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Traveling waves bifurcating from plane Poiseuille flow of the compressible Navier-Stokes equation

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Abstract

Plane Poiseuille flow in viscous compressible fluid is known to be asymptotically stable if Reynolds number R and Mach number M are sufficiently small. On the other hand, for R and M being not necessarily small, an instability criterion for plane Poiseuille flow is known; and the criterion says that, when R increases, a pair of complex conjugate eigenvalues of the linearized operator cross the imaginary axis. In this paper it is proved that a spatially periodic traveling wave bifurcates from plane Poiseuille flow when the critical eigenvalues cross the imaginary axis.

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1 Introduction

This paper is concerned with the bifurcation of traveling waves from plane Poiseuille flow of the compressible Navier-Stokes equation. We consider the following system of equations

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div}\, v + \nabla P(\rho) = \rho \boldsymbol{g} \tag{1.2}$$

in a 2-dimensional infinite layer $\Omega_\ell = \mathbb{R} \times (0, \ell)$:

$$\Omega_\ell = \{x = (x_1, x_2) \,:\, x_1 \in \mathbb{R},\, 0 < x_2 < \ell\}.$$

Here $\rho = \rho(x,t)$ and $v = {}^\top(v^1(x,t), v^2(x,t))$ denote the density and velocity, respectively, at time $t \geq 0$ and position $x \in \Omega_\ell$; $P = P(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$P'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are the viscosity coefficients that are assumed to be constants and satisfy

$$\mu > 0, \quad \mu + \mu' \geq 0;$$

div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x; and $\boldsymbol{g}$ is a given external force. Here and in what follows ${}^\top\cdot$ stands for the transposition.

We assume that the external force $\boldsymbol{g}$ takes the form

$$\boldsymbol{g} = g\boldsymbol{e}_1,$$

where g is a positive constant and $\boldsymbol{e}_1 = {}^\top(1,0) \in \mathbb{R}^2$.

The system (1.1)–(1.2) is considered under the boundary condition

$$v|_{x_2=0,\ell} = 0. \tag{1.3}$$

We also require periodicity of ρ and v in x_1:

$$\rho(x_1 + \tfrac{2\pi}{\alpha}, x_2) = \rho(x_1, x_2), \quad v(x_1 + \tfrac{2\pi}{\alpha}, x_2) = v(x_1, x_2), \tag{1.4}$$

where $\alpha > 0$ is a given wave number.

It is easily seen that (1.1)–(1.4) has a stationary solution $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$ satisfying

$$\overline{\rho}_s = \rho_*, \quad \overline{v}_s = \frac{\rho_* g}{2\mu} x_2(\ell - x_2)\boldsymbol{e}_1,$$

that is the so-called plane Poiseuille flow.

The aim of this paper is to show the bifurcation of traveling wave solutions from plane Poiseuille flow.

The function $\overline{v}_s$ also gives a stationary solution representing parallel flow of the incompressible Navier-Stokes equation. It is known that stationary parallel flow of the incompressible Navier-Stokes equation is stable under any initial perturbations in L^2 if the Reynolds number R is sufficiently small. Furthermore, plane Poiseuille flow is stable under sufficiently small initial perturbations if $\mathrm{R} < \mathrm{R}_c$ for a critical number $\mathrm{R}_c \sim 5772$, and unstable if $\mathrm{R} > \mathrm{R}_c$ ([9]).

As for the compressible case, the stability of parallel flow in the infinite layer Ω_ℓ was studied in [7]; and it was proved that parallel flow is asymptotically stable under perturbations sufficiently small in some Sobolev space over Ω_ℓ if the Reynolds and Mach numbers are sufficiently small. In [8] an instability criterion was established; plane Poiseuille flow of the compressible Navier-Stokes equation (1.1)–(1.4) is linearly unstable if $\alpha \ll 1$ and

$$\frac{1}{280} > \gamma^2, \quad \frac{1}{280} - \gamma^2 > \frac{\nu}{30\gamma^2}(3\nu + \nu'), \tag{1.5}$$

where $\nu = \frac{\mu}{8\rho_* \ell V_0}$, $\nu' = \frac{\mu'}{8\rho_* \ell V_0}$ and $\gamma = \frac{\sqrt{P'(\rho_*)}}{8V_0}$ with $V_0 = \frac{\rho_* g \ell}{8\mu}$ being the maximum velocity of plane Poiseuille flow $\overline{v}_s$. More precisely, the spectrum of the linearized operator $-L$ consists of simple eigenvalues $\lambda_{\alpha k}$ ($|k| = 1, \cdots, n_0$) for some $n_0 \in \mathbb{R}$ such that

$$\lambda_{\alpha k} = -\frac{i}{6}(\alpha k) + \kappa_0(\alpha k)^2 + O(|\alpha k|^3) \quad (\alpha k \to 0).$$

Here κ_0 is the number given by

$$\kappa_0 = \frac{1}{12\nu}\left[\left(\frac{1}{280} - \gamma^2\right) - \frac{\nu}{30\gamma^2}(3\nu + \nu')\right].$$

As a consequence, if $\alpha \ll 1$ and (1.5) is satisfied, then $\kappa_0 > 0$ and plane Poiseuille flow $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$ is linearly unstable. Note that the Reynolds number R and Mach number M are given by $\mathrm{R} = \frac{1}{16\nu}$ and $\mathrm{M} = \frac{1}{8\gamma}$, respectively. Instability condition (1.5) is thus restated as

$$\mathrm{M} > \sqrt{\frac{35}{8}} \sim 2.09, \quad \frac{1}{35} - \frac{1}{8\mathrm{M}^2} > \frac{\mathrm{M}^2}{15\mathrm{R}}\left(\frac{3}{\mathrm{R}} + \frac{1}{\mathrm{R}'}\right), \tag{1.6}$$

where $\mathrm{R}' = \frac{1}{16\nu'}$. Therefore, Reynolds and Mach numbers are not small when (1.5) (i.e., (1.6)) is satisfied. For example, if $\mathrm{M} = 2.5$, $\mathrm{R} = \frac{173}{16} \sim 10.81$ and $\frac{1}{\mathrm{R}'} = -\frac{2}{3\mathrm{R}}$ (i.e., $\nu' = -\frac{2\nu}{3}$), then instability condition (1.6) (i.e., (1.5)) is satisfied.[1]

When the instability described above occurs, there seems to appear the Hopf bifurcation. In fact, if γ^2 is fixed so that $\frac{1}{280} - \gamma^2 > 0$, one can find the value $\nu_1 > 0$ such that $\kappa_0 < 0$ for $\nu = \nu_1$. When ν is decreased from ν_1, complex conjugate eingenvalues $\lambda_{\pm\alpha}$ cross the imaginary axis at some $\nu = \nu_0$. We will show that there are traveling wave solutions, which are periodic in x_1 and t, bifurcating from plane Poiseuille flow for $\nu \sim \nu_0$, provided that

$$\sigma(-L) \cap \{\lambda; \mathrm{Re}\,\lambda = 0\} = \{\lambda_\alpha, \lambda_{-\alpha}\} \text{ at } \nu = \nu_0. \tag{1.7}$$

Since Iooss and Padula ([6]) proved that $\sigma(-L) \cap \{\lambda; \mathrm{Re}\,\lambda > -c\}$ consists of finite number of eigenvalues with finite multiplicities for some constant $c > 0$, it seems very unlikely that the assumption (1.7) is not satisfied for all $\alpha \ll 1$. We also note that we construct bifurcating solutions from Poiseuille flow when ν and γ are small, which implies that Poiseuille flow is large, in other words, we show the bifurcation from large stationary solution.

The bifurcation problem for compressible fluid was firstly treated by Nishida-Padula-Teramoto [11] (cf., [10]); and the existence of the bifurcating convection solutions was proved for thermal convection problem. The main difficulty in the proof of the bifurcation arises from the convection term $v \cdot \nabla\rho$ in (1.1) which may cause the derivative-loss, in other words, it is not Frechét differentiable in a standard setting. In [11], the effective viscous flux was used to overcome this difficulty

[1] The definition of M in [8] should be corrected as the one defined in this paper; in [8], M is defined as $\mathrm{M} = \sqrt{P'(\rho_*)}/V_0$; and, in [8, Remark 3.2], the value $\mathrm{M} = 8/\gamma = 160$ should be corrected as $\mathrm{M} = 1/(8\gamma) = 2.5$ as in the example given here.

and establish the necessary estimates for the proof of the bifurcation of stationary convective patterns. (Cf., [1, 5].) In this paper we will not use the effective viscous flux but employ the iterative method in which the convection term $v \cdot \nabla \rho$ in (1.1) is regarded as a part of the principal part as in the proof of the local solvability of the time evolution problem. The method of this paper will be widely applicable to the bifurcation problem for certain classes of quasilinear hyperbolic-parabolic systems.

To prove the existence of bifurcating traveling waves, we rewrite the time evolution problem to a stationary problem in a moving coordinates. We then decompose the stationary problem into the null space of the linearized operator and its complementary subspace. One of the points of the proof is to establish the solvability in the complementary subspace, for which we apply the Matsumura-Nishida energy method [12] and the results on the resolvent problem for transport equation by Heywood and Padula [4] for a linear system which includes the convective term $v \cdot \nabla \rho$ as in (1.1) with a given velocity v.

This paper is organized as follows. In section 2 we derive a non-dimensional form of system (1.1)–(1.2) and rewrite it into the system of equations for the perturbation. We also introduce notation used in this paper. In section 3 we state the instability result of Poiseuille flow obtained in [8], and in section 4, we state the main result of this paper on the existence of bifurcating traveling waves. Sections 5–8 are devoted to the proof of the main result. In section 5 we first formulate the problem. We then rewrite the time evolution problem to a stationary problem in a moving coordinates, and we give a proof of the main result. In section 6 we prove the solvability in the complementary subspace. Section 7 is devoted to a proof of a periodic version of Bogovskii's lemma. In section 8 we present a proof of the solvability in the null space of the linearized operator.

2 Preliminaries

In this section we first derive a non-dimensional form of system (1.1)–(1.2) and then give the system of equations for the perturbation. In the end of this section we introduce notations used in this paper.

2.1 Non-dimensionalization

We introduce the following non-dimensional variables:

$$x = \ell \tilde{x}, \ t = \frac{\ell}{V}\tilde{t}, \ v = V\tilde{v}, \ \rho = \rho_* \tilde{\rho}, \ P = \rho_* V^2 p$$

with

$$V = \frac{\rho_* g \ell^2}{\mu}.$$

Under this transformation, (1.1) and (1.2) on Ω_ℓ are written, by omitting tildes, as

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0, \tag{2.1}$$

$$\rho(\partial_t v + v\cdot\nabla v) - \nu\Delta v - (\nu+\nu')\nabla \mathrm{div}\, v + \nabla p(\rho) = \nu\rho \boldsymbol{e}_1 \tag{2.2}$$

on the infinite layer $\Omega = \Omega_1$:

$$\Omega = \{x = (x_1, x_2) \,:\, x_1 \in \mathbb{R},\, 0 < x_2 < 1\}.$$

Here and in what follows we denote $\boldsymbol{e}_1 = {}^\top(1,0) \in \mathbb{R}^2$; ν and ν' are the non-dimensional parameters given by

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}.$$

The assumption $P'(\rho_*) > 0$ is restated as

$$p'(1) > 0.$$

To derive (2.2) we have used the relation $\frac{\ell g}{V^2} = \nu$.

We will show the existence of traveling wave solutions of (2.1)–(2.2) bifurcating from Poiseuille flow. Due to the above non-dimensionalization, the Poiseuille flow is transformed to

$$u_s = {}^\top(\rho_s, v_s),$$

where

$$\rho_s = 1, \quad v_s = {}^\top(v_s^1(x_2), 0), \quad v_s^1(x_2) = \frac{1}{2}(-x_2^2 + x_2).$$

We next derive the system of equations for the perturbation. We substitute $u(t) = {}^\top(\phi(t), w(t)) \equiv {}^\top(\gamma^2(\rho(t) - \rho_s), v(t) - v_s)$ into (2.1) and (2.2), where γ is the non-dimensional number given by

$$\gamma = \sqrt{p'(1)} = \frac{\sqrt{P'(\rho_*)}}{V}.$$

Noting that $\rho_s = 1$, $v_s = {}^\top(v_s^1(x_2), 0)$ and $-\Delta v_s = \boldsymbol{e}_1$, we obtain the following system of equations

$$\partial_t \phi + v_s^1 \partial_{x_1}\phi + \gamma^2 \mathrm{div}\, w = f^0, \tag{2.3}$$

$$\partial_t w - \nu\Delta w - \tilde{\nu}\nabla \mathrm{div}\, w + \nabla\phi - \frac{\nu}{\gamma^2}\phi \boldsymbol{e}_1 + v_s^1 \partial_{x_1} w + (\partial_{x_2} v_s^1) w^3 \boldsymbol{e}_1 = f. \tag{2.4}$$

Here $\tilde{\nu} = \nu + \nu'$; and f^0 and $f = {}^\top(f^1, f^2)$ denote the nonlinearities:

$$f^0 = -\mathrm{div}\,(\phi w),$$

$$f = -w\cdot\nabla w - \frac{\phi}{\gamma^2 + \phi}\left(\nu\Delta w + \frac{\nu}{\gamma^2}\phi \boldsymbol{e}_1 + \tilde{\nu}\nabla \mathrm{div}\, w\right) + P^{(1)}(\phi)\phi\nabla\phi$$

where

$$P^{(1)}(\phi) = \frac{1}{\gamma^2 + \phi}\left(1 - \frac{1}{\gamma^2}\int_0^1 P''(1 + \theta\gamma^{-2}\phi)\, d\theta\right).$$

We consider (2.3)–(2.4) under the boundary conditions

$$w|_{x_2 = 0,1} = 0, \quad \phi,\ w\text{: } \tfrac{2\pi}{\alpha}\text{-periodic in } x_1, \tag{2.5}$$

and the initial condition

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{2.6}$$

Here α is a given positive number.

2.2 Notation

We introduce some notations used in this paper. For given $\alpha > 0$, we denote the basic period cell by

$$\mathcal{P}_\alpha = \left[-\tfrac{\pi}{\alpha}, \tfrac{\pi}{\alpha}\right).$$

We set

$$\Omega_\alpha = \mathcal{P}_\alpha \times (0,1).$$

We denote by $C^\infty_{per}(\overline{\Omega}_\alpha)$ the space of restrictions to $\overline{\Omega}_\alpha$ of functions in $C^\infty(\overline{\Omega})$ which are $\mathcal{P}_\alpha$-periodic in x_1. We also denote by $C^\infty_{per,0}(\Omega_\alpha)$ the space of restrictions to $\overline{\Omega}_\alpha$ of functions in $C^\infty(\Omega)$ which are $\mathcal{P}_\alpha$-periodic in x_1 and vanish near $x_2 = 0, 1$.

We set

$$L^2_{per}(\Omega_\alpha) = \text{the } L^2(\Omega_\alpha)\text{-closure of } C^\infty_{per,0}(\Omega_\alpha),$$

$$H^k_{per}(\Omega_\alpha) = \text{the } H^k(\Omega_\alpha)\text{-closure of } C^\infty_{per}(\overline{\Omega}_\alpha),$$

$$H^1_{per,0}(\Omega_\alpha) = \text{the } H^1(\Omega_\alpha)\text{-closure of } C^\infty_{per,0}(\Omega_\alpha).$$

We note that if $f \in H^1_{per,0}(\Omega_\alpha)$, then $f|_{x_1=-\pi/\alpha} = f|_{x_1=\pi/\alpha}$ and $f|_{x_2=0,1} = 0$. $H^{-1}_{per}(\Omega_\alpha)$ stands for the dual space of $H^1_{per,0}(\Omega_\alpha)$. The inner product of $f_j \in L^2_{per}(\Omega_\alpha)$ $(j = 1, 2)$ is denoted by

$$(f_1, f_2) = \int_{\Omega_\alpha} f_1(x)\overline{f_2(x)}\, dx,$$

where $\overline{z}$ denotes the complex conjugate of z.

The mean value of a function $\phi(x)$ over Ω_α is denoted by $\langle \phi \rangle$:

$$\langle \phi \rangle = \frac{1}{|\Omega_\alpha|} \int_{\Omega_\alpha} \phi(x)\, dx.$$

The set of all $\phi \in L^2_{per}(\Omega_\alpha)$ with $\langle \phi \rangle = 0$ is denoted by $L^2_{per,*}(\Omega_\alpha)$, i.e.,

$$L^2_{per,*}(\Omega_\alpha) = \{\phi \in L^2_{per}(\Omega_\alpha) : \langle \phi \rangle = 0\}.$$

Furthermore, we set

$$H^k_{per,*}(\Omega_\alpha) = H^k_{per}(\Omega_\alpha) \cap L^2_{per,*}(\Omega_\alpha).$$

For simplicity the set of all vector fields whose components are in $L^2_{per}(\Omega_\alpha)$ (resp. $H^1_{per,0}(\Omega_\alpha)$, $H^k_{per}(\Omega_\alpha)$) is also denoted by $L^2_{per}(\Omega_\alpha)$ (resp. $H^1_{per,0}(\Omega_\alpha)$, $H^k_{per}(\Omega_\alpha)$) if no confusion will occur.

We also use notation $L^2_{per}(\Omega_\alpha)$ for the set of all $u = {}^\top(\phi, w)$ with $\phi \in L^2_{per}(\Omega_\alpha)$ and $w = {}^\top(w^1, w^2) \in L^2_{per}(\Omega_\alpha)$ if no confusion will occur. The inner product of $u_j = {}^\top(\phi_j, w_j) \in L^2_{per}(\Omega_\alpha)$ $(j = 1, 2)$ is defined by

$$\langle u_1, u_2 \rangle = \frac{1}{\gamma^2} \int_{\Omega_\alpha} \phi_1(x)\overline{\phi_2(x)}\, dx + \int_{\Omega_\alpha} w_1(x) \cdot \overline{w_2(x)}\, dx.$$

In what follows we abbreviate Ω_α in $L^2_{per}(\Omega_\alpha)$, $H^k_{per}(\Omega_\alpha)$, $\cdots$, and etc., and write them as L^2_{per}, H^k_{per}, $\cdots$, and etc.

