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出版情報：MI lecture note series. 64, pp.103-112, 2015-09-18. 九州大学マス・フォア・インダストリ研究所

MATHEMATICAL PROGRESS IN EXPRESSIVE IMAGE SYNTHESIS 2015
Sep 25-27, 2015, Nishijin Plaza Kyushu University, Fukuoka

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Abstract Bézier, B-Spline and NURBS are de facto flexible curves developed for various design intent. However, these curves possess complex curvature function complicating the process of aesthetic shape design. In order to shorten the process, we introduce the fundmental equations of aesthetic curves and surfaces. This paper elaborates on the technicalities of Log-Aesthetic (LA) curves and surfaces along with its practical application for industrial design. It is anticipated that the emerging LA curves and surfaces have good prospects to be the standards for designing aesthetic shapes.

Keywords: fair curves and surfaces, log-aesthetic curve and surface, general equations of aesthetic curves, logarithmic curvature graph

1 Introduction

Fairness metric is a conventional term used to describe the quality of curves and surfaces. According to Farin, a planar curve is fair if it has continuous curvature and consists of few monotonic curvature pieces [2]. In the past, researchers formulated high-quality curves and surfaces using polynomial representation i.e. Bézier, B-spline and NURBS. The underlying principle was to obtain suitable functionals in order to minimize the oscillation of a curvature. Numerous principles are elaborated as chapters in a book edited by Sapidis entitled "Designing Fair Curves and Surfaces"[1].

A well defined Cesáro equation can be employed to produce high quality curves and surfaces. This equation expresses intrinsic properties of curves and surfaces without the presence of traditional polynomials. In this paper, we follow suit to distinguish the terms "fair" and "aesthetic". In brief, free-form curves and surfaces are constructed with specific formulations rather than polynomials or rationals to represent high quality shapes denoted as aesthetic curves and surfaces.

In recent years, researchers are perfecting the works on aesthetic curves contributing to exponential increase of findings and publications in this arena. However, research pertaining to the formulation of aesthetic surfaces are still at an early stage and offers bright opportunity for CAD community to contribute significantly. Figure 1 shows an Euler diagram depicting the classifications of fair and aesthetic curves. The members in the green circle comprises of the family which involves traditional curves. The orange circle encloses aesthetic curves including the log-aesthetic (LA) curves; which is one of the main focus of this paper. In this paper, we define aesthetic curves as non-polynomial

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functions used for aesthetic shape designs. The intersection of these circles represent the curves expressed by traditional formulations as well as specific methods of construction.

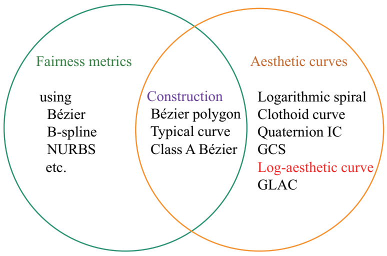


Figure 1: Classifications of the researches on fair and aesthetic curves.

Recent advancement on the LA curve has been promising and it has now matured for industrial and graphical design practices. In 2009, Levien and Séquin [3] indicated that LA curves are the most promising curve for aesthetic design. Gobithaasan and Miura [5] formulated the generalized log-aesthetic curve (GLAC) in a standard form by representing the gradient of the logarithmic curvature graph (LCG) as a function of its arc length. They also reported that the LCG gradient of Generalized Cornu spiral [6] can be written as a linear function of the LCG gradient [7]. In 2012, Miura et al. reformulated the LA curve using variational principles to obtain minimized functionals for free-form surface design [4]. In the same year, Ziatdnov et al. [8] showed that some LA curves can be expressed by incomplete Gamma functions which shortens the computation time up to 10 times. Recently Meek et al. [9] proved that an unique solution exists for G^1 interpolation by using an LA curve segment when $\alpha < 0$. In 2015, Miura et al.[10] proposed another type of aesthetic curve called polar-aesthetic curve for scissors blade design.

