Drawing Curves

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出版情報：MI lecture note series. 64, pp.117-120, 2015-09-18. 九州大学マス・フォア・インダストリ研究所
バージョン：
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MATHEMATICAL PROGRESS IN EXPRESSIVE IMAGE SYNTHESIS 2015
Sep 25-27, 2015, Nishijin Plaza Kyushu University, Fukuoka

Drawing Curves

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Abstract A method drawing curves is proposed. A program drawing graphs of mathematical functions using this method is realized in a computer algebra and outputs the graphs in a source file of TEX and then transforms it into a PDF file.

Keywords: Bézier curve, spline curve, computer algebra, Risa/Asir, TEX, Ti*k*Z, 3D graph

1 Introduction

Since the last year I have a class of calculus in my university and show graphs of functions such as $f(x, y) = x^2 - y^2$. I have been developing a library os_muldif.rr [1] of a computer algebra Risa/Asir [3] to realize my research explained in [2] and then I added some functions in the library for such educational purpose including calculus, linear algebra and elementary number theory. The library is an open source and can be equally executed by a personal computer with any one of the operating systems Windows, Mac and UNIX.

In fact, a function in the library executes the following procedure to get the graphs.

$$\text{Risa/Asir} \xrightarrow{\text{output}} \text{Ti}k\text{Z or XY-pic in a TEX file} \xrightarrow{\text{LaTeX}} \text{DVI file} \xrightarrow{\text{dvipdfm}} \text{PDF file} \qquad \text{(TEX file} \xrightarrow[\text{pdfLaTeX}]{} \text{PDF file)} \tag{1}$$

2 Curves

Consider a curve

$$C : [a, b] \ni t \mapsto \gamma(t) = \big(x(t), y(t)\big) \in \mathbb{R}^2. \tag{2}$$

We choose points in $[a, b]$, namely, $P_j = \gamma(t_j) \in C$ with $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ and draw a certain curve C' starting from P_0, exactly passing through $P_1, \ldots, P_{N-1}$ in this order and ending at P_N. We request the following conditions.

- C' is determined only by $\{P_0, P_1, \ldots, P_N\}$.
- C' is a good approximation of C and it is free from its final resolution in drawing.
- Smaller size of data (i.e. the number N) and an output in a popular format are desirable.
- The curve can be described in a usual TEX source file.

One of the way to realize it is to connect the points by Bézier curves and use Ti*k*Z and/or XY-pic which are in a package of a TEX system (cf. (1)).

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2.1 Smooth curves

A Bézier curve of degree n is

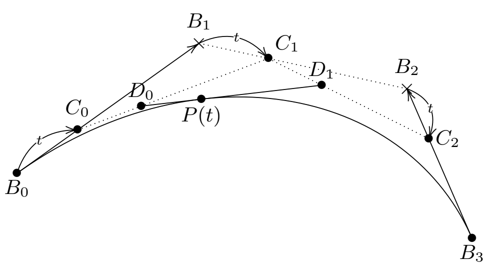
$$[0,1] \ni t \mapsto P(t) = P(B_0, \ldots, B_n; t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} B_i \tag{3}$$

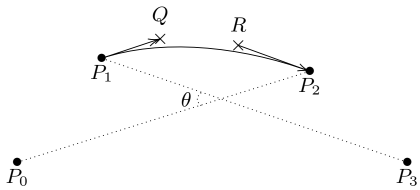
determined by $(n+1)$ points $B_0, \ldots, B_n$.

Note that $P(B, B'; t)$ is the point internally dividing the line segment BB' by $t : 1-t$. Since $P(B_0, \ldots, B_n; t) = P(P(B_0, B_1; t), P(B_1, B_2; t), \ldots, P(B_{n-1}, B_n; t); t)$, the point $P(t)$ is geometrically described. For example, the **cubic Bézier curve** is

$$\begin{aligned} P(t) &= P(B_0, B_1, B_2, B_3; t) = P\big(P(B_0, B_1; t), P(B_1, B_2; t), P(B_2, B_3; t); t\big) \\ &= P\Big(P\big(P(B_0, B_1; t), P(B_1, B_2; t); t\big), P\big(P(B_1, B_2; t), P(B_2, B_3; t); t\big); t\Big). \end{aligned}$$

The curve starts from B_0 to the direction $\overrightarrow{B_0B_1}$ and ends at B_3 to the direction $\overrightarrow{B_2B_3}$. It does not necessarily pass through B_1 nor B_2.





$$\overrightarrow{P_1Q} = c\overrightarrow{P_0P_2},\ \overrightarrow{RP_2} = c\overrightarrow{P_1P_3},$$
$$\frac{\overline{P_1Q} + \overline{P_2R}}{\overline{P_1P_2}} = \frac{2}{3(1+\cos\frac{\theta}{2})}$$