We denote by $L^2(0,1)$ the usual L^2 space on $(0,1)$ with norm $|\cdot|_{L^2}$, and, likewise, by $H^k(0,1)$ the kth order L^2-Sobolev space on $(0,1)$ with norm $|\cdot|_{H^k}$. The H^1-closure of $C_0^\infty(0,1)$ is denoted by $H_0^1(0,1)$. As in the case of functions on Ω_α, function spaces of vector fields $w = {}^\top(w^1, w^2)$ and, also, those of $u = {}^\top(\phi, w)$, are simply denoted by $L^2(0,1)$, $H_0^1(0,1)$, and so on. We define an inner product $\langle\langle u_1, u_2\rangle\rangle$ of $u_j = {}^\top(\phi_j, w_j) \in L^2(0,1)$ $(j = 1, 2)$, by

$$\langle\langle u_1, u_2\rangle\rangle = \frac{1}{\gamma^2}\int_0^1 \phi_1(x_2)\overline{\phi_2(x_2)}\,dx_2 + \int_0^1 w_1(x_2)\cdot\overline{w_2(x_2)}\,dx_2.$$

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. The null space and the range of A are denoted by $N(A)$ and $R(A)$, respectively.

3 Instability of Poiseuille flow

In this section we consider the instability of Poisueille flow.

Let us consider the linearized problem

$$\partial_t\phi + v_s^1\partial_{x_1}\phi + \gamma^2\mathrm{div}\, w = 0, \tag{3.1}$$

$$\partial_t w - \nu\Delta w - \tilde{\nu}\nabla\mathrm{div}\, w + \nabla\phi - \frac{\nu}{\gamma^2}\phi\boldsymbol{e}_1 + v_s^1\partial_{x_1}w + (\partial_{x_2}v_s^1)w^2\boldsymbol{e}_1 = 0, \tag{3.2}$$

$$w|_{x_2=0,1} = 0, \quad \phi, w:\ \tfrac{2\pi}{\alpha}\text{-periodic in } x_1, \tag{3.3}$$

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{3.4}$$

We set

$$X = L^2_{per,*} \times (L^2_{per})^2.$$

We define the operator L on X by

$$D(L) = \left\{u = {}^\top(\phi, w) \in X;\ w \in (H^1_{per,0})^2,\ Lu \in X\right\},$$

$$L = \begin{pmatrix} v_s^1\partial_{x_1} & \gamma^2\mathrm{div} \\ \nabla & -\nu\Delta - \tilde{\nu}\nabla\mathrm{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{\nu}{\gamma^2}\boldsymbol{e}_1 & v_s^1\partial_{x_1} + (\partial_{x_2}v_s^1)\boldsymbol{e}_1{}^\top\boldsymbol{e}_2 \end{pmatrix}.$$

Recall that $\tilde{\nu} = \nu + \nu' \geq 0$. As in [6] one can show that $-L$ generates a C_0-semigroup in X.

We state an instability criterion for Poiseuille flow.

Theorem 3.1. ([8]) *There exist constants $r_0 > 0$ and $\eta_0 > 0$ such that if $\alpha \leq r_0$, then*

$$\sigma(-L) \cap \left\{\lambda \in \mathbb{C} : |\lambda| \leq \eta_0\right\} = \{\lambda_{\alpha k} :\ |k| = 1, \cdots, n_0\}$$

for some $n_0 \in \mathbb{N}$, where $\lambda_{\alpha k}$ are simple eigenvalues of $-L$ that satisfies

$$\lambda_{\alpha k} = -\frac{i}{6}(\alpha k) + \kappa_0(\alpha k)^2 + O(|\alpha k|^3)$$

as $\alpha k \to 0$. Here κ_0 is the number given by

$$\kappa_0 = \frac{1}{12\nu}\left[\left(\frac{1}{280} - \gamma^2\right) - \frac{\nu}{30\gamma^2}(3\nu + \nu')\right].$$

As a consequence, if $\gamma^2 < \frac{1}{280}$ and $\nu(3\nu + \nu') < 30\gamma^2\left(\frac{1}{280} - \gamma^2\right)$, then $\kappa_0 > 0$ and plane Poiseuille flow $u_s = {}^\top(\phi_s, v_s)$ is linearly unstable.

Remark 3.2. The eigenspace for $\lambda_{\alpha k}$ is spanned by a function of the form $u(x_2)e^{i\alpha k x_1}$ where $u(x_2)$ is an eigenfunction for $\lambda_{\alpha k}$ of $-L_{\eta,k}$. Here $L_{\eta,k}$ is an operator appearing in (5.2) below. See [8, Sections 4–6].

4 Traveling wave solutions

In this section we state the result on the existence of traveling wave solutions bifurcating from the Poiseuille flow when it becomes unstable as in Theorem 3.1.

We fix γ such that $\frac{1}{280} - \gamma^2 > 0$. We will take ν as a bifurcation parameter, and therefore, denote the eigenvalue $\lambda_{\alpha k}$ by $\lambda_{\alpha k}(\nu)$:

$$\lambda_{\alpha k} = \lambda_{\alpha k}(\nu),$$

and the linearized operator L by L_ν:

$$L = L_\nu.$$

Let $\tilde{\nu}_0 > 0$ be the number satisfying $\kappa_0 = 0$, where κ_0 is the coefficient of $(\alpha k)^2$ in $\lambda_{\alpha k}(\nu)$ given in Theorem 3.1. Then, by a perturbation argument, one can see that for each $0 < \alpha \ll 1$, there exists $\nu_0 > 0$ such that

$$\begin{aligned} \operatorname{Re}\lambda_{\pm\alpha}(\nu_0) &= 0; \\ \operatorname{Re}\lambda_{\pm\alpha}(\nu) < 0 \ &\Leftrightarrow\ \nu > \nu_0; \\ \operatorname{Re}\lambda_{\pm\alpha}(\nu) > 0 \ &\Leftrightarrow\ \nu < \nu_0. \end{aligned}$$

From [8, Section 6], one can see that $\operatorname{Re}\lambda_\alpha(\nu)$ is analytic in α^2. Setting $\zeta(\alpha^2, \nu) = \operatorname{Re}\lambda_\alpha(\nu)/\alpha^2$, we see that $\partial_\nu\zeta(\alpha^2,\nu) = -\frac{1}{12\nu^2}\left[\left(\frac{1}{280} - \gamma^2\right) + \frac{\nu^2}{10\gamma^2}\right] + O(\alpha^2) < 0$ for $\alpha \ll 1$, and so $\operatorname{Re}\lambda_\alpha(\nu)$ crosses the imaginary axis from left to right at $\nu = \nu_0$ when ν is decreased.

We make the following assumption:

$$\sigma(-L_{\nu_0}) \cap \{\lambda; \operatorname{Re}\lambda = 0\} = \{\lambda_\alpha(\nu_0), \lambda_{-\alpha}(\nu_0)\}. \tag{4.1}$$

Theorem 4.1. *Assume that* (4.1) *holds true. Then there is a solution branch* $\{\nu, u\} = \{\nu_\varepsilon, u_\varepsilon\}$ $(|\varepsilon| \ll 1)$ *such that*

$$\begin{aligned}
&\nu_\varepsilon = \nu_0 + O(\varepsilon),\\
&u_\varepsilon = u_\varepsilon(x_1 - c_\varepsilon t, x_2),\\
&u_\varepsilon(x_1 + \tfrac{2\pi}{\alpha}, x_2) = u_\varepsilon(x_1, x_2),\\
&u_\varepsilon(x_1, x_2) = \varepsilon \begin{pmatrix} 1 \\ \frac{1}{2\gamma^2}(-x_2^2 + x_2) \\ 0 \end{pmatrix} \tfrac{\sqrt{2}}{2} \cos \alpha x_1 (1 + O(\alpha)) + O(\varepsilon^2),\\
&c_\varepsilon = \tfrac{1}{6} + O(\varepsilon).
\end{aligned}$$

Remark 4.2. Iooss and Padula ([6]) showed that, for each ν, there exists a positive number c such that the set

$$\sigma(-L_\nu) \cap \{\lambda; \operatorname{Re}\lambda \geq -c\}$$

consists of a finite number of eigenvalues with finite multiplicities. (See Lemma 6.10 below.) Therefore, it seems very unlikely that assumption (4.1) does not hold true for all $\alpha \ll 1$.

5 Proof of Theorem 4.1

In this section we give a proof of Theorem 4.1.

We set $\eta = \nu - \nu_0$ that will be taken as a new bifurcation parameter. For simplicity, we write L_η for $L_{\eta+\nu_0}$ omitting ν_0.

5.1 Spectrum of $-L_0$

We first make an observation of the spectrum of $-L_\eta$. Let us consider the resolvent problem

$$\lambda u + Lu = F. \tag{5.1}$$

We expand u and F into the Fourier series in x_1:

$$u = \sqrt{\frac{\alpha}{2\pi}} \sum_{k\in\mathbb{Z}} u_k(x_2) e^{i\alpha k x_1}, \quad u_k = {}^\top(\phi_k, w_k),$$

$$F = \sqrt{\frac{\alpha}{2\pi}} \sum_{k\in\mathbb{Z}} F_k(x_2) e^{i\alpha k x_1}, \quad F_k = {}^\top(f_k^0, f_k)$$

with $\int_0^1 \phi_0(x_2)\, dx_2 = \int_0^1 f_0^0(x_2)\, dx_2 = 0$. Then the problem is reduce to the following problems for $k \in \mathbb{Z}$:

$$(\lambda + L_{\eta,k}) u_k = F_k. \tag{5.2}$$

Here $L_{\eta,k}$ is the operator on $L^2_k(0,1) \times L^2(0,1)^2$ obtained by replacing ∂_{x_1} in L by $i\alpha k$ with domain $D(L_{\eta,k}) = \{u_k = {}^\top(\phi_k, w_k) \in L^2_k(0,1) \times L^2(0,1)^2; w_k \in$

$H^1_0(0,1), L_{\eta,k}u_k \in L^2_k(0,1) \times L^2(0,1)^2\}$, where $L^2_k(0,1) = L^2(0,1)$ when $k \neq 0$ and $L^2_0(0,1) = L^2(0,1) \cap \{\phi; \int_0^1 \phi(x_2)\,dx_2 = 0\}$.

Let $\tilde{X} = L^2_{per} \times (L^2_{per})^2$. We denote by $\tilde{L}$ the extension of L to $\tilde{X}$, more precisely, $\tilde{L}$ is an operator on $\tilde{X}$ with domain $D(\tilde{L}) = \{u = {}^\top(\phi, w) \in \tilde{X};\ w \in (H^1_{per,0})^2,\ \tilde{L}u \in \tilde{X}\}$ and $\tilde{L}$ has the same form as L. Similarly, we define an operator $\tilde{L}_{\eta,k}$ on $L^2(0,1) \times L^2(0,1)^2$ by the extension of $L_{\eta,k}$ to $L^2(0,1) \times L^2(0,1)^2$. Note that $\tilde{L}_{\eta,k} = L_{\eta,k}$ when $k \neq 0$ and $L_{\eta,0}$ is the restriction of $\tilde{L}_{0,\eta}$ to $L^2_0(0,1) \times L^2(0,1)^2$. We also introduce the adjoint operator $\tilde{L}^*$ (with respect to the inner product $\langle \cdot, \cdot \rangle$) which is given by

$$\tilde{L}^* = \begin{pmatrix} -v_s^1 \partial_{x_1} & -\nu^\top \boldsymbol{e}_1 - \gamma^2 \mathrm{div} \\ -\nabla & -\nu\Delta - \tilde{\nu}\nabla \mathrm{div} - v_s^1 \partial_{x_1} + (\partial_{x_2} v_s^1)\boldsymbol{e}_2{}^\top \boldsymbol{e}_1 \end{pmatrix}.$$

Similarly, the adjoint operators $\tilde{L}^*_{\eta,k}$ of $\tilde{L}_{\eta,k}$ are defined.

Since X is an invariant set of $\tilde{L}$, we see that if λ is an eigenvalue of $-L$, then the eigenprojection for λ of $-L$ is the restriction of the eigenprojection for λ of $-\tilde{L}$. The same also holds for eigenprojections of $L_{\eta,0}$ and $\tilde{L}_{\eta,0}$.

Under the assumption (4.1), the following claims are concluded. In what follows we denote the critical eigenvalues $\lambda_{\pm\alpha}(\nu_0)$ by $\pm ia$ with $a = -\frac{\alpha}{6}(1 + O(\alpha^2)) \in \mathbb{R}\setminus\{0\}$:

$$\lambda_{\pm\alpha}(\nu_0) = \pm ia.$$

As for $\sigma(-L_{0,k})$, we have

- $k = \pm 1$:
$$\begin{aligned} &\sigma(-L_{0,\pm1}) \cap \{\lambda; \mathrm{Re}\,\lambda = 0\} = \{\pm ia\}, \\ &\pm ia \text{ are isolated simple eigenvalues of } -L_{0,\pm1}, \\ &N(\pm ia + L_{0,\pm1}) = \mathrm{span}\{v_{\pm1}\}, \quad v_{-1} = \overline{v_{+1}}. \end{aligned}$$
- $k \neq \pm 1$: there exists a constant $\beta > 0$ such that $\sigma(-L_{0,k}) \subset \{\lambda; |\mathrm{Re}\,\lambda| \geq \beta\}$ for all $k \in \mathbb{Z}$ with $k \neq \pm 1$.

The eigenprojections for $\pm ia$ are given in terms of eigenfunctions of the adjoint operator $\tilde{L}^*_{0,k}$. Namely, we have

$$\begin{aligned} &\text{the eigenprojections } \Pi_\pm \text{ for } \pm ia \text{ are given by } \Pi_\pm u = \langle\langle u, v^*_{\pm1} \rangle\rangle v_{\pm1}\ , \\ &\text{where } N(\mp ia + \tilde{L}^*_{0,\pm1}) = \mathrm{span}\{v^*_{\pm1}\},\ \langle\langle v_{\pm1}, v^*_{\pm1} \rangle\rangle = 1\ . \end{aligned}$$

It then follows that $\sigma(-L_0)$ satisfies

$$\begin{aligned} &\sigma(-L_0) \cap \{\lambda; \mathrm{Re}\,\lambda = 0\} = \{\pm ia\}, \\ &\pm ia \text{ are isolated simple eigenvalues of } -L_0, \\ &N(\pm ia + L_0) = \mathrm{span}\{V_\pm\}, \\ &\text{where } V_\pm = v_{\pm1}(x_2) e^{\pm i\alpha x_1}. \end{aligned}$$

Furthermore, $V_{\pm}^{*} = \frac{\alpha}{2\pi} v_{\pm 1}^{*}(x_2) e^{\pm i\alpha x_1}$ satisfy

$$-\tilde{L}_0^{*} V_{\pm}^{*} = \mp i a V_{\pm}^{*},\ \langle V_{\pm}, V_{\pm}^{*}\rangle = 1,\ \langle V_{\pm}, V_{\mp}^{*}\rangle = 0,$$

and the eigenprojections $P_{\pm}$ for $\pm ia$ of $-L$ are given by

$$P_{\pm} V = \ \langle V, V_{\pm}^{*}\rangle V_{\pm}.$$

It was proved in [7] that eigenfunctions $V_{\pm}$ and $V_{\pm}^{*}$ are smooth and, for each nonnegative integer k, eigenprojections $P_{\pm}$ are bounded from $L^2_{per,*} \times L^2_{per}$ to $H^k_{per,*} \times H^k_{per}$:

$$\|P_{\pm} u\|_{H^k \times H^k} \leq C_k \|u\|_2.$$

See [7, Lemma 4.3]. These boundedness properties of $P_{\pm}$ will be employed later.

5.2 Traveling wave solution

Let us consider the nonlinear problem

$$\partial_t \tilde{u} + L_\eta \tilde{u} = F(\eta, \tilde{u}), \tag{5.3}$$

where $F(\eta, \tilde{u})$ denotes the nonlinear term.

We look for a solution in the form

$$\tilde{u}(x_1, x_2, t) = u(x_1 - ct, x_2).$$

We substitute this into (5.3). Then the problem is rewritten as

$$\mathcal{L}_{c,\eta} u = F(\eta, u), \tag{5.4}$$

where

$$\mathcal{L}_{c,\eta} = L_\eta - c\partial_{x_1}.$$

We first investigate the spectrum of $-\mathcal{L}_{c_0,0}$.

5.3 Spectrum of $-\mathcal{L}_{c_0,0}$

The following proposition on the spectrum of $\mathcal{L}_{c_0,0}$ follows from the observation in section 5.1.