2 Log-aesthetic Curves

This section discusses about the details of LA curves and surfaces in a greater depth. We show that the formulation of LA curves has been perfected along with its fundamental properties. In the following sections we further illustrate that its applications for industrial design which proven to be promising. Albeit, the formulation and representation of aesthetic surfaces has been progressing and much efforts are anticipated for practical design. Next section is dedicated to dissect the foundation of LCG which led to the general equations of aesthetic curves.

2.1 Logarithmic Curvature Graphs

Harada's formulation of the Logarithmic Distribution Diagram of Curvature (LDDC) is based on a quantitative approach which involves tedious curvature radius and its arc length frequency calculation. They highlighted that an aesthetic curve has a linear LDDC plot. However, the length frequency of the curve can not be evaluated at certain positions on the curve or for arbitrary radius of curvature. Thus, it was not given much attention as a shape interrogation tool.

In 2003, Kanaya et al. [12] showed that for a given curve $\boldsymbol{C}(t) = (x(t), y(t))$, the derivative of the arc length s with respect to the logarithm of the radius of curvature $R = \log \rho$ is given by

$$\frac{ds}{dR} = \frac{(x'y'' - x''y')(x'^2 + y'^2)^{\frac{3}{2}}}{3(x'x'' + y'y'')(x'y'' - x''y') - (x'^2 + y'^2)(x'y''' - x'''y')} \tag{1}$$

where $'$ denotes the derivative with respect to parameter t. The quantitative LDDC approach is mathematically equivalent to equation (1) where horizontal and vertical coordinates represent R and $\log|ds/dR|$, respectively. Thus, equation (1) is sufficient to analytically define the LDDC plot. However, it does not provide concrete conditions in order to approximate LDDC with a straight line or it does indicate the slope of the approximated line to represent aesthetic shapes. Furthermore, for a curve whose shape is obtained from its image data, only discrete data are available and the partial arc length s_j must be a finite value to calculate the length frequency.

Hence algebraic manipulation was carried out to reformulate the LHS of Eq. (1). Since the LDDC graph is expressed by $\log|ds/d(\log\rho)|$ and both s and ρ are functions of the parameter t, we can further simplify as follows

$$\log\left|\frac{ds}{d(\log\rho)}\right| = \log\left|\frac{\frac{ds}{dt}}{\frac{d(\log\rho)}{dt}}\right| = \log\left|\rho\frac{\frac{ds}{dt}}{\frac{d\rho}{dt}}\right|$$
$$= \log\rho + \log s_d - \log\left|\frac{d\rho}{ds}\right| \tag{2}$$

where $s_d = ds/dt$. Equation (2) is defined by the radius of curvature and its derivative. It describes the relationship between the radius of curvature and the derivative of the arc length more explicitly as compared to Eq. (1). This new analytical approach is denoted as Logarithmic Curvature Graph (LCG).

2.2 First & Second Fundamental Equations of Aesthetic Curves

In this section, we derive the equation of a curve that produces LCG strictly as a straight line. The curves obtained satisfying such a condition are defined as aesthetic curves and it is the fundamental equations of aesthetic curves [11].

If we assume that the LCG graph of a given curve is strictly expressed by a straight line, on the LHS of Eq. (2) there is a constant α and

$$\log\left|\rho\frac{ds}{d\rho}\right| = \alpha\log\rho + C \tag{3}$$

where C is a constant. By transforming Eq. (3), we obtain

$$\frac{1}{\rho^{\alpha-1}}\frac{ds}{d\rho} = e^C = C_0 \tag{4}$$

Hence,

$$\frac{ds}{d\rho} = C_0\rho^{\alpha-1} \tag{5}$$

If $\alpha \neq 0$,

$$s = \frac{C_0}{\alpha}\rho^\alpha + C_1 \tag{6}$$

In the above equation, C_1 is an integral constant. Therefore

$$\rho^\alpha = C_2 s + C_3 \tag{7}$$

where $C_2 = \alpha/C_0$ and $C_3 = -(C_1\alpha)/C_0$. Here we rename C_2 and C_3 to c_0 and c_1, respectively. Then

$$\rho^\alpha = c_0 s + c_1 \tag{8}$$

The above equation indicates that the α-th power of the radius of curvature ρ is given by a linear function of the arc length s. The above equation is named an the first fundamental equation of aesthetic curves [11].

