left(\int_{\mathbb{R}}e^{\epsilon|y|}\mu_2(dy)\right)^2<\infty.$$

For $1\leq k\leq N$,

$$\limsup_{n\to\infty}\frac{(\nu_{(1)})^{k*}*\mu_2^{2*}(x_n+\Delta)}{\mu(x_n+\Delta)}$$
$$\leq\limsup_{n\to\infty}\frac{(\nu_{(1)})^{k*}(x_n+\Delta)}{\mu(x_n+\Delta)}$$
$$\times\int_{\mathbb{R}}\frac{(\nu_{(1)})^{k*}(x_n-y+\Delta)}{(\nu_{(1)})^{k*}(x_n+\Delta)}\mu_2^{2*}(dy)$$
$$\leq\limsup_{n\to\infty}\frac{(\nu_{(1)})^{k*}(x_n+\Delta)}{\mu(x_n+\Delta)}\times M\int_{\mathbb{R}}e^{\epsilon|y|}\mu_2^{2*}(dy)$$
$$=0.$$

Now we see from Theorem 26.8 of [13] that

$$\overline{\mu_2^{2*}}(x)=o(\exp(-\alpha x\log x))$$

for some $\alpha > 0$. Furthermore, $\mu \in \mathcal{L}_\Delta$ yields

$$\lim_{x\to\infty} e^{\alpha x \log x}\mu(x+\Delta) = \infty.$$

Thus, for $k = 0$,

$$\lim_{n\to\infty} \frac{(\nu_{(1)})^{k*} * \mu_2^{2*}(x_n+\Delta)}{\mu(x_n+\Delta)} = \lim_{n\to\infty} \frac{\mu_2^{2*}(x_n+\Delta)}{\mu(x_n+\Delta)} = 0.$$

Notice that

$$\mu_1^{2*} = e^{-2c}\sum_{k=0}^{\infty} \frac{(2c)^k}{k!}(\nu_{(1)})^{k*}$$

and recall that $e^{-c}2^N > 1$. Here we have

$$\begin{aligned} (23) \quad 2 &= \lim_{n\to\infty} \frac{\mu^{2*}(x_n+\Delta)}{\mu(x_n+\Delta)} \\ &= \lim_{n\to\infty} \frac{e^{-2c}\displaystyle\sum_{k=N+1}^{\infty} \frac{(2c)^k}{k!}(\nu_{(1)})^{k*} * \mu_2^{2*}(x_n+\Delta)}{\mu(x_n+\Delta)}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &\frac{\mu * \mu_2(x_n+\Delta)}{\mu(x_n+\Delta)} \\ &\le e^{-c}\sum_{k=0}^{N} \frac{c^k}{k!}\frac{(\nu_{(1)})^{k*} * \mu_2^{2*}(x_n+\Delta)}{\mu(x_n+\Delta)} \\ &+ e^c 2^{-(N+1)} \frac{e^{-2c}\displaystyle\sum_{k=N+1}^{\infty} \frac{(2c)^k}{k!}(\nu_{(1)})^{k*} * \mu_2^{2*}(x_n+\Delta)}{\mu(x_n+\Delta)}. \end{aligned}$$

Hence it follows that

$$(24) \qquad \limsup_{n\to\infty} \frac{\mu * \mu_2(x_n+\Delta)}{\mu(x_n+\Delta)} \le e^c 2^{-N}.$$

Therefore we obtain from (22), (23) and (24) that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{\mu^{2*}(x_n+\Delta)}{\mu^{2*}(x_n+\Delta)} \\ &\ge 2^{-1}\liminf_{n\to\infty} \frac{\mu^{2*}(x_n+\Delta)/\mu(x_n+\Delta)}{\mu * \mu_2(x_n+\Delta)/\mu(x_n+\Delta)} \\ &\ge 2^{-1}\cdot \frac{2}{e^c 2^{-N}} > 1. \end{aligned}$$

This is a contradiction. Thus,

$$\liminf_{x\to\infty} \frac{(\nu_{(1)})^{N*}(x+\Delta)}{\mu(x+\Delta)} > 0$$

and thereby we see from (21) that

$$(\nu_{(1)})^{N*}(x+\Delta) \asymp \mu(x+\Delta).$$

As $(\nu_{(1)})^{N*} \in \mathcal{L}_\Delta$ and $\mu \in \mathcal{S}_\Delta$, it follows from Lemma 2.2 that $(\nu_{(1)})^{N*} \in \mathcal{S}_\Delta$. Furthermore, we have $\nu_{(1)} \in \mathcal{S}_\Delta$ by Lemma 2.5. □

Remark 3.1. If we can show that $\nu_{(1)} \in \mathcal{L}_\Delta$ implies $\mu \in \mathcal{L}_\Delta$, the proof of this proposition becomes simple. We could not do it, but find the way to avoid using it. Here we pose an open problem:
Problem. If $\nu_{(1)} \in \mathcal{L}_\Delta$, then does it hold that $\mu \in \mathcal{L}_\Delta$?

We have prepared for the proof of the theorem. Now we prove Theorem 1.2.

Proof of Theorem 1.2. We see from Propositions 3.1 and 3.2 that (iii) implies (ii), and that (i) implies (ii). Suppose that (ii) holds. It follows from Corollary 1.2 that $\mu_1 \in \mathcal{S}_\Delta$ and

$$\mu_1(x+\Delta) \sim \nu(x+\Delta).$$

By virtue of Theorem 26.8 of [13], we have

$$\mu_2(x+\Delta) = o(\exp(-\alpha x\log x)) \quad \text{for some } \alpha > 0.$$

Furthermore, as $\nu_{(1)} \in \mathcal{L}_\Delta$, we have

$$\lim_{x\to\infty} e^{\alpha x}\nu_{(1)}(x+\Delta) = \infty.$$

These yield that

$$\lim_{x\to\infty} \frac{\mu_2(x+\Delta)}{\nu_{(1)}(x+\Delta)} = 0.$$

Here (11) implies that

$$\int_{\mathbb{R}} e^{-\epsilon y}\mu_j(dy) < \infty \quad \text{for } j = 1, 2.$$

Hence it follows from Lemma 2.4 that

$$\lim_{x\to\infty} \frac{\mu_1 * \mu_2(x+\Delta)}{\nu(x+\Delta)} = 1.$$

This is assertion (iii) and thereby, we see that $\mu \in \mathcal{L}_\Delta$. We obtain from Lemma 2.2 that $\mu \in \mathcal{S}_\Delta$ too. Assertion (i) also has been proved. □

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