Proposition 5.1. *Set $c_0 = -\frac{a}{\alpha}$. Then*

$$\sigma(-\mathcal{L}_{c_0,0}) \cap \{\lambda; \mathrm{Re}\,\lambda = 0\} = \{0\},$$

0 *is an isolated semisimple eigenvalue of* $-\mathcal{L}_{c_0,0}$,

$$N(-\mathcal{L}_{c_0,0}) = \mathrm{span}\{V_+, V_-\}, \quad V_- = \overline{V_+}.$$

Let us next introduce the eigenprojection for the eigenvalue 0 of $-\mathcal{L}_{c_0,0}$. We set

$$V_1 = \sqrt{2}\mathrm{Re}\, V_+, \quad V_2 = \sqrt{2}\mathrm{Im}\, V_+,$$

$$V_1^* = \sqrt{2}\mathrm{Re}\, V_+^*, \quad V_2^* = \sqrt{2}\mathrm{Im}\, V_+^*.$$

Then

$$N(-\mathcal{L}_{c_0,0}) = \mathrm{span}\{V_1, V_2\},$$

$$\langle V_j, V_k^* \rangle = \delta_{jk}, \quad j,k = 1,2.$$

We introduce the following notation $[\![u]\!]_j$ $(j = 1,2)$:

$$[\![u]\!]_j = \langle u, V_j^* \rangle.$$

Proposition 5.2. *Define P, P_1 and P_2 by*

$$Pu = P_1 u + P_2 u, \quad P_j u = [\![u]\!]_j V_j \quad (j = 1,2).$$

Then P is the eigenprojection for eigenvalue 0 of $-\mathcal{L}_{c_0,0}$; and

$$R(P_j) = \mathrm{span}\{V_j\}, \quad P_j^2 = P_j, \quad P_j P_k = O \ (j \neq k).$$

For each nonnegative integer k, P_j are bounded from $L^2_{per,*} \times L^2_{per}$ to $H^k_{per,*} \times H^k_{per}$:

$$\|P_j u\|_{H^k \times H^k} \leq C \|u\|_2.$$

Furthermore, $u \in R(I - P_j)$ if and only if $[\![u]\!]_j = 0$.

5.4 Formulation of the problem

We look for solutions of (5.4) in a neighborhood of $\{c, \eta, u\} = \{c_0, 0, 0\}$ in the form:

$$u = \varepsilon(V_1 + \varepsilon V), \quad V \in R(Q), \quad Q = I - P,$$

$$c = c_0 + \varepsilon\sigma.$$

Here ε is a small parameter. Note that $P_2 u = 0$.

We set

$$K_0 = \frac{1}{\eta}(L_\eta - L_0) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\gamma^2}\boldsymbol{e}_1 & -\Delta - \nabla \mathrm{div} \end{pmatrix}.$$

Then

$$L_\eta = L_0 + \eta K_0,$$

and

$$\mathcal{L}_{c,\eta} = \mathcal{L}_{c_0,0} - \varepsilon\sigma\partial_{x_1} + \eta K_0.$$

We scale η as

$$\eta = \varepsilon\omega.$$

Problem (5.4) is then written as

$$\mathcal{L}_{c_0,0}V - \sigma\partial_{x_1}(V_1 + \varepsilon V) + \omega K_0(V_1 + \varepsilon V) = \frac{1}{\varepsilon^2}F(\varepsilon\omega, \varepsilon(V_1 + \varepsilon V)). \tag{5.5}$$

We denote the right-hand side by

$$\frac{1}{\varepsilon^2}F(\varepsilon\omega, \varepsilon(V_1 + \varepsilon V)) = -N[V_1 + \varepsilon V](V_1 + \varepsilon V) + G(\varepsilon, \varepsilon\omega, V_1 + \varepsilon V),$$

where

$$N[\tilde{u}]u = {}^\top(\mathrm{div}\,(\phi\tilde{w}), 0)$$

for $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ and $u = {}^\top(\phi, w)$, and

$$G(\varepsilon, \omega, u) = {}^\top(0, g(\varepsilon, \omega, u))$$

with

$$\begin{aligned} g(\varepsilon, \omega, u) \quad = \quad & -w\cdot\nabla w - \tfrac{\phi}{\gamma^2+\varepsilon\phi}\Big((\nu_0+\omega)\Delta w + \tfrac{(\nu_0+\omega)}{\gamma^2}\phi\boldsymbol{e}_1 + (\tilde{\nu}_0+\omega)\nabla\mathrm{div}\,w\Big) \\ & + P^{(1)}(\varepsilon\phi)\phi\nabla\phi \end{aligned}$$

for $u = {}^\top(\phi, w)$, where $\tilde{\nu}_0 = \nu_0 + \nu'$.

We decompose (5.5) into the P_j-parts ($j = 1, 2$) and Q-part. Here and in what follows we set

$$Q = I - P = I - P_1 - P_2.$$

We take the inner product of (5.5) with V_j^* ($j = 1, 2$) and apply Q to (5.5). Since

$$[\![\partial_{x_1}(V_1 + \varepsilon V)]\!]_1 = 0, \quad [\![\partial_{x_1}(V_1 + \varepsilon V)]\!]_2 = -\alpha,$$

we find that

$$\begin{aligned} \omega[\![K_0V_1]\!]_1 \quad &= \quad -\varepsilon\omega[\![K_0V]\!]_1 - [\![N[V_1 + \varepsilon V](V_1 + \varepsilon V)]\!]_1 \\ & \qquad + [\![G(\varepsilon, \varepsilon\omega, V_1 + \varepsilon V)]\!]_1, \end{aligned}$$

$$\begin{aligned} \omega[\![K_0V_1]\!]_2 + \alpha\sigma \quad &= \quad -\varepsilon\omega[\![K_0V]\!]_2 - [\![N[V_1 + \varepsilon V](V_1 + \varepsilon V)]\!]_2 \\ & \qquad + [\![G(\varepsilon, \varepsilon\omega, V_1 + \varepsilon V)]\!]_2, \end{aligned}$$

$$\begin{aligned} & \omega QK_0V_1 + (\mathcal{L}_{c_0,0} - \varepsilon\sigma Q\partial_{x_1} + \varepsilon QN[V_1 + \varepsilon V])\,V \\ & \quad = \quad -\varepsilon\omega QK_0V - QN[V_1 + \varepsilon V]V_1 + QG(\varepsilon, \varepsilon\omega, V_1 + \varepsilon V). \end{aligned}$$

We thus arrive at the following problem:

$$T(\varepsilon, \sigma, V)U = \mathcal{F}(\varepsilon, U), \tag{5.6}$$

where

$$U = {}^\top(\omega, \sigma, V) \in \mathbb{R} \times \mathbb{R} \times X^2.$$

Here X^ℓ denotes the function space

$$X^\ell = H^\ell_{per,*} \times [H^{\ell+1}_{per} \cap H^1_{per,0}], \quad \ell = 1, 2,$$

and, for a given $(\tilde{\sigma}, \tilde{V}) \in \mathbb{R} \times X^2$, $T(\varepsilon, \tilde{\sigma}, \tilde{V})$ is the linear map defined by

$$T(\varepsilon, \tilde{\sigma}, \tilde{V}) : \mathbb{R} \times \mathbb{R} \times QX^\ell \;\to\; \mathbb{R} \times \mathbb{R} \times Q(H^\ell \times H^{\ell-1}),\ \ell = 1, 2,$$

$$T(\varepsilon, \tilde{\sigma}, \tilde{V}) \;=\; \begin{pmatrix} [\![K_0 V_1]\!]_1 & 0 & 0 \\ [\![K_0 V_2]\!]_1 & \alpha & 0 \\ QK_0 V_1 & 0 & \mathcal{L}_{c_0,0} - \varepsilon\tilde{\sigma} Q\partial_{x_1} + \varepsilon QN[V_1 + \varepsilon\tilde{V}] \end{pmatrix}.$$

$\mathcal{F}(\varepsilon, U)$ is the nonlinear map given by

$$\mathcal{F}(\varepsilon, U) = {}^\top(\mathcal{F}_1(\varepsilon, U), \mathcal{F}_2(\varepsilon, U), \mathcal{F}_3(\varepsilon, U)) \quad (U = {}^\top(\omega, \sigma, V)),$$

where

$$\begin{aligned} \mathcal{F}_j(\varepsilon, U) \;&=\; -\varepsilon\omega[\![K_0 V]\!]_j - [\![N[V_1 + \varepsilon V](V_1 + \varepsilon V)]\!]_j + [\![G(\varepsilon, \varepsilon\omega, V_1 + \varepsilon V)]\!]_j, \\ &\quad\ (j = 1, 2), \\ \mathcal{F}_3(\varepsilon, U) \;&=\; -\varepsilon\omega QK_0 V - QN[V_1 + \varepsilon V]V_1 + QG(\varepsilon, \varepsilon\omega, V_1 + \varepsilon V). \end{aligned}$$

Concerning $T(\varepsilon, \tilde{\sigma}, \tilde{V})$ we have the following

Proposition 5.3. (i) $[\![K_0 V_1]\!]_1 > 0$.

(ii) *For given $M > 0$, there exists $\varepsilon_1 > 0$ such that if $|\varepsilon| \leq \varepsilon_1$ and $|\tilde{\sigma}| + \|\tilde{V}\|_{X^2} \leq M$, then $\mathcal{L}_{c_0,0} - \varepsilon\tilde{\sigma} Q\partial_{x_1} + \varepsilon QN[V_1 + \varepsilon\tilde{V}]$ has a bounded inverse from $Q(H^\ell_{per,*} \times H^{\ell-1}_{per})$ to QX^ℓ ($\ell = 1, 2$).*

(iii) *Under the assumption of (ii), $T(\varepsilon, \tilde{\sigma}, \tilde{V})$ has a bounded inverse from $\mathbb{R} \times \mathbb{R} \times Q(H^\ell_{per,*} \times H^{\ell-1}_{per})$ to $\mathbb{R} \times \mathbb{R} \times QX^\ell$ ($\ell = 1, 2$), and it holds that for $U = {}^\top(\tilde{\eta}, \sigma, V)$,*

$$\|T(\varepsilon, \tilde{\sigma}, \tilde{V})^{-1} U\|_{\mathbb{R}\times\mathbb{R}\times X^\ell} \leq C_1 \|U\|_{\mathbb{R}\times\mathbb{R}\times H^\ell \times H^{\ell-1}}, \quad \ell = 1, 2.$$

We will give a proof of Proposition 5.3 (ii) and (iii) in section 6, and a proof of (i) will be given in section 8.

As for $\mathcal{F}(\varepsilon, U)$, using Sobolev inequalities, we have the following estimates by a straightforward computation.

Proposition 5.4. *For given $M \in (0, \frac{\gamma^2}{2C_S}]$, there exists $\varepsilon_2 > 0$ such that if $|\varepsilon| \leq \varepsilon_2$, $\|U\|_{\mathbb{R}\times\mathbb{R}\times X^2} \leq M$ and $\|U^{(j)}\|_{\mathbb{R}\times\mathbb{R}\times X^2} \leq M$ ($j = 1, 2$), then the following estimates hold:*

$$\|\mathcal{F}(\varepsilon, U) - \mathcal{F}(0, 0)\|_{\mathbb{R}\times\mathbb{R}\times H^2 \times H^1} \leq C(M) M |\varepsilon|,$$

$$\|\mathcal{F}(\varepsilon, U^{(1)}) - \mathcal{F}(\varepsilon, U^{(2)})\|_{\mathbb{R}\times\mathbb{R}\times H^1 \times H^0} \leq C(M) |\varepsilon| \|U^{(1)} - U^{(2)}\|_{\mathbb{R}\times\mathbb{R}\times X^1},$$

where $C(M) > 0$ is a nondecreasing continuous function of M.

5.5 Iteration

The desired solution branch in Theorem 4.1 can now be obtained by an iteration argument.

We define $U^{(n)} = {}^\top(\omega^{(n)}, \sigma^{(n)}, V^{(n)})$ $(n \geq 1)$ in the following way. $U^{(1)}$ is the solution of

$$\begin{aligned} T(0,0,0)U^{(1)} &= \mathcal{F}(0,0) \\ &= {}^\top(\llbracket F(0,V_1)\rrbracket_1, \llbracket F(0,V_1)\rrbracket_2, QF(0,V_1)). \end{aligned}$$

Note that $F(0,V_1) = -N[V_1]V_1 + G(0,0,V_1)$. By Propositions 5.3 we have

$$\|U^{(1)}\|_{\mathbb{R}\times\mathbb{R}\times X^2} \leq C_1\|\mathcal{F}(0,0)\|_{\mathbb{R}\times\mathbb{R}\times H^2\times H^1} < \infty. \tag{5.7}$$

We set

$$M = 2C_1\|\mathcal{F}(0,0)\|_{\mathbb{R}\times\mathbb{R}\times H^2\times H^1}. \tag{5.8}$$

Let $\varepsilon > 0$ satisfy $|\varepsilon| \leq \min\{\varepsilon_1, \varepsilon_2, \frac{1}{2C_1C(M)}\}$. Then for $n \geq 2$ we can define $U^{(n)}$ by the solution of

$$T(\varepsilon, \sigma^{(n-1)}, V^{(n-1)})U^{(n)} = \mathcal{F}(\varepsilon, U^{(n-1)}), \tag{5.9}$$

and $U^{(n)}$ satisfies

$$\|U^{(n)}\|_{\mathbb{R}\times\mathbb{R}\times X^2} \leq M$$

for all $n \geq 1$. In fact, assume that $\|U^{(n-1)}\|_{\mathbb{R}\times\mathbb{R}\times X^2} \leq M$. Then, $\mathcal{F}(\varepsilon, U^{(n-1)}) \in \mathbb{R}\times\mathbb{R}\times Q(H^2_{per,*}\times H^1_{per})$, and thus, Proposition 5.3 implies that (5.9) has a solution $U^{(n)} \in \mathbb{R}\times\mathbb{R}\times X^2$. Furthermore, since

$$T(\varepsilon, \sigma^{(n-1)}, V^{(n-1)})U^{(n)} = \mathcal{F}(0,0) + (\mathcal{F}(\varepsilon, U^{(n-1)}) - \mathcal{F}(0,0))$$

and $|\varepsilon| \leq \min\{\varepsilon_1, \varepsilon_2, \frac{1}{2C_1C(M)}\}$, we see from Propositions 5.3 and 5.4 that

$$\|U^{(n)}\|_{\mathbb{R}\times\mathbb{R}\times X^2} \leq \frac{M}{2} + C_1C(M)M|\varepsilon| \leq M.$$

Therefore, with this observation and (5.7), we conclude by induction that $\|U^{(n)}\|_{\mathbb{R}\times\mathbb{R}\times X^2} \leq M$ for all $n \geq 1$.

We next prove that $\{U^{(n)}\}$ is a Cauchy sequence in $\mathbb{R}\times\mathbb{R}\times X^1$. We set

$$\mathcal{D}V = {}^\top(0,0,\partial_{x_1}V), \quad \mathcal{N}_Q[\tilde{V}]V = {}^\top(0,0,QN[\tilde{V}]V).$$

Since

$$\begin{aligned} &T(\varepsilon, \sigma^{(n)}, V^{(n)})U^{(n+1)} - T(\varepsilon, \sigma^{(n-1)}, V^{(n-1)})U^{(n)} \\ &= T(\varepsilon, \sigma^{(n)}, V^{(n)})(U^{(n+1)} - U^{(n)}) - \varepsilon(\sigma^{(n)} - \sigma^{(n-1)})\mathcal{D}V^{(n)} \\ &\quad +\varepsilon^2\mathcal{N}_Q[V^{(n)} - V^{(n-1)}]V^{(n)}, \end{aligned}$$

we have

$$\begin{aligned} &T(\varepsilon, \sigma^{(n)}, V^{(n)})(U^{(n+1)} - U^{(n)}) \\ &\quad = \varepsilon(\sigma^{(n)} - \sigma^{(n-1)})\mathcal{D}V^{(n)} - \varepsilon^2\mathcal{N}_Q[V^{(n)} - V^{(n-1)}]V^{(n)} \\ &\qquad +(\mathcal{F}(\varepsilon, U^{(n)}) - \mathcal{F}(\varepsilon, U^{(n-1)})), \end{aligned}$$

and by Propositions 5.3 and 5.4,

$$\begin{aligned}
&\|U^{(n+1)}-U^{(n)}\|_{\mathbb{R}\times\mathbb{R}\times X^1}\\
&\le\ C_1\Big\{|\varepsilon||\sigma^{(n)}-\sigma^{(n-1)}|\|\partial_{x_1}V^{(n)}\|_{H^1\times H^0}+|\varepsilon|^2\|QN[V^{(n)}-V^{(n-1)}]V^{(n)}\|_{H^1\times H^0}\\
&\quad+\|\mathcal{F}(\varepsilon,U^{(n)})-\mathcal{F}(\varepsilon,U^{(n-1)})\|_{\mathbb{R}\times\mathbb{R}\times H^1\times H^0}\Big\}\\
&\le\ C_1\Big\{CM|\varepsilon||\sigma^{(n)}-\sigma^{(n-1)}|+CM|\varepsilon|^2\|V^{(n)}-V^{(n-1)}\|_{X^1}\\
&\quad+C(M)|\varepsilon|\|U^{(n)}-U^{(n-1)}\|_{\mathbb{R}\times\mathbb{R}\times X^1}\Big\}\\
&\le\ \tfrac{1}{2}\|U^{(n)}-U^{(n-1)}\|_{\mathbb{R}\times\mathbb{R}\times X^1}
\end{aligned}$$

if $|\varepsilon|\le\frac{1}{2C_1(2CM+C(M))}$. It then follows that there exists $\varepsilon_0>0$ such that if $|\varepsilon|\le\varepsilon_0$, then $\{U^{(n)}\}$ is a Cauchy sequence in $\mathbb{R}\times\mathbb{R}\times X^1$. We thus conclude if $|\varepsilon|\le\varepsilon_0$, there exists $U={}^{\top}(\omega,\sigma,V)\in\mathbb{R}\times\mathbb{R}\times X^2$ satisfying

$$T(\varepsilon,\sigma,V)U=\mathcal{F}(\varepsilon,U).$$

With this $U={}^{\top}(\omega,\sigma,V)$, setting

$$\nu=\nu_0+\varepsilon\omega,\quad u=\varepsilon V_1(x_1-ct,x_2)+\varepsilon^2V(x_1-ct,x_2),\quad c=c_0+\varepsilon\sigma,$$

we have the desired traveling wave solutions.

To complete the proof of Theorem 4.1, it remains to prove Proposition 5.3.

6 Proof of Proposition 5.3 (ii), (iii)

In this section we give a proof of Proposition 5.3 (ii), (iii).