For the case of $\alpha = 0$,

$$s = C_0 \log \rho + C_1 \tag{9}$$

Hence,

$$\rho = C_2 e^{C_3 s} \tag{10}$$

where $C_2 = e^{-C_1/C_0}$ and $C_3 = 1/C_0$. We rename C_2 and C_3 as c_0 and c_1, respectively. We get

$$\rho = c_0 e^{c_1 s} \tag{11}$$

The ρ is given by an exponential function of s. The above equation is named as the second fundamental equation of aesthetic curves [11].

It is know that logarithmic spiral and clothoid are regarded as high quality curves as explained in the next section. One of the principal characters of the logarithmic spiral is that its radius of curvature and arc length are proportional. Hence, the logarithmic spiral satisfies Eq. (8) and its α is equal to 1. Additionally , the main property of clothoid is that its radius of curvature is inversely proportion to its arc length. Thus, Eq.(8) is satisfied for the clothoid if α is given by -1.

In summary, the general equations of aesthetic curves expressed by Eq. (8) encompasses beautiful curves such as logarithmic spirals and clothoids. In fact Nielsen's spiral [13] is also expressed by Eq.(11). The curves which satisfies the first and second fundamental equations of aesthetic curves are denoted as Log-aesthetic (LA) curves, which was coined by Professor Carlo H. Séquin from University of California, Berkeley during the presentation of this work at the International CAD Conference & Exhibition 2007 at Honolulu, Hawaii.

2.3 Parametric Expression of LA Curve

In this section, we describe the parametric expressions of the fundamental equations for aesthetic curves as Eqs. (8) and (11). Let a curve $\boldsymbol{C}(s)$ satisfies Eq. (8). Then

$$\rho(s) \quad = \quad (c_0 s + c_1)^{\frac{1}{\alpha}} \tag{12}$$

Since s is the arc length, $|s_d| = 1$ (refer to, for example, [2]) and there exists $\theta(s)$ satisfying the following two equations:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta \tag{13}$$

Since $\rho(s) = 1/(d\theta/ds)$,

$$\frac{d\theta}{ds} = (c_0 s + c_1)^{-\frac{1}{\alpha}} \tag{14}$$

Hence, if $\alpha \neq 1$,

$$\theta(s) = \frac{\alpha(c_0 s + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha - 1)c_0} + c_2 \tag{15}$$

If the starting point of the curve is given by $\boldsymbol{P}_0 = \boldsymbol{C}(0)$, thus LA curve can be represented in a complex plane as follows

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{i \frac{\alpha (c_0 u + c_1)^{\frac{\alpha - 1}{\alpha}}}{(\alpha - 1) c_0}} du \tag{16}$$

The above expression can be regarded as an extension of the clothoid curve where power of e in its definition is changed from 2 to $\alpha + 1$ and its LCG gradient can be specified to be equal to any value except for 0.

Similarly, the second fundamental equation of aesthetic curves are expressed by Eq. (11),

$$\frac{d\theta}{ds} = \frac{1}{c_0} e^{-c_1 s} \tag{17}$$

$$\theta = -\frac{1}{c_0 c_1} e^{-c_1 s} + c_2 \tag{18}$$

Therefore the curve is given by

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{-\frac{i}{c_0 c_1} e^{-c_1 s}} ds \tag{19}$$

Figure 2 shows LA curves with various α values.

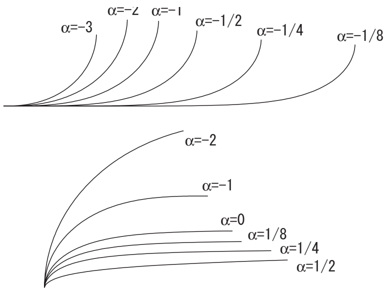


Figure 2: LA curves whose LCG gradients are given by various α, top:curves with negative α, bottom: those with positive α and two negative values -2 and -1 for comparisons.