Consider a curve C passing through P_0, P_1, P_2, P_3 in this order. We simulate the curve segment of C connecting P_1 to P_2 by the cubic Bézier curve $P(P_1, Q, R, P_2; t)$ with the control points Q, R defined as above. Then the number c is determined by

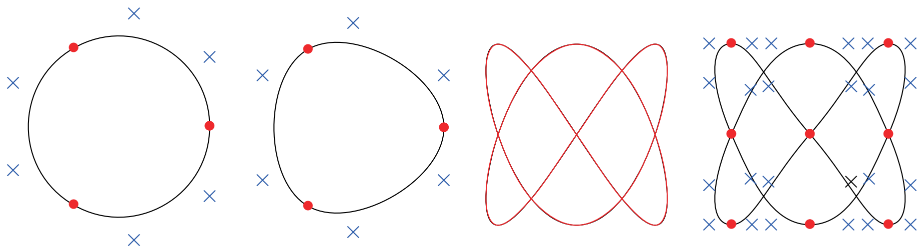
$$c = \frac{4\overline{P_1P_2}}{3(\overline{P_0P_2} + \overline{P_1P_3})} \frac{1}{1 + \sqrt{\frac{1 + \frac{(\overrightarrow{P_0P_2}, \overrightarrow{P_1P_3})}{\overline{P_0P_2} \cdot \overline{P_1P_3}}}{2}}} \tag{4}$$

so that the curve simulates the arc if $P_0, \ldots, P_3$ are on a circle with a center O and moreover $\overline{P_0P_1} = \overline{P_1P_2} = \overline{P_2P_3}$. In this case, the error $\left|\frac{\overline{OP(t)}}{\overline{OP_1}} - 1\right|$ is less than $\frac{1}{640}$ (resp. $\frac{1}{3600}$) if $\angle P_1OP_2 \leq 120°$ (resp. $\leq 90°$). Note that a Bézier curve never coincides with an exact arc. For a closed curve C passing through points $R_0, R_1, \ldots, R_N = R_0$ in this order we draw a curve segment between R_j and R_{j+1} by putting $P_i = R_{i+j-1}$ for $i = 0, 1, 2$ and 3 as in the above and $R_{\nu \pm N} = R_\nu$ ($\nu = 1, \ldots, N$). Then the resulting curve C' we draw is a smooth closed curve (of class C^1) which simulates C.

When the number c is fixed to be $\frac{1}{6}$ in the above, the corresponding curve is known as the (uniform) Catmull-Rom spline curve. This curve is invariant under affine transformations and our curve is invariant under conformal affine transformations.

The following first example is the curve drawn by the three points $(\cos t, \sin t)$ with $t = 0$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ indicated by $\bullet$. The other 6 points calculated by using (4) are indicated by $\times$. In the final PDF file the positions of these 9 points are only written and the real rendering of the Bézier curve is done by a viewer of the file and therefore the size of the PDF file is small. The second example is the (uniform/centripetal) Catmull-Rom spline curve passing through these three points.

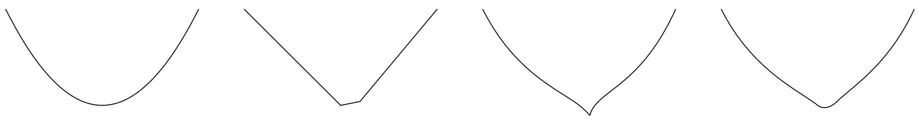
The other examples below are the Lissajous curve $\gamma(t) = (\sin 2t, \sin 3t)$ drawn by the points corresponding to $t = \frac{2\pi j}{N}$ for $j = 0, \ldots, N$.



$N = 3$ (3 points) Catmull-Rom spline $N = 24$, $N = 96$ $N = 12$ (9 points)

If the points $P_j = \gamma(t_j)$ are not suitably chosen, the resulting curve drawn by the points may be not good. Even in this case our curve is better than the corresponding Catmull-Rom spline curve as in the following example.

Taking the points on the curve $\gamma(t) = (t, t^2)$ corresponding to $t = -2, -1, 0, 0.2, 1, 2$, we draw curve for $-1 \le t \le 1$ by these 6 points.



$y = x^2$ $(-1 \le x \le 1)$ piecewise linear Catmull-Rom spline Our curve

2.2 Singularities

We consider a curve $\gamma(t)$ $(t \in [a, b])$ which has singular points or discontinuous points. We assume that the curve is a finite union of smooth curves but we do not know the singular points of the curve.