By a perturbation argument for $\alpha\ll 1$, one can compute $u_{\pm1}$ and $u_{\pm1}^*$ to see assertion (i) $[\![K_0V_1]\!]_1>0$ for $\alpha\ll 1$. See section 8 for the proof of (i). If assertion (ii) holds, then $T(\varepsilon,\tilde\sigma,\tilde V)$ has a bounded inverse $T(\varepsilon,\tilde\sigma,\tilde V)^{-1}$ which is given by

$$T(\varepsilon,\tilde\sigma,\tilde V)^{-1}=\begin{pmatrix}\mathscr{A}^{-1} & 0\\ -\mathscr{L}(\varepsilon,\tilde\sigma,\tilde V)^{-1}\mathscr{B}\mathscr{A}^{-1} & \mathscr{L}(\varepsilon,\tilde\sigma,\tilde V)^{-1}\end{pmatrix},$$

where

$$\mathscr{A}=\begin{pmatrix}[\![K_0V_1]\!]_1 & 0\\ [\![K_0V_2]\!]_2 & \alpha\end{pmatrix},$$

$$\mathscr{B}=\begin{pmatrix}QK_0V_1 & 0\end{pmatrix},$$

$$\mathscr{L}(\varepsilon,\tilde\sigma,\tilde V)=\mathbf{\mathcal{L}}_{c_0,0}-\varepsilon\tilde\sigma Q\partial_{x_1}+\varepsilon QN[V_1+\varepsilon\tilde V].$$

Therefore, in the rest of this section we will prove assertion (ii), i.e, $\mathscr{L}(\varepsilon,\tilde\sigma,\tilde V)$ has a bounded inverse.

6.1 Basic estimates

From now on, we simply write $N[\tilde{w}]u$ for $N[\tilde{u}]u$ with $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$:

$$N[\tilde{w}]u = {}^\top(\operatorname{div}(\phi\tilde{w}), 0), \quad u = {}^\top(\phi, w).$$

In this subsection we establish basic a priori estimates of solution u to

$$\lambda u + Lu + N[\tilde{w}]u = F, \quad u \in X^\ell, \tag{6.1}$$

where $\tilde{w}$ is a given function in $H^3_{per} \cap H^1_{per,0}$ with $\tilde{w}(x) \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}$ is a parameter.

We introduce some notations. We define the new norm $|||\cdot|||_2$ of L^2_{per} by

$$|||u|||_2 = \left(\frac{1}{\gamma^2}\|\phi\|_2^2 + \|w\|_2^2\right)^{\frac{1}{2}}$$

for $u = {}^\top(\phi, w)$. We also define $D[w]$ and $\dot{\phi}_\lambda$ by

$$D[w] = \nu\|\nabla w\|_2^2 + \tilde{\nu}\|\operatorname{div} w\|_2^2$$

and

$$\dot{\phi}_\lambda = \lambda\phi + v_s^1\partial_{x_1}\phi + \operatorname{div}(\phi\tilde{w}),$$

respectively. For operators A and B, we denote by $[A, B]$ the commutator of A and B:

$$[A, B]f = A(Bf) - B(Af).$$

We will prove the following

Proposition 6.1. *There exists a number Λ satisfying $0 < \Lambda \leq \frac{1}{2}\frac{\gamma^2}{\nu+\tilde{\nu}}$ such that if $\operatorname{Re}\lambda \geq -\Lambda$, then*

$$\begin{aligned}
&\Big(\operatorname{Re}\lambda + \Lambda\Big)^2\|u\|_2^2 + \Big(\operatorname{Re}\lambda + \Lambda\Big)\|\partial_x u\|_2^2 + \|\partial_x^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 + \|\dot{\phi}_\lambda\|_{H^1}^2 \\
&\quad \leq \ C\Big\{\|F\|_{H^1\times L^2}^2 + \|\tilde{w}\|_{H^3}(1 + \|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2 + \|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\},
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
&\Big(\operatorname{Re}\lambda + \Lambda\Big)^2\|u\|_2^2 + \Big(\operatorname{Re}\lambda + \Lambda\Big)(\|\partial_x u\|_2^2 + \|\partial_x^2\phi\|_2^2 + \|\partial_{x_1}^2 w\|_2^2 + |\lambda|^2\|u\|_2^2) \\
&\quad + \|\partial_x^2 w\|_2^2 + \|\partial_x^3 w\|_2^2 + |\lambda|^2\|\nabla u\|_2^2 + \|\dot{\phi}_\lambda\|_{H^2}^2 \\
&\quad \leq \ C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2 + \|\tilde{w}\|_{H^3}(1 + \|\tilde{w}\|_{H^3})(\|\phi\|_{H^2}^2 + |\lambda|^2\|\phi\|_2^2) \\
&\qquad\qquad + \|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}.
\end{aligned} \tag{6.3}$$

To prove Proposition 6.1, we will employ the following Bogovskii lemma.

Lemma 6.2. ([2]) *There exists a bouded operator* $\mathcal{B} : L^2_{per,*} \to H^1_{per,0}$ *such that*

$$\operatorname{div} \mathcal{B} g = g, \quad g \in L^2_{per,*}$$

$$\|\nabla \mathcal{B} g\|_2 \le C_B \|g\|_2,$$

where C_B *is a positive constant depending only on* α. *Furthermore, if* $g = \operatorname{div} \boldsymbol{g}$ *with* $\boldsymbol{g} = {}^\top(g^1, g^2)$ *satisfying* $g^1|_{x_1 = -\frac{\pi}{\alpha}} = g^1|_{x_1 = \frac{\pi}{\alpha}}$, $g^2|_{x_2 = 0,1} = 0$, *then*

$$\operatorname{div} \mathcal{B}(\operatorname{div} \boldsymbol{g}) = \operatorname{div} \boldsymbol{g},$$

$$\|\mathcal{B}(\operatorname{div} \boldsymbol{g})\|_2 \le C_B \|\boldsymbol{g}\|_2.$$

An outline of the proof of Lemma 6.2 will be given in Section 7. We will also employ the Poincaré inequalities

$$\|\phi\|_2 \le C \|\nabla \phi\|_2, \quad \|w\|_2 \le \|\nabla w\|_2$$

for $\phi \in H^1_{per,*}$ and $w \in H^1_{per,0}$, and the Sobolev inequality

$$\|f\|_\infty \le C \|f\|_{H^2}$$

for $f \in H^2_{per}$. Here C is a positive constant depending only on α.

We begin with the following L^2 energy estimates.

Proposition 6.3. *There exists a positive number* Λ_0 *such that the following inequalities hold uniformly for* $\operatorname{Re} \lambda \ge \Lambda_0$.

$$\begin{aligned} &(\operatorname{Re} \lambda + \Lambda_0) |\lambda|^{2k} |||u|||_2^2 + \frac{1}{4} |\lambda|^{2k} D[w] \\ &\le \ C |\lambda|^{2k} \Big\{ |||F|||_2 |||u|||_2 + \|\partial_{x_2} v_s^1\|_\infty \|w\|_2^2 + \frac{\nu}{\gamma^4} \|\phi\|_{H^{-1}}^2 + \frac{\|\tilde{w}\|_{H^3}}{\gamma} \|\phi\|_2 |||u|||_2 \Big\} \end{aligned} \tag{6.4}$$

for $k = 0, 1$,

$$\begin{aligned} &\Big(\operatorname{Re} \lambda + \frac{1}{2} \Lambda_0\Big) |||u|||_2^2 + \frac{1}{8} D[w] + \frac{\nu + \tilde{\nu}}{32 \gamma^4} \|\dot{\phi}_\lambda\|_2^2 \\ &\le \ C \Big\{ \Big(\frac{1}{\gamma^2 \Lambda_0} + \frac{\nu + \tilde{\nu}}{\gamma^4} \Big) \|f^0\|_2^2 + \frac{1}{\nu} \|f\|_{H^{-1}}^2 + \frac{\|\tilde{w}\|_{H^3}}{\gamma^2} \Big(1 + \frac{\|\tilde{w}\|_{H^3}}{\nu} \Big) \|\phi\|_2^2 \\ &\qquad + \|\partial_{x_2} v_s^1\|_\infty \|w\|_2^2 + \frac{\nu}{\gamma^4} \|\phi\|_{H^{-1}}^2 \Big\}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} &\Big(\operatorname{Re} \lambda + \frac{1}{2} \Lambda_0\Big) |||\partial_{x_1}^j u|||_2^2 + \frac{1}{16} D[\partial_{x_1}^j w] + \frac{\nu + \tilde{\nu}}{32 \gamma^4} \|\partial_{x_1}^j \dot{\phi}_\lambda\|_2^2 \\ &\le \ C \Big\{ \Big(\frac{1}{\gamma^2 \Lambda_0} + \frac{\nu + \tilde{\nu}}{\gamma^4} \Big) \|\partial_{x_1}^j f^0\|_2^2 + \frac{1}{\nu} \|\partial_{x_1}^j f\|_{H^{-1}}^2 + \frac{\|\tilde{w}\|_{H^3}}{\gamma^2} \Big(1 + \frac{\|\tilde{w}\|_{H^3}}{\nu} \Big) \|\phi\|_{H^j}^2 \\ &\qquad + \|\partial_{x_2} v_s^1\|_\infty \|\partial_{x_1}^j w\|_2^2 + \frac{\nu}{\gamma^4} \|\partial_{x_1}^j \phi\|_{H^{-1}}^2 \Big\} \end{aligned} \tag{6.6}$$

for $j = 1, 2$.

Proof. We follow the argument in [6]. We take the weighted inner product of (6.1) with u. Since

$$\mathrm{Re}\,\langle Lu, u\rangle = D[w] + \mathrm{Re}\left\{-\frac{\nu}{\gamma^2}(\phi, w^1) + (\partial_{x_2} v_s^1 w^2, w^1)\right\}$$

and

$$\frac{1}{\gamma^2}\mathrm{Re}\,(\mathrm{div}\,(\phi\tilde{w}), \phi) = \frac{1}{2\gamma^2}(\mathrm{div}\,\tilde{w}, |\phi|^2),$$

we have

$$\begin{aligned}
\mathrm{Re}\,\lambda|||u|||_2^2 + D[w] &= \mathrm{Re}\,\langle F, u\rangle - \mathrm{Re}\left\{\tfrac{1}{2\gamma^2}(\mathrm{div}\,\tilde{w}, |\phi|^2) - \tfrac{\nu}{\gamma^2}(\phi, w^1) + (\partial_{x_2} v_s^1 w^2, w^1)\right\} \\
&\le |\langle F, u\rangle| + \tfrac{\nu}{\gamma^2}\|\phi\|_{H^{-1}}\|\nabla w\|_2 + \|\partial_{x_2} v_s^1\|_\infty \|w\|_2^2 \\
&\quad + \tfrac{1}{2\gamma^2}\|\mathrm{div}\,\tilde{w}\|_\infty \|\phi\|_2^2.
\end{aligned} \tag{6.7}$$

We next introduce a new inner product $((u_1, u_2))$ defined by

$$((u_1, u_2)) = \langle u_1, u_2\rangle - \delta\left[(w_1, \mathcal{B}\phi_2) + (\mathcal{B}\phi_1, w_2)\right]$$

for $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, 2)$. Here δ is a positive number to be determined later. Note that $((u,u))^{\frac{1}{2}}$ is equivalent to $|||u|||_2$ if $\delta \le \frac{1}{2C_B\gamma}$. We also write the density and velocity components of Lu as $(Lu)_d$ and $(Lu)_v$, respectively, i.e., $Lu = {}^\top((Lu)_d, (Lu)_v)$. Then, by Lemma 6.2,

$$\begin{aligned}
((Lu)_v, \mathcal{B}\phi) &= \nu(\nabla w, \nabla\mathcal{B}\phi) + \tilde{\nu}(\mathrm{div}\,w, \mathrm{div}\,\mathcal{B}\phi) - (\phi, \mathrm{div}\,\mathcal{B}\phi) \\
&\quad - \tfrac{\nu}{\gamma^2}(\phi\boldsymbol{e}_1, \mathcal{B}\phi) - (v_s^1 w, \partial_{x_1}\mathcal{B}\phi) + (\partial_{x_2} v_s^1 w^2 \boldsymbol{e}_1, \mathcal{B}\phi) \\
&= \nu(\nabla w, \nabla\mathcal{B}\phi) + \tilde{\nu}(\mathrm{div}\,w, \phi) - \|\phi\|_2^2 \\
&\quad - \tfrac{\nu}{\gamma^2}(\phi\boldsymbol{e}^1, \mathcal{B}\phi) - (v_s^1 w, \partial_{x_1}\mathcal{B}\phi) + (\partial_{x_2} v_s^1 w^2 \boldsymbol{e}_1, \mathcal{B}\phi).
\end{aligned}$$

Applying Lemma 6.2 again, we have

$$\begin{aligned}
-\mathrm{Re}\,((Lu)_v, \mathcal{B}\phi) &\ge \|\phi\|_2^2 - \nu C_B\|\nabla w\|_2\|\phi\|_2 - \tilde{\nu}\|\mathrm{div}\,w\|_2\|\phi\|_2 \\
&\quad - \tfrac{\nu}{\gamma^2}\|\phi\|_{H^{-1}}\|\nabla\mathcal{B}\phi\|_2 - \|v_s^1\|_{C^1}(\|w\|_2 + \|w\|_{H^{-1}})\|\nabla\mathcal{B}\phi\|_2 \\
&\ge \tfrac{3}{4}\|\phi\|_2^2 - C\Big\{\nu^2 C_B^2\|\nabla w\|_2^2 + \tilde{\nu}^2\|\mathrm{div}\,w\|_2^2 \\
&\qquad - \tfrac{\nu^2 C_B^2}{\gamma^4}\|\phi\|_{H^{-1}}^2 - C_B^2\|v_s^1\|_{C^1}^2\|\nabla w\|_2^2\Big\}.
\end{aligned} \tag{6.8}$$

Since $(Lu)_d = \mathrm{div}\,(\phi v_s + \gamma^2 w + \phi\tilde{w})$, we see from Lemma 6.2 that

$$\begin{aligned}
|(\mathcal{B}(Lu)_d, w)| &\le C_B\|\phi v_s + \gamma^2 w + \phi\tilde{w}\|_2\|w\|_2 \\
&\le C_B(\|v_s^1\|_\infty + \|\tilde{w}\|_\infty)\|\phi\|_2\|w\|_2 + C_B\gamma^2\|\nabla w\|_2^2 \\
&\le \tfrac{1}{4}\|\phi\|^2 + C\Big\{(C_B^2\|v_s^1\|_\infty^2 + C_B\gamma^2)\|\nabla w\|_2^2 + C_B\|\tilde{w}\|_{H^3}\|\phi\|_2\|w\|_2\Big\}.
\end{aligned} \tag{6.9}$$

Taking $\delta > 0$ such that $\delta \le \delta_1$ with $\delta_1 = \min\{\frac{1}{2C_B\gamma}, \frac{1}{16CC_B^2\nu}, \frac{\nu}{16CC_B^2\|v_s\|_{C^1}^2}, \frac{\nu}{16CC_B\gamma^2}, \frac{1}{2C\tilde{\nu}}\}$, we deduce from (6.7)–(6.9) that

$$\begin{aligned}
&\mathrm{Re}\,\lambda((u,u)) + \frac{1}{2}D[w] + \frac{\delta}{2}\|\phi\|_2^2 \\
&\le C\Big\{|\langle F, u\rangle| + \delta\left(|(f, \mathcal{B}\phi)| + |(\mathcal{B}f^0, w)|\right) \\
&\qquad + \tfrac{\nu}{\gamma^4}\|\phi\|_{H^{-1}}^2 + \|\partial_{x_2} v_s^1\|_\infty \|w\|_2^2 + \tfrac{\|\tilde{w}\|_{H^3}}{\gamma}\|\phi\|_2 |||u|||_2\Big\}.
\end{aligned} \tag{6.10}$$

By using the Poincaré inequalities, (6.4) follows from (6.10). As for (6.5), we have

$$\begin{aligned}
&|\langle F, u\rangle| + \delta\left(|(f, \mathcal{B}\phi)| + |(\mathcal{B}f^0, w)|\right) \\
&\le \frac{1}{\gamma^2}\|f^0\|_2\|\phi\|_2 + \|f\|_{H^{-1}}\|\nabla w\|_2 + \delta\left\{\|f\|_{H^{-1}}\|\nabla \mathcal{B}\phi\|_2 + \|\mathcal{B}f^0\|_2\|w\|_2\right\} \\
&\le \frac{\delta}{4}\|\phi\|_2^2 + \frac{\nu}{16}\|\nabla w\|_2^2 + C\left\{\left(\frac{1}{\delta\gamma^2} + \frac{\nu}{\gamma^4}\right)\|f^0\|_2^2 + \frac{1}{\nu}\|f\|_{H^{-1}}^2\right\}, \\
&\frac{\|\tilde{w}\|_{H^3}}{\gamma}\|\phi\|_2 |||u|||_2 \le \frac{16C\|\tilde{w}\|_{H^3}}{\gamma^2}\left(1 + \frac{\|\tilde{w}\|_{H^3}}{\nu}\right)\|\phi\|_2^2 + \frac{\nu}{32C}\|\nabla w\|_2^2
\end{aligned}$$

and

$$\|\dot{\phi}_\lambda\|_2^2 = \| - \gamma^2 \mathrm{div}\, w + f^0\|_2^2 \le 2\{\gamma^4\|\mathrm{div}\, w\|_2^2 + \|f^0\|_2^2\}.$$

Combining these inequalities with (6.10), we obtain (6.5). As for (6.6), we observe that

$$\partial_{x_1}^j (Lu)_d = \mathrm{div}\,(\partial_{x_1}^j \phi v_s + \gamma^2 \partial_{x_1}^j w + \partial_{x_1}^j \phi \tilde{w} + [\partial_{x_1}^j, \tilde{w}]\phi)$$

and

$$\|\mathrm{div}\,([\partial_{x_1}^j, \tilde{w}]\phi)\|_2 + \|[\partial_{x_1}^j, \tilde{w}]\phi\|_2 \le C\|\tilde{w}\|_{H^3}\|\phi\|_{H^j} \quad (j = 1, 2).$$