3 Log-aesthetic Spline

In this section, we describe a method to simultaneously specify endpoints, tangent vectors and its curvatures (G^2 Hermite data) using an LA spline which consists three LA segments. The idea was obtained from Lan et al.'s work [14] where they solve the G^2 Hermite interpolation problem using triple clothoids. Miura et al. [15] used the shape parameter α as an additional parameter to make the end curvature as 0. Since α is related to impressions of the shape [16], hence α is fixed as a constant to produce G^1 Hermite interpolation using a single LA curve segment and a C^3 continuous compound-rhythm LA curve is connected with two LA segments.

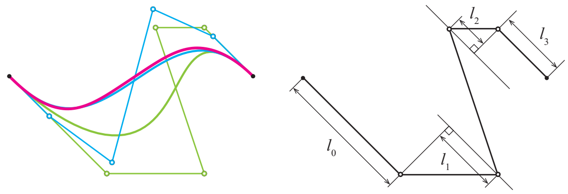
α is usually regarded as a parameter which can be controlled by designers. Thus, we do not use it to satisfy any geometric constraints to shape LA curves. Note that the degree of freedom (DoF) of the LA spline with three segments is similar to triple clothoids. It is a common practice to fix the value of α to design aesthetic shapes using LA curves. An LA spline consists of three LA curve segments with different α values which are joined with G^2 continuity [17].

3.1 Initial Value Estimation

We need initial values for parameters to define an LA spline. To obtain these initial values, we estimate curvatures at the two joints of the LA curve segments. We may use a Bézier curve of degree 5 for the estimation of the initial values to shape an LA spline. Let the total length of the Bézier curve as h. In general, Bézier curves are not uniquely determined by endpoints, tangent directions and its curvatures; these conditions do not necessarily yield a suitable curve for the initial value estimation. Hence we use an objective function which is modified to be independent from the total length h as proposed by Miura at al. [17]:

$$J_{LAC} = \frac{\sum_{i=1}^{3} \int_{S_{i-1}h}^{S_i h} \sqrt{1 + \alpha_i^2 \rho^{2\alpha_i - 2} \rho_s^2} ds}{h} \tag{20}$$

where $S_0 = 0$, $S_1 = 0.25$, $S_2 = 0.75$ and $S_3 = 1$. The above function is minimized to generate an appropriate initial Bézier curve. Fig. 3(a) shows a Bézier curve of degree 5 and its initial control points in green and those after optimization in blue for $\alpha_i = -0.5$, $i = 1, 2, 3$.



(a) A triple LA curve (b) Parameters for optimization

Figure 3: Optimization of the approximation curve for initial values.

Upon using the Bézier curve after optimization, an LA spline curve with three segments shown in red is determined. To be precise, the LA spline curve does not strictly minimize the objective function in Eq. (20), but note that the shapes of the Bézier curve and LA spline are almost in a similar shape. In this example, parameters calculated from the input of Bézier curve are not appropriate, hence the numerical calculation diverges because the total length h becomes negative. If we use the Bézier curve after optimization, we may obtain these values without calculation failure and generate an LA spline successfully. Since an optimization process is necessary only for the initial value estimation, we propose a simpler method as shown in Fig. 3(b). Let the lengths between the first and the second control points and the fifth and sixth control points be l_0 and l_3, respectively. Furthermore let parameters to determine the positions of the 3^{rd} and 4^{th} be l_1 and l_2, respectively. We change these parameters independently in the range of $0.05 \leq l_i/h \leq 0.5$ for $i = 0, 1, \cdots$ by 0.05 where h is the total arc length of the input Bézier curve. We may now obtain parameter values easily which minimizes the objective function in Eq.(20).

Figure 4 depicts Bézier curve of degree 5 (black) and an LA spline (red) generated using the proposed method. The Bézier curve is generated and deformed by built-in commands of a commercial CAD system to satisfy G^2 Hermite data posed at its endpoints. The LA spline is generated with similar G^2 Hermite data to achieve G^2 continuity at the joints. These curves are also shown separately for visual clarity. The curvature plot of these curves is drawn in blue. It is visually clear that the curvature of the Bézier curve varies drastically in order to satisfy G^2 continuity at its endpoints whereas the LA spline joins gradually.