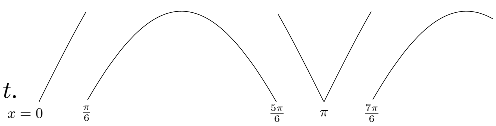
First we choose points $P_j = \gamma(t_j)$ with $t_0 = a < t_1 < \cdots < t_N = b$ on the curve. We put $t_j = \frac{(N-j)a+jb}{N}$ in most cases (or as default)[1]. Then we recursively choose more points on the curve up to certain times if necessary as in the following way.

- If the number $1 - \frac{(\overrightarrow{P_{j-1}P_j}, \overrightarrow{P_jP_{j+1}})}{\overline{P_{j-1}P_j} \cdot \overline{P_jP_{j+1}}}$ exceeds a given threshold value, we add the points $\gamma(\frac{t_{j-1}+t_j}{2})$ and $\gamma(\frac{t_j+t_{j+1}}{2})$.
- If the length $\overline{P_{j-1}P_j}$ exceeds a given threshold value, we add the point $\gamma(\frac{t_{j-1}+t_j}{2})$.

If the length $\overline{P_{j-1}P_j}$ still exceeds a given threshold value after this procedure, we cut our curve between two points P_{j-1} and P_j.

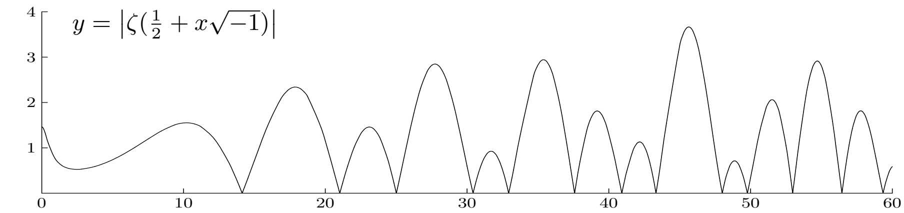
We show examples:

$y = |2\sin x| - \left[|2\sin x|\right]$ $\quad (0 \le x \le 5)$
discontinuous at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$
not smooth at $x = \pi$
$[t]$: the largest integer which does not exceed t.



[1] Moreover if the curve is defined outside $[a, b]$, we use the points P_{-1} and P_{N+1} to define Bézier curves.

It takes less than a second to get the following graph of Riemann's zeta function in a PDF file.

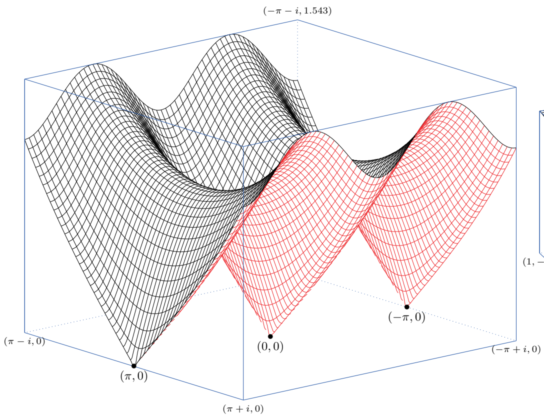


3 3D graphs

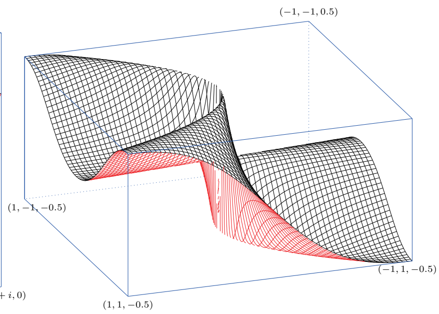
Our main purpose is to draw graphs of surfaces defined by $z = f(x, y)$ with mathematical functions $f(x, y)$. Using our method (1), we draw curves on a surface defined by the condition that x is constant or y is constant. It takes $10 \sim 30$ seconds to get a required PDF file after a command in Risa/Asir if $f(x, y)$ is a simple rational function. We can use Ti*k*Z and XY-pic. In contrast to XY-pic the source text in Ti*k*Z is more readable, easy to be edited and has stronger abilities such that it supports coloring and filling region by a pattern but is takes a little longer time to be transformed into a PDF file. Hence our library supports both of them.

We give two examples:

$|\sin z| \quad (z = x + yi,\ -\pi \le x \le \pi,\ -1 \le y \le 1)$
angle $(50°, 15°)$ ratio $1 : 3 : 3$



$z = \frac{xy^2}{x^2 + y^4} \quad (-1 \le x \le 1,\ -1 \le y \le 1)$
angle $(70°, 20°)$
This function is discontinuous at $(x, y) = (0, 0)$



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[3] Risa/Asir, an open source general computer algebra system, `http://www.math.kobe-u.ac.jp/Asir/asir.html`