Therefore, as in the case of (6.4) and (6.5), we can obtain (6.6). This completes the proof. $\square$

Proposition 6.4. *There holds the inequality*

$$\begin{aligned}
&\mathrm{Re}\,\lambda D[w] + \frac{1}{2}|\lambda|^2 |||u|||_2^2 \\
&\le C\Big\{|||f|||_2^2 + \frac{\|v_s^1\|_\infty^2}{\gamma^2}\|\partial_{x_1}\phi\|_2^2 + \frac{\|\tilde{w}\|_{H^3}^2}{\gamma^2}\|\nabla\phi\|_2^2 + \frac{\nu^2}{\gamma^4}\|\phi\|_2^2 \\
&\qquad + (\|v_s^1\|_{C^1}^2 + \gamma^2)\|\nabla w\|_2^2\Big\}.
\end{aligned} \tag{6.11}$$

Proof. We take the inner product of (6.1) with λu. Then the real part of the resulting equation yields

$$\begin{aligned}
|\lambda|^2 |||u|||_2^2 + \mathrm{Re}\,\lambda D[w] = \mathrm{Re}\Big\{&\tfrac{\bar{\lambda}}{\gamma^2}(f^0, \phi) + \bar{\lambda}(f, w) - \tfrac{\bar{\lambda}}{\gamma^2}(v_s^1 \partial_{x_1}\phi, \phi) \\
&- \tfrac{\bar{\lambda}}{\gamma^2}(\mathrm{div}\,(\phi\tilde{w}), \phi) - \bar{\lambda}(\mathrm{div}\, w, \phi) + \bar{\lambda}(\phi, \mathrm{div}\, w) \\
&+ \tfrac{\nu}{\gamma^2}\bar{\lambda}(\phi, w^1) - \bar{\lambda}(v_s^1 \partial_{x_1} w, w) - \bar{\lambda}(\partial_{x_2} v_s^1 w^3, w^1)\Big\}.
\end{aligned}$$

By a direct computation, the right-hand side is bounded by

$$\frac{|\lambda|^2}{2}|||u|||_2^2 + C\Big\{|||f|||_2^2 + \frac{\|v_s^1\|_\infty^2}{\gamma^2}\|\partial_{x_1}\phi\|_2^2 + \frac{\|\tilde{w}\|_{H^3}^2}{\gamma^2}\|\nabla\phi\|_2^2 + \frac{\nu^2}{\gamma^4}\|\phi\|_2^2$$
$$+(\|v_s^1\|_{C^1}^2 + \gamma^2)\|\nabla w\|_2^2\Big\}.$$

We thus obtain the desired estimate. This completes the proof. $\square$

Proposition 6.5. *Let j and k be integers satisfying $0 \le j+k \le 1$. Then there holds the inequality*

$$\begin{aligned}
&\Big|\mathrm{Re}\,\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\Big|\|\partial_{x_1}^j\partial_{x_2}^{k+1}\phi\|_2 \\
&\le C\Big\{\|\partial_{x_1}^j\partial_{x_2}^{k+1}f^0\|_2 + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^k f^2\|_2 \\
&\quad +\|\partial_{x_2}v_s^1\|_{C^k}\|\partial_{x_1}^{j+1}\phi\|_{H^k} + \|\tilde{w}\|_{H^3}\|\phi\|_{H^{j+k+1}} \\
&\quad +\frac{\gamma^2}{\nu+\tilde{\nu}}\Big(|\lambda|\|\partial_{x_1}^j\partial_{x_2}^k w\|_2 + \nu\|\partial_{x_1}^{j+1}\partial_{x_2}^k\nabla w\|_2 + \|v_s^1\|_{C^k}\|\partial_{x_1}^{j+1}w\|_{H^k}\Big)\Big\}.
\end{aligned} \tag{6.12}$$

Furthermore, if $\mathrm{Re}\,\lambda \ge -\frac{1}{2}\frac{\gamma^2}{\nu+\tilde{\nu}}$, *then*

$$\begin{aligned}
&\|\partial_{x_1}^j\partial_{x_2}^{k+1}\dot{\phi}_\lambda\|_2 \\
&\le C\Big\{\|\partial_{x_1}^j\partial_{x_2}^{k+1}f^0\|_2 + \frac{\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_1}^j\partial_{x_2}^k f^2\|_2 \\
&\quad +\|\partial_{x_2}v_s^1\|_{C^k}\|\partial_{x_1}^{j+1}\phi\|_{H^k} + \|\tilde{w}\|_{H^3}\|\phi\|_{H^{j+k+1}} \\
&\quad +\frac{\gamma^2}{\nu+\tilde{\nu}}\Big(|\lambda|\|\partial_{x_1}^j\partial_{x_2}^k w\|_2 + \nu\|\partial_{x_1}^{j+1}\partial_{x_2}^k\nabla w\|_2 + \|v_s^1\|_{C^k}\|\partial_{x_1}^{j+1}w\|_{H^k}\Big)\Big\}.
\end{aligned} \tag{6.13}$$

Proof. Applying $\partial_{x_1}^j\partial_{x_2}^{k+1}$ to the first component of equation (6.1), we have

$$\begin{aligned}
&\lambda\partial_{x_1}^j\partial_{x_2}^{k+1}\phi + v_s^1\partial_{x_1}\partial_{x_1}^j\partial_{x_2}^{k+1}\phi + \mathrm{div}\,((\partial_{x_1}^j\partial_{x_2}^{k+1}\phi)\tilde{w}) + \gamma^2\partial_{x_1}^j\partial_{x_2}^{k+2}w^2 \\
&= \partial_{x_1}^j\partial_{x_2}^{k+1}f^0 - \Big\{[\partial_{x_2}^{k+1}, v_s^1]\partial_{x_1}^{j+1}\phi + \mathrm{div}\,([\partial_{x_1}^j\partial_{x_2}^{k+1},\tilde{w}]\phi) + \gamma^2\partial_{x_1}^{j+1}\partial_{x_2}^{k+1}w^1\Big\}.
\end{aligned} \tag{6.14}$$

We also apply $\partial_{x_1}^j\partial_{x_2}^k$ to the third component of equation (6.1) to obtain

$$\begin{aligned}
&-(\nu+\tilde{\nu})\partial_{x_1}^j\partial_{x_2}^{k+2}w^2 + \partial_{x_1}^j\partial_{x_2}^{k+1}\phi \\
&= \partial_{x_1}^j\partial_{x_2}^k f^2 - \Big\{\lambda\partial_{x_1}^j\partial_{x_2}^k w^2 - \nu\partial_{x_1}^{j+2}\partial_{x_2}^k w^2 - \tilde{\nu}\partial_{x_1}^{j+1}\partial_{x_2}^{k+1}w^1 + \partial_{x_2}^k(v_s^1\partial_{x_1}^{j+1}w^2)\Big\}.
\end{aligned} \tag{6.15}$$

By adding (6.14) and $\frac{\gamma^2}{\nu+\tilde{\nu}} \times$ (6.15) we obtain

$$\lambda\partial_{x_1}^j\partial_{x_2}^{k+1}\phi + \frac{\gamma^2}{\nu+\tilde{\nu}}\partial_{x_1}^j\partial_{x_2}^{k+1}\phi + v_s^1\partial_{x_1}\partial_{x_1}^j\partial_{x_2}^{k+1}\phi + \mathrm{div}\,((\partial_{x_1}^j\partial_{x_2}^{k+1}\phi)\tilde{w}) = H, \tag{6.16}$$

where

$$H = \partial_{x_1}^j \partial_{x_2}^{k+1} f^0 + \tfrac{\gamma^2}{\nu+\tilde{\nu}} \partial_{x_1}^j \partial_{x_2}^k f^2 - \Big\{ [\partial_{x_2}^{k+1}, v_s^1] \partial_{x_1}^{j+1} \phi + \mathrm{div}\,([\partial_{x_1}^j \partial_{x_2}^{k+1}, \tilde{w}]\phi) \Big\}$$
$$- \tfrac{\gamma^2}{\nu+\tilde{\nu}} \Big\{ \lambda \partial_{x_1}^j \partial_{x_2}^k w^2 - \nu \partial_{x_1}^{j+2} \partial_{x_2}^k w^2 + \nu \partial_{x_1}^{j+1} \partial_{x_2}^{k+1} w^1 + \partial_{x_2}^k (v_s^1 \partial_{x_1}^{j+1} w^2) \Big\}.$$

Taking the inner product of (6.16) with $\partial_{x_1}^j \partial_{x_2}^{k+1} \phi$, we have

$$\mathrm{Re}\,\lambda \|\partial_{x_1}^j \partial_{x_2}^{k+1} \phi\|_2^2 + \frac{\gamma^2}{\nu+\tilde{\nu}} \|\partial_{x_1}^j \partial_{x_2}^{k+1} \phi\|_2^2 = -\frac{1}{2}(\mathrm{div}\,\tilde{w}, |\partial_{x_1}^j \partial_{x_2}^{k+1} \phi|^2) + \mathrm{Re}\,(H, \partial_{x_1}^j \partial_{x_2}^{k+1} \phi),$$

from which estimate (6.12) is obtained. As for (6.13), we rewrite (6.16) as

$$\partial_{x_1}^j \partial_{x_2}^{k+1} \dot{\phi}_\lambda = -\frac{\gamma^2}{\nu+\tilde{\nu}} \partial_{x_1}^j \partial_{x_2}^{k+1} \phi + \tilde{H},$$

where

$$\tilde{H} = H + [\partial_{x_2}^{k+1}, v_s^1] \partial_{x_1}^{j+1} \phi + \mathrm{div}\,([\partial_{x_1}^j \partial_{x_2}^{k+1}, \tilde{w}]\phi).$$

This, together with (6.12), yields (6.13). This completes the proof. $\square$

We next prepare the following estimate for the Stokes system to estimate the higher order derivatives.

Lemma 6.6. *Let $(\phi, w) \in H^{k+1}_{per,*} \times [H^{k+2}_{per} \cap H^1_{per,0}]$ be a solution of*

$$\mathrm{div}\, w = h^0,$$

$$-\Delta w + \nabla \phi = h$$

for $(h^0, h) \in H^{k+1}_{per,*} \times H^k_{per}$. Then

$$\|\partial_x^{k+2} w\|_2 + \|\partial_x^{k+1} \phi\|_2 \le C\Big\{ \|h^0\|_{H^{k+1}} + \|h\|_{H^k} \Big\}.$$

See, e.g., [3, 13] for the proof. Applying Lemma 6.6 we have the following

Proposition 6.7. *Let j and k be integers satisfying $0 \le j+k \le 1$. Then*

$$\begin{aligned} &\|\partial_x^{k+2} \partial_{x_1}^j w\|_2 + \frac{1}{\nu} \|\partial_x^{k+1} \partial_{x_1}^j \phi\|_2 \\ &\le C\Big\{ \frac{\nu+\tilde{\nu}}{\nu\gamma^2} \|\partial_{x_1}^j f^0\|_{H^{k+1}} + \frac{1}{\nu} \|\partial_{x_1}^j f\|_{H^k} + \frac{\nu+\tilde{\nu}}{\nu\gamma^2} \|\partial_{x_1}^j \dot{\phi}_\lambda\|_{H^{k+1}} \\ &\quad + \frac{|\lambda|}{\nu} \|\partial_{x_1}^j w\|_{H^k} + \frac{1}{\gamma^2} \|\partial_{x_1}^j \phi\|_{H^k} \\ &\quad + \frac{1}{\nu} \|v_s^1\|_{C^k} \|\nabla \partial_{x_1}^j w\|_{H^k} + \frac{1}{\nu} \|\partial_{x_2} v_s^1\|_{C^k} \|\partial_{x_1}^j w\|_{H^k} \Big\}. \end{aligned} \tag{6.17}$$

Proof. We apply $\partial_{x_1}^j$ to (6.1) and write the resulting equation as

$$\operatorname{div} \partial_{x_1}^j w = \frac{1}{\gamma^2}\partial_{x_1}^j h^0,$$

$$-\Delta \partial_{x_1}^j w + \nabla\left(\frac{1}{\nu}\partial_{x_1}^j \phi\right) = \frac{1}{\nu}\partial_{x_1}^j h,$$

where

$$h^0 = f^0 - \dot{\phi}_\lambda,$$

$$h = f - \left\{\lambda w - \frac{\tilde{\nu}}{\gamma^2}\nabla h^0 - \frac{\nu}{\gamma^2}\phi \boldsymbol{e}_1 + v_s^1 \partial_{x_1} w + \partial_{x_2} v_s^1 w^2 \boldsymbol{e}_1\right\}.$$

Applying Lemma 6.6 we have the desired estimate. This completes the proof. $\square$

The following proposition follows from the first equation of (6.1).

Proposition 6.8. *There holds the inequality*

$$|\lambda|\|\partial_x^k \phi\|_2 \le C\left\{\|\partial_x^k f^0\|_2 + \|v_s^1\|_{C^k}\|\partial_{x_1}\phi\|_{H^k} + \|\tilde{w}\|_{H^3}\|\partial_x \phi\|_{H^k} + \gamma^2\|\partial_x^k \operatorname{div} w\|_2\right\} \quad (6.18)$$

for $k = 0, 1$.

We are now in a position to prove Proposition 6.1.

Proof of Proposition 6.1. Observe first that $\|\partial_{x_1} g\|_{H^{-1}} \le \|g\|_2$. We see from (6.4) with $k = 0$ that

$$\begin{aligned}
&(\operatorname{Re}\lambda + \Lambda_0)^2 |||u|||_2^2 \\
&\le \ C\left\{|||F|||_2^2 + \|\partial_{x_2} v_s^1\|_\infty^2 \|w\|_2^2 + \frac{\nu^2}{\gamma^6}\|\phi\|_{H^{-1}}^2 + \frac{\|\tilde{w}\|_{H^3}^2}{\gamma^2}\|\phi\|_2^2\right\}.
\end{aligned} \quad (6.19)$$

We compute (6.19) + (6.5) + $b_1 \times$ (6.6)$|_{j=1}$. Taking b_1 suitably small, we see that if $\operatorname{Re}\lambda > -\Lambda_0/2$, then

$$\begin{aligned}
&\left(\operatorname{Re}\lambda + \frac{1}{2}\Lambda_0\right)^2 \|u\|_2^2 + \left(\operatorname{Re}\lambda + \frac{1}{2}\Lambda_0\right)\|\partial_{x_1} u\|_2^2 + \sum_{j=0}^{1}(D[\partial_{x_1}^j w] + \|\partial_{x_1}^j \dot{\phi}_\lambda\|_2^2) \\
&\le C\left\{\|F\|_{H^1\times L^2}^2 + \|\tilde{w}\|_{H^3}(1 + \|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2 + \|w\|_2^2 + \|\phi\|_{H^{-1}}^2\right\}.
\end{aligned} \quad (6.20)$$

We next consider (6.20) + $b_2 \times$ (6.11). Taking b_2 suitably small, we see that there exists a positive number Λ_1 such that if $\operatorname{Re}\lambda > -\Lambda_1$, then

$$\begin{aligned}
&\left(\operatorname{Re}\lambda + \Lambda_1\right)^2 \|u\|_2^2 + \left(\operatorname{Re}\lambda + \Lambda_1\right)(\|\partial_{x_1}\phi\|_2^2 + \|\partial_x w\|_2^2) \\
&+ D[\partial_{x_1} w] + |\lambda|^2 \|u\|_2^2 + \sum_{j=0}^{1}\|\partial_{x_1}^j \dot{\phi}_\lambda\|_2^2 \\
&\le C\left\{\|F\|_{H^1\times L^2}^2 + \|\tilde{w}\|_{H^3}(1 + \|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2 + \|w\|_2^2 + \|\phi\|_{H^{-1}}^2\right\}.
\end{aligned} \quad (6.21)$$

We then compute (6.21) + $b_3 \times \{(6.12)|_{j=k=0} + (6.13)|_{j=k=0}\}^2$. Taking b_3 suitably small, we see that there exists a positive number Λ_2 such that if $\mathrm{Re}\,\lambda > -\Lambda_2$, then

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \Lambda_2\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_2\Big)\|\partial_x u\|_2^2 + D[\partial_{x_1} w] + |\lambda|^2\|u\|_2^2 + \|\dot{\phi}_\lambda\|_{H^1}^2 \\ &\leq \ C\Big\{\|F\|_{H^1\times L^2}^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2 + \|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}. \end{aligned} \tag{6.22}$$

We next compute $(6.22) + b_4 \times \{(6.17)|_{j=k=0}\}^2$. We take b_4 suitably small to see that there exists a positive number Λ_3 such that if $\mathrm{Re}\,\lambda > -\Lambda_3$, then

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \Lambda_3\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_3\Big)\|\partial_x u\|_2^2 + \|\partial_x^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 + \|\dot{\phi}_\lambda\|_{H^1}^2 \\ &\leq \ C\Big\{\|F\|_{H^1\times L^2}^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2 + \|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}. \end{aligned} \tag{6.23}$$

This shows (6.2).