Figure 5 shows a practical design of a car using LA splines. Figure 5(a) shows iso-parametric lines of the free-form surface generated using LA splines and its corresponding zebra maps. Figure 5(b) depicts the CAD model with a special lighting condition and 5(c) are photos of its mock-up

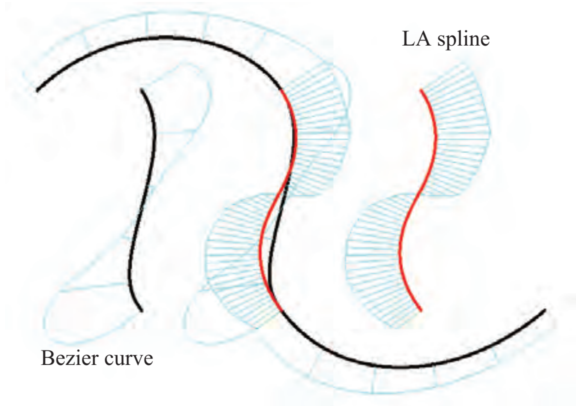


Figure 4: A generation of G^2 continuous Bézier curve of degree five satisfying traditional G^2 Hermite data by a built-in command with and an LA spline in red color with the same G^2 Hermite data .

manufactured based on the CAD model. To note, the roof of the car is designed by a LA spline curve with three segments and its zebra maps indicates the surface is of high quality. Based on our experience on various aesthetic shape design, LA splines can be used to satisfy any G^2 Hermite data.

(a) Iso-lines and zebra mapping (b) Rendering (c) Mock-up

Figure 5: A car model designed by using LA spline and its mock-up.

4 Variational Formulation of Aesthetic Shapes

In this section, we show on approximating aesthetic shapes with regard to the functional which LA curve minimizes [19]. Then, we extend the functional to formulate log-aesthetic surfaces [4].

4.1 Variational Formulation of LA Curves [19]

If we substitute ρ^α with σ in Eq. (8), the equation is then given by

$$\sigma = cs + d \tag{21}$$

The above equation indicates that LA curves are the representation of a straight line in the $s - \sigma$ plane where the horizontal and vertical axes are the arc length s and $\sigma = \rho^{\alpha}$ respectively which connects two given points (s_1, β_1) and (s_2, β_2). In this case, the following objective functional is minimized.

$$J_{LAC} = \int_{s_1}^{s_2} \sqrt{1 + \sigma_s^2} ds = \int_{s_1}^{s_2} \sqrt{1 + \alpha^2 \rho^{2\alpha - 2} \rho_s^2} ds \tag{22}$$

4.2 Variational Formulation of LA Surfaces [4]

Here, we apply the variational principle to the surface formulation. We let the curvature of the curve κ and the arc length s correspond to the Gaussian curvature K and the surface area S, respectively. From Eq. (22), when $\alpha = -1$, $\kappa_s = -\rho_s/\rho^2$ and we obtain the following equation.

$$J_{LAC} = \int_{s_1}^{s_2} \sqrt{1 + \kappa_s^2} ds \tag{23}$$

By reparameterizing the above equation with $s = s(t)$, it becomes

$$J_{LAC} = \int_{t_1}^{t_2} \sqrt{x_t^2 + y_t^2 + \kappa_t^2} dt = \int_{t_1}^{t_2} \sqrt{\lambda_C + \kappa_t^2} dt \tag{24}$$

where $s_1 = s(t_1)$, $s_2 = s(t_2)$, and $\lambda_C = \sqrt{x_t^2 + y_t^2}$. Note that $ds/dt = \lambda_C$.