Let us prove (6.3). We compute $(6.23) + b_5 \times (6.6)|_{j=2}$. Taking b_5 suitably small, we see that there exists a positive number Λ_4 such that if $\mathrm{Re}\,\lambda > -\Lambda_4$, then

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \Lambda_4\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_4\Big)(\|\partial_x u\|_2^2 + \|\partial_{x_1}^2 u\|_2^2) \\ &+\|\partial_x^2 w\|_2^2 + \|\nabla\partial_{x_1}^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 + \|\dot{\phi}_\lambda\|_{H^1}^2 + \|\partial_{x_1}^2\dot{\phi}_\lambda\|_2^2 \\ &\leq \ C\Big\{\|F\|_{H^2\times H^1}^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})\|\phi\|_{H^2}^2 + \|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}. \end{aligned} \tag{6.24}$$

We next compute $(6.24) + b_6 \times (6.4)|_{k=1}$. Taking b_6 suitably small, we see that there exists a positive number Λ_5 such that if $\mathrm{Re}\,\lambda > -\Lambda_5$, then

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \Lambda_5\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_5\Big)(\|\partial_x u\|_2^2 + \|\partial_{x_1}^2 u\|_2^2 + |\lambda|^2\|u\|_2^2) \\ &+\|\partial_x^2 w\|_2^2 + \|\nabla\partial_{x_1}^2 w\|_2^2 + |\lambda|^2\|\nabla w\|_2^2 + \|\dot{\phi}_\lambda\|_{H^1}^2 + \|\partial_{x_1}^2\dot{\phi}_\lambda\|_2^2 \\ &\leq \ C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})(\|\phi\|_{H^2}^2 + |\lambda|^2\|\phi\|_2^2) \\ &\qquad +\|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}. \end{aligned} \tag{6.25}$$

We next consider (6.25) + $b_7 \times \{(6.12)|_{j=1,k=0} + (6.13)|_{j=1,k=0}\}^2$. Taking b_7 suitably small, we see that there exists a positive number Λ_6 such that if $\mathrm{Re}\,\lambda > -\Lambda_6$, then

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \Lambda_6\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_6\Big)(\|\partial_x u\|_2^2 + \|\partial_{x_1}^2 u\|_2^2 + \|\partial_{x_1}\partial_{x_2}\phi\|_2^2 + |\lambda|^2\|u\|_2^2) \\ &+\|\partial_x^2 w\|_2^2 + \|\nabla\partial_{x_1}^2 w\|_2^2 + |\lambda|^2\|\nabla w\|_2^2 + \|\dot{\phi}_\lambda\|_{H^1}^2 + \|\nabla\partial_{x_1}\dot{\phi}_\lambda\|_2^2 \\ &\leq \ C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})(\|\phi\|_{H^2}^2 + |\lambda|^2\|\phi\|_2^2) \\ &\qquad +\|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}. \end{aligned} \tag{6.26}$$

It then follows from (6.26) $+ b_8 \times \{(6.17)|_{j=1,k=0}\}^2$ with suitably small b_8 that there exists a positive number Λ_7 such that if $\mathrm{Re}\,\lambda > -\Lambda_7$, then

$$
\begin{aligned}
&\Big(\mathrm{Re}\,\lambda + \Lambda_7\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_7\Big)(\|\partial_x u\|_2^2 + \|\partial_{x_1}^2 u\|_2^2 + \|\partial_{x_1}\partial_{x_2}\phi\|_2^2 + |\lambda|^2\|u\|_2^2) \\
&\quad +\|\partial_x^2 w\|_2^2 + \|\partial_x^2\partial_{x_1} w\|_2^2 + |\lambda|^2\|\nabla w\|_2^2 + \|\dot{\phi}_\lambda\|_{H^1}^2 + \|\nabla\partial_{x_1}\dot{\phi}_\lambda\|_2^2 \\
&\quad \leq \; C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})(\|\phi\|_{H^2}^2 + |\lambda|^2\|\phi\|_2^2) \\
&\qquad\qquad +\|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}.
\end{aligned}
\tag{6.27}
$$

We then compute (6.27) $+ b_9 \times \{(6.12)|_{j=0,k=1} + (6.13)|_{j=0,k=1}\}^2$ and take b_9 suitably small so that there exists a positive number Λ_8 such that if $\mathrm{Re}\,\lambda > -\Lambda_8$, then

$$
\begin{aligned}
&\Big(\mathrm{Re}\,\lambda + \Lambda_8\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_8\Big)(\|\partial_x u\|_2^2 + \|\partial_x^2 \phi\|_2^2 + \|\partial_{x_1}^2 w\|_2^2 + |\lambda|^2\|u\|_2^2) \\
&\quad +\|\partial_x^2 w\|_2^2 + \|\partial_x^2\partial_{x_1} w\|_2^2 + |\lambda|^2\|\nabla w\|_2^2 + \|\dot{\phi}_\lambda\|_{H^2}^2 \\
&\quad \leq \; C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})(\|\phi\|_{H^2}^2 + |\lambda|^2\|\phi\|_2^2) \\
&\qquad\qquad +\|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}.
\end{aligned}
\tag{6.28}
$$

Finally, consider (6.28) $+ b_{10} \times \{(6.17)|_{j=0,k=1} + (6.18)|_{k=1}\}^2$. Taking b_{10} suitably small, we deduce that there exists a positive number Λ_9 such that if $\mathrm{Re}\,\lambda > -\Lambda_9$, then

$$
\begin{aligned}
&\Big(\mathrm{Re}\,\lambda + \Lambda_9\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_9\Big)(\|\partial_x u\|_2^2 + \|\partial_x^2 \phi\|_2^2 + \|\partial_{x_1}^2 w\|_2^2 + |\lambda|^2\|u\|_2^2) \\
&\quad +\|\partial_x^2 w\|_2^2 + \|\partial_x^3 w\|_2^2 + |\lambda|^2\|\nabla u\|_2^2 + \|\dot{\phi}_\lambda\|_{H^2}^2 \\
&\quad \leq \; C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2 + \|\tilde{w}\|_{H^3}(1+\|\tilde{w}\|_{H^3})(\|\phi\|_{H^2}^2 + |\lambda|^2\|\phi\|_2^2) \\
&\qquad\qquad +\|w\|_2^2 + \|\phi\|_{H^{-1}}^2\Big\}.
\end{aligned}
\tag{6.29}
$$

We thus obtain (6.3). This completes the proof. □

6.2 A priori estimates

We consider

$$
\lambda u + \mathscr{L}(\varepsilon, \tilde{\sigma}, \tilde{V})u = F, \quad u \in QX^\ell \;\; (\ell = 1, 2), \tag{6.30}
$$

where $F \in Q(H_{per}^\ell \times H_{per}^{\ell-1})$ and

$$
\mathscr{L}(\varepsilon, \tilde{\sigma}, \tilde{V}) = \mathcal{L}_{c_0,0} - \varepsilon\tilde{\sigma}Q\partial_{x_1} + \varepsilon QN[V_1 + \varepsilon\tilde{V}]
$$

with $\tilde{\sigma} \in \mathbb{R}$ and $\tilde{V} \in X^2$ satisfying $|\tilde{\sigma}| + \|\tilde{V}\|_{X^2} \leq M$. In this subsection we show the a priori estimates for solution u of (6.30).

We show the following a priori estimates.

Proposition 6.9. *Let $M > 0$ and assume that $|\tilde{\sigma}| + \|\tilde{V}\|_{X^2} \leq M$. Then there exist $\varepsilon_3 > 0$, $r_0 > 0$, $\Lambda > 0$ and $\{\lambda_j\}_{j=1}^{K} \subset \mathbb{C}$ with $|\lambda_j| \geq 2r_0$ such that if $|\varepsilon| \leq \varepsilon_3$ and*

$$\lambda \in \Sigma_0 = \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -\Lambda,\ |\lambda - \lambda_j| \geq r_0,\ j = 1, \cdots, K\},$$

the solution $u \in QX^1$ of (6.30) satisfies the estimate

$$\Big(\mathrm{Re}\,\lambda + \Lambda\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda\Big)\|\partial_x u\|_2^2 + \|\partial_x^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 \leq C\|F\|_{H^1\times L^2}^2 \quad (6.31)$$

uniformly for $\lambda \in \Sigma_0$. In addition, if $u \in QX^2$, then

$$\begin{aligned}&\Big(\mathrm{Re}\,\lambda + \Lambda\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda\Big)(\|\partial_x u\|_2^2 + \|\partial_x^2 \phi\|_2^2 + |\lambda|^2\|u\|_2^2)\\ &+\|\partial_x^2 w\|_{H^1}^2 + |\lambda|^2\|\nabla u\|_2^2 \leq C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2\Big\}\end{aligned} \quad (6.32)$$

uniformly for $\lambda \in \Sigma_0$.

We note that $0 \in \Sigma_0$.

Proof. We first introduce frequency cut off operators. We expand $f \in L^2_{per}$ into the Fourier series $f = \sqrt{\frac{\alpha}{2\pi}}\sum_{k\in\mathbb{Z}} f_k(x_2)e^{i\alpha k x_1}$. We define $\Pi_{\leq N}$ and $\Pi_{\geq N}$ by

$$\Pi_{\leq N} f = \sqrt{\frac{\alpha}{2\pi}}\sum_{|k|\leq N} f_k(x_2)e^{i\alpha k x_1}$$

and

$$\Pi_{\geq N} f = \sqrt{\frac{\alpha}{2\pi}}\sum_{|k|\geq N} f_k(x_2)e^{i\alpha k x_1}$$

respectively. $\Pi_{<N}$ and $\Pi_{>N}$ are defined similarly. Observe that they are orthogonal projections on L^2_{per} and

$$\|w\|_2 \leq \frac{1}{\alpha N}\|\nabla w\|_2, \quad \|\phi\|_{H^{-1}} \leq \frac{1}{\alpha N}\|\phi\|_2 \quad (6.33)$$

for $w \in \Pi_{\geq N} H^1_{per}$ and $\phi \in \Pi_{\geq N} L^2_{per}$ with $N \geq 1$.

We first prove (6.31). We write (6.30) as

$$\lambda u + \mathcal{L}_{c_0,0} u - \varepsilon\tilde{\sigma} QJu + \varepsilon QN[\tilde{w}]u = F. \quad (6.34)$$

Here $\tilde{w}$ is the function defined by

$$\tilde{w} = -\tilde{\sigma}\boldsymbol{e}_1 + W_1 + \varepsilon\tilde{W}$$

with W_1 and $\tilde{W}$ being the velocity components of V_1 and $\tilde{V}$ respectively; and Ju and $N[\tilde{w}]u$ are defined by $Ju = {}^\top(0, \partial_{x_1} w)$ and $N[\tilde{w}]u = {}^\top(\mathrm{div}\,(\phi\tilde{w}), 0)$ for $u = {}^\top(\phi, w)$, respectively. Since $Q = I - P$, (6.34) is rewritten as

$$\lambda u + \mathcal{L}_{c_0,0} u + \varepsilon N[\tilde{w}]u = F + \varepsilon\tilde{\sigma} QJu + \varepsilon PN[\tilde{w}]u. \quad (6.35)$$

Note that

$$\|QJu\|_{H^{\ell+1}\times H^{\ell}} \le C\|\partial_{x_1} w\|_{H^{\ell}} \quad (\ell = 0, 1)$$

and

$$\|PN[\tilde{w}]u\|_{H^2\times H^1} \le C\|N[\tilde{w}]u\|_2 \le C\|\tilde{w}\|_{H^3}\|\phi\|_{H^1}.$$

Applying (6.2) with v_s, $\tilde{w}$ and F replaced by $v_s - c_0\boldsymbol{e}_1$, $\varepsilon\tilde{w}$ and $F + \varepsilon\tilde{\sigma}QJu + \varepsilon PN[\tilde{w}]u$, respectively, we see that

$$\begin{aligned}
&\Big(\mathrm{Re}\,\lambda + \Lambda\Big)^2\|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda\Big)\|\partial_x u\|_2^2 + \|\partial_x^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 \\
&\le \ C\Big\{\|F\|_{H^1\times L^2}^2 + |\varepsilon|^2|\tilde{\sigma}|^2\|QJu\|_{H^1\times L^2}^2 + |\varepsilon|^2\|PN[\tilde{w}]u\|_{H^1\times L^2}^2 + \|w\|_2^2 \\
&\quad + \|\phi\|_{H^{-1}}^2 + |\varepsilon|\|w\|_{H^3}(1 + |\varepsilon|\|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2\Big\} \\
&\le \ C\Big\{\|F\|_{H^1\times L^2}^2 + \|u_{<N}\|_2^2 + \frac{1}{\alpha^2N^2}\Big(\|\nabla w_{\ge N}\|_2^2 + \|\phi_{\ge N}\|_2^2\Big) \\
&\quad + |\varepsilon|^2|\tilde{\sigma}|^2\|\partial_{x_1} w\|_2 + |\varepsilon|\|\tilde{w}\|_{H^3}(1 + |\varepsilon|\|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2\Big\}.
\end{aligned}$$

It then follows that there exists $N_0 \in \mathbb{N}$ such that the inequality

$$\begin{aligned}
&\Big(\mathrm{Re}\,\lambda + \Lambda_{10}\Big)^2\|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_{10}\Big)\|\partial_x u\|_2^2 + \|\partial_x^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 \\
&\quad \le C\Big\{\|F\|_{H^1\times L^2}^2 + \|u_{<N}\|_2^2 + |\varepsilon|^2|\tilde{\sigma}|^2\|\partial_{x_1} w\|_2^2 \qquad (6.36) \\
&\quad + |\varepsilon|\|\tilde{w}\|_{H^3}(1 + |\varepsilon|\|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2\Big\}
\end{aligned}$$

holds with $\Lambda_{10} = \frac{1}{2}\Lambda$ uniformly for $N \ge N_0$.

To proceed further, we apply the following result on the spectral distribution proved by Iooss and Padula [6].

Lemma 6.10. ([6]) *There exists a constant $\tilde{\Lambda} > 0$ with $\tilde{\Lambda} \le \Lambda_{10}$ such that*

$$\sigma(-\mathcal{L}_{c_0,0}) \cap \{\lambda; \mathrm{Re}\,\lambda \ge -\tilde{\Lambda}\} = \{\lambda_j\}_{j=0}^K,$$

where λ_j $(j = 0, 1, \cdots, K)$ are eigenvalues of $-\mathcal{L}_{c_0,0}$ with finite multiplicities.

We may assume $N_0 \ge 2$. Furthermore, by assumption (4.1), we may assume that $\lambda_0 = 0$ and $\lambda_j \ne 0$ for $j = 1, \cdots, K$. By Lemma 6.10, we see that there is a positive number r_0 such that

$$|\lambda_j - \lambda_k| \ge 4r_0,\ j \ne k,\ j, k = 0, 1, \cdots, K,$$

$$\rho(-\mathcal{L}_{c_0,0}|_{\Pi_{\le N_0}QX}) \supset \Sigma_0 \equiv \{\lambda; \mathrm{Re}\,\lambda \ge -\tilde{\Lambda},\ |\lambda - \lambda_j| \ge r_0,\ j = 1, \cdots, K\}$$

and

$$(|\lambda| + 1)\|(\lambda + \mathcal{L}_{c_0,0}|_{\Pi_{\le N_0}QX})^{-1}F\|_2 \le C\|F\|_2 \qquad (6.37)$$

uniformly for $\lambda \in \Sigma_0$. Note that $\Sigma_0 \ni 0$ since $\lambda_0 = 0$.

Let us estimate $\|u_{<N}\|_2$. Applying $\Pi_{<N_0}$ to (6.35), we have

$$\lambda u_{<N_0} + \mathcal{L}_{c_0,0} u_{<N_0} = F_{<N_0} - \varepsilon \Pi_{<N_0} N[\tilde{w}]u + \varepsilon\tilde{\sigma} QJu + \varepsilon PN[\tilde{w}]u.$$

Here we have used the fact $\Pi_{<N_0}P = P$. It then follows from (6.37) that

$$\begin{aligned} \|u_{<N_0}\|_2 &\leq C\Big\{\|F_{<N_0}\|_2 + |\varepsilon|\|\Pi_{<N_0}N[\tilde{w}]u\|_2 + |\varepsilon||\tilde{\sigma}|\|QJu\|_2 + |\varepsilon|\|PN[\tilde{w}]u\|_2\Big\} \\ &\leq C\Big\{\|F\|_2 + |\varepsilon||\tilde{\sigma}|\|\partial_{x_1} w\|_2 + |\varepsilon|\|\tilde{w}\|_\infty \|\nabla\phi\|_2\Big\} \\ &\leq C\Big\{\|F\|_2 + |\varepsilon||\tilde{\sigma}|\|\partial_{x_1} w\|_2 + |\varepsilon|\|\tilde{w}\|_{H^3} \|\nabla\phi\|_2\Big\}. \end{aligned} \tag{6.38}$$

We see from (6.36) and (6.38) that

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \tilde{\Lambda}\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \tilde{\Lambda}\Big)\|\partial_x u\|_2^2 + \|\partial_x^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 \\ &\leq C\Big\{\|F\|_{H^1\times L^2}^2 + |\varepsilon|^2|\tilde{\sigma}|^2\|\partial_{x_1} w\|_2^2 + |\varepsilon|\|\tilde{w}\|_{H^3}(1 + |\varepsilon|\|\tilde{w}\|_{H^3})\|\phi\|_{H^1}^2\Big\} \end{aligned} \tag{6.39}$$

uniformly for $\lambda \in \Sigma_0$. Since $|\tilde{\sigma}| \leq M$ and $\|\tilde{w}\|_{H^3} \leq C(\|V_1\|_{H^3} + M)$, we conclude that there exists $\varepsilon_3 > 0$ such that if $|\varepsilon| \leq \varepsilon_3$, then

$$\Big(\mathrm{Re}\,\lambda + \Lambda_{11}\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_{11}\Big)\|\partial_x u\|_2^2 + \|\partial_x^2 w\|_2^2 + |\lambda|^2\|u\|_2^2 \leq C\|F\|_{H^1\times L^2}^2 \tag{6.40}$$

uniformly for $\lambda \in \Sigma_0$ with $\Lambda_{11} = \frac{1}{2}\tilde{\Lambda}$. This shows (6.31).

As for (6.32), by (6.3) and (6.40), we have

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \Lambda_{11}\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_{11}\Big)(\|\partial_x u\|_2^2 + \|\partial_x^2 \phi\|_2^2 + |\lambda|^2\|u\|_2^2) \\ &\quad + \|\partial_x^2 w\|_{H^1}^2 + |\lambda|^2\|\nabla u\|_2^2 \\ &\leq C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2 + |\varepsilon|^2|\tilde{\sigma}|^2(\|\partial_{x_1} w\|_{H^1}^2 + |\lambda|^2\|\partial_{x_1} w\|_2^2) \\ &\qquad + |\varepsilon|\|\tilde{w}\|_{H^3}(1 + |\varepsilon|\|\tilde{w}\|_{H^3})(\|\nabla\phi\|_{H^1}^2 + |\lambda|^2\|\phi\|_{H^1}^2)\Big\} \end{aligned}$$

uniformly for $\lambda \in \Sigma_0$. Therefore, if $|\varepsilon| \leq \varepsilon_3$ (by taking ε_3 smaller if necessary), then

$$\begin{aligned} &\Big(\mathrm{Re}\,\lambda + \Lambda_{12}\Big)^2 \|u\|_2^2 + \Big(\mathrm{Re}\,\lambda + \Lambda_{12}\Big)(\|\partial_x u\|_2^2 + \|\partial_x^2 \phi\|_2^2 + |\lambda|^2\|u\|_2^2) \\ &+ \|\partial_x^2 w\|_{H^1}^2 + |\lambda|^2\|\nabla u\|_2^2 \\ &\leq C\Big\{\|F\|_{H^2\times H^1}^2 + |\lambda|^2\|F\|_2^2\Big\} \end{aligned} \tag{6.41}$$

uniformly for $\lambda \in \Sigma_0$ with $\Lambda_{12} = \frac{1}{2}\Lambda_{11}$. This completes the proof. □

6.3 Invertibility

We finally prove the invertibility of $\mathscr{L}(\varepsilon, \tilde{\sigma}, \tilde{V})$. We first show the existence of solution of (6.30) in QX^ℓ ($\ell = 1, 2$) for sufficiently large $\lambda > 0$.

Proposition 6.11. *Let $\ell = 1, 2$ and assume that $|\varepsilon| \le \varepsilon_1$. There exists $\mu_0 > 0$ such that if $\lambda \ge \mu_0$, then for any $F = {}^\top(f^0, f) \in Q(H^\ell_{per,*} \times H^{\ell-1}_{per})$, there exists a unique solution $u = {}^\top(\phi, w) \in QX^\ell$ of (6.30), and $u = {}^\top(\phi, w)$ satisfies*

$$\lambda \|\phi\|_{H^\ell} + \sum_{j=0}^{\ell+1} \lambda^{\frac{\ell+1-j}{2}} \|\partial_x^j w\|_2 \le C\|f^0\|_{H^\ell} + C\sum_{j=0}^{\ell-1} \lambda^{\frac{\ell-1-j}{2}} \|\partial_x^j f\|_2.$$

Proof. We consider (6.35) instead of (6.30). Suppose that $u \in X^\ell$ is a solution of (6.35). Then

$$\lambda u + \mathcal{L}_{c_0,0} u = \varepsilon\tilde{\sigma} QJu - \varepsilon QN[\tilde{w}]u + F.$$

Applying P to both sides, we see that $\lambda Pu = 0$. Since $\lambda > 0$, we have $Pu = 0$, and hence $u \in QX^\ell$. Therefore, it suffices to show the existence of solution of (6.35) in X^ℓ.

Hereafter in the proof, we simply denote the density and velocity components of $PJu = P\,{}^\top(0, \partial_{x_1} w)$ ($u = {}^\top(\phi, w)$) by $P_d(\partial_{x_1} w)$ and $P_v(\partial_{x_1} w)$ respectively, i.e.,

$$PJu = P\,{}^\top(0, \partial_{x_1} w) = {}^\top(P_d(\partial_{x_1} w), P_v(\partial_{x_1} w)),$$

and likewise, we denote the density and velocity components of $PN[\tilde{w}]u = P\,{}^\top(\mathrm{div}\,(\phi\tilde{w}), 0)$ with $u = {}^\top(\phi, w)$ by $P_d(\mathrm{div}\,(\phi\tilde{w}))$ and $P_v(\mathrm{div}\,(\phi\tilde{w}))$ respectively, i.e.,

$$PN[\tilde{w}]u = P\,{}^\top(\mathrm{div}\,(\phi\tilde{w}), 0) = {}^\top(P_d(\mathrm{div}\,(\phi\tilde{w})), P_v(\mathrm{div}\,(\phi\tilde{w}))).$$

We write (6.35) as

$$\lambda\phi + \mathrm{div}\,((\tilde{v}_s + \varepsilon\tilde{w})\phi) = -\varepsilon\tilde{\sigma} P_d(\partial_{x_1} w) + \varepsilon P_d(\mathrm{div}\,(\phi\tilde{w})) - \gamma^2 \mathrm{div}\, w + f^0, \tag{6.42}$$

$$\lambda w + Aw = B\phi - \varepsilon\tilde{\sigma} P_v(\partial_{x_1} w) + \varepsilon P_v(\mathrm{div}\,(\phi\tilde{w})) + f. \tag{6.43}$$

Here $\tilde{v}_s = v_s - (c_0 + \varepsilon\tilde{\sigma})\boldsymbol{e}_1$; A denotes the elliptic operator on L^2_{per} defined by

$$Aw = -\nu\Delta w - \tilde{\nu}\nabla \mathrm{div}\, w + \tilde{v}_s^1 \partial_{x_1} w + (\partial_{x_2} \tilde{v}_s^1)(w \cdot \boldsymbol{e}_2)\boldsymbol{e}_1$$

with domain $D(A) = H^2_{per} \cap H^1_{per,0}$; B is the operator on H^1_{per} defined by

$$B\phi = -\nabla\phi + \frac{\nu}{\gamma^2}\phi\boldsymbol{e}_1.$$

By [4], there exists $\mu_1 > 0$ such that if $\lambda \ge \mu_1$, then, for any $f^0 \in H^\ell_{per,*}$, there exists a unique solution $\Phi \in H^\ell_{per,*}$ of

$$\lambda\Phi + \mathrm{div}\,(\Phi(\tilde{v}_s + \varepsilon\tilde{w})) = f^0, \tag{6.44}$$

and Φ satisfies the estimate

$$\|\Phi\|_{H^\ell} \le \frac{C}{\lambda}\|f^0\|_{H^\ell}.$$

We denote by Φ_λ the solution map $f^0 \mapsto \Phi$ for (6.44). Then Φ_λ is a bounded linear operator on $H^\ell_{per,*}$ and

$$\|\Phi_\lambda f^0\|_{H^\ell} \le \frac{C}{\lambda}\|f^0\|_{H^\ell}. \tag{6.45}$$

It then follows that (6.42) is equivalent to

$$\Psi_\lambda \phi = \Phi_\lambda(-\varepsilon\tilde{\sigma}P_d(\partial_{x_1}w) - \gamma^2 \mathrm{div}\, w + f^0), \tag{6.46}$$

where Ψ_λ is the operator defined by

$$\Psi_\lambda \phi = \phi - \varepsilon\Phi_\lambda P_d(\mathrm{div}\,(\phi\tilde{w})).$$

To solve (6.46), we show that the map $\Psi_\lambda : H^\ell_{per,*} \to H^\ell_{per,*}$ has a bounded inverse. By (6.45), we have

$$\|\varepsilon\Phi_\lambda P_d(\mathrm{div}\,(\phi\tilde{w}))\|_{H^\ell} \le \frac{|\varepsilon|C}{\lambda}\|\mathrm{div}\,(\phi\tilde{w})\|_2 \le \frac{\varepsilon_3 C}{\lambda}(\|V_1\|_{C^1} + M)\|\phi\|_{H^1}.$$

This implies that if $\lambda \ge \mu_2 = \max\{\mu_1, 2C\varepsilon_3(\|V_1\|_{C^1}+M)\}$, then $\|\varepsilon\Phi_\lambda P_d(\mathrm{div}\,(\phi\tilde{w}))\|_{H^\ell} \le \frac{1}{2}\|\phi\|_{H^\ell}$ for $\ell = 1, 2$, and hence, $\Psi_\lambda : H^\ell_{per,*} \to H^\ell_{per,*}$ has a bounded inverse Ψ_λ^{-1}, and Ψ_λ^{-1} satisfies

$$\|\Psi_\lambda^{-1}\phi\|_{H^\ell} \le 2\|\phi\|_{H^\ell}. \tag{6.47}$$

In terms of Ψ_λ^{-1}, the solution ϕ of (6.46) is written as

$$\phi = \Psi_\lambda^{-1}\Phi_\lambda(-\varepsilon\tilde{\sigma}P_d(\partial_{x_1}w) - \gamma^2 \mathrm{div}\, w + f^0), \tag{6.48}$$

and, by (6.45) and (6.47), ϕ satisfies

$$\|\phi\|_{H^\ell} \le \frac{C}{\lambda}\Big\{\|w\|_{H^{\ell+1}} + \|f^0\|_{H^\ell}\Big\}. \tag{6.49}$$

From (6.43) and (6.48), we have

$$(\lambda + A)w = B_1[\tilde{w}]\Psi_\lambda^{-1}\Phi_\lambda(-\varepsilon\tilde{\sigma}P_d(\partial_{x_1}w) - \gamma^2 \mathrm{div}\, w + f^0) - \varepsilon\tilde{\sigma}P_v(\partial_{x_1}w) + f$$

with

$$B_1[\tilde{w}]\phi = B\phi + \varepsilon P_v(\mathrm{div}\,(\phi\tilde{w})).$$

This is equivalent to

$$(I - \Gamma_\lambda)w = (\lambda + A)^{-1}(B_1[\tilde{w}]\Psi_\lambda^{-1}\Phi_\lambda f^0 + f), \tag{6.50}$$

where Γ_λ is the operator defined by

$$\Gamma_\lambda w = (\lambda + A)^{-1}\left(B_1[\tilde{w}]\Psi_\lambda^{-1}\Phi_\lambda(-\varepsilon\tilde{\sigma}P_d(\partial_{x_1}w) - \gamma^2 \mathrm{div}\, w) - \varepsilon\tilde{\sigma}P_v(\partial_{x_1}w)\right).$$

Since A is strongly elliptic, there exists $\mu_3 > 0$ such that if $\lambda \geq \mu_3$, then $(\lambda + A)^{-1} f \in H^{\ell+1}_{per} \cap H^1_{per,0}$ for $f \in H^{\ell-1}$ and it holds that

$$\sum_{j=0}^{\ell+1} \lambda^{\frac{\ell+1-j}{2}} \|\partial_x^j (\lambda + A)^{-1} f\|_2 \leq C \sum_{j=0}^{\ell-1} \lambda^{\frac{\ell-1-j}{2}} \|\partial_x^j f\|_2. \tag{6.51}$$

Furthermore, for $j = 1, 2$, we have

$$\|B_1[\tilde{w}]\phi\|_{H^{j-1}} \leq C\Big\{\|\phi\|_{H^j} + |\varepsilon| \|\mathrm{div}\,(\phi\tilde{w})\|_2\Big\} \leq C\|\phi\|_{H^j}. \tag{6.52}$$

We now introduce the norm $|||w|||_{(\lambda)} = \sum_{j=0}^{\ell+1} \lambda^{\frac{\ell+1-j}{2}} \|\partial_x^j w\|_2$ of $H^{\ell+1}_{per}$ and show that the map $\Gamma_\lambda : H^{\ell+1}_{per} \cap H^1_{per,0} \to H^{\ell+1}_{per} \cap H^1_{per,0}$ has a bounded inverse Γ_λ^{-1}. By (6.49) with $f^0 = 0$, (6.51) and (6.52), we see that if $\lambda \geq \max\{\mu_3, 1\}$, then

$$|||\Gamma_\lambda w|||_{(\lambda)} \leq C \sum_{j=0}^{\ell-1} \lambda^{\frac{\ell-1-j}{2}} \frac{1}{\lambda} \|w\|_{H^{j+2}} \leq \frac{C}{\lambda} \sum_{j=0}^{\ell+1} \lambda^{\frac{\ell+1-j}{2}} \|\partial_x^j w\|_2.$$

Therefore, there exists $\mu_4 > 0$ such that if $\lambda \geq \mu_4$, then

$$|||\Gamma_\lambda w|||_{(\lambda)} \leq \frac{1}{2} |||w|||_{(\lambda)},$$

and hence, $I - \Gamma_\lambda$ has a bounded inverse $(I - \Gamma_\lambda)^{-1}$, and $(I - \Gamma_\lambda)^{-1}$ satisfies the estimate

$$|||(I - \Gamma_\lambda)^{-1} f|||_{(\lambda)} \leq 2|||f|||_{(\lambda)}.$$

In terms of $(I - \Gamma_\lambda)^{-1}$, the solution $w \in H^{\ell+1} \cap H^1_{per,0}$ of (6.50) is given by

$$w = (I - \Gamma_\lambda)^{-1} (\lambda + A)^{-1} (B_1[\tilde{w}] \Psi_\lambda^{-1} \Phi_\lambda f^0 + f)$$

and w satisfies the estimate

$$\sum_{j=0}^{\ell+1} \lambda^{\frac{\ell+1-j}{2}} \|\partial_x^j w\|_2 \leq C\|f^0\|_{H^\ell} + C \sum_{j=0}^{\ell-1} \lambda^{\frac{\ell-1-j}{2}} \|\partial_x^j f\|_2. \tag{6.53}$$

With this w, we define ϕ by (6.48). Then, by (6.49) and (6.53), we see that $\phi \in H^\ell_{per,*}$ and it holds that

$$\lambda \|\phi\|_{H^\ell} \leq C\Big\{\|f^0\|_{H^\ell} + \sum_{j=0}^{\ell-1} \lambda^{\frac{\ell-1-j}{2}} \|\partial_x^j f\|_2\Big\}.$$

This completes the proof. $\square$

We are now in a position to prove Proposition 5.3 (ii).

Proof of Proposition 5.3 (ii). Let $|\varepsilon| \leq \varepsilon_3$ and $|\tilde{\sigma}| + \|\tilde{V}\|_{X^2} \leq M$. Define the operator $\mathscr{L}$ on $Q(H^\ell_{per,*} \times H^{\ell-1}_{per})$ ($\ell = 1, 2$) by

$$D(\mathscr{L}) = QX^\ell,$$

$$\mathscr{L} = \mathscr{L}(\varepsilon, \tilde{\sigma}, \tilde{V}) = \mathcal{L}_{c_0,0} - \varepsilon\tilde{\sigma}Q\partial_{x_1} + \varepsilon QN[V_1 + \varepsilon\tilde{V}].$$

Set

$$\Sigma_1 = \Sigma_0 \cap \{\lambda; |\lambda| \le \mu_0\}.$$

It follows from Proposition 6.9 that there exists a positive constant C_2 such that if $\lambda \in \rho(-\mathscr{L}) \cap \Sigma_1$, then

$$\|(\lambda + \mathscr{L})^{-1}F\|_{X^\ell} \le C_2\|F\|_{H^\ell \times H^{\ell-1}}. \tag{6.54}$$

Assume that $\mu \in \rho(-\mathscr{L}) \cap \Sigma_1$. Then, by (6.54), we have

$$\{\lambda; |\lambda - \mu| < \tfrac{1}{C_2}\} \cap \Sigma_1 \subset \rho(-\mathscr{L}) \tag{6.55}$$

and the estimate (6.54) holds for $\lambda \in \Sigma_1$ with $|\lambda - \mu| < \frac{1}{C_2}$.

Since Σ_1 is compact, there exists a finite number of balls B_j $(j = 1, \cdots, N_1)$ with radius $\frac{1}{2C_2}$ such that $\Sigma_1 \subset \cup_{j=1}^{N_1} B_j$. By Proposition 6.11, we have $\lambda_0 \in \rho(-\mathscr{L}) \cap \Sigma_1$, and hence, $\mu_0 \in B_j$ for some j. Since Σ_1 is connected, we see from (6.55) that $\Sigma_1 \subset \rho(-\mathscr{L})$. Since $0 \in \Sigma_1$, we conclude that $0 \in \rho(-\mathscr{L})$ and the estimate (6.54) holds for $\lambda = 0$. This completes the proof. □

7 Proof of Lemma 6.2

In this section we give an outline of the proof of Lemma 6.2.

Proof of Lemma 6.2. Let $a = \frac{2\pi}{\alpha}$. In this section we write $\Omega_a = (0, a) \times (0, 1)$ instead of Ω_α. We set

$$G_1 = \left(-\frac{a}{4}, \frac{a}{4}\right), \quad G_2 = \left(\frac{a}{8}, \frac{7}{8}a\right),$$

and take ψ_1, $\psi_2 \in C^\infty$ satisfying

$$\begin{aligned} &\psi_1 \ge 0, \quad \left(-\tfrac{3}{16}a, \tfrac{3}{16}a\right) \subset \operatorname{supp}\psi_1 \subset G_1, \\ &\psi_2 \ge 0, \quad \left(\tfrac{5}{32}a, \tfrac{27}{32}a\right) \subset \operatorname{supp}\psi_2 \subset G_2. \end{aligned}$$

We define $\eta(x_1)$ by

$$\eta(x_1) = \sum_{j=1,2,k\in\mathbb{Z}} \psi_j(x_1 - ak).$$

Then $\eta \in C^\infty(\mathbb{R})$, $\eta(x_1 + a) = \eta(x_1)$ and $\eta(x_1) > 0$ for all $x_1 \in \mathbb{R}$. Setting

$$\phi_{j,k}(x_1) = \frac{\psi_j(x_1 - ak)}{\eta(x_1)},$$

we see that

$$\phi_{j,k} \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}\phi_{j,k} \subset G_j + ak\boldsymbol{e}_1 \quad (j = 1, 2, k \in \mathbb{Z}),$$

$$\phi_{j,k}(x_1) = \frac{\psi_j(x_1 - ak)}{\eta(x_1 - ak)} = \phi_{j,0}(x_1 - ak) \quad (j = 1, 2, k \in \mathbb{Z}),$$

$$\sum_{j=1,2,k\in\mathbb{Z}} \phi_{j,k}(x_1) = 1 \quad (x_1 \in \mathbb{R}).$$

Let us consider the problem

$$\operatorname{div} v = f$$

for a given $f \in C^\infty_{per,0}(\Omega_a)$ with $\int_{\Omega_a} f(x)\,dx = 0$.