By extending Eq. (24) into the surface, we define the objective functional for the surface J_{LAS} as follows:

$$J_{LAS} = \int_{u_1}^{u_2} \int_{v_1}^{v2} \sqrt{\det(\boldsymbol{I}) + K_u^2 + K_v^2} du dv \tag{25}$$

where $\boldsymbol{I}$ is a matrix expressed with the first fundamental form by

$$\boldsymbol{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{26}$$

where $E = \boldsymbol{S}_u \cdot \boldsymbol{S}_u$, $F = \boldsymbol{S}_u \cdot \boldsymbol{S}_v$ and $G = \boldsymbol{S}_v \cdot \boldsymbol{S}_v$. Note that the area of the surface S is given by

$$S = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \sqrt{\det(\boldsymbol{I})} du dv \tag{27}$$

We assume local parameterization (s, t) around a point on the surface so that the tangent vectors with respect to the parameters are orthogonal to each other, their directions are the same as the principal direction, and the norm of each tangent vector is equal to 1. With this parameterization, $\boldsymbol{I}$ becomes the 2×2 unit matrix. By performing integration around the point $\boldsymbol{S}(s_1, t_1)$, Eq. (24) can be rewritten as

$$\Delta J_{LAS} = \int_{s_1}^{s_1 + \Delta s} \int_{t_1}^{t_1 + \Delta t} \sqrt{1 + K_s^2 + K_t^2} ds dt \tag{28}$$

According to the general principle of variational principle, to minimize the following functional

$$J = \int_{s_1}^{s_2} \int_{t_1}^{t_2} g(K, K_s, K_t, s, t) ds dt \tag{29}$$

the following equation should be satisfied.

$$\frac{\partial g}{\partial K} - \frac{\partial}{\partial s}\frac{\partial g}{\partial K_s} - \frac{\partial}{\partial t}\frac{\partial g}{\partial K_t} = 0 \tag{30}$$

Note that $g = \sqrt{1 + K_s^2 + K_t^2}$ does not explicitly depend on K. Eq. (30) yields

$$(1 + K_t^2)K_{ss} - 2K_sK_tK_{st} + (1 + K_s^2)K_{tt} = 0 \tag{31}$$

The above equation is called the minimal surface or Lagrange's equation and the surface $\boldsymbol{S}(s,t) = (s, t, K(s,t))$ is given by a minimal surface. Therefore, in a case where the Gaussian curvature on the boundary is specified, the Gaussian curvature should be given by a minimal surface interpolating the boundary values. The above discussion assumes local isometric parameterization whereby global isometric parameterization does not exist in general. It is not possible to deal with the case where the functional is defined globally as in Eq. (25). In such cases, some optimization technique should be adopted to minimize the functional to generate a desired surface.

According to Bernstein's theorem [18], if the boundary of the surface is located infinitely far, the minimal surface is given by a plane. Therefore, the Gaussian curvature is given by

$$K(s,t) = c_0 s + c_1 t + c_2 \tag{32}$$

where c_0, c_1, and c_2 are constants.

For further extension, we may use the mean curvature H instead of the Gaussian curvature K. In this paper we do not elaborate the effects of the power α. To take into account the effects of the power, we may use $\kappa_{max}^{\alpha}\kappa_{min}^{\beta}$ where κ_{max} and κ_{min} are the maximum and minimum normal curvatures, respectively. For example, an objective functional may be defined by

$$J_{LAS} = \int_{u_1}^{u_2}\int_{v_1}^{v_2} \sqrt{\det(\boldsymbol{I}) + \{(\kappa_{max}^{\alpha}\kappa_{min}^{\beta})_u\}^2 + \{(\kappa_{max}^{\alpha}\kappa_{max}^{\beta})_v\}^2} du dv$$

These extensions are proposed as topics for future research.

5 Conclusions

This paper reviews on fair curves and surfaces leading towards the formalization of aesthetic curves and surfaces. LA curves depict promising properties for practical applications and we hope it will be used as one of the standard curves for industrial and graphical design in the near future. Much effort for theoretical as well as practical researches are anticipated in order to define and utilize LA surface for various design feats.

Acknowledgements

This work was supported in part by JSPS Grant-in-Aid for Scientific Research (B) Grant Number 25289021, JSPS Grant-in-Aid for Challenging Exploratory Research Grant Number 26630038, JST RISTEX Service Science, Solutions and Foundation Integrated Research Program and FRGS grant (FRGS: 59265) provided by University Malaysia Terengganu and Ministry of Education Malaysia.

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