We set $Q_0 = G_1 \cup G_2$ and define f_0 by

$$f_0(x) = \phi_{1,0}(x_1)f(x) + \phi_{2,0}(x_1)f(x) \quad (x \in Q_0).$$

It then follows that $f_0 \in C^\infty_0(Q_0)$. Furthermore,

$$\begin{aligned}
\int_{Q_0} f_0(x)\,dx &= \int_{G_1} \phi_{1,0}(x_1)f(x)\,dx + \int_{G_2} \phi_{2,0}(x_1)f(x)\,dx \\
&= \int_0^1 \left(\int_{-\frac{a}{4}}^0 \phi_{1,0}(x_1)f(x)\,dx_1 \right) dx_2 \\
&\quad + \int_0^1 \left(\int_0^{\frac{3}{4}a} (\phi_{1,0}(x_1) + \phi_{2,0}(x_1))f(x)\,dx_1 \right) dx_2 \\
&\quad + \int_0^1 \left(\int_{\frac{3}{4}a}^{\frac{7}{8}a} \phi_{2,0}(x_1)f(x)\,dx_1 \right) dx_2 \\
&= \int_0^1 \left(\int_{\frac{3}{4}a}^{a} \phi_{1,0}(x_1 - a)f(x - a\boldsymbol{e}_1)\,dx_1 \right) dx_2 + \int_0^1 \left(\int_0^{\frac{3}{4}a} f(x)\,dx_1 \right) dx_2 \\
&\quad + \int_0^1 \left(\int_{\frac{3}{4}a}^{\frac{7}{8}a} \phi_{2,0}(x_1)f(x)\,dx_1 \right) dx_2 \\
&= \int_0^1 \left(\int_{\frac{3}{4}a}^{a} (\phi_{1,1}(x_1) + \phi_{2,0}(x_1))f(x)\,dx_1 \right) dx_2 + \int_0^1 \left(\int_0^{\frac{3}{4}a} f(x)\,dx_1 \right) dx_2 \\
&= \int_{\Omega_a} f(x)\,dx = 0.
\end{aligned}$$

Therefore, from [3, Theorem III. 3.2] and its proof, we see that there exist $v_j \in C^\infty_0(\mathbb{R})$ $(j = 1, 2)$ such that $\operatorname{supp} v_j \subset G_j$ $(j = 1, 2)$ and $v_0 = v_1 + v_2 \in C^\infty_0(Q_0)$ satisfies

$$\operatorname{div} v_0 = f_0,$$

$$\|\nabla v_0\|_{L^2(Q_0)} \le C\|f_0\|_{L^2(Q_0)} \le C\|f\|_{L^2(\Omega_a)}.$$

Let $\tilde{v}_0$ and $\tilde{f}_0$ be the zero extensions of v_0 and f_0 on $\mathbb{R}^2$, respectively, and define v by

$$v(x) = \sum_{k\in\mathbb{Z}} \tilde{v}_0(x - ak\boldsymbol{e}_1).$$

Then $v \in C^\infty_{per,0}$ and

$$\operatorname{div} v(x) = \sum_{k\in\mathbb{Z}} \operatorname{div} \tilde{v}_0(x - ak\boldsymbol{e}_1) = \sum_{k\in\mathbb{Z}} \tilde{f}_0(x - ak\boldsymbol{e}_1).$$

For $x \in \Omega_a \cap G_1$, we have

$$\sum_{k\in\mathbb{Z}} \tilde{f}_0(x - ak\boldsymbol{e}_1) = \sum_{j=1,2,k\in\mathbb{Z}} \phi_{j,k}(x_1) f_0(x - ak\boldsymbol{e}_1) = \sum_{j=1,2} \phi_{j,0}(x_1) f(x) = f(x).$$

Furthermore, for $x \in \left[\frac{a}{4}, \frac{3}{4}a\right) \times (0,1)$, we have

$$\sum_{k\in\mathbb{Z}} \tilde{f}_0(x - ak\boldsymbol{e}_1) = \sum_{j=1,2,k\in\mathbb{Z}} \phi_{j,k}(x_1) f(x - ak\boldsymbol{e}_1) = \phi_{2,0}(x_1) f(x) = f(x),$$

and, for $x \in \left[\frac{3}{4}a, a\right) \times (0,1)$, we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \tilde{f}_0(x - ak\boldsymbol{e}_1) &= \sum_{j=1,2,k\in\mathbb{Z}} \phi_{j,k}(x_1) f_0(x - ak\boldsymbol{e}_1) \\ &= \phi_{1,1}(x_1) f(x - a\boldsymbol{e}_1) + \phi_{2,0}(x_1) f(x) \\ &= (\phi_{1,1}(x_1) + \phi_{2,0}(x_1)) f(x) = f(x). \end{aligned}$$

We thus conclude that $\operatorname{div} v(x) = f(x)$ for $x \in \Omega_a$. Moreover,

$$\|\nabla v\|_{L^2(\Omega_a)} \le 2\|\nabla v_0\|_{L^2(Q_0)} \le 2C\|f\|_{L^2(\Omega_a)}.$$

We next consider the case $f = \operatorname{div} \boldsymbol{g}$ with $\boldsymbol{g} = {}^\top(g_1, g_2)$, $g_j \in C^\infty_{per}(\overline{\Omega}_a)$ $(j = 1, 2)$ and $\operatorname{div} \boldsymbol{g} \in C^\infty_{per,0}(\Omega_a)$. Following the proofs of [3, Lemma III. 3.5] and [3, Theorem III.3.3], one can show that v_0 satisfies

$$\|v_0\|_{L^2(Q_0)} \le C\|\boldsymbol{g}\|_{L^2(Q_0)} \le C\|\boldsymbol{g}\|_{L^2(\Omega_a)},$$

$$\|\nabla v_0\|_{L^2(Q_0)} \le C\|f_0\|_{L^2(Q_0)} \le C\|\operatorname{div} \boldsymbol{g}\|_{L^2(\Omega_a)}.$$

It then follows that

$$\|v\|_{L^2(\Omega_a)} \le C\|\boldsymbol{g}\|_{L^2(\Omega_a)},$$
$$\|\nabla v\|_{L^2(\Omega_a)} \le C\|\operatorname{div} \boldsymbol{g}\|_{L^2(\Omega_a)}.$$

This completes the proof. □

8 Proof of Proposition 5.3 (i)

In this section we will give a proof of Proposition 5.3 (i). We denote $\tilde{L}_{\eta,k}$ and $\tilde{L}^*_{\eta,k}$ with $k = +1$ by $L(\alpha)$ and $L(\alpha)^*$. Then $L(\alpha)$ is expanded as

$$L(\alpha) = L^{(0)} + \alpha L^{(1)} + \alpha^2 L^{(2)},$$

where

$$L^{(0)} = \begin{pmatrix} 0 & 0 & \gamma^2 \partial_{x_2} \\ -\frac{\nu}{\gamma^2} & -\nu \partial_{x_2}^2 & \partial_{x_2} v_s^1 \\ \partial_{x_2} & 0 & -(\nu + \tilde{\nu}) \partial_{x_2}^2 \end{pmatrix},$$

$$L^{(1)} = \begin{pmatrix} i v_s^1 & i\gamma^2 & 0 \\ i & i v_s^1 & -i\tilde{\nu}\partial_{x_2} \\ 0 & -i\tilde{\nu}\partial_{x_2} & i v_s^1 \end{pmatrix},$$

$$L^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu + \tilde{\nu} & 0 \\ 0 & 0 & \nu \end{pmatrix}.$$

Similarly, $L(\alpha)^*$ is expanded as

$$L(\alpha)^* = L^{(0)*} + \alpha L^{(1)*} + \alpha^2 L^{(2)*},$$

where

$$L^{(0)*} = \begin{pmatrix} 0 & -\nu & -\gamma^2 \partial_{x_2} \\ 0 & -\nu \partial_{x_2}^2 & 0 \\ -\partial_{x_2} & \partial_{x_2} v_s^1 & -(\nu + \tilde{\nu}) \partial_{x_2}^2 \end{pmatrix},$$

$$L^{(1)*} = \begin{pmatrix} -i v_s^1 & -i\gamma^2 & 0 \\ -i & -i v_s^1 & -i\tilde{\nu}\partial_{x_2} \\ 0 & -i\tilde{\nu}\partial_{x_2} & -i v_s^1 \end{pmatrix},$$

$$L^{(2)*} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu + \tilde{\nu} & 0 \\ 0 & 0 & \nu \end{pmatrix}.$$

Lemma 8.1. *There exists a positive number r_1 such that if $\alpha \leq r_1$, then $V_\pm$ and $V^*_\pm$ given in section 5.1 are represented as*

$$V_+(x) = \left(v^{(0)}(x_2) + \alpha v^{(1)}(x_2) + O(\alpha^2) \right) e^{i\alpha x_1}, \quad V_- = \overline{V_+},$$

$$V_+^*(x) = \frac{\alpha}{2\pi}\left(v^{(0)*}(x_2) + \alpha v^{(1)*}(x_2) + O(\alpha^2)\right)e^{i\alpha x_1}, \quad V_-^* = \overline{V_+^*},$$

where $v^{(0)} = {}^\top(\phi^{(0)}, w^{(0),1}, 0)$ *with*

$$\phi^{(0)} = 1, \quad w^{(0),1} = \frac{1}{2\gamma^2}(-x_2^2 + x_2),$$

$v^{(1)} = {}^\top(\phi^{(1)}, w^{(1),1}, w^{(1),2})$ *with*

$$\begin{aligned}
\phi_1^{(1)}(x_2) &= -i\left(\tfrac{\nu}{\gamma^2} + \tfrac{\tilde{\nu}}{2\gamma^2}\right)\left(-x_2^2 + x_2 - \tfrac{1}{6}\right),\\
w_1^{(1),1}(x_2) &= -i\left(\tfrac{\nu}{\gamma^4} + \tfrac{\tilde{\nu}}{2\gamma^4}\right)\left(\tfrac{1}{12}x_2^4 - \tfrac{1}{6}x_2^3 + \tfrac{1}{12}x_2^2\right)\\
&\quad -\tfrac{i}{12\nu\gamma^2}\left(\tfrac{1}{30}x_2^6 - \tfrac{1}{10}x_2^5 + \tfrac{1}{12}x_2^4 - \tfrac{1}{60}x_2\right) - \tfrac{i}{2\nu}(-x_2^2 + x_2),\\
w_1^{(1),2}(x_2) &= -\tfrac{i}{\gamma^2}\left(-\tfrac{1}{3}x_2^3 + \tfrac{1}{2}x_2^2 - \tfrac{1}{6}x_2\right),
\end{aligned}$$

$v^{(0)*} = {}^\top(\phi^{(0)*}, 0, 0)$ *with* $\phi^{(0)*} = \gamma^2$, *and* $v^{(1)*} = {}^\top(\phi^{(1)*}, w^{(1),1*}, w^{(1),2*})$ *with*

$$w^{(1),1*} = \frac{i\gamma^2}{2\nu}(-x_2^2 + x_2).$$

Remark 8.2. Note that we will not use the explicit form of $\phi^{(1)*}$ and $w^{(1),2*}$.

Proof. We see from [8, Lemma 5.1] that $v^{(0)}$ and $v^{(0)*}$ are eigenfunctions for eigenvalue 0 of $-L^{(0)}$ and $-L^{(0)*}$, respectively, and the corresponding eigenprojections $\Pi^{(0)}$ and $\Pi^{(0)*}$ are given by

$$\Pi^{(0)}u = \langle\langle u, v^{(0)*}\rangle\rangle v^{(0)}, \quad \Pi^{(0)*}u = \langle\langle u, v^{(0)}\rangle\rangle v^{(0)*}.$$

Let P_α be the eigenprojection for λ_α. Then

$$P_\alpha = \Pi^{(0)} - \alpha(SL^{(1)}\Pi^{(0)} + \Pi^{(0)}L^{(1)}S) + O(\alpha^2),$$

where $S = [(I - \Pi^{(0)})L^{(0)}(I - \Pi^{(0)})]^{-1}$. Set $v_{+1} = P_\alpha v^{(0)}$. We see that v_{+1} is an eigenfunction for λ_α and

$$v_{+1} = v^{(0)} - \alpha SL^{(1)}v^{(0)} + O(\alpha^2).$$

Therefore, setting $v^{(1)} = -SL^{(1)}v^{(0)}$, we have the desired expression of $v^{(1)}$ from [8, Proposition 6.5], where $SL^{(1)}v^{(0)}$ is computed.

As for V_+^*, let P_α^* be the eigenprojection for $\lambda_\alpha^* = \overline{\lambda_\alpha}$. Then

$$P_\alpha^* = \Pi^{(0)*} - \alpha(S^*L^{(1)*}\Pi^{(0)*} + \Pi^{(0)*}L^{(1)*}S^*) + O(\alpha^2),$$

where $S^* = [(I - \Pi^{(0)*})L^{(0)*}(I - \Pi^{(0)*})]^{-1}$. Set $\tilde{v}_{+1}^* = P_\alpha^* v^{(0)*}$. Then $\tilde{v}_{+1}^*$ is an eigenfunction for λ_α^* and

$$\tilde{v}_{+1}^* = v^{(0)*} - \alpha S^*L^{(1)*}v^{(0)*} + O(\alpha^2).$$

Let us compute $\tilde{v}^{(1)*} = -S^* L^{(1)*} v^{(0)*}$ which is the solution of

$$L^{(0)*} u = -(I - \Pi^{(0)*}) L^{(1)*} v^{(0)*}, \quad \langle\langle u, v^{(0)} \rangle\rangle = 0.$$

By [8, Proposition 6.3], we have $\langle\langle L^{(1)} v^{(0)}, v^{(0)*} \rangle\rangle = \frac{i}{6}$, and hence,

$$\Pi^{(0)*} L^{(1)*} v^{(0)*} = \langle\langle L^{(1)*} v^{(0)*}, v^{(0)} \rangle\rangle v^{(0)*} = \overline{\langle\langle L^{(1)} v^{(0)}, v^{(0)*} \rangle\rangle} v^{(0)*} = -\frac{i}{6} v^{(0)*}.$$

We set $f = {}^\top(f^{0*}, f^{1*}, f^{2*}) = -(I - \Pi^{(0)*}) L^{(1)*} v^{(0)*}$. By a direct computation we have

$$f^{0*} = i\gamma^2 v_s^1 - \frac{i}{6}\gamma^2, \quad f^{1*} = i\gamma^2, \quad f^{2*} = 0.$$

It then follows that

$$\partial_{x_2}^2 w^1 = -\frac{i\gamma^2}{\nu}, \quad w^1|_{x_2=0,1} = 0. \tag{8.1}$$

This gives $w^1 = \frac{i\gamma^2}{2\nu}(-x_2^2 + x_2)$, and then w^2 and ϕ are given by

$$\partial_{x_2} w^2 = -\frac{1}{\gamma^2}(\nu w^1 + f^{0*}), \tag{8.2}$$

$$\partial_{x_2} \phi = (\partial_{x_2} v_s^1) w^1 - (\nu + \tilde{\nu}) \partial_{x_2}^2 w^2, \quad \int_0^1 \phi \, dx_2 = -\gamma^2 (w^1, w^{(0),1}).$$

Since $\langle\langle v^{(1)}, v^{(0)*} \rangle\rangle = \langle\langle \tilde{v}^{(1)*}, v^{(0)} \rangle\rangle = 0$, we have $\langle\langle v_{+1}, \tilde{v}_{+1}^* \rangle\rangle = 1 + O(\alpha^2)$. Therefore, setting $v_{+1}^* = \tilde{v}_{+1}^* / \overline{\langle\langle v_{+1}, \tilde{v}_{+1}^* \rangle\rangle}$, we have the desired result. This completes the proof. $\square$

We are now in a position to prove Proposition 5.3 (i).

Proof of Proposition 5.3 (i). By Lemma 8.1 and the relation that $-\partial_{x_2}^2 w^{(0),1} = \frac{1}{\gamma^2}\phi^{(0)}$, we have

$$\begin{aligned}
[\![K_0 V_1]\!]_1 &= \alpha^2 \Big\{ \left(\partial_{x_2} w^{(0),1}, i w^{(1),2*} \right) + \left(-\frac{i}{\gamma^2}\phi^{(1)} - i\partial_{x_2}^2 w^{(1),1}, i w^{(1),1*} \right) \\
&\quad -2 \left(i\partial_{x_2}^2 w^{(1),3}, i w^{(1),2*} \right) \Big\} + O(\alpha^3) \\
&= \alpha^2 \Big\{ -\left(w^{(0),1}, i\partial_{x_2} w^{(1),2*} \right) - \left(\frac{i}{\gamma^2}\phi^{(1)}, i w^{(1),1*} \right) - \left(i w^{(1),1}, i\partial_{x_2}^2 w^{(1),1*} \right) \\
&\quad +2 \left(i\partial_{x_2} w^{(1),2}, i\partial_{x_2} w^{(1),2*} \right) \Big\} + O(\alpha^3).
\end{aligned}$$

By using (8.1), (8.2) and Lemma 8.1, we find that

$$[\![K_0 V_1]\!]_1 = \frac{\alpha^2}{12\nu^2} \left\{ \left(\frac{1}{280} - \gamma^2 \right) + \frac{\nu^2}{10\gamma^2} \right\} + O(\alpha^3) > 0$$

for $\alpha \ll 1$. This completes the proof. $\square